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MEASURE AND INTEGRATION

On Ordinary and Standard Lebesgue Measures on $\mathbb{R}^\infty$

by

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Summary. New concepts of Lebesgue measure on $\mathbb{R}^\infty$ are proposed and some of their realizations in the ZFC theory are given. Also, it is shown that Baker's both measures [1], [2], Mankiewicz and Preiss–Tišer generators [6] and the measure of [4] are not α-standard Lebesgue measures on $\mathbb{R}^\infty$ for $\alpha = (1, 1, \ldots)$.

We discuss the problem of existence of an analog of Lebesgue measure on the vector space $\mathbb{R}^\infty = \prod_{i=1}^\infty \mathbb{R}$ of all real-valued sequences equipped with the Tikhonov topology.

R. Baker [1] introduced the notion of "Lebesgue measure" on $\mathbb{R}^\infty$ as follows: a measure λ which is the completion of a translation-invariant Borel measure on $\mathbb{R}^\infty$ is called a *Lebesgue measure* on $\mathbb{R}^\infty$ if for any measurable rectangle $\prod_{i=1}^\infty (a_i, b_i)$ with $-\infty < a_i < b_i < \infty$ and $0 \le \prod_{i=1}^\infty (b_i - a_i) < \infty$, we have

$$\lambda\Big(\prod_{i=1}^\infty (a_i, b_i)\Big) = \prod_{i=1}^\infty (b_i - a_i),$$

where

$$\prod_{i=1}^\infty (b_i - a_i) := \lim_{n\to\infty} \prod_{i=1}^n (b_i - a_i).$$

Subsequently, Baker [2] extended this notion as follows: a measure λ which is the completion of a translation-invariant Borel measure on $\mathbb{R}^\infty$ is called a *Lebesgue measure* if for any measurable rectangle $\prod_{i=1}^\infty R_i$ with $R_i \in \mathcal{B}(\mathbb{R})$

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and $0 \le \prod_{i=1}^{\infty} m(R_i) < \infty$, we have

$$\lambda\Big(\prod_{i=1}^{\infty} R_i\Big) = \prod_{i=1}^{\infty} m(R_i),$$

where m denotes the linear Lebesgue measure on $\mathbb{R}$.

In [1] and [2] Baker constructed examples of Lebesgue measures in the respective sense.

To propose a new concept of Lebesgue measure on $\mathbb{R}^\infty$ we point out the following two simple facts.

Fact 1. *Let μ be a probability measure defined on a measure space (E, S). Then the product measure $\mu^{\mathbb{N}}$ defined on $(E^{\mathbb{N}}, S^{\mathbb{N}})$ has the following property: if f is any permutation of $\mathbb{N}$ and $A_f((x_k)_{k\in\mathbb{N}}) := (x_{f(k)})_{k\in\mathbb{N}}$ for $(x_k)_{k\in\mathbb{N}} \in E^{\mathbb{N}}$, then $\mu^{\mathbb{N}}(A_f(X)) = \mu^{\mathbb{N}}(X)$ for every $X \in S^{\mathbb{N}}$.*

Fact 2. *The n-dimensional Lebesgue measure ℓ_n on $\mathbb{R}^n$ has the following property: if f is any permutation of $\{1, \ldots, n\}$ and*

$$A_f((x_k)_{1\le k\le n}) = (x_{f(k)})_{1\le k\le n} \qquad ((x_k)_{1\le k\le n} \in \mathbb{R}^n),$$

then $\ell_n(A_f(X)) = \ell_n(X)$ for every $X \in \mathcal{B}(\mathbb{R}^n)$.

In view of these facts we can say that Baker's measures of [1], [2] do not have the essential property of a product measure of being invariant under the group of all canonical permutations (1) of $\mathbb{R}^\infty$.

Indeed, if we consider the infinite-dimensional rectangular set

$$X = \prod_{k=1}^{\infty} [0, e^{(-1)^k/k}],$$

then for every non-zero real number a there exists a permutation f_a of $\mathbb{N}$ such that $\lambda(A_{f_a}(X)) = a$, where λ is any of Baker's measures of [1], [2].

To introduce new concepts of Lebesgue measure on $\mathbb{R}^\infty$, we need some definitions.

Let $(\beta_j)_{j\in\mathbb{N}} \in [0, +\infty]^{\mathbb{N}}$.

Definition 1. We say that $\beta \in [0, +\infty]$ is the *ordinary product* of numbers $(\beta_j)_{j\in\mathbb{N}}$ if

$$\beta = \lim_{n\to\infty} \prod_{i=1}^{n} \beta_i.$$

The ordinary product of $(\beta_j)_{j\in\mathbb{N}}$ is denoted by $(\mathbf{O}) \prod_{i\in\mathbb{N}} \beta_i$.

(1) Let f be any permutation of $\mathbb{N}$. The mapping $A_f : \mathbb{R}^\infty \to \mathbb{R}^\infty$ defined by $A_f((x_k)_{k\in\mathbb{N}}) = (x_{f(k)})_{k\in\mathbb{N}}$ for $(x_k)_{k\in\mathbb{N}} \in \mathbb{R}^\infty$ is called a *canonical permutation* of $\mathbb{R}^\infty$.

DEFINITION 2. The *standard product* of numbers $(\beta_i)_{i\in\mathbb{N}}$ is denoted by $(\mathbf{S})\prod_{i\in\mathbb{N}}\beta_i$ and defined as follows:

$$(\mathbf{S})\prod_{i\in\mathbb{N}}\beta_i=\begin{cases}0 & \text{if } \sum_{i\in\mathbb{N}^-}\ln(\beta_i)=-\infty,\\ & \qquad\qquad \text{where } \mathbb{N}^-=\{i:\ln(\beta_i)<0\}\ (^2),\\ e^{\sum_{i\in\mathbb{N}}\ln(\beta_i)} & \text{if } \sum_{i\in\mathbb{N}^-}\ln(\beta_i)\neq-\infty.\end{cases}$$

Let $\alpha=(n_k)_{k\in\mathbb{N}}\in(\mathbb{N}\setminus\{0\})^{\mathbb{N}}$. We set

$$F_0=[0,n_0]\cap\mathbb{N},\quad F_1=[n_0+1,n_0+n_1]\cap\mathbb{N},\quad \ldots,$$
$$F_k=[n_0+\cdots+n_{k-1}+1,n_0+\cdots+n_k]\cap\mathbb{N},\quad \ldots.$$

DEFINITION 3. We say that $\beta\in[0,+\infty]$ is the *ordinary α-product* of numbers $(\beta_i)_{i\in\mathbb{N}}$ if β is the ordinary product of the numbers $(\prod_{i\in F_k}\beta_i)_{k\in\mathbb{N}}$. The ordinary α-product of $(\beta_i)_{i\in\mathbb{N}}$ is denoted by $(\mathbf{O},\alpha)\prod_{i\in\mathbb{N}}\beta_i$.

