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CALCULATION OF THE AVOIDING IDEAL FOR $\Sigma^{1,1}$

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Abstract. We calculate the mapping $H^*(BO;\mathbb{Z}_2) \to H^*(K^{1,0};\mathbb{Z}_2)$ and obtain a generating system of its kernel. As a corollary, bounds on the codimension of fold maps from real projective spaces to Euclidean space are calculated and the rank of a singular bordism group is determined.

1. Introduction and definitions. We work with homologies and cohomologies with $\mathbb{Z}_2$ coefficients, even when the coefficient ring is not indicated. We will investigate the spaces of the Kazarian construction (see [5]) and the maps in cohomology induced by the natural embeddings of these spaces into one another. In Section 2 these maps are calculated explicitly, this result is then used in Section 3 to provide bounds on the codimension of fold maps of real projective spaces into Euclidean spaces. Another application is demonstrated in Section 4, where we obtain a description of the rank of the unoriented right-left fold bordism group ($C^{1,0}(n,k)$ in the notation of [3]).

To reach these goals, the Kazarian construction will be considered for immersions, locally stable maps without $\Sigma^{1,1}$ singularities and maps without any constraints on their singularities. The fine details of the construction are presented in [5]; we briefly recall its properties relevant to the aims of this paper. The Kazarian spaces of the classes of maps defined above, denoted here by $K^0 \approx BO(k)$, $K^{1,0}$ and $K^\infty \approx BO$ respectively, can be thought about as subspaces of the bundle of jets over BO cut out by the appropriate restrictions, so we have natural embeddings $K^0 \xrightarrow{u} K^{1,0} \xrightarrow{g} K^\infty$, the composition of which will be denoted by $\overline{u} : K^0 \to K^\infty$; it is known to be homotopic to the standard embedding $BO(k) \to BO$. Whenever we have a mapping $f : M^n \to P^{n+k}$ with all singularities in a class τ (in our case: regular points only; regular points and folds only; and any singularities), the mapping inducing the stable normal bundle of f, $\nu_f \oplus \varepsilon^N : M \to BO(N+k) \to BO$ can be chosen to lie in K^τ (K^0, $K^{1,0}$, K^∞ respectively).

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Using an alternative construction that gives a space homotopically equivalent to $K^{1,0}$, we can obtain $K^{1,0}$ as the total space of the vector bundle ξ over the base $B = BG_{\Sigma^{1,0}} \approx \mathbb{R}P^\infty \times BO(k)$, which has the form

$$\xi = l \oplus \gamma$$

with l and γ being the pullbacks of the tautological bundle over $\mathbb{R}P^\infty$ and $BO(k)$, respectively, glued to K^0. This gives us an embedding $b : B \to K^{1,0}$, and after factoring out K^0 by the projection $p : K^{1,0} \to (K^{1,0}, K^0)$ we obtain an embedding $\overline{b} : B \to (K^{1,0}, K^0)$. By excision, for all cohomological purposes $\overline{b}$ is the embedding of B into the pair of the unit ball and unit sphere bundles of ξ for a suitable metric, $(D\xi, S\xi)$. For the calculations, we will need to be able to identify the restrictions of the elements of $H^*(K^\infty)$ to B, which are the corresponding characteristic classes of the restriction of the virtual normal bundle ν over K^∞ to B; it can be shown that stably, $\nu|_B \approx l \otimes \gamma \ominus l$.

$$\begin{tikzcd}
& B \arrow[d, "b"] \arrow[dr, "\overline{b}"] & \\
K^0 \arrow[r, "u"] \arrow[dr, "\overline{u}"'] & K^{1,0} \arrow[r, "p"'] \arrow[d, "g"] & (K^{1,0}, K^0) \\
& K^\infty &
\end{tikzcd}$$

$$\begin{tikzcd}
& \langle v_I, c \mid \max I \le k\rangle & \\
\langle w_I \mid \max I \le k\rangle & H^*(K^{1,0}) \arrow[l, "u^*"'] \arrow[u, "b^*"] & U\langle v_I, c \mid \max I \le k\rangle \arrow[l, "p^*"] \arrow[ul, "Ua \mapsto a w_{k+1}(\xi)"'] \\
& \langle w_I\rangle \arrow[u, "g^*"] \arrow[ul, "w_{>k} \mapsto 0"] &
\end{tikzcd}$$

