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A NOTE ON GENERALIZED EQUIVARIANT HOMOTOPY GROUPS

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Abstract. In this paper, we generalize the equivariant homotopy groups or equivalently the Rhodes groups. We establish a short exact sequence relating the generalized Rhodes groups and the generalized Fox homotopy groups and we introduce Γ-Rhodes groups, where Γ admits a certain co-grouplike structure. Evaluation subgroups of Γ-Rhodes groups are discussed.

1. Introduction. In 1966, F. Rhodes [8] introduced the fundamental group of a transformation group (X, G) for a topological space on which a group G acts. This group, denoted by $\sigma_1(X, x_0, G)$, is the equivariant analog of the classical fundamental group $\pi_1(X, x_0)$. Rhodes showed that $\sigma_1(X, x_0, G)$ is a group extension of $\pi_1(X, x_0)$ with quo-

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tient G. Thus, $\sigma_1(X, x_0, G)$ incorporates the G-action as well as the action of $\pi_1(X, x_0)$ on the universal cover $\tilde{X}$ of the space X. This group has been used in [10] to study the Nielsen fixed point theory for equivariant maps. In 1969, F. Rhodes [9] extended $\sigma_1(X, x_0, G)$ to $\sigma_n(X, x_0, G)$, which is the equivariant higher homotopy group of (X, G). Like $\sigma_1(X, x_0, G)$, $\sigma_n(X, x_0, G)$ is an extension of the Fox torus homotopy group $\tau_n(X, x_0)$ but not of the classical homotopy group $\pi_n(X, x_0)$ by G. The Fox torus homotopy groups were first introduced by R. Fox [2] in 1948 in order to give a geometric interpretation of the classical Whitehead product. Recently, a modern treatment of $\tau_n(X, x_0)$ and of $\sigma_n(X, x_0, G)$ has been given in [4] and in [5], respectively. In [5], we further investigated the relationships between the Gottlieb groups of a space and of its orbit space, analogous to the similar study in [3]. Further properties of the Fox torus homotopy groups, their generalizations, and Jacobi identities were studied in [6]. It is therefore natural to generalize $\sigma_n(X, x_0, G)$ to more general constructions with respect to general spaces and to co-grouplike spaces Γ other than the 1-sphere $\mathbb{S}^1$.

The main objective of this paper is to generalize $\sigma_n(X, x_0, G)$ of a G-space X with respect to a space W and also with respect to a pair (W, Γ), where W is a space and Γ satisfies a suitable notion of the classical co-grouplike space. We prove in section 1 that the Rhodes exact sequence of [9] can be generalized to $\sigma_W(X, x_0, G) := \{[f; g] \mid f : (\widehat{\Sigma} W, v_1, v_2) \to (X, x_0, gx_0)\}$, the W-Rhodes group, with the generalized Fox torus homotopy group $\tau_W(X, x_0)$ as the kernel. In section 2, we further extend the construction of Rhodes groups to $\sigma_W^\Gamma(X, x_0, G) := \{[f; g] \mid f : (\Gamma(W), \bar{\gamma}_1, \bar{\gamma}_2) \to (X, x_0, gx_0)\}$, the W-Γ-Rhodes groups, where Γ admits a co-grouplike structure with *two* basepoints. Under such assumptions, we obtain a W-Γ-generalization of the Rhodes exact sequence [9]. In the last section, we generalize the notion of the Gottlieb (evaluation) subgroup to that of a W-Γ-Rhodes group and we establish a short exact sequence generalizing [5, Theorem 2.2]. Throughout, G denotes a group acting on a compactly generated Hausdorff path-connected space X with a basepoint x_0. The associated pair (X, G) is called in the literature a transformation group.