DEFINITION 4. We say that $\beta\in[0,+\infty]$ is the *standard α-product* of $(\prod_{i\in F_k}\beta_i)_{k\in\mathbb{N}}$ if β is the standard product of $(\prod_{i\in F_k}\beta_i)_{k\in\mathbb{N}}$. The standard α-product of $(\beta_i)_{i\in\mathbb{N}}$ is denoted $(\mathbf{S},\alpha)\prod_{i\in\mathbb{N}}\beta_i$.

DEFINITION 5. Let $\alpha=(n_k)_{k\in\mathbb{N}}\in(\mathbb{N}\setminus\{0\})^{\mathbb{N}}$. Let $(\alpha)\mathcal{OR}$ be the class of all infinite-dimensional measurable rectangles $R=\prod_{i\in\mathbb{N}}R_i$ $(R_i\in\mathcal{B}(\mathbb{R}^{n_i}))$ for which the ordinary α-product of $(m^{n_i}(R_i))_{i\in\mathbb{N}}$ exists and is finite.

We say that a measure λ which is the completion of a translation-invariant Borel measure is an *ordinary α-Lebesgue measure* (or, briefly, $\lambda\in$ O(α)LM) if for every $R\in(\alpha)\mathcal{OR}$ we have

$$\lambda(R)=(\mathbf{O})\prod_{k\in\mathbb{N}}m^{n_k}(R_k).$$

DEFINITION 6. Let $\alpha=(n_k)_{k\in\mathbb{N}}\in(\mathbb{N}\setminus\{0\})^{\mathbb{N}}$. Let $(\alpha)\mathcal{SR}$ be the class of all infinite-dimensional measurable rectangles $R=\prod_{i\in\mathbb{N}}R_i$ $(R_i\in\mathcal{B}(\mathbb{R}^{n_i}))$ for which the standard α-product of $(m^{n_i}(R_i))_{i\in\mathbb{N}}$ exists and is finite.

We say that a measure λ which is the completion of a translation-invariant Borel measure is a *standard α-Lebesgue measure* on $\mathbb{R}^\infty$ (or, briefly, $\lambda\in$ S(α)LM) if for every $R\in(\alpha)\mathcal{SR}$ we have

$$\lambda(R)=(\mathbf{S})\prod_{k\in\mathbb{N}}m^{n_k}(R_k).$$

PROPOSITION 1. *For every* $\alpha=(n_k)_{k\in\mathbb{N}}\in(\mathbb{N}\setminus\{0\})^{\mathbb{N}}$ *we have the strict inclusion*

$$(\alpha)\mathcal{OR}\subset(\alpha)\mathcal{SR}.$$

(^{2}) We set $\ln(0)=-\infty$.

Proof. Suppose that $R = \prod_{i\in\mathbb{N}} R_i \in (\alpha)\mathcal{OR}$. This means that

$$0 \le \lim_{n\to\infty} \prod_{k=1}^{n} m^{n_k}(R_k) < \infty.$$

Three cases are possible:

(1) $\sum_{i=1}^{\infty} \ln(m^{n_k}(R_k))$ is convergent to $-\infty$;
(2) $\sum_{i=1}^{\infty} \ln(m^{n_k}(R_k))$ is conditionally convergent to a finite real number;
(3) $\sum_{i=1}^{\infty} \ln(m^{n_k}(R_k))$ is absolutely convergent to a finite real number.

Conditions (1) and (2) each imply that

$$(\mathbf{S}) \prod_{k\in\mathbb{N}} m^{n_k}(R_k) = 0.$$

Condition (3) implies that

$$0 < (\mathbf{S}) \prod_{k\in\mathbb{N}} m^{n_k}(R_k) < \infty. \ \blacksquare$$

The main purpose of the present paper is to give a new construction of translation-invariant Borel measures on $\mathbb{R}^\infty$ which will be different from the construction of [2] in the sense that it does not apply the metric properties of $\mathbb{R}^\infty$. It will be an adaptation of a construction from general measure theory which will allow us to construct interesting examples of analogs of Lebesgue measure on the entire space.

Let (E, S) be a measurable space and let $\mathcal{R}$ be any subclass of the σ-algebra S. Let $(\mu_B)_{B\in\mathcal{R}}$ be a family of σ-finite measures such that for $B \in \mathcal{R}$ we have $\mathrm{dom}(\mu_B) = S \cap \mathcal{P}(B)$, where $\mathcal{P}(B)$ denotes the power set of B.

Definition 7. The family $(\mu_B)_{B\in\mathcal{R}}$ is called *consistent* if

$$(\forall X)(\forall B_1, B_2)(X \in S \ \&\ B_1, B_2 \in \mathcal{R} \to \mu_{B_1}(X \cap B_1 \cap B_2) = \mu_{B_2}(X \cap B_1 \cap B_2)).$$

The following assertion plays a key role in our investigations.

Lemma 1. *Let $(\mu_B)_{B\in\mathcal{R}}$ be a consistent family of σ-finite measures. Then there exists a measure $\mu_{\mathcal{R}}$ on (E, S) such that*

(i) *$\mu_{\mathcal{R}}(B) = \mu_B(B)$ for every $B \in \mathcal{R}$;*
(ii) *if there exists an uncountable family of pairwise disjoint sets $\{B_i : i \in I\} \subseteq \mathcal{R}$ such that $0 < \mu_{B_i}(B_i) < \infty$, then the measure $\mu_{\mathcal{R}}$ is non-σ-finite;*
(iii) *if G is a group of measurable transformations of E such that $G(\mathcal{R}) = \mathcal{R}$ and*

$$(\forall B)(\forall X)(\forall g)((B \in \mathcal{R} \ \&\ X \in S \cap \mathcal{P}(B) \ \&\ g \in G) \to \mu_{g(B)}(g(X)) = \mu_B(X)),$$

then the measure $\mu_{\mathcal{R}}$ is G-invariant.

