

A Note on Hamiltonian Lie Group Actions and Massey Products

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Summary. We show that the property of having only vanishing triple Massey products in equivariant cohomology is inherited by the set of fixed points of hamiltonian circle actions on closed symplectic manifolds. This result can be considered in a more general context of characterizing homotopic properties of Lie group actions. In particular it can be viewed as a partial answer to a question posed by Allday and Puppe about finding conditions ensuring the “formality” of G -actions.

1. Introduction. In this article we prove the following theorem.

THEOREM 1.1. *Let (M, ω) be a closed symplectic manifold endowed with a hamiltonian circle action $S^1 \times M \rightarrow M$. Let F be any connected component of the fixed point set of this action. If there exists a non-vanishing triple Massey product in $H^*(F)$, then the same is valid for the equivariant cohomology $H_{S^1}^*(M)$.*

Obviously, the above theorem can be rephrased as follows. If (M, ω) is a closed symplectic manifold endowed with a hamiltonian circle action $S^1 \times M \rightarrow M$ such that all the triple Massey products in the equivariant cohomology $H_{S^1}^*(M)$ vanish, then any connected component F of the fixed point set M^{S^1} also has only vanishing Massey products in $H^*(F)$.

Thus, the vanishing of the triple Massey products is inherited by the connected components of the fixed point set. This result can be considered in a more general context of the cohomology theory of Lie transformation groups [AP]. Allday and Puppe asked about possible characterizations of “formal” Lie group actions. Recall that a topological space X is called *formal*

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if it has a minimal model \mathcal{M}_X quasi-isomorphic to the cohomology algebra $H^*(\mathcal{M}_X)$ (the latter is isomorphic to the cohomology algebra $H^*(X)$ of X ; we do not give any details here, referring to [AP, TO]). It is natural to ask if one can characterize (in terms of the invariants of the given G -action) the group actions with formal fixed point set X^G . Since the non-vanishing of Massey products is an obstruction to formality [TO] (see also references therein), Theorem 1.1 is a partial answer to the above question. It appears that the weaker property of vanishing of the triple Massey products is indeed inherited by the fixed points of *hamiltonian* circle actions.

On the other hand, this result can be viewed as a characterization of hamiltonian circle actions themselves, and in this respect develops the theory along the lines of Kirwan's result [K] that for a closed (M, ω) endowed with a hamiltonian action of a compact Lie group G the Leray–Serre spectral sequence associated with the Borel fibration

$$M \rightarrow EG \times_G M \rightarrow BG$$

collapses at E_2 . Note that characterizing hamiltonian and non-hamiltonian actions is an important question in symplectic topology (cf. [Au, G, JK, LO, McD, McDS]). For example, it is well known that in dimension 4, a circle action is hamiltonian if and only if it has fixed points [Au, McD]. This fact does not hold in higher dimensions, but the counterexample of McDuff is quite subtle. It is still not known if there exist non-hamiltonian circle actions with *discrete* set of fixed points [G].

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2. Preliminaries and notation

Massey products and the Cartan model. Here we recall briefly the notion of Massey products in the form suitable for our considerations. In this paper all algebras and cohomologies are considered over the reals. To get a more detailed exposition of this topic, one can consult [RT]. Note however, that we use the term “non-vanishing” or “non-zero” Massey product instead of “essential” Massey product in [RT]. Let (A, d) be a commutative differential graded algebra. The cohomology algebra of (A, d) is denoted by $H^*(A)$. If $a \in A$ is a cocycle, we write $[a]$ for the corresponding cohomology class. For a homogeneous element $a \in A$ of degree p we write $\bar{a} = (-1)^p a$. Assume that we are given a triple of cohomology classes $[a], [b], [c]$ such that $[a][b] =$

$[b][c] = 0$. Consider $x \in A$ and $y \in A$ such that $dx = \bar{a}b$ and $dy = \bar{b}c$. One can check that $\bar{a}y + \bar{x}c$ is a cocycle and therefore determines a cohomology class $[\bar{a}y + \bar{x}c] \in H^*(A)$. Note that this class depends on the choice of x and y . By definition the set of all cohomology classes $[\bar{a}y + \bar{x}c]$ is denoted by $\langle [a], [b], [c] \rangle$ and is called the *triple Massey product*.

