DIFFERENTIAL AND INTEGRAL CALCULUS FOR A SCHAUDER BASIS ON A FRACTAL SET (I) (SCHAUDER BASIS 80 YEARS AFTER)

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Abstract. In this paper we introduce a concept of Schauder basis on a self-similar fractal set and develop differential and integral calculus for them. We give the following results:

(1) We introduce a Schauder/Haar basis on a self-similar fractal set (Theorems I and I').

(2) We obtain a wavelet expansion for the L^2-space with respect to the Hausdorff measure on a self-similar fractal set (Theorems II and II').

(3) We introduce a product structure and derivation on a self-similar fractal set (Theorem III).

(4) We give the Taylor expansion theorem on a fractal set (Theorem IV and IV').

(5) By use of the Taylor expansion for wavelet functions, we introduce basic functions, for example, exponential and trigonometrical functions, and discuss the relationship between the usual and introduced corresponding special functions (Theorem V).

(6) Finally we discuss the relationship between the wavelet functions and the generating functions of the dynamical systems on a fractal set and show that our wavelet expansions can describe several fluctuations observed in nature.

[115]

1. Introduction. In 1927, Schauder [10, 11] introduced a system of functions which is called the Schauder basis on the interval and made a uniform approximation of a continuous function. In this paper we shall introduce a concept of Schauder basis on a self-similar fractal set and make differential and integral calculus for them. Hence we may say that Schauder's observation will supply an important method for new fields of mathematics still now and furthermore in the future.

Fractal structure can be observed in many places and it is expected that it will supply an important method for complex systems, especially the description of fluctuations in music, stock price behaviour, and neural systems of brains. The fundamental material is already well prepared in the famous book by Falconer [2]: the geometric measure theory with respect to the Hausdorff measure is well exposed and its applications are given. The concept of self-similar fractal sets was introduced by Hutchinson in [3] and the geometric theory of the Hausdorff measure is treated there in detail. Many authors have contributed to the calculus of the Hausdorff dimensions.

On the other hand, the analysis on a fractal set also has started and became one of the most interesting topics. One of important and beatiful results is supplied by the theory of Laplacian on the Sierpiński's gasket [3]. We may expect further developments in this direction.

In this paper we introduce differential and integral calculus on a self-similar fractal set. The basic idea is to introduce a wavelet expansion on a fractal set and to define a derivation on the Hilbert space of L^2-space with respect to the Hausdorff measures on it. This can be performed by introducing the "∗-product" on the space. Then we can see that the derivation can be defined by the shift operator on the space and develop differential calculus on a self-similar fractal set with the Taylor expansion theorem on it. By use of the Taylor expansion, we can introduce basic functions, for example, exponential, trigonometrical, and geometric functions on a fractal set. Here we wish to stress that we can give an embedding theorem of the usual corresponding functions into our basic functions on the fractal sets which may be regarded as the *noncommutative embedding* of derivations. We call such an embedding the *fluctuation mapping*. By this mapping we can get functions with complicated fluctuations from smooth functions without fluctuations. We can find a new approach to the problem of the basic fluctuations in the nature by such a mapping [6, 8].

This is the first part of our study on a fractal set. In the second part we shall introduce differential forms and chains on fractal sets and develop the vector anlysis on a fractal set.

2. Self-similar fractal set. In this section we recall some basic material on fractal sets [2, 3]. Here we restrict ourselves only to a system of self-similar mappings $\{\sigma_j : j = 1, 2, \ldots, N\}$ of a compact set K_0. Then we can see that there exist positive numbers λ_i $(0 < \lambda_i < 1)$ satisfying the conditions:

$$d(\sigma_i(x), \sigma_i(y)) = \lambda_i d(x, y) \quad (i = 1, 2, \ldots, N).$$

Then we give the definition of a self-similar fractal set:

DEFINITION 1 (self-similar fractal set). For a system of self-similar mappings $\sigma_j : K_0 \to K_0$ $(j = 1, 2, \ldots, N)$, we put

$$(2.1) \qquad K = \bigcap_{n=1}^{\infty} K_n, \text{ where } K_n = \bigcup_{j=1}^{N} \sigma_j(K_{n-1}),$$

and call K a *self-similar fractal set*.

In this paper we always assume the following separation condition:

DEFINITION 2 (separation condition). A system of mappings $\sigma_j : K_0 \to K_0$ $(j = 1, 2, \ldots, N)$ is said to satisfy the *separation condition* whenever

$$(2.2) \qquad \sigma_i(\overset{\circ}{K_0}) \cap \sigma_j(\overset{\circ}{K_0}) = \emptyset \quad (i \neq j),$$

where $\overset{\circ}{K_0}$ is the open kernel of K_0.

Some basic properties of fractal sets. Under the condition (2.1), we observe the following facts:

(1) We have the invariance condition:

$$(2.3) \qquad K = \bigcup_{j=1}^{N} \sigma_j(K).$$

(2) In the case of a self-similar fractal set, we can calculate the Hausdorff dimension D $(= \dim_H K)$ by the following formula:

$$(2.4) \qquad \sum_{j=1}^{N} \lambda_j^D = 1.$$

The proof will be given later.

Here we give examples of a fractal set.

EXAMPLE 1 (Cantor set). For a system of self-similar mappings $\sigma_1 = \frac{1}{3}x$, $\sigma_2 = \frac{1}{3}x + \frac{2}{3}$, we choose

$$(2.5) \qquad C = \bigcap_{n=1}^{\infty} C_n,$$

where (see Fig. 1):

$$(2.6) \qquad \begin{aligned} C_0 &= I, \\ C_n &= \sigma_1(C_{n-1}) \cup \sigma_2(C_{n-1}). \end{aligned}$$

The Hausdorff dimension D is

$$(2.7) \qquad D = \log 2 / \log 3.$$

EXAMPLE 2 (self-similar fractal set of Cantor type). In a similar manner we can define a fractal set of Cantor type, removing subintervals from the original interval: Let I be

the interval $I = [0,1]$ and let $\sigma_i : I \to I$ $(i = 1, \ldots, N)$ be a system of contractions with contraction ratios λ_i which satisfy the separation conditions (2.2). Then we put

$$(2.8) \qquad C(\lambda_1, \ldots, \lambda_N) = \bigcap_{n=0}^{\infty} C_n, \quad C_n = \bigcup_{j=1}^{N} \sigma_j(C_{n-1})$$

and call C *self-similar fractal set of Cantor type*.

We denote the original Cantor set by $C(\frac{1}{3}, \frac{1}{3})$. In the case where $\lambda_1 = \cdots = \lambda_N = 1/M$ $(N \leq M)$ we have the Hausdorff dimension

$$(2.9) \qquad D = \log N / \log M.$$

We give the graph of the fractal set in the case of $C(\frac{1}{3}, \frac{1}{9}, \frac{1}{9})$; see Fig. 2.
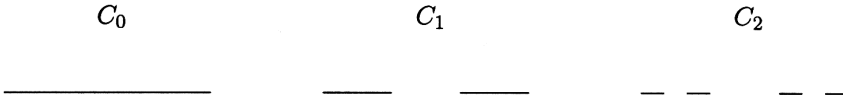
$$C_0 \qquad\qquad\qquad C_1 \qquad\qquad\qquad C_2$$

Fig. 1. The Cantor set $C\left(\frac{1}{3}, \frac{1}{3}\right)$.

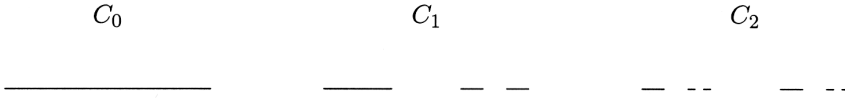$$C_0 \qquad\qquad\qquad C_1 \qquad\qquad\qquad C_2$$

Fig. 2. Self-similar fractal set $C\left(\frac{1}{3}, \frac{1}{9}, \frac{1}{9}\right)$ of Cantor type.

3. The Hausdorff measure and integration on a fractal set. We are going to deal with the integral calculus on a fractal set. First we prepare the Hausdorff measure. Next we give the integral calculus with respect to that measure.

A collection of compact sets $\{U_1, U_2, \ldots, U_m\}$ is called a ρ-*covering* of X if the diameter satisfies $d(U_i) \leq \rho$ $(i = 1, \ldots, m)$. We make the following definition [3, 5]:

DEFINITION 3 (k-dimensional Hausdorff measure). The *k-dimensional Hausdorff measure* is defined by

$$(3.10) \qquad \mu^k(X) = \lim_{\rho \to 0} \mu_\rho^k(X),$$

where

$$\mu_\rho^k(X) \equiv \inf\left\{ \sum_{i=1}^{n} d_i^k \,\middle|\, 0 < d_i \leq \rho, X \subseteq \bigcup_{i=1}^{n} U_i \right\} \quad (d_i = d(U_i)).$$

We notice the following

PROPOSITION 1 (Hausdorff-Besicovitch Theorem) [2, 3]. *There exists a unique D $(D \geq 0)$ such that*

$$(3.11) \qquad \begin{cases} \mu^k(X) = \infty \ (k < D), \\ \mu^k(X) = 0 \ (k > D). \end{cases}$$

In the case $D = 0$, we take only the latter condition.

We call D the *Hausdorff dimension* of X and μ^D the *Hausdorff measure*, respectively.

