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FUNCTIONAL ANALYSIS

Truncation and Duality Results for Hopf Image Algebras

by

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Summary. Associated to an Hadamard matrix $H \in M_N(\mathbb{C})$ is the spectral measure $\mu \in \mathcal{P}[0,N]$ of the corresponding Hopf image algebra, $A = C(G)$ with $G \subset S_N^+$. We study a certain family of discrete measures $\mu^r \in \mathcal{P}[0,N]$, coming from the idempotent state theory of G, which converge in Cesàro limit to μ. Our main result is a duality formula of type $\int_0^N (x/N)^p \, d\mu^r(x) = \int_0^N (x/N)^r \, d\nu^p(x)$, where μ^r, ν^r are the truncations of the spectral measures μ, ν associated to H, H^t. We also prove, using these truncations μ^r, ν^r, that for any deformed Fourier matrix $H = F_M \otimes_Q F_N$ we have $\mu = \nu$.

Introduction. A complex Hadamard matrix is a square matrix H in $M_N(\mathbb{C})$ whose entries are on the unit circle, $|H_{ij}| = 1$, and whose rows are pairwise orthogonal. The basic example of such a matrix is the Fourier one, $F_N = (w^{ij})$ with $w = e^{2\pi i/N}$:

$$F_N = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{N-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & w^{N-1} & w^{2(N-1)} & \cdots & w^{(N-1)^2} \end{pmatrix}.$$

In general, the theory of complex Hadamard matrices can be regarded as a "non-standard" branch of discrete Fourier analysis. For a number of potential applications to quantum physics and quantum information theory, see [4], [8], [10].

Each Hadamard matrix $H \in M_N(\mathbb{C})$ is known to produce a subfactor $M \subset R$ of the Murray–von Neumann hyperfinite factor R, having index $[R : M] = N$. The associated planar algebra $P = (P_k)$ has a direct description

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in terms of H, worked out in [7], and a key problem is that of computing the corresponding Poincaré series, given by

$$f(z) = \sum_{k=0}^{\infty} \dim(P_k) z^k.$$

An alternative approach to this question is via quantum groups [11], [12]. The idea is that associated to $H \in M_N(\mathbb{C})$ is a quantum subgroup $G \subset S_N^+$ of Wang's quantum permutation group [9], constructed by using the Hopf image method, developed in [2]. More precisely, $G \subset S_N^+$ appears via a factorization diagram, as follows:

$$\begin{tikzcd} C(S_N^+) \arrow[rr, "\pi"] \arrow[dr] & & M_N(\mathbb{C}) \\ & C(G) \arrow[ur, "\rho"'] & \end{tikzcd}$$

Here the upper arrow is defined by $\pi : u_{ij} \to P_{ij} = \mathrm{Proj}(H_i/H_j)$, where u_{ij} are the standard generators of $C(S_N^+)$, and where $H_1, \ldots, H_N \in \mathbb{T}^N$ are the rows of H. The lower left arrow is by definition transpose to the embedding $G \subset S_N^+$, and the quantum group $G \subset S_N^+$ itself is by definition the minimal one producing such a factorization.

With this notion in hand, the problem is that of computing the spectral measure μ of the main character $\chi : G \to \mathbb{C}$. This is indeed the same problem as above, because by Woronowicz's Tannakian duality [12], f is the Stieltjes transform of μ:

$$f(z) = \int_G \frac{1}{1 - z\chi}.$$

Here and in what follows, we use the integration theory developed in [11].

For a Fourier matrix F_N the associated quantum group $G \subset S_N^+$ is the cyclic group $\mathbb{Z}_N$, and we therefore have $\mu = (1 - 1/N)\delta_0 + (1/N)\delta_N$ in this case. In general, however, the computation of μ is a difficult question (see [3]).

In this paper we discuss a certain truncation procedure for the main spectral measure, coming from the idempotent state theory of the associated quantum group [3], [6]. Consider the following functionals:

$$\int_G^r = (\mathrm{tr} \circ \rho)^{*r}$$

where $*$ is convolution, $\psi * \phi = (\psi \otimes \phi)\Delta$.

The point with these functionals is that, as explained in [3], we have the following Cesàro limiting result, coming from the general results of

Woronowicz [11]:

$$\int_G \varphi = \lim_{k\to\infty} \frac{1}{k} \sum_{r=1}^{k} \int_G^r \varphi.$$

This formula can of course be used to estimate or exactly compute various integrals over G, and doing so will be the main idea in the present paper.

At the level of the main character, we have the following result:

THEOREM A. *The law χ with respect to $\int_G^r$ equals the law of the Gram matrix*

$$X_{i_1\dots i_r, j_1\dots j_r} = \langle \xi_{i_1\dots i_r}, \xi_{j_1\dots j_r}\rangle$$

of the norm one vectors

$$\xi_{i_1\dots i_r} = \frac{1}{\sqrt{N}} \cdot \frac{H_{i_1}}{H_{i_2}} \otimes \dots \otimes \frac{1}{\sqrt{N}} \cdot \frac{H_{i_r}}{H_{i_1}}.$$

Here the law of X is by definition its spectral measure, with respect to the trace.

Observe that as $r \to \infty$, via the above-mentioned Cesàro limiting procedure, we obtain from the laws in Theorem A the spectral measure μ we are interested in.

Our second and main theoretical result is as follows:

THEOREM B. *We have the moment/truncation duality formula*

$$\int_{G_H}^r \left(\frac{\chi}{N}\right)^p = \int_{G_{H^t}}^p \left(\frac{\chi}{N}\right)^r$$

where G_H, G_{H^t} are the quantum groups associated to H, H^t.

This formula, which is quite non-trivial, is probably of interest in connection with the duality between the quantum groups G_H, $G_{\overline{H}}$, G_{H^t}, G_{H^*} studied in [1].

As an illustration for the above methods, we will work out the case of the deformed Fourier matrices, $H = F_N \otimes_Q F_M$, with the following result:

THEOREM C. *For $H = F_N \otimes_Q F_M$ we have the self-duality formula*

$$\int_{G_H} \varphi(\chi) = \int_{G_{H^t}} \varphi(\chi)$$

for any parameter matrix $Q \in M_{M\times N}(\mathbb{T})$.

The paper is organized as follows: Sections 1–2 are preliminary, and in Sections 3–5 we present the truncation procedure and prove Theorems A–C above.

1. Hadamard matrices. A *complex Hadamard matrix* is a matrix $H \in M_N(\mathbb{C})$ whose entries are on the unit circle, and whose rows are pairwise orthogonal. The basic example is the Fourier matrix, $F_N = (w^{ij})$ with $w = e^{2\pi i/N}$. A more general example is the Fourier matrix $F_G = F_{N_1} \otimes \cdots \otimes F_{N_k}$ of any finite abelian group $G = \mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_k}$ (see [8]).

Complex Hadamard matrices are usually regarded modulo equivalence:

Definition 1.1. Two complex Hadamard matrices $H, K \in M_N(\mathbb{C})$ are called *equivalent*, written $H \sim K$, if one can pass from one to the other by permuting rows and columns, or by multiplying rows and columns by numbers in $\mathbb{T}$.

