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NUMBER THEORY

Dirichlet Series and Gamma Function Associated with Rational Functions

by

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Summary. We investigate zeta regularized products of rational functions. As an application, we obtain the asymptotic expansion of the Euler Gamma function associated with a rational function.

1. Introduction. Let $r(z) = cp_h(z)/q_k(z)$ be a rational function of z, where p_h and q_k are monic polynomials with real coefficients of degree h and k, respectively, and $c \neq 0$ is a real number. Factoring $r(z)$ into the product

$$r(z) = c\frac{p_h(z)}{q_k(z)} = c\frac{(z+a_1)\cdots(z+a_h)}{(z+b_1)\cdots(z+b_k)},$$

it is clear (see for example [8, 12.13]) that the infinite product

$$\prod_{n=1}^{\infty} c\frac{p_h(n)}{q_k(n)}$$

converges if $c = 1$, $h = k$, $a_1 + \cdots + a_h - b_1 - \cdots - b_h = 0$, and assuming that no factor in the denominator vanishes. If this is the case, it is a result of Euler that

$$\prod_{n=1}^{\infty} \frac{p_h(n)}{q_h(n)} = \frac{\Gamma(1+b_1)\cdots\Gamma(1+b_h)}{\Gamma(1+a_1)\cdots\Gamma(1+a_h)}.$$

M. Eie [4, Main theorem II] proved that this result generalizes to zeta regularized products when $q_k(z) = 1$ and $|a_l| < 1$. Recall that if $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ is a sequence of complex numbers with a unique accumulation point at infin-

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ity and genus g (see for example [2, 7.5], or [6, Section 2] for the definition), and if Λ is contained in some suitable sector of the complex plane, then the *zeta regularization* of the infinite product

$$\prod_{n=1}^{\infty} \lambda_n$$

is by definition $\mathrm{e}^{-\zeta'(0,\Lambda)}$, where the zeta function associated to Λ is defined by the Dirichlet series

$$\zeta(s,\Lambda) = \sum_{n=1}^{\infty} \lambda_n^{-s}$$

when $\mathrm{Re}(s) > \mathsf{g}$, and by analytic continuation elsewhere, and where by $\zeta'(0,\Lambda)$ we mean the finite part of $\zeta(s,\Lambda)$ if $\zeta(s,\Lambda)$ has a pole at $s=0$ (we refer to [6] for details). If the unique accumulation point of Λ is zero, then we define the associated zeta function by $\zeta(s,\Lambda) = \zeta(-s, 1/\Lambda)$.

If $\Lambda = \{cp_h(n)/q_k(n)\}_{n=1}^{\infty}$, we denote by $\zeta(s, cp_h/q_k)$ the associated zeta function, and we call it the *zeta function associated with a rational function.* The polynomial zeta function $\zeta(s, p_k)$ has been studied in the cited work of Eie. Subsequently, the construction has been generalized by studying multiple polynomial zeta functions in [5], and introducing polynomial multiplicity in [3].

In this note, we extend this construction to the case of rational functions. Our first result is the following proposition, which also gives an elementary proof of Main Theorem II in [4].

Proposition 1. *Let $c \neq 0$, and $a_1, \dots, a_h$ and $b_1, \dots, b_k$ be complex numbers with $|a_j| < 1$, $|b_j| < 1$, and $h \neq k$. Then the zeta regularization of the infinite product*

$$\prod_{n=1}^{\infty} c\frac{p_h(n)}{q_k(n)} = \prod_{n=1}^{\infty} c\frac{(n+a_1)\cdots(n+a_h)}{(n+b_1)\cdots(n+b_k)}$$

is

$$\mathrm{e}^{-\zeta'(0,cp_h/q_k)} = (2\pi)^{\frac{h-k}{2}} c^{-\frac{a_1+\cdots+a_h-b_1-\cdots-b_k}{h-k}-\frac{1}{2}} \frac{\Gamma(1+b_1)\cdots\Gamma(1+b_k)}{\Gamma(1+a_1)\cdots\Gamma(1+a_h)}.$$

As a second result, we present the following natural application of Proposition 1. Define the Euler Gamma function associated with the rational function $r(z) = cp_h(z)/q_k(z)$, with $h > k > 0$, to be the Weierstrass product

$$\Gamma(z, cp_h/q_k) = \prod_{n=1}^{\infty} \frac{\mathrm{e}^{\mathsf{g}z/r(n)}}{1 + z/r(n)},$$

where we put $\mathsf{g} = 1$ if $h = k+1$, and $\mathsf{g} = 0$ otherwise (note that g is the genus of the sequence Λ). Then we have the following asymptotic expansion, where the notation $\operatorname*{Res}_{s=s_0}{}_{l} f(s)$ denotes the coefficient of the term $(s-s_0)^{-l}$ in the Laurent expansion of $f(s)$ at $s=s_0$ (see for example [1, p. 420]).

