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FUNCTIONS OF A COMPLEX VARIABLE

On Uniqueness of Meromorphic Functions Sharing Three Sets with Finite Weights

by

Abhijit BANERJEE and Molla Basir AHAMED

Presented by Józef SICIAK

Summary. We prove the uniqueness of meromorphic functions sharing some three sets with finite weights.

1. Introduction, definitions and results. In the paper we will denote by $\mathbb{C}$ the set of all complex numbers, by $\mathbb{N}$ the set of all positive integers and write $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, $\overline{\mathbb{N}} := \mathbb{N} \cup \{0, \infty\}$. Throughout the paper the letters n, m are reserved for elements of $\mathbb{N}$, while $k, l, p \in \overline{\mathbb{N}}$, $z, w \in \mathbb{C}$. Also it is tacitly assumed that all meromorphic functions considered are defined on $\mathbb{C}$ and that they are non-constant.

For such a function f and $a \in \overline{\mathbb{C}}$, each z with $f(z) = a$ will be called an *a-point* of f. For a meromorphic function f and a set $S \subset \overline{\mathbb{C}}$ we define $E_f(S)$ (resp. $\overline{E}_f(S)$) as the set of all a-points of f, when $a \in S$, together with their multiplicity (resp. without their multiplicity). If $E_f(S) = E_g(S)$ (resp. $\overline{E}_f(S) = \overline{E}_g(S)$) then we simply say f, g *share* S Counting Multiplicities or CM (resp. Ignoring Multiplicities or IM).

More formally we define

Definition 1.1. If f is a meromorphic function and $S \subset \overline{\mathbb{C}}$ then if $z_0 \in f^{-1}(S)$, the value of $E_f(S)$ at the point z_0 is denoted by $E_f(S)(z_0) : f^{-1}(S) \to \mathbb{N}$ and is equal to the multiplicity of zero of the function $f(z) - f(z_0)$ at z_0, i.e. the order of the pole of the function $(f(z) - f(z_0))^{-1}$ at z_0 if $f(z_0) \in \mathbb{C}$ (resp. of the function $f(z)$ if z_0 is a pole for f).

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The following notion of weighted sharing of values and sets was introduced by Lahiri [8, 9]. It expedited new directions of research in value distribution theory.

Definition 1.2. For $k \in \overline{\mathbb{N}}$ and $z_0 \in f^{-1}(S)$ we put $E_f(S,k)(z_0) = \min\{E_f(S)(z_0), k+1\}$. Given $S \subset \overline{\mathbb{C}}$, we say that meromorphic functions f and g *share* the set S *up to multiplicity* k (or *share* S *with weight* k, or simply *share* (S,k)) if $f^{-1}(S) = g^{-1}(S)$ and for each $z_0 \in f^{-1}(S)$ we have $E_f(S,k)(z_0) = E_g(S,k)(z_0)$, which is represented by the notation $E_f(S,k) = E_g(S,k)$.

The subject of the paper is closely related to a problem posed by H. X. Yi [13]. The problem was to find three, possibly small, finite subsets S_1, S_2, S_3 of $\overline{\mathbb{C}}$ such that for any two meromorphic functions f, g which share each of the three sets S_i, $i = 1,2,3$ CM, we have $f \equiv g$. The problem has drawn attention of many mathematicians. It was solved by W. C. Lin and H. X. Yi [10] who proved that the sets $S_1 = \{0\}$, $S_2 = \{z \in \mathbb{C} : az^n - n(n-1)z^2 + 2n(n-2)bw = (n-1)(n-2)b^2\}$ and $S_3 = \{\infty\}$ have the above property, for $n \geq 5$, where a and b are complex numbers satisfying $ab^{n-2} \neq 2{,}0$. Later the result was strengthened by H. Y. Xu, H. X. Zhang and C. F. Yi [11] and the first author of the present paper [2]–[3].

In this paper we modify the sets S_1, S_2 so that $S_1 = \{0,1\}$, and the number of elements in the new set S_2 is decreased by 1 in the optimal case. Moreover the conditions on the sharing sets S_i, $i = 1,2,3$, are relaxed to the conditions of sharing (S_i, k_i), $i = 1,2,3$, where k_1, k_2, k_3 are relatively small.

The main result of the paper is the following.

Theorem 1.1. *Let* $S_1 = \{0,1\}$,
$$S_2 = \left\{z : \frac{(n-1)(n-2)}{2} z^n - n(n-2)z^{n-1} + \frac{n(n-1)}{2} z^{n-2} - c = 0\right\},$$
where $n \geq 4$, $c \in \mathbb{C}$, $c \neq 0, 1, 1/2$, *and* $S_3 = \{\infty\}$. *If two meromorphic functions* f *and* g *share* (S_1,p), (S_2,m) *and* (S_3,k), *where* $p \leq 1$, $2 \leq m < \infty$ *and*
$$0 < \frac{9 - 4p/3 - 2m}{m+1} < 2 - \frac{4 - 2p/3}{k+2},$$
then $f \equiv g$.

