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NUMBER THEORY

Fibonacci Numbers with the Lehmer Property

by

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Summary. We show that if $m > 1$ is a Fibonacci number such that $\phi(m) \mid m-1$, where ϕ is the Euler function, then m is prime.

Let $\phi(n)$ be the Euler function of the positive integer n. Clearly, $\phi(n) = n-1$ if n is a prime. Lehmer [9] (see also B37 in [7]) conjectured that if $\phi(n) \mid n-1$, then n is prime. To this day, no counterexample to this conjecture (and no proof of it either) has been found. Let us say that n has the *Lehmer property* if n is composite and $\phi(n) \mid n-1$. Thus, Lehmer's conjecture is that there is no number with the Lehmer property.

Pomerance (see [14], [15]) showed that if $\mathcal{L}(x)$ denotes the number of numbers $n \le x$ with the Lehmer property then the estimate

$$\mathcal{L}(x) = O(x^{1/2}(\log x)^{3/4}(\log\log x)^{-1/2})$$

holds, where $\log x$ stands for the natural logarithm of x. The exponent $3/4$ of $\log x$ in the above bound was successively lowered to $1/2$ by Zhun [18] and to 0 (at the cost of some extra power of $\log\log x$) by Banks and Luca [2].

In the recent paper [6], Diaconescu studied numbers with the Lehmer property and some extra structure and concluded that there should be only finitely many of them. For example, he showed that if $k \ge 1$ is a fixed positive integer then there are only finitely many positive integers n with the Lehmer property which also satisfy the congruence $\phi(n)^k \equiv 1 \pmod n$.

Here, we study the numbers with the Lehmer property which belong to a familiar subset of positive integers, namely the Fibonacci numbers. Recall

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that the sequence of Fibonacci numbers $(F_n)_{n\geq 0}$ has $F_0 = 0,\ F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Our result is the following.

THEOREM 1. *There is no Fibonacci number with the Lehmer property.*

Throughout this paper, we use p with or without subscripts for a prime number. For a positive integer m we write $\omega(m)$ and $\tau(m)$ for the number of distinct prime divisors of m and the total number of positive integer divisors of m, respectively. Recall that if $m = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, where $p_1, \ldots, p_k$ are distinct primes and $\alpha_1, \ldots, \alpha_k$ are positive integer exponents, then $\omega(m) = k$ and $\tau(m) = (\alpha_1 + 1) \cdots (\alpha_k + 1)$.

We also recall that if we write $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$, then $F_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ for all $n \geq 0$. This is sometimes called the Binet formula. Furthermore, if we write $(L_n)_{n\geq 0}$ for the Lucas sequence given by $L_0 = 2,\ L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$ for all $n \geq 0$, then both the Binet formula $L_n = \alpha^n + \beta^n$ and

$$L_n^2 - 5F_n^2 = 4(-1)^n \tag{1}$$

hold for all $n \geq 0$.

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1. The proof. Assume that $n > 2$ and that F_n is a composite positive integer such that $\phi(F_n) \mid F_n - 1$. Lehmer [9] showed that $\omega(F_n) \geq 7$ and this was subsequently improved to 11 by Lieuwens [10], to 13 by Kishore [8], and to 14 by Cohen and Hagis [5]. When $3 \mid F_n$, Lieuwens [10] showed that in fact $F_n > 5.5 \cdot 10^{570}$. Since certainly $\phi(F_n)$ is even, we infer that F_n is odd. Thus, either $3 \mid F_n$ and $F_n \geq 5.5 \cdot 10^{570}$, or

$$F_n \geq 5 \cdot 7 \cdot 11 \cdot 13 \cdots 53.$$

In both cases, we see that $n \geq 50$. Let $K = \omega(F_n)$. By Theorem 4 in [15], we have $F_n < K^{2^K}$. It is easy to check by induction that $F_s > 2^{s/2}$ for all $s > 10$. Since $n > 50$, we have $K^{2^K} > F_n > 2^{n/2}$, therefore