PROPOSITION 2. *For large z with $|\arg(z)|<\pi$,*

$$\log \varGamma(z, cp_h/q_k) = \begin{cases} \dfrac{\pi c^{\frac{1}{k-h}}}{\sin\frac{\pi}{k-h}} z^{\frac{1}{h-k}} & \text{if } h>k+1,\\ \dfrac{1}{c} z\log z + \left(\operatorname*{Res}_{s=1}{}_0\, \zeta(s, cp_h/q_k) - \dfrac{1}{c}\right) z & \text{if } h=k+1, \end{cases}$$
$$+\left(\frac{1}{2}+\frac{a_1+\dots+a_h-b_1-\dots-b_k}{h-k}\right)\log z$$
$$-\left(\frac{1}{2}+\frac{a_1+\dots+a_h-b_1-\dots-b_k}{h-k}\right)\log c$$
$$+\frac{h-k}{2}\log 2\pi + \log\frac{\varGamma(1+b_1)\cdots\varGamma(1+b_k)}{\varGamma(1+a_1)\cdots\varGamma(1+a_h)} + o(1).$$

The proofs of these propositions are presented in the next two sections.

2. The proof of Proposition 1. Expanding the powers of the binomials we obtain, for large $\operatorname{Re}(s)$,

$$c^s\zeta(s, cp_h/q_k) = \sum_{j,l=0}^{\infty}\binom{-s}{j}\binom{s}{l} a^j b^l \zeta((h-k)s+|j|+|l|),$$

where $\zeta(s)$ is the Riemann zeta function, we use the multi-indices $j=(j_1,\dots,j_h)$, $l=(l_1,\dots,l_k)$, and $|j|=j_1+\dots+j_h$, $|l|=l_1+\dots+l_k$, $a_j^j=a_1^{j_1}\cdots a_h^{j_h}$, $b_k^k=b_1^{l_1}\cdots b_k^{l_k}$, $\binom{-s}{j}=\binom{-s}{j_1}\cdots\binom{-s}{j_h}$, $\binom{s}{l}=\binom{s}{l_1}\cdots\binom{s}{l_k}$. Thus,

$$c^s\zeta(s, cp_h/q_k) = \zeta((h-k)s) + \sum_{\alpha=1}^{h}\sum_{j_\alpha=1}^{\infty}\binom{-s}{j_\alpha} a_\alpha^{j_\alpha}\zeta((h-k)s+j_\alpha)$$
$$+\sum_{\beta=1}^{k}\sum_{l_\beta=1}^{\infty}\binom{s}{l_\beta} b_\beta^{l_\beta}\zeta((h-k)s+l_\beta)+\varphi(s),$$

where $\varphi(s)$ has a zero of degree 2 at $s=0$. Isolating the singular terms, we obtain

$$c^s\zeta(s, cp_h/q_k) = \zeta((h-k)s) - s\zeta((h-k)s+1)\Big(\sum_{\alpha=1}^{h} a_\alpha - \sum_{\beta=1}^{k} b_\beta\Big) \tag{1}$$
$$+\sum_{\alpha=1}^{h}\sum_{j_\alpha=2}^{\infty}\binom{-s}{j_\alpha} a_\alpha^{j_\alpha}\zeta((h-k)s+j_\alpha)$$
$$+\sum_{\beta=1}^{k}\sum_{l_\beta=2}^{\infty}\binom{s}{l_\beta} b_\beta^{l_\beta}\zeta((h-k)s+l_\beta)+\varphi(s).$$

The analytic continuation at $s = 0$ of the function on the right side of (1) is given by that of the Riemann zeta function, and is regular at $s = 0$. In particular,

$$\begin{aligned}\frac{d}{ds}\Big|_{s=0}\sum_{j_\alpha=2}^{\infty}\binom{-s}{j_\alpha}\zeta((h-k)s+j_\alpha)a_\alpha^{j_\alpha} &= \sum_{j_\alpha=2}^{\infty}\frac{(-1)^{j_\alpha}}{j_\alpha}\zeta(j_\alpha)a_\alpha^{j_\alpha}\\ &= \log\Gamma(1+a_\alpha)+\gamma a_\alpha,\\ \frac{d}{ds}\Big|_{s=0}\sum_{l_\beta=2}^{\infty}\binom{s}{l_\beta}\zeta((h-k)s+l_\beta)b_\beta^{l_\beta} &= -\sum_{l_\beta=2}^{\infty}\frac{(-1)^{l_\beta}}{l_\beta}\zeta(l_\beta)b_\beta^{l_\beta}\\ &= -\log\Gamma(1+b_\beta)-\gamma b_\beta.\end{aligned}$$