Corollary 1.1. *If* (p,m,k) *is one of the triplets* $(0,2,11)$, $(0,3,2)$, $(0,4,1)$, $(1,2,3)$, $(1,3,1)$ *then the conclusion of Theorem* 1.1 *holds.*

2. Auxiliary definitions and lemmas.

The proofs of the main theorems depend heavily on the value distribution of meromorphic functions, as in [6]. We will use standard definitions and notations from this theory. In particular $N(r,a;f)$ (resp. $\overline{N}(r,a;f)$) denotes the counting function (resp.

reduced counting function) of a-points of a meromorphic function f, $T(r, f)$ is the Nevanlinna characteristic function of f, and $S(r, f)$ is used to denote each function which is of smaller order than $T(r, f)$ when $r \to \infty$. Moreover we will need the following notation.

Definition 2.1 ([7]). For $a \in \overline{\mathbb{C}}$ we denote by $N(r, a; f \mid = 1)$ the counting function of simple a-points of f. For a positive integer m we denote by $N(r, a; f \mid \geq m)$ the counting function of those a-points of f whose multiplicities are not less than m, where each a-point is counted according to its multiplicity. We denote by $\overline{N}(r, a; f \mid \geq m)$ the reduced form of $N(r, a; f \mid \geq m)$.

Definition 2.2 ([14]). Let f and g be meromorphic functions sharing $(a, 0)$ where $a \in \mathbb{C} \cup \{\infty\}$. We denote by $\overline{N}_L(r, a; f > g)$ the reduced counting function of those a-points of f whose multiplicity corresponding to f is greater than that corresponding to g.

Definition 2.3 ([8, 9]). Let f, g share $(a, 0)$. We denote

$$\overline{N}_*(r, a; f, g) = \overline{N}_*(r, a; g, f) = \overline{N}_L(r, a; f > g) + \overline{N}_L(r, a; g > f).$$

For fixed $n \geq 3$ and $c \in \mathbb{C} \setminus \{0, 1, 1/2\}$ we set

$$Q(z) := \frac{(n-1)(n-2)}{2} z^2 - n(n-2)z + \frac{n(n-1)}{2} \quad \text{and} \quad P(z) := z^{n-2} Q(z).$$

To meromorphic functions f, g we associate F, G by

$$F = \frac{P(f)}{c}, \qquad G = \frac{P(g)}{c}, \tag{2.1}$$

and to F, G we associate H by the formula

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1} \right). \tag{2.2}$$

Lemma 2.1 ([9, Lemma 1]). *Let F, G be meromorphic functions sharing $(1, 1)$ and let H be given by (2.2). If $H \not\equiv 0$, then*

$$N(r, 1; F \mid = 1) = N(r, 1; G \mid = 1) \leq N(r, H) + S(r, F) + S(r, G).$$

Lemma 2.2. *Let F, G, H be as in (2.1), (2.2) and let S_i $i = 1, 2, 3$, be as defined in Theorem 1.1. If $H \not\equiv 0$ and f, g share (S_1, p), $(S_2, 0)$ and $(S_3, 0)$, where $p < \infty$, then*

$$\begin{aligned} N(r, H) \leq \overline{N}(r, 0; f \mid \geq p+1) + \overline{N}(r, 1; f \mid \geq p+1) + \overline{N}_*(r, 1; F, G) \\ + \overline{N}_*(r, \infty; f, g) + \overline{N}_0(r, 0; f') + \overline{N}_0(r, 0; g'), \end{aligned}$$

where $\overline{N}_0(r, 0; f')$ is the reduced counting function for the points $\{z \in \mathbb{C} : f'(z) = 0,\ f(z) \neq 0, 1;\ F(z) \neq 1\}$, and $\overline{N}_0(r, 0; g')$ is defined similarly.

Proof. Since

$$F-1=\frac{P(f)-c}{c},\qquad G-1=\frac{P(g)-c}{c}$$

and $E_f(S_2,0)=E_g(S_2,0)$ we see that F and G share $(1,0)$. It is easy to check that

$$H=\frac{2f'}{f-1}-\frac{2g'}{g-1}+\frac{(n-3)f'}{f}-\frac{(n-3)g'}{g}+\frac{f''}{f'}-\frac{g''}{g'}-\left(\frac{2F'}{F-1}-\frac{2G'}{G-1}\right).$$

Since $E_f(S_1,p)=E_g(S_1,p)$ we deduce that $z\in f^{-1}(\{0,1\})$ if and only if $z\in g^{-1}(\{0,1\})$. Hence

$$\begin{aligned}&\overline{N}(r,0;f\mid\geq p+1)+\overline{N}(r,1;f\mid\geq p+1)\\&\qquad=\overline{N}(r,0;g\mid\geq p+1)+\overline{N}(r,1;g\mid\geq p+1).\end{aligned}$$

It can also be easily verified that possible poles of H occur at (i) zeros (or 1-points) of f and g with multiplicity greater than p, (ii) poles of f and g with different multiplicities, (iii) 1-points of F and G with different multiplicities, (iv) zeros of f' which are not zeros of $f(f-1)$ and $F-1$, (v) zeros of g which are not zeros of $g(g-1)$ and $G-1$.

