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FINITE ORDER SYMMETRIES OF J_{10}

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1. Introduction. We require the following to begin.

DEFINITION 1. Let $g = (g', g'')$ be a transformation of $\mathbf{C}^n \times \mathbf{C}$ such that $f \circ g' = g'' \circ f$ for some function f on $\mathbf{C}^n$. We call g a *symmetry* of f and we say that f is *g-equivariant*. If however $g'' = \mathrm{id}$ then we say that f is *g-invariant* and identify g with g'.

In [6] and [8] finite order symmetries of simple function singularities were found to be related to finite complex reflection groups. To be more precise, the monodromy groups of the symmetric singularities were shown to coincide with certain Shephard-Todd groups.

A natural extension to this work stems from asking about what information one could obtain from Arnol′d's *parabolic* singularities [1] and moreover, what role complex affine reflection groups might play for these particular unimodal singularities.

2. Formulation. In this paper we only examine representatives of the J_{10} class of singularities

$$J_{10} : x^3 + pxy^4 + qy^6, \qquad 4p^3 + 27q^2 \neq 0$$

where $p, q \in \mathbf{C}$. One of the two parameters here can be normalized to 1. The zero level of any representative f from this class consists of three distinct parabolas and hence one can write $f = (x + \alpha y^2)(x + \beta y^2)(x + \gamma y^2)$, where $\alpha, \beta, \gamma \in \mathbf{C}$ and chosen so that $\alpha + \beta + \gamma = 0$.

Now consider a general quasihomogeneous diffeomorphism

$$g' : \begin{cases} x \mapsto ax + by^2 \\ y \mapsto cy \end{cases}$$

where $a, b, c \in \mathbf{C}$, $ac \neq 0$. Here the pair of maps g', g'' constitute a symmetry if and only if $f(g'(x, y)) = g''(f(x, y))$. The existence of such a symmetry implies that g' must permute

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the parabolas. Indeed, under the prescribed f and g' we see that $g'' \circ f = a^3 f$. These permutations can happen in one of three ways: (I) each parabola is sent back to itself, (II) a pair of parabolas swop and, (III) all 3 parabolas swop. Equating the coefficients of x^2y^2 results in finding $b = 0$, unanimous for all three options. On closer examination one can reduce g' and f to more convenient and simplified forms:

$$
\begin{array}{lll}
\text{(I)} & f = x^3 + pxy^4 + y^6, & g' : \begin{cases} x \mapsto c^2 x \\ y \mapsto cy \end{cases} \\
\text{(II)} & f = x^3 + xy^4, & g' : \begin{cases} x \mapsto -c^2 x \\ y \mapsto cy \end{cases} \\
\text{(III)} & f = x^3 + y^6, & g' : \begin{cases} x \mapsto \varepsilon_3 c^2 x \\ y \mapsto cy \end{cases}
\end{array}
$$

where $p = \frac{\alpha\beta+\beta\gamma+\alpha\gamma}{(\alpha\beta\gamma)^{2/3}}$ is the modulus and ε_3 is a primitive third root of unity. The simplified forms of f shall henceforth be referred to as *stems*.

3. Classification. We anticipate that Popov groups [10]—complex *affine* reflection groups—can be constructed from the monodromy of smooth Milnor fibres obtained from g-equivariant deformations of the three stems described above. However, as in [6]–[8], we restrict ourselves to *smoothable* symmetries of g (below) to ensure straightforward computation of the intersection forms on the Milnor fibres.

DEFINITION 2. A symmetry g of f is called *smoothable* if there exists a g-equivariant deformation germ f_t of f, $f_0 = f$, $t \in \mathbf{C}$, such that the curve $f_t = 0$ is smooth for all $t \neq 0$.

PROPOSITION 1. *If a symmetry g of f is smoothable then g must be a symmetry of at least one of the functions* 1*,* x*, or* y*.*

The condition above is *necessary* since if none of 1, x or y is in the deformation then no constant term will appear for a fixed $t \neq 0$ in either $f_t = 0$, $\frac{\partial f_t}{\partial x} = 0$ or $\frac{\partial f_t}{\partial y} = 0$ (we keep in mind that g', in (I)–(III) above, acts by multiplying monomials by individual constants). In these cases the origin will be a singular point thus implying that $f_t = 0$ is not smooth.