2. Generalized Rhodes groups. For $n \geq 1$, F. Rhodes [9] defined higher homotopy groups $\sigma_n(X, x_0, G)$ of a pair (X, G) which is an extension of $\tau_n(X, x_0)$ by G so that

$$1 \to \tau_n(X, x_0) \to \sigma_n(X, x_0, G) \to G \to 1 \tag{1}$$

is exact. Here, $\tau_n(X, x_0)$ denotes the n-th torus homotopy group of X introduced by R. Fox [2]. The group $\tau_n = \tau_n(X, x_0)$ is defined to be the fundamental group of the function space $X^{\mathbb{T}^{n-1}}$ and is uniquely determined by the groups $\tau_1, \tau_2, \ldots, \tau_{n-1}$ and the Whitehead products, where $\mathbb{T}^{n-1}$ is the $(n-1)$-dimensional torus. The group τ_n is non-abelian in general.

Now we recall the construction of $\sigma_n(X, x_0, G)$ presented in [9]. Suppose that X is a G-space with a basepoint $x_0 \in X$ and let $C_n = I \times \mathbb{T}^{n-1}$. We say that a map $f : C_n \to X$ is of *order* $g \in G$ provided $f(0, t_2, \ldots, t_n) = x_0$ and $f(1, t_2, \ldots, t_n) = g(x_0)$ for $(t_2, \ldots, t_n) \in \mathbb{T}^{n-1}$. Two maps $f_0, f_1 : C^n \to X$ of order g are said to be homotopic if there exists a continuous map $F : C^n \times I \to X$ such that:

- $F(t, t_2, \dots, t_n, 0) = f_0(t, t_2, \dots, t_n)$;
- $F(t, t_2, \dots, t_n, 1) = f_1(t, t_2, \dots, t_n)$;
- $F(0, t_2, \dots, t_n, s) = x_0$;
- $F(1, t_2, \dots, t_n, s) = gx_0$ for all $(t_2, \dots, t_n) \in \mathbb{T}^{n-1}$ and $s, t \in$I.

Denote by $[f; g]$ the homotopy class of a map $f : C_n \to X$ of order g and by $\sigma_n(X, x_0, G)$ the set of all such homotopy classes. We define an operation $*$ on the set $\sigma_n(X, x_0, G)$ by

$$[f'; g'] * [f; g] := [f' + g'f; g'g].$$

This operation makes $\sigma_n(X, x_0, G)$ a group.

We have generalized the Fox torus homotopy groups in [4]. In this section, we give a similar generalization of Rhodes groups. In a special case, we obtain an extension group of the Abe group considered in [1].

Let X be a path-connected space with a basepoint x_0. For any space W, we let

$$\sigma_W(X, x_0, G) := \{[f; g] \mid f : (\widehat{\Sigma} W, v_1, v_2) \to (X, x_0, gx_0)\}$$

where $[f; g]$ denotes the homotopy class of the map f of order $g \in G$, v_1 and v_2 are the vertices of the cones C^+W and C^-W, respectively and $\widehat{\Sigma} W = C^+W \cup C^-W$. Under the operation $[f_1; g_1] * [f_2; g_2] := [f_1 + g_1 f_2; g_1 g_2]$, σ_W is a group called a *W-Rhodes group.*

Write $C(W, X)$ for the mapping space of all continuous maps from W to X with the compact-open topology. We point out that $\sigma_W(X, x_0, G) = \sigma_1(C(W, X), \bar{x_0}, G)$ provided W is a locally-compact space, where $(gf)(x) = gf(x)$ for $f \in C(W, X)$, $g \in G$ and $\bar{x_0}$ denotes the constant map from $C(W, X)$ determined by the point $x_0 \in X$.

The canonical projection $\sigma_W(X, x_0, G) \to G$ given by $[f; g] \mapsto g$ has the kernel $\{[f; 1] \mid f : (\widehat{\Sigma} W, v_1, v_2) \to (X, x_0, x_0)\}$. It is easy to see that this kernel is isomorphic to the generalized Fox torus group $[\Sigma(W \sqcup *), X] = \tau_W(X, x_0)$ defined in [4]. Therefore, we get the following result.