Since H has only simple poles, clearly the lemma follows from the above explanations. ■

LEMMA 2.3 ([12]). *If f is a meromorphic function and R a polynomial of degree n then*

$$T(r,R(f))=nT(r,f)+O(1).$$

LEMMA 2.4 ([4, Lemma 2.10]). *If meromorphic functions f, g share $(1,m)$, then*

$$\begin{aligned}\overline{N}(r,1;f)+\overline{N}(r,1;g)-N(r,1;f\mid=1)+\left(m-\frac{1}{2}\right)\overline{N}_*(r,1;f,g)&\\\leq\frac{1}{2}[N(r,1;f)+N(r,1;g)].&\end{aligned}$$

LEMMA 2.5. *If meromorphic functions f, g share $(\{0,1\},0)$ and $(\infty,0)$ then $P(f)P(g)$ is not a constant.*

Proof. On the contrary, assume that

$$(n-1)^2(n-2)^2f^{n-2}(f-\gamma)(f-\delta)g^{n-2}(g-\gamma)(g-\delta)\equiv 4c^2, \tag{2.3}$$

where γ and δ are the roots of the equation $Q(z)=0$.

If f has a pole then g will also have a pole, which is impossible by (2.3). So f and g have no poles. Similarly f (resp. g) cannot have any zero, γ-points or δ-points as they can only be neutralized by poles of g (resp. f). So f and g omit 0, ∞ as well as γ, δ, which is impossible. ■

LEMMA 2.6 ([5, p. 192]). *Let*

$$R(z) = (n-1)^2(z^n - 1)(z^{n-2} - 1) - n(n-2)(z^{n-1} - 1)^2.$$

Then $R(z) = (z-1)^4 W(z)$ *and all the* $2n-6$ *roots of the polynomial* W *are distinct and different from* 0, 1.

LEMMA 2.7. *If* $n \geq 4$ *and meromorphic functions* f, g *share* $(\{0,1\}, 0)$ *and* $P(f) \equiv P(g)$ *then* $f \equiv g$.

Proof. From the assumption we can write

$$f^{n-2}(f-\gamma)(f-\delta) \equiv g^{n-2}(g-\gamma)(g-\delta). \tag{2.4}$$

Clearly (2.4) implies that f and g share (∞, ∞). Since $E_f(\{0,1\}, 0) = E_g(\{0,1\}, 0)$ it follows that if z_0 is a zero of f (resp. g) then it cannot be a 1-point of g (resp. f) as none of γ and δ is zero. So f and g share $(0, \infty)$ and $(1, \infty)$. Suppose $h = f/g$. Clearly h has no zero and no pole. Substituting $f = hg$ in (2.4) we get

$$\begin{aligned} \frac{(n-1)(n-2)}{2}(h^n - 1)g^2 - n(n-2)(h^{n-1} - 1)g \\ + \frac{n(n-1)}{2}(h^{n-2} - 1) \equiv 0. \end{aligned} \tag{2.5}$$

Suppose h is not a constant. Then by a simple calculation we deduce from (2.5) that

$$\{(n-1)(n-2)(h^n - 1)g - n(n-2)(h^{n-1} - 1)\}^2 \equiv -n(n-2)R(h), \tag{2.6}$$

where $R(z)$ is as in Lemma 2.6. So using Lemma 2.6 we have

$$\begin{aligned} &\{(n-1)(n-2)(h^n - 1)g - n(n-2)(h^{n-1} - 1)\}^2 \\ &\qquad \equiv -n(n-2)(h-1)^4(h-\beta_1)\dots(h-\beta_{2n-6}), \end{aligned} \tag{2.7}$$

where $\beta_j \in \mathbb{C} - \{0,1\}$ $(j = 1, \dots, 2n-6)$ are distinct. From (2.7) we see that $h - \beta_j$ $(j = 1, \dots, 2n-6)$ each have multiplicity at least 2. So by the Second Fundamental Theorem we get

$$\begin{aligned} (2n-6)T(r,h) &\leq \overline{N}(r,\infty;h) + \overline{N}(r,0;h) + \sum_{j=1}^{2n-6} \overline{N}(r,\beta_j;h) + S(r,h) \\ &\leq \frac{1}{2}\sum_{j=1}^{2n-6} N(r,\beta_j;h) + S(r,h) \\ &\leq (n-3)T(r,h) + S(r,h), \end{aligned}$$

which is a contradiction for $n \geq 4$. So h is a constant. From (2.5) we have $h^n - 1 = 0$, $h^{n-1} - 1 = 0$. It follows that $h \equiv 1$ and so $f \equiv g$. ∎