This gives the expansions near $s = 0$ of the different terms:

$$\begin{aligned}-s\zeta((h-k)s+1)\Big(\sum_{\alpha=1}^{h}a_\alpha-\sum_{\beta=1}^{k}b_\beta\Big) &= -\frac{1}{h-k}\Big(\sum_{\alpha=1}^{h}a_\alpha-\sum_{\beta=1}^{k}b_\beta\Big)\\ &\quad-\gamma\Big(\sum_{\alpha=1}^{h}a_\alpha-\sum_{\beta=1}^{k}b_\beta\Big)s+O(s^2),\end{aligned}$$

$$\sum_{\alpha=1}^{h}\sum_{j_\alpha=2}^{\infty}\binom{-s}{j_\alpha}a_\alpha^{j_\alpha}\zeta((h-k)s+j_\alpha)=\Big(\log\prod_{\alpha=1}^{h}\Gamma(1+a_\alpha)+\gamma\sum_{\alpha=1}^{h}a_\alpha\Big)s+O(s^2),$$

$$\sum_{\beta=1}^{k}\sum_{l_\beta=2}^{\infty}\binom{s}{l_\beta}b_\beta^{l_\beta}\zeta((h-k)s+l_\beta)=-\Big(\log\prod_{\beta=1}^{k}\Gamma(1+b_\beta)+\gamma\sum_{\beta=1}^{k}b_\beta\Big)s+O(s^2),$$

and hence

$$\begin{aligned}\zeta(0,p_h/q_k) &= \zeta(0)-\frac{1}{h-k}\Big(\sum_{\alpha=1}^{h}a_\alpha-\sum_{\beta=1}^{k}b_\beta\Big),\\ \zeta(0,p_h/q_k) &= (h-k)\zeta'(0)+\log\frac{\prod_{\alpha=1}^{h}\Gamma(1+a_\alpha)}{\prod_{\beta=1}^{k}\Gamma(1+b_\beta)}.\end{aligned}$$

3. The proof of Proposition 2. We shall use some notations and results from [6] and [7], concerning sequences of spectral type. Let us indicate the main steps of the proof.

Let S be any sequence of positive real numbers, which is simple regular of spectral type with genus g. Let $F(z,S)$ denote the Fredholm determinant associated to S (see [6, p. 866]; also note that its inverse is called the Gamma function in [7]). There exists an expansion (use Definition 2.1 and Lemma 2.7 in [6] or Definitions 2.1 and 2.7 in [7])

$$-\log F(z,S) = \sum_{j=0}^{\mathsf{g}} a_{j,1} z^j \log z + \sum_{j=0}^{J} a_{\alpha_j,0} z^{\alpha_j} + o(z^{\alpha_J})$$

for large z with $|\arg(z)| < \pi$ and $\alpha_0 > \alpha_1 > \cdots > \alpha_J$. Observe that our sequence Λ is a simply regular sequence of spectral type with genus g, where $\mathsf{g} = 1$ if $h = k+1$ and $\mathsf{g} = 0$ otherwise. Indeed, Λ is a sequence of spectral type by Lemma 2.5 in [6], and is simply regular by Proposition 2.11 in [7], since the possible poles of the zeta function $\zeta(s,\Lambda)$ are at most simple. Moreover, by the same proposition the possible poles of the zeta function are located at $s = \alpha_j$, and by Remark 2.9 in [7], $\alpha_0 < \mathsf{g}+1$. Now, using the expansion given in the previous section, it is easy to see that $\zeta(s,\Lambda)$ has at most one simple pole on the positive part of the real axis, and this pole is at $s = 1$ if $\mathsf{g} = 1$, and at $s = 1/(h-k)$ otherwise. It follows that the unique possible positive value of the α_j is either $\alpha_0 = 1$, if $\mathsf{g} = 1$, or $\alpha_0 = \frac{1}{h-k}$, if $\mathsf{g} = 0$; and that $\alpha_1 = 0$ for any g. Also note that $\log F(z,\Lambda) = -\log \Gamma(z,\Lambda)$. This means that

$$\log \Gamma(z,\Lambda) = \sum_{j=0}^{\mathsf{g}} a_{j,1} z^j \log z + \sum_{j=0}^{1} a_{\alpha_j,0} z^{\alpha_j} + o(1).$$

