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WEAKLY COMPACT OPERATORS ON KÖTHE-BOCHNER SPACES WITH THE MIXED TOPOLOGY

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Abstract. Let E be a Banach function space and let X be a real Banach space. We examine weakly compact linear operators from a Köthe-Bochner space $E(X)$ endowed with some natural mixed topology $\gamma_{E(X)}$ (in the sense of Wiweger) to a Banach space Y.

1. Introduction and preliminaries. The paper is devoted to the study of weakly compact operators acting from a Köthe-Bochner space $E(X)$ endowed with the mixed topology $\gamma_{E(X)}$ to a Banach space Y.

The problem of characterizing weakly compact operators has been considered by many authors (see [Ga], [G], [D], [Ru]). In [G, III. 3, Corollary 1 of Theorem 11] A. Grothendieck has proved that every linear continuous operator from a quasinormable Hausdorff locally convex space X into a Banach space Y, which transforms bounded sets into relatively weakly compact sets is weakly compact. Grothendieck's result was partially extended by D. van Dulst to the case when Y is Fréchet space, but under the additional assumption about the space X (see [D] for more details). Both those results were later extended by W. Ruess in [Ru] for operators acting between gDF-spaces and Fréchet spaces. gDF spaces have been introduced and studied by K. Noureddine in [No_1], [No_2] under the name of D_b-spaces ("espaces D_b" in French). Since not only many properties of classical DF-spaces of Grothendieck carry over to D_b-spaces, but also some fruitful DF-techniques can be applied to them, W. Ruess decided in [Ru] to change their original name of Noureddine to the gDF-spaces (generalized DF-spaces), in order to stress their close relationship with DF-spaces.

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Given a topological vector space (L,ξ), by $(L,\xi)^*$ or L^*_ξ we will denote its topological dual and by $Bd(L,\xi)$ we will denote the family of all ξ-bounded subsets of L. We denote by $\sigma(L,K)$ and $\tau(L,K)$ the weak topology and the Mackey topology on L with respect to a dual system $\langle L,K\rangle$ respectively. Recall that a subset Z of L is said to be conditionally $\sigma(L,K)$-compact (resp. relatively $\sigma(L,K)$-sequentially compact) whenever each sequence in Z contains a $\sigma(L,K)$-Cauchy subsequence (resp. each sequence in Z contains a subsequence which is $\sigma(L,K)$-convergent to some element of L).

For terminology concerning Riesz spaces and function spaces we refer to [AB], [KA]. Throughout the paper let (Ω,Σ,μ) be an atomless, complete and σ-finite measure space and let L^0 denote the corresponding space of μ-equivalence classes of Σ-measurable real valued functions. Then L^0 is super Dedekind complete Riesz space under the ordering $u_1\le u_2$ whenever $u_1(\omega)\le u_2(\omega)$ a.e. on Ω. A Banach space $(E,\|\cdot\|_E)$ is assumed to be a *Banach function space*, that is, E is an ideal of L^0 with $\operatorname{supp} E=\Omega$ and $\|\cdot\|_E$ is a Riesz norm. The Köthe dual of E is defined by

$$E'=\left\{v\in L^0:\int_\Omega |u(\omega)v(\omega)|\,d\mu<\infty \text{ for all } u\in E\right\}.$$

The associated norm $\|\cdot\|_{E'}$ on E' is defined for $v\in E'$ by

$$\|v\|_{E'}=\sup\left\{\left|\int_\Omega u(\omega)v(\omega)\,d\mu\right| : u\in E, \|u\|_E\le 1\right\}.$$

It is known that $\operatorname{supp} E'=\Omega$ (see [KA, Theorem 6.1.3]). Recall that a Banach function space $(E,\|\cdot\|_E)$ is said to be *perfect* if $E''=E$ and $\|u\|_{E''}=\|u\|_E$ for $u\in E$. It is well known that E is perfect if and only if the norm $\|\cdot\|_E$ satisfies both the σ-Fatou property (i.e., $0\le u_n\uparrow u$ in E implies $\|u_n\|_E\uparrow\|u\|_E$) and the σ-Levy property (i.e., if $0\le u_n\uparrow$ in E and $\sup\|u_n\|_E<\infty$, then there exists u in E such that $u_n\uparrow u$) (see [KA, Theorem 6.1.7]). We denote by E_a the ideal of elements of order continuous norm in E, i.e.,

$$E_a=\{u\in E: |u|\ge u_n\downarrow 0 \text{ in } E \text{ imply } \|u_n\|\to 0\}.$$

From now on in this paper we assume that $(E,\|\cdot\|_E)$ is a perfect Banach function space and $\operatorname{supp}(E')_a=\Omega$. Note that $(L^1)'=L^\infty$ and $(L^\infty)_a=\{0\}$. Hence the space L^1 is excluded.