Lemma 2.8. *Let $n \geq 3$ and S_i, $i = 1, 2, 3$, be as in Theorem* 1.1. *Also let meromorphic functions f and g share (S_1, p), (S_2, m), (S_3, k), where $p < \infty$. If F, G are given by* (2.1) *and*

$$\Phi := \frac{F'}{F-1} - \frac{G'}{G-1} \not\equiv 0,$$

then

$$\begin{aligned}\min\{(n-2)p + (n-3),\ 3p+2\}\{\overline{N}(r, 0; f \mid \geq p+1) + \overline{N}(r, 1; f \mid \geq p+1)\} \\ \leq \overline{N}_*(r, 1; F, G) + \overline{N}_*(r, \infty; f, g) + S(r, f) + S(r, g).\end{aligned}$$

Proof. By the assumptions, F and G share $(1, m)$. Also we see that

$$\Phi = \frac{n(n-1)(n-2)f^{n-3}(f-1)^2 f'}{2c(F-1)} - \frac{n(n-1)(n-2)g^{n-3}(g-1)^2 g'}{2c(G-1)}.$$

Let z_0 be a zero or a 1-point of f with multiplicity r. Since $E_f(S_1, p) = E_g(S_1, p)$, z_0 is a zero of Φ of multiplicity

$$\min\{(n-3)r + r - 1, 2r + r - 1\} = \min\{(n-2)r - 1,\ 3r - 1\},$$

if $r \leq p$, and of multiplicity at least

$$\min\{(n-3)(p+1) + p, 2(p+1) + p\} = \min\{(n-2)p + (n-3), 3p+2\}$$

if $r > p$. So by a simple calculation we can write

$$\begin{aligned}\min\{(n-2)p + (n-3),\ 3p+2\}&\{\overline{N}(r, 0; f \mid \geq p+1) + \overline{N}(r, 1; f \mid \geq p+1)\} \\ &\leq N(r, 0; \Phi) \leq T(r, \Phi) \\ &\leq N(r, \infty; \Phi) + S(r, F) + S(r, G) \\ &\leq \overline{N}_*(r, 1; F, G) + \overline{N}_*(r, \infty; f, g) + S(r, f) + S(r, g). \quad \blacksquare\end{aligned}$$

Lemma 2.9. *Let S_i, $i = 1, 2, 3$, be as in Theorem* 1.1 *and F, G, H be given by* (2.1) *and* (2.2). *If meromorphic functions f and g share (S_1, p), (S_2, m) and (S_3, k), where $p < \infty$, $2 \leq m < \infty$ and $H \not\equiv 0$, then*

$$\begin{aligned}&(n+1)\{T(r, f) + T(r, g)\} \\ &\quad \leq 2\{\overline{N}(r, 0; f) + \overline{N}(r, 1; f)\} + \overline{N}(r, 0; f \mid \geq p+1) + \overline{N}(r, 1; f \mid \geq p+1) \\ &\qquad + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + \overline{N}_*(r, \infty; f, g) \\ &\qquad + \frac{1}{2}[N(r, 1; F) + N(r, 1; G)] \\ &\qquad - \left(m - \frac{3}{2}\right)\overline{N}_*(r, 1; F, G) + S(r, f) + S(r, g).\end{aligned}$$

Proof. By the Second Fundamental Theorem we get

$$\begin{aligned}(2.8)\qquad &(n+1)\{T(r,f)+T(r,g)\}\\ &\le \overline{N}(r,1;F)+\overline{N}(r,0;f)+\overline{N}(r,1;f)+\overline{N}(r,\infty;f)+\overline{N}(r,1;G)+\overline{N}(r,0;g)\\ &\quad+\overline{N}(r,1;g)+\overline{N}(r,\infty;g)-N_0(r,0;f')-N_0(r,0;g')+S(r,f)+S(r,g).\end{aligned}$$

Using Lemmas 2.1–2.4 we see that

$$\begin{aligned}(2.9)\qquad &\overline{N}(r,1;F)+\overline{N}(r,1;G)\\ &\le \frac{1}{2}[N(r,1;F)+N(r,1;G)]+N(r,1;F\mid=1)-\left(m-\frac{1}{2}\right)\overline{N}_*(r,1;F,G)\\ &\le \frac{1}{2}[N(r,1;F)+N(r,1;G)]+\overline{N}(r,0;f\mid\ge p+1)+\overline{N}(r,1;f\mid\ge p+1)\\ &\quad+\overline{N}_*(r,\infty;f,g)-\left(m-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g')\\ &\quad+S(r,f)+S(r,g).\end{aligned}$$

Applying (2.9) in (2.8) and noting that

$$\overline{N}(r,0;f)+\overline{N}(r,1;f)=\overline{N}(r,0;g)+\overline{N}(r,1;g),$$

the lemma follows. ∎

LEMMA 2.10 ([14, Lemma 6]). *If $H\equiv 0$, then F, G share $(1,\infty)$. If further F, G share $(\infty,0)$ then they share (∞,∞).*

