

ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO THE COMPRESSIBLE NAVIER-STOKES EQUATIONS ON THE HALF SPACE

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We consider the initial boundary value problem for the compressible Navier-Stokes equation on the half space \mathbf{R}_+^n :

$$(0.1) \quad \begin{aligned} \partial_t \rho + \operatorname{div} m &= 0, \\ \partial_t m + \operatorname{div} \left(\frac{m \otimes m}{\rho} \right) + \nabla P(\rho) &= \nu \Delta \left(\frac{m}{\rho} \right) + (\nu + \tilde{\nu}) \nabla \operatorname{div} \left(\frac{m}{\rho} \right), \end{aligned}$$

$$(0.2) \quad \begin{aligned} m|_{x_n=0} &= 0, \\ \rho(0, x) &= \rho_0(x), \quad m(0, x) = m_0(x). \end{aligned}$$

Here

$$\mathbf{R}_+^n = \{x = {}^T(T x', x_n); x' = {}^T(x_1, \dots, x_{n-1}) \in \mathbf{R}^{n-1}, x_n > 0\},$$

$n \geq 2$, and the superscript T stands for the transposition; $\rho = \rho(t, x)$ and $m = {}^T(m_1(t, x), \dots, m_n(t, x))$ denote the unknown density and momentum at time $t \geq 0$ and position $x \in \mathbf{R}_+^n$, respectively; $P = P(\rho)$ is the pressure; ν and $\tilde{\nu}$ are the viscosity coefficients that satisfy $\nu > 0$, $\frac{2}{n}\nu + \tilde{\nu} \geq 0$; $\operatorname{div} m$ denotes the usual divergence in x of m ; and ∇f denotes the usual gradient in x of a scalar function f . (e.g., j -th component of $\nabla P(\rho)$ is $\partial_{x_j} P(\rho)$.) The notation $\operatorname{div} \left(\frac{m \otimes m}{\rho} \right)$ means that its j -th component is given by $\operatorname{div} \left(\frac{m_j m}{\rho} \right)$.

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In this paper we are interested in the large time behavior of solutions to problem (0.1)–(0.2) around a constant equilibrium $T(\rho, Tm) = T(\rho^*, 0)$, where ρ^* is a given positive number. We will investigate large time behavior in L^p spaces for $2 \leq p \leq \infty$.

We will assume that $P(\rho)$ is smooth and that $\partial_\rho P(\rho^*) > 0$.

Notations. We denote the $k \times k$ identity matrix by I_k . In particular, when $k = n + 1$, we simply write I for I_{n+1} . We also define $(n + 1) \times (n + 1)$ diagonal matrices \check{I} and Q_0 by $\check{I} = \text{diag}(1, \dots, 1, -1)$ and $Q_0 = \text{diag}(1, 0, \dots, 0)$. Namely,

$$(0.3) \quad I = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \quad \check{I} = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{pmatrix}, \quad Q_0 = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}.$$

For a function $f = f(x')$ ($x' \in \mathbf{R}^{n-1}$), we denote its Fourier transform by $\mathcal{F}_{x'} f$:

$$(\mathcal{F}_{x'} f)(\xi') = \int_{\mathbf{R}^{n-1}} f(x') e^{-ix' \cdot \xi'} dx'.$$

We also define $q(\ell)$ for $\ell \geq 0$ by

$$(0.4) \quad q(\ell) = \begin{cases} \min \left\{ \frac{\ell}{2}, \frac{\ell}{4} + \frac{1}{8} \right\} & (n = 2), \\ \min \left\{ \frac{\ell}{2}, \frac{\ell}{4} + \frac{3}{8} \right\} & (n \geq 3), \end{cases}$$

and $k_p(\ell)$ for $\ell \in \mathbf{R}$ and $2 \leq p \leq \infty$ by

$$(0.5) \quad k_p(\ell) = \begin{cases} \max \left\{ 0, \ell + n \left(\frac{1}{2} - \frac{1}{p} \right) \right\} & \text{when } p = 2, \\ \max \left\{ 0, \left[\ell + n \left(\frac{1}{2} - \frac{1}{p} \right) \right] \right\} + 1 & \text{when } 2 < p \leq \infty, \end{cases}$$

where $[q]$ denotes the greatest integer less than or equal to q .