The mappings defined above commute in a natural manner, implying the commutativity of the corresponding diagram of cohomology groups in all dimensions. The elements of those groups will be expressed in the terms of the usual generators $w_I \in H^*(BO)$ in case of K^0 and K^∞, while the elements of $H^*(B)$ (and subsequently $H^*(D\xi, S\xi)$) will be expressed in the terms of the generators $c \in H^1(\mathbb{R}P^\infty)$ and $v_I = w_I(\gamma) \in H^*(BO(k))$.

2. Calculation. The Stiefel-Whitney characteristic classes of the tensor product $l \otimes \gamma$ are easily calculated using the splitting lemma to be

$$w_i(l \otimes \gamma) = \sum_{j=0}^{i} \binom{k-j}{i-j} v_j c^{i-j}.$$

Inverting the total Stiefel-Whitney class of l we have

$$w(-l) = w(l)^{-1} = 1 + c + c^2 + ..., \; w_i(-l) = c^i,$$

so the characteristic classes of the sum are

$$w_i(\nu|_B) = \sum_{s=0}^{i} w_s(l\otimes\gamma)v_{i-s}(-l) = \sum_{s=0}^{i}\sum_{j=0}^{s}\binom{k-j}{s-j}v_jc^{s-j}c^{i-s} = \sum_{j=0}^{i} v_jc^{i-j}\sum_{s=j}^{i}\binom{k-j}{s-j}.$$

If additionally $i \geq k$, then the inner sum takes the form

$$\sum_{s=j}^{i}\binom{k-j}{s-j} = \binom{k-j}{0} + \cdots + \binom{k-j}{k-j} + 0 + \cdots + 0 = 2^{k-j},$$

if $j \leq k$ and is 0 otherwise, so for these values of i we have

$$w_i(\nu|_B) = \sum_{j=0}^{k} 2^{k-j}v_jc^{i-j} = v_kc^{i-k}.$$

Consider now the mapping $b^*\circ g^* : w_I \mapsto w_I(\nu|_B)$ on monomials with $\max I > k$. By the formula derived above, the image will be divisible by $w_{k+1}(\xi) = v_kc$, let $a_I \in H^*(B)$ be such that $b^*g^*w_I = v_kca_I$ (if $I = I^+ \cup I^-$ with $\max I^- \leq k$, $\min I^+ \geq k+1$ and $\sum_{i\in I^+}(i-k) = S$, then $a_I = w_{I^-}(\nu)v_k^{|I^+|-1}c^{S-1}$). The element g^*w_I is sent to 0 by u^* since $u^*g^* = \overline{u}^*$ annihilates all w_i with $i > k$, so by exactness of the horizontal row of our diagram (it is a fragment of the cohomology long exact sequence of the pair $(K^{1,0}, K^0)$) there is a class $Ub_I \in H^*(D\xi, S\xi)$ such that

$$g^*w_I = p^*(Ub_I).$$

Applying b^* to both sides of this equation, we get that $v_kca_I = b^*g^*w_I = b^*p^*(Ub_I) = \overline{b}^*(Ub_I) = v_kcb_I$. Since $H^*(B)$ has no zero divisors, this implies $a_I = b_I$ and hence

$$g^*w_I = p^*(Ua_I). \tag{1}$$

The mapping p^* is injective given that even $\overline{p}^* = b^*\circ p^*$ is injective, so

$$g^*\sum_{I\in\mathcal{I}} w_I = p^*\sum_{I\in\mathcal{I}} Ua_I = 0 \Leftrightarrow \sum_{I\in\mathcal{I}} a_I = 0$$

if all of the index sets I satisfied $\max I > k$ to begin with. However, for $\max I \leq k$ we have $u^*g^*w_I = \overline{u}^*w_I = w_I$, so if a class in $H^*(K^\infty)$ lies in the kernel of g^*, then all of its monomials (with non-zero coefficients) have to satisfy $\max I > k$.

