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WEAK AND STRONG MOMENTS OF RANDOM VECTORS

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Abstract. We discuss a conjecture about comparability of weak and strong moments of log-concave random vectors and show the conjectured inequality for unconditional vectors in normed spaces with a bounded cotype constant.

1. Introduction. Let X be a random vector with values in some normed space $(F, \| \ \|)$. The question we will discuss is how to estimate $\|X\|_p = (\mathbb{E}\|X\|^p)^{1/p}$ for $p \geq 1$. Obviously $\|X\|_p \geq \|X\|_1 = \mathbb{E}\|X\|$ and for any continuous linear functional φ on F with $\|\varphi\|_* \leq 1$ we have $\|X\|_p \geq (\mathbb{E}|\varphi(X)|^p)^{1/p}$. It turns out that in some situations one may reverse these obvious estimates and show that for an absolute constant C and any $p \geq 1$,

$$(\mathbb{E}\|X\|^p)^{1/p} \leq C\Big(\mathbb{E}\|X\| + \sup_{\|\varphi\|_* \leq 1} (\mathbb{E}|\varphi(X)|^p)^{1/p}\Big).$$

This is for example the case when X has Gaussian or product exponential distribution. In this note we will concentrate on the more general case of log-concave vectors.

A measure μ on $\mathbb{R}^n$ is called *logarithmically concave* (*log-concave* in short) if for any compact nonempty sets $A, B \subset \mathbb{R}^n$ and $\lambda \in (0,1)$,

$$\mu\big(\lambda A + (1-\lambda)B\big) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}.$$

By the result of Borell [3] a measure μ on $\mathbb{R}^n$ with full dimensional support is log-concave if and only if it is absolutely continuous with respect to the Lebesgue measure and has a density of the form e^{-f}, where $f : \mathbb{R}^n \to (-\infty, \infty]$ is a convex function. Log-concave measures are frequently studied in convex geometry, since by the Brunn-Minkowski inequality

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uniform distributions on convex bodies as well as their lower dimensional marginals are log-concave. In fact the class of log-concave measures on $\mathbb{R}^n$ is the smallest class of probability measures closed under linear transformation and weak limits that contains uniform distributions on convex bodies. Vectors with logarithmically concave distributions are called log-concave.

In the sequel we discuss the following conjecture posed in a stronger form in [7] about the comparison of strong and weak moments for log-concave vectors.

CONJECTURE 1.1. *For any n-dimensional log-concave random vector and any norm $\|\ \|$ on $\mathbb{R}^n$ we have for $1 \le p < \infty$,*

$$(\mathbb{E}\|X\|^p)^{1/p} \le C_1 \mathbb{E}\|X\| + C_2 \sup_{\|\varphi\|_* \le 1} (\mathbb{E}|\varphi(X)|^p)^{1/p}, \tag{1}$$

where C_1 and C_2 are absolute constants.

In Section 2 we gather known results about validity of (1) in special cases. Section 3 is devoted to the unconditional vectors. In particular we show that Conjecture 1.1 is satisfied under additional assumption of unconditionality of X and bounded cotype constant of the underlying normed space.

Notation. Let (ε_i) be a Bernoulli sequence, i.e. a sequence of independent symmetric random variables taking values ± 1. We assume that (ε_i) are independent of other random variables.

By $(\mathcal{E}_i)$ we denote a sequence of independent symmetric exponential random variables with variance 1 (i.e. the density $2^{-1/2}\exp(-\sqrt{2}|x|)$). We set $\mathcal{E} = \mathcal{E}^{(n)} = (\mathcal{E}_1, \ldots, \mathcal{E}_n)$ for an n-dimensional random vector with product exponential distribution and identity covariance matrix.

By $\langle \cdot, \cdot \rangle$ we denote the standard scalar product on $\mathbb{R}^n$ and by (e_i) the standard basis of $\mathbb{R}^n$. We set B_p^n for a unit ball in ℓ_p^n, i.e. $B_p^n = \{x \in \mathbb{R}^n : \|x\|_p \le 1\}$. For a random variable Y and $p > 0$ we write $\|Y\|_p = (\mathbb{E}|Y|^p)^{1/p}$.