1. Linearized problem. The linearized problem takes the form, after a suitable change of variables,

$$(1.1) \quad \begin{aligned} \partial_t \rho + \gamma \text{div } m &= 0, \\ \partial_t m - \alpha \Delta m - \beta \nabla \text{div } m + \gamma \nabla \rho &= 0, \end{aligned}$$

$$(1.2) \quad m|_{x_n} = 0; \quad \begin{pmatrix} \rho \\ m \end{pmatrix} \Big|_{t=0} = \begin{pmatrix} \rho_0 \\ m_0 \end{pmatrix} \equiv u_0,$$

where α, β and γ are positive constants. Hereafter we denote the solution $\begin{pmatrix} \rho(t) \\ m(t) \end{pmatrix}$ of problem (1.1)–(1.2) by $\mathcal{U}(t)u_0$:

$$\mathcal{U}(t)u_0 = \begin{pmatrix} \rho(t) \\ m(t) \end{pmatrix}.$$

To understand the behavior of the solution to problem (1.1)–(1.2), we briefly observe the Cauchy problem for (1.1) on the whole space \mathbf{R}^n . Let us denote by $\mathcal{E}(t, x)$ the fundamental solution of the Cauchy problem. Then the Fourier transform of $\mathcal{E}(t, x)$, which

is denoted by $\mathcal{A}(t, \xi)$ ($\xi \in \mathbf{R}^n$), is given by

$$\begin{aligned} \mathcal{A}(t, \xi) = e^{-\alpha|\xi|^2 t} & \begin{pmatrix} 0 & 0 \\ 0x & I_n - \frac{\xi^T \xi}{|\xi|^2} \end{pmatrix} \\ & + \left\{ \frac{e^{\lambda_+ t}}{\lambda_+ - \lambda_-} \begin{pmatrix} -\lambda_- & -i\gamma^T \xi \\ -i\gamma \xi & \lambda_+ \frac{\xi^T \xi}{|\xi|^2} \end{pmatrix} + \frac{e^{\lambda_- t}}{\lambda_- - \lambda_+} \begin{pmatrix} -\lambda_+ & -i\gamma^T \xi \\ -i\gamma \xi & \lambda_- \frac{\xi^T \xi}{|\xi|^2} \end{pmatrix} \right\}, \end{aligned}$$

Here

$$\lambda_{\pm} = -a_1 |\xi|^2 \pm \frac{1}{2} \sqrt{(\alpha + \beta)^2 |\xi|^4 - 4\gamma^2 |\xi|^2}, \quad a_1 = \frac{1}{2}(\alpha + \beta).$$

Since

$$\lambda_{\pm} \sim -a_1 |\xi|^2 \pm i\gamma |\xi| \quad \text{for } |\xi| \ll 1$$

and

$$\lambda_+ \sim -a_0, \quad \lambda_- \sim -2a_1 |\xi|^2 \quad \text{for } |\xi| \gg 1,$$

where $a_0 = \frac{\gamma^2}{\alpha + \beta}$, we see that

$$\mathcal{A} \sim e^{-\alpha|\xi|^2 t} \widehat{E}_D(\xi) + \frac{1}{2} e^{-a_1 |\xi|^2 t} \{e^{i\gamma |\xi| t} \widehat{E}_{DW,+}(\xi) + e^{-i\gamma |\xi| t} \widehat{E}_{DW,-}(\xi)\}$$

for $|\xi| \ll 1$, and

$$\mathcal{A} \sim e^{-a_0 t} \widehat{E}_{\infty}(\xi)$$

for $|\xi| \gg 1$, where

$$(1.3) \quad \widehat{E}_D(\xi) = \begin{pmatrix} 0 & 0 \\ 0 & I_n - \frac{\xi^T \xi}{|\xi|^2} \end{pmatrix}, \quad \widehat{E}_{DW,\pm}(\xi) = \begin{pmatrix} 1 & \mp \frac{\xi^T \xi}{|\xi|} \\ \mp \frac{\xi}{|\xi|} & \frac{\xi^T \xi}{|\xi|^2} \end{pmatrix},$$

and

$$\widehat{E}_{\infty}(\xi) = \begin{pmatrix} 1 & \frac{i\gamma}{2a_1} \frac{\xi^T \xi}{|\xi|^2} \\ \frac{i\gamma}{2a_1} \frac{\xi}{|\xi|^2} & -\frac{a_0}{2a_1} \frac{\xi^T \xi}{|\xi|^4} \end{pmatrix}.$$

Therefore, the asymptotically leading part is given by the low frequency part $|\xi| \ll 1$. The first term for $|\xi| \ll 1$ gives the heat kernel. The second term for $|\xi| \ll 1$ is the Gaussian with oscillatory factor, and, thus, yields the convolution of the heat kernel and the fundamental solution of the wave equation, which is the so-called *diffusion wave*.