Some basic properties of the Hausdorff measure.
(1) μ^D is a σ-additive measure [2]:

$$(3.12) \qquad \mu^D\Big(\sum_{j=1}^{\infty} A_j\Big) = \sum_{j=1}^{\infty} \mu^D(A_j)$$

for measurable sets A_j $(j = 1, 2, \ldots, n)$ with $A_j \cap A_k = \emptyset$ $(j \neq k)$.
(2) For a measurable set A, we have

$$(3.13) \qquad \mu^D(A) = \sum_{j=1}^{N} \mu^D(\sigma_j(A)).$$

Hence we can express the measure in terms of the so-called mass distribution [2]. Here we notice the following important fact: $\mu^D(K) > 0$ [3]. After normalization $\mu^D(K) = 1$, we have

$$(3.14) \qquad \mu^D(K_{j_n \cdots j_1}) = \lambda_{j_n}^D \cdots \lambda_{j_1}^D,$$

where

$$K_{j_n \cdots j_1} = \sigma_{j_n} \circ \cdots \circ \sigma_{j_1}(K).$$

In the case of $n = 1$, we arrive at the formula (2.4).
(3) The Borel algebra is generated by $\{K_{j_n \cdots j_1}\}$.
We can develop a theory of integral with respect to this measure.

4. Representations of Cuntz algebras.
We introduce representations of the Cuntz algebras on the Hilbert space $L^2(K, d\mu^D)$.

The *Cuntz algebra* $\mathcal{O}(N)$ is a C^*-algebra with generators $\{S_j\}$ $(j = 1, 2, .., N)$ and the following commutation relations [1, 7]:

$$(4.15) \qquad (1) \ S_j^* S_j = 1 (j = 1, 2, .., N), \qquad (2) \ \sum S_j S_j^* = 1.$$

These commutation relations give an algebraic description of the division of the total space into N parts. Then we can make the representation of the Cuntz algebras on fractal sets in a well known manner:

PROPOSITION 2 (Hausdorff representations on fractal sets of flower type) [7].
(1) *Let C be a self-similar fractal set defined by $\{\sigma_j\}$ $(j = 1, 2, .. N)$. Then we have the following representation $\pi : \mathcal{O}(N) \to B(L^2(C))$:*

$$\pi(S_j)f(x) = \begin{cases} J(\sigma_j)^{1/2}(\sigma_j^{-1}(x))f(\sigma_j^{-1}(x)), & x \in \sigma_j(K), \\ 0, & x \notin \sigma_j(K)(j = 1, 2, \ldots, N), \end{cases}$$

$$\pi(S_j^*)f(x) = J(\sigma_j)^{-1/2}(x)f(\sigma_j(x)) \quad (j = 1, 2, \ldots, N),$$

where $J(\sigma_i)$ is the Radon-Nikodym derivative of σ_i.

The representation defined above is called *Hausdorff representation*.

(2) *Let C_1 and C_2 be two self-similar fractal sets with the same number of generators N. The Hausdorff representations are equivalent if and only if they have the same Kakutani-Jørgensen invariant*

(4.16) $$\{\lambda_j^D \ (j = 1, 2, \ldots, N)\}.$$

Here two Hausdorff representations: $\pi_i : \mathcal{O}(N) \to B(L^2(C_i))$ $(i = 1, 2)$ *are unitarily equivalent if we can find a unitary operator* $U : L^2(C_1) \to L^2(C_2)$ *such that* $\pi_1(S)U = U\pi_2(S)$ *holds for any* $S \in \mathcal{O}(N)$.

5. The Schauder basis on a fractal set. In this section we recall the Schauder basis on the interval and introduce a basis of Schauder type on a fractal set [4] [10, 11].

We put

$$G^{(0)}(x) = \begin{cases} x, & x \in [0, \frac{1}{2}], \\ 1 - x, & x \in (\frac{1}{2}, 1], \end{cases}$$

and consider self-similar mappings $\sigma_1(x) = \frac{1}{2}x$, $\sigma_2(x) = \frac{1}{2}x + \frac{1}{2}$. For an integer m, putting

(5.17) $$G^{(0)}_{j_m \cdots j_1}(x) = \begin{cases} \frac{1}{2^{\frac{1}{2}m}}G^{(0)}(\sigma_{j_1}^{-1} \circ \cdots \circ \sigma_{j_m}^{-1}(x)) & (x \in I_{j_m \cdots j_1}), \\ 0 & \text{otherwise}, \end{cases}$$

we can see that these functions give a linear basis of $L^2(I, d\mu)$. Here we notice that

(5.18) $$(G^{(0)}_{i_m i_{m-1} \cdots i_1}, G^{(0)}_{j_m j_{m-1} \cdots j_1}) = \delta_{i_m j_m}\delta_{i_{m-1} j_{m-1}} \cdots \delta_{i_1 j_1}.$$

Yet, we can see that the different m and m' are not necessarily orthonormal, for example (Fig. 3):

(5.19) $$(G^{(0)}, G^{(0)}_{j_1}) \neq 0.$$

PROPOSITION 3. *A continuous function f with $f(0) = f(1) = 0$ on I $(= [0, 1])$ is expanded in the Schauder basis:*

$$f(x) = a_0 G^{(0)}(x) + \sum_{m=1}^{\infty} \sum_{j_1=1}^{2} \cdots \sum_{j_m=1}^{2} a_{j_m \cdots j_1} G_{j_m \cdots j_1}(x).$$

Proof. First we take a continuous function $f(x)$ $(f(0) = f(1) = 0)$. Next we consider

$$f_1(x) = f(x) - a_0 G^{(0)}(x) \quad \left(a_0 = 2f(\frac{1}{2})\right)$$

on $I(= [0, 1])$. Then we have $f_1(\frac{1}{2}) = 0$. In a similar manner, we choose a_{j_1} $(j_1 = 1, 2)$ in

$$f_2(x) = f(x) - a_0 G^{(0)}(x) - \sum_{j_1=1}^{2} a_{j_1} G^{(0)}_{j_1}(x),$$

so that

$$f_2\left(\frac{1}{2^2}\right) = f_2\left(\frac{3}{2^2}\right) = 0.$$

$$G^{(0)}(x)$$

$$G_1^{(0)}(x) \qquad\qquad G_2^{(0)}(x)$$

etc.

$$G_{11}^{(0)}(x) \qquad\qquad G_{21}^{(0)}(x)$$

$$G_{12}^{(0)}(x) \qquad\qquad G_{21}^{(0)}(x)$$

etc.
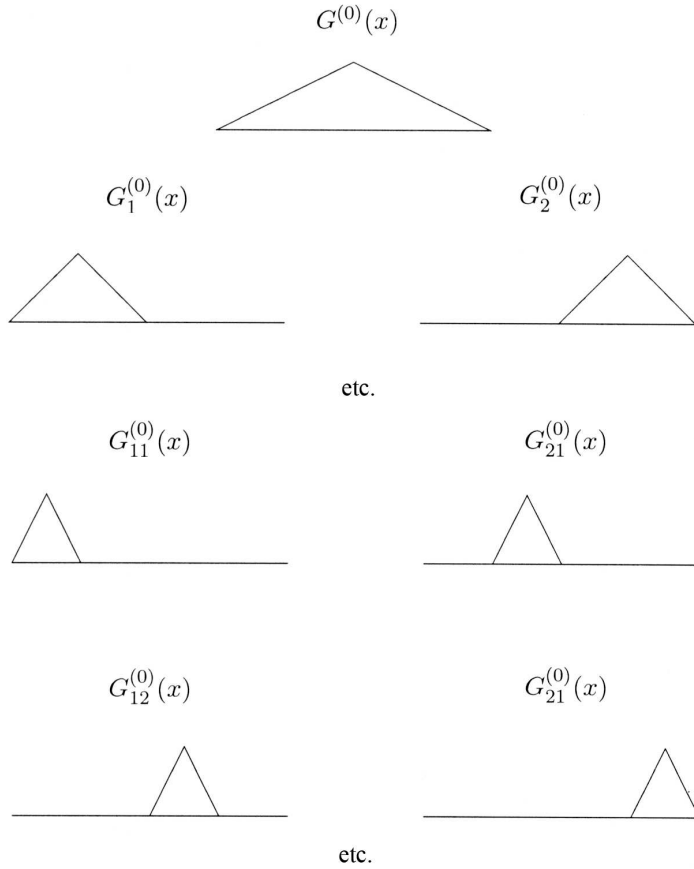
Fig. 3. A basis of Schauder type on a fractal set.

Hence we have

$$f_2\left(\frac{i}{2^2}\right) = 0 \qquad (i = 1, \ldots, 3).$$

In a similar manner, we have

$$f_n(x) = f(x) - a_0 G^{(0)}(x) - \sum_{m=1}^{n} \sum_{j_1=1}^{2} \cdots \sum_{j_m=1}^{2} a_{j_m \cdots j_1} G_{j_m \cdots j_1}^{(0)}(x),$$

$$f_n\left(\frac{i}{2^n}\right) = 0 \quad (i = 1, \ldots, 2^n - 1).$$

Therefore we can see the following: For any $\epsilon > 0$, there exists n_0 such that

$$|f_n(x)| < \epsilon \quad (\forall n > n_0, \forall x \in I),$$

which implies that f_n converges to the constant zero function uniformly. Hence we have completed the proof. ∎

Next we give several kinds of bases of Schauder type.

EXAMPLE 3 (the Weierstrass basis) [2, 12].

In a similar manner to Example 1, putting

$$(5.20) \qquad\qquad W^{(0)}(x) = 4x(1-x) \quad (0 \leq x \leq 1)$$

and choosing σ_1, σ_2 in Example 1, we have an orthonormal basis in $L^2(I, d\mu^D)$.

EXAMPLE 4 (the Haar-Schauder type).

Putting

$$H^{(0)}(x) = \chi_K(x),$$

$$(5.21)$$
$$H^{(0)}_{j_n \cdots j_1}(x) = \begin{cases} \lambda_{j_1}^{-D/2} \cdots \lambda_{j_n}^{-D/2} H^{(0)}(\sigma_{j_1}^{-1} \circ \cdots \circ \sigma_{j_n}^{-1}(x)) & \text{on } K_{j_n \cdots j_1}, \\ 0 & \text{otherwise,} \end{cases}$$

we have a linear basis in $L^2(K, d\mu^D)$ which satisfies (5.18).