Notice that if f is g-invariant then g is always smoothable: just set $f_t = f - t$ in the above.

With the prescribed function f and transformation g' from the previous section, $g = (g', g'')$ is a symmetry of $1, x$ or y if and only if $g''(1) = 1$, $g''(x) = ax$ or $g''(y) = cy$ respectively, that is $a^3 = 1, a, c$. Taking each in turn gives rise to the explicit forms for our symmetries. Symmetries that are powers of each other correspond to the same overall group action. Taking one representative symmetry for each group leaves us with 19 different cases. These are listed in our main result.

THEOREM 1. *The complete list of finite cyclic symmetry groups of any member of the* J_{10} *family, for which the conclusion of Proposition* 1 *holds, is given in Tables* 1–3.

In the Tables, $\epsilon_k = e^{2\pi i/k}$.

Case	1	2	3	4	5	6
$g:\begin{pmatrix}x\\y\\f\end{pmatrix}\mapsto$	$\begin{pmatrix}x\\y\\f\end{pmatrix}$	$\begin{pmatrix}\epsilon_3x\\\epsilon_6y\\f\end{pmatrix}$	$\begin{pmatrix}\epsilon_3x\\\epsilon_3^2y\\f\end{pmatrix}$	$\begin{pmatrix}x\\-y\\f\end{pmatrix}$	$\begin{pmatrix}-x\\iy\\-f\end{pmatrix}$	$\begin{pmatrix}\epsilon_5^2x\\\epsilon_5y\\\epsilon_5f\end{pmatrix}$
Action of g	id	$\mathbf{Z}_6-$inv	$\mathbf{Z}_3-$inv	$\mathbf{Z}_2-$inv	$\mathbf{Z}_4-$equ	$\mathbf{Z}_5-$equ

Table 1. $f = x^3 + pxy^4 + y^6$, arbitrary p

Case	7	8	9	10	11
$g:\begin{pmatrix}x\\y\\f\end{pmatrix}\mapsto$	$\begin{pmatrix}\epsilon_3^2x\\\epsilon_{12}y\\f\end{pmatrix}$	$\begin{pmatrix}x\\iy\\f\end{pmatrix}$	$\begin{pmatrix}-x\\y\\-f\end{pmatrix}$	$\begin{pmatrix}-x\\-y\\-f\end{pmatrix}$	$\begin{pmatrix}\epsilon_{10}^7x\\\epsilon_{10}y\\\epsilon_{10}f\end{pmatrix}$
Action of g	$\mathbf{Z}_{12}-$inv	$\mathbf{Z}_4-$inv	$\mathbf{Z}_2-$equ	$\mathbf{Z}_2-$equ	$\mathbf{Z}_{10}-$equ

Table 2. $f = x^3 + xy^4$

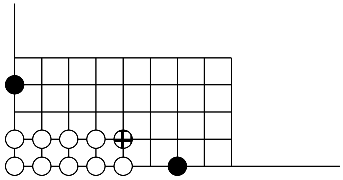
Case	12	13	14	15
$g:\begin{pmatrix}x\\y\\f\end{pmatrix}\mapsto$	$\begin{pmatrix}\epsilon_3x\\y\\f\end{pmatrix}$	$\begin{pmatrix}\epsilon_3^2x\\\epsilon_6y\\f\end{pmatrix}$	$\begin{pmatrix}x\\\epsilon_3y\\f\end{pmatrix}$	$\begin{pmatrix}\epsilon_3x\\-y\\f\end{pmatrix}$
Action of g	$\mathbf{Z}_3-$inv	$\mathbf{Z}_6-$inv	$\mathbf{Z}_3-$inv	$\mathbf{Z}_6-$inv