Proof. If $X \in S$ is covered by a countable family $(A_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{R}$, then we put

$$\mu_{\mathcal{R}}(X) = \sum_{n\in\mathbb{N}} \mu_{A_n}\Big(\Big(A_n \setminus \bigcup_{k=1}^{n-1} A_k\Big) \cap X\Big).$$

We set $\mu_{\mathcal{R}}(X) = +\infty$ if X is not covered by any countable family of elements of $\mathcal{R}$.

Let us show the correctness of the definition of the functional $\mu_{\mathcal{R}}$.

If X is not covered by any countable family of elements of $\mathcal{R}$, then the correctness is obvious.

Now let X be covered by two countable families $(A_n)_{n\in\mathbb{N}}, (B_n)_{n\in\mathbb{N}} \subset \mathcal{R}$. We have to show that

$$\sum_{n\in\mathbb{N}} \mu_{A_n}\Big(\Big(A_n \setminus \bigcup_{k=1}^{n-1} A_k\Big) \cap X\Big) = \sum_{n\in\mathbb{N}} \mu_{B_n}\Big(\Big(B_n \setminus \bigcup_{k=1}^{n-1} B_k\Big) \cap X\Big).$$

Indeed, we have

$$\begin{aligned}
&\sum_{n\in\mathbb{N}} \mu_{A_n}\Big(\Big(A_n \setminus \bigcup_{k=1}^{n-1} A_k\Big) \cap X\Big) \\
&\qquad = \sum_{n\in\mathbb{N}} \mu_{A_n}\Big(\Big(A_n \setminus \bigcup_{k=1}^{n-1} A_k\Big) \cap \Big(\bigcup_{m\in\mathbb{N}}\Big(B_m \setminus \bigcup_{l=1}^{m-1} B_l\Big)\Big) \cap X\Big) \\
&\qquad = \sum_{n\in\mathbb{N}} \mu_{A_n}\Big(\bigcup_{m\in\mathbb{N}}\Big(\Big(A_n \setminus \bigcup_{k=1}^{n-1} A_k\Big) \cap \Big(B_m \setminus \bigcup_{l=1}^{m-1} B_l\Big)\Big) \cap X\Big) \\
&\qquad = \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} \mu_{A_n}\Big(\Big(A_n \setminus \bigcup_{k=1}^{n-1} A_k\Big) \cap \Big(B_m \setminus \bigcup_{l=1}^{m-1} B_l\Big) \cap X\Big) \\
&\qquad = \sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{N}} \mu_{A_n}\Big(\Big(A_n \setminus \bigcup_{k=1}^{n-1} A_k\Big) \cap \Big(B_m \setminus \bigcup_{l=1}^{m-1} B_l\Big) \cap X\Big) \\
&\qquad = \sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{N}} \mu_{B_m}\Big(\Big(A_n \setminus \bigcup_{k=1}^{n-1} A_k\Big) \cap \Big(B_m \setminus \bigcup_{l=1}^{m-1} B_l\Big) \cap X\Big) \\
&\qquad = \sum_{m\in\mathbb{N}} \mu_{B_m}\Big(\bigcup_{n\in\mathbb{N}}\Big(A_n \setminus \bigcup_{k=1}^{n-1} A_k\Big) \cap \Big(B_m \setminus \bigcup_{l=1}^{m-1} B_l\Big) \cap X\Big) \\
&\qquad = \sum_{m\in\mathbb{N}} \mu_{B_m}\Big(\Big(B_m \setminus \bigcup_{l=1}^{m-1} B_l\Big) \cap X\Big).
\end{aligned}$$

Thus the correctness is proved.

Let us prove that the functional $\mu_{\mathcal{R}}$ is σ-additive.

Let $(X_k)_{k\in\mathbb{N}}$ be a countable family of pairwise disjoint elements of S.

Case I. Each X_k is covered by a countable family of elements of $\mathcal{R}$. Then so will be their union. Let $(A_m)_{m\in\mathbb{N}}$ be a family of elements of $\mathcal{R}$ that covers $\bigcup_{k\in\mathbb{N}} X_k$. We have

$$\begin{aligned}\mu_{\mathcal{R}}\Big(\bigcup_{k\in\mathbb{N}} X_k\Big) &= \sum_{n\in\mathbb{N}} \mu_{A_n}\Big(\Big(A_n \setminus \bigcup_{k=1}^{n-1} A_k\Big) \cap \Big(\bigcup_{k\in\mathbb{N}} X_k\Big)\Big)\\ &= \sum_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}} \mu_{A_n}\Big(\Big(A_n \setminus \bigcup_{k=1}^{n-1} A_k\Big) \cap X_k\Big)\\ &= \sum_{k\in\mathbb{N}}\sum_{n\in\mathbb{N}} \mu_{A_n}\Big(\Big(A_n \setminus \bigcup_{k=1}^{n-1} A_k\Big) \cap X_k\Big) = \sum_{k\in\mathbb{N}} \mu_{\mathcal{R}}(X_k).\end{aligned}$$

Case II. Let us assume that not every element of the family $(X_k)_{k\in\mathbb{N}}$ is covered by a countable family of elements of $\mathcal{R}$. Then neither will be their union and we get

$$\mu_{\mathcal{R}}\Big(\bigcup_{k\in\mathbb{N}} X_k\Big) = +\infty = \sum_{k\in\mathbb{N}} \mu_{\mathcal{R}}(X_k).$$

Proof of (i). We set $A_k = B$ for $k \in \mathbb{N}$. Then the family $(A_k)_{k\in\mathbb{N}}$ covers B and by the definition of $\mu_{\mathcal{R}}$ we have

$$\mu_{\mathcal{R}}(B) = \mu_B(B) + \mu_B((B \setminus B) \cap B) + \cdots = \mu_B(B).$$

The proof of (ii) is obvious and we omit it.

Proof of (iii). Let G be a group of measurable transformations of E such that $G(\mathcal{R}) = \mathcal{R}$ and

$$(\forall B)(\forall X)(\forall g)((B \in \mathcal{R} \ \&\ X \in B \cap S \ \&\ g \in G) \to \mu_{g(B)}(g(X)) = \mu_B(X)).$$

We are to show that the measure $\mu_{\mathcal{R}}$ is G-invariant.