It then turns out that the solution $\mathcal{E}(t, \cdot) * u_0$ of the Cauchy problem is decomposed as

$$\mathcal{E}(t, \cdot) * u_0 = \mathcal{E}_D(t, \cdot) * u_0 + \mathcal{E}_{DW}(t, \cdot) * u_0 + \mathcal{E}_{\infty}(t, \cdot) * u_0,$$

where $*$ denotes the convolution in $x \in \mathbf{R}^n$,

$$\begin{aligned} \mathcal{E}_D(t, x) & \sim \mathcal{F}_x^{-1} [e^{-\alpha|\xi|^2 t} \widehat{E}_D(\xi)], \\ \mathcal{E}_{DW}(t, x) & \sim \mathcal{F}_x^{-1} \left[\frac{1}{2} e^{-a_1 |\xi|^2 t} \{e^{i\gamma |\xi| t} \widehat{E}_{DW,+}(\xi) + e^{-i\gamma |\xi| t} \widehat{E}_{DW,-}(\xi)\} \right], \\ \mathcal{E}_{\infty}(t, x) & \sim e^{-a_0 t} \mathcal{F}_x^{-1} [\widehat{E}_{\infty}(\xi)], \end{aligned}$$

and each term is estimated as follows for $\ell \neq 0$:

$$\|\partial_x^{\ell} \mathcal{E}_D(t, \cdot) * u_0\|_{L^p(\mathbf{R}^n)} \leq C t^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\ell|}{2}} \|u_0\|_{L^1(\mathbf{R}^n)},$$

$$\begin{aligned} \|\partial_x^\ell \mathcal{E}_{DW}(t, \cdot) * u_0\|_{L^p(\mathbf{R}^n)} &\leq C t^{-\frac{n}{2}(1-\frac{1}{p})-\frac{n-1}{4}(1-\frac{2}{p})-\frac{|\ell|}{2}} \|u_0\|_{L^1(\mathbf{R}^n)}, \\ \|\partial_x^\ell \mathcal{E}_\infty(t, \cdot) * u_0\|_{L^p(\mathbf{R}^n)} &\leq C e^{-c_0 t} (\|m_0\|_{H^{k_p(|\ell|-1)}(\mathbf{R}^n)} + \|\rho_0\|_{H^{k_p(|\ell|)}(\mathbf{R}^n)}). \end{aligned}$$

Here, note that the factor $t^{-\frac{n-1}{4}(1-\frac{2}{p})}$ in the estimate for the diffusion wave $\mathcal{E}_{DW}(t, \cdot) * u_0$ arises from the spreading effect of the oscillatory factor $e^{\pm i\gamma|\xi|t}$. Also when $\ell = 0$, the similar results hold. See [2, 10] for more details.

Let us now consider the half space problem (1.1)–(1.2). For this problem, as we will see below, the solution $\mathcal{U}(t)u_0$ is written in the sum of $\mathcal{U}^{(C)}(t)u_0$ and $\mathcal{U}^{(H)}(t)u_0$, where $\mathcal{U}^{(C)}(t)u_0$ can be obtained by a simple reflection argument from the solution of the Cauchy problem, while $\mathcal{U}^{(H)}(t)u_0$ cannot be obtained by reflection argument from the Cauchy problem. We will see that the asymptotically leading part of $\mathcal{U}(t)u_0$ is given by the sum of two terms, similarly to the Cauchy problem, one of which behaves like purely diffusively and the other one behaves like diffusion waves. However, there appear some aspects different from the Cauchy problem, due to $\mathcal{U}^{(H)}(t)u_0$.

THEOREM 1.1. *Let $\mathcal{U}(t)u_0$ be the solution of problem (1.1)–(1.2). Then $\mathcal{U}(t)u_0$ is decomposed as*

$$(1.4) \quad \mathcal{U}(t)u_0 = \mathcal{U}_D(t)u_0 + \mathcal{U}_{DW}(t)u_0 + \mathcal{U}_\infty(t)u_0,$$

where $\mathcal{U}_D(t)u_0$ and $\mathcal{U}_{DW}(t)u_0$ take the form in (i) and (ii), respectively, and each term on the right-hand side has the estimates in (iii).

(i) $\mathcal{U}_D(t)u_0$ behaves like purely diffusively and it is written in the form

$$\begin{aligned} \mathcal{U}_D(t)u_0 &= \mathcal{F}_{x'}^{-1}[\widehat{\mathcal{U}}_D(t)\widehat{u}_0], \\ \widehat{\mathcal{U}}_D(t)\widehat{u}_0 &= \int_0^\infty \widehat{U}_D(t, \xi', x_n, y_n)\widehat{u}_0(\xi', y_n) dy_n \end{aligned}$$

with

$$\widehat{U}_D(t, \xi', x_n, y_n) = \frac{1}{2\pi} \int_{\mathbf{R}} e^{-\alpha|\xi|^2 t} \widehat{L}_D(\xi', \xi_n, x_n, y_n) d\xi_n,$$

for some function $\widehat{L}_D(\xi', \xi_n, x_n, y_n)$.