Case	16	17	18	19
$g:\begin{pmatrix}x\\y\\f\end{pmatrix}\mapsto$	$\begin{pmatrix}\epsilon_3x\\\epsilon_3y\\f\end{pmatrix}$	$\begin{pmatrix}x\\\epsilon_6y\\f\end{pmatrix}$	$\begin{pmatrix}-x\\\epsilon_{12}y\\-f\end{pmatrix}$	$\begin{pmatrix}\epsilon_{15}x\\\epsilon_5y\\\epsilon_5f\end{pmatrix}$
Action of g	$\mathbf{Z}_3-$inv	$\mathbf{Z}_6-$inv	$\mathbf{Z}_{12}-$equ	$\mathbf{Z}_{15}-$equ

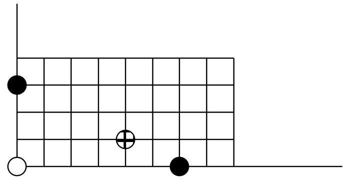
Table 3. $f = x^3 + y^6$

4. Versal deformations. From the local ring([1]) of f, the set of monomials multiplied by g' by the same factor as f provides a g-equivariant miniversal deformation F of f. We represent such deformations on diagrams below. Each case number from the tables corresponds exactly to the diagram of the same number. Black monomials belong to the stem f, white monomials represents a monomial in the base of the deformation, a hatched monomial (whose coefficient is the modulus) is one that is both black and white.

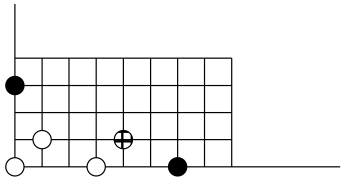
([1]) We take the spanning set $\{1, x, y, y^2, y^3, y^4, xy, xy^2, xy^3, xy^4\}$ for permutations of types (I) and (III), and $\{1, x, y, y^2, y^3, y^4, y^5, y^6, xy, xy^2\}$ for type (II).



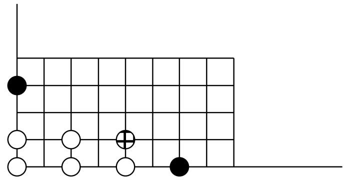
Case 1



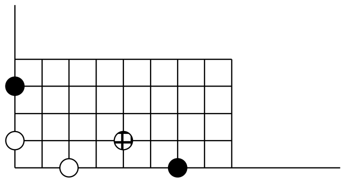
Case 2



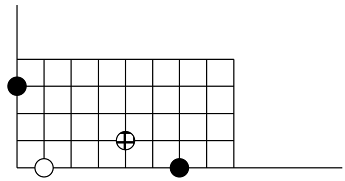
Case 3



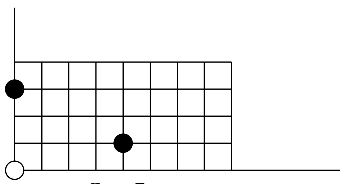
Case 4



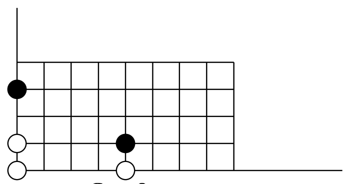
Case 5



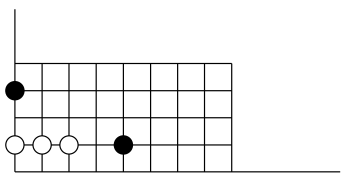
Case 6



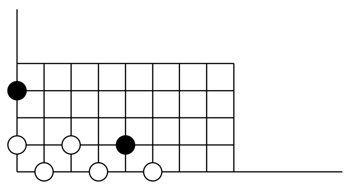
Case 7



Case 8



Case 9

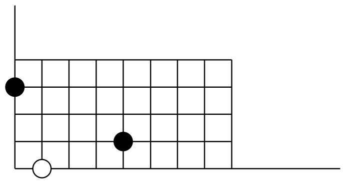


Case 10

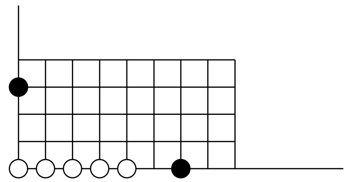
Recall from Proposition 1 the condition that 1, x or y must be in the deformation is a necessary condition. It is not a sufficient one and so, in consequence, we may pick up symmetries that are not smoothable. From the tables, only Case 9 is not smoothable. The zero level of any deformation from it is reducible as it will always contain, as a factor, the line $x = 0$. We therefore dismiss Case 9.