Let Z be a $\sigma(E,(E')_a)$-bounded subset of E. Then Z is also $|\sigma|(E,(E')_a)$-bounded (see [AB, Theorem 2.33]), so one can define a Riesz seminorm p_Z on $(E')_a$ by

$$p_Z(v)=\sup\left\{\int_\Omega |u(\omega)v(\omega)|\,d\mu : u\in Z\right\}.$$

Then by [N_2, Theorem 1.1] we have

$$Bd(E,\sigma(E,(E')_a))=Bd(E,\|\cdot\|_E).$$

Hence for every $\sigma(E,(E')_a)$-bounded subset Z of E the seminorm p_Z on $(E')_a$ is order continuous. Making use of [N_4, Proposition 1.1] and [BD, Corollary 5.2] we get

Proposition 1.1. *For a subset Z of E the following statements are equivalent:*

(i) *Z is conditionally $\sigma(E,(E')_a)$-compact.*
(ii) *Z is relatively $\sigma(E,(E')_a)$-sequentially compact.*

(iii) Z *is* $\sigma(E,(E')_a)$*-bounded.*
(iv) Z *is* $\|\cdot\|_E$*-bounded.*

Now we establish terminology and some basic results concerning vector-valued function spaces (see [Bu$_1$], [Bu$_2$], [L, Chap. 3]). Let $(X,\|\cdot\|_X)$ be a real Banach space and let X^* stand for the Banach dual of X. Let B_X denote the closed unit ball in X. By $L^0(X)$ we denote the set of μ-equivalence classes of all strongly Σ-measurable functions $f:\Omega\to X$. The F-norm

$$\|f\|_{L^0(X)}=\int_\Omega \frac{\|f(\omega)\|_X}{1+\|f(\omega)\|_X}w(\omega)d\mu \quad \text{for} \quad f\in L^0(X),$$

where $w:\Omega\to(0,\infty)$ is a Σ-measurable function with $\int_\Omega w(\omega)d\mu=1$, determines the topology $\mathcal{T}_0(X)$ on $L^0(X)$ of convergence in measure on sets of finite measure.

For $f\in L^0(X)$ let us set $\widetilde{f}(\omega)=\|f(\omega)\|_X$ for $\omega\in\Omega$. The linear space

$$E(X)=\{f\in L^0(X):\widetilde{f}\in E\}$$

provided with the norm $\|f\|_{E(X)}:=\|\widetilde{f}\|_E$ is a Banach space and is called a *Köthe-Bochner space* (see [L]). For $r>0$ we will write

$$B_{E(X)}(r)=\{f\in E(X):\|f\|_{E(X)}\le r\}.$$

Let $L^0(X^*,X)$ be the set of weak*-equivalence classes of all weak*-measurable functions $g:\Omega\to X^*$. One can define the so-called *abstract norm* $\vartheta: L^0(X^*,X)\to L^0$ by $\vartheta(g)=\sup\{|g_x|:x\in B_X\}$, where $g_x(\omega)=g(\omega)(x)$ for $\omega\in\Omega$ and $x\in X$. Then for $f\in L^0(X)$ and $g\in L^0(X^*,X)$ the function $\langle f,g\rangle:\Omega\to\mathbb{R}$ defined by $\langle f,g\rangle(\omega)=\langle f(\omega),g(\omega)\rangle$ is measurable and $|\langle f,g\rangle|\le\widetilde{f}\vartheta(g)$. Moreover, $\vartheta(g)=\widetilde{g}$ for $g\in L^0(X^*)$. For an ideal M of E' let

$$M(X^*,X)=\{g\in L^0(X^*,X):\vartheta(g)\in M\}.$$

Then $M(X^*,X)$ is an ideal of $E'(X^*,X)$, i.e., if $\vartheta(g_1)\le\vartheta(g_2)$ with $g_1\in E'(X^*,X)$ and $g_2\in M(X^*,X)$, then $g_1\in M(X^*,X)$. $M(X^*,X)$ can be provided with the norm

$$\|g\|_{M(X^*,X)}:=\|\vartheta(g)\|_{E'} \quad \text{for} \quad g\in M(X^*,X).$$

