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INVARIANTS OF EQUIDIMENSIONAL CORANK-1 MAPS

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Abstract. To a given complex-analytic equidimensional corank-1 germ f, one can associate a set of integer $\mathcal{A}$-invariants such that f is $\mathcal{A}$-finite if and only if all these invariants are finite. An analogous result holds for corank-1 germs for which the source dimension is smaller than the target dimension.

1. Introduction and notation. Let $f : \mathbf{C}^n, 0 \to \mathbf{C}^n, 0$ be a complex-analytic corank-1 germ given by the pre-normal form $(x, y) \mapsto (x, g(x, y))$, where (x, y) belongs to $\mathbf{C}^{n-1} \times \mathbf{C}$, and let $\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_s) : \mathbf{C}^n, S \to \mathbf{C}^n, \tilde{f}(S) = q$, $\tilde{f}_i(x, y_i) = (x, \tilde{g}_i(x, y_i))$, $i = 1, \ldots, s := |S|$, be an s-germ appearing in a deformation of f (here and in what follows S denotes a finite set of source points being mapped to a common point q in the target). The corank-1 $\mathcal{K}$-classes of equidimensional germs are those of type A_k, with representatives (x, y^{k+1}), and the $\mathcal{K}$-classes of s-germs $A_{(k_1,\ldots,k_s)}$ have an A_{k_i}-singularity at the ith source point. The stable equidimensional corank-1 multi-germs are those being transverse to their $\mathcal{K}$-class $A_{(k_1,\ldots,k_s)}$, and the isolated stable singularities amongst these are those with $\sum_{i=1}^{s} k_i = n$.

In the present note we define a set of $\mathcal{A}$-invariants $v_{(k_1,\ldots,k_s)}(f)$, $1 \leq \sum_i k_i \leq n$, of a germ $f : \mathbf{C}^n, 0 \to \mathbf{C}^n, 0$, which, roughly speaking, measure the failure of transversality of the multi-jet extension of f to the closures of the $A_{(k_1,\ldots,k_s)}$-orbits, and show that their finiteness is necessary and sufficient for the $\mathcal{A}$-finiteness of the germ f (see Theorem 2.6 below). The definition of these invariants is based on the defining equations for the closures of the $\mathcal{K}$-classes $A_{(k_1,\ldots,k_s)}$ in the jet-space J_s^ℓ of corank-1 s-germs in [11].

Let $r_{\mathbf{k}}(f) := r_{(k_1,\ldots,k_s)}(f)$, where $\sum_{i=1}^{s} k_i = n$, denote the number of isolated stable $A_{(k_1,\ldots,k_s)}$-points in a generic deformation of f, these are related to a subset of the above $\mathcal{A}$-invariants in a simple way:

$$r_{\mathbf{k}}(f) = c^{-1}(v_{\mathbf{k}}(f) + 1),$$

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where $c = \prod_{i=1}^{t}(m_i!)$ is an overcount factor caused by those permutations of the s source points that permute subsets of m_i points of the same type A_{k_i}, $s = \sum_{i=1}^{t} m_i$. In dimension $n \geq 3$, the finiteness of the invariants $r_{\mathbf{k}}(f)$ alone does, in general, not ensure the $\mathcal{A}$-finiteness of f (see Example 2.8). Marar, Montaldi and Ruas [7] have given formulas for the invariants $r_{(k_1,\ldots,k_s)}(f)$, $\sum_{i=1}^{s} k_i = n$, in the case of weighted homogeneous corank-1 germs f. The defining equations for the closures of the $\mathcal{K}$-classes $A_{(k_1,\ldots,k_s)}$ in the jet-space of corank-1 s-germs J_s^ℓ in [11] also provide such formulas for general f (not necessarily weighted homogeneous), see Lemma 2.2 below.

The geometric meaning of the invariants $v_{\mathbf{k}}(f)$ for $\sum_{i=1}^{s} k_i =: m < n$ is less clear than in the case where $m = n$. In the special case, where f is $\mathcal{A}$-equivalent to a weighted homogeneous germ, $v_{\mathbf{k}}(f)$ is the number of spheres in the wedge of $(n-m)$-spheres of $\bar{A}_{\mathbf{k}}$-points in the source $(\mathbf{C}^n)^s$ of a generic deformation of f. In that case results of Aleksandrov [1] give formulas for our invariants in terms of the weights and weighted degrees of f. The weighted homogeneous case, and the case of corank-1 germs from $\mathbf{C}^n$ to $\mathbf{C}^p$ with $n < p$, will be briefly discussed in the concluding section of the present note (this yields simplified proofs of the results in [7] and of Theorem 2.14 in [6]).

Apart from standard notation and results on determinacy theory, for which we refer to the survey article by Wall [12], we use the following notation for $\mathcal{K}$-orbits of corank-1 s-germs. Let $\mathbf{k}$ be a partition of m with s summands, for which we use three different notations (each being useful in different contexts):

1. $(k_1, \ldots, k_s)$, where $k_i \geq k_{i+1}$,
2. $(k_1^{m_1}, \ldots, k_t^{m_t})$, where $k_i^{m_i} := k_i, \ldots, k_i$ (m_i times) and $\sum_{i=1}^{t} m_i = s$,
3. $k(s, m)$.