Let $X \in S$ be covered by a countable family $(A_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{R}$. Then $g(X)$ will be covered by $(g(A_n))_{n\in\mathbb{N}}$, which is a countable family of elements of $\mathcal{R}$.

We have

$$\begin{aligned}\mu_{\mathcal{R}}(g(X)) &= \sum_{n\in\mathbb{N}} \mu_{g(A_n)}\Big(\Big(g(A_n) \setminus \bigcup_{k=1}^{n-1} g(A_k)\Big) \cap g(X)\Big)\\ &= \sum_{n\in\mathbb{N}} \mu_{g(A_n)}\Big(g\Big(\Big(A_n \setminus \bigcup_{k=1}^{n-1} A_k\Big) \cap X\Big)\Big)\\ &= \sum_{n\in\mathbb{N}} \mu_{A_n}\Big(\Big(A_n \setminus \bigcup_{k=1}^{n-1} A_k\Big) \cap X\Big) = \mu_{\mathcal{R}}(X).\end{aligned}$$

If X is not covered by any countable family of elements of $\mathcal{R}$, then the same is true for $g(X)$ and we get

$$\mu_{\mathcal{R}}(g(X)) = \mu_{\mathcal{R}}(X) = +\infty. \ \blacksquare$$

LEMMA 2. *Let* $\alpha = (n_i)_{i\in\mathbb{N}} \in (\mathbb{N}\setminus\{0\})^{\mathbb{N}}$. *Set* $\mathcal{R} = (\alpha)\mathcal{OR}$. *Suppose that* $R = \prod_{i\in\mathbb{N}} R_i \in \mathcal{R}$ *with* $R_i \in \mathcal{B}(\mathbb{R}^{n_i})$ *for* $i \in \mathbb{N}$.

For $X \in \mathcal{B}(R)$, *set* $\mu_R(X) = 0$ *if*

$$(\mathbf{O})\prod_{i\in\mathbb{N}} m^{n_i}(R_i) = 0,$$

and

$$\mu_R(X) = (\mathbf{O})\prod_{i\in\mathbb{N}} m^{n_i}(R_i) \times \left(\prod_{i\in\mathbb{N}} \frac{m^{n_i}{}_{R_i}}{m^{n_i}(R_i)}\right)(X)$$

otherwise, where $\frac{m^{n_i}{}_{R_i}}{m^{n_i}(R_i)}$ *is a Borel probability measure defined on* R_i *as follows:*

$$\frac{m^{n_i}{}_{R_i}}{m^{n_i}(R_i)}(X) = \frac{m^{n_i}(Y\cap R_i)}{m^{n_i}(R_i)} \quad \text{for } X \in \mathcal{B}(R_i).$$

Then the family $(\mu_R)_{R\in\mathcal{R}}$ *of measures is consistent.*

Proof. Let $R_1 = \prod_{i=1}^\infty R_i^{(1)}$ and $R_2 = \prod_{i=1}^\infty R_i^{(2)}$ be two elements of $\mathcal{R}$.

Without loss of generality it can be assumed that $0 < (\mathbf{O})\prod_{i\in\mathbb{N}} m^{n_i}(R_i^{(1)}) < \infty$ and $0 < (\mathbf{O})\prod_{i\in\mathbb{N}} m^{n_i}(R_i^{(2)}) < \infty$.

We will show that $\mu_{R_1}(X) = \mu_{R_2}(X)$ for every $X \in \mathcal{B}(R_1\cap R_2)$. In this case it is sufficient to show that $\mu_{R_1}(Y) = \mu_{R_2}(Y)$ for every elementary measurable rectangle $Y = \prod_{i=1}^\infty Y_i$ in $R_1 \cap R_2$. Note that by an elementary measurable rectangle $Y = \prod_{i=1}^\infty Y_i$ in $R_1\cap R_2$ we mean a subset of $R_1 \cap R_2$ such that $Y_i \in \mathcal{B}(R_i^{(1)} \cap R_i^{(2)})$ for every $i \in \mathbb{N}$ and, in addition, there exists a natural number n such that $Y_i = R_i^{(1)} \cap R_i^{(2)}$ for $i \geq n$.

For every $i \in \mathbb{N}$ and every $Y_i \in \mathcal{B}(R_i^{(1)} \cap R_i^{(2)})$ we have

$$m^{n_i}(Y_i \cap R_i^{(1)} \cap R_i^{(2)}) = m^{n_i}(Y_i \cap R_i^{(1)}) = m^{n_i}(Y_i \cap R_i^{(2)}).$$

This implies that

$$\begin{aligned}(\mathbf{O})\prod_{i=1}^\infty m^{n_i}(Y_i\cap R_i^{(1)}\cap R_i^{(1)}) &= \lim_{n\to\infty}\prod_{i=1}^n m^{n_i}(Y_i\cap R_i^{(1)}\cap R_i^{(1)}) \\ &= \lim_{n\to\infty}\prod_{i=1}^n m^{n_i}(Y_i\cap R_i^{(1)}) = (\mathbf{O})\prod_{i\in\mathbb{N}} m^{n_i}(Y_i\cap R_i^{(1)}).\end{aligned}$$

Analogously, we have

$$(\mathbf{O})\prod_{i\in\mathbb{N}} m^{n_i}(Y_i\cap R_i^{(1)}\cap R_i^{(1)}) = \lim_{n\to\infty}\prod_{i=1}^{n} m^{n_i}(Y_i\cap R_i^{(1)}\cap R_i^{(1)})$$
$$= \lim_{n\to\infty}\prod_{i=1}^{n} m^{n_i}(Y_i\cap R_i^{(2)}) = (\mathbf{O})\prod_{i\in\mathbb{N}} m^{n_i}(Y_i\cap R_i^{(2)}).$$

Hence we get

$$\mu_{R_1}\Big(\prod_{i=1}^{\infty} Y_i\Big) = (\mathbf{O})\prod_{i\in\mathbb{N}} m^{n_i}(Y_i\cap R_i^{(1)}) = (\mathbf{O})\prod_{i\in\mathbb{N}} m^{n_i}(Y_i\cap R_i^{(1)}\cap R_i^{(1)})$$
$$= (\mathbf{O})\prod_{i\in\mathbb{N}} m^{n_i}(Y_i\cap R_i^{(2)}) = \mu_{R_2}\Big(\prod_{i=1}^{\infty} Y_i\Big).$$

Since the class $\mathcal{A}(R_1\cap R_2)$ of all finite disjoint unions of elementary measurable rectangles in $R_1\cap R_2$ is a ring, and since, by definition, the class $\mathcal{B}(R_1\cap R_2)$ of Borel measurable sets of $R_1\cap R_2$ is the minimal σ-ring generated by $\mathcal{A}(R_1\cap R_2)$, we claim (cf. [7, Theorem B, p. 27]) that the class of all sets in $R_1\cap R_2$ for which this equality holds coincides with $\mathcal{B}(R_1\cap R_2)$.