We write C (resp. $C(\alpha)$) to denote universal constants (resp. constants depending only on parameter α). Value of a constant C may differ at each occurrence.

2. Known results. Since any norm on $\mathbb{R}^n$ may be approximated by a supremum of exponential number of functionals we get

PROPOSITION 2.1 (see [7, Proposition 3.20]). *For any n-dimensional random vector X inequality* (1) *holds for $p \ge n$ with $C_1 = 0$ and $C_2 = 10$.*

It is also easy to reduce Conjecture 1.1 to the case of symmetric vectors.

PROPOSITION 2.2. *Suppose that* (1) *holds for all symmetric n-dimensional log-concave vectors X. Then it is also satisfied with constants $4C_1 + 1$ and $4C_2$ by all log-concave vectors X.*

Proof. Assume first that X has a log-concave distribution and $\mathbb{E}X = 0$. Let X' be an independent copy of X, then $X - X'$ is symmetric and log-concave. Moreover for $p \geq 1$,

$$(\mathbb{E}\|X\|^p)^{1/p} = (\mathbb{E}\|X - \mathbb{E}X'\|^p)^{1/p} \leq (\mathbb{E}\|X - X'\|^p)^{1/p},$$
$$\mathbb{E}\|X - X'\| \leq \mathbb{E}\|X\| + \mathbb{E}\|X'\| = 2\mathbb{E}\|X\|$$

and for any functional φ,

$$(\mathbb{E}|\varphi(X - X')|^p)^{1/p} \leq (\mathbb{E}|\varphi(X)|^p)^{1/p} + (\mathbb{E}|\varphi(X')|^p)^{1/p} = 2(\mathbb{E}|\varphi(X)|^p)^{1/p}.$$

Hence (1) holds for X with constants $2C_1$ and $2C_2$.

If X is arbitrary log-concave then $X - \mathbb{E}X$ is log-concave with mean zero. We have for any $p \geq 1$,

$$(\mathbb{E}\|X\|^p)^{1/p} \leq (\mathbb{E}\|X - \mathbb{E}X\|^p)^{1/p} + \mathbb{E}\|X\|, \quad \mathbb{E}\|X - \mathbb{E}X\| \leq 2\mathbb{E}\|X\|$$

and for any functional φ,

$$(\mathbb{E}|\varphi(X - \mathbb{E}X)|^p)^{1/p} \leq (\mathbb{E}|\varphi(X)|^p)^{1/p} + |\varphi(\mathbb{E}X)| \leq 2(\mathbb{E}|\varphi(X)|^p)^{1/p}. \ \blacksquare$$

REMARK. Estimating $\|X\|_p$ is strictly connected with bounding tails of $\|X\|$. Indeed by Chebyshev's inequality we have

$$\mathbb{P}\big(\|X\| \geq e\|X\|_p\big) \leq e^{-p}$$

and by the Paley–Zygmund inequality and the fact that $\|X\|_{2p} \leq C\|X\|_p$ for $p \geq 1$ we get

$$\mathbb{P}\Big(\|X\| \geq \frac{1}{C}\|X\|_p\Big) \geq \min\Big\{\frac{1}{C}, e^{-p}\Big\}.$$

Gaussian concentration inequality easily implies (1) for Gaussian vectors X (see for example Chapter 3 of [8]). For Rademacher sums comparability of weak and strong moments was established by Dilworth and Montgomery-Smith [4]. More general statement was shown in [5].

THEOREM 2.3. *Suppose that $X = \sum_i v_i\xi_i$, where $v_i \in F$ and ξ_i are independent symmetric random variables with logarithmically concave tails. Then for any $p \geq 1$ inequality* (1) *holds with absolute constants C_1 and C_2.*

This immediately implies

COROLLARY 2.4. *Conjecture* 1.1 *holds under additional assumption that coordinates of X are independent.*

Proof. We have $X = \sum_{i=1}^n e_iX_i$ with X_i independent log-concave real random variables. It is enough to notice that variables X_i have log-concave tails and in the symmetric case apply Theorem 2.3. General independent case may be reduced to the symmetric one as in the proof of Proposition 2.2. ∎

The crucial tool in the proof of Theorem 2.3 is the Talagrand two-level concentration inequality for the product exponential distribution [12]:

$$\nu^n(A) \geq \frac{1}{2} \quad \Rightarrow \quad 1 - \nu^n(A + \sqrt{t}B_2^n + tB_1^n) \leq e^{-t/C}, \ t > 0,$$

where ν is the symmetric exponential distribution, i.e. $d\nu(x) = \frac{1}{2}\exp(-|x|)\,dx$.