The corresponding $\mathcal{K}$-class will be denoted by $A_{\mathbf{k}}$, where $\mathbf{k}$ stands for one of the three notations above, and $\bar{A}_{\mathbf{k}}$ denotes the closure of this $\mathcal{K}$-class. For multi-jet spaces we use the following notation: let $\pi : J_s^\ell \to (\mathbf{C}^n)^s$ be the projection onto the source, $\Delta \subset (\mathbf{C}^n)^s$ the diagonal and $(\mathbf{C}^n)^{(s)} := (\mathbf{C}^n)^s \setminus \Delta$. Setting $J_{(s)}^\ell := \pi^{-1}\big((\mathbf{C}^n)^{(s)}\big) \subset J_s^\ell$, we have jet-extension maps $j_{(s)}^\ell f : (\mathbf{C}^n)^{(s)} \to J_{(s)}^\ell$ and $j_s^\ell f : (\mathbf{C}^n)^s \to J_s^\ell$. For corank-1 s-germs we can identify $(\mathbf{C}^n)^s$ with $\mathbf{C}^{n+s-1}$, with coordinates $(x, y_1, \ldots, y_s) = (x_1, \ldots, x_{n-1}, y_1, \ldots, y_s)$. For the latter $\mathbf{C}^{n+s-1}$ we also use coordinates $(x, y_1, \epsilon_2, \ldots, \epsilon_s)$, where $\epsilon_{j+1} := y_{j+1} - y_j$ for $j = 1, \ldots, s-1$. The coordinates in $\mathbf{C}^{n+s-1}$ are related by an origin-preserving linear coordinate change $\lambda(x, y_1, \ldots, y_s) = (x, y_1, y_2 - y_1, \ldots, y_s - y_{s-1})$ with inverse $\lambda^{-1}(x, y_1, \epsilon_2, \ldots, \epsilon_s) = \big(x, y_1, y_1 + \epsilon_2, \ldots, y_1 + \sum_{i=2}^{s} \epsilon_i\big)$. By the diagonal in the target of λ we mean the image of $\bigcup_{i<j}\{y_i - y_j = 0\}$ under λ, which is $\bigcup_{i<j}\big\{\sum_{l=i+1}^{j} \epsilon_l = 0\big\}$. Permutations $\sigma(x, y_1, \ldots, y_s) = (x, y_{\sigma(1)}, \ldots, y_{\sigma(s)})$ in the source of λ correspond to linear origin-preserving coordinate changes $\lambda \circ \sigma$ in the target. Given an ideal $\mathcal{I}$ in $\mathcal{O}_{n+s-1}$, the local algebras $\mathcal{O}_{n+s-1}/\mathcal{I}$ and $\mathcal{O}_{n+s-1}/(\lambda \circ \sigma)^*(\mathcal{I})$ are isomorphic, we therefore change coordinate systems without explicitly mentioning λ. Hence we shall tacitly identify the three source-spaces of s-fold points $(\mathbf{C}^n)^s$ with coordinates $(x, y_1, \ldots, x, y_s)$, $\mathbf{C}^{n+s-1}$ with coordinates $(x, y_1, \ldots, y_s)$ and $\mathbf{C}^{n+s-1}$ with coordinates $(x, y_1, \epsilon_2, \ldots, \epsilon_s)$ and also their jet-spaces J_s^ℓ. Furthermore, we will not distinguish permutations σ of source points and the diagonal Δ in the first two source-spaces from their λ-images $\lambda \circ \sigma$ and $\lambda(\Delta)$ in the

third source-space. Finally, the ℓ in J^ℓ_s is assumed to be sufficiently large (one can take $\ell = \sum_{i=1}^s (k_i + 1)$).

2. Invariants and $\mathcal{A}$-finiteness. First, we give formulas for the number of transverse $A_{(k_1,\dots,k_s)}$-points, $\sum_{i=1}^s k_i = n$, appearing in generic deformations of $\mathcal{A}$-finite corank-1 germs $\mathbf{C}^n \to \mathbf{C}^n$ (for weighted-homogeneous germs such formulas, in terms of weights and degrees, may be found in [7]). Let $W \subset J^\ell_s$ be a closed $\mathcal{A}$-invariant subvariety and let $i_W(f)$ denote the intersection multiplicity of W and the image of the ℓ-jet extension $j^\ell_s f$ at $j^\ell_s f(0)$. If the local ring $R_W := \mathcal{O}_{J^\ell_s, j^\ell_s f(0)}/\mathcal{I}(W)$ is Cohen-Macaulay then

$$i_W(f) = \dim_{\mathbf{C}} \mathcal{O}_{n+s-1}/(j^\ell_s f)^*(\mathcal{I}(W))$$

(in general the intersection number is less than or equal to the dimension on the right).