As explained in the introduction, each complex Hadamard matrix produces a subfactor $M \subset R$ of the Murray–von Neumann hyperfinite factor R, having index $[R : M] = N$, which can be understood in terms of quantum groups. Indeed, let a *magic matrix* be any square matrix $u = (u_{ij})$ whose entries are projections ($p = p^2 = p^*$), summing up to 1 along each row and each column. We then have the following key definition, due to Wang [9]:

Definition 1.2. $C(S_N^+)$ is the universal C^*-algebra generated by the entries of an $N \times N$ magic matrix $u = (u_{ij})$, with comultiplication, counit and antipode maps defined on the standard generators by $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$, $\varepsilon(u_{ij}) = \delta_{ij}$ and $S(u_{ij}) = u_{ji}$.

As explained in [9], this algebra satisfies Woronowicz's axioms in [11], and so S_N^+ is a compact quantum group, called the *quantum permutation group*. Since the functions $v_{ij} : S_N \to \mathbb{C}$ given by $v_{ij}(\sigma) = \delta_{i\sigma(j)}$ form a magic matrix, we have a quotient map $C(S_N^+) \to C(S_N)$, which corresponds to an embedding $S_N \subset S_N^+$. This embedding is an isomorphism for $N = 1, 2, 3$, but not for $N \geq 4$, where S_N^+ is not finite (see [9]).

The link with Hadamard matrices comes from:

Definition 1.3. Associated to an Hadamard matrix $H \in M_N(\mathbb{T})$ is the minimal quantum group $G \subset S_N^+$ producing a factorization of type

$$\begin{tikzcd} C(S_N^+) \arrow[rr, "\pi"] \arrow[dr] & & M_N(\mathbb{C}) \\ & C(G) \arrow[ur, "\rho"'] & \end{tikzcd}$$

where $\pi : u_{ij} \to P_{ij} = \mathrm{Proj}(H_i/H_j)$, where $H_1, \ldots, H_N \in \mathbb{T}^N$ are the rows of H.

Here π is indeed well-defined because $P = (P_{ij})$ is magic, which comes from the fact that the rows of H are pairwise orthogonal. The existence and uniqueness of the quantum group $G \subset S_N^+$ as in the statement comes from Hopf algebra theory, by dividing $C(S_N^+)$ by a suitable ideal (see [2]).

At the level of examples, it is known that the Fourier matrix F_G produces the group G itself. In general, the computation of G is a quite difficult problem (see [3]).

At a theoretical level, it is known that the above-mentioned subfactor $M \subset R$ associated to H appears as a fixed point subfactor associated to G (see [1]).

In what follows we will rather use a representation-theoretic formulation of this latter result. Let $u = (u_{ij})$ be the fundamental representation of G.

Definition 1.4. We let $\mu \in \mathcal{P}[0, N]$ be the law of the variable $\chi = \sum_i u_{ii}$ with respect to the Haar integration functional of $C(G)$.

Note that the main character $\chi = \sum_i u_{ii}$ being a sum of N projections, we have the operator-theoretic formula $0 \leq \chi \leq N$, and so $\mathrm{supp}(\mu) \subset [0, N]$, as stated above.

Observe also that the moments of μ are integers, because we have the following computation, based on Woronowicz's general Peter–Weyl type results in [11]:

$$\int_0^N x^k \, d\mu(x) = \int_G \mathrm{Tr}(u)^k = \int_G \mathrm{Tr}(u^{\otimes k}) = \dim(\mathrm{Fix}(u^{\otimes k})).$$

The above moments, or rather the fixed point spaces appearing on the right, can be computed by using the following fundamental result from [2]:

Theorem 1.5. *We have an equality of complex vector spaces*

$$\mathrm{Fix}(u^{\otimes k}) = \mathrm{Fix}(P^{\otimes k})$$

where for $X \in M_N(A)$ *we set* $X^{\otimes k} = (X_{i_1 j_1} \ldots X_{i_k j_k})_{i_1 \ldots i_k, j_1 \ldots j_k}$.

Now back to subfactor problems, it is known from [7] that the planar algebra associated to H is given by $P_k = \mathrm{Fix}(P^{\otimes k})$. Thus, Theorem 1.5 tells us that the Poincaré series $f(z) = \sum_{k=0}^\infty \dim(P_k) z^k$ is nothing but the Stieltjes transform of μ:

$$f(z) = \int_G \frac{1}{1 - z\chi}.$$

Summarizing, modulo some standard correspondences, the main subfactor problem regarding H consists in computing the spectral measure μ in Definition 1.4.

2. Finiteness and duality. In this section we discuss a key issue, namely the formulation of the duality between the quantum permutation groups associated to the matrices $H, \overline{H}, H^t, H^*$. Our claim is that the general scheme for this duality is, roughly speaking, as follows:

$$\begin{tikzcd} H \arrow[r, dash] \arrow[d, dash] & H^t \arrow[d, dash] \\ \overline{H} \arrow[r, dash] & H^* \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} G \arrow[r, dash] \arrow[d, dash] & \widehat{G} \arrow[d, dash] \\ G^\sigma \arrow[r, dash] & \widehat{G}^\sigma \end{tikzcd}$$

More precisely, this scheme fully works when the quantum groups are finite. In the general case the situation is more complicated, as explained in [1].

The results in [1], written some time ago, in the general context of vertex models, and without using the Hopf image formalism of [2], are in fact not very enlightening in the Hadamard matrix case. Below we will present an updated approach.

First, we have:

Proposition 2.1. *The matrices $P=(P_{ij})$ for $H, \overline{H}, H^t, H^*$ are related by*

$$\begin{tikzcd} H \arrow[r, dash] \arrow[d, dash] & H^t \arrow[d, dash] \\ \overline{H} \arrow[r, dash] & H^* \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} (P_{ij})_{kl} \arrow[r, dash] \arrow[d, dash] & (P_{kl})_{ij} \arrow[d, dash] \\ (P_{ji})_{kl} \arrow[r, dash] & (P_{kl})_{ji} \end{tikzcd}$$

In addition, we have $(P_{ij})_{kl} = (P_{ji})_{lk}$.

Proof. The magic matrix associated to H is given by $P_{ij} = \mathrm{Proj}(H_i/H_j)$. Now since $H \to \overline{H}$ transforms $H_i/H_j \to H_j/H_i$, we conclude that the magic matrices $P^H, P^{\overline{H}}$ associated to $H, \overline{H}$ are related by the formula $P^H_{ij} = P^{\overline{H}}_{ji}$, as stated above.

In matrix notation, the formula for the matrix P^H is as follows:

$$(P^H_{ij})_{kl} = \frac{1}{N} \cdot \frac{H_{ik}H_{jl}}{H_{il}H_{jk}}.$$

Now by replacing $H \to H^t$, we obtain

$$(P^{H^t}_{ij})_{kl} = \frac{1}{N} \cdot \frac{H_{ki}H_{lj}}{H_{li}H_{kj}} = (P^H_{kl})_{ij}.$$

Finally, the last assertion is clear from the above formula for P^H. ∎

Let us now compute Hopf images. First, regarding the operation $H \to \overline{H}$, we have:

Proposition 2.2. *The quantum groups associated to $H, \overline{H}$ are related by*

$$G_{\overline{H}} = G^\sigma_H$$

where the Hopf algebra $C(G^\sigma)$ is $C(G)$ with comultiplication $\Sigma\Delta$, where Σ is the flip.