THEOREM 1. *The sequence*

$$1 \to \tau_W(X, x_0) \to \sigma_W(X, x_0, G) \to G \to 1 \tag{2}$$

is exact.

REMARK 1. When $W = \mathbb{T}^{n-1}$, the $(n-1)$-dimensional torus, σ_W coincides with the n-th Rhodes group σ_n and (2) reduces to (1). When $W = \mathbb{S}^{n-1}$, the $(n-1)$-sphere, τ_W becomes κ_n, the n-th Abe group (see [2] or [4]). Thus, by Theorem 1, we have the exact sequence

$$1 \to \pi_n(X, x_0) \rtimes \pi_1(X, x_0) \cong \kappa_n(X, x_0) \to \sigma_{\mathbb{S}^{n-1}}(X, x_0, G) \to G \to 1. \tag{3}$$

One can also generalize the split exact sequence for Rhodes groups from [9] as follows.

THEOREM 2. *Let W be a space with a basepoint w_0. Then, for any space V, the sequence*

$$1 \to [(V \times W)/V, \Omega X] \to \sigma_{V \times W}(X, x_0, G) \overset{\leftarrow}{\to} \sigma_V(X, x_0, G) \to 1 \tag{4}$$

is split exact.

Proof. By [4, Theorem 3.1], we have the split exact sequence

$$1 \to [(V \times W)/V, \Omega X] \to \tau_{V \times W}(X) \overset{\leftarrow}{\to} \tau_V(X) \to 1. \tag{5}$$

Given $[F; g] \in \sigma_{V\times W}(X, x_0, G)$, where $F : \widehat{\Sigma}(V \times W) \to X$, let $f : \widehat{\Sigma}V \to X$ be the composite map of $\widehat{\Sigma}V \approx \widehat{\Sigma}(V \times \{w_0\}) \to \widehat{\Sigma}(V \times W)$ with F. This map gives rise to a homomorphism $\sigma_{V\times W}(X, x_0, G) \to \sigma_V(X, x_0, G)$. Likewise, using the projection $V \times W \to V$, one obtains a section $\sigma_V(X, x_0, G) \to \sigma_{V\times W}(X, x_0, G)$. We have the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \tau_{V\times W}(X, x_0) & \longrightarrow & \sigma_{V\times W}(X, x_0, G) & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \tau_V(X, x_0) & \longrightarrow & \sigma_V(X, x_0, G) & \longrightarrow & G & \longrightarrow & 1, \end{array} \tag{6}$$

where the first two vertical homomorphisms have sections. Combining with (5), the assertion follows. ■

As an immediate corollary of Theorem 2, we have the following:

COROLLARY 3. *The sequence*

$$1 \to [W, \Omega X] \to \sigma_W(X, x_0, G) \overset{\leftarrow}{\to} \sigma_1(X, x_0, G) \to 1 \tag{7}$$

is split exact.

Proof. The result follows from Theorem 2 by letting V be a point. ■

REMARK 2. For any space W, Corollary 3 asserts that $\sigma_1(X, x_0, G)$ acts on $[\Sigma W, X] = [W, \Omega X]$ according to the splitting. Furthermore, when $W = \mathbb{S}^{n-1}$, this corollary gives an alternate description of the action of σ_1 on $\pi_n(X)$ as described in [5, Remark 1.4]. In this case, $\sigma_W(X, x_0, G) = \sigma_{\mathbb{S}^{n-1}}(X, x_0, G)$ is the extension group of the n-th Abe group $\kappa_n(X, x_0)$ [1] as in (3). Thus, one can either embed σ_1 in σ_n as in [5, Remark 1.4] or in $\sigma_{\mathbb{S}^{n-1}}(X, x_0, G)$.