DEFINITION. We say that the triple Massey product $\langle [a], [b], [c] \rangle$ *does not vanish* if the set of all cohomology classes $[\bar{a}y + \bar{x}c]$ does not contain zero. In the opposite case we say that $\langle [a], [b], [c] \rangle$ vanishes.

One can easily describe the vanishing of the triple Massey product in the following way (see [RT, Prop. 1.5]). Let $\langle [a], [b], [c] \rangle$ be a triple Massey product in $H^*(A)$. Denote by $([a], [c])$ the ideal in $H^*(A)$ generated by $[a]$ and $[c]$. The product $\langle [a], [b], [c] \rangle$ does not vanish if and only if there exists a cohomology class $x \in \langle [a], [b], [c] \rangle$ such that $x \notin ([a], [c])$.

We also need the following formulas (see [RT, Prop. 1.4]):

$$\begin{aligned}
 (2.1) \quad & \xi \langle a_1, a_2, a_3 \rangle \subset \langle \xi a_1, a_2, a_3 \rangle, \\
 & \xi \langle a_1, a_2, a_3 \rangle \subset \langle a_1, \xi a_2, a_3 \rangle, \\
 & \xi \langle a_1, a_2, a_3 \rangle \subset \langle a_1, a_2, \xi a_3 \rangle,
 \end{aligned}$$

which are valid for any $a_1, a_2, a_3 \in H^*(A)$ and for any ξ represented by a central element in A .

Let $f : (A, d) \rightarrow (A', d')$ be a morphism of differential graded algebras. Then

$$(2.2) \quad f^* \langle [a], [b], [c] \rangle \subset \langle f^*[a], f^*[b], f^*[c] \rangle$$

(see Prop. 1.3 in [RT]).

Consider now the case of a G -manifold, i.e. a manifold endowed with a smooth action of a Lie group G . We use the *equivariant* cohomology of the G -manifold, i.e. the cohomology of the total space of the Borel fibration

$$M \rightarrow EG \times_G M \rightarrow BG$$

associated with the universal principal G -bundle $G \rightarrow EG \rightarrow BG$ over the classifying space BG of the Lie group G . Thus, $H_G^*(M) = H^*(EG \times_G M)$ (see [Au, GS]).

In the proof of Theorem 1.1 we will calculate Massey products with respect to the *Cartan model*. Recall that with any G -manifold M one can associate the following differential graded algebra. Let $\Omega^*(M)$ be the de Rham algebra of M , and $S(\mathfrak{g}^*)$ the symmetric algebra over the dual to the Lie algebra \mathfrak{g} of G . Then G acts on \mathfrak{g}^* by the coadjoint representation and hence there is a natural G -action on the tensor product $\Omega^*(M) \otimes S(\mathfrak{g}^*)$. We consider the subalgebra $\Omega_G^*(M) = (\Omega^*(M) \otimes S(\mathfrak{g}^*))^G$ of the fixed points of this action. The details of this construction can be found in [BV, GS, JK, McDS]. We use the fact that there is a natural differential $D : \Omega_G^*(M) \rightarrow$

$\Omega_G^*(M)$ and that

$$H^*(\Omega_G^*(M), D) \cong H_G^*(M).$$

Hamiltonian G -actions. Finally, recall the definition of the hamiltonian action of a Lie group on a symplectic manifold [Au, GS, K, McDS]. Let (M, ω) be a symplectic manifold and let G be a Lie group acting on M by symplectomorphisms. A smooth map $\mu : M \rightarrow \mathfrak{g}^*$ is called a *moment map* of the given G -action if

- (1) μ is G -equivariant with respect to the given G -action on M and with respect to the coadjoint G -action on \mathfrak{g}^* ;
- (2) for any $m \in M$, $X \in \mathfrak{g}$ and $v \in T_m M$ the following equality holds:

$$T_m \mu(v)(X) = \omega(v, \tilde{X}_m)$$

(where \tilde{X} denotes the fundamental vector field on M generated by X).