$$2^K \log K > \frac{n \log 2}{2} > \frac{n}{3}. \tag{2}$$

We now check that the above inequality (2) implies that

$$2^K > \frac{n}{4 \log\log n}. \tag{3}$$

Indeed, assume that the reverse inequality

$$2^K \leq \frac{n}{4 \log\log n}$$

holds. Then

$$K \log 2 < \log n - \log 4 - \log\log\log n < \log n.$$

In the rightmost inequality above we used the fact that $n > 50 > e^e$, so $\log\log n > 1$, therefore $\log\log\log n$ is positive. Thus, $K < (\log n)/\log 2 < 2\log n$, therefore

$$2^K \log K < \frac{n \log(2\log n)}{4\log\log n} = \frac{n}{4} + \frac{\log 2}{4 \log\log n}.$$

Comparing the last inequality above with (2), we get

$$\frac{n}{3} < \frac{n}{4} + \frac{\log 2}{4\log\log n},$$

therefore

$$n < \frac{3\log 2}{\log\log n} < \frac{3 \log 2}{\log\log 50} < 2,$$

which is impossible. Thus, inequality (2) holds.

In what follows, we will use the following well-known relations (see, for example, Lemma 2 in [11]):

$$\begin{aligned} F_{4m} - 1 &= F_{2m+1}L_{2m-1}, \qquad & F_{4m+1} - 1 &= F_{2m}L_{2m+1}, \\ F_{4m+2} - 1 &= F_{2m}L_{2m+2}, \qquad & F_{4m+3} - 1 &= F_{2m+2}L_{2m+1}, \end{aligned} \tag{4}$$

which can be easily verified using the Binet formulae.

We split the remaining analysis in two cases.

Case 1: *n is odd.* Let p be any prime factor of F_n. Clearly, p is odd. Reducing relation (1) modulo p we get $L_n^2 \equiv -4 \pmod{p}$, so we infer that -1 is a quadratic residue modulo p. In particular, $p \equiv 1 \pmod 4$. Since this is true for all prime factors p of F_n, we conclude that $2^{2K} \mid \phi(F_n)$. Since $n = 2m+1$ is odd, formulae (4) tell us that $F_n - 1 = F_{(n-1)/2}L_{(n+1)/2}$ or $F_{(n+1)/2}L_{(n-1)/2}$ according as m is even or odd. Thus, we get

$$2^{2K} \mid \phi(F_n) \mid F_n - 1 \mid F_{(n-\varepsilon)/2}L_{(n+\varepsilon)/2} \quad \text{for some } \varepsilon \in \{\pm 1\}.$$

The period of the sequence $(L_s)_{s\geq 0}$ modulo 8 is 12. Furthermore, listing the first twelve members of $(L_s)_{s\geq 0}$ one notices that none of them is a multiple of 8. Thus, the above divisibility condition certainly implies that 2^{2K-2} divides either $F_{(n-1)/2}$ or $F_{(n+1)/2}$. It is well-known and easy to check by induction that if $\ell \geq 3$ and $2^\ell \mid F_s$, then $2^{\ell-2}\cdot 3 \mid s$. Since $2K-2 \geq 2\cdot 14 - 2 > 3$, we find that $2^{2K-4}\cdot 3$ divides one of $(n-1)/2$ or $(n+1)/2$. Thus, using also inequality (3), we have

$$\frac{n+1}{2} \geq 2^{2K-4} \cdot 3 \geq \frac{3}{16}\left(\frac{n}{4\log\log n}\right)^2,$$

therefore

$$n^2 < \frac{128}{3}\,(n+1)(\log\log n)^2, \tag{5}$$

leading to $n < 101$.

Case 2: *n is even.* Here, we write $n = 2m$, so $F_n = F_{2m} = F_m L_m$. Relation (1) together with the fact that F_n is odd implies that F_m and L_m are coprime, so $\phi(F_n) = \phi(F_m L_m) = \phi(F_m)\phi(L_m)$. Let $m = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ be the factorization of n, where $p_1 < \cdots < p_k$ are distinct primes. We shall spend a lot of time bounding p_1.