LEMMA 2.11. *Let F, G be given by (2.1) and suppose they share $(1,m)$. Also let $\alpha_1,\dots,\alpha_n$ be the distinct elements of the set*

$$\left\{z:\frac{(n-1)(n-2)}{2}z^n-n(n-2)z^{n-1}+\frac{n(n-1)}{2}z^{n-2}-c=0\right\},$$

where $c\ne 0,1,1/2$ is a complex number and $n\ge 3$. Then

$$\overline{N}_L(r,1;F>G)\le\frac{1}{m+1}[\overline{N}(r,0;f)+\overline{N}(r,\infty;f)-N_\otimes(r,0;f')]+S(r,f),$$

where $N_\otimes(r,0;f')$ is the counting function of those 0-points of f' which are not in $f^{-1}(\{0,\alpha_1,\dots,\alpha_n\})$.

Proof. The proof can be carried out along the lines of the proof of [1, Lemma 2.14]. ∎

3. Proof of the theorem

Proof of Theorem 1.1. Let F, G be given by (2.1) and (2.2). Then F, G share $(1,m)$ and f, g share (∞,k) . We consider two cases, each of them split into several subcases.

CASE 1. Suppose that $\Phi\not\equiv 0$.

SUBCASE 1.1. Let $H \not\equiv 0$. First suppose $p = 0$.

In view of Definition 2.3 we observe that

$$\begin{aligned}\overline{N}_*(r,\infty;f,g) &= \overline{N}_L(r,\infty;f) + \overline{N}_L(r,\infty;g)\\ &\le \overline{N}(r,\infty;f \mid \ge k+2) + \overline{N}(r,\infty;g \mid \ge k+2)\\ &\le \frac{1}{k+2}\{N(r,\infty;f) + N(r,\infty;g)\}.\end{aligned}$$

Then using Lemma 2.3, Lemma 2.8 with $p = 0$ and Lemma 2.11 we deduce that

$$\begin{aligned}(3.1)\qquad & (n+1)\,\{T(r,f)+T(r,g)\}\\ &\le 3\{\overline{N}(r,0;f)+\overline{N}(r,1;f)\} + \left\{1+\frac{1}{k+2}\right\}\{N(r,\infty;f)+N(r,\infty;g)\}\\ &\quad + \frac{1}{2}[N(r,1;F)+N(r,1;G)] - \left(m-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)\\ &\quad + S(r,f)+S(r,g)\\ &\le 3\overline{N}_*(r,\infty;f,g) + \left\{1+\frac{1}{k+2}\right\}\{N(r,\infty;f)+N(r,\infty;g)\}\\ &\quad + \frac{n}{2}\{T(r,f)+T(r,g)\}\\ &\quad - \left(m-\frac{9}{2}\right)\overline{N}_*(r,1;F,G) + S(r,f)+S(r,g)\\ &\le \left\{\frac{n}{2}+1+\frac{4}{k+2}\right\}\{T(r,f)+T(r,g)\}\\ &\quad - \frac{2m-9}{2(m+1)}\{\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)\}\\ &\quad + S(r,f)+S(r,g)\\ &\le \left\{\frac{n}{2}+1+\frac{4}{k+2}+\frac{9-2m}{m+1}\right\}\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g).\end{aligned}$$

Since $2-\frac{4}{k+2} > \frac{9-2m}{m+1} > 0$, (3.1) gives a contradiction for $n \ge 4$.

Next suppose $p = 1$.

Using Lemma 2.3, Lemma 2.8 for $p = 0$ and again for $p = 1$, and Lemma 2.11, we get

$$\begin{aligned}(3.2)\qquad & (n+1)\{T(r,f)+T(r,g)\}\\ &\le \frac{7}{3}\{\overline{N}_*(r,\infty;f,g)+\overline{N}_*(r,1;F,G)\}\\ &\quad + \left\{1+\frac{1}{k+2}\right\}\{N(r,\infty;f)+N(r,\infty;g)\}\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}[N(r,1;F)+N(r,1;G)]\\
&-\left(m-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
\le&\left\{1+\frac{10}{3(k+2)}\right\}\{N(r,\infty;f)+N(r,\infty;g)\}+\frac{n}{2}\{T(r,f)+T(r,g)\}\\
&-\left(m-\frac{23}{6}\right)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
\le&\left\{\frac{n}{2}+1+\frac{10}{3(k+2)}\right\}\{T(r,f)+T(r,g)\}-\frac{6m-23}{6(m+1)}\{2T(r,f)+2T(r,g)\}\\
&+S(r,f)+S(r,g)\\
\le&\left\{\frac{n}{2}+1+\frac{10}{3(k+2)}+\frac{23-6m}{3(m+1)}\right\}\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g).
\end{aligned}
$$

Since the assumption for $p=1$ implies $2-\frac{10}{3(k+2)}>\frac{23-6m}{3(m+1)}>0$, (3.8) gives a contradiction for $n\ge 4$.