Unlike the reduced suspension Σ which has the loop functor Ω as its right adjoint, the un-reduced suspension $\widehat{\Sigma}$ does *not* admit a right adjoint. Nevertheless, one can describe the adjoint property for the W-Rhodes groups as follows. Recall that a typical element in $\sigma_W(X, x_0, G)$ is a homotopy class $[f; g]$ where $f : (\widehat{\Sigma}W, v_1, v_2) \to (X, x_0, gx_0)$. Thus, σ_W is a subset of $[\widehat{\Sigma}W, X]_0 \times G$, where $[\widehat{\Sigma}W, X]_0$ denotes the homotopy classes of maps $f : \widehat{\Sigma}W \to X$ such that $f(v_1) = x_0$ and $f(v_2)$ is independent of the homotopy class of f. Then, σ_W is also a subset of $[W, \mathcal{P}_{x_0}]^* \times G$, where $[W, \mathcal{P}_{x_0}]^*$ denotes the set of homotopy classes of unpointed maps from W to the space $\mathcal{P}_{x_0}$ of paths originating from x_0. In the special case when $G = \{1\}$, $\sigma_W = [\Sigma(W \cup *), X] = \sigma^*_W = [W, \Omega Y]^* = [W \cup *, \Omega X]$.

3. Generalized W-Γ-Rhodes groups. In the definition of the generalized Rhodes group $\sigma_W(X, x_0, G)$, the two cone points from the un-reduced suspension $\widehat{\Sigma}W = C^+W \cup C^-W$ play an important role. Therefore in replacing $\mathbb{S}^1$ with arbitrary co-grouplike space, we require that the space has two distinct basepoints.

Let Γ be a space and $\gamma_1, \gamma_2 \in \Gamma$ satisfying the following conditions:

(I) there exists a map $\nu : (\Gamma, \gamma_1, \gamma_2) \to (\Gamma \times \{\gamma_1\} \cup \{\gamma_2\} \times \Gamma, (\gamma_1, \gamma_1), (\gamma_2, \gamma_2))$ such that $\mathrm{proj}_i \circ \nu \simeq \mathrm{id}$ as maps of triples for each $i = 1, 2$, where $\mathrm{proj}_1, \mathrm{proj}_2 : (\Gamma \times \{\gamma_1\} \cup \{\gamma_2\} \times \Gamma, (\gamma_1, \gamma_1), (\gamma_2, \gamma_2)) \to (\Gamma, \gamma_1, \gamma_2)$ are the canonical projections;

(II) there exists a map $\eta : \Gamma \to \Gamma$ such that:

(a) $\eta(\gamma_1) = \gamma_2, \eta(\gamma_2) = \gamma_1$;

(b) $\nabla \circ (\overline{\mathrm{id}} \vee \overline{\eta}) \circ \nu$ is homotopic to the constant map at γ_1, where

$$\overline{\mathrm{id}} \vee \overline{\eta} : \Gamma \times \{\gamma_1\} \cup \{\gamma_2\} \times \Gamma, (\gamma_1, \gamma_1), (\gamma_2, \gamma_2) \to \Gamma \times \{\gamma_1\} \cup \{\gamma_2\} \times \Gamma, (\gamma_1, \gamma_2), (\gamma_2, \gamma_1)$$

with $\overline{\mathrm{id}}(\gamma, \gamma_1) = (\gamma, \gamma_2)$, $\overline{\eta}(\gamma_2, \gamma) = (\gamma_2, \eta(\gamma))$ for $\gamma \in \Gamma$ and $\nabla : (\Gamma \times \{\gamma_2\} \cup \{\gamma_2\} \times \Gamma, (\gamma_1, \gamma_2), (\gamma_2, \gamma_1)) \to (\Gamma, \gamma_1, \gamma_2)$ is the folding map;

(c) similarly, $\nabla \circ (\tilde{\mathrm{id}} \vee \tilde{\eta}) \circ \nu$ is homotopic to the constant map at γ_2, where

$$\tilde{\mathrm{id}} \vee \tilde{\eta} : \Gamma \times \{\gamma_1\} \cup \{\gamma_2\} \times \Gamma, (\gamma_1, \gamma_1), (\gamma_2, \gamma_2) \to \Gamma \times \{\gamma_1\} \cup \{\gamma_2\} \times \Gamma, (\gamma_2, \gamma_1), (\gamma_1, \gamma_2)$$

with $\tilde{\mathrm{id}}(\gamma_2, \gamma) = (\gamma_1, \gamma)$, $\tilde{\eta}(\gamma, \gamma_1) = (\eta(\gamma), (\gamma_1))$ for $\gamma \in \Gamma$;