We start by noticing that m has to be odd (so $p_1 > 2$). Indeed, assume that $m = 2m_0$ is even. Then formula (4) tells us that $\phi(F_n) \mid F_{2m_0+1}L_{2m_0-1}$. As we have said before, 8 cannot divide L_s for any value of the positive integer s. Furthermore, if $8 \mid F_s$, then $6 \mid s$, and in particular s is even. Since $2m_0 + 1$ is odd, we conclude that F_{2m_0+1} is not a multiple of 8. Thus, 32 cannot divide $F_{2m_0+1}L_{2m_0-1}$, but this is impossible since $2^K \mid \phi(F_n)$ and $K \geq 14$. Hence, m is odd, therefore $p_1 > 2$. If $p_1 = 3$, then F_n is even, which is not the case. Thus, $p_1 \geq 5$.

By the primitive divisor theorem for the Fibonacci and Lucas numbers (see [4]), for each divisor $d > 1$ of m there exists a prime $p \mid L_d$ such that $p \nmid L_{d_1}$ for all $0 < d_1 < d$. Since m is odd, Binet's formula implies that $p \mid L_m$. Reducing relation (1) modulo p, we get $-5F_m^2 \equiv -4 \pmod{p}$, therefore $5F_m^2 \equiv 4 \pmod{p}$. This shows that 5 is a quadratic residue modulo p, so by quadratic reciprocity, p is a quadratic residue modulo 5 also. Thus, $p \equiv 1 \pmod{d}$, therefore $d \mid p - 1$. Now let d be an arbitrary divisor of m which is a multiple of p_1. The number of such divisors is at least $\tau(m/p_1)$. For each such d, there is a primitive prime factor p_d of L_d such that $p_1 \mid d \mid p_d - 1 \mid \phi(L_d) \mid \phi(L_m)$. This shows that the exponent ℓ_1 of p_1 in $\phi(L_m)$ is at least $\tau(m/p_1)$. Thus,

$$p_1^{\tau(m/p_1)} \mid p_1^{\ell_1} \mid \phi(L_m) \mid \phi(L_n) \mid F_{2m} - 1 \mid F_{m-1}L_{m+1},$$

and $p_1 \mid m$. Let $z(p_1)$ be the order of appearance of p_1 in the Fibonacci sequence, i.e., the smallest positive integer s such that $p_1 \mid F_s$. It is known that $z(p_1) \mid p_1 - e$, where e is the Legendre symbol $(5/p_1)$; hence, it is 1 if $p_1 \equiv \pm 1 \pmod{5}$, it is -1 if $p_1 \equiv \pm 2 \pmod{5}$, and it is 0 if $p_1 = 5$. Let a_1 be the exponent of p_1 in $F_{z(p_1)}$. Since $p_1 \mid F_{m-1}L_{m+1}$, we find that either $p_1 \mid F_{m-1}$ or $p_1 \mid L_{m+1}$. Since $L_{m+1} \mid F_{2(m+1)}$, we further deduce that either $p_1 \mid F_{m-1}$ or $p_1 \mid F_{2(m+1)}$. Let us notice that p_1 can divide only one but not both of the above numbers. Indeed, since $F_{m-1} \mid F_{2(m-1)}$, it follows that if p_1 divides both the above numbers, then it divides both F_{2m-2} and F_{2m+2}. But then $p_1 \mid F_{\gcd(2m-2,2m+2)}$, and $\gcd(2m-2, 2m+2) \mid 4$. However, $F_4 = 3$ and we have already seen that $p_1 > 3$. Thus, only one of $m-1$ or $2(m+1)$ is divisible by $z(p_1)$, therefore $p_1^{a_1}$ divides either F_{m-1} or $F_{2(m+1)}$. It is well-known that if $\ell > a_1$ and $p_1^{\ell} \mid F_s$, then $p_1 z(p_1) \mid s$. Since $p_1 \mid m$ and p_1 is odd, it follows that $p_1 z(p_1)$ can divide neither $m-1$ nor $2m+2$. The conclusion is that $\ell_1 \leq a_1$, therefore $p_1^{\ell_1} \leq p_1^{a_1}$. In particular, $\ell_1 = a_1 = 1$ if $p_1 = 5$ or $p_1 = 7$.