SUBCASE 1.2. Suppose $H\equiv 0$. Then

$$F\equiv\frac{AG+B}{CG+D}, \tag{3.3}$$

where A, B, C, D are constants such that $AD-BC\ne 0$. Also $T(r,F)=T(r,G)+O(1)$, i.e.,

$$T(r,f)=T(r,g)+O(1). \tag{3.4}$$

In view of Lemma 2.10 it follows that F and G share $(1,\infty)$ and (∞,∞), that is, f and g share (∞,∞). So in view of Lemma 2.8, $\overline{N}(r,0;f)+\overline{N}(r,1;f)=S(r,f)+S(r,g)$. Since $P(1)=1$, by a simple computation it can be easily seen that 1 is a zero with multiplicity 3 of $F-\frac{1}{c}=\frac{P(f)-1}{c}$ and hence

$$F-\frac{1}{c}=(f-1)^3Q_{n-3}(f),$$

where $Q_{n-3}(f)$ is a polynomial in f of degree $n-3$ and thus

$$
\begin{aligned}
\overline{N}\left(r,\frac{1}{c};F\right)&\le\overline{N}(r,1;f)+\overline{N}(r,0;Q_{n-3}(f))\\
&\le\overline{N}(r,1;f)+(n-3)T(r,f)+S(r,f).
\end{aligned}
$$

We now consider the following cases.

SUBCASE 1.2.1. Let $AC\ne 0$. From (3.3) we get

$$\overline{N}(r,\infty;G)=\overline{N}\left(r,\frac{A}{C};F\right). \tag{3.5}$$

Since F and G share $(1,\infty)$, it follows that $A/C \neq 1$. Suppose $A/C \neq 1/c$. Then in view of Lemma 2.3 and (3.4), by the Second Fundamental Theorem we get

$$\begin{aligned}(n+1)T(r,f) &\leq \overline{N}(r,0;f)+\overline{N}(r,1;f)+\overline{N}(r,\infty;f)\\ &\quad +\overline{N}\left(r,\frac{A}{C};F\right)+S(r,f)+S(r,g)\\ &= \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+S(r,f)\\ &\leq 2T(r,f)+S(r,f),\end{aligned}$$

which gives a contradiction for $n \geq 4$.

Next suppose $A/C = 1/c$. Then

$$F-\frac{A}{C} \equiv \frac{BC-AD}{C(CG+D)} \quad \text{i.e.,} \quad (f-1)^3 Q_{n-3}(f) \equiv \frac{BC-AD}{C(CG+D)}.$$

Suppose

$$Q_{n-3}(f) = (f-\alpha_1')\dots(f-\alpha_{n-3}'),$$

where α_i''s, $i=1,\dots,n-3$ are distinct. Then the above expression implies that any α_i'-point of f of order p (say) will be a pole of order q (say) of g. Consequently, we have

$$p = nq \geq n.$$

Noting that $\overline{N}(r,0;f)+\overline{N}(r,1;f) = S(r,f)+S(r,g)$, in view of (3.4) the Second Fundamental Theorem yields

$$\begin{aligned}&(n-2)T(r,f)\\ &\quad \leq \overline{N}(r,0;f)+\overline{N}(r,1;f)+\overline{N}(r,\infty;f)+\sum_{i=1}^{n-3}\overline{N}(r,\alpha_i';f)+S(r,f)\\ &\quad \leq \overline{N}(r,\infty;f)+\frac{n-3}{n}T(r,f)+S(r,f)\\ &\quad \leq \left(1+\frac{n-3}{n}\right)T(r,f)+S(r,f),\end{aligned}$$

which is a contradiction for $n \geq 4$.

Subcase 1.2.2. Let $A \neq 0$ and $C = 0$. Then $F \equiv \alpha_0 G + \beta_0$, where $\alpha_0 = A/D$ and $\beta_0 = B/D$.

We note that 1 cannot be a Picard exceptional value (P.e.v.) of F (or G). For, if it happens, then f (resp. g) omits $n \geq 4$ values, which is a contradiction.

So F and G have some 1-points. Then $\alpha_0+\beta_0 = 1$ and so

$$F \equiv \alpha_0 G + 1 - \alpha_0. \tag{3.6}$$

Suppose $\alpha_0 \neq 1$. If $1-\alpha_0 \neq 1/c$ then using Lemma 2.3, (3.4) and the Second Fundamental Theorem we get

$$\begin{aligned}
&2nT(r,f)\\
&\qquad \le \overline{N}(r,0;F) + \overline{N}(r,1-\alpha_0;F) + \overline{N}\left(r,\frac{1}{c};F\right) + \overline{N}(r,\infty;F) + S(r,F)\\
&\qquad \le \overline{N}(r,0;f) + 2T(r,f) + \overline{N}(r,0;G) + \overline{N}(r,1;f)\\
&\qquad\qquad + (n-3)T(r,f) + \overline{N}(r,\infty;f) + S(r,f)\\
&\qquad \le (n-1)T(r,f) + 3T(r,g) + \overline{N}(r,\infty;f) + S(r,f) + S(r,g)\\
&\qquad \le (n+3)T(r,f) + S(r,f),
\end{aligned}$$

which implies a contradiction since $n \ge 4$.