5. Interesting cases. In this section we extract cases from our remaining 18 for which a standard construction can be applied to obtain complex affine reflection groups on $\mathbf{C}^{\mu}$. The construction begins with a semi-definite Hermitian form on $\mathbf{C}^{\mu+1}$ (the addition of an extra variable is detailed below) whose kernel is one-dimensional. We fix a non-zero vector $\mathbf{v}$ from the kernel and take the set $H \subset \mathbf{C}^{\mu+1,*}$ of all the linear forms φ on $\mathbf{C}^{\mu+1}$ for which $\varphi(\mathbf{v}) = \delta$ for some fixed non-zero $\delta \in \mathbf{C}$. The duals of the Picard-

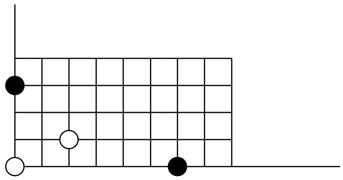
Lefschetz operators are then affine reflections that generate a Popov group on H (details in a forthcoming paper).



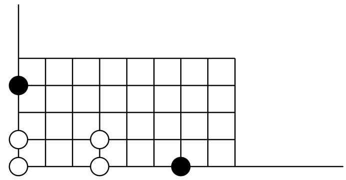
Case 11



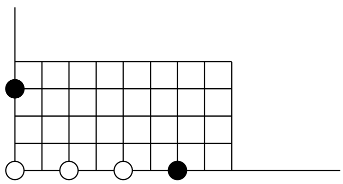
Case 12



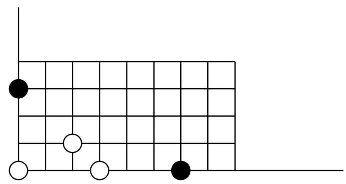
Case 13



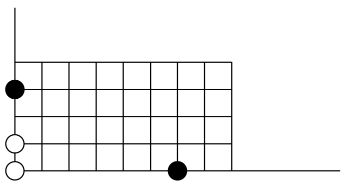
Case 14



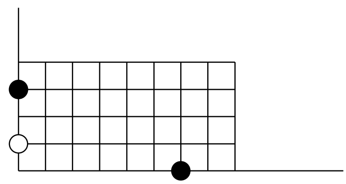
Case 15



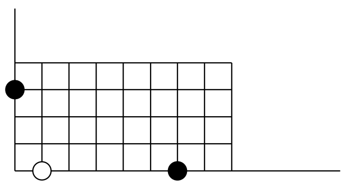
Case 16



Case 17



Case 18



Case 19

Before continuing we must establish a general set up. First of all, in order to switch to the Hermitian intersection form on the homology, instead of the skew-Hermitian, we shall consider the one-variable stabilizations: $\tilde{f} = f + z^2$ and $\tilde{F} = F + z^2$.

To incorporate the new variable z into the symmetry, we extend g to $\tilde{g} = (\tilde{g}', g'')$ by setting $\tilde{g}'(x, y, z) = (g'(x, y), a^{3/2}z)$. In all the invariant cases, that is when $a^3 = 1$, we choose $a^{3/2} = 1$.

Let $\tilde{V}$ be the Milnor fibre $\tilde{F}_* = 0$ where $\tilde{F}_*$ is a generic member of the $\tilde{g}$-equivariant versal family. The kernel of the intersection form on $H_2(\tilde{V}, \mathbf{C})$ has rank 2. Since $\tilde{g}'$ acts on $\tilde{V}$ and thus on its homology, we have the splitting

$$H_2(\tilde{V}, \mathbf{C}) = \bigoplus_{\chi} H_{\chi}, \qquad \chi^m = 1$$

where m is the order of the transformation $\tilde{g}'$ (which coincides with the order of $\tilde{g}$). Here $\tilde{g}'$ acts on H_{χ} as a multiplication by χ. The subspaces H_{χ} are called *character subspaces.*