EXAMPLE 5 (Schauder basis on a fractal set of Cantor type).

We can introduce a basis of Schauder type for a self-similar fractal set of Cantor type. First we consider a set which satisfies the division condition:

$$(5.22) \qquad\qquad I = \bigcup_{j=1}^{N} \sigma_j(I).$$

We can give a Schauder basis in the following manner: First we define the basic functions:

$$(5.23) \qquad G^{(\rho)} = \begin{cases} \dfrac{1}{x_\rho - x_{\rho-1}}(x - x_\rho), & x \in I_{\rho-1,\rho}, \\[2ex] \dfrac{1}{x_\rho - x_{\rho+1}}(x - x_\rho), & x \in I_{\rho,\rho+1} \quad (\rho = 1, 2, \ldots, N-1), \\[2ex] 0 & \text{otherwise.} \end{cases}$$

Then we can introduce the Schauder basis defining $G^{(\rho)}_{j_n, j_{n-1}, \ldots, j_1}$ by

$$(5.24) \qquad G^{(\rho)}_{j_m \cdots j_1}(x) = \begin{cases} G^{(\rho)}(\sigma_{j_1}^{-1} \circ \cdots \circ \sigma_{j_m}^{-1}(x)) & (x \in I_{j_m \cdots j_1}), \\ 0 & \text{otherwise;} \end{cases}$$

we define a basis of Schauder type, where

$$I_{j_m, j_{m-1}, \ldots, j_1} = \sigma_{j_m} \circ \sigma_{j_{m-1}} \ldots \circ \sigma_{j_1}(I).$$

We can prove the uniform approximation theorem of continuous functions for this case:

PROPOSITION 3. *For any continuous function f with $f(0) = f(1)$, we can approximate f by $G_{j_n, j_{n-1}, \ldots, j_1}$ uniformly:*

(5.25)
$$f = \sum_{\rho=1}^{N-1} a_0^{(\rho)} G^{(\rho)} + \sum_{\rho=1}^{N-1} \sum a_{j_n, j_{n-1}, \ldots, j_1}^{(\rho)} G_{j_n, j_{n-1}, \ldots, j_1}^{(\rho)}.$$

In a similar manner we make a basis of Schauder type on a (general) self-similar fractal set of Cantor type C (Fig. 4).

$G_1^{(0)}(x)$ $\qquad\qquad$ $G_2^{(0)}(x)$ $\qquad\qquad$ $G_3^{(0)}(x)$

$G_{11}^{(0)}(x)$ $\qquad\qquad$ $G_{21}^{(0)}(x)$

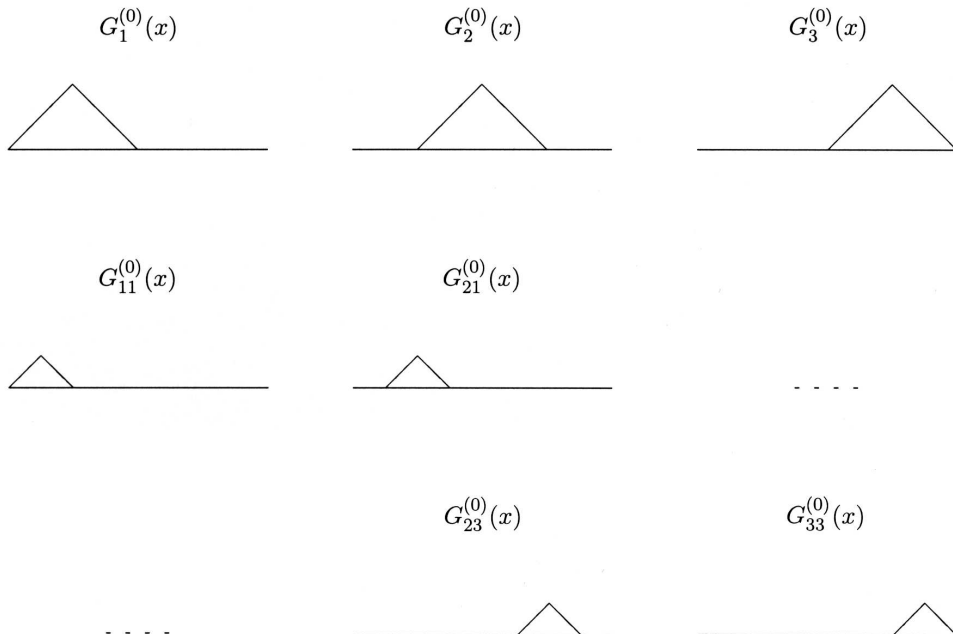$G_{23}^{(0)}(x)$ $\qquad\qquad$ $G_{33}^{(0)}(x)$

Fig. 4. A basis of Schauder type on a (general) self-similar fractal set of Cantor type.

DEFINITION 4 (Schauder basis on a self-similar fractal set of Cantor type). Let C $(= C(\lambda_1, \lambda_2, \ldots, \lambda_N))$ be a fractal set of Cantor type which is generated by $\{\sigma_j : j = 1, 2, \ldots, N\}$ on the interval I. We set

(5.26)
$$G_{j_m \cdots j_1}^{(\rho)}(x) = \begin{cases} C_{j_m \cdots j_1}^{(\rho)} G^{(\rho)}(\sigma_{j_1}^{-1} \circ \cdots \circ \sigma_{j_m}^{-1}(x)) & (x \in C_{j_m \cdots j_1}), \\ 0 & \text{otherwise,} \end{cases}$$

where the constants $C_{j_m \cdots j_1}^{(\rho)}$ are determined by the orthonormality conditions:

(5.27)
$$\int_C G_{i_m \cdots i_1}^{(\rho)} G_{j_m \cdots j_1}^{(\rho')} d\mu^D = \delta_{\rho, \rho'} \delta_{i_m, j_m} \delta_{i_{m-1}, j_{m-1}} \cdots \delta_{i_1, j_1}.$$

We call $\{G^{(\rho)}, G_{j_m \cdots j_1}^{(\rho)}\}$ the *Schauder basis on K*.

Then we can prove the following

THEOREM I (on Schauder basis on a self-similar fractal set of Cantor type). *Let C ($=$ $C(\lambda_1, \lambda_2, \ldots, \lambda_N)$ be a fractal set of Cantor type. Then the system $\{G_0^{(\rho)}, G_{j_m \cdots j_1}^{(\rho)(0)}\}$ constitutes orthonormal basis of $L^2(C, d\mu^D)_*$* (cf. (7.41) below).

Proof. We take a fractal set (2.1). Next we consider the interval with the fractal structure defined by $\tilde{\sigma}_j : I \to I (j = 1, 2, \ldots, N)$ with the contraction ratios $\{\lambda_j^D (j = 1, 2, \ldots, N)\}$. From (2.4), we have a fractal structure on I satisfying the condition (5.22) and it is a fractal set of Cantor type: $I = I(\lambda_1^D, \lambda_2^D, \ldots, \lambda_N^D)$. We construct the desired Schauder basis on the interval I. For an integer m, we put

$$(5.28) \qquad \tilde{G}_{j_m \cdots j_1}^{(\rho)}(x) = \begin{cases} C_{j_m \cdots j_1}^{(\rho)} G^{(\rho)}(\tilde{\sigma}_{j_1}^{-1} \circ \cdots \circ \tilde{\sigma}_{j_m}^{-1}(x)) & (x \in I_{j_m \cdots j_1}), \\ \\ 0 & \text{otherwise.} \end{cases}$$

Next we consider a representation of the central extension of the Cuntz algebra [1]. Setting

$$(5.29) \qquad \begin{cases} S_j(G_{j_1, j_2, \ldots, j_m}^{(\rho)}) = (G_{j, j_1, j_2, \cdots, j_m}^{(\rho)}), \\ \\ S_j^*(G_{j_1, j_2, \ldots, j_m}^{(\rho)}) = \delta_{j, j_1}(G_{j_2, \ldots, j_m}^{(\rho)}) \end{cases}$$

we can construct representations of the Cuntz algebra on both spaces. Comparing the Kakutani-Jørgensen invariants [4] of both representations, we have an intertwining unitary operator

$$(5.30) \qquad\qquad U : L^2(I, d\tilde{\mu}) \to L^2(C, d\mu^D),$$

and this suffices to prove the assertion. ∎

6. An orthonormal basis of Haar type on a fractal set. We are going to introduce a basis of Haar type on a fractal set. This gives an orthonormal system of the class $L^2(K, d\mu^D)$. We begin with the Haar basis on the interval [11, 12]. We consider the following mappings:

$$(6.31) \qquad\qquad \sigma_1(x) = \frac{1}{2}x, \qquad \sigma_2(x) = \frac{1}{2}x + \frac{1}{2}.$$

Then we have the interval I ($= [0, 1]$) as the self-similar fractal set. Choosing

$$H^{(0)}(x) = 1 \quad \text{on } [0, 1], \quad H^{(1)}(x) = \begin{cases} 1 & x \in \left[0, \frac{1}{2}\right], \\ \\ -1 & x \in \left(\frac{1}{2}, 1\right], \end{cases}$$

we get an orthonormal system which is called *Haar basis*. Putting

$$(6.32) \qquad H^{(1)}{}_{j_m \cdots j_1}(x) = \begin{cases} \frac{1}{2^{\frac{1}{2}m}} H^{(1)}(\sigma_{j_1}^{-1} \circ \cdots \circ \sigma_{j_m}^{-1}(x)) & (x \in C_{j_m \cdots j_1}), \\ 0 & \text{otherwise,} \end{cases}$$

we have an orthonormal system on $L^2(I, d\mu)$; see Fig. 5.