Assume now that $p_1 \geq 11$. Then $p_1^{a_1} \mid F_{p_1-e} = F_{(p_1-e)/2} L_{(p_1-e)/2}$. The greatest common divisor of $F_{(p_1-e)/2}$ and $L_{(p_1-e)/2}$ is at most 2 (by (1) for $n = (p_1-e)/2$) and p_1 is odd, so either $p_1^{a_1} \mid F_{(p_1-e)/2}$ or $p_1^{a_1} \mid L_{(p_1-e)/2}$. Since $F_s < \alpha^s$ for all positive integers s, as can be easily verified by induction, we see that when $p_1^{a_1} \mid F_{(p_1-e)/2}$, we have

$$p_1^{a_1} \leq F_{(p_1-e)/2} \leq F_{(p_1+1)/2} \leq \alpha^{(p_1+1)/2},$$

so

$$\tau(m/p_1) \leq \ell_1 \leq a_1 \leq \frac{(p_1+1)\log\alpha}{2\log p_1}. \tag{6}$$

The same conclusion, namely that $p_1^{a_1} < \alpha^{(p_1+1)/2}$, is also reached when $p_1^{a_1} \mid L_{(p_1-e)/2}$, in the following way. First observe that the above inequality is certainly true when $a_1 = 1$ since $p_1 \geq 11$. Now assume that $a_1 > 1$. If $L_{(p_1-e)/2} = p_1^{a_1}$, then, in particular, $L_{(p_1-e)/2}$ is a perfect power. However, by the recent results from [3], there is no perfect power of the form L_s for $s > 3$. Hence,

$$p_1^{a_1} \leq \frac{1}{2} L_{(p_1-e)/2} < \frac{1}{2}\left(\alpha^{(p_1-e)/2} + 1\right) < \alpha^{(p_1+1)/2},$$

which implies inequality (6). Now note that

$$\tau(m/p_1) = \alpha_1(\alpha_2+1)\cdots(\alpha_k+1) \geq \left(\frac{\alpha_1+1}{2}\right)(\alpha_2+1)\cdots(\alpha_k+1) = \frac{\tau(m)}{2},$$

therefore

$$\tau(m) \leq 2\tau(m/p_1) \leq \frac{(p_1+1)\log\alpha}{\log p_1}. \tag{7}$$

Now observe that $\phi(F_n) \mid F_n - 1$ and $\phi(F_n) < F_n - 1$. Thus, $F_n - 1 \geq 2\phi(F_n)$, therefore

$$2 \leq \frac{F_n}{\phi(F_n)} \leq \prod_{p \mid F_n}\left(1 + \frac{1}{p-1}\right) < \exp\left(\sum_{p \mid F_n} \frac{1}{p-1}\right),$$

so

$$\log 2 \leq \sum_{p \mid F_n} \frac{1}{p-1}. \tag{8}$$

In what follows, we shall exploit the above relation. Since our ultimate goal is to bound p_1, we shall from now on assume that $p_1 > 1000$.