Theorem 1. *The avoiding ideal $\mathcal{A}$ for the singularity $\Sigma^{1,1}$ is generated as an $H^*(K^\infty)$ ideal by the set*

$$\{w_{k+l}w_{k+m} + w_{k+q}w_{k+r} | l,m,q,r \geq 0 \text{ and } l+m = q+r \geq 2\}.$$

Proof. Denote by

$$\mathcal{B} = (w_{k+l}w_{k+m} + w_{k+q}w_{k+r} | l,m,q,r \geq 0 \text{ and } l+m = q+r \geq 2)_{H^*(K^\infty)}$$

the ideal generated by the elements given in the statement of the theorem. It is easy to see that $\mathcal{B} \subset \ker\overline{u}^*$ and

$$a_{\{k+l,k+m\}} = v_k^2c^{l+m} = v_k^2c^{q+r} = a_{\{k+q,k+r\}}$$

holds for all the quartuples (l,m,q,r) involved, so by equality (1)

$$\mathcal{B} \subseteq \mathcal{A}.$$

To finish the proof, it is sufficient to verify that

$$\operatorname{rank}\mathcal{A}^n \le \operatorname{rank}\mathcal{B}^n \text{ for all } n. \tag{2}$$

The left hand side can be calculated from the fact that $\ker g^* = \ker b^* g^*$.

Indeed, if $b^*g^*\alpha = 0$ for some $\alpha \in H^*(BO)$, then set $\alpha^- = \overline{u}^*\alpha \in H^*(BO(k)) \subset H^*(BO)$ and $\alpha^+ = \alpha - \alpha^-$. We have $b^*g^*\alpha = \alpha(\nu) = \alpha^-(\nu) + \alpha^+(\nu)$. Observe that the mapping $H^*(BO(k)) \ni \alpha^- \mapsto \alpha^-(\nu) \in H^*(B)$ is the sum of coordinate maps $w_I \mapsto w_I(l \otimes \gamma - l) = w_I(\gamma) + c \cdot (\dots)$, so $\alpha^-(\nu)$ written in the basis we use will contain every v_I for which α^- contains w_I. On the other hand, all of the monomials of $\alpha^+(\nu)$ contain c (since all $w_{k+1+a}(\nu) = cc^a v_k$ do), so if $b^*g^*\alpha = 0$, then $\alpha^- = 0$. By (1) we then have $g^*\alpha = g^*\alpha^+ = p^*(U\alpha(\nu)/v_k c) = 0$ and $\alpha \in \ker g^*$.

To calculate the image of b^*g^*, we know that $b^*g^*w_I = w_I(\nu)$, in particular,

$$b^*g^*w_{k+a}w_I = v_k c^a w_I(\nu).$$

If we choose any I with $\max I \le k$, then $w_I(\nu) = v_I + c \cdot (\dots)$ shows that b^*g^* is onto the factor ring $H^*(BO(k)) = H^*(B)/(c)$, and $w_{k+a}w_I(\nu) = v_k v_I c^a + c^{a+1} \cdot (\dots)$ shows that the image of b^*g^* in the slice $c^a H^*(BO(k)) = c^a H^*(B)/(c^{a+1})$ contains exactly the elements divisible by v_k. Thus the image of b^*g^* is spanned by w_I, $\max I \le k$ and $c^a w_I$, $\max I = k$.