Next we proceed to an orthonormal basis of Haar type on a self-similar fractal set of general type. We can prove the following

THEOREM I' (on an orthonormal basis of Haar type on a self-similar fractal set). *Take an orthonormal system on a fractal set K of Cantor type. Putting*

$$\begin{cases} H^{(0)}(x) = \chi_K & (\chi_K = characteristic\ function\ of\ K), \\ H^{(\rho)}(x) = \epsilon_\rho \chi_{K_\rho} - \epsilon'_\rho \sum_{k=\rho+1}^{N-1} \chi_{K_k} & (1 \le \rho \le N-1), \end{cases}$$
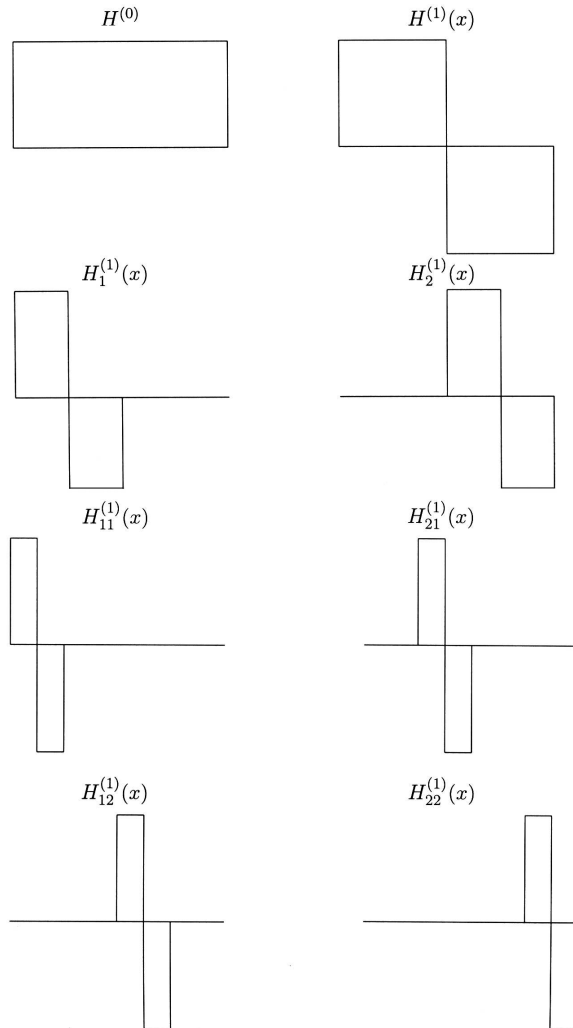

Fig. 5. An orthonormal basis of Haar type on a fractal set.

where

$$\epsilon_\rho = \sqrt{\frac{(1 - \lambda_1^D - \cdots - \lambda_\rho^D)}{\lambda_\rho^D(1 - \lambda_1^D - \cdots - \lambda_{\rho-1}^D)}},$$

(6.33)

$$\epsilon_\rho' = \sqrt{\frac{\lambda_\rho^D}{(1 - \lambda_1^D - \cdots - \lambda_\rho^D)(1 - \lambda_1^D - \cdots - \lambda_{\rho-1}^D)}},$$

we get a complete orthonormal system on $L^2(K, d\mu^D)$:

(6.34) $H^{(0)}(x),\ H^{(\rho)}(x),\ H_{j_n \cdots j_1}^{(\rho)}(x)\quad (\rho = 1, 2, \ldots, N-1),$

where

$$H_{j_n \cdots j_1}^{(\rho)}(x) = \lambda_{j_1}^{-D/2} \cdots \lambda_{j_n}^{-D/2} H^{(\rho)}(\sigma_{j_1}^{-1} \circ \cdots \circ \sigma_{j_n}^{-1}(x)),$$

(6.35)

$$(x \in K_{j_n \cdots j_1}, \rho = 1, 2, \ldots, N-1).$$

Proof. We choose a system of functions $\{H_j'\ (j = 1, 2, \ldots, N)\}$ by

(6.36) $H_j' = \chi_{K_j}\quad (K_j = \sigma_j(K)).$

We include the constant function on K:

(6.37) $H^{(0)} = \sum_{j=1}^{N} H_j'.$

Putting

(6.38) $H^{(1)} = \epsilon_1 H_1' + \epsilon_1'(H_2' + H_3' + \cdots + H_N')$

and choosing ϵ_1 and ϵ_1' in (6.33) we can see that the orthonormality conditions hold:

$$(H^{(0)}, H^{(1)}) = 0 \quad \text{and} \quad (H^{(1)}, H^{(1)}) = 1.$$

Next putting

$$H^{(2)} = \epsilon_2 H_2' + \epsilon_2'(H_3' + H_4' + \cdots + H_N')$$

and choosing ϵ_2, and ϵ_2' in (6.33), we have the orthonormality conditions:

$$(H^{(0)}, H^{(2)}) = 0 \quad \text{and} \quad (H^{(2)}, H^{(2)}) = 1.$$

Moreover, we can see that the condition

$$(H^{(1)}, H^{(2)}) = 0$$

holds automatically by the construction. Repeating this process we can get the first N basic functions:

(6.39) $H^{(0)}, H^{(1)}, \ldots, H^{(N-1)}.$

Next we are going to prolongate these functions to a complete orthonormal basis: Putting (6.33), we have the orthonormality condition:

$$\int_K H_{i_n \cdots i_1}^{(\rho)} H_{j_m \cdots j_1}^{(\rho')} d\mu^D = \delta_{\rho\rho'} \delta_{nm} \delta_{i_n j_n} \cdots \delta_{i_1 j_1} \quad (\rho = 1, 2, \ldots, N-1).$$

We can prove the completeness condition as follows. First we notice that every characteristic function of K_j $(j = 1, 2, \ldots, N)$ can be expressed as a linear combination of (6.33). By this we can see that the characteristic function of every generator of the Borel algebra can be written with the use of

$$H^{(\rho)} \quad (\rho = 0, \ldots, N-1), \quad H^{(\rho)}_{j_n \cdots j_1} (\rho = 1, 2, \ldots, N-1).$$

Hence we can see that the completeness condition holds for our basis as desired. ∎

Here we can give two examples (Figs. 6 and 7).

EXAMPLE 6 (the Haar basis on the Cantor set) (Fig. 6).
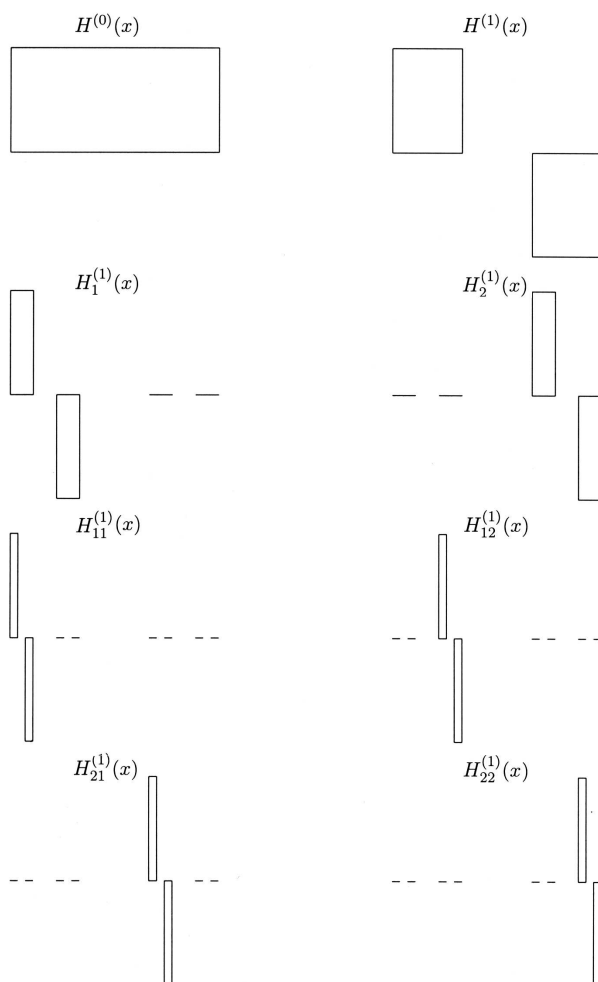
Fig. 6. The Haar basis on the Cantor set.

EXAMPLE 7 (the Haar basis on a self-similar fractal set) (Fig. 7).

$H^{(1)}(x)$ $H^{(2)}(x)$

$H_1^{(1)}(x)$ $H_2^{(1)}(x)$ $H_3^{(1)}(x)$

$H_2^{(1)}(x)$ $H_3^{(1)}(x)$

$H_1^{(2)}(x)$ $H_2^{(2)}(x)$ $H_3^{(2)}(x)$
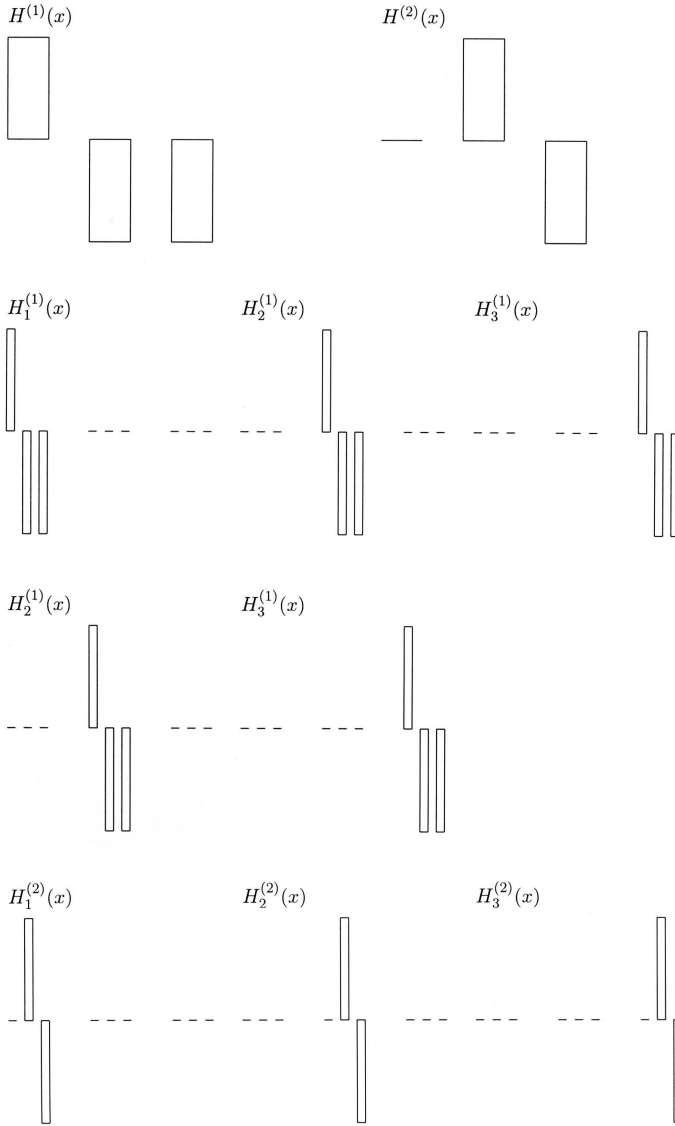
Fig. 7. The Haar basis on a self-similar fractal set.