In order to apply this to $W = \bar{A}_{(k_1,\dots,k_s)}$ ($\bar{X}$ closure of X), we have to "fill-in" the missing points on the diagonal in the closure of $A_{(k_1,\dots,k_s)}$. This can be done as follows ([11]). Set $y := y_1$ and

$$g_1^{(i)} := \partial^i g/\partial y_1^i, \qquad i \geq 1$$

and define by iteration for $j = 1, \dots, s-1$,

$$g_{j+1}^{(0)} := \sum_{\alpha \geq k_j+1} g_j^{(\alpha)} \epsilon_{j+1}^{\alpha-k_j-1}/\alpha!\,, \qquad g_{j+1}^{(i)} := \partial^i g_{j+1}^{(0)}/\partial \epsilon_{j+1}^i, \quad i \geq 1.$$

Then

$$\bar{A}_{(k_1,\dots,k_s)} := \{g_1^{(1)} = \dots = g_1^{(k_1)} = g_j^{(0)} = \dots = g_j^{(k_j)} = 0 : j = 2, \dots, s\}.$$

These conditions and the obvious "naive" recognition conditions for a singularity of type $A_{(k_1,\dots,k_s)}$ define the same ideal off the diagonal in the source, where the $\Delta_{ij} := \sum_{\ell=i+1}^j \epsilon_\ell$, $i < j$, are units (see Remark 2.1 below). Furthermore, the following properties of these recognition conditions can be checked easily:

(i) the conditions are *additive* on the diagonal with respect to the multiplicities $m(A_{k_i}) = k_i + 1$ of the component-germs (i.e. the multiplicities of a set of coalescing source-points have to add),

(ii) $\bar{A}_{(k_1,\dots,k_s)} \cap \Delta$ has codimension 1 in $\bar{A}_{(k_1,\dots,k_s)}$,

(iii) $R_{\bar{A}_{(k_1,\dots,k_s)}}$ is a regular local ring (hence Cohen-Macaulay) and $\bar{A}_{(k_1,\dots,k_s)} \subset J^\ell_s$ is smooth and has codimension $(\sum_{i=1}^s k_i) + s - 1$.

Remark 2.1. Here is a brief discussion of the relation between the above recognition conditions and the "naive" conditions for an $A_{(k_1,\dots,k_s)}$-singularity (the conditions and their properties (i) to (iii) have been used in [11], see also 2.5 for a simple example). Setting $g^{(r)} := \partial^r g/\partial y^r$, the "naive" conditions for an $A_{(k_1,\dots,k_s)}$-singularity at *distinct* points $p_1 := (x, y_1)$, $p_j := (x, y_1 + \sum_{i=2}^j \epsilon_i)$, $j = 2, \dots, s$ are given by:

$$g(p_j) - g(p_1) = 0, \qquad j = 2, \dots, s,$$
$$g^{(r)}(p_i) = 0, \qquad r = 1, \dots, k_i, \qquad i = 1, \dots, s.$$

For all $i < j$, $g(p_j) - g(p_i) = \sum_{\alpha \geq k_i+1} \Delta_{ij}^\alpha g^{(\alpha)}(p_i)/\alpha!$ (modulo $g^{(r)}(p_i) = 0$, $r = 1, \dots, k_i$) is divisible by the unit $\Delta_{ij}^{k_i+1}$. Taking $i = j-1$ (so that $\Delta_{j-1,j} = \epsilon_j$) we can obtain the defining equations of $\bar{A}_{(k_1,\dots,k_s)}$ above by induction on j and k_j: working modulo $\mathcal{I}(\bar{A}_{(k_1,\dots,k_{j-1})})$ and dividing by powers of ϵ_j we can reduce $g(p_j)-g(p_1)$ to $g_j^{(0)}$, and similarly we can reduce $g^{(r)}(p_j)$ to $g_j^{(r)}$ modulo $\mathcal{I}(\bar{A}_{(k_1,\dots,k_{j-1},r-1)})$. (Notice that, although e.g. $g_y(p_j)$ can be obtained by substituting $y_1+\epsilon_2+\ldots+\epsilon_j$ for y in g and by differentiating with respect to any one of these j variables, the definition of the $g_j^{(i)}$ requires derivatives with respect to the last variable ϵ_j. The reduction of $g(p_j)-g(p_1)$ to $g_j^{(0)}$ has removed the symmetry in these variables, as can be seen in Example 2.5.) Properties (i) and (ii) concerning the diagonal $p_i = p_j$, $i \neq j$, become obvious after applying a permutation such that $p_{j+1} = p_{\sigma(i)}$ and setting $\epsilon_{j+1} = 0$ in the equations of $\bar{A}_{(k_1,\dots,k_s)}$. Also note that these equations can be solved for the $\partial^r g/\partial y_1^r$ coordinates, $r = 1,\dots,\sum_{i=1}^s k_i + s - 1$, in J_s^ℓ, which implies property (iii) above.