In [7] more general concentration inequalities were investigated. For a probability measure μ on $\mathbb{R}^n$ define

$$\Lambda_\mu(y) = \log \int e^{\langle y,z\rangle}\, d\mu(z), \quad \Lambda_\mu^*(x) = \sup_y(\langle y, x\rangle - \Lambda_\mu(y))$$

and

$$B_\mu(t) = \{x \in \mathbb{R}^n : \Lambda_\mu(x) \le t\}.$$

One may show that $B_{\nu^n}(t) \sim \sqrt{t}B_2^n + tB_1^n$. The argument presented in [7, Section 3.3] gives

PROPOSITION 2.5. *Suppose that for some $\alpha \ge 1$ and $\beta > 0$ and any convex symmetric compact set $K \subset \mathbb{R}^n$ we have*

$$\mu(K) \ge \frac{1}{2} \;\Rightarrow\; 1 - \mu(\alpha K + B_\mu(t)) \le e^{-t/\beta}, \quad \textit{for all } t > 0. \tag{2}$$

Then inequality (1) *holds with $C_1 = \alpha$ and $C_2 = C\beta$.*

In [7] it was shown that the concentration inequality (2) holds with $\alpha = 1$ for symmetric product log-concave measures and for uniform distributions on B_r^n balls. This gives

COROLLARY 2.6. *Inequality* (1) *holds with $C_1 = 1$ and universal C_2 for uniform distributions on B_r^n balls, $1 \le r \le \infty$.*

Comparability of weak and strong moments of log-concave vectors in Euclidean spaces follows from Paouris' results [11] (see [1] for details and applications):

THEOREM 2.7. *If X is a log-concave n-dimensional random vector then for any Euclidean norm $\|\ \|$ on $\mathbb{R}^n$ we have*

$$(\mathbb{E}\|X\|^p)^{1/p} \le C\Big(\mathbb{E}\|X\| + \sup_{\|\varphi\|_* \le 1}(\mathbb{E}|\varphi(X)|^p)^{1/p}\Big).$$

3. Unconditional case. We say that a random vector $X = (X_1, \ldots, X_n)$ has *unconditional distribution* if the distribution of $(\eta_1 X_1, \ldots, \eta_n X_n)$ is the same as X for any choice of signs $\eta_1, \ldots, \eta_n$. A random vector X is called *isotropic* if it has identity covariance matrix, i.e. $\mathrm{Cov}(X_i, X_j) = \delta_{i,j}$.

THEOREM 3.1. *Suppose that X is an n-dimensional isotropic, unconditional, log-concave vector. Then for any norm $\|\ \|$ on $\mathbb{R}^n$ and $p \ge 1$,*

$$(\mathbb{E}\|X\|^p)^{1/p} \le C\Big(\mathbb{E}\|\mathcal{E}\| + \sup_{\|\varphi\|_* \le 1}(\mathbb{E}|\varphi(X)|^p)^{1/p}\Big). \tag{3}$$

Proof. Let $T = \{t \in \mathbb{R}^n : \|t\|_* \le 1\}$ be the unit ball in the space $(\mathbb{R}^n, \|\ \|_*)$ dual to $(\mathbb{R}^n, \|\ \|)$. Then $\|x\| = \sup_{t\in T}\langle t, x\rangle$. By the result of Talagrand [13] (see also [14]) there exist subsets $T_n \subset T$ and functions $\pi_n : T \to T_n$, $n = 0, 1, \ldots$, such that $\pi_n(t) \to t$ for all $t \in T$, $\#T_0 = 1$, $\#T_n \le 2^{2^n}$ and

$$\sum_{n=0}^{\infty} \big\|\langle \pi_{n+1}(t) - \pi_n(t), \mathcal{E}\rangle\big\|_{2^n} \le C\mathbb{E}\sup_{t\in T}\langle t, \mathcal{E}\rangle = C\mathbb{E}\|\mathcal{E}\|. \tag{4}$$