DEFINITION 2.2. We call a symplectic G -action on (M, ω) *hamiltonian* if it has a moment map.

The moment map need not be unique. It is not difficult to check that any symplectic action of a compact Lie group on a closed symplectic manifold M such that $H_1(M) = 0$ is hamiltonian. On the other hand, in the case of manifolds with large fundamental groups, e.g. aspherical symplectic manifolds, the notion of hamiltonian action is related to interesting and non-trivial topological questions (cf. [LO, McD]).

3. Proof of Theorem 1.1. Consider the triple (M, ω, G) . Recall that if the G -action is symplectic, then the fixed point set M^G is a symplectic submanifold and it is a finite union of connected closed symplectic submanifolds:

$$M^G = \bigcup_{i=1}^p F_i.$$

Choose one of the connected components, say F_1 , and denote it by F . We have a symplectic embedding

$$i_F : F \rightarrow M.$$

Consider F as a symplectic G -manifold with a trivial action of G . In particular, $H_G^*(F) = H^*(F) \otimes S(\mathfrak{g}^*)^G$. In case $G = S^1$ we have

$$(3.1) \quad H_{S^1}^*(F) = H^*(F) \otimes \mathbb{R}[h],$$

where $\mathbb{R}[h]$ denotes the free polynomial algebra with one generator h of degree 2. Passing to the Borel fibrations we obtain the following commutative

diagram:

$$\begin{array}{ccc}
 F & \xrightarrow{i_F} & M \\
 \downarrow & & \downarrow \\
 EG \times_G F & \xrightarrow{(i_F)_G} & EG \times_G M \\
 \downarrow & & \downarrow \\
 BG & \xrightarrow{=} & BG
 \end{array}$$

On the cohomology level we get the maps $(i_F)_G^* : H_G^*(M) \rightarrow H_G^*(F)$ and $i_F^* : H^*(M) \rightarrow H^*(F)$. Let ν denote the normal bundle of the symplectic embedding i_F . Since G acts on this bundle fiberwise, one can define the *equivariant* normal bundle $\nu_G = EG \times_G \nu$. In this way we obtain the vector bundle

$$(3.2) \quad EG \times_G \nu \rightarrow EG \times_G F.$$

In particular, the Euler class of (3.2) is called the *equivariant Euler class* of ν . Throughout the paper, it is denoted by $\chi \in H_G^*(F)$.

The following facts can be found in [McDS, pp. 192–193] or in [JK].

PROPOSITION 3.1. *Let $G = S^1$ act on a closed symplectic manifold (M, ω) and let F denote a connected component of M^G . Then:*

(i) *the normal bundle ν splits into the sum of complex line bundles*

$$\nu = \bigoplus_{j=1}^m L_j, \quad m = \text{codim}_M F,$$

invariant with respect to the S^1 -action, where the circle group acts on each L_j with weight k_j ;

(ii) *the equivariant Euler class has the form*

$$(3.3) \quad \chi = \prod_{j=1}^m (c_1(L_j) + k_j h),$$

where $c_1(L_j)$ denotes the first Chern class of L_j .

Consider now the whole set $M^G = \bigcup_{k=1}^p F_k$ and apply the described constructions to each connected component F_k . In this way one gets the maps $(i_{F_k})_G^*$ for all k , determining the map $\sum_k (i_{F_k})_G^* : H_G^*(M) \rightarrow H_G^*(M^G)$. The following properties of these maps are valid for any *hamiltonian* G -action.