Let $k_i^{m_i}$ denote $k_i,\dots,k_i$ (m_i times) and $\sum_{i=1}^t m_i = s$, $\sum_{i=1}^t m_i k_i = n$, then the number of $A_{(k_1^{m_1},\dots,k_t^{m_t})}$-points in a generic deformation of a germ f is given by

$$r_{(k_1^{m_1},\dots,k_t^{m_t})}(f) := \frac{1}{\prod_{i=1}^t (m_i!)} \cdot \dim_{\mathbf{C}} \mathcal{O}_{n+s-1}/(j_s^\ell f)^*(\mathcal{I}(\bar{A}_{(k_1^{m_1},\dots,k_t^{m_t})})),$$

where $\prod_{i=1}^t (m_i!)$ is an overcount factor (caused by permutations of A_{k_i}-points in the source) and the second term is equal to the intersection multiplicity $i := i_{\bar{A}_{(k_1^{m_1},\dots,k_t^{m_t})}}(f)$ (by (iii) above the relevant local ring is regular). The conservation of i under deformations then implies that a generic deformation of f has precisely $r_{(k_1^{m_1},\dots,k_t^{m_t})}(f)$ transverse $A_{(k_1^{m_1},\dots,k_t^{m_t})}$-points (note that, by a result of Mather, any $\mathcal{K}$-finite germ has a stable unfolding whose jet-extension is transverse to any given submanifold in multi-jet space). Hence we have the following.

LEMMA 2.2. *Any generic deformation of a $\mathcal{K}$-finite corank-1 germ $f : \mathbf{C}^n, 0 \to \mathbf{C}^n, 0$, with $r_{(k_1^{m_1},\dots,k_t^{m_t})}(f) < \infty$, has precisely $r_{(k_1^{m_1},\dots,k_t^{m_t})}(f)$ transverse $A_{(k_1^{m_1},\dots,k_t^{m_t})}$-points.*

From now on we use the following notation for partitions of m with s summands:

$$k(s,m) := (k_1,\dots,k_s), \qquad k_i \geq k_{i+1}, \qquad \sum_i k_i = m.$$

Viewing the generators of $(j_s^\ell f)^*(\mathcal{I}(\bar{A}_{k(s,m)}))$ as a map

$$G_{k(s,m)} = (G_1,\dots,G_{m+s-1}) : \mathbf{C}^{n+s-1} \to \mathbf{C}^{m+s-1},$$

where $2 \leq m \leq n$, and using this notation we have

$$r_{k(s,n)}(f) := c^{-1} \cdot \dim_{\mathbf{C}} \mathcal{O}_{n+s-1}/G^*_{k(s,n)}\mathcal{M}_{n+s-1},$$

where $c = \prod_{i=1}^t (m_i!)$.

The following lemma states multi-germ versions of some results in Section 2 of [9].

LEMMA 2.3.

(i) *Given a pair of $\mathcal{A}$-equivalent, equidimensional corank-1 germs f and f', the corresponding pairs of germs $G_{k(s,m)}$ and $G'_{k(s,m)}$ are $\mathcal{K}$-equivalent.*

(ii) *Let X be the inverse-image of $\bar{A}_{k(s,m)} \subset J^\ell_s$ under the multi-jet extension of a stable d-parameter unfolding F of f and $\pi : \mathbf{C}^d \times \mathbf{C}^{n+s-1} \to \mathbf{C}^d$ the projection, then $G_{k(s,m)}$ and $\pi|_X$ are $\mathcal{K}$-equivalent (up to a suspension).*

Proof. (i) Let $f = l \circ f' \circ h$ with $(l, h^{-1}) \in \mathcal{A}$ be a pair of diffeomorphisms defined on neighborhoods U and V of 0 in the source and target, and let $S = \{p_1, \ldots, p_s\} \subset U$ be a finite set of source points. There is an induced diffeomorphism $L : J^\ell_s, j^\ell_s f'(S) \to J^\ell_s, j^\ell_s f(S)$, given by $j^\ell \rho_i(q_i) \mapsto j^\ell(l \circ \rho_i \circ h_i)(h_i^{-1}(q_i))$, $i = 1, \ldots, s$, such that $L(j^\ell_s f'(\mathbf{C}^{ns})) = j^\ell_s f(\mathbf{C}^{ns})$. The sets $\bar{A}_{k(s,m)}$ are smooth submanifolds of J^ℓ_s (see [11]) and clearly $\mathcal{A}$-invariant (i.e. $L(\bar{A}_{k(s,m)}) = \bar{A}_{k(s,m)}$). The contact of $\bar{A}_{k(s,m)}$ with $j^\ell_s f(\mathbf{C}^{ns})$ at $j^\ell_s f(S)$ and with $j^\ell_s f'(\mathbf{C}^{ns})$ at $j^\ell_s f'(S)$ is therefore the same. The corresponding maps $G_{k(s,m)}$ and $G'_{k(s,m)}$ are therefore $\mathcal{K}$-equivalent.