Let us fix $p \geq 1$ and choose $n_0 \geq 1$ such that $2^{n_0-1} < 2p \leq 2^{n_0}$. We have

$$\|X\| = \sup_{t\in T}\langle t, X\rangle \leq \sup_{t\in T}\big|\langle \pi_{n_0}(t), X\rangle\big| + \sup_{t\in T}\sum_{n=n_0}^{\infty}\big|\langle \pi_{n+1}(t)-\pi_n(t), X\rangle\big|. \tag{5}$$

We get

$$\Big(\mathbb{E}\sup_{t\in T}|\langle \pi_{n_0}(t), X\rangle|^p\Big)^{1/p} \leq \Big(\mathbb{E}\sum_{s\in T_{n_0}}|\langle s, X\rangle|^p\Big)^{1/p} \leq (\#T_{n_0})^{1/p}\sup_{s\in T_{n_0}}(\mathbb{E}|\langle s, X\rangle|^p)^{1/p}$$
$$\leq 16\sup_{t\in T}(\mathbb{E}|\langle t, X\rangle|^p)^{1/p} = 16\sup_{\|\varphi\|_*\leq 1}(\mathbb{E}|\varphi(X)|^p)^{1/p}. \tag{6}$$

To estimate the last term in (5) notice that for $u \geq 16$ we have by Chebyshev's inequality

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{t\in T}\sum_{n=n_0}^{\infty}\big|\langle \pi_{n+1}(t)-\pi_n(t), X\rangle\big| \geq u\sup_{t\in T}\sum_{n=n_0}^{\infty}\big\|\langle \pi_{n+1}(t)-\pi_n(t), X\rangle\big\|_{2^n}\Big)\\
&\quad\leq \mathbb{P}\Big(\exists n\geq n_0\ \exists t\in T\ \big|\langle \pi_{n+1}(t)-\pi_n(t), X\rangle\big| \geq u\big\|\langle \pi_{n+1}(t)-\pi_n(t), X\rangle\big\|_{2^n}\Big)\\
&\quad\leq \sum_{n=n_0}^{\infty}\sum_{s\in T_{n+1}}\sum_{s'\in T_n}\mathbb{P}\big(|\langle s-s', X\rangle| \geq u\|\langle s-s', X\rangle\|_{2^n}\big) \leq \sum_{n=n_0}^{\infty}\#T_{n+1}\#T_n u^{-2^n}\\
&\quad\leq \sum_{n=n_0}^{\infty}\Big(\frac{8}{u}\Big)^{2^n} \leq 2\Big(\frac{8}{u}\Big)^{2^{n_0}} \leq 2\Big(\frac{8}{u}\Big)^{2p}.
\end{aligned}$$

Integrating by parts this gives

$$\begin{aligned}
&\Big(\mathbb{E}\Big(\sup_{t\in T}\sum_{n=n_0}^{\infty}\big|\langle \pi_{n+1}(t)-\pi_n(t), X\rangle\big|\Big)^p\Big)^{1/p}\\
&\quad\leq \sup_{t\in T}\sum_{n=n_0}^{\infty}\big\|\langle \pi_{n+1}(t)-\pi_n(t), X\rangle\big\|_{2^n}\Big(16 + \Big(2p\int_0^{\infty}u^{p-1}\Big(\frac{8}{u+16}\Big)^{2p}\Big)^{1/p}\Big)\\
&\quad\leq 32\sup_{t\in T}\sum_{n=n_0}^{\infty}\big\|\langle \pi_{n+1}(t)-\pi_n(t), X\rangle\big\|_{2^n}.
\end{aligned} \tag{7}$$

The result of Bobkov and Nazarov [2] gives

$$\|\langle t, X\rangle\|_r \leq C\|\langle t, \mathcal{E}\rangle\|_r \quad \text{for any } t\in\mathbb{R}^n \text{ and } r\geq 1. \tag{8}$$

Thus the statement follows by (4)–(7). ∎

Remark. The only property of the vector X that was used in the above proof was estimate (8). Thus inequality (3) holds for all n-dimensional random vectors satisfying (8).