Proof. Our claim is that, starting from a factorization for H as in Definition 1.3 above, we can construct a factorization for $\overline{H}$, as follows:

$$\begin{tikzcd} u_{ij} \arrow[rr] \arrow[dr] & & P_{ij} \\ & v_{ij} \in C(G) \arrow[ur] & \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} u_{ij} \arrow[rr] \arrow[dr] & & P_{ji} \\ & v_{ji} \in C(G^\sigma) \arrow[ur] & \end{tikzcd}$$

Indeed, observe first that since $v_{ij} \in C(G)$ are the coefficients of a corepresentation, so are the elements $v_{ji} \in C(G^\sigma)$. Thus, in order to produce the factorization on the right, it is enough to take the diagram on the left, and compose at top left with the canonical map $C(S_N^+) \to C(S_N^{+\sigma})$ given by $u_{ij} \to u_{ji}$. ∎

Let us now investigate the operation $H \to H^t$. We use the notion of dual of a finite quantum group (see e.g. [11]). The result here is as follows:

THEOREM 2.3. *The quantum groups associated to H, H^t are related by the usual duality,*

$$G_{H^t} = \widehat{G}_H$$

provided that the quantum group G_H is finite.

Proof. Our claim is that, starting from a factorization for H as in Definition 1.3 above, we can construct a factorization for H^t, as follows:

$$\begin{tikzcd} C(S_N^+) \arrow[rr, "\pi_H"] \arrow[dr] & & M_N(\mathbb{C}) \\ & C(G) \arrow[ur, "\rho"'] & \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} C(S_N^+) \arrow[rr, "\pi_{H^t}"] \arrow[dr] & & M_N(\mathbb{C}) \\ & C(G)^* \arrow[ur, "\eta"'] & \end{tikzcd}$$

More precisely, having a factorization as the one on the left, let us set

$$\eta(\varphi) = (\varphi(v_{kl}))_{kl},$$
$$w_{kl}(x) = (\rho(x))_{kl}.$$

Our claim is that η is a representation, w is a corepresentation, and the factorization on the right holds indeed. Let us first check that η is a representation:

$$\begin{aligned} \eta(\varphi\psi) &= (\phi\psi(v_{kl}))_{kl} = ((\varphi \otimes \psi)\Delta(v_{kl}))_{kl} \\ &= \Big(\sum_a \varphi(v_{ka})\psi(v_{al})\Big)_{kl} = \eta(\varphi)\eta(\psi), \\ \eta(\varepsilon) &= (\varepsilon(v_{kl}))_{kl} = (\delta_{kl})_{kl} = 1, \\ \eta(\varphi^*) &= (\varphi^*(v_{kl}))_{kl} = (\overline{\varphi(S(v_{kl}^*))})_{kl} = (\overline{\varphi(v_{lk})})_{kl} = \eta(\varphi)^*. \end{aligned}$$

Let us now check that w is a corepresentation:

$$\begin{aligned}(\Delta w_{kl})(x\otimes y)&=w_{kl}(xy)=\rho(xy)_{kl}=\sum_i\rho(x)_{ki}\rho(y)_{il}\\&=\sum_i w_{ki}(x)w_{il}(y)=\Big(\sum_i w_{ki}\otimes w_{il}\Big)(x\otimes y),\\ \varepsilon(w_{kl})&=w_{kl}(1)=1_{kl}=\delta_{kl}.\end{aligned}$$

We now check that the above diagram commutes on the generators u_{ij}:

$$\eta(w_{ab})=(w_{ab}(v_{kl}))_{kl}=(\rho(v_{kl})_{ab})_{kl}=((P^H_{kl})_{ab})_{kl}=((P^{H^t}_{ab})_{kl})_{kl}=P^{H^t}_{ab}.$$

It remains to prove that w is magic. We have the following formula:

$$\begin{aligned}w_{a_0a_p}(v_{i_1j_1}\dots v_{i_pj_p})&=(\Delta^{(p-1)}w_{a_0a_p})(v_{i_1j_1}\otimes\dots\otimes v_{i_pj_p})\\&=\sum_{a_1\dots a_{p-1}}w_{a_0a_1}(v_{i_1j_1})\dots w_{a_{p-1}a_p}(v_{i_pj_p})\\&=\frac{1}{N^p}\sum_{a_1\dots a_{p-1}}\frac{H_{i_1a_0}H_{j_1a_1}}{H_{i_1a_1}H_{j_1a_0}}\dots\frac{H_{i_pa_{p-1}}H_{j_pa_p}}{H_{i_pa_p}H_{j_pa_{p-1}}}.\end{aligned}$$

In order to check that each w_{ab} is an idempotent, observe that

$$\begin{aligned}&w^2_{a_0a_p}(v_{i_1j_1}\dots v_{i_pj_p})\\&\quad=(w_{a_0a_p}\otimes w_{a_0a_p})\sum_{k_1\dots k_p}v_{i_1k_1}\dots v_{i_pk_p}\otimes v_{k_1j_1}\dots v_{k_pj_p}\\&\quad=\frac{1}{N^{2p}}\sum_{k_1\dots k_p}\sum_{a_1\dots a_{p-1}}\sum_{\alpha_1\dots\alpha_{p-1}}\frac{H_{i_1a_0}H_{k_1a_1}}{H_{i_1a_1}H_{k_1a_0}}\dots\frac{H_{i_pa_{p-1}}H_{k_pa_p}}{H_{i_pa_p}H_{k_pa_{p-1}}}\\&\qquad\qquad\cdot\frac{H_{k_1a_0}H_{j_1\alpha_1}}{H_{k_1\alpha_1}H_{j_1a_0}}\dots\frac{H_{k_p\alpha_{p-1}}H_{j_pa_p}}{H_{k_pa_p}H_{j_p\alpha_{p-1}}}.\end{aligned}$$

The point now is that when summing over k_1 we obtain $N\delta_{a_1\alpha_1}$, then when summing over k_2 we obtain $N\delta_{a_2\alpha_2}$, and so on until we sum over k_{p-1}, where we obtain $N\delta_{a_{p-1}\alpha_{p-1}}$. Thus, after performing all these summations, we are left with

$$\begin{aligned}&w^2_{a_0a_p}(v_{i_1j_1}\dots v_{i_pj_p})\\&\quad=\frac{1}{N^{p+1}}\sum_{k_p}\sum_{a_1\dots a_{p-1}}\frac{H_{i_1a_0}H_{j_1a_1}}{H_{i_1a_1}H_{j_1a_0}}\dots\frac{H_{i_pa_{p-1}}H_{k_pa_p}}{H_{i_pa_p}H_{k_pa_{p-1}}}\cdot\frac{H_{k_pa_{p-1}}H_{j_pa_p}}{H_{k_pa_p}H_{j_pa_{p-1}}}\\&\quad=\frac{1}{N^{p+1}}\sum_{k_p}\sum_{a_1\dots a_{p-1}}\frac{H_{i_1a_0}H_{j_1a_1}}{H_{i_1a_1}H_{j_1a_0}}\dots\frac{H_{i_pa_{p-1}}H_{j_pa_p}}{H_{i_pa_p}H_{j_pa_{p-1}}}\end{aligned}$$

$$= \frac{1}{N^p} \sum_{a_1 \dots a_{p-1}} \frac{H_{i_1 a_0} H_{j_1 a_1}}{H_{i_1 a_1} H_{j_1 a_0}} \cdots \frac{H_{i_p a_{p-1}} H_{j_p a_p}}{H_{i_p a_p} H_{j_p a_{p-1}}}$$
$$= w_{a_0 a_p}(v_{i_1 j_1} \dots v_{i_p j_p}).$$