In particular, we will consider the dual pair $\langle E(X),(E')_a(X^*,X)\rangle$ with the duality:

$$\langle f,g\rangle=\int_\Omega\langle f(\omega),g(\omega)\rangle d\mu \quad \text{for} \quad f\in E(X),\ g\in(E')_a(X^*,X).$$

2. Mixed topologies on Köthe-Bochner spaces. In this section we consider the mixed topology $\gamma[\mathcal{T}_E(X),\mathcal{T}_0(X)|_{E(X)}]$ on $E(X)$ (briefly $\gamma_{E(X)}$), where $\mathcal{T}_E(X)$ stands for the topology on $E(X)$ of the norm $\|\cdot\|_{E(X)}$. For the definition and basic properties of $\gamma_{E(X)}$ see [F], [W]. In case when $X=\mathbb{R}$ the mixed topology γ_E $(=\gamma_{E(\mathbb{R})})$ on E has been studied in [N$_1$] and [N$_2$]. It is known that $\mathcal{T}_0(X)|_{E(X)}\subset\gamma_{E(X)}\subset\mathcal{T}_E(X)$. We will need the following:

Proposition 2.1. *We have* $Bd(E(X),\gamma_{E(X)})=Bd(E(X),\|\cdot\|_{E(X)})$.

Proof. Using [KA, Lemma 4.3.4] we see that $B_{E(X)}(1)$ is closed in $(E(X),\mathcal{T}_0(X)|_{E(X)})$. Hence by [W, Theorem 2.4.1] we get $Bd(E(X),\gamma_{E(X)})=Bd(E(X),\|\cdot\|_{E(X)})$. ■

The mixed topology $\gamma_{E(X)}$ is a Hausdorff locally convex-solid topology on $E(X)$ (see [F, §3]) and it is the finest locally convex topology on $E(X)$ which agrees with $\mathcal{T}_0(X)$ on $\|\cdot\|_{E(X)}$-bounded sets in $E(X)$ (see [W, 2.2.2]). Since $(B_{E(X)}(2^n) : n \in \mathbb{N})$ is a fundamental sequence of $\gamma_{E(X)}$-bounded sets in $E(X)$, $(E(X), \gamma_{E(X)})$ is *a generalized DF space* (see [Ru, Definition 1.1]).

Recall that a locally solid topology τ on $E(X)$ is said to be *uniformly Lebesgue* if $f_n \to 0$ for τ in $E(X)$ whenever $\|f_n\|_{L^0(X)} \to 0$ with $\sup_n \|f_n\|_{E(X)} < \infty$ (see [F, Definition 2.2]).

The basic properties of $\gamma_{E(X)}$ are given in the following (see [F, Theorem 3.1]):

PROPOSITION 2.2. *We have*

(i) $\gamma_{E(X)}$ *is the finest uniformly Lebesgue toplogy on* $E(X)$.
(ii) $f_n \to 0$ *in* $E(X)$ *for* $\gamma_{E(X)}$ *if and only if* $\|f_n\|_{L^0(X)} \to 0$ *and* $\sup_n \|f_n\|_{E(X)} < \infty$.

A linear operator $T : E(X) \to Y$ is said to be γ-*linear* if $\|T(f_n)\|_Y \to 0$ whenever $\|f_n\|_{L^0(X)} \to 0$ and $\sup_n \|f_n\|_{E(X)} < \infty$.

PROPOSITION 2.3. *For a linear operator* $T : E(X) \to Y$ *the following statements are equaivalent:*

(i) T *is* $(\gamma_{E(X)}, \|\cdot\|_Y)$-*continuous.*
(ii) T *is sequentially* $(\gamma_{E(X)}, \|\cdot\|_Y)$-*continuous.*
(iii) T *is* γ-*linear.*
(iv) T *is* $(\gamma_{E(X)}|_{B_{E(X)}(r)}, \|\cdot\|_Y)$-*continuous for every* $r > 0$.

Proof. (i)⇒(ii) It is obvious.

(ii)⇒(iii) It follows from Proposition 2.2(i).

(iii)⇒(iv) It is obvious, because $\gamma_{E(X)}|_{B_{E(X)}(r)} = \mathcal{T}_0(X)|_{B_{E(X)}(r)}$ for $r > 0$.