PROPOSITION 3.2 ([Au, p. 139, JK, p. 5]). *Assume that G is a torus and acts on a closed symplectic manifold (M, ω) in a hamiltonian way. Then*

(i) *the map $\sum_{F_k} (i_{F_k})_G^* : H_G^*(M) \rightarrow H_G^*(M^G)$ is injective;*

- (ii) *there exists a linear map (the Gysin homomorphism) $((i_F)_G)_* : H_G^*(F) \rightarrow H_G^*(M)$ with the property*

$$(i_F)_G^*((i_F)_G)_*(x) = \chi x \quad \text{for any } x \in H_G^*(F);$$
- (iii) χ *is not a zero divisor in $H_G^*(F)$.*

The Gysin map was introduced in [Au, JK] in topological terms. We need an alternative way of describing it in the language of the Cartan complex. This was done in [GS, Chapter 10]. However, to use it, we need to prove (ii), since we do not know of any source where it is proved that the definitions of $((i_F)_G)_*$ in [Au, J] and [GS] coincide on the level of equivariant cohomology.

Let τ denote the equivariant Thom form of the equivariant normal bundle [GS]. Recall that $\tau \in \Omega_G^*(\nu)_c$, where $\Omega_G^*(\nu)_c$ denotes the Cartan complex of equivariant differential forms on the normal bundle ν with compact supports. If one identifies ν with the tubular neighbourhood of F in M , one can extend τ onto M by zero. Consider the natural projection $\pi : \nu \rightarrow F$. For any $\theta \in \Omega_G^*(F)$, define $i_*\theta$ as

$$(3.4) \quad i_*\theta = \pi^*\theta \wedge \tau.$$

Extending $i_*\theta$ onto the whole M by zero, and passing to the equivariant cohomology, we get the *Gysin map*

$$((i_F)_G)_* : H_G^*(F) \rightarrow H_G^*(M).$$

Now the only thing we need is to verify (ii). Since in [GS, formula 10.12] it is shown that $(i_F)_G^*\tau = \chi$, the result follows by applying $(i_F)_G^*$ to both sides of (3.4).

The proof of Theorem 1.1 will now follow from the two lemmas below. Note that in both lemmas we keep the same notation and we assume that G is a torus acting on (M, ω) in a hamiltonian way.

LEMMA 3.1. *Let $\langle u, v, w \rangle \subset H_G^*(F)$ be a triple Massey product. Then the following Massey product is defined in $H_G^*(F)$:*

$$(3.5) \quad \langle \chi u, \chi v, \chi w \rangle.$$

If the Massey product (3.5) does not contain zero, then the Massey product

$$(3.6) \quad \langle ((i_F)_G)_*u, ((i_F)_G)_*v, ((i_F)_G)_*w \rangle \subset H_G^*(M)$$

is defined and does not contain zero.

Proof. To avoid clumsy notation let us temporarily denote $((i_F)_G)_*$ as $(i_F)_*$ and $(i_F)_G^*$ as $(i_F)^*$. Now we will need equivariant differential forms representing the equivariant cohomology. Let U_l denote an equivariant tubular neighbourhood of F_l (identified, as usual, with the equivariant normal bundle of the embedding), and let $[\tau]$ denote the equivariant Thom class represented by the equivariant differential form τ with compact support in U_l . Note that from the very definition of the Thom form and the Gysin map, the image

of $(i_{F_l})_*$ is contained in U_l (see (3.4)). This obviously means that for any $x \in H_G^*(F_l)$ its image under the Gysin map in fact lies in the cohomology of U_l with compact supports, and therefore its pullback by $i_{F_k}^*$ must be zero. Finally, we have obtained the equality

$$(3.7) \quad i_{F_k}^*(i_{F_l})_*x = 0 \quad \text{for any } x \in H_G^*(F_l), \quad l \neq k.$$

Now we are ready to show that (3.5) is defined. By (3.7),

$$\sum_{F_k} i_{F_k}^*((i_F)_*u \cdot (i_F)_*v) = i_F^*(i_F)_*u \cdot i_F^*(i_F)_*v = \chi u \cdot \chi v = 0,$$

which obviously implies $(i_F)_*u(i_F)_*v = 0$ because of Proposition 3.2. Obviously, the same could be written for $(i_F)_*v(i_F)_*w$. Note that the Gysin map is not multiplicative, but i_F^* is, and this allows us to complete the proof. The following formulae show that (3.6) does not contain zero:

$$i_F^*\langle (i_F)_*u, (i_F)_*v, (i_F)_*w \rangle \subset \langle i_F^*(i_F)_*u, i_F^*(i_F)_*v, i_F^*(i_F)_*w \rangle = \langle \chi u, \chi v, \chi w \rangle.$$