Regarding the involutivity, the check is simple:

$$\begin{aligned} w^*_{a_0 a_p}(v_{i_1 j_1} \dots v_{i_p j_p}) &= \overline{w_{a_0 a_p}(S(v_{i_p j_p} \dots v_{i_1 j_1}))} \\ &= \overline{w_{a_0 a_p}(v_{j_1 i_1} \dots v_{j_p i_p})} \\ &= w^*_{a_0 a_p}(v_{i_1 j_1} \dots v_{i_p j_p}). \end{aligned}$$

Finally, to check the first "sum 1" condition, observe that

$$\sum_{a_0} w_{a_0 a_p}(v_{i_1 j_1} \dots v_{i_p j_p}) = \frac{1}{N^p} \sum_{a_0 \dots a_{p-1}} \frac{H_{i_1 a_0} H_{j_1 a_1}}{H_{i_1 a_1} H_{j_1 a_0}} \cdots \frac{H_{i_p a_{p-1}} H_{j_p a_p}}{H_{i_p a_p} H_{j_p a_{p-1}}}.$$

The point now is that when summing over a_0 we obtain $N\delta_{i_1 j_1}$, then when summing over a_1 we obtain $N\delta_{i_2 j_2}$, and so on until we sum over a_{p-1}, where we obtain $N\delta_{i_p j_p}$. Thus, after performing all these summations, we are left with

$$\sum_{a_0} w_{a_0 a_p}(v_{i_1 j_1} \dots v_{i_p j_p}) = \delta_{i_1 j_1} \dots \delta_{i_p j_p} = \varepsilon(v_{i_1 j_1} \dots v_{i_p j_p}).$$

The proof of the other "sum 1" condition is similar, and this finishes the proof of Theorem 2.3. ∎

3. The truncation procedure. Let us now go back to the factorization in Definition 1.3. Regarding the Haar functional of the quantum group G, we have the following key result from [3]:

PROPOSITION 3.1. *We have the Cesàro limiting formula*

$$\int_G = \lim_{k \to \infty} \frac{1}{k} \sum_{r=1}^{k} \int_G^r$$

where the functionals on the right are by definition given by $\int_G^r = (\mathrm{tr} \circ \rho)^{*r}$.

The evaluation of the functionals $\int_G^r$ is a linear algebra problem. Several formulations of the problem were proposed in [3], and we will use here the following formula, which appears in [3], but in a somewhat technical form:

PROPOSITION 3.2. *The functionals* $\int_G^r = (\mathrm{tr} \circ \rho)^{*r}$ *are given by*

$$\int_G^r u_{a_1 b_1} \dots u_{a_p b_p} = (T_p^r)_{a_1 \dots a_p, b_1 \dots b_p}$$

where $(T_p)_{i_1 \dots i_p, j_1 \dots j_p} = \mathrm{tr}(P_{i_1 j_1} \dots P_{i_p j_p})$, *with* $P_{ij} = \mathrm{Proj}(H_i/H_j)$.

Proof. With $a_s = i_s^0$ and $b_s = i_s^{r+1}$, we have the following computation:

$$\begin{aligned}\int_G^r u_{a_1b_1}\dots u_{a_pb_p} &= (\mathrm{tr}\circ\rho)^{\otimes r}\Delta^{(r)}(u_{i_1^0 i_1^{r+1}}\dots u_{i_p^0 i_p^{r+1}})\\ &= (\mathrm{tr}\circ\rho)^{\otimes r}\sum_{i_1^1\dots i_p^r} u_{i_1^0 i_1^1}\dots u_{i_p^0 i_p^1}\otimes\dots\otimes u_{i_1^r u_1^{r+1}}\dots u_{i_p^r i_p^{r+1}}\\ &= \mathrm{tr}^{\otimes r}\sum_{i_1^1\dots i_p^r} P_{i_1^0 i_1^1}\dots P_{i_p^0 i_p^1}\otimes\dots\otimes P_{i_1^r i_1^{r+1}}\dots P_{i_p^r i_p^{r+1}}.\end{aligned}$$

On the other hand, we also have the following computation:

$$\begin{aligned}(T_p^r)_{a_1\dots a_p,b_1\dots b_p} &= \sum_{i_1^1\dots i_p^r}(T_p)_{i_1^0\dots i_p^0,i_1^1\dots i_p^1}\dots(T_p)_{i_1^r\dots i_p^r,i_1^{r+1}\dots i_p^{r+1}}\\ &= \sum_{i_1^1\dots i_p^r}\mathrm{tr}(P_{i_1^0 i_1^1}\dots P_{i_p^0 i_p^1})\dots\mathrm{tr}(P_{i_1^r i_1^{r+1}}\dots P_{i_p^r i_p^{r+1}})\\ &= \mathrm{tr}^{\otimes r}\sum_{i_1^1\dots i_p^r} P_{i_1^0 i_1^1}\dots P_{i_p^0 i_p^1}\otimes\dots\otimes P_{i_1^r i_1^{r+1}}\dots P_{i_p^r i_p^{r+1}}.\end{aligned}$$

Thus we have obtained the formula in the statement. ∎

We can now define the truncations of μ, as follows:

PROPOSITION 3.3. *Let μ^r be the law of χ with respect to $\int_G^r = (\mathrm{tr}\circ\rho)^{*r}$.*

(1) *μ^r is a probability measure on $[0,N]$.*
(2) *$\mu = \lim_{k\to\infty} k^{-1}\sum_{r=1}^k \mu^r$.*
(3) *The moments of μ^r are $c_p^r = \mathrm{Tr}(T_p^r)$.*

Proof. (1) The fact that μ^r is indeed a probability measure follows from the fact that the linear form $(\mathrm{tr}\circ\rho)^{*r} : C(G)\to\mathbb{C}$ is a positive unital trace, and the assertion on the support comes from the fact that the main character χ is a sum of N projections.

(2) This follows from Proposition 3.1, i.e. from the main result in [3].

(3) This follows from Proposition 3.2 above, by summing over $a_i = b_i$. ∎

Let us now recall that associated to a complex Hadamard matrix H in $M_N(\mathbb{C})$ is its profile matrix, given by

$$Q_{ab,cd} = \frac{1}{N}\left\langle\frac{H_a}{H_b},\frac{H_c}{H_d}\right\rangle = \frac{1}{N}\sum_i\frac{H_{ia}H_{id}}{H_{ib}H_{ic}}.$$

With this notation, we have the following result:

PROPOSITION 3.4. *The measures μ^r have the following properties:*

(1) *$\mu^0 = \delta_N$.*
(2) *$\mu^1 = \big(1-\frac{1}{N}\big)\delta_0 + \frac{1}{N}\delta_N$.*

(3) $\mu^2 = \mathrm{law}(S)$, *where* $S_{ab,cd} = |Q_{ab,cd}|^2$.
(4) *For a Fourier matrix* F_G *we have* $\mu^1 = \mu^2 = \cdots = \mu$.