Let us now take a closer look at the right hand side of inequality (8). For each divisor $d > 1$ of m, let $\mathcal{P}_d$ be the set of primitive prime factors of $F_{2d} = F_d L_d$. All these primes are $\equiv \pm 1 \pmod d$ and are odd. In particular, the smallest one is $\geq 2d-1$. Assume that $\ell_d = \#\mathcal{P}_d$ is their number. Then

$$(2d-1)^{\ell_d} \leq F_{2d} < \alpha^{2d},$$

so

$$\ell_d < \frac{2d\log\alpha}{\log(2d-1)}.$$

We next show that the estimate

$$\sum_{p\in\mathcal{P}_d} \frac{1}{p-1} \leq \frac{1.8}{d} + \frac{4.3\log\log d}{d} \tag{9}$$

holds for our ranges of variables. Observe that

$$\begin{aligned}\sum_{p\in\mathcal{P}_d} \frac{1}{p-1} &= \sum_{p\in\mathcal{P}_d} \frac{1}{p} + \sum_{p\in\mathcal{P}_d} \frac{1}{p(p-1)} \leq \sum_{p\in\mathcal{P}_d} \frac{1}{p} + \frac{\ell_d}{(2d-2)(2d-1)}\\ &\leq \frac{1}{2d-1} + \frac{1}{2d+1} + \sum_{3d<p<d^2} \frac{1}{p} + \ell_d\left(\frac{1}{d^2} + \frac{1}{(2d-2)(2d-1)}\right).\end{aligned}$$

For coprime integers a and b and a positive real number t let $\pi(t;a,b)$ be the number of primes $p \leq t$ with $p \equiv a \pmod b$. The large sieve inequality of Montgomery and Vaughan [13] tells us that

$$\pi(t;a,b) \leq \frac{2t}{\phi(b)\log(t/b)}$$

for all $t > b$ and all a coprime to b. Since the set of primes $p \in (3d, d^2)$ which belong to $\mathcal{P}_d$ is contained in the set of primes $p \equiv \pm 1 \pmod d$, it follows, by Abel's summation formula, that

$$\begin{aligned}\sum_{3d<p<d^2} \frac{1}{p} &\leq \sum_{\substack{3d<p\leq d^2\\ p\equiv -1 \pmod d}} \frac{1}{p} + \sum_{\substack{3d<p\leq d^2\\ p\equiv 1 \pmod d}} \frac{1}{p}\\ &\leq \frac{\pi(d^2;-1,d)+\pi(d^2;1,d)}{d^2} + \int_{3d}^{d^2} \frac{\pi(t;-1,d)+\pi(t;1,d)}{t^2}\,dt\\ &\leq \frac{4d^2}{\phi(d)d^2\log(d^2/d)} + \frac{4}{\phi(d)}\int_{3d}^{d^2}\frac{dt}{t\log(t/d)}\\ &= \frac{4}{\phi(d)\log d} + \frac{4}{\phi(d)}\log\log(t/d)\Big|_{t=3d}^{t=d^2}\\ &< \frac{4}{\phi(d)\log d} + \frac{4\log\log d}{\phi(d)}\end{aligned}$$

because $\log\log 3 > 0$. As for $\phi(d)$ versus d, note that, by inequality (7),

$$\frac{d}{\phi(d)} \le \prod_{p\,|\,m}\left(1+\frac{1}{p-1}\right) \le \left(1+\frac{1}{p_1-1}\right)^{\tau(m)}$$
$$\le \exp\left(\frac{\tau(m)}{p_1-1}\right) \le \exp\left(\frac{(\log\alpha)(p_1+1)}{(p_1-1)\log p_1}\right) < 1.073$$

because $p_1 > 10^3$. Thus, $d/\phi(d) \le 1.073$, so putting all of the above estimates together we get

$$\sum_{p\in\mathcal{P}_d}\frac{1}{p-1} \le \frac{1}{2d-1}+\frac{1}{2d+1}+\ell_d\left(\frac{1}{d^2}+\frac{1}{(2d-2)(2d-1)}\right) + \frac{4.3}{d\log d}+\frac{4.3\log\log d}{d}. \tag{10}$$

Since $d \ge p_1 > 10^3$, we have

$$\frac{1}{2d-1}+\frac{1}{2d+1}+\ell_d\left(\frac{1}{d^2}+\frac{1}{(2d-2)(2d-1)}\right)+\frac{4.3}{d\log d}$$
$$\le \frac{1}{d}\left(\frac{4\cdot 10^6}{4\cdot 10^6-1}+\frac{2\log\alpha}{\log(2\cdot 10^3-1)}\left(1+\frac{1}{(2-2/10^3)(2-1/10^3)}\right)+\frac{4.3}{\log(10^3)}\right)$$
$$< \frac{1.8}{d},$$

which together with inequality (10) gives

$$\sum_{d\in\mathcal{P}_d}\frac{1}{p-1} < \frac{1.8}{d}+\frac{4.3\log\log d}{d},$$

which is the promised inequality (9).