(III) Moreover, we have co-associativity so that the diagram

$$\begin{tikzcd}
(\Gamma, \gamma_1, \gamma_2) \arrow[r, "\nu"] \arrow[d, "\nu"'] & (\Gamma \times \{\gamma_1\} \cup \{v_2\} \times \Gamma, (\gamma_1, \gamma_1), (v_2, v_2)) \arrow[d, "\tilde{\mathrm{id}} \vee \tilde{\nu}"] \\
\Gamma \times \{\gamma_1\} \cup \{v_2\} \times \Gamma, (\gamma_1, \gamma_1), (v_2, v_2) \arrow[r, "\overline{\mathrm{id}} \vee \overline{\nu}"] & \Gamma \times \{(\gamma_1, \gamma_1)\} \cup \{v_2\} \times (\Gamma \times \{\gamma_1\} \cup \{v_2\} \times \Gamma), \gamma_1^*, v_2^*
\end{tikzcd}$$

is commutative up to homotopy, where $\gamma_1^* = (\gamma_1, (\gamma_1, \gamma_1))$, $\gamma_2^* = (\gamma_2, (\gamma_2, \gamma_2))$, and $\overline{\mathrm{id}}(\gamma, \gamma_1) = (\gamma, (\gamma_1, \gamma_1))$, $\overline{\nu}(\gamma_2, \gamma) = (\gamma_2, \nu(\gamma))$, $\tilde{\mathrm{id}}(\gamma_2, \gamma) = ((\gamma_2, \gamma_2), \gamma))$, $\tilde{\nu}(\gamma, \gamma_1) = (\nu(\gamma), \gamma_1)$ for $\gamma \in \Gamma$.

Now, we generalize the notion of a co-grouplike space presented e.g. in [7]. A *co-grouplike space with two basepoints* $\Gamma = (\Gamma, \gamma_1, \gamma_2; \nu, \eta)$ consists of a topological space Γ together with basepoints γ_1, γ_2 and maps ν, η satisfying conditions (I)–(III). For any space W, the *smash product* is given by

$$\Gamma(W) := W \times \Gamma / \{(w, \gamma_1) \sim (w', \gamma_1), (w, \gamma_2) \sim (w', \gamma_2)\}$$

for any $w, w' \in W$.

For instance, if $\Gamma = ([0,1], 0, 1; \nu, \eta)$ with $\nu(t) = \begin{cases} (2t, 0) & \text{if } 0 \le t \le \frac{1}{2}, \\ (1, 2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$ and $\eta(t) = 1 - t$ for $t \in [0,1]$ then $\Gamma(W) = \widehat{\Sigma} W$, the un-reduced suspension of W.

Remark 3. Note that if $\gamma_1 = \gamma_2$, we obtain the usual co-grouplike structure and $\Gamma_0 := \Gamma / \sim$ given by identifying the basepoints γ_1 and γ_2 is a co-grouplike space as well.

Next, we define the W-Γ-Rhodes groups.

Let Γ be a co-grouplike space with two basepoints, (X, G) a G-space and W a space. The W-Γ-*Rhodes group* of X with respect to W is defined to be

$$\sigma_W^\Gamma(X, x_0, G) = \{[f; g] \mid f : (\Gamma(W), \bar{\gamma}_1, \bar{\gamma}_2) \to (X, x_0, gx_0)\}.$$

Write $\tau_W^{\Gamma_0}(X, x_0)$ for the Γ_0-W-Fox group considered in [6].