Remark. Estimate (8) gives $(\mathbb{E}|\varphi(X)|^p)^{1/p} \leq C(\mathbb{E}|\varphi(\mathcal{E})|^p)^{1/p}$ for any functional φ, therefore Theorem 3.1 is stronger than the estimate from [6]:

$$(\mathbb{E}\|X\|^p)^{1/p} \leq C\big(\mathbb{E}\|\mathcal{E}\|^p\big)^{1/p} \sim C\Big(\mathbb{E}\|\mathcal{E}\| + \sup_{\|\varphi\|_*\leq 1}(\mathbb{E}|\varphi(\mathcal{E})|^p)^{1/p}\Big).$$

In some situation one may show that $\mathbb{E}\|\mathcal{E}\| \leq C\mathbb{E}\|X\|$. This is the case of spaces with bounded cotype constant.

Corollary 3.2. *Suppose that $2 \le q < \infty$, $F = (\mathbb{R}^n, \|\ \|)$ is a finite-dimensional space with a q-cotype constant bounded by $\beta < \infty$. Then for any n-dimensional unconditional, log-concave vector X and $p \ge 1$,*

$$(\mathbb{E}\|X\|^p)^{1/p} \le C(q,\beta)\Big(\mathbb{E}\|X\| + \sup_{\|\varphi\|_* \le 1} (\mathbb{E}|\varphi(X)|^p)^{1/p}\Big),$$

where $C(q,\beta)$ is a constant that depends only on q and β.

Proof. Applying diagonal transformation (and appropriately changing the norm) we may assume that X is also isotropic.

By the result of Maurey and Pisier [9] (see also Appendix II in [10]) one has

$$\mathbb{E}\|\mathcal{E}\| = \mathbb{E}\Big\|\sum_{i=1}^n e_i\mathcal{E}_i\Big\| \le C_1(q,\beta)\mathbb{E}\Big\|\sum_{i=1}^n e_i\varepsilon_i\Big\|.$$

By the unconditionality of X and Jensen's inequality we get

$$\mathbb{E}\|X\| = \mathbb{E}\Big\|\sum_{i=1}^n e_i\varepsilon_i|X_i|\Big\| \ge \mathbb{E}\Big\|\sum_{i=1}^n e_i\varepsilon_i\mathbb{E}|X_i|\Big\|.$$

We have $\mathbb{E}|X_i| \ge \frac{1}{C}(\mathbb{E}|X_i|^2)^{1/2} = \frac{1}{C}$, therefore

$$\mathbb{E}\|\mathcal{E}\| \le CC_1(q,\beta)\mathbb{E}\|X\|$$

and the statement follows by Theorem 3.1. ∎

For general norm on $\mathbb{R}^n$ one has

$$\mathbb{E}\|\mathcal{E}\| = \mathbb{E}\Big\|\sum_{i=1}^n e_i\varepsilon_i|\mathcal{E}_i|\Big\| \le \mathbb{E}\sup_i|\mathcal{E}_i|\mathbb{E}\Big\|\sum_{i=1}^n e_i\varepsilon_i\Big\| \le C\log n\,\mathbb{E}\Big\|\sum_{i=1}^n e_i\varepsilon_i\Big\|.$$

This together with the similar argument as in the proof of Corollary 3.2 gives the following.

Corollary 3.3. *For any n-dimensional unconditional, log-concave vector X, any norm $\|\ \|$ on $\mathbb{R}^n$ and $p \ge 1$ one has*

$$(\mathbb{E}\|X\|^p)^{1/p} \le C\Big(\log n\,\mathbb{E}\|X\| + \sup_{\|\varphi\|_* \le 1} (\mathbb{E}|\varphi(X)|^p)^{1/p}\Big).$$

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