The consistency of the family $(\mu_R)_{R\in\mathcal{R}}$ of measures is proved. ∎

LEMMA 3. *Let $\alpha = (n_i)_{i\in\mathbb{N}} \in (\mathbb{N}\setminus\{0\})^{\mathbb{N}}$. Set $\mathcal{R} = (\alpha)\mathcal{SR}$. Suppose that $R = \prod_{i\in\mathbb{N}} R_i \in \mathcal{R}$ with $R_i\in\mathcal{B}(\mathbb{R}^{n_i})$ for $i\in\mathbb{N}$ and $R\in(\alpha)\mathcal{SR}$. For $X\in\mathcal{B}(R)$, set $\mu_R(X)=0$ if*

$$(\mathbf{S})\prod_{i\in\mathbb{N}} m^{n_i}(R_i) = 0,$$

and

$$\mu_R(X) = (\mathbf{S})\prod_{i\in\mathbb{N}} m^{n_i}(R_i)\times\Big(\prod_{i\in\mathbb{N}}\frac{m^{n_i}{}_{R_i}}{m^{n_i}(R_i)}\Big)(X)$$

otherwise, where $\frac{m^{n_i}{}_{R_i}}{m^{n_i}(R_i)}$ is the Borel probability measure defined on R_i as in Lemma 2. Then the family $(\mu_R)_{R\in\mathcal{R}}$ of measures is consistent.

The proof of Lemma 3 can be obtained by the scheme applied in the proof of Lemma 2.

Let us consider some corollaries of Lemmas 1–3.

THEOREM 1. *For every $\alpha = (n_i)_{i\in\mathbb{N}} \in (\mathbb{N}\setminus\{0\})^{\mathbb{N}}$, there exists a Borel measure μ_α on $\mathbb{R}^\infty$ which is in* $\mathrm{O}(\alpha)\mathrm{LM}$.

Proof. By Lemma 2, the class $(\mu_R)_{R\in(\alpha)\mathcal{OR}}$ of measures is consistent. Since the class $(\alpha)\mathcal{OR}$ is translation-invariant and condition (iii) in Lemma 1 is satisfied with respect to the group of all translations of $\mathbb{R}^\infty$, Lemma 1 shows that $\mu_\alpha := \lambda_{(\alpha)\mathcal{OR}} \in \mathrm{O}(\alpha)\mathrm{LM}$. ∎

THEOREM 2. *For every* $\alpha = (n_i)_{i\in\mathbb{N}} \in (\mathbb{N}\setminus\{0\})^{\mathbb{N}}$, *there exists a Borel measure* ν_α *on* $\mathbb{R}^\infty$ *which is in* S(α)LM.

Proof. By Lemma 3, the class $(\mu_R)_{R\in(\alpha)\mathcal{SR}}$ of measures is consistent. Since the class $(\alpha)\mathcal{SR}$ is translation-invariant and condition (iii) in Lemma 1 is satisfied with respect to the group of all translations of $\mathbb{R}^\infty$, by Lemma 1 we conclude that $\nu_\alpha := \lambda_{(\alpha)\mathcal{SR}} \in$ S(α)LM. ∎

Let μ_1 and μ_2 be two measures defined on a measurable space $(\mathbb{E}, \mathbb{S})$.

DEFINITION 8 ([4, p. 124]). We say that μ_1 is *absolutely continuous* with respect to μ_2, in symbols $\mu_1 \ll \mu_2$, if

$$(\forall X)(X \in \mathbb{S} \;\&\; \mu_2(X) = 0 \to \mu_1(X) = 0).$$

DEFINITION 9 ([4, p. 126]). Two measures μ_1 and μ_2 for which both $\mu_1 \ll \mu_2$ and $\mu_2 \ll \mu_1$ are called *equivalent*, in symbols $\mu_1 \equiv \mu_2$.

We have the following assertion.

THEOREM 3. *For every* $\alpha = (n_i)_{i\in\mathbb{N}} \in (\mathbb{N}\setminus\{0\})^{\mathbb{N}}$, *we have* $\nu_\alpha \ll \mu_\alpha$ *and the measures* ν_α *and* μ_α *are not equivalent.*

Proof. Suppose that $\mu_\alpha(D) = 0$ for some $D \in \mathcal{B}(\mathbb{R}^\infty)$. This means that D is covered by a countable family $(D_k)_{k\in\mathbb{N}}$ of elements of $(\alpha)\mathcal{OR}$ such that $D_k = \prod_{i\in\mathbb{N}} D_i^{(k)}, D_i^{(k)} \in \mathcal{B}(\mathbb{R}^{n_i})$ $(k, i \in \mathbb{N})$ and $\mu_{D_k}(D \cap D_k) = 0$ for each k.

We have to show that $\nu_\alpha(D) < \epsilon$ for all $\epsilon > 0$.

If $\mu_{D_k}(D_k) = 0$, then it is obvious that $\mu_{D_k}(D \cap D_k) = 0 < \epsilon/2^{k+1}$.