7. A wavelet expansion of Schauder type (respectively: of Haar type) on a fractal set. In this section we give the wavelet expansion of Schauder/Haar type on a self-similar fractal set [12]. The Fourier expansion supplies an important tool in the analysis on the Euclidean space. This expansion may be regarded as the Fourier expansion on a fractal set. We can easily prove the following

THEOREM II (Wavelet expansion theorem for the Haar basis on a self-similar fractal set). *Every function of $L^2(K, d\mu^D)$ can be expressed as a linear combination of*

$$\{H^{(\rho)}, H^{(\rho)}_{j_n,\ldots,j_1}\}.$$

Namely, for any function $f \in L^2(K, d\mu^D)$, we can find a sequence of complex numbers $\{a^{(\rho)}, a^{(\rho)}_{j_n\cdots j_1}\}$ so that

$$f = \sum_{\rho=1}^{N-1} a^{(\rho)} H^{(\rho)} + \sum_{n=1}^{\infty} \sum_{\rho=1}^{N-1} \sum_{j_1=1}^{N} \cdots \sum_{j_n=1}^{N} a^{(\rho)}_{j_n\cdots j_1} H^{(\rho)}_{j_n\cdots j_1},$$

(7.40)

$$a^{(\rho)} = (f, H^{(\rho)}) \quad and \quad a^{(\rho)}_{j_n\cdots j_1} = (f, H^{(\rho)}_{j_n\cdots j_1}).$$

Next we proceed to wavelet expansions for Schauder basis on a self-similar fractal set. First we introduce a concept of degrees for Schauder basis. We call elements $G^{(0)}_{j_m\cdots j_1}(x)$ *elements of degree m* (or m-forms) which we denote by $L^2_m(K, d\mu^D)$. Then we can see that these elements constitute an orthonormal basis of this space. Here we notice that elements of different degree are not always orthogonal, for example G^0 is not orthogonal to elements of degree m ($m > 0$). Here we introduce a new inner product defined by the orthonormalization conditions concerned with

(7.41)
$$L^2_m(K, d\mu^D) \quad and \quad L^2_{m'}(K, d\mu^D),$$

which is called *orthonormalized inner product* with respect to degrees. The newly introduced inner product is denoted by $(\ ,\ )_*$ and the space endowed with the inner product is denoted by $L^2_*(K, d\mu^D)$, respectively.

In a similar manner to Theorem II', we have the following

THEOREM II' (wavelet expansion theorem for the Schauder basis on a self-similar fractal set). *Every function of $L_*^2(K, d\mu^D)$ can be expressed as a linear combination of*

$$\{G^{(\rho)}, G^{(\rho)}_{j_n,\ldots,j_1}\}.$$

Namely, for any function $f \in L_*^2(K, d\mu^D)$, we can find a sequence of complex numbers $\{a^{(\rho)}, a^{(\rho)}_{j_n\cdots j_1}\}$ so that

$$f = \sum_{\rho=1}^{N-1} a^{(\rho)} G^{(\rho)} + \sum_{\rho=1}^{N-1} \sum_{n=1}^{\infty} \sum_{j_1=1}^{N} \cdots \sum_{j_n=1}^{N} a^{(\rho)}_{j_n\cdots j_1} G^{(\rho)}_{j_n\cdots j_1},$$

(7.42)

$$a^{(\rho)} = (f, G^{(\rho)})_* \quad and \quad a^{(\rho)}_{j_n\cdots j_1} = (f, G^{(\rho)}_{j_n\cdots j_1})_*.$$

Examples of expansions will be given in Section 9.

8. The $*$-products and the derivations on a fractal set. We are going to introduce a product structure on $L^2(K, d\mu)$ and then to define a derivation on the space. First we describe our idea on the introduction of the desired derivations. We consider the Taylor expansion of an analytic function:

$$f = \sum_{n=0}^{\infty} \frac{a_n}{n!}(z-a)^n \quad (a_n = f^{(n)}(a)).$$

Putting $f = (a_0, a_1, \ldots)$, we have $f' = (a_1, a_2, \ldots)$. Hence we see that the derivation is just the shift operator. Based on this fact we can get a derivation on space of sequences of numbers. We choose

$$\Sigma = \{s = (s_0, s_1, \ldots, s_n, \ldots) | s_j \in \mathbf{C}\}$$

and introduce the product structure on the space by

$$s * t = (u_0, u_1, \ldots, u_n, \ldots) \quad (u_n = \sum_{j=0}^{n} s_j t_{n-j})$$

for two elements

$$s = (s_0, s_1, \ldots, s_n, \ldots) \text{ and } t = (t_0, t_1, \cdots, t_n, \ldots).$$

Then we have the following

PROPOSITION 5. *The shift operation σ ($\sigma : \Sigma(\mathbf{C}) \to \Sigma(\mathbf{C})$) defines a derivation on Σ. Namely, we have*

$$\sigma(s * t) = \sigma(s) * t + s * \sigma(t).$$

Proof: easy and can be omitted.

Next we proceed to introducing a derivation on $L^2(K, d\mu)$. First we define the product structure by the following bilinear mapping:

$$* : L^2(K, d\mu) \times L^2(K, d\mu) \to L^2(K, d\mu)$$

which is determined by

$$\begin{cases} H^{(\rho)} * H^{(\rho')} & = & H^{(\rho')} * H^{(\rho)} & = H^{(\rho)} \cdot H^{(\rho')}, \\ H^{(\rho)} * H^{(\rho')}_{j_n \cdots j_1} & = & \delta_{\rho\rho'} H^{(\rho)} \cdot H^{(\rho)}_{j_n \cdots j_1}, \\ H^{(\rho)}_{j_n \cdots j_1} * H^{(\rho')} & = & \delta_{\rho\rho'} H^{(\rho)}_{j_n \cdots j_1} \cdot H^{(\rho)}, \\ H^{(\rho)}_{i_n \cdots i_1} * H^{(\rho')}_{j_n \cdots j_1} & = & \delta_{\rho,\rho'} H^{(\rho)}_{i_n \cdots i_1 j_n \cdots j_1}. \end{cases}$$

Then, for $f, g \in L^2(K, d\mu^D)$:

$$f = \sum_{\rho=0}^{N} a^{(\rho)} H^{(\rho)} + \sum_{n=1}^{\infty} \sum_{\rho=1}^{N-1} \sum_{i_1=1}^{N} \cdots \sum_{i_n=1}^{N} a^{(\rho)}_{i_n \cdots i_1} H^{(\rho)}_{i_n \cdots i_1},$$

and

$$g = \sum_{\rho=0}^{N} b^{(\rho)} H^{(\rho')} + \sum_{n=1}^{\infty} \sum_{\rho=1}^{N-1} \sum_{j_1=1}^{N} \cdots \sum_{j_n=1}^{N} b^{(\rho')}_{j_n \cdots j_1} H^{(\rho')}_{j_n \cdots j_1},$$

and hence we have

$$f * g = \sum_{\rho=0}^{N} a^{(\rho)} b^{(\rho')} H^{(\rho)} \cdot H^{(\rho')} + \sum_{n=1}^{\infty} \sum_{\rho=1}^{N-1} \sum_{i_1,j_1=1}^{N} \cdots \sum_{i_n,j_n=1}^{N} a^{(\rho)}_{i_n \cdots i_1} b^{(\rho')}_{j_n \cdots j_1} H^{(\rho)}_{i_n \cdots i_1} * H^{(\rho')}_{j_n \cdots j_1}$$

$$= \sum_{\rho=0}^{N} a^{(\rho)} b^{(\rho)} H^{(\rho)} \cdot H^{(\rho')} + \sum_{m=1}^{\infty} \sum_{\rho=1}^{N-1} \sum_{i_1=1}^{N} \cdots \sum_{i_m=1}^{N} c^{(\rho)}_{i_m \cdots i_1} H^{(\rho)}_{i_m \cdots i_1}$$

$$(c^{(\rho)}_{i_m \cdots i_1} = \delta_{\rho\rho'} \sum a^{(\rho)}_{i_{n-k} \cdots i_1} b^{(\rho')}_{i_m \cdots i_{n-k+1}}),$$

which we call the *-*product on $L^2(K, d\mu)$ with respect to the basis* $\{H\}$. We notice the following

PROPOSITION 6. *We have*

$$\begin{cases} (1) & (f_1 + f_2) * g = f_1 * g + f_2 * g \text{ and } f * (g_1 + g_2) = f * g_1 + f * g_2, \\ (2) & (f * g) * h = f * (g * h). \end{cases}$$

Proof: easy and can be omitted.