If $1-\alpha_0 = 1/c$, then from (3.6) we have $cF \equiv (c-1)G + 1$.

Noting that $c \neq 1/2$ and $\overline{N}(r,0;f) + \overline{N}(r,1;f) = \overline{N}(r,0;g) + \overline{N}(r,1;g)$, using Lemma 2.3, (3.4) and (3.6) we obtain, by the Second Fundamental Theorem,

$$\begin{aligned}
&2nT(r,g)\\
&\quad \le \overline{N}(r,0;G) + \overline{N}\left(r,\frac{1}{c};G\right) + \overline{N}\left(r,\frac{1}{1-c};G\right) + \overline{N}(r,\infty;G) + S(r,G)\\
&\quad \le 2T(r,g) + \overline{N}(r,0;g) + (n-3)T(r,g) + \overline{N}(r,1;g) + 2T(r,f) + \overline{N}(r,0;f)\\
&\qquad + \overline{N}(r,\infty;g) + S(r,g)\\
&\quad \le 3T(r,f) + nT(r,g) + S(r,f) + S(r,g)\\
&\quad \le (n+3)T(r,g) + S(r,g),
\end{aligned}$$

which implies a contradiction as $n \ge 4$. Therefore $\alpha_0 = 1$ and hence $F \equiv G$. This implies $\Phi \equiv 0$, a contradiction to the initial assumption.

Subcase 1.2.3. Let $A = 0$ and $C \neq 0$. Then

$$F \equiv \frac{1}{\gamma_0 G + \delta_0},$$

where $\gamma_0 = C/B$ and $\delta_0 = D/B$.

Clearly 1 cannot be a P.e.v. of F and so of G. Since F and G have some 1-points we have $\gamma_0 + \delta_0 = 1$ and so

$$F \equiv \frac{1}{\gamma_0 G + 1 - \gamma_0}. \tag{3.7}$$

Suppose $\gamma_0 \neq 1$. If $\gamma_0 \neq 1-c$, then noting that

$$\overline{N}(r,0;G) = \overline{N}\left(r,\frac{1}{1-\gamma_0};F\right) \neq \overline{N}\left(r,\frac{1}{c};F\right),$$

by the Second Fundamental Theorem, using Lemma 2.3 we can again deduce a contradiction as above when $n \ge 4$.

If $\gamma_0 = 1 - c$, from (3.7) we have

$$F \equiv \frac{1}{(1-c)G + c}.$$

If possible suppose that $\frac{1}{c} \neq \frac{c}{c-1}$. Now in the same way as above using (3.4), Lemma 2.3, and the Second Fundamental Theorem yields

$$\begin{aligned}
&2nT(r,g) \\
&\quad \leq \overline{N}(r,0;G) + \overline{N}\left(r,\frac{1}{c};G\right) + \overline{N}\left(r,\frac{c}{c-1};G\right) + \overline{N}(r,\infty;G) + S(r,G) \\
&\quad \leq \overline{N}(r,0;g) + \overline{N}(r,1;g) + 2T(r,g) + (n-3)T(r,g) + \overline{N}(r,\infty;F) \\
&\qquad + \overline{N}(r,\infty;G) + S(r,f) + S(r,g) \\
&\quad \leq nT(r,g) + N(r,\infty;f) + S(r,f) + S(r,g),
\end{aligned}$$

which implies a contradiction for $n \geq 4$.

Next suppose $\frac{1}{c} = \frac{c}{c-1}$. Then

$$F \equiv \frac{1}{-c^2(G - \frac{1}{c})}, \quad \text{i.e.,} \quad F\left(G - \frac{1}{c}\right) \equiv \frac{1}{-c^2}.$$

Since F, G share (∞,∞), it follows that 0 is a P.e.v. of F, which implies f omits three distinct complex numbers, which is impossible. So we must have $\gamma_0 = 1$, i.e., $FG \equiv 1$, which is impossible by Lemma 2.5.

Case 2. Suppose that $\Phi \equiv 0$. On integration we get $F - 1 \equiv A(G-1)$ for some non-zero constant A. So in view of Lemma 2.3, (3.4) is satisfied. Since by the assumption of the theorem $E_f(S_1,0) = E_g(S_1,0)$, we consider the following cases.

Subcase 2.1. First assume f and g share $(0,0)$ and $(1,0)$. If none of 0 and 1 is a P.e.v. of f and g, then we have $A = 1$. Similarly if one of 0 or 1 is a P.e.v. of f and g, then we get $A = 1$ and so in both cases we have $F \equiv G$, which in view of Lemma 2.7 implies $f \equiv g$. If both 0 and 1 are P.e.v. of f as well as of g then noting that here $F \equiv AG + (1-A)$ which is similar to (3.6), we can handle the situation as in Subcase 1.2.2. So we omit the details.