Now assume $\mu_{D_k}(D_k) > 0$. We have $\mu_{D_k}(D\cap D_k) = 0$. By Carathéodory's well known theorem there exists a sequence $(A_s^{(k,\epsilon)})_{s\in\mathbb{N}} = (\prod_{i\in\mathbb{N}} A_i^{(s)})_{s\in\mathbb{N}}$ of elementary measurable rectangles in D_k for which $A_i^{(s)} \in R^{n_i}$ for $s, i \in \mathbb{N}$, $D \cap D_k \subseteq \bigcup_{s\in\mathbb{N}} A_s^{(k,\epsilon)}$ and

$$\sum_{s\in\mathbb{N}} \mu_{D_k}\Big(\prod_{i\in\mathbb{N}} A_i^{(s)}\Big) < \frac{\epsilon}{2^{k+1}}.$$

We set

$$A = \Big\{s : \sum_{i\in\mathbb{N}} \ln(m^{n_i}(A_i^{(s)})) \text{ is not absolutely convergent}\Big\}.$$

Then we get

$$\begin{aligned}
\nu_\alpha(D \cap D_k) &\le \nu_\alpha\Big(\bigcup_{s\in\mathbb{N}} A_s^{(k,\epsilon)}\Big) \le \sum_{s\in\mathbb{N}} \nu_\alpha(A_s^{(k,\epsilon)}) \\
&= \sum_{s\in A} \nu_\alpha(A_s^{(k,\epsilon)}) + \sum_{s\in\mathbb{N}\setminus A} \nu_\alpha(A_s^{(k,\epsilon)}) \\
&= \sum_{s\in A} (\mathbf{S})\prod_{i\in\mathbb{N}} m^{n_i}(A_i^{(s)}) + \sum_{s\in\mathbb{N}\setminus A} (\mathbf{S})\prod_{i\in\mathbb{N}} m^{n_i}(A_i^{(s)})
\end{aligned}$$

$$= 0 + \sum_{s \in \mathbb{N} \setminus A} (\mathbf{0}) \prod_{i \in \mathbb{N}} m^{n_i}(A_i^{(s)})$$
$$= \sum_{s \in \mathbb{N} \setminus A} \mu_\alpha(A_s^{(k,\epsilon)}) \leq \sum_{s \in \mathbb{N}} \mu_\alpha(A_s^{(k,\epsilon)}) \leq \frac{\epsilon}{2^{k+1}}.$$

Finally, we get

$$\nu_\alpha(D) \leq \sum_{k \in \mathbb{N}} \nu_\alpha(D \cap D_k) \leq \sum_{k \in \mathbb{N}} \frac{\epsilon}{2^{k+1}} = \epsilon.$$

The proof of the fact that the measures ν_α and μ_α are not equivalent can be obtained as follows: Let $D = \prod_{i \in \mathbb{N}} D_i$ with $D_i \in \mathcal{B}(\mathbb{R}^{n_i})$ $(i \in \mathbb{N})$ be such that $\mu^{n_0}(D_0) = 1$ and $\mu^{n_i}(D_i) = e^{(-1)^i/i}$ for $i \geq 1$. Then we get

$$\mu_\alpha(D) = (\mathbf{O}) \prod_{i \in \mathbb{N}} m^{n_i}(D_i) = 2$$

and

$$\nu_\alpha(D) = (\mathbf{S}) \prod_{i \in \mathbb{N}} m^{n_i}(D_i) = 0. \ \blacksquare$$

Remark 1. Note that μ_α coincides with Baker's measure of [2] for $\alpha = (1, 1, \ldots)$. By Lemmas 1 and 2 we can get the construction of Baker's measure of [1]. To do this we consider the class $\mathcal{R}_B$ of all measurable rectangles $\prod_{i=1}^{\infty}(a_i, b_i)$ with $-\infty < a_i < b_i < \infty$ and $0 \leq (\mathbf{O}) \prod_{i \in \mathbb{N}}(b_i - a_i) < \infty$. Since $\mathcal{R}_B$ is translation-invariant and the family $(\mu_R)_{R \in \mathcal{R}_B}$ of measures is consistent as a subfamily of the consistent family of measures constructed in Lemma 2, we claim that Baker's measure of [1] coincides with $\lambda_{\mathcal{R}_B}$. Note also that for every $\beta = (m_i)_{i \in \mathbb{N}} \in (\mathbb{N} \setminus \{0\})^{\mathbb{N}}$, the measure μ_β coincides with the measure of [8, Theorem 2, p. 7].

Definition 10. Let $\alpha = (n_i)_{i \in \mathbb{N}} \in (\mathbb{N} \setminus \{0\})^{\mathbb{N}}$ be such that $n_i = n_j$ for every $i, j \in \mathbb{N}$. We set $F_i = (a_1^{(i)}, \ldots, a_{n_0}^{(i)})$ for every $i \in \mathbb{N}$ (see notations introduced before Definition 3). Let f be any permutation of $\mathbb{N}$ such that for every $i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that $f(F_i) = F_j$. Then the map $A_f : \mathbb{R}^\infty \to \mathbb{R}^\infty$ defined by $A_f((z_k)_{k \in \mathbb{N}}) = (z_{f(k)})_{k \in \mathbb{N}}$ for $(z_k)_{k \in \mathbb{N}} \in \mathbb{R}^\infty$ is called a *canonical α-permutation* of $\mathbb{R}^\infty$.

The group of transformations generated by all α-permutations and shifts of $\mathbb{R}^\infty$ is denoted by $\mathcal{G}_\alpha$.

Corollary 1. *For every $\alpha = (n_i)_{i \in \mathbb{N}} \in (\mathbb{N} \setminus \{0\})^{\mathbb{N}}$ for which $n_i = n_j$ $(i, j \in \mathbb{N})$, the measure ν_α is $\mathcal{G}_\alpha$-invariant.*

One can easily prove the following propositions.

Proposition 2. *For every $\alpha = (n_i)_{i \in \mathbb{N}} \in (\mathbb{N} \setminus \{0\})^{\mathbb{N}}$ there exists $\beta \in (\mathbb{N} \setminus \{0\})^{\mathbb{N}}$ such that μ_α and μ_β are different.*

Proposition 3. *For every* $\alpha = (n_i)_{i\in\mathbb{N}} \in (\mathbb{N}\setminus\{0\})^{\mathbb{N}}$ *there exists* $\beta \in (\mathbb{N}\setminus\{0\})^{\mathbb{N}}$ *such that* ν_α *and* ν_β *are different.*

As a corollary of Propositions 2–3 we get

Corollary 2. *There does not exist a translation-invariant Borel measure* λ *on* $\mathbb{R}^\infty$ *such that* $\lambda(D) = \mu_\alpha(D)$ *for every* $\alpha = (n_i)_{i\in\mathbb{N}} \in (\mathbb{N}\setminus\{0\})^{\mathbb{N}}$ *and every* $D \in \mathcal{B}(\mathbb{R}^\infty)$.