Then we can prove the following

THEOREM III (derivation on a self-similar fractal set). *Putting*

$$(8.43) \quad \begin{cases} (1) & \delta_i H^{(\rho)} = 0, \\\\ (2) & \delta_i(H^{(\rho)} * H^{(\rho)}_{j_n \cdots j_1}) = H^{(\rho)} * \delta_i H^{(\rho)}_{j_n \cdots j_1} \quad (i = 1, \ldots, N), \\\\ (3) & \delta_i(H^{(\rho)}_{i_n, \ldots, i_1}) = \sum_{k=1}^n \delta_{i,i_k} H^{(\rho)}_{i_n, \cdots, \breve{i}_k, \ldots, i_1}, \end{cases}$$

where the notation $\breve{i}_k$ *means omitting the index* i_k, *we have the derivations on* $L^2(K, d\mu)$:

$$\delta_i(f * g) = \delta_i(f) * g + f * \delta_i(g) \quad (i = 1, \ldots, N).$$

Proof. First we prove the assertion for functions of the form

$$F = \sum_{k=1}^\infty \sum a^{(\rho)}_{i_k \cdots i_1} H^{(\rho)}_{i_k \cdots i_1}, \quad G = \sum_{l=1}^\infty \sum b^{(\rho)}_{j_l \cdots j_1} H^{(\rho)}_{j_l \cdots j_1}.$$

Then we have

$$F * G = \sum_{m=1}^\infty \sum c^{(\rho)}_{p_m \cdots p_1} H^{(\rho)}_{p_m \cdots p_1}$$

$$(c^{(\rho)}_{p_m \cdots p_1} = a_1 b_{p_m \cdots p_1} + a_{p_1} b_{p_m \cdots p_2} + \cdots + a_{p_m \cdots p_1} b_1).$$

Next we introduce the derivations

$$\delta_x F = \sum_{k=1}^\infty \sum_{\alpha=1}^k \sum a^{(\rho)}_{i_k \cdots i_1} \delta_{x,i_\alpha} H^{(\rho)}_{i_k \cdots \breve{i}_\alpha \cdots i_1},$$

$$\delta_x G = \sum_{l=1}^\infty \sum_{\beta=1}^l \sum b^{(\rho)}_{j_l \cdots j_1} \delta_{x,j_\beta} H^{(\rho)}_{j_l \cdots \breve{j}_\beta \cdots j_1}.$$

Then we have

$$\delta_x(F * G) = \sum_{m=1}^\infty \sum_{\alpha=1}^m \sum a^{(\rho)}_{p_m \cdots p_k} b^{(\rho)}_{p_{k+1} \cdots p_1} \delta_{x,p_\alpha} H^{(\rho)}_{p_m \cdots \breve{p}_\alpha \cdots p_{k+1} p_k \cdots p_1}$$

$$+ \sum_{m=1}^\infty \sum_{\alpha=1}^m \sum a^{(\rho)}_{p_m \cdots p_k} b^{(\rho)}_{p_{k+1} \cdots p_1} \delta_{x,p_\alpha} H^{(\rho)}_{p_m \cdots p_{k+1} p_k \cdots \breve{p}_\alpha \cdots p_1}.$$

On the other hand, we get

$$(\delta_x F) * G = \Big(\sum_{k=1}^{\infty}\sum_{\alpha=1}^{k}\sum a_{i_k\cdots i_1}^{(\rho)}\delta_{x,i_\alpha} H_{i_k\cdots \check{i_\alpha}\cdots i_1}^{(\rho)}\Big) * \Big(\sum_{l=1}^{\infty}\sum b_{j_l\cdots j_1}^{(\rho')} H_{j_l\cdots j_1}^{(\rho')}\Big)$$

$$= \sum_{k=1}^{\infty}\sum_{\alpha=1}^{k}\sum_{l=1}^{\infty}\sum\sum a_{i_k\cdots i_1}^{(\rho)}b_{j_l\cdots j_1}^{(\rho)}\delta_{x,i_\alpha}H_{i_k\cdots\check{i_\alpha}\cdots i_1 j_l\cdots j_1}^{(\rho)},$$

$$F * (\delta_x G) = \Big(\sum_{k=1}^{\infty}\sum\sum a_{i_k\cdots i_1}^{(\rho)} H_{i_k\cdots i_1}^{(\rho)}\Big) * \Big(\sum_{l=1}^{\infty}\sum_{\beta=1}^{l}\sum b_{j_l\cdots j_1}^{(\rho')}\delta_{x,j_\beta} H_{j_l\cdots\check{j_\beta}\cdots j_1}^{(\rho')}\Big)$$

$$= \sum_{k=1}^{\infty}\sum_{\beta=1}^{l}\sum_{l=1}^{\infty}\sum\sum a_{i_k\cdots i_1}^{(\rho)}b_{j_l\cdots j_1}^{(\rho')}\delta_{x,j_\beta}H_{i_k\cdots i_1 j_l\cdots\check{j_\beta}\cdots j_1}^{(\rho)}.$$

which proves the assertion. ∎

Next we prove the assertion of Theorem II in the general case. Choosing

$$f = f_0 + F, \quad g = g_0 + G \quad \Big(f_0 = \sum_{\rho=0}^{N-1} a^{(\rho)}H^{(\rho)}, \quad g_0 = \sum_{\rho'=0}^{N-1} b^{(\rho')}H^{(\rho')}\Big),$$

and using $\delta_x(f_0) = 0$, we easily prove the assertion, as desired.

REMARK 1. We can define a derivation for a linear basis which is not necessarily orthonormal, for example, for a Schauder basis.

9. The Taylor expansion theorem on a fractal set. In this section we give the Taylor expansion theorem on a self-similar fractal set including several examples of Taylor expansions. First we take an orthonormal basis $\{H^{(\rho)}, H_{j_n\cdots j_1}^{(\rho)}\}$ of Haar type and state the Taylor expansion. We introduce a special class of functions:

DEFINITION 5 (symmetric functions).

(1) A function f of $L^2(K, d\mu)$ is called *symmetric* if

$$f = \sum a_{i_m\cdots i_1}^{(\rho)} H_{i_m\cdots i_1}^{(\rho)}, \quad a_{i_m\cdots i_1}^{(\rho)} = a_{\tau(i_m)\tau(i_{m-1})\cdots\tau(i_1)}^{(\rho)},$$

where τ is a permutation of N symbols.

(2) A function f of $L^2(K, d\mu)$ is called *of one variable type*, if

$$(9.44) \qquad a_{i_m\cdots i_1}^{(\rho)} = a_{j_m\cdots j_1}^{(\rho)}$$

for any pair of $(i_m\cdots i_1)$ and $(j_m\cdots j_1)$. In that case we have

$$(9.45) \qquad f = \sum a_n^{(\rho)} R_n^{(\rho)}, \quad R_n^{(\rho)} = \sum R_{i_m\cdots i_1}^{(\rho)}.$$

Then we can prove the following

THEOREM IV (Taylor expansion theorem). *A symmetric function f of $L^2(K, d\mu)$ has the following Taylor expansion:*

$$f = \sum a_{i_n \cdots i_1}^{(\rho)} H_{i_n \cdots i_1}^{(\rho)},$$

$$a_{i_n \cdots i_1} = \frac{1}{n_1! n_2! \ldots n_N!} \delta_1^{m_1} \cdots \delta_N^{n_N} f(H^{(\rho)}) \quad (n_1 + n_2 + \ldots + n_N = n),$$

where $f(H^{(\rho)}) = (f, H^{(\rho)})$.

Proof. We take an element f of $L^2(K, d\mu^D)$ with symmetric coefficients. Then we can see that

$$
\begin{aligned}
(\delta_i f, H^{(\rho)}) &= a_i, \\
(\delta_i \delta_j f, H_{(\rho)}) &= a_{ij} + a_{ji}, \\
&\cdots \qquad\qquad \cdots \\
(\delta_{i_m} \cdots \delta_{i_1} f, H^{(\rho)}) &= \sum_{\tau \in G_m} a_{\tau(i_m) \cdots \tau(i_1)}, \\
&\cdots \qquad\qquad \cdots
\end{aligned}
$$

where G_m is the m-th symmetric group. ∎

Hence, for a symmetric function, we have

$$a_{i_m \cdots i_1} = \frac{1}{n_1! n_2! \ldots n_N!} \delta_1^{m_1} \cdots \delta_N^{n_N} f(H^{(\rho)}).$$

We give several examples.

EXAMPLE 8 (Haar functions) [12]. The Haar basis constitutes a complete orthonormal basis of $L^2(I, d\mu)$ and every function can be expanded into the Taylor expansion in our sense.

Next we give the Taylor expansion for a non-orthonormal basis. We take the Schauder basis (5.26). To state the Taylor expansion theorem, we prepare the following notation: For

(9.46) $$f = \sum a_{j_n \cdots j_1} G_{j_n \cdots j_1},$$

we define

(9.47) $$\langle f, G_{j_n \cdots j_1} \rangle := a_{j_n \cdots j_1}.$$

Following the discussion in the case of an orthonormal basis, we can prove the Taylor expansion theorem:

THEOREM IV' (Taylor expansion for Schauder basis). *A symmetric function f of $L^2(K, d\mu)$ has the following Taylor expansion:*

$$
\begin{aligned}
f &= \sum a_{j_n \cdots j_1}^{(\rho)} G_{j_n \cdots j_1}^{(\rho)}, \\
a_{j_n \cdots j_1} &= \frac{1}{n_1! n_2! \ldots n_N!} \delta_1^{m_1} \cdots \delta_N^{n_N} f(H^{(\rho)}),
\end{aligned}
$$

where $f(G^{(\rho)}) = \langle f, G^{(\rho)} \rangle$.