Here we used (2.2) and Proposition 3.2. ■

LEMMA 3.2. *Assume $G = S^1$ and that $\langle u, v, w \rangle$ is a non-vanishing triple Massey product in $H^*(F) \subset H_{S^1}^*(F) = H^*(F) \otimes \mathbb{R}[h]$. Then*

$$\langle \chi u, \chi v, \chi w \rangle \neq 0.$$

Proof. According to Section 2, we need only prove that there exists $z \in \langle \chi u, \chi v, \chi w \rangle$ such that $z \notin (\chi u, \chi w)$. Use the equivariant cohomology $H_{S^1}^*(F)$ calculated with respect to the Cartan model. Take a non-trivial Massey triple product $\langle u, v, w \rangle$ considered as a non-trivial Massey product in the equivariant cohomology (one can easily check by straightforward calculation that $\langle u, v, w \rangle$ cannot become zero in the tensor product $H^*(F) \otimes \mathbb{R}[h]$ by writing the corresponding cocycles in the Cartan model). From Lemma 3.1, $\langle \chi u, \chi v, \chi w \rangle$ is defined. Since $\langle u, v, w \rangle \neq 0$, there exists $x \in \langle u, v, w \rangle$ such that $x \notin (u, w)$. Note that $\chi^3 x \in \langle \chi u, \chi v, \chi w \rangle$ (by (2.1)). Assume that

$$\langle \chi u, \chi v, \chi w \rangle = 0.$$

This means that any $z \in \langle \chi u, \chi v, \chi w \rangle$ belongs to the ideal $(\chi u, \chi w) \subset H_{S^1}^*(F)$. In particular, $\chi^3 x \in (\chi u, \chi w)$. Hence

$$\chi^3 x = \chi u a + \chi w b, \quad a, b \in H_{S^1}^*(F).$$

Therefore

$$\chi(\chi^2 x - u a - w b) = 0.$$

Recalling that χ is not a zero divisor (Proposition 3.2(iii)) one can write

$$\chi^2 x = u a + w b.$$

Taking into consideration that $u, w \in H^*(F) \subset H_{S^1}^*(F)$ and the expression for the Euler class (3.3) one obtains

$$\prod_{j=1}^m (c_1(L_j) + k_j h)^2 x = u(a_0 + a_1 h + \cdots + a_{2m} h^{2m}) + w(b_0 + b_1 h + \cdots + b_{2m} h^{2m}).$$

Note that h is a free generator, and $k_1 \cdots k_m \neq 0$, because all k_j are the weights of the normal representation of the circle. Using this and comparing the coefficients of h^{2m} on both sides of the latter equation yields $x \in (u, w)$, a contradiction. Finally, $z = \chi^3 x$ is the required element. ■

Now we can complete the proof of Theorem 1.1. If $\langle u, v, w \rangle$ is a non-trivial triple Massey product in $H^*(F)$, then Lemma 3.2 implies that $\langle \chi u, \chi v, \chi w \rangle$ is a non-vanishing triple Massey product in $H_{S^1}^*(F)$. From Lemma 3.1 we get a non-trivial triple Massey product in $H_G^*(M)$ expressed by formula (3.6). ■

REMARK. In [McD] an example of a non-hamiltonian circle action on some closed 6-dimensional manifold was given. This action had a 2-dimensional submanifold of fixed points. Theorem 1.1 may yield another method of constructing non-hamiltonian circle actions with “big” sets of fixed points. Indeed, it would be sufficient to construct a closed symplectic G -manifold M with vanishing triple Massey products but with M^G having a non-vanishing one. However, this does not seem easy.

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