With the above one can now begin extracting the cases that give way to the standard affine construction mentioned earlier. Such cases are called *interesting.*

DEFINITION 3. A case from the tables is *interesting* if

1. the dimension of the base of its $\tilde{g}'$-equivariant miniversal deformation is greater than one whenever the case has no modulus, and greater than two otherwise,

2. if two different character subspaces in $H_2(\tilde{V}, \mathbf{C})$, each of dimension at least 2, share the kernel of the intersection form. That is, the restriction of the form to each of the two character subspaces has a one-dimensional kernel.

REMARK 1. A reason for the first requirement is that a deformation which does not satisfy it can provide no more than one Picard-Lefschetz operator, not enough for us to obtain an affine reflection group from (since these require at least two generators).

Let us show how we can check the splittings against condition 2 of the above definition. For example, the character splitting of $H_2(\tilde{V}, \mathbf{C})$ in any of the g-invariant cases of type (I) or (III) is the same as that in the smaller g-invariant sub-family

$$x^3 + y^6 + z^2 + \lambda$$

where $\lambda \in \mathbf{C}$. However, this small family is much more symmetric: its members are invariant with respect to the group $\mathbf{Z}_3 \times \mathbf{Z}_6 \times \mathbf{Z}_2$ generated by transformations acting on individual variables only:

$$\begin{aligned} g_1 &: x \mapsto xe^{2\pi i/3} \\ g_2 &: y \mapsto ye^{2\pi i/6} \\ g_3 &: z \mapsto -z. \end{aligned}$$

Respectively we have a finer splitting $H_2(\tilde{V}, \mathbf{C}) = \bigoplus H_{\chi_1,\chi_2,\chi_3}$ where $\chi_1^3 = \chi_2^6 = \chi_3^2 = 1$, $\chi_j \neq 1$. Each of the summands here is one-dimensional, and each g_j acts on such a summand as a multiplication by χ_j. If $\tilde{g}' = g_1^r g_2^s g_3^t$ where $r, s, t \in \mathbf{Z}$, so that $\chi = \chi_1^r \chi_2^s \chi_3^t$ then $H_{\chi_1,\chi_2,\chi_3} \in H_{\chi}$ (see Section 8 for an example of this splitting). We describe the setting as a *multi-character* one.

This type of setup is due to Pham. His paper [9] contains the self-intersection indices of the multi-character cycles of a Brieskorn singularity $x_1^{a_1} + \ldots + x_n^{a_n}$. For such a singularity one would have a set of the g_j where $j = 1, \ldots, n$. More notably from [9], the self-intersection number of a multi-character cycle is a non-zero multiple of $1 - \Pi\chi_j$. The values $(\chi_1, \ldots \chi_n)$ for which this expression is zero correspond to the kernel elements in the homology. For our g-invariant cases of type (I) and (III), this happens precisely when $(\chi_1, \chi_2, \chi_3) = (e^{2\pi i/3}, e^{2\pi i/6}, -1)$ and $(e^{-2\pi i/3}, e^{-2\pi i/6}, -1)$. To satisfy the second condition of Definition 3, we just need these two triples to give distinct $\chi = \chi_1^r \chi_2^s \chi_3^t$.

Methods similar to Pham's can be applied to Case 8, too. This is the only case of type (II) satisfying the first condition of Definition 3. However, we do not give the details of the modifications involved here.

With regards to the strict *equivariant* cases, these can also be treated with appropriate adjustments of Pham's methods. Again the details are a bit too technical for our short exposition.

Using the setups described above one can write out the homology, locate the splitting of the J_{10} kernel and thus decide which cases are interesting.

THEOREM 2. *Cases that are interesting arise only from Tables* 2 *and* 3*. Moreover, only symmetries* 8*,* 10*,* 12–17 *are interesting.*

So, none of the moduli cases is interesting.

REMARK 2. A more careful analysis shows that, for our singularities, condition 1 from Definition 3 follows from condition 2.