Proof. We use the formula $c_p^r = \mathrm{Tr}(T_p^r)$ from Proposition 3.3(3) above.
(1) For $r = 0$ we have $c_p^0 = \mathrm{Tr}(T_p^0) = \mathrm{Tr}(\mathrm{Id}_{N^p}) = N^p$, so $\mu^0 = \delta_N$.
(2) For $r = 1$, if we denote by J the flat matrix $(1/N)_{ij}$, we have indeed

$$\begin{aligned} c_p^1 = \mathrm{Tr}(T_p) &= \sum_{i_1\ldots i_p} \mathrm{tr}(P_{i_1i_1}\ldots P_{i_pi_p}) = \sum_{i_1\ldots i_p} \mathrm{tr}(J^p) \\ &= \sum_{i_1\ldots i_p} \mathrm{tr}(J) = N^{p-1}. \end{aligned}$$

(3) This can be checked directly, and is also a consequence of Theorem 3.5 below.

(4) For a Fourier matrix the representation ρ producing the factorization in Definition 1.3 is faithful, and this gives the result. ■

In the general case, we have the following result:

THEOREM 3.5. *We have* $\mu^r = \mathrm{law}(X)$, *where*

$$X_{a_1\ldots a_r, b_1\ldots b_r} = Q_{a_1b_1,a_2b_2}\ldots Q_{a_rb_r,a_1b_1}$$

where Q *denotes as usual the profile matrix.*

Proof. We compute the moments of μ^r. We first have

$$\begin{aligned} c_p^r = \mathrm{Tr}(T_p^r) &= \sum_{i^1\ldots i^r} (T_p)_{i^1i^2}\ldots(T_p)_{i^ri^1} \\ &= \sum_{i_1^1\ldots i_p^r} (T_p)_{i_1^1\ldots i_p^1, i_1^2\ldots i_p^2}\ldots(T_p)_{i_1^r\ldots i_p^r, i_1^1\ldots i_p^1} \\ &= \sum_{i_1^1\ldots i_p^r} \mathrm{tr}(P_{i_1^1i_1^2}\ldots P_{i_p^1i_p^2})\ldots\mathrm{tr}(P_{i_1^ri_1^1}\ldots P_{i_p^ri_p^1}). \end{aligned}$$

In terms of H, we obtain the following formula:

$$\begin{aligned} c_p^r &= \frac{1}{N^r}\sum_{i_1^1\ldots i_p^r}\sum_{a_1^1\ldots a_p^r} (P_{i_1^1i_1^2})_{a_1^1a_2^1}\ldots(P_{i_p^1i_p^2})_{a_p^1a_1^1}\ldots(P_{i_1^ri_1^1})_{a_1^ra_2^r}\ldots(P_{i_p^ri_p^1})_{a_p^ra_1^r} \\ &= \frac{1}{N^{(p+1)r}}\sum_{i_1^1\ldots i_p^r}\sum_{a_1^1\ldots a_p^r} \frac{H_{i_1^1a_1^1}H_{i_1^2a_2^1}}{H_{i_1^1a_2^1}H_{i_1^2a_1^1}}\ldots\frac{H_{i_p^1a_p^1}H_{i_p^2a_1^1}}{H_{i_p^1a_1^1}H_{i_p^2a_p^1}}\ldots\frac{H_{i_1^ra_1^r}H_{i_1^1a_2^r}}{H_{i_1^ra_2^r}H_{i_1^1a_1^r}} \\ &\qquad \ldots\frac{H_{i_p^ra_p^r}H_{i_p^1a_1^r}}{H_{i_p^ra_1^r}H_{i_p^1a_p^r}}. \end{aligned}$$

Now by changing the order of the summation, we obtain

$$c_p^r = \frac{1}{N^{(p+1)r}} \sum_{a_1^1 \dots a_p^r} \sum_{i_1^1} \frac{H_{i_1^1 a_1^1} H_{i_1^1 a_2^r}}{H_{i_1^1 a_2^1} H_{i_1^1 a_1^r}} \dots \sum_{i_1^r} \frac{H_{i_1^r a_2^{r-1}} H_{i_1^r a_1^r}}{H_{i_1^r a_1^{r-1}} H_{i_1^r a_2^r}} \dots \sum_{i_p^1} \frac{H_{i_p^1 a_p^1} H_{i_p^1 a_1^r}}{H_{i_p^1 a_1^1} H_{i_p^1 a_p^r}} \dots \sum_{i_p^r} \frac{H_{i_p^r a_1^{r-1}} H_{i_p^r a_p^r}}{H_{i_p^r a_p^{r-1}} H_{i_p^r a_1^r}}.$$

In terms of Q, and then of the matrix X in the statement, we get

$$\begin{aligned} c_p^r &= \frac{1}{N^r} \sum_{a_1^1 \dots a_p^r} (Q_{a_1^1 a_2^1, a_1^r a_2^r} \dots Q_{a_1^r a_2^r, a_1^{r-1} a_2^{r-1}}) \dots (Q_{a_p^1 a_1^1, a_p^r a_1^r} \dots Q_{a_p^r a_1^r, a_p^{r-1} a_1^{r-1}}) \\ &= \frac{1}{N^r} \sum_{a_1^1 \dots a_p^r} X_{a_1^1 \dots a_1^r, a_2^1 \dots a_2^r} \dots X_{a_p^1 \dots a_p^r, a_1^1 \dots a_1^r} = \frac{1}{N^r} \operatorname{Tr}(X^p) = \operatorname{tr}(X^p). \end{aligned}$$

But this gives the formula in the statement. ∎

Observe that the above result covers the previous computations of μ^0, μ^1, μ^2, and in particular the formula for μ^2 in Proposition 3.4(3). Indeed, for $r = 2$ we have

$$X_{ab,cd} = Q_{ac,bd} Q_{bd,ac} = Q_{ab,cd} \overline{Q_{ab,cd}} = |Q_{ab,cd}|^2.$$

In the next section we will discuss some further interpretations of μ^r.

4. Basic properties and examples. Let us first take a closer look at the matrices X appearing in Theorem 3.5. These are in fact Gram matrices, of certain norm one vectors:

Proposition 4.1. *We have $\mu^r = \operatorname{law}(X)$, with*

$$X_{a_1 \dots a_r, b_1 \dots b_r} = \langle \xi_{a_1 \dots a_r}, \xi_{b_1 \dots b_r} \rangle,$$

where

$$\xi_{a_1 \dots a_r} = \frac{1}{\sqrt{N}} \cdot \frac{H_{a_1}}{H_{a_2}} \otimes \dots \otimes \frac{1}{\sqrt{N}} \cdot \frac{H_{a_r}}{H_{a_1}}.$$

In addition, the vectors $\xi_{a_1 \dots a_r}$ are all of norm one.