(ii) Choosing coordinates $(u, p) \in \mathbf{C}^d \times \mathbf{C}^{n+s-1}$, consider the germ $\pi|_X : X, (0,0) \to \mathbf{C}^d, 0$. The hypothesis on F implies that $X \subset \mathbf{C}^d \times \mathbf{C}^{n+s-1}$ is a smooth submanifold of dimension $d+n-m$, and that $\pi|_X$ is a germ of a complete intersection with (possibly) an isolated singular point at $(0,0)$, hence $\mathcal{K}$-finite. Now one checks that $\mathcal{O}_{X,(0,0)}/(\pi|_X)^* \mathcal{M}_d$ is isomorphic to $\mathcal{O}_{n+s-1}/G^*_{k(s,m)} \mathcal{M}_{m+s-1}$. ∎

Now note that f is stable as an s-germ at $p = (x, y_1, \epsilon_2, \ldots, \epsilon_s) \iff j^{n+1}_{(s)} f$ is transverse to its $\mathcal{K}^{n+1}$-orbit $A_{k(s,m)}$ at $j^{n+1}_{(s)} f(p)$ (this is a formulation of Proposition 1.1 in [8] in terms of transversality). Hence, f is unstable as an s-germ $\iff j^\ell_s f$ fails to be transverse to some $\bar{A}_{k(s,m)} \subset J^\ell_s$ for $\ell := m + s \iff G_{k(s,m)}$ fails to be a submersion (note that the recognition conditions defining $\bar{A}_{k(s,m)}$ depend on the $(m+s)$-jet of f, and their composition with $j^\ell_s f$ yields $G_{k(s,m)}$). Let $J(G_{k(s,m)})$ denote the ideal of $(m+s-1) \times (m+s-1)$ minors of $dG_{k(s,m)}$ and $\mathcal{M}_{m+s-1}$ the maximal ideal in $\mathcal{O}_{m+s-1}$, then

$$v_{k(s,m)}(f) := \dim_{\mathbf{C}} \mathcal{O}_{n+s-1}/G^*_{k(s,m)} \mathcal{M}_{m+s-1} + J(G_{k(s,m)})$$

is a $\mathcal{K}$-invariant of $G_{k(s,m)}$ "measuring" the failure of transversality of $j^\ell_s f$ to $\bar{A}_{k(s,m)}$ at $j^\ell_s f(0)$.

Remarks 2.4.

(i) Note that $c \cdot r_{A_{k(s,n)}}(f)$ is the local multiplicity of the equidimensional germ $G_{k(s,n)}$, hence by Theorem 4.5.1 of [12] we have that

$$v_{k(s,n)}(f) = c \cdot r_{k(s,n)}(f) - 1$$

(assuming that the RHS is non-negative).

(ii) For $m < n$ the geometric meaning of the invariants $v_{k(s,m)}(f)$ is less clear. For a weighted homogeneous $\mathcal{K}$-finite germ $G_{k(s,m)} : \mathbf{C}^{n+s-1}, 0 \to \mathbf{C}^{m+s-1}, 0$—e.g. in the case when f is weighted homogeneous and $v_{k(s,m)}(f) < \infty$—this invariant is equal to the Milnor number of $G_{k(s,m)}$ by a result of Greuel (Korollar 5.8 in [4], see also Chapter 5.B of [5]). Therefore, by part (ii) of Lemma 2.3, the fibre $(\pi|_X)^{-1}(u)$ over a generic $u \in \mathbf{C}^d, 0$ is homotopy equivalent to a wedge of $v_{k(s,m)}(f)$ spheres of dimension $n - m$, where X is the inverse image of $\bar{A}_{k(s,m)} \subset J^\ell_s$ under the multi-jet extension-map of a stable unfolding F of f.

EXAMPLE 2.5. For the series of germs $f_k = (x, y^4 + xy^2 + x^k y)$, $k \geq 2$, from the plane to the plane the corresponding map $G_{(1,1)} = (g_1^{(1)}, g_2^{(0)}, g_2^{(1)})$ is given by $g_1^{(1)} = 4y_1^3 + 2xy_1 + x^k$, $g_2^{(0)} = 6y_1^2 + x + 4y_1\epsilon_2 + \epsilon_2^2$ and $g_2^{(1)} = 4y_1 + 2\epsilon_2$ and is $\mathcal{K}$-equivalent to $(y_1^{2k}, x, \epsilon_2)$. Hence $r_{(1,1)}(f_k) = k$ (this is the double-fold number of the series of germs 11_{2k+1} in [10], which are $\mathcal{A}$-equivalent to f_k) and $v_{(1,1)}(f_k) = 2k-1$ (the overcount factor being $c = 2$).

Recall that the summands of the partitions $k(s,m)$ are non-increasing. Consider the partial order on the partitions with s summands, where $k(s,m) \leq k'(s,m') \iff k_i \leq k'_i$ for all $1 \leq i \leq s$. The following is the main result of the present note.