Therefore

$$\begin{aligned}
\operatorname{rank}\mathcal{A}^n &= \operatorname{rank}\ker g^n = \operatorname{rank}\ker b^n g^n = \operatorname{rank} H^n(K^\infty) - \operatorname{rank}\operatorname{im} b^n \circ g^n \\
&= |\{a_0 \ge \dots \ge a_m \ge 0 | n = a_0 + \dots + a_m\}| \\
&\qquad -|\{k \ge a_0 \ge \dots \ge a_m \ge 0 | n = a_0 + \dots + a_m\}| \\
&\qquad\qquad -|\{k = a_0 \ge \dots \ge a_m \ge 0 | n \ge a_0 + \dots + a_m\}| \\
&= |\{a_0 \ge \dots \ge a_m \ge 0 | a_0 > k \text{ and } n = a_0 + \dots + a_m\}| \\
&\qquad -|\{a'_0 > k \ge a_1 \ge \dots \ge a_m \ge 0 | n = a'_0 + a_1 + \dots + a_m\}| \\
&= |\{a_0 \ge \dots \ge a_m \ge 0 | a_1 > k \text{ and } n = a_0 + \dots + a_m\}|.
\end{aligned}$$

The right hand side of (2) can be estimated similarly, once we observe that the elements of $H^n(BO)/\mathcal{B}^n$ can be represented as sums of monomials w_I or $w_{k+a}w_I$ with $\max I \le k$: indeed, if a monomial has the form $w_{k+a}w_{k+b}\hat{w}$, we can change it by $(w_{k+a}w_{k+b} + w_{k+a+b}w_k)\hat{w} \in \mathcal{B}$ to get an equivalent representation $w_{k+a+b}w_k\hat{w}$ with less indices larger than k. Thus we have $\operatorname{rank} H^n(K^\infty)/\mathcal{B}^n \le \operatorname{rank}\operatorname{im} b^n \circ g^n$ (the number of words $c^a w_k w_I$ with $\max I \le k$ is the same as the number of words $w_{k+a}w_I$ with $\max I \le k$), implying

$$\operatorname{rank}\mathcal{B}^n = \operatorname{rank} H^n(BO) - \operatorname{rank} H^n(BO)/\mathcal{B}^n \ge \operatorname{rank} H^n(K^\infty) - \operatorname{rank} b^n \circ g^n = \operatorname{rank}\mathcal{A}^n$$

and (2) holds, completing the proof. ■

As an immediate consequence, we obtain the following corollary which allows us to efficiently decide whether a characteristic number lies in the avoiding ideal or not:

Corollary 2. *The avoiding ideal for the singularity* $\Sigma^{1,1}$ *consists of elements*

$$\sum_{I \in \mathcal{I}} w_I \textit{ such that } \sum_{I \in \mathcal{I}} c^S w_k^{|I^+|} w_{I \setminus I^+} = 0,$$

where $\mathcal{I}$ contains only index sets I with $\max I > k$*, I^+ denotes* $\bigcup\{J \subseteq I | \min J > k\}$ *and* $S = \sum_{i \in I^+}(i-k)$.

3. Fold maps of projective spaces. As an application of Theorem 1, we will consider maps of projective spaces into Euclidean space. If we have a mapping $f : M^n \to \mathbb{R}^{n+k}$ with only regular points and folds, then the classifying map of its stable normal bundle $\nu_f : M \to BO$ is homotopic to a composition of a suitable $\tilde{\nu}_f : M \to K^{1,0}$ and the canonical embedding $g : K^{1,0} \to K^\infty$. Hence the induced mapping in cohomology $-(\nu_f) : H^*(BO) \to H^*(M)$ decomposes as $-(\nu_f) = (\tilde{\nu}_f)^* \circ g^*$ and consequently $\ker -(\nu_f) \supseteq \ker g^*$. In particular, all elements $\alpha(l,m,q,r) = w_l w_m + w_q w_r$ with $l+m = q+r \geq 2k+2$, $l,m,q,r \geq k$, must evaluate to 0 on ν_f. When we choose $M = \mathbb{R}P^n$, then this evaluation is particularly easy to compute since if we denote by a the generator of $H^1(\mathbb{R}P^n)$ and $n = 2^s + t$ is the unique decomposition of n such that s and $m < 2^s$ are nonnegative integers, then $a^{n+1} = 0$ and hence