Proof. The first assertion follows from the following computation:

$$\begin{aligned} X_{a_1 \dots a_r, b_1 \dots b_r} &= \frac{1}{N^r} \left\langle \frac{H_{a_1}}{H_{b_1}}, \frac{H_{a_2}}{H_{b_2}} \right\rangle \dots \left\langle \frac{H_{a_r}}{H_{b_r}}, \frac{H_{a_1}}{H_{b_1}} \right\rangle \\ &= \frac{1}{N^r} \left\langle \frac{H_{a_1}}{H_{a_2}}, \frac{H_{b_1}}{H_{b_2}} \right\rangle \dots \left\langle \frac{H_{a_r}}{H_{a_1}}, \frac{H_{b_r}}{H_{b_1}} \right\rangle \\ &= \frac{1}{N^r} \left\langle \frac{H_{a_1}}{H_{a_2}} \otimes \dots \otimes \frac{H_{a_r}}{H_{a_1}}, \frac{H_{b_1}}{H_{b_2}} \otimes \dots \otimes \frac{H_{b_r}}{H_{b_1}} \right\rangle. \end{aligned}$$

The second assertion is clear from the formula for $\xi_{a_1 \dots a_r}$. ∎

At the level of concrete examples, we first have:

PROPOSITION 4.2. *For a Fourier matrix $H = F_G$ we have:*

(1) $Q_{ab,cd} = \delta_{a+d,b+c}$.
(2) $X_{a_1\ldots a_r,b_1\ldots b_r} = \delta_{a_1-b_1,\ldots,a_r-b_r}$.
(3) $X^2 = NX$*, so* X/N *is a projection.*

Proof. We use the formulae $H_{ij}H_{ik} = H_{i,j+k}$, $\overline{H}_{ij} = H_{i,-j}$ and $\sum_i H_{ij} = N\delta_{j0}$.

(1) Indeed,

$$Q_{ab,cd} = \frac{1}{N}\sum_i H_{i,a+d-b-c} = \delta_{a+d,b+c}.$$

(2) This follows from the following computation:

$$\begin{aligned}X_{a_1\ldots a_r,b_1\ldots b_r} &= \delta_{a_1+b_2,b_1+a_2}\ldots\delta_{a_r+b_1,b_r+a_1}\\ &= \delta_{a_1-b_1,a_2-b_2}\ldots\delta_{a_r-b-r,a_1-b_1}\\ &= \delta_{a_1-b_1,\ldots,a_r-b_r}.\end{aligned}$$

(3) By using the formula in (2) above, we obtain

$$\begin{aligned}(X^2)_{a_1\ldots a_r,b_1\ldots b_r} &= \sum_{c_1\ldots c_r} X_{a_1\ldots a_r,c_1\ldots c_r}X_{c_1\ldots c_r,b_1\ldots b_r}\\ &= \sum_{c_1\ldots c_r}\delta_{a_1-c_1,\ldots,a_r-c_r}\delta_{c_1-b_1,\ldots,c_r-b_r}\\ &= N\delta_{a_1-b_1,\ldots,a_r-b_r} = NX_{a_1\ldots a_r,b_1\ldots b_r}.\end{aligned}$$

Thus $(X/N)^2 = X/N$, and since X/N is self-adjoint as well, it is a projection. ∎

Another elementary situation is for the tensor product:

PROPOSITION 4.3. *Let* $L = H \otimes K$*. Then*

(1) $Q^L_{iajb,kcld} = Q^H_{ij,kl}Q^K_{ab,cd}$.
(2) $X^L_{i_1a_1\ldots i_ra_r,j_1b_1\ldots j_rb_r} = X^H_{i_1\ldots i_r,j_1\ldots j_r}X^K_{a_1\ldots a_r,b_1\ldots b_r}$.
(3) $\mu^r_L = \mu^r_H * \mu^r_K$ *for any* $r \geq 0$.

Proof. (1) Indeed,

$$\begin{aligned}Q^L_{iajb,kcld} &= \frac{1}{NM}\sum_{me}\frac{L_{me,ia}L_{me,ld}}{L_{me,kc}L_{me,jb}}\\ &= \frac{1}{NM}\sum_{me}\frac{H_{mi}K_{ea}H_{ml}K_{ld}}{H_{mk}K_{ec}H_{mj}K_{eb}}\\ &= \frac{1}{N}\sum_m\frac{H_{mi}H_{ml}}{H_{mk}H_{mj}}\cdot\frac{1}{M}\sum_e\frac{K_{ea}K_{ed}}{K_{ec}K_{eb}} = Q^H_{ij,kl}Q^K_{ab,cd}.\end{aligned}$$

(2) This follows from (1) above, because

$$\begin{aligned} X^L_{i_1a_1\dots i_ra_r,j_1b_1\dots j_rb_r} &= Q^L_{i_1a_1j_1b_1,1_2a_2j_2b_2}\dots Q^L_{i_ra_rj_rb_r,i_1a_1j_1b_1}\\ &= Q^H_{i_1j_1,i_2j_2}Q^K_{a_1b_1,a_2b_2}\dots Q^H_{i_rj_r,i_1j_1}Q^K_{a_rb_r,a_1b_1}\\ &= X^H_{i_1\dots i_r,j_1\dots j_r}X^K_{a_1\dots a_r,b_1\dots b_r}. \end{aligned}$$

(3) This follows from (2), which tells us that, modulo certain standard identifications, we have $X^L = X^H \otimes X^K$. ∎

We will return to concrete examples in Section 5. Now let us discuss some general duality issues.

Theorem 4.4. *We have the moment/truncation duality formula*

$$\int_{G_H}^r \left(\frac{\chi}{N}\right)^p = \int_{G_{H^t}}^p \left(\frac{\chi}{N}\right)^r$$

where G_H, G_{H^t} are the quantum groups associated to H, H^t.

Proof. We use the following formula from the proof of Theorem 3.5:

$$c_p^r = \frac{1}{N^{(p+1)r}}\sum_{i_1^1\dots i_p^r}\sum_{a_1^1\dots a_p^r}\frac{H_{i_1^1a_1^1}H_{i_1^2a_2^1}}{H_{i_1^1a_2^1}H_{i_1^2a_1^1}}\dots\frac{H_{i_p^1a_p^1}H_{i_p^2a_1^1}}{H_{i_p^1a_1^1}H_{i_p^2a_p^1}}\dots\frac{H_{i_1^ra_1^r}H_{i_1^1a_2^r}}{H_{i_1^ra_2^r}H_{i_1^1a_1^r}}\dots\frac{H_{i_p^ra_p^r}H_{i_p^1a_1^r}}{H_{i_p^ra_1^r}H_{i_p^1a_p^r}}.$$

By interchanging $p \leftrightarrow r$, and by transposing as well all the summation indices according to the rules $i_x^y \to i_y^x$ and $a_x^y \to a_y^x$, we obtain the following formula:

$$c_r^p = \frac{1}{N^{(r+1)p}}\sum_{i_1^1\dots i_p^r}\sum_{a_1^1\dots a_p^r}\frac{H_{i_1^1a_1^1}H_{i_2^1a_1^2}}{H_{i_1^1a_1^2}H_{i_2^1a_1^1}}\dots\frac{H_{i_1^ra_1^r}H_{i_2^ra_1^1}}{H_{i_1^ra_1^1}H_{i_2^ra_1^r}}\dots\frac{H_{i_p^1a_p^1}H_{i_1^1a_p^2}}{H_{i_p^1a_p^2}H_{i_1^1a_p^1}}\dots\frac{H_{i_p^ra_p^r}H_{i_1^ra_p^1}}{H_{i_p^ra_p^1}H_{i_1^ra_p^r}}.$$

Now by interchaging all the summation indices, $i_x^y \leftrightarrow a_x^y$, we obtain

$$c_r^p = \frac{1}{N^{(r+1)p}}\sum_{i_1^1\dots i_p^r}\sum_{a_1^1\dots a_p^r}\frac{H_{a_1^1i_1^1}H_{a_2^1i_1^2}}{H_{a_1^1i_1^2}H_{a_2^1i_1^1}}\dots\frac{H_{a_1^ri_1^r}H_{a_2^ri_1^1}}{H_{a_1^ri_1^1}H_{a_2^ri_1^r}}\dots\frac{H_{a_p^1i_p^1}H_{a_1^1i_p^2}}{H_{a_p^1i_p^2}H_{a_1^1i_p^1}}\dots\frac{H_{a_p^ri_p^r}H_{a_1^ri_p^1}}{H_{a_p^ri_p^1}H_{a_1^ri_p^r}}.$$