(iv)⇔(i) See [W, 2.2.4]. ∎

Recall that a Banach space X is said to be *almost reflexive* if every norm-bounded subset of X is conditionally weakly compact (see [C], [H]). The fundamental l^1-Rosenthal theorem [R] says that X is almost reflexive if and only if it contains no isomorphic copy of l^1.

From now on for a subset H of $E(X)$ we will denote

$$\widetilde{H} = \{\widetilde{f} : f \in H\}.$$

The following result extends Proposition 1.1 to the vector-valued setting.

PROPOSITION 2.4. *Let* X *be an almost reflexive Banach space. Then for a subset* H *of* $E(X)$ *the following statements are equivalent:*

(i) $\sup_{f\in H} \|f\|_{E(X)} < \infty$.
(ii) H *is conditionally* $\sigma(E(X), (E')_a(X^*, X))$-*compact.*

Moreover, if X *is a reflexive Banach space, then the statements* (i)–(ii) *are equivalent to the following:*

(iii) H *is relatively* $\sigma(E(X), (E')_a(X^*, X))$-*compact.*

Proof. (i)⇒(ii) Assume that $\sup_{f\in H}\|f\|_{E(X)}<\infty$, i.e., $\widetilde{H}$ is a $\|\cdot\|_E$-bounded subset of E. Then by Proposition 1.1 $\widetilde{H}$ is conditionally $\sigma(E,(E')_a)$-compact. Making use of [N$_4$, Corollary 2.3] we obtain that H is conditionally $\sigma(E(X),\,(E')_a(X^*,X))$-compact.

(ii)⇒(i) Assume that H is conditionally $\sigma(E(X),(E')_a(X^*,X))$-compact. Then $\widetilde{H}$ is conditionally $\sigma(E,(E')_a)$-compact in E (see [N$_4$, Theorem 2.2]), so by Proposition 1.1 $\sup_{f\in H}\|f\|_{E(X)}<\infty$.

(i)⇔(iii) See [N$_3$, Corollary 2.4]. ∎

3. Weakly compact operators on Köthe-Bochner spaces. In this section we examine linear operators $T:E(X)\to Y$ whenever $E(X)$ is provided with the mixed topology $\gamma_{E(X)}$. Recall that a linear operator $T:E(X)\to Y$ is said to be $(\gamma_{E(X)},\|\cdot\|_Y)$-*weakly compact* if there exists a neighbourhood V of 0 for $\gamma_{E(X)}$ such that $T(V)$ is a relatively $\sigma(Y,Y^*)$-compact subset of Y.

THEOREM 3.1. *For a linear operator $T:E(X)\to Y$ the following statements are equivalent:*

(i) *T is $(\gamma_{E(X)},\|\cdot\|_Y)$-weakly compact.*
(ii) *T is $(\gamma_{E(X)},\|\cdot\|_Y)$-continuous and $(\|\cdot\|_{E(X)},\|\cdot\|_Y)$-weakly compact.*

Proof. (i)⇒(ii) It is obvious.

(ii)⇒(i) Assume that T is $(\gamma_{E(X)},\|\cdot\|_Y)$-continuous and $(\|\cdot\|_{E(X)},\|\cdot\|_Y)$-weakly compact. Hence for every $r>0$, $T(B_{E(X)}(r))$ is a relatively $\sigma(Y,Y^*)$-compact subset of Y. Since $Bd(E(X),\gamma_{E(X)})=Bd(E(X),\|\cdot\|_{E(X)})$ (see Proposition 2.1), in view of [Ru, Theorem 3.1] T is $(\gamma_{E(X)},\|\cdot\|_{E(X)})$-weakly compact, as desired. ∎

REMARK. For the proof of the implication (ii)⇒(i) one can also use the earlier result of Grothendieck. Indeed, since $(E(X),\gamma_{E(X)})$ is a generalized DF-space, it is as well a quasinormable Hausdorff locally convex space. Moreover, T is $(\gamma_{E(X)},\|\cdot\|_Y)$-continuous and $(\|\cdot\|_{E(X)},\|\cdot\|)$-weakly compact, so T transforms $\|\cdot\|_{E(X)}$-bounded sets into relatively $\sigma(Y,Y^*)$-compact sets in Y. Thus by [G, III. 3, Corollary 1 of Theorem 11] T is $(\gamma_{E(X)},\|\cdot\|_Y)$-weakly compact, because $Bd(E(X),\gamma_{E(X)})=Bd(E(X),\|\cdot\|_{E(X)})$.