THEOREM 2.6. *Let $f : \mathbf{C}^n, 0 \to \mathbf{C}^n, 0$ be a corank-1 germ.*

(i) *The following conditions are equivalent*:

(a) *f is $\mathcal{A}$-finite,*

(b) *$v_{k(s,m)}(f) < \infty$ for all partitions $k(s,m)$ with $2 \leq m \leq n$ and $m+s \leq m_f(0)$, where $m_f(0) := \dim_{\mathbf{C}} \mathcal{O}_n/f^*\mathcal{M}_n$,*

(c) *$v_{k(s,m)}(f) < \infty$ for all partitions of $m = 2, \ldots, n$ consisting of ones and twos and satisfying $m + s \leq m_f(0)$.*

(ii) *The numbers $v_{k(s,m)}(f)$ are $\mathcal{A}$-invariant.*

Proof. (i) The vanishing ideal of the set of $\mathcal{K}$-unstable points of $G_{k(s,m)}$, i.e. of $G^{-1}_{k(s,m)}(0) \cap \Sigma_{G_{k(s,m)}}$, is

$$\mathcal{I} := G^*_{k(s,m)}\mathcal{M}_{m+s-1} + J(G_{k(s,m)}),$$

and, by the analytic Nullstellensatz, $\mathcal{I} \subset \mathcal{O}_{n+s-1}$ has finite codimension if and only if $V(\mathcal{I}) \subset \{0\}$. Hence, $v_{k(s,m)}(f) < \infty \iff G_{k(s,m)}$ is a submersion on some open set $U \setminus \{0\}$ of the origin $0 \in \mathbf{C}^{n+s-1} \iff j^\ell_s f$ is transverse to $\bar{A}_{k(s,m)}$ at $j^\ell_s f(p)$ for all $p \in U \setminus \{0\}$.

Now we claim that

$$(*) \qquad v_{k(s,m)}(f) < \infty \quad \forall k(s,m), \qquad 1 \leq m \leq n,$$

if and only if, for any finite set S of source points in a sufficiently small neighborhood $V \setminus \{0\}$ of $0 \in \mathbf{C}^n$, the s-germ of f at S is transverse to its $\mathcal{K}$-orbit, and hence $\mathcal{A}$-stable. This follows from the above transversality condition for the finiteness of the $v_{k(s,m)}(f)$, $m \leq n$, and the observation that if 0 is not an isolated $\bar{A}_{k(s,r)}$-point of f, where $r > n$, then some $v_{k(s',n)}(f)$, where $s' \leq n$, must be infinite. Let C be a set of non-isolated $\bar{A}_{k(s,r)}$-points containing 0 in its closure. For $s \leq n$ there is a partition $k(s,n) < k(s,r)$ of n (which is smaller, in the partial order on the set of partitions with s summands, than $k(s,r)$), and $\dim_{\mathbf{C}} \mathcal{O}_{n+s-1}/G^*_{k(s,n)}\mathcal{M}_{n+s-1} = \infty$ (because $C \subset G^{-1}_{k(s,n)}(0)$), hence $v_{k(s,n)}(f) = \infty$. On the other hand, for $s > n$ there is always a suitable permutation of the s source points such that $\pi(C)$, where $\pi : \mathbf{C}^{n+s-1} \to \mathbf{C}^{2n-1}$ is the projection onto the first $2n-1$ coordinates (corresponding to the projection onto the first n source points), is a set of non-isolated points containing $0 \in \mathbf{C}^{n+s-1}$ in its closure. Clearly $\pi(C) \subset G^{-1}_{k(n,n)}(0)$, hence $v_{k(n,n)}(f) = \infty$. Therefore, for V sufficiently small, the s-germ of f at $S \subset V \setminus \{0\}$ has $\mathcal{K}$-type $A_{k(s,m)}$, $1 \leq m \leq n$, if $v_{k(s,n)}(f) < \infty$, for all partitions $k(s,n)$ of n.

The above finiteness condition $(*)$ can be restricted to certain subsets of the set of partitions of m, $1 \leq m \leq n$. First, note that $v_{(1)}(f) = \infty$ implies $v_{(2)}(f) = \infty$, because any non-transverse $\bar{A}_1$-point p must lie in $\bar{A}_2$ (the $\mathcal{K}$-orbit A_1 contains the generalized fold-maps as the only $\mathcal{A}$-orbit) and, in fact, must be a non-transverse $\bar{A}_2$-point (note that $\Sigma_{G_{(1)}} \subset \Sigma_{G_{(2)}}$). Next, the additivity of the recognition conditions for $\bar{A}_{k(s,m)}$ with respect to the local multiplicities of the component-germs implies that the image of the jet-extension map $j^\ell_s f$ and $\bar{A}_{k(s,m)}$ have non-empty intersection at $j^\ell_s f(0)$ precisely for $m + s \leq \dim_{\mathbf{C}} \mathcal{O}_n / f^* \mathcal{M}_n$—this yields condition (b).