Corollary 3. *There does not exist a translation-invariant Borel measure* λ *on* $\mathbb{R}^\infty$ *such that* $\lambda(D) = \nu_\alpha(D)$ *for every* $\alpha = (n_i)_{i\in\mathbb{N}} \in (\mathbb{N}\setminus\{0\})^{\mathbb{N}}$ *and every* $D \in \mathcal{B}(\mathbb{R}^\infty)$.

Corollary 4. *Set*

$$\mathcal{R} = \{R : R = [0,1]^{\mathbb{N}} + a \text{ for some } a \in \mathbb{R}^\infty\}$$

and

$$\mu_R(X) = \lambda((X-a)\cap[0,1]^{\mathbb{N}})$$

for every $X \in \mathcal{B}(R)$, *where* $\lambda = \mu^{\mathbb{N}}$ *and* μ *is a linear probability Lebesgue measure on* $[0,1]$. *Then the family* $(\mu_R)_{R\in\mathbb{R}}$ *and the class* $\mathcal{R}$, *being invariant under the group* $\mathcal{G}$, *satisfy all conditions of Lemma* 1. *Hence* $\mu_{\mathcal{R}}$ *is a* $\mathcal{G}$-*invariant measure on* $\mathbb{R}^\infty$.

Corollary 5. *Let* $(L_i^{(n)})_{i\in I}$ *be the family of all* n-*dimensional vector subspaces of* $\mathbb{R}^\infty$ *and let* $\ell_n^{(i)}$ *be the* n-*dimensional Lebesgue measure on* L_i. *Set*

$$\mathcal{R} = \{L_i^{(n)} + a : a \in \mathbb{R}^\infty,\ i \in I\}$$

and

$$\mu_{L_i^{(n)}+a}(X) = \ell_n^{(i)}((X-a)\cap L_i^{(n)})$$

for every $X \in \mathcal{B}(\mathbb{R}^\infty)$. *Then the class* $\mathcal{R}$, *the family of measures* $(\mu_R)_{R\in\mathbb{R}}$ *and the group of all translations of* $\mathbb{R}^\infty$ *satisfy all conditions of Lemma* 1. *Hence there exists a translation-invariant Borel measure* $\mu_{\mathcal{R}}$ *such that* $\mu_{\mathcal{R}}(X) = \mu_{L_i^{(n)}+a}(X)$ *for every Borel subset* $X \subset L_i^{(n)} + a$.

Though the next three examples are not the particular realizations of Lemma 1, they are of some interest.

Example 1. The Mankiewicz generator G_M [7] is the usual completion of the functional μ defined by

$$\mu(X) = \sum_{a\in\ell_1^{\perp}} \mu_{[0,1]^{\mathbb{N}}}((X-a)\cap B_{[0,1]^{\mathbb{N}}})$$

for every $X \in \mathcal{B}(\mathbb{R}^\infty)$, where

(i) $\mu_{[0,1]^{\mathbb{N}}}$ denotes Kharazishvili's quasi-generator of shy sets on $\mathbb{R}^\infty$ (see [7]),

(ii) $B_{[0,1]^{\mathbb{N}}} = \bigcup_{n\in\mathbb{N}}(\mathbb{R}^n \times [0,1]^{\mathbb{N}\setminus\{1,\dots,n\}})$,
(iii) $\ell_1^{\perp}$ denotes a linear complement of the vector subspace ℓ_1 in $\mathbb{R}^\infty$.

This measure G_M is $\mathcal{G}$-invariant and has the property that X is a standard cube null set iff X is of G_M-measure zero for every $X \subset \mathbb{R}^\infty$.

The measure described in Corollary 4 is different from the Mankiewicz generator G_M. Indeed, if we consider the set $(2\mathbb{Z})^{\mathbb{N}}$, then we observe that it is not covered by the union of a countable family of elements of the class $\mathcal{R}$, and hence $\mu_{\mathcal{R}}(2\mathbb{Z}^{\mathbb{N}}) = +\infty$ whenever $G_M(2\mathbb{Z}^{\mathbb{N}}) = 0$.

Example 2. Let $(L_i)_{i\in I}$ be the family of all n-dimensional vector subspaces of $\mathbb{R}^\infty$ and let $\ell_n^{(i)}$ be the n-dimensional Lebesgue measure on L_i. For $i \in I$, denote by $L_i^{\perp}$ a linear complement of L_i. Then the functional $G_{P\&T}$ defined by

$$G^{(n)}_{P\&T}(X) = \sum_{i\in I} \sum_{a\in L_i^{\perp}} \ell_n^{(i)}((X-a)\cap L_i)$$

for $X \in \mathcal{B}(\mathbb{R}^\infty)$ is a $\mathcal{G}$-invariant Borel measure and $G_{P\&T}(Y) = 0$ iff Y is n-dimensional null in the sense of [9] for every $Y \subset \mathbb{R}^\infty$ (see [7]).

Note that $G^{(n)}_{P\&T}$ and the measure $\mu_{\mathcal{R}}$ described in Corollary 5 are different. Indeed, for $n > 1$, let S_n be an n-dimensional sphere lying in an $n+1$-dimensional vector subspace of $\mathbb{R}^\infty$. Then $G^{(n)}_{P\&T}(S_n) = 0$, while $\mu_{\mathcal{R}}(S_n) = +\infty$ because it is not covered by a countable family of elements of $\mathcal{R}$.

Remark 2. For a set $\prod_{k\in\mathbb{N}} X_k$, where $X_k = [0,1/2]$ for even k and $X_k = [0,k]$ for odd k, we have

$$+\infty = \lambda\Big(\prod_{n\in\mathbb{N}} X_k\Big) \neq (\mathbf{S}) \prod_{k\in\mathbb{N}} m(X_k) = 0$$

for Baker's measures λ of [1], [2].