COROLLARY 3.2. *Assume that a linear operator $T:E(X)\to Y$ is $(\|\cdot\|_{E(X)},\|\cdot\|_Y)$-weakly compact. Then the following statements are equivalent:*

(i) *T is $(\gamma_{E(X)},\|\cdot\|_Y)$-weakly compact.*
(ii) *T is $(\gamma_{E(X)},\|\cdot\|_Y)$-continuous.*

COROLLARY 3.3. *Assume that a linear operator $T:E(X)\to Y$ is $(\gamma_{E(X)},\|\cdot\|_Y)$-continuous. Then the following statements are equivalent:*

(i) *T is $(\gamma_{E(X)},\|\cdot\|_Y)$-weakly compact.*
(ii) *T is $(\|\cdot\|_{E(X)},\|\cdot\|_Y)$-weakly compact.*

Now we are ready to present our main results.

THEOREM 3.4. *Let X be an almost reflexive Banach space and let Y be a weakly sequentially complete Banach space. Then for a linear operator $T:E(X)\to Y$ the following* *statements are equivalent:*

(i) *T is $(\gamma_{E(X)}, \|\cdot\|_Y)$-continuous.*
(ii) *T is $(\gamma_{E(X)}, \|\cdot\|_Y)$-weakly compact.*

Proof. (i)⇒(ii) Assume that T is $(\gamma_{E(X)}, \|\cdot\|_Y)$-continuous. In view of Corollary 3.3 it is enough to show that T transforms $\|\cdot\|_{E(X)}$-bounded sets in $E(X)$ into relatively $\sigma(Y, Y^*)$-compact sets in Y. Indeed, let H be a $\|\cdot\|_{E(X)}$-bounded set in $E(X)$. Then by Proposition 2.4 H is conditionally $\sigma(E(X), (E')_a(X^*, X))$-compact. In view of [F, Theorem 3.2] it is seen that the topology $\gamma_{E(X)}$ is coarser than the Mackey topology $\tau(E(X), (E')_a(X^*, X))$. Hence T is also $(\tau(E(X), (E')_a(X^*, X)), \|\cdot\|_Y)$-continuous and it follows that T is $(\sigma(E(X), (E')_a(X^*, X)), \sigma(Y, Y^*))$-continuous. Thus $T(H)$ is a conditionally $\sigma(Y, Y^*)$-compact subset of Y, and since Y is $\sigma(Y, Y^*)$-sequentially complete, we obtain that $T(H)$ is relatively $\sigma(Y, Y^*)$-sequentially compact in Y. It follows that $T(H)$ is relatively $\sigma(Y, Y^*)$-compact in Y, as desired.

(ii)⇒(i) See Theorem 3.1. ■

THEOREM 3.5. *Let X be a reflexive Banach space. Then for a linear operator $T : E(X) \to Y$ the following statements are equivalent:*

(i) *T is $(\gamma_{E(X)}, \|\cdot\|_Y)$-continuous.*
(ii) *T is $(\gamma_{E(X)}, \|\cdot\|_Y)$-weakly compact.*

Proof. (i)⇒(ii) Assume that T is $(\gamma_{E(X)}, \|\cdot\|_Y)$-continuous. Since $\gamma_{E(X)} \subset \tau(E(X), (E')_a(X^*, X))$, T is also $(\tau(E(X), (E')_a(X^*, X)), \|\cdot\|_Y)$-continuous. It follows that T is also $(\sigma(E(X), (E')_a(X^*, X)), \sigma(Y, Y^*))$-continuous. Note that T transforms $\|\cdot\|_{E(X)}$-bounded sets in $E(X)$ into relatively $\sigma(Y, Y^*)$-compact sets in Y. Indeed, let H be a $\|\cdot\|_{E(X)}$-bounded set in $E(X)$. Then by Proposition 2.4, H is relatively $(\sigma(E(X), (E')_a(X^*, X))$-compact, and consequently $T(H)$ is relatively $\sigma(Y, Y^*)$-compact in Y. By Corollary 3.3, T is $(\gamma_{E(X)}, \|\cdot\|_Y)$-weakly compact.

(ii)⇒(i) It is obvious. ■

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