$$\begin{aligned} w(\nu_f) = w(-\tau_{\mathbb{R}P^n}) &= (1+a)^{-n-1} = (1+a^{2^{s+1}})(1+a)^{-n-1} = (1+a)^{2^{s+1}}(1+a)^{-n-1} \\ &= \sum_{j=0}^{n} \binom{2^{s+1}-n-1}{j} a^j = \sum_{j=0}^{n} \binom{2^s-t-1}{j} a^j. \end{aligned} \tag{3}$$

Therefore $\alpha(l,m,q,r)(\nu_f) = \left(\binom{2^s-t-1}{k+l}\binom{2^s-t-1}{k+m} + \binom{2^s-t-1}{k+q}\binom{2^s-t-1}{k+r}\right) a^{2k+l+m}$ is null if and only if $\binom{2^s-t-1}{k+l}\binom{2^s-t-1}{k+m} + \binom{2^s-t-1}{k+q}\binom{2^s-t-1}{k+r}$ is even or $2k+l+m > n$. If we produce an $\alpha(l,m,q,r)$ that does not evaluate to 0, then the first k for which this element will be in $\mathcal{A}$ is the minimum of $\{l,m,q,r\}$, so we need to maximize this quantity in order to optimize our estimate on k.

If $t > \frac{2^s}{3}$, then the maximal j in the sum (3) for which the corresponding term is nonzero is $2^s - t - 1 < \frac{n}{2}$, so the best α which does not evaluate to 0 is $\alpha(2^s-t-2, 2^s-t, 2^s-t-1, 2^s-t-1) = \left(0 + \binom{2^s-t-1}{2^s-t-1}\right)a^{2^{s+1}-2t} = a^{2^{s+1}-2t} \neq 0$, and we have to consider this element if $k \leq 2^s - t - 2$. Hence in this case, the existence of a fold map from $\mathbb{R}P^n$ to $\mathbb{R}^{n+k}$ implies $k \geq 2^s - t - 1 = 2^{s+1} - n - 1$.

If $t < \frac{2^s}{3}$, the calculation is less obvious. All $\alpha(l,m,q,r)$ with $l+m > n$ evaluate to 0 by virtue of being elements of $H^{l+m}(\mathbb{R}P^n)$, so we can assume that $l + m \leq n$. Start listing the values

$$\binom{2^s-t-1}{\lfloor \frac{n}{2} \rfloor}, \binom{2^s-t-1}{\lfloor \frac{n}{2} \rfloor - 1}, \binom{2^s-t-1}{\lfloor \frac{n}{2} \rfloor - 2}, \dots, \binom{2^s-t-1}{\lfloor \frac{n}{2} \rfloor - h}$$

and assume that the first h elements of this sequence have the same parity while the next one has the opposite parity. If the sequence starts with even elements, then it is clear that any term $w_b w_c$ which does not evaluate to zero has $\min\{b,c\} \leq \lfloor \frac{n}{2} \rfloor - h$, and an optimal α is either $\alpha(j, j+i, j+1, j+i-1)$ with an i such that $h < i \leq n-j$ and $\binom{2^s-t-1}{j+i}$ is odd, or $\alpha(j-1, j+1, j, j)$ with $j = \lfloor \frac{n}{2} \rfloor - h$ if there is no such i. If the sequence starts with odd elements, then the same argument with the roles of 0 and a^{b+c} reversed gives us that an optimal α is either $\alpha(j, j+i, j+1, j+i-1)$ with an i such that $h < i \leq n-j$ and $\binom{2^s-t-1}{j+i}$ is even, or $\alpha(j-1, j+1, j, j)$ with $j = \lfloor \frac{n}{2} \rfloor - h$ if there is no such i.