EXAMPLE 9 (Takagi function) [2, 10, 11]. We consider a fractal structure on I which is defined by the mappings (6.31) and we choose the function G_0. The Takagi function is given by:

$$(9.48) \qquad T(x) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n G_n(x),$$

where

$$(9.49) \qquad G_n(x) = \sum G_0 \left(\sigma_{j_1}^{-1} \circ \cdots \circ \sigma_{j_n}^{-1}(x)\right).$$

Hence we can see that the function is of one variable type and we have

$$(9.50) \qquad \delta^n T(G_0) = n! \left(\frac{1}{2}\right)^n.$$

EXAMPLE 10 (Schauder expansion) [7]. We notice that we can regard the Schauder expansion as the Taylor expansion. We can expand any continuous function f on I ($= [0,1]$) with $f(0) = f(1) = 0$ as the Schauder expansion:

$$(9.51) \qquad f(x) = \sum_{n=1}^{\infty} a_{j_n \cdots j_1} G_{j_n \cdots j_1}(x),$$

where

$$(9.52) \qquad G_{j_n \cdots j_1}(x) = G_0(\sigma_{j_1}^{-1} \circ \cdots \circ \sigma_{j_n}^{-1}(x)).$$

Then, in the case where f is symmetric, we can see that

$$(9.53) \qquad a_{j_n \cdots j_1} = \frac{1}{n!} \delta_{j_n} \cdots \delta_{j_1} f(G_0).$$

EXAMPLE 11 (Weierstrass function) [2, 12]. We consider the fractal set I defined by

$$(9.54) \qquad \sigma_1(x) = \frac{1}{2}(1 + \sqrt{x+1}), \quad \sigma_2(x) = \frac{1}{2}(1 - \sqrt{x+1}).$$

We put

$$(9.55) \qquad W_0(x) = 4x(1-x),$$

and

$$(9.56) \qquad W_{j_n \cdots j_1}(x) = W_0 \left(\sigma_{j_1}^{-1} \circ \cdots \circ \sigma_{j_n}^{-1}(x)\right).$$

The Weierstrass function is given by

$$(9.57) \qquad W(x) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n W_n(x),$$

where W_n are symmetrizations of $W_{j_n \cdots j_1}$. Hence we can see that

$$(9.58) \qquad \delta^n W(W_0) = n! \left(\frac{1}{2}\right)^n.$$

10. Basic functions on a self-similar fractal set.

We are going to introduce basic functions, i.e., exponential, trigonometrical, and geometric functions on a self-similar fractal set and derive differential equations of them. Then we give a close relationship between these functions and the corresponding functions on the Euclidean space. We

can show that the usual exponential, trigonometrical, and geometric functions can be embedded into the fractal basic functions.

We take a self-similar fractal set K and choose a basis of Schauder type

$$(10.59) \qquad \{G_0^{(\rho)}, G_{j_m, j_{m-1}, \dots, j_1}^{(\rho)}\}.$$

We make the symmetrization of these functions with respect to a fixed degree m and obtain functions of one variable: $\{R^0, R^m\}$. Then we can introduce basic functions in terms of the Taylor expansion theorem:

$$
\begin{cases}
(1) \quad \dfrac{1}{1-R} = R^0 + R^1 + R^2 + \dots + R^n + \dots, \\[2mm]
(2) \quad \sin R = R^1 - \dfrac{1}{3!}R^3 + \dfrac{1}{5!}R^5 - \dots + (-1)^n \dfrac{1}{2n-1!}R^{2n-1} + \dots, \\[2mm]
(3) \quad \cos R = R^0 - \dfrac{1}{2!}R^2 + \dfrac{1}{4!}R^4 - \dots + (-1)^n \dfrac{1}{2n-1!}R^{2n} + \dots, \\[2mm]
(4) \quad e^R = R^0 + \dfrac{1}{1!}R^1 + \dfrac{1}{2!}R^2 + \dots + \dfrac{1}{n!}R^n + \dots.
\end{cases}
$$

Here we understand the series as formal series and the convergences are not considered. Then we can see that they satisfy the differential equations as in the case of the corresponding basic functions: Putting

$$\delta = \frac{1}{N}\sum_{\rho=1}^{N-1} \delta^{(\rho)}$$

we have

$$
\begin{cases}
(1) \quad \delta\dfrac{1}{1-R} & = & 1/(1-R)^2, \\
(2) \quad \delta\sin R & = & \cos R, \\
(3) \quad \delta\cos R & = & -\sin R, \\
(4) \quad \delta e^R & = & e^R.
\end{cases}
$$

In the final section we shall show that *these basic functions can describe fluctuations which can be observed in the nature.*

In turn we give an intimate relationship between the usual basic functions and fractal basic functions mentioned above. Namely, we can prove the following embedding theorem:

THEOREM V (on the fluctuation mapping). *Let K be a self-similar fractal set and let* (10.59) *be a Schauder basis on it. Let $\{R^0, R^m\}$ $(= \mathbb{C}[R])$ be the one-dimensional reduction. Then we can get the following embedding which preserves the derivation structure: Putting $\iota(x) = R$, we have a ring isomorphism satisfying the commutative diagram*

$$
\begin{array}{c}
\iota : \mathbb{C}[x] \to \mathbb{C}[R] \\[1mm]
\dfrac{d}{dx} \downarrow \qquad \downarrow \delta \\[1mm]
\iota : \mathbb{C}[x] \to \mathbb{C}[R]
\end{array}
$$

(10.60)

DEFINITION 6 (fluctuation mapping). We call the embedding in Theorem V the *fluctuation mapping*.

REMARK 2. Here we have to notice the following fact: Our embedding $\iota(x) = R$ should be understood as a *noncommutative isomorphism* between the function spaces $\mathbb{C}[x]$ and $\mathbb{C}[R]$. In the case where the fractal set K is a boundary of some domain D in the complex plane $\mathbb{C}$, we can see that the derivation on K *cannot* be obtained as the restriction of the usual derivation:

$$(10.61) \qquad \frac{d}{dz}|_K \neq \delta.$$

This will be discussed in the next section.

We conclude this section with a comment on the relationship between the Schauder expansion and the discrete Laplace operator. First we give

DEFINITION 7 (discrete Laplace operator on a self-similar fractal set). Let K be a self-similar fractal set which is defined by a system of contractions $\{\sigma_j : j = 1, 2, \ldots, N\}$. Then we define the *Laplace operator*

$$\Delta : L^2(K, d\mu^D) \to L^2(K, d\mu^D)$$

of the fractal set by

$$(10.62) \qquad \Delta f(p) = \sum_{j=1}^{N} f(\sigma_j(p)) - f(p).$$

We notice that it can be expressed in terms of the derivations:

$$(10.63) \qquad \Delta f(p) = \sum_{j=1}^{N} \delta_j f(p) - f(p).$$

Then we may expect to prove the following formula: For a function f which is the restriction of a continuous function on the ambient space, we have

$$(10.64) \qquad f = \sum_{\rho=1}^{N-1} a^{(\rho)} R^{(\rho)} + \sum_{\rho=1}^{N-1} \sum_{n=1}^{\infty} \sum_{j_1=1}^{N} \cdots \sum_{j_n=1}^{N} a^{(\rho)}_{j_n \cdots j_1} R^{(\rho)}_{j_n \cdots j_1}.$$

In consequence,

$$(10.65) \qquad a^{(\rho)}_{j_n \cdots j_1} = < \Delta f, R^{(\rho)}_{j_n \cdots j_1} > .$$

REMARK 3. In the case where $K = I$, I being the interval with the fractal structure defined by (5.26), we can see that the above mentioned formula holds [12].

EXAMPLE 12 (Takagi function). The Takagi function can be obtained by solving the equation

$$(10.66) \qquad T(G^{(0)}(x)) = 2T(x) + G^{(0)}(x)$$

Putting $T(x) = \sum a_n R^n$, we can obtain the solution by solving the equations

$$(10.67) \qquad a_n = \langle \Delta f, R^n \rangle \left(= \frac{1}{2^{n-1}} \right).$$

EXAMPLE 13 (Weierstrass function). The Weierstrass functions can be treated in a completely similar manner. In the next section we shall consider the dynamical system for a

self-similar fractal set and the corresponding generating functions. These functions can be treated analogously; see [12].

11. Dynamical systems of a fractal set and differential equations for their generating functions. In this section we introduce the dynamical system for a self-similar fractal set and show that the corresponding generating functions can be expressed in terms of the generators of the one-dimensional part. We take a self-similar fractal set K which is defined by contractions $\{\sigma_j : j = 1, 2, \ldots, N\}$. Then with $\Phi : K \to K$ given by

$$(11.68) \qquad\qquad \Phi = \sigma_j^{-1} \text{ on } K_j$$

we can define the dynamical system $\{x_n\}$ with an initial value $x_0 \in K$ by

$$(11.69) \qquad\qquad x_n = \Phi(x_{n-1}) \ (n = 1, 2, ..).$$

Then we can see that the dynamical system becomes a *chaotic* dynamical system. Moreover,

   (1) We have a dense subset of periodic orbits

and

   (2) There exists a dense orbit

and

   (3) The behaviour of the dynamical system is sensitive to the choice of initial values.

   Next we consider the generating function of the dyamical system with respect to the base function $g \in L^2(K, d\mu^D)$:

$$(11.70) \qquad \begin{cases} G(g : x, t) = \sum_{n=0}^{\infty} t^n g(\Phi^n(x)), \\[2em] \hat{G}(g : x, t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} g(\Phi^n(x)). \end{cases}$$

In the case where $g(x) = G_0(x)$, we call g the *generating function* of the Schauder basis. In this case we can see that

$$(11.71) \qquad \begin{cases} G(G_0 : x, t) = \sum_{n=0}^{\infty} t^n \Phi^n(x), \\[2em] \hat{G}(G_0 : x, t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \Phi^n(x). \end{cases}$$

We arrive at

PROPOSITION 7. *We have*

   (1) $\Phi^n(x) = R^n(x)$.