We can easily show:

Proposition 4. *Let $\pi : \sigma_W^\Gamma(X, x_0, G) \to G$ be the projection sending $[f; g] \mapsto g$. By identifying the two basepoints of $\Gamma(W)$, the quotient space $\Gamma(W)/\sim$ is canonically homeomorphic to $\Gamma_0 \wedge (W \cup \{*\})$. Furthermore,*

$$\mathrm{Ker}\, \pi \cong [\Gamma(W)/\sim, X] = \tau_W^{\Gamma_0}(X, x_0).$$

Then we obtain a general Γ-Rhodes exact sequence, generalizing (2).

THEOREM 5. *The sequence*

$$1 \to \tau_W^{\Gamma_0}(X, x_0) \to \sigma_W^{\Gamma}(X, x_0, G) \xrightarrow{\pi} G \to 1$$

is exact.

We now derive the following generalized split exact sequence for the W-Γ-Rhodes groups.

COROLLARY 6. *Let W be a space with a basepoint w_0 and Γ be a co-grouplike space with two basepoints. The sequence*

$$1 \to [\Gamma_0 \wedge ((V \times W)/V), \Omega X] \to \sigma_{V\times W}^{\Gamma}(X, x_0, G) \overset{\leftarrow}{\to} \sigma_V^{\Gamma}(X, x_0, G) \to 1 \tag{8}$$

is split exact.

Proof. From Theorem 5, we have the short exact sequences

$$1 \to \tau_{V\times W}^{\Gamma_0}(X, x_0) \to \sigma_{V\times W}^{\Gamma}(X, x_0, G) \xrightarrow{\pi} G \to 1$$

and

$$1 \to \tau_V^{\Gamma_0}(X, x_0) \to \sigma_V^{\Gamma}(X, x_0, G) \xrightarrow{\pi} G \to 1.$$

Moreover, the following split exact sequence was shown in [6, Theorem 4.1]:

$$1 \to [\Gamma_0 \wedge ((V \times W)/V), \Omega X] \to \tau_{V\times W}^{\Gamma_0}(X, x_0, G) \overset{\leftarrow}{\to} \tau_V^{\Gamma_0}(X, x_0, G) \to 1.$$

A straightforward diagram chasing argument involving these short exact sequences yields the desired split exact sequence. ∎

4. Evaluation subgroups of W-Γ-Rhodes groups. We end this note by extending a result concerning the evaluation subgroups of the Rhodes groups and the Fox torus homotopy groups obtained in [5, Theorem 2.2].

Given a G-space X, the function space X^X is also a G-space where the action is pointwise, that is, $(gf)(x) = gf(x)$ for $f \in X^X$, $g \in G$ and $x \in X$. Let Γ be a co-grouplike space with two basepoints and W be a space.

The *evaluation subgroup* of the W-Γ-Rhodes group of X is defined by

$$\mathcal{G}\sigma_W^{\Gamma}(X, x_0, G) := \mathrm{Im}(ev_* : \sigma_W^{\Gamma}(X^X, \mathrm{id}_X, G) \to \sigma_W^{\Gamma}(X, x_0, G)).$$

Similarly, the *evaluation subgroup* of $\tau_W^{\Gamma_0}(X, x_0)$ is defined by

$$\mathcal{G}\tau_W^{\Gamma_0}(X, x_0) := \mathrm{Im}(ev_* : \tau_W^{\Gamma_0}(X^X, \mathrm{id}_X) \to \tau_W^{\Gamma_0}(X, x_0)).$$

It is straightforward to see that the proof of [5, Theorem 2.2] is also valid in the setting of W-Γ-Rhodes groups. Therefore, we have the following generalization.

THEOREM 7. *Let G_0 be the subgroup of G consisting of elements g considered as homeomorphisms of X which are freely homotopic to the identity map id_X. Then the sequence*

$$1 \to \mathcal{G}\tau_W^{\Gamma_0}(X, x_0) \to \mathcal{G}\sigma_W^{\Gamma}(X, x_0, G) \to G_0 \to 1$$

is exact.

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