Subcase 2.2. Next suppose that f, g do not share $(0,0)$, $(1,0)$. Here we have to consider the following subcases.

Subcase 2.2.1. Suppose there exist z_0, z_1 such that

$$f(z_0) = 0, \quad g(z_0) = 1, \quad f(z_1) = 1, \quad g(z_1) = 0.$$

i.e., none of 0 and 1 is a P.e.v. of f and g. We note that from $F - 1 \equiv A(G-1)$ we get $P(f) - c(1-A) \equiv AP(g)$. If $A \neq 1$, then $c(1-A) \neq 0$. If $c(1-A) = 1$, then $A = \frac{c-1}{c}$. So $F - \frac{1}{c} \equiv \frac{c-1}{c}G$. We have $F(z_0) = 0$ and $G(z_0) = 1/c$. Putting these values we obtain $\frac{-1}{c} = \frac{c-1}{c^2}$, which implies $c = \frac{1}{2}$, a contradiction. So $c(1-A) \neq 0, 1$. Hence $P(f) - c(1-A)$ has simple zeros and so

$$(f-\omega_1)\dots(f-\omega_n) \equiv A\frac{(n-1)(n-2)}{2}g^{n-2}(g-\gamma)(g-\delta),$$

where ω_i $(i=1,\dots,n)$ are the distinct zeros of $P(f)-c(1-A)$. Since f, g share the set S_1, from the above we see that 0 is a P.e.v. of g, a contradiction.

SUBCASE 2.2.2. If no such z_0 exists, i.e., if 0 is a P.e.v. of f and 1 is a P.e.v. of g, then again as above from $\Phi \equiv 0$ we get

$$F \equiv AG + 1 - A, \tag{3.8}$$

i.e.,

$$\frac{P(f)}{A} \equiv P(g) - \frac{c(A-1)}{A}. \tag{3.9}$$

Clearly, $\frac{c(A-1)}{A} \neq 0$ as $c \neq 0$ and $A \neq 1$. Now if $\frac{c(A-1)}{A} = 1$ then $A = \frac{c}{c-1}$. Since any 1-point of f is 0-point of g, from (3.8) we have $\frac{1}{c} = 1 - A$, i.e., $A = \frac{c-1}{c}$. Therefore

$$\frac{c-1}{c} = \frac{c}{c-1},$$

which implies $c = \frac{1}{2}$, a contradiction. This implies $\frac{c(A-1)}{A} \neq 1$ and so $P(g) - \frac{c(A-1)}{A}$ has n distinct zeros β'_j, say $(j=1,\dots,n)$. Hence from (3.9) we have

$$\frac{(n-1)(n-2)}{2A}f^{n-2}(f-\gamma)(f-\delta) \equiv (g-\beta'_1)\dots(g-\beta'_n).$$

Now by the Second Fundamental Theorem and (3.4) we get

$$\begin{aligned} nT(r,g) &\le \overline{N}(r,0;g) + \overline{N}(r,1;g) + \sum_{j=1}^{n}\overline{N}(r,\beta'_j;g) + S(r,g) \\ &\le \overline{N}(r,0;g) + \overline{N}(r,\gamma;f) + \overline{N}(r,\delta;f) + S(r,g) \\ &\le 3T(r,g) + S(r,g), \end{aligned}$$

which is a contradiction for $n \ge 4$.

SUBCASE 2.2.3. If no such z_0, z_1 exist at all, i.e., 0 and 1 are both Picard exceptional values of f and g then again we can obtain either (3.9) or

$$P(f) - c(1-A) \equiv AP(g). \tag{3.10}$$

We prove that either the right hand side of (3.9) or the left hand side of (3.10) will have n distinct factors. Now if $\frac{c(A-1)}{A} = 1$, i.e., the right hand side of (3.9) does not have n distinct factors, then $A = \frac{c}{c-1}$ and hence $c(1-A) = -A = \frac{c}{1-c} \neq 1$ as $c \neq \frac{1}{2}$. So $P(f) - c(1-A)$ has simple zeros and consequently we have $(f-\omega_1)\dots(f-\omega_n) \equiv A\frac{(n-1)(n-2)}{2}g^{n-2}(g-\gamma)(g-\delta)$. Therefore by the Second Fundamental Theorem and (3.4),

$$
\begin{aligned}
nT(r,f) &\le \sum_{i=1}^{n} \overline{N}(r,\omega_i;f) + \overline{N}(r,0;f) + \overline{N}(r,1;f) + S(r,f) \\
&\le \overline{N}(r,\gamma;g) + \overline{N}(r,\delta;g) + S(r,f),
\end{aligned}
$$

which is a contradiction for $n \geq 3$. ■

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Abhijit Banerjee, Molla Basir Ahamed
Department of Mathematics
University of Kalyani
Nadia, West Bengal 741235, India
E-mail: abanerjee_kal@yahoo.co.in, abanerjee_kal@rediffmail.com,
abanerjeekal@gmail.com
bsrhmd117@gmail.com, bsrhmd116@gmail.com

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