With $H\to H^t$, we obtain the following formula, this time for H^t:

$$c_r^p=\frac{1}{N^{(r+1)p}}\sum_{i_1^1\ldots i_p^r}\sum_{a_1^1\ldots a_p^r}\frac{H_{i_1^1a_1^1}H_{i_1^2a_2^1}}{H_{i_1^2a_1^1}H_{i_1^1a_2^1}}\cdots\frac{H_{i_1^ra_1^r}H_{i_1^1a_2^r}}{H_{i_1^1a_1^r}H_{i_1^ra_2^r}}\cdots\frac{H_{i_p^1a_p^1}H_{i_p^2a_1^1}}{H_{i_p^2a_p^1}H_{i_p^1a_1^1}}\cdots\frac{H_{i_p^ra_p^r}H_{i_p^1a_1^r}}{H_{i_p^1a_p^r}H_{i_p^ra_1^r}}.$$

The point now is that, modulo a permutation of terms, the quantity on the right is exactly the one in the above formula for c_p^r. Thus, if we denote this quantity by α, then

$$c_p^r(H)=\frac{\alpha}{N^{(p+1)r}},\qquad c_r^p(H^t)=\frac{\alpha}{N^{(r+1)p}}.$$

Hence $N^rc_p^r(H)=N^pc_r^p(H^t)$, and by dividing by N^{p+r}, we obtain

$$\frac{c_p^r(H)}{N^p}=\frac{c_r^p(H^t)}{N^r}.$$

But this gives the formula in the statement. ∎

The above result shows that the normalized moments $\gamma_p^r=c_p^r/N^p$ are subject to the condition $\gamma_p^r(H)=\gamma_r^p(H^t)$. We have the following table of γ_p^r numbers for H:

$p\setminus r$	1	2	r	∞
1	$1/N$	$1/N$	$1/N$	$1/N$
2	$1/N$	$\mathrm{tr}(S/N)^2$	$\mathrm{tr}(S/N)^r$	c_2
p	$1/N$	$\mathrm{tr}(S/N)^p$	?	c_p
∞	$1/N$	c_2	$\mu^r(1)$	$\mu(1)$

Here we have used the well-known fact that for $\mathrm{supp}(\mu)\subset[0,1]$ we have $c_p\to\mu(1)$, a fact which is clear for discrete measures, and for continuous measures too.

Since the table for H^t is transpose to the table of H, we obtain:

Proposition 4.5. $\mu_H(1)=\mu_{H^t}(1)$.

Proof. This follows indeed from Theorem 4.4 by letting $p,r\to\infty$. ∎

Observe that this result recovers a bit of Theorem 2.3, because we have:

Proposition 4.6. *For $G\subset S_N^+$ finite we have $\mu(1)=1/|G|$.*

Proof. The idea is to use the principal graph. So, let first Γ be an arbitrary finite graph, with a distinguished vertex denoted 1, let $A\in M_M(0,1)$ with $M=|\Gamma|$ be its adjacency matrix, set $N=\|\Gamma\|$, and let $\xi\in\mathbb{R}^M$ be a Perron–Frobenius eigenvector for A, known to be unique up to multiplication by a scalar. Our claim is that

$$\lim_{p\to\infty}\frac{(A^p)_{11}}{N^p}=\frac{\xi_1^2}{\|\xi\|^2}.$$

Indeed, if we choose an orthonormal basis (ξ^i) of eigenvectors, with $\xi^1=\xi/\|\xi\|$, and write $A=UDU^t$ with $U=[\xi^1\ldots\xi^M]$ and D diagonal, then we have, as claimed:

$$(A^p)_{11}=(UD^pU^t)_{11}=\sum_k U_{1k}^2D_{kk}^p\simeq U_{11}^2N^p=\frac{\xi_1^2}{\|\xi\|^2}N^p.$$

Now back to our quantum group $G\subset S_N^+$, let Γ be its principal graph, having as vertices the elements $r\in\mathrm{Irr}(G)$. The moments of μ being the numbers $c_p=(A^p)_{11}$, we have

$$\mu(1)=\lim_{p\to\infty}\frac{c_p}{N^p}=\lim_{p\to\infty}\frac{(A^p)_{11}}{N^p}=\frac{\xi_1^2}{\|\xi\|^2}.$$

On the other hand, it is known that with the normalization $\xi_1=1$, the entries of the Perron–Frobenius eigenvector are simply $\xi_r=\dim(r)$. Thus we have

$$\frac{\xi_1^2}{\|\xi\|^2}=\frac{1}{\sum_r\dim(r)^2}=\frac{1}{|G|}.$$

Together with the above formula for $\mu(1)$, this finishes the proof. ■

5. Deformed Fourier matrices. In this section we study the deformed Fourier matrices, $L=F_M\otimes_Q F_N$, constructed by Diţă [5]. They are defined by $L_{ia,jb}=Q_{ib}(F_M)_{ij}(F_N)_{ab}$.

We first have the following technical result:

PROPOSITION 5.1. *Let $H=F_M\otimes_Q F_N$, and set*

$$R^x_{ab,cd}=\frac{1}{M}\sum_m w^{mx}\frac{Q_{ma}Q_{md}}{Q_{mc}Q_{mb}}.$$

Then:

(1) $Q_{iajb,kcld}=\delta_{a-b,c-d}R^{i+l-k-j}_{ab,cd}$.

(2) $X_{i_1a_1\ldots i_ra_r,j_1b_1\ldots j_rb_r}=\delta_{a_1-b_1,\ldots,a_r-b_r}R^{i_1+j_2-j_1-i_2}_{a_1b_1,a_2b_2}\ldots R^{i_r+j_1-j_r-i_1}_{a_rb_r,a_1b_1}$.

Proof. First, for a general deformation $H = K \otimes_Q L$, we have

$$\begin{aligned} Q_{iajb,kcld} &= \frac{1}{MN} \sum_{me} \frac{H_{me,ia} H_{me,ld}}{H_{me,kc} H_{me,jb}} \\ &= \frac{1}{MN} \sum_{me} \frac{Q_{ma} K_{mi} L_{ea} Q_{md} K_{ml} L_{ld}}{Q_{mc} K_{mk} L_{ec} Q_{mb} K_{mj} L_{eb}} \\ &= \frac{1}{M} \sum_{m} \frac{Q_{ma} Q_{md}}{Q_{mc} Q_{mb}} \cdot \frac{K_{mi} K_{ml}}{K_{mk} K_{mj}} \cdot \frac{1}{N} \sum_{e} \frac{L_{ea} L_{ed}}{L_{ec} L_{eb}}. \end{aligned}$$

Thus for a deformed Fourier matrix $H = F_M \otimes_Q F_N$ we have

$$Q_{iajb,kcld} = \delta_{a+d,b+c} \frac{1}{M} \sum_{m} \frac{Q_{ma} Q_{md}}{Q_{mc} Q_{mb}} w^{m(i+l-k-j)}.$$

But this gives (1), and then (2), and we are done. ∎

With the above formulae in hand, we can now prove:

THEOREM 5.2. *For the matrix $H = F_M \otimes_Q F_N$ we have*

$$\mu_H = \mu_{H^t}$$

for any value of the parameter matrix $Q \in M_{M \times N}(\mathbb{T})$.