6. Dynkin diagrams. Securing the degenerate intersection forms on the H_χ subspaces of all invariant interesting cases, that is, all with the exception of Case 10, give rise a table of Dynkin diagrams (Fig. 1). The third column lists the values of χ that are associated to the Dynkin diagrams in the second column. It is for these values of χ that the corresponding intersection forms are degenerate.

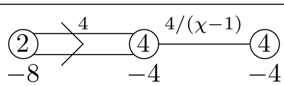
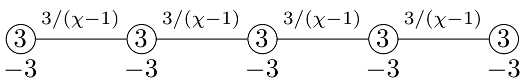
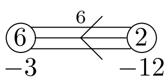
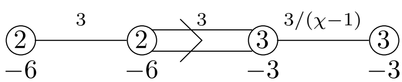
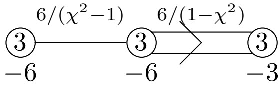
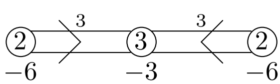
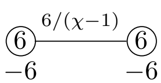
Case	Dynkin diagram	χ	Discriminant type
8		$i, -i$	C_3
12		ε_3	A_5
13		ε_6	G_2
14		ε_3	F_4
15		ε_6	B_3
16		ε_3	–
17		ε_6	A_2

Figure 1. Dynkin diagrams for interesting invariant cases

The conventions for the table are

1. Each node represents a *generator* of H_χ, or rather, a Picard-Lefschetz operator, corresponding to a vanishing χ-cycle (constructed similar to those in [6]–[8]).

2. The order of each Picard-Lefschetz operator is written inside the node.

3. The self-intersection number of the cycle is written below each node.

4. The intersection index of two cycles is written above the edge connecting the corresponding nodes. This number is zero for nodes with no connecting edge.

5. An inequality sign is open towards nodes corresponding to longer cycles.

6. Primitive k-th roots of unity are denoted by ε_k.

7. A simple edge is of weight 3, a double edge is of weight 4 and a triple edge is of weight 6. These weights, r say, correspond to the length of a braiding relation as follows: for $r = 4, 6$ the braid relation $(ab)^{r/2} = (ba)^{r/2}$ applies where a, b are the monodromy operators corresponding to a pair of nodes connected by an edge. Simple edges, $r = 3$, have the associated braid relation $aba = bab$. Lastly, no edge between nodes indicates commutativity: $ab = ba$.

Remark 3. The ambiguity of the convention for reading an edge $e_1 \overset{\alpha}{\text{---}} e_2$ as either $(e_1, e_2) = \alpha$ or $(e_2, e_1) = \alpha$ is easily fixed by the freedom in the choice of the vanishing cycles, up to multiplication by powers of χ and reorientation. All the graphs being trees also help at this point.

7. Weyl groups and discriminants. The study of the discriminants shows that in all the interesting invariant cases, except for Case 16, the discriminants coincide with the discriminants of certain Weyl groups. The Dynkin diagrams of these Weyl groups are obtained from those in the table (Fig. 1) by erasing all the numerical information. The Weyl groups concerned are listed in the last column of the table. We must remark that in each of the cases the set of quasihomogeneous weights of the parameters of the g-invariant miniversal deformation coincide, up to a common factor, with the set of degrees of basic invariants of the Weyl group.

Returning to Case 16 one finds that for the versal family $x^3 + y^6 + \frac{\lambda_2}{2}y^3 + \frac{\lambda_1}{3}xy^2 + \lambda_0$ the discriminant is given by $\lambda_0((\lambda_0 + (\lambda_1^3 - \lambda_2^2))^2 + 4\lambda_1^3\lambda_2^2) = 0$. The two components are the strata $D_4 : \{\lambda_0 = 0\}$ and $3A_1 : \{(\lambda_0 + (\lambda_1^3 - \lambda_2^2))^2 + 4\lambda_1^3\lambda_2^2 = 0\}$.