Since the function $x \mapsto (\log\log x)/x$ is decreasing for $x > 10$, we have

$$\sum_{p|F_n}\frac{1}{p-1} = \sum_{d|m}\sum_{p\in\mathcal{P}_d}\frac{1}{p-1} \le \sum_{d|m,\,d>1}\left(\frac{1.8}{d}+\frac{4.3\log\log d}{d}\right)$$
$$\le \left(\frac{1.8}{p_1}+\frac{4.3\log\log p_1}{p_1}\right)\tau(m)$$
$$\le (\log\alpha)\,\frac{(p_1+1)}{\log p_1}\cdot\left(\frac{1.8}{p_1}+\frac{4.3\log\log p_1}{p_1}\right), \tag{11}$$

and comparing (11) with (8), we get

$$\log p_1 \le \frac{\log\alpha}{\log 2}\,\frac{p_1+1}{p_1}\,(1.8+4.3\log\log p_1). \tag{12}$$

Since $p_1 > 10^3$, we get

$$\frac{\log \alpha}{\log 2}\left(1 + \frac{1}{p_1}\right) < 0.7.$$

Hence, with $x = \log p_1$, we get $x < 0.7(1.8 + 4.3 \log x)$, which implies that $x < 7.21$, therefore $p_1 = e^x < e^{7.21} < 1400$. Thus, $p_1 < 1400$. We have finally bounded p_1.

At this point, we recall that D. D. Wall [17] conjectured that $p \,\|\, F_{z(p)}$ for all primes p. No counterexample to this conjecture (nor a proof of it either) has been found. Sun and Sun [16] deduced that the so-called first case of Fermat's Last Theorem is impossible under Wall's conjecture. We checked with Mathematica that Wall's conjecture is true for all $p < 1400$. In fact, in [1] it is mentioned that recently McIntosh and Roettger [12] verified Wall's conjecture for all $p < 10^{14}$ and found it to be true. In particular, it is true for p_1. This shows that $a_1 = 1$ for all possible values of p_1, therefore $\tau(m/p_1) = 1$, so $m = p_1$ and L_{p_1} is a prime. But in this case, $F_m = F_{p_1}$ has $K - 1$ prime factors and m is odd, so by the arguments from Case 1 each prime factor of F_m is congruent to 1 modulo 4. Thus, $2^{2K-1} \mid \phi(F_n) \mid F_{m-1}L_{m+1}$. Since 8 cannot divide L_{m+1}, we infer that $2^{2K-3} \mid F_{m-1}$, therefore $2^{2K-5} \cdot 3 \mid m - 1$. We thus find, using inequality (3), that

$$\frac{n}{2} > \frac{n}{2} - 1 = m - 1 \geq 2^{2K-5} \cdot 3 \geq \frac{3}{32}\left(\frac{n}{4 \log\log n}\right)^2,$$

therefore

$$n < \frac{256}{3} (\log\log n)^2,$$

leading to $n < 250$.

Thus, in both cases of n odd or n even we arrived at the conclusion that $n < 250$. We now checked that there is no Fibonacci number F_n with $n < 250$ having the Lehmer property in the following way. We used Mathematica to show that if $\omega(F_n) \geq 14$ and $n < 250$, then $n \in \{180, 210, 240\}$. Then we used again Mathematica and checked that for these three values of n, the ratio $(F_n - 1)/\phi(F_n)$ is not an integer.

This completes the proof of Theorem 1.

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