Proof. We use the matrices X, R constructed in Proposition 5.1. According to Proposition 5.1(2), we have

$$\begin{aligned} c_p^r &= \frac{1}{N^r} \sum_{a_1^1 \ldots a_p^r} X_{a_1^1 \ldots a_1^r, a_2^1 \ldots a_2^r} \ldots X_{a_p^1 \ldots a_p^r, a_1^1 \ldots a_1^r} \\ &= \frac{1}{N^r} \sum_{a_1^1 \ldots a_p^r} \sum_{i_1^1 \ldots i_p^r} \delta_{a_1^1 - a_2^1, \ldots, a_1^r - a_2^r} R^{i_1^1 + i_2^2 - i_1^2 - i_2^1}_{a_1^1 a_2^1, a_1^2 a_2^2} \ldots R^{i_1^r + i_2^1 - i_1^1 - i_2^r}_{a_1^r a_2^r, a_1^1 a_2^1} \\ &\qquad \ldots \delta_{a_p^1 - a_1^1, \ldots, a_p^r - a_1^r} R^{i_p^1 + i_1^2 - i_1^1 - i_p^2}_{a_p^1 a_1^1, a_p^2 a_1^2} \ldots R^{i_p^r + i_1^1 - i_p^1 - i_1^r}_{a_p^r a_1^r, a_p^1 a_1^1}. \end{aligned}$$

Observe that the conditions on the a indices, coming from the Kronecker symbols, state that the columns of $a = (a_i^j)$ must differ by vertical vectors of type $(s, \ldots, s)$.

Now let us compute the sum over the i indices, obtained by neglecting the Kronecker symbols. According to the formula for $R^x_{ab,cd}$ in Proposition 5.1, this is

$$\begin{aligned} S = \frac{1}{N^{pr}} \sum_{i_1^1 \ldots i_p^r} \sum_{m_1^1 \ldots m_p^r} w^{E(i,m)} \frac{Q_{m_1^1 a_1^1} Q_{m_1^1 a_2^2}}{Q_{m_1^1 a_2^1} Q_{m_1^1 a_1^2}} \ldots \frac{Q_{m_1^r a_1^r} Q_{m_1^r a_2^1}}{Q_{m_1^r a_2^r} Q_{m_1^r a_1^1}} \\ \ldots \frac{Q_{m_p^1 a_p^1} Q_{m_p^1 a_1^2}}{Q_{m_p^1 a_1^1} Q_{m_p^1 a_p^2}} \ldots \frac{Q_{m_p^r a_p^r} Q_{m_p^r a_1^1}}{Q_{m_p^r a_1^r} Q_{m_p^r a_p^1}}. \end{aligned}$$

Here the exponent appearing on the right is given by

$$\begin{aligned} E(i,m) &= m_1^1(i_1^1+i_2^2-i_1^2-i_2^1)+\cdots+m_1^r(i_1^r+i_2^1-i_1^1-i_2^r)\\ &\quad+\cdots+m_p^1(i_p^1+i_1^2-i_p^2-i_1^1)+\cdots+m_p^r(i_p^r+i_1^1-i_p^1-i_1^r). \end{aligned}$$

Now observe that this exponent can be written as

$$\begin{aligned} E(i,m) &= i_1^1(m_1^1-m_1^r-m_p^1+m_p^r)+\cdots+i_1^r(m_1^r-m_1^{r-1}-m_p^r+m_p^{r-1})\\ &\quad+\cdots+i_p^1(m_p^1-m_p^r-m_{p-1}^1+m_{p-1}^r)\\ &\quad+\cdots+i_p^r(m_p^r-m_p^{r-1}-m_{p-1}^r+m_{p-1}^{r-1}). \end{aligned}$$

With this formula in hand, we can perform the sum over the i indices, and the point is that the resulting condition on the m indices will be exactly the same as the above-mentioned condition on the a indices. Thus, we obtain a formula as follows, where $\Delta(\cdot)$ is a certain product of Kronecker symbols:

$$\begin{aligned} c_p^r = \frac{1}{N^r}\sum_{a_1^1\ldots a_p^r}\sum_{m_1^1\ldots m_p^r}\Delta(a)\Delta(m)\frac{Q_{m_1^1a_1^1}Q_{m_1^1a_2^2}}{Q_{m_1^1a_2^1}Q_{m_1^1a_1^2}}\cdots\frac{Q_{m_1^ra_1^r}Q_{m_1^ra_2^1}}{Q_{m_1^ra_2^r}Q_{m_1^ra_1^1}}\\ \cdots\frac{Q_{m_p^1a_p^1}Q_{m_p^1a_1^2}}{Q_{m_p^1a_1^1}Q_{m_p^1a_p^2}}\cdots\frac{Q_{m_p^ra_p^r}Q_{m_p^ra_1^1}}{Q_{m_p^ra_1^r}Q_{m_p^ra_p^1}}. \end{aligned}$$

The point now is that when replacing $H = F_M \otimes_Q F_N$ with its transpose matrix, $H^t = F_N \otimes_{Q^t} F_M$, we will obtain exactly the same formula, with Q replaced by Q^t. But, with $a_x^y \leftrightarrow m_x^y$, this latter formula will be exactly the one above, and we are done. ■

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References

[1] T. Banica, *Compact Kac algebras and commuting squares*, J. Funct. Anal. 176 (2000), 80–99.

[2] T. Banica and J. Bichon, *Hopf images and inner faithful representations*, Glasgow Math. J. 52 (2010), 677–703.

[3] T. Banica, U. Franz and A. Skalski, *Idempotent states and the inner linearity property*, Bull. Polish Acad. Sci. Math. 60 (2012), 123–132.

[4] I. Bengtsson, *Three ways to look at mutually unbiased bases*, in: Foundations of Probability in Physics—4, AIP Conf. Proc. 889, Amer. Inst. Phys., Melville, NY, 2007, 40–51.

[5] P. Diţă, *Some results on the parametrization of complex Hadamard matrices*, J. Phys. A 37 (2004), 5355–5374.

[6] U. Franz and A. Skalski, *On idempotent states on quantum groups*, J. Algebra 322 (2009), 1774–1802.

[7] V. F. R. Jones and V. S. Sunder, *Introduction to Subfactors*, Cambridge Univ. Press, 1997.
[8] W. Tadej and K. Życzkowski, *A concise guide to complex Hadamard matrices*, Open Systems Information Dynam. 13 (2006), 133–177.
[9] S. Wang, *Quantum symmetry groups of finite spaces*, Comm. Math. Phys. 195 (1998), 195–211.
[10] R. F. Werner, *All teleportation and dense coding schemes*, J. Phys. A 34 (2001), 7081–7094.
[11] S. L. Woronowicz, *Compact matrix pseudogroups*, Comm. Math. Phys. 111 (1987), 613–665.
[12] S. L. Woronowicz, *Tannaka–Krein duality for compact matrix pseudogroups. Twisted* $\mathrm{SU}(N)$ *groups*, Invent. Math. 93 (1988), 35–76.

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