Let 1_l denote a sequence of l ones. Using the additivity of the recognition conditions on the diagonal we have that (by "specializing to the diagonal")

$$(G_{(k_1,\ldots,k_{r-1},1_l,k_{r+l},\ldots,k_s)}, \epsilon_{r+1}, \ldots, \epsilon_{r+l-1}) = G_{(k_1,\ldots,k_{r-1},2l-1,k_{r+l},\ldots,k_s)}$$

and

$$(G_{(k_1,\ldots,k_{r-1},2,1_{l-1},k_{r+l},\ldots,k_s)}, \epsilon_{r+1}, \ldots, \epsilon_{r+l-1}) = G_{(k_1,\ldots,k_{r-1},2l,k_{r+l},\ldots,k_s)}.$$

$G := G_{k(s,m)}$, where $k(s,m)$ is one of the partitions in condition (c), defines an isolated complete intersection singularity (or a regular complete intersection), both referred to as ICIS for short. We claim that (G, ϵ_{r+1}) also defines an ICIS, and so does, by induction, $(G, \epsilon_{r+1}, \ldots, \epsilon_{r+l-1})$. Notice that the ideals $J(G)$ and $J(G, \epsilon_{r+1})$ are equal modulo (G, ϵ_{r+1}), hence $G^* \mathcal{M}_{m+s-1} + J(G)$ is contained in $(G, \epsilon_{r+1})^* \mathcal{M}_{m+s-1} + J(G, \epsilon_{r+1})$, which implies the claim. By specializing the partitions in (c) to the diagonal and by permuting source points (so that the new sequence of k_is obtained after specializing to the diagonal becomes non-increasing again, i.e. a partition) we can generate all partitions in condition (b).

Finally, note that the $\mathcal{A}$-stability of the s-germ of f at all $S \subset U \setminus \{0\}$ is equivalent to the $\mathcal{A}$-finiteness of the germ f (Mather-Gaffney criterion, see e.g. Theorem 2.1 in [12]), which implies the first statement in the theorem.

(ii) The $\mathcal{A}$-invariance of the numbers $v_{k(s,m)}(f)$ follows from Lemma 2.3, part (i), and the fact that they are $\mathcal{K}$-invariants of the maps $G_{k(s,m)}$. ∎

REMARK 2.7. There are at most $(\frac{n}{2})^2 + n - 1$ (for even n) and at most $(\frac{n-1}{2})^2 + 3\frac{n-1}{2}$ (for odd n) invariants in (c), and for $m_f(0) \geq 2n$ these upper bounds are attained.

EXAMPLE 2.8. The germ $f : \mathbf{C}^3, 0 \to \mathbf{C}^3, 0$, $(x, y, z) \mapsto (x, y, z^3 + x^2 z)$ fails to be $\mathcal{A}$-finite. The numbers of isolated stable singularities in a deformation of f, given by

$$r_{(3)}(f) = r_{(2,1)}(f) = r_{(1,1,1)}(f) = 0,$$

do not detect this, but $v_{(2)}(f) = \infty$ does. The local multiplicity of f is three, hence (2) is the only partition satisfying the conditions in (c).

3. Concluding remarks on the weighted homogeneous case and the case $n < p$. We conclude with a couple of remarks.

(i) In the weighted homogeneous case the invariants $v_{k(s,m)}(f)$ are equal to the Milnor numbers of the maps $G_{k(s,m)}$. Hence one can express them in terms of weights and weighted degrees of f.

(ii) The characterization of $\mathcal{A}$-finite equidimensional corank-1 germs has an analogue in the case of corank-1 germs from $\mathbf{C}^n$ to $\mathbf{C}^p$, where $n < p$, whose proof is essentially identical.

First suppose that $f = (x, g(x,y))$ is weighted-homogeneous, and that for some given set of weights for $x_1, \ldots, x_{n-1}, y$ the last component function g has weighted degree d. Then, by using the weights $\mathrm{wt}(\epsilon_j) = \mathrm{wt}(y) =: w$ for $j = 2, \ldots, s$, the ith component function of $G_{k(s,m)}$ has weighted degree $d - iw$. Therefore the invariants $v_{k(s,m)}(f)$ are equal to the Milnor number (and also to the Tjurina number) of $G_{k(s,m)}$, and we can use the formula of Aleksandrov ([1], see also p. 36 of [3]) to express $v_{k(s,m)}(f)$ in terms of d and the weights of the variables of f.

For $m = n$ the above recovers the formulas for the number of 0-stable invariants of weighted-homogeneous germs f in terms of weights and degrees in [7] (recall that $r_{k(s,n)}(f) = c^{-1}(v_{k(s,n)} + 1))$. But for $m = n$ we can relax the condition of weighted homogeneity: if $f = f_0 + f_1$, where f_0 is $\mathcal{A}$-finite and weighted homogeneous and where f_1 has higher weighted degree (with respect to the same weights) then $G_{k(s,n)}$ is semi-weighted homogeneous. We can then use the generalized Bezout formula (see e.g. p. 39 of [2]) for the local multiplicity of $G_{k(s,n)}$ to obtain a formula for $v_{k(s,n)}(f)$ in terms of weights and weighted degrees of f_0.

The explicit defining equations for the closures of $A_{(k_1,\ldots,k_s)}$ in Section 2 hold also in a slightly modified form for map-germs $f : \mathbf{C}^n, 0 \to \mathbf{C}^p, 0$ with $n < p$. In this case it is also necessary to consider the closures of the sets $A_{(0,\ldots,0)}$. Using the additivity of these defining equations on the diagonal one can easily recover Theorem 2.14 of Marar and Mond [6], see statement (c) in the theorem below.