For $Y_k = [0,1]$ $(k \in \mathbb{N})$, the condition

$$+\infty = \mu_{\mathcal{R}}\Big(\prod_{n\in\mathbb{N}} Y_k\Big) = G^{(n)}_{P\&T}\Big(\prod_{n\in\mathbb{N}} Y_k\Big) > 1 = (\mathbf{S}) \prod_{k\in\mathbb{N}} m(Y_k)$$

implies that the measures described in Corollary 5 and Example 2 are not α-standard Lebesgue measures for $\alpha = (1,1,\dots)$.

For the Mankiewicz generator G_M described in Example 1 we have

$$G_M\Big(\prod_{k\in\mathbb{N}} X_k\Big) = 0,$$

but for the set $\prod_{k\in\mathbb{N}} Z_k = \prod_{k\in\mathbb{N}}([0,1/2] \cup [1,3/2])$ we get

$$0 = G_M\Big(\prod_{n\in\mathbb{N}} Z_k\Big) \neq (\mathbf{S}) \prod_{k\in\mathbb{N}} m(Z_k) = 1.$$

EXAMPLE 3 ([5]). For $k \in \mathbb{N}$, let S_k be the unit circle in the Euclidean plane $\mathbb{R}^2$. We may identify S_k with the compact group of all rotations of $\mathbb{R}^2$ around the origin. Let $\lambda_\mathbb{N}$ be the probability Haar measure defined on the compact group $\prod_{k\in\mathbb{N}} S_k$. For $k \in \mathbb{N}$, define $f_k(x) = \exp\{2\pi x i\}$ for every $x \in \mathbb{R}$.

For $E \subset \mathbb{R}^\mathbb{N}$ and $g \in \prod_{k\in\mathbb{N}} S_k$, put

$$f_E(g) = \begin{cases} \mathrm{card}((\prod_{k\in\mathbb{N}} f_k)^{-1}(g) \cap E) & \text{if this is finite,} \\ +\infty & \text{otherwise.} \end{cases}$$

In the Solovay model [10], we define the functional $\mu_\mathbb{N}$ by

$$\mu_\mathbb{N}(E) = \int\limits_{\prod_{k\in\mathbb{N}} S_k} f_E(g)\, d\lambda_\mathbb{N}(g) \quad \text{for } E \subseteq \mathbb{R}^\infty.$$

It was established in [5] that $\mu_\mathbb{N}$ is a translation-invariant Borel measure on $\mathbb{R}^\infty$ which takes the value one on the set $[0,1]^\mathbb{N}$.

Let us show that $\mu_\mathbb{N}$ is not an α-standard Lebesgue measure on $\mathbb{R}^\infty$ for $\alpha = (1,1,\ldots)$. Indeed, consider an infinite-dimensional measurable rectangle $R \in \mathcal{B}(\mathbb{R}^\infty)$ of the form

$$R = \prod_{i\in\mathbb{N}} R_i, \quad \text{where} \quad R_i = \bigcup_{k=1}^{i} [k, k+1/i[$$

for every $i \in \mathbb{N}$. It is obvious that $m(R_i) = 1$ for every $i \in \mathbb{N}$, which implies that

$$0 < 1 = (\mathbf{S}) \prod_{k\in\mathbb{N}} m(R_k) < \infty.$$

Note that $f_{\prod_{i\in\mathbb{N}} R_i}(g) = +\infty$ if $g \in \prod_{k\in\mathbb{N}} f_k([0,1/k[)$, and $= 0$ otherwise. Hence

$$\begin{aligned}\mu_\mathbb{N}\Big(\prod_{i\in\mathbb{N}} R_i\Big) &= \int\limits_{\prod_{k\in\mathbb{N}} S_k} f_{\prod_{i\in\mathbb{N}} R_i}(g)\, d\lambda_\mathbb{N}(g)) \\ &= +\infty \times \lambda_\mathbb{N}\Big(\prod_{k\in\mathbb{N}} f_k([0,1/k[)\Big) + 0 \times \lambda_\mathbb{N}\Big(\prod_{k\in\mathbb{N}} S_k \setminus \prod_{k\in\mathbb{N}} f_k([0,1/k[)\Big) \\ &= 0 < 1 = (\mathbf{S}) \prod_{k\in\mathbb{N}} m(R_k).\end{aligned}$$

REMARK 3. Example 3 shows that Conjecture 1 of [8, p. 9] is not valid, i.e. $\mu_\mathbb{N}(D) \neq \nu(D)$ for every $\nu \in \mathrm{O}(\alpha)\mathrm{LM}$ ($\alpha \in (\mathbb{N} \setminus \{0\})^\mathbb{N}$) and every $D \in \mathcal{B}(\mathbb{R}^\infty)$ with $0 \leq \nu(D) < \infty$. Corollary 2 contains a more precise result, in particular, it answers negatively Problem 2 of [8, p. 9].

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References

[1] R. Baker, *"Lebesgue measure" on* $\mathbb{R}^\infty$, Proc. Amer. Math. Soc. 113 (1991), 1023–1029.
[2] —, *"Lebesgue measure" on* $\mathbb{R}^\infty$*. II*, ibid. 132 (2004), 2577–2591.
[3] J. Cichoń, A. Kharazishvili and B. Węglorz, *Subsets of the Real Line*, Wydawnictwo Uniwersytetu Łódzkiego, Łódź, 1995.
[4] P. R. Halmos, *Measure Theory*, Van Nostrand, Princeton, 1950.
[5] G. R. Pantsulaia, *Relations between shy sets and sets of* ν_p*-measure zero in Solovay's model*, Bull. Polish Acad. Sci. Math. 52 (2004), 63–69.
[6] —, *Invariant and Quasiinvariant Measures in Infinite-Dimensional Topological Vector Spaces*, Nova Science, 2007.
[7] —, *On generators of shy sets on Polish topological vector spaces*, New York J. Math. 14 (2008), 235–261.
[8] —, *Change of variable formula for "Lebesgue measures" on* $\mathbb{R}^\mathbb{N}$, J. Math. Sci. Adv. Appl. 2 (2009), 1–12.
[9] D. Preiss and J. Tišer, *Two unexpected examples concerning differentiability of Lipschitz functions on Banach spaces*, in: Geometric Aspects of Functional Analysis (Israel, 1992–1994), Oper. Theory Adv. Appl. 77, Birkhäuser, Basel, 1995, 219–238.
[10] R. M. Solovay, *A model of set theory in which every set of reals is Lebesgue measurable*, Ann. of Math. 92 (1970), 1–56.

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