Therefore, we have to investigate the parity of $F(j) = \binom{2^s-t-1}{j}$ for values of j close to $n/2$, and for that, we need to look at the binary expansions of $2^s - t - 1$ and j: by [4], a binomial coefficient $\binom{b}{c}$ is odd precisely when the binary expansion of b has digits 1 at all the places where the binary expansion of c has digits 1. Given the binary expansion of $n = 2^s + t$ we can obtain the binary expansion of $2^{s+1} - n - 1 = \underbrace{1...1}_{s+1}{}_2 - n$ by bitwise negation, and the binary expansion of $\lfloor \frac{n}{2} \rfloor$ is obtained by shifting right by one position. This implies that $F(\lfloor n/2 \rfloor)$ is odd precisely when the binary expansion of n does not contain the substring ...11..., and according to this we have two cases.

n contains ...11..., first at position u: $n = 2^u(8a+3)+b$ with u maximal and $0 \le b < 2^u$. Then decreasing j starting from $\lfloor n/2 \rfloor$ we will get even values of $F(j)$ until the decrease does not affect the u^{th} digit since this is the highest 1 at a place where $2^s - t - 1$ has a 0; once that location is reached, the highest value of j for which $F(j)$ is odd has to copy the rest of the string from $2^s - t - 1$, that is, $j = 2^{u+2}a + 2^u - 1 - b > \frac{n}{2} - 2^{u+1}$. Increasing j, on the other hand, does not change the parity of $F(j)$ as long as $j < 2^{u+2}(a+1)$ due to either the u^{th} or the $u+1^{st}$ digit, which is more than 2^{u+1} steps so we don't get a better estimate on k than $k \ge j - 1 = 2^{u+2}a + 2^u - b - 2$.

n does not contain ...11.... In this case we first deal with the case of n odd; $2^s - t - 1$ is even and both $\lfloor n/2 \rfloor - 1$ and $\lfloor n/2 \rfloor + 1$ are odd, so the sharpest possible estimate holds, $k \ge \frac{n-3}{2}$. And if $n \in 2^{p+1}\mathbb{Z} + 2^p$, $p > 0$, it is easy to see that increasing j first produces a parity change after 2^{p-1} steps and decreasing j does the same after $2^{p-1} + 1$ steps, so the estimate is $k \le \frac{n}{2} - 2^{p-1} - 1$.

We have thus proved the following result:

Theorem 3. *If there exists a fold map $\mathbb{R}P^n \to \mathbb{R}^{n+k}$, then*

$$k \ge \begin{cases} 2^{s+1} - n - 2 & \text{if } \frac{4}{3}2^s < n < 2^{s+1}, \\ \lfloor \frac{n}{2} \rfloor - 1 & \text{if } 2^s < n < \frac{4}{3}2^s \text{ is odd and } \forall u \quad \lfloor \frac{n}{2^u} \rfloor \not\equiv 3 \bmod 4, \\ \frac{n}{2} - 2^{p-1} & \text{if } 2^{s-1} < n = 2^p m < 2^s \frac{4}{3} \text{ with } p > 0, \text{ odd } m \\ & \quad \text{and } \forall u \quad \lfloor \frac{n}{2^u} \rfloor \not\equiv 3 \bmod 4, \\ 2^{u+2}a + 2^u - b - 2 & \text{if } n = 2^u(8a+3) + b \text{ with } 0 \le b < 2^u \text{ and } u \text{ maximal.} \end{cases}$$

Remark. When $t > 2^s/3$, our estimate on the codimension is one less than the geometric dimension of the stable normal bundle of $\mathbb{R}P^n$, so allowing fold singularities does not decrease the necessary codimension for a mapping to $\mathbb{R}^{n+k}$ to exist by more than 1 compared to the analogous estimate for immersions. In the case $0 < t < 2^s/3$ this is no longer the case, but our estimate stays close to the sharpest possible value $k = \lfloor n/2 \rfloor$, for which any generic mapping can only have fold singularities anyway: the restriction on t implies $p \le s - 2 \Rightarrow k > 3n/8$ in the second and third cases as well as $a \ge 2 \Rightarrow 2^{u+2}a + 2^u - b - 2 = \frac{n}{2} - \left(2^{u-1} + \frac{3b}{2} + 2\right) \ge \frac{n}{2} - \left(2^{u+1} + \frac{1}{2}\right) > \frac{3n}{8}$ in the last case with the sole exception of $n = 19$ (alternatively, $k \ge 7n/19$ for all n).