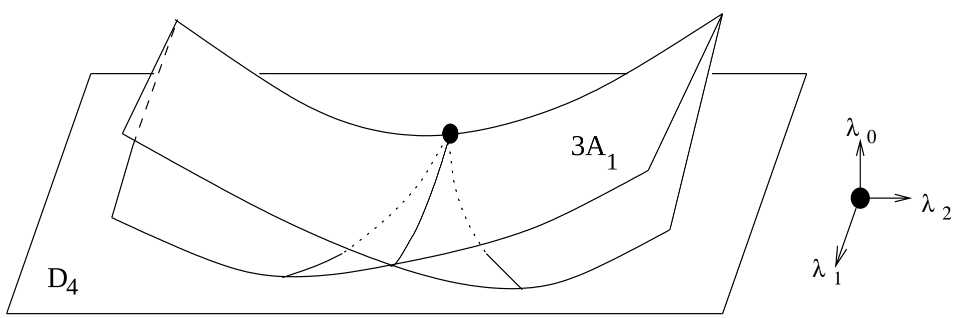


Figure 2. Discriminant for Case 16

The folded Whitney umbrella $3A_1$ is tangent to the plane D_4 along $\lambda_1^3+\lambda_2^2=0$ which in the notation of Goryunov [7] corresponds to the singularity $D_7/\mathbf{Z}_3$; squaring the transformation of the penultimate singularity as listed in Table 1 of [7] ($D_7/\mathbf{Z}_6$, so with $m=3$ and $k=2$) gives the inverse of the transformation $\tilde{g}'$ for Case 16. One should notice that a discriminant similar to the one in this case, but with an additional smooth component $\lambda_1=0$, has already occurred in singularity theory (see [3], p. 62).

8. An example. For illustration purposes, in ascertaining the Dynkin diagrams etc., we consider in some detail Case 13. All the other invariant cases are similar.

Splitting of the homology. Here $f=x^3+y^6$, $\tilde{F}=f+\lambda_1xy^2+\lambda_0+z^2$ and $\tilde{g}'$ is of order 6. In terms of the Pham basis outlined in Section 5 we have $\tilde{g}'=g_1^2g_2$ and so $\chi=\chi_1^2\chi_2$. Setting $u=e^{2\pi i/3}$ and $v=e^{2\pi i/6}$ (recall g_1,g_2,g_3 multiply x,y and z respectively by u,v and -1) we have representations for χ that, because $\tilde{g}$ is of order 6, must be a sixth root of unity:

$$\begin{array}{lcl}
\chi=1 & : & (u)^2v^2,(u^2)^2v^4\\
\chi=v & : & (u)^2v^3,(u^2)^2v^5\\
\chi=v^2 & : & (u)^2v^4\\
\chi=v^3 & : & (u)^2v^5,(u^2)^2v\\
\chi=v^4 & : & (u^2)^2v^2\\
\chi=v^5 & : & (u)^2v,(u^2)^2v^3.
\end{array}$$

Hence we have the splitting

$$H_2(\tilde{V},\mathbf{C})=\bigoplus_{\chi}H_\chi,\qquad \chi^6=1,$$

that one can write in full as

$$\begin{array}{lcl}
H_{\chi=1} & = & \mathrm{sp}\{H_{u,v^2,-1},H_{u^2,v^4,-1}\}\\
H_{\chi=v} & = & \mathrm{sp}\{H_{u,v^3,-1},\underline{H_{u^2,v^5,-1}}\}\\
H_{\chi=v^2} & = & \mathrm{sp}\{H_{u,v^4,-1}\}\\
H_{\chi=v^3} & = & \mathrm{sp}\{H_{u,v^5,-1},H_{u^2,v,-1}\}\\
H_{\chi=v^4} & = & \mathrm{sp}\{H_{u^2,v^2,-1}\}\\
H_{\chi=v^5} & = & \mathrm{sp}\{\underline{H_{u,v,-1}},H_{u^2,v^3,-1}\}
\end{array}$$

where $H_{u^j,v^k,-1}$ is the multi-character subspace of the triple g_1,g_2,g_3.

This case is interesting because the kernel lines, underlined above, lie one in each of the $H_{\chi=v}$ and $H_{\chi=v^5}$ subspaces that are both of dimension 2. This is somewhat confirmed from the derivation of the intersection matrix below.