For $n < p$ we have to replace each of the defining equations $g_j^{(l)}$ of the closure of $A_{(k_1,\ldots,k_s)}$ by $p - n + 1$ equations $g_{j,i}^{(l)}$, $i = 1, \ldots, p - n + 1$, and to also allow $k_i = 0$ (i.e. non-singular source-points). Letting

$$G_s := G_{(0,\ldots,0)} : \mathbf{C}^{n+s-1}, 0 \to \mathbf{C}^{(s-1)(p-n+1)}$$

denote the map whose 0-set is the closure of $A_{(0,\ldots,0)}$ (s times 0), $v_s(f)$ the codimension of the ideal $(G_s, J(G_s))$ and $m_f(0) := \dim_{\mathbf{C}} \mathcal{O}_n / f^* \mathcal{M}_p$, we have the following (note that we should write partitions in quotes, because the $k(s,m)$ can contain summands that are 0).

THEOREM 3.1. *Let $f : \mathbf{C}^n, 0 \to \mathbf{C}^p, 0$, $n < p$ be a corank-1 germ. The following conditions are equivalent*:

(a) *f is $\mathcal{A}$-finite.*

(b) *$v_{k(s,m)}(f) < \infty$ for all partitions $k(s,m)$ such that $k_1 \geq 1$ (for $s = 1$) and $k_i \geq 0$ (for $s > 1$), and $(m + s - 1)(p - n + 1) \leq n + s - 1$ and $m + s \leq m_f(0)$. Furthermore, for the partitions $k(s,m)$ not satisfying these conditions the ideals generated by $G_{k(s,m)}$ have finite codimension.*

(c) $v_s(f) < \infty$ *for all* $s = 2, \ldots, \min([p/(p-n)], m_f(0))$. *If* p *is not divisible by* $p-n$ *and* $m_f(0) > p/(p-n)$ *then we need the extra condition that the codimension of the ideal generated by* G_s, *for* $s = [p/(p-n)] + 1$, *be finite.*

Proof. Apart from the following remarks the proof is the same as that of Theorem 2.6.

In statement (b): $(m + s - 1)(p - n + 1)$, where $m = \sum_i k_i$ and $k_i \geq 0$, is the codimension of the closure of $A_{(k_1,\ldots,k_s)}$ and, depending on $p-n$ and p, there may be no partitions $k(s,m)$ for which $(m+s-1)(p-n+1) = n+s-1$ (corresponding to 0-stable invariants). If $k(s,m)$ corresponds to a 0-stable invariant and $v_{k(s,m)}(f) < \infty$ then the local multiplicities of the maps $G_{k'(s',m')}$ (where $s' \geq s$) are finite for all $k'(s',m') = (k'_1, \ldots, k'_s, k'_{s+1}, \ldots, k'_{s'})$ for which $(k'_1, \ldots, k'_s) \geq k(s,m)$. But if there is no such 0-stable invariant, we need the extra condition in (b). Also note that the partition (0) is not needed, because any non-transverse $\bar{A}_{(0)}$-point is in fact an $\bar{A}_{(1)}$-point (there is only one $\mathcal{A}$-orbit within the $\mathcal{K}$-orbit of non-singular source-points).

In statement (c): we can generate all the partitions in the first statement of (b) by specializing those in the first statement of (c) to the diagonal. If $p - n$ divides p or $m_f(0) \leq p/(p-n)$ then all G_s with $s > p/(p-n)$ have finite local multiplicity, otherwise this follows from the extra condition in (c). ∎

Remarks 3.2.

(i) The equivalence of (a) and (c) basically corresponds to Theorem 2.14 of Marar and Mond [6]: the set $\tilde{D}^s(f)$ in this theorem is, up to a linear origin preserving coordinate change, equal to $G_s^{-1}(0)$, and $v_s(f) < \infty \iff G_s$ is $\mathcal{K}$-finite $\iff \tilde{D}^s(f)$ is an ICIS. Furthermore, G_s generates in $\mathcal{O}_{n+s-1}$ an ideal of finite codimension $\iff G_s^{-1}(0) \subset \{0\}$ (the formulation of the extra condition in (c) is slightly sharper in the following sense: if the extra condition is not needed or if it holds for $s = [p/(p-n)] + 1$ then $\tilde{D}^s(f) \subset \{0\}$ for the range of s stated in [6]).

(ii) In the 0-stable case, where $k(s,m)$ is such that $(m+s-1)(p-n+1) = n+s-1$, the number of transverse $A_{k(s,m)}$-points in a stabilization of f is given by $r_{k(s,m)}(f) = c^{-1}(v_{k(s,m)}(f) + 1)$, where c is the same overcount factor as in the equidimensional case $n = p$. For example, a double-point formula for f in dimension $p = 2n$ is given by $r_{(0,0)}(f)$, which is equal to $1/2$ times the local multiplicity of $G_{(0,0)}$—for $(n,p) = (1,2)$ this is the δ-invariant of f.

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