4. Fold bordism groups. We will simply combine several previously known results. The trivial adaptation of [6, Theorem 14] to the case of unoriented manifolds reduces the calculation of the group $C^{1,0}(n,k)$ to the calculation of the bordism group $\mathfrak{N}_{n+k}(X_{\Sigma^{1,0}})$.

By [1, Theorem 1.9],

$$\mathfrak{N}_{n+k}(X_{\Sigma^{1,0}}) \approx (H_*(X_{\Sigma^{1,0}};\mathbb{Z}_2) \otimes \mathfrak{N}_*)_{n+k}.$$

[6, Corollary 72] expresses $X_{\Sigma^{1,0}}$ as $\Gamma T\nu^k$ with ν^k a bundle over $K^{1,0}$, so we can apply the results of [2] to calculate the ranks of $H_*(X_{\Sigma^{1,0}};\mathbb{Z}_2)$ from an additive basis of $\overline{H}_*(T\nu^k) \approx H_{*-k}(K^{1,0})$ (and Theorem 1 provides us with one). As a result, we obtain the following:

THEOREM 4. *$C^{1,0}(n,k) \approx \mathbb{Z}_2{}^{r_{n,k}}$, where $r_{n,k}$ is the number of different sets of multiindex pairs and a partition $\{(I_1,J_1),...,(I_s,J_s),d_1,...,d_u\}$ such that*

- *Each $J_m = (j_{m,0} \geq j_{m,1} \geq ... \geq j_{m,s_m})$ consists of positive indices with $j_{m,1} \leq k$.*
- *Each I_m is either empty or $I_m = (i_{m,1} \geq ... \geq i_{m,t_m} > 0)$ with $i_{m,1} \leq 2i_{m,2}$, ..., $i_{m,t_m-1} \leq 2i_{m,t_m}$.*
- *If I_m is not empty, then $i_{m,1} - i_{m,2} - ... - i_{m,t_m} > j_{m,0} + ... + j_{m,s_m} + k$.*
- *$\sum_m (i_{m,1} + ... + i_{m,t_m} + j_{m,0} + ... + j_{m,s_m}) = n - d_1 - ... - d_u$.*
- *$d_1 \geq d_2 \geq ... \geq d_u > 0$ do not contain numbers of form $2^v - 1$.*

Proof. We have

$$C^{1,0}(n,k) \approx \mathfrak{N}_{n+k}(X_{\Sigma^{1,0}}) \approx \bigoplus \mathfrak{N}_d \otimes H_{n+k-d}(\Gamma T\nu).$$

$\mathfrak{N}_d$ is a free $\mathbb{Z}_2$-module with a basis enumerated by partitions of d into natural numbers not of the form $2^v - 1$, and $\overline{H}_*(\Gamma T\nu)$ has an additive basis consisting of products of admissible elements of the form $Q^I(\Phi a)$ with Φ the Thom isomorphism of ν and a chosen from an additive basis of $H_*(K^{1,0};\mathbb{Z}_2) \approx H^*(K^{1,0};\mathbb{Z}_2)$ with all a homogeneous. Choosing this basis to be the elements of the form g^*w_J with $\max(J \setminus \{\max J\}) \leq k$ as in the proof of Theorem 1, we obtain exactly our claim. ∎

REMARK. While this rank is clearly calculable for every n and k, it does not seem to have a closed form that would ease its handling.

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