Discriminant. Consider the miniversal family

$$F=x^3+y^6+\lambda_1xy^2+\lambda_0.$$

Elementary calculations lead us to the discriminant $\lambda_0(\lambda_1^3+27\lambda_0)=0$ consisting of two strata: $\{\lambda_0=0\}$ corresponding to the critical point of the D_4 type at the origin, and the cubic $\{\lambda_1^3+27\lambda_0=0\}$ which is the $6A_1$ stratum, see Fig. 3.

Vanishing cycles. So we have a D_4 singularity at the origin and 6 standard Morse singularities on a $\mathbf{Z}_6$ orbit out of the origin. The D_4 singularity provides us with a *short cycle* and the cluster of six A_1 points makes up a *long cycle*. Such cycles are formally defined in [6].

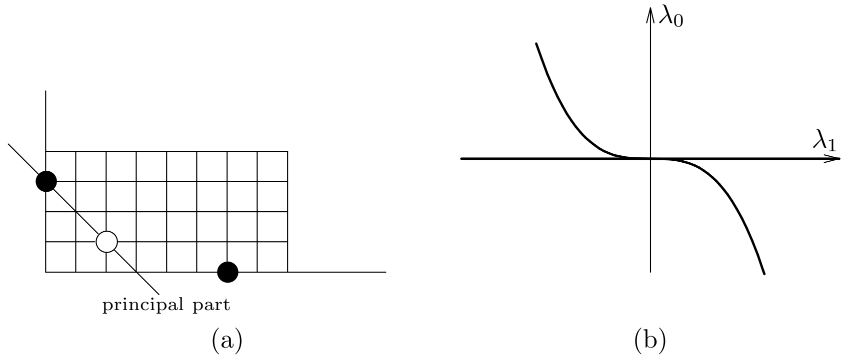


Figure 3. Case 13: (a) principal part for the critical point at the origin if $\lambda_0 = 0$, (b) discriminant.

Intersection matrix. From Goryunov's paper [7] the self-intersection index of the short cycle is $3(-2+\chi+\overline{\chi})$ and by [6] the self-intersection index of the long cycle is -12. The intersection form on H_χ for $\chi = e^{\pm 2\pi i/6}$ is found to be

$$\begin{pmatrix} 3(-2+\chi+\overline{\chi}) & 6 \\ 6 & -12 \end{pmatrix} = \begin{pmatrix} -3 & 6 \\ 6 & -12 \end{pmatrix}$$

since it must be degenerate for these characters. For the remainder of the section we shall only use these particular values of χ.

Picard-Lefschetz operators. These are complex linear reflections that generate the monodromy group. Again from [6] and [7] respectively, we have for the short cycle,

$$h_{c_1} : a \mapsto a + \frac{\langle a, c_1 \rangle}{3(1-\chi)} c_1$$

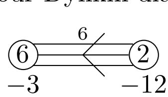
and for the long cycle,

$$h_{c_2} : a \mapsto a + \frac{\langle a, c_2 \rangle}{6} c_2.$$

In matrix form these are

$$h_{c_1} : \begin{pmatrix} \overline{\chi} & \frac{2}{1-\chi} \\ 0 & 1 \end{pmatrix}, \qquad h_{c_2} : \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}.$$

Dynkin diagram. The Picard-Lefschetz operators h_1 and h_2 are of order 6 and 2 respectively and from the discriminant we see that the braid relation for these generators $(h_{c_1} h_{c_2})^3 = (h_{c_2} h_{c_1})^3$ applies. Thus our Dynkin diagram is



9. Last words. A sequel to this paper devoted to the remaining strict equivariant interesting case, Case 10 (that shall be called $J_{10}/\mathbf{Z}_2$ following similar notation to [8]), and the realizations of complex affine reflection groups is in preparation. Popov groups that arise are $[G(6,2,2)]^*$, $[K_3(6)]$ (twice), $[K_5], [K_8], [K_{26}]_1, [K_{31}]$ and $[K_{32}]$. The first of these makes its debut here, having been accidentically omitted from the tables of [10].

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