   (2) *The shift operation σ given by $\sigma g(\Phi^n(x)) = g(\Phi^{n+1}(x))$ satisfies the relation*

$$(11.72) \qquad\qquad \sigma^* G(z, t) = \frac{1}{t}(G(z : x, t) - \Phi(z)).$$

(3) *In the case of a generating function of the Schauder basis,*

$$\text{(11.73)} \qquad \delta(\hat{G}(z,t)) = t\hat{G}(z,t).$$

Proof.

(1) Choosing the base function $\Phi(z)$, we get

$$\Phi^{(1)}(z) = \Phi(\sigma^{-1}(z)) \text{ on } K_j,$$
$$\Phi^{(2)}(z) = \Phi(\sigma_{j_1}\sigma_{j_2}) \text{ on } K_{j_2,j_1},$$

$$\text{(11.74)} \qquad \ldots\ldots\ldots\ldots$$
$$\Phi^{(n)}(z) = \Phi(\sigma_{j_1}\sigma_{j_2}\ldots\sigma_{j_n}) \text{ on } K_{j_n,\ldots,j_2,j_1}.$$

Hence the assertion follows.

(2) We calculate:

$$\text{(11.75)} \qquad \begin{aligned} G(\Phi : x,t) &= \sum_{n=0}^{\infty} t^n \Phi(\Phi^n(x)) = \sum_{n=0}^{\infty} t^n \Phi^{n+1}(x)) \\ &= \frac{1}{t}\sum_{n=0}^{\infty} t^{n+1}\Phi^{n+1}(x)) = \frac{1}{t}\sum_{n=1}^{\infty} t^n \Phi^n(x)) \\ &= \frac{1}{t}(G(z:x,t) - \Phi(z)). \end{aligned}$$

(3) This is a well known property of generating functions. ∎

12. Realization of fluctuations by fluctuation mappings. In the final section we are going to show that *fluctuations observed in the nature can be realized as the image of fluctuation mappings of smooth functions without fluctuations.*

Several physists have discussed the appearance of background fluctuations in the nature. Recently Nottale [8] has introduced the concept of the scale symmetry and tried to understand the hierarchy structure of the universe and the fluctuations. One of the authors of this paper has discussed the "background fluctuations of the nature" in terms of Hurwitz pairs. He has introduced additional dimensions for fluctuations and made a trial of including fluctuations in mathematics.

First we notice that important functions, for example, Takagi and Weierstrass functions are nothing but the generating functions of the Schauder basis (see (9.48) and (9.57)). Hence we may expect getting functions with big fluctuations as images of holomorphic functions without fluctuations by the fluctuation mappings. Here we shall show that we can realize in such a way fluctuations observed in nature.

We consider a real analytic function

$$\text{(12.76)} \qquad f(z) = \sum_{n=0}^{\infty} c_n z^n \quad (|z| \leq 1).$$

Choosing a fractal set K with a Schauder basis, making its one-dimensional part $\{R^n\}$ and using the mapping $\iota : \mathbb{C}[z] \to \mathbb{C}[R]$, we have

(12.77)
$$\iota(f) = \sum_{n=0}^{\infty} c_n R^n,$$

and then we can realize fluctuations by a suitable choice of $\{c_n\}$.

Here we are concerned with the following two kinds of wavelets:

(1) *Wavelet expansions of Weierstrass functions type*

We consider the dynamical system on the unit circle $S = \partial \mathbf{D}$ defined by

(12.78)
$$\Phi(\theta) = b\theta,$$

where b is an integer, $b > 1$. This dynamical system can be realized as the boundary values of the holomorphic dynamical system defined by $\tilde{\Phi}(z) = z^b$:

(12.79)
$$z, z^b, z^{b^2}, \ldots, z^{b^n}, \ldots$$

In that case the generating function can be written as

(12.80)
$$\tilde{G}(z, t : z) = \sum_{i=1}^{\infty} t^n z^{b^n}.$$

Taking the boundary value of the function and restricting its real part we get the Weierstrass function. We notice that the derivation is not the restriction to the boundary. Namely, we have the formula

(12.81)
$$\delta(z^{b^n}) = n z^{b^{n-1}}.$$

Choosing coefficients $\{a^{(\rho)}, a^{(\rho)}_{j_n \cdots j_1}\}$, we can realize several fluctuations, for instance in music and geography.

(2) *Wavelet expansions of exponential logistic type*

Finally we are concerned with fluctuations which can be obtained by the dynamical system of the logistic type on the unit circle $S = \partial \mathbf{D}$:

(12.82)
$$\Phi(\theta) = a\theta(2\pi - \theta),$$

where $\mathbf{D}$ is the unit disc and a is a constant. We can realize it as the boundary values of the following non-holomorphic dynamical system on $\mathbf{D}$:

(12.83)
$$\tilde{\Phi}^n(z) = r^n e^{i\Phi^n(\theta)}.$$

We notice that this is not realized as the boundary values of a holomorphic dynamical system on $\mathbf{D}$. Choosing coefficients $\{a^{(\rho)}, a^{(\rho)}_{j_n \cdots j_1}\}$, we can realize several fluctuations in the behaviours of the stock prices, neuron systems and music. For examples, we refer to our paper [9].

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TEORYA

FUNKCYJ ANALITYCZNYCH

NAPISAŁ

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TOM II.

[Z 121 figurami w tekscie.]

LWÓW.

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TOM I

THÉORIE
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OPÉRATIONS LINÉAIRES
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TOM VI

THEORIE
DER
ORTHOGONALREIHEN
VON
STEFAN KACZMARZ
DOZENT AN DER UNIVERSITÄT LWÓW
UND
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TOM VIII

STEFAN BANACH
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MECHANIKA
W ZAKRESIE SZKÓŁ AKADEMICKICH

Tom 1

Wydanie drugie niezmienione

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TOM XXIV

STEFAN BANACH

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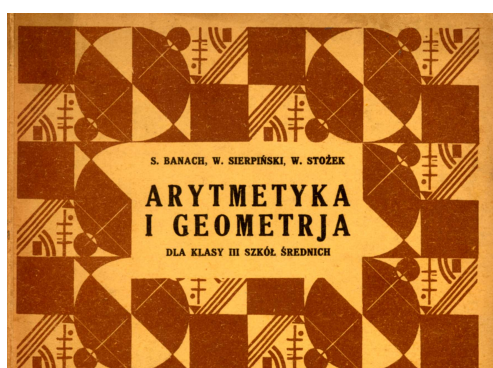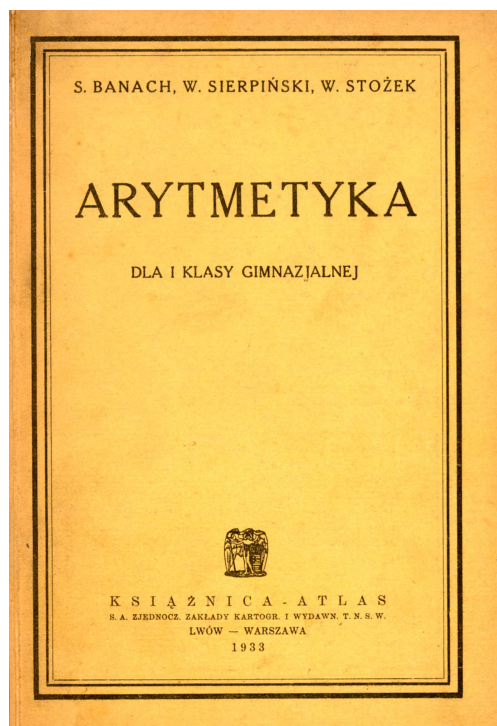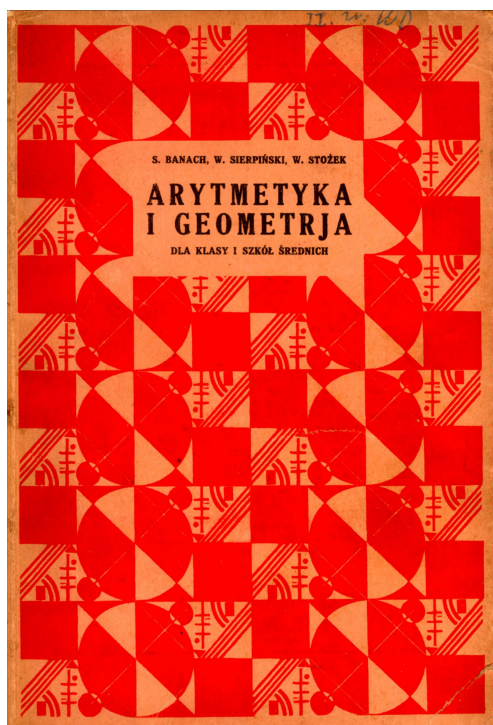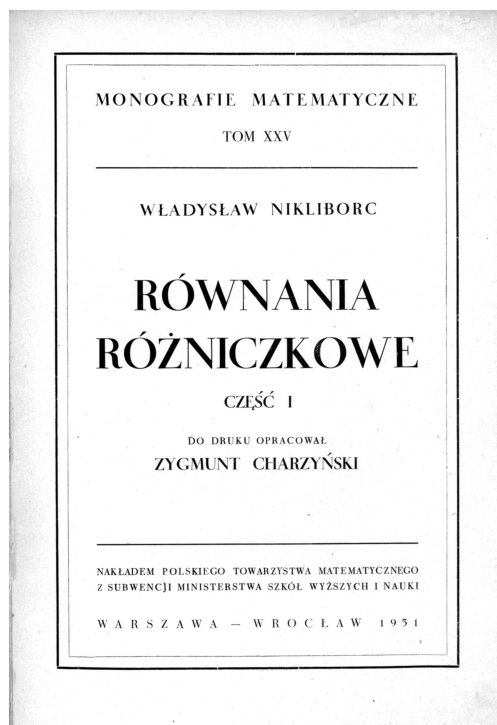
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