The values of $a_{x,k}$'s can be calculated explicitly as follows (see Propositions 2.11 and 2.14 in [7], and Proposition 2.6 in [6]):

$$\begin{aligned}
a_{0,0} &= -\operatorname*{Res_0}_{s=0} \zeta'(s,\Lambda),\\
a_{0,1} &= -\operatorname*{Res_0}_{s=0} \zeta(s,\Lambda),\\
a_{\alpha_0,0} &= \begin{cases} a_{\frac{1}{h-k},0} = \Gamma\left(\dfrac{1}{h-k}\right)\Gamma\left(\dfrac{1}{k-h}\right) \operatorname*{Res_1}_{s=\frac{1}{h-k}} \zeta(s,\Lambda) = \dfrac{\pi c^{\frac{1}{k-h}}}{\sin \frac{\pi}{k-h}}, & \mathsf{g} = 0,\\[2ex] a_{1,0} = \operatorname*{Res_0}_{s=1} \zeta(s,\Lambda) - \operatorname*{Res_1}_{s=1} \zeta(s,\Lambda) = \operatorname*{Res_0}_{s=1} \zeta(s,\Lambda) - \dfrac{1}{c}, & \mathsf{g} = 1,\end{cases}\\
a_{\alpha_0,1} &= \begin{cases} a_{\frac{1}{h-k},1} = 0, & \mathsf{g} = 0,\\[1ex] a_{1,1} = \operatorname*{Res_1}_{s=1} \zeta(s,\Lambda) = \dfrac{1}{c}, & \mathsf{g} = 1.\end{cases}
\end{aligned}$$

Applying Proposition 1, we are done.

4. Remarks. In this section we investigate the case $h = k$. Before, we discuss multiplicativity of zeta regularization. This appears in the interpretation of the infinite product $\prod \lambda_n$ as the determinant of the infinite diagonal matrix Λ with entries λ_n. Namely, we set

$$\det_\zeta \Lambda = \mathrm{e}^{-\zeta'(0,\Lambda)}.$$

Then it is natural to ask: given two infinite diagonal matrices Λ_1 and Λ_2, is $\det_\zeta \Lambda_1\Lambda_2 = \det_\zeta \Lambda_1 \det_\zeta \Lambda_2$? Multiplicativity of determinants clearly corresponds to additivity of the derivative at zero of the associated zeta functions. Now, it is clear that $\zeta(0, c\Lambda) = \zeta(0,\Lambda)$ for any $c \neq 0$, and that

$$\zeta'(0, c\Lambda) = -\zeta(0,\Lambda)\log c + \zeta'(0,\Lambda).$$

Restricting to the case where the λ_n are rational functions of n, it follows from Proposition 1 and the formula

$$\zeta(0, p_h) = -\frac{1}{2} - \frac{1}{h}(a_1 + \cdots + a_h)$$

that $\zeta'(0, c_1c_2p_{1,h_1}p_{2,h_2})$ is not equal to $\zeta'(0, c_1p_{1,h_1}) + \zeta'(0, c_2p_{2,h_2})$ (however, note this is the case when $h_1 = h_2$ and $p_{1,h_1} = p_{2,h_2}$). Therefore, we further restrict to monic polynomials, and in this case it is easy to see that (if $h_j \neq k_j$)

$$\zeta'(0, p_{1,h_1}p_{2,h_2}/q_{1,k_1}q_{2,k_2}) = \zeta'(0, p_{1,h_1}/q_{1,k_1}) + \zeta'(0, p_{2,h_2}/q_{2,k_2}).$$

Thus we consider the case $k = h$ only for monic polynomials. It is clear that a zeta function for the sequence $\{p_h(n)/q_h(n)\}_{n=1}^\infty$ cannot be defined, since the exponent of convergence is not finite. However, we can introduce the following regularization of the infinite product: we define the regularized product

$$\begin{aligned}\prod_{n=1}^{\infty}{}^{\mathrm{R}} \frac{p_h(n)}{q_h(n)} &= \prod_{n=1}^{\infty}{}^{\mathrm{R}} \frac{(n+a_1)\cdots(n+a_h)}{(n+b_1)\cdots(n+b_h)} \\ &= \mathrm{e}^{a_1+\cdots+a_h-b_1-\cdots-b_h} \prod_{n=1}^{\infty} \frac{(n+a_1)\cdots(n+a_h)}{(n+b_1)\cdots(n+b_h)} \mathrm{e}^{-(a_1+\cdots+a_h-b_1-\cdots-b_h)/n}.\end{aligned}$$

It is then easy to see that the product on the right side converges, as desired, to

$$\frac{\Gamma(1+b_1)\cdots\Gamma(1+b_h)}{\Gamma(1+a_1)\cdots\Gamma(1+a_h)} = \frac{\mathrm{e}^{-\zeta'(0,p_h)}}{\mathrm{e}^{-\zeta'(0,q_h)}}.$$

We conclude by observing that the method described in this note can be used to obtain a formula for the derivative at zero of the zeta function studied by Dąbrowski in [3], where some polynomial multiplicity has been introduced.

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