(ii) $\mathcal{U}_{DW}(t)u_0$ behaves like diffusion waves and it is written in the form

$$(1.5) \quad \mathcal{U}_{DW}(t)u_0 \sim \mathcal{U}_{DW}^{(0)}(t)u_0 \quad (t \rightarrow \infty),$$

where

$$\begin{aligned} \mathcal{U}_{DW}^{(0)}(t)u_0 &= \mathcal{F}_{x'}^{-1}[\widehat{\mathcal{U}}_{DW}^{(0)}(t)\widehat{u}_0], \\ \widehat{\mathcal{U}}_{DW}^{(0)}(t)\widehat{u}_0 &= \int_0^\infty \widehat{U}_{DW}^{(0)}(t, \xi', x_n, y_n)\widehat{u}_0(\xi', y_n) dy_n, \end{aligned}$$

and

$$\begin{aligned} &\widehat{U}_{DW}^{(0)}(t, \xi', x_n, y_n) \\ &= \frac{1}{2\pi} \int_{\mathbf{R}} e^{-a_1|\xi|^2 t} \{e^{i\gamma|\xi|t} \widehat{G}_{DW,+}(\xi', \xi_n, x_n, y_n) + e^{-i\gamma|\xi|t} \widehat{G}_{DW,-}(\xi', \xi_n, x_n, y_n)\} d\xi_n \\ &\quad + e^{-\tilde{a}_1|\xi'|^2 t} \{e^{i\gamma|\xi'|t} \widehat{G}_{DW,+}^{(Res)}(\xi', x_n, y_n) + e^{-i\gamma|\xi'|t} \widehat{G}_{DW,-}^{(Res)}(\xi', x_n, y_n)\}. \end{aligned}$$

Here $a_1 = \frac{1}{2}(\alpha + \beta)$, $\tilde{a}_1 = \frac{1}{2}(2\alpha + \beta)$,

$$\begin{aligned} \widehat{G}_{DW,\pm}(\xi', \xi_n, x_n, y_n) &= \frac{1}{2} \widehat{E}_{DW,\pm}(\xi', \xi_n) \{ e^{i\xi_n(x_n - y_n)} I + e^{i\xi_n(x_n + y_n)} \check{I} \} \\ &\quad + \widehat{G}_{DW,\pm}^{(1)}(\xi', \xi_n) e^{i\xi_n x_n} e^{-a_2^{(\pm)} |\xi|^{1/2} y_n} (I - Q_0) \\ &\quad + \{ \widehat{G}_{DW,\pm}^{(2)}(\xi', \xi_n) e^{i\xi_n x_n} + \widehat{G}_{DW,\pm}^{(3)}(\xi', \xi_n) e^{-a_2^{(\pm)} |\xi|^{1/2} x_n} \} \\ &\quad \times \{ e^{i\xi_n y_n} Q_0 + (e^{-a_2^{(\pm)} |\xi|^{1/2} y_n} - e^{i\xi_n y_n}) (I - Q_0) \} \end{aligned}$$

with $\xi = T(T\xi', \xi_n)$, $a_2^{(\pm)} = e^{\pm i\frac{\pi}{4}} \sqrt{\frac{\gamma}{\alpha}}$, $\widehat{E}_{DW,\pm}(\xi, \xi_n)$ being defined in (1.3), and I , \check{I} and Q_0 being defined (0.3), and

$$\widehat{G}_{DW,\pm}^{(1)}(\xi', \xi_n) = \begin{pmatrix} 0 & \pm \frac{T\xi'}{|\xi|} & 0 \\ 0 & -\frac{\xi'^T \xi'}{|\xi|^2} & 0 \\ 0 & -\frac{\xi_n^T \xi'}{|\xi|^2} & 0 \end{pmatrix},$$

$$\widehat{G}_{DW,\pm}^{(2)}(\xi', \xi_n) = -\frac{|\xi'|^2}{ia_2^{(\pm)} |\xi|^{1/2} \xi_n + |\xi'|^2} \begin{pmatrix} 1 & \pm \frac{T\xi'}{|\xi|} & \mp \frac{\xi_n}{|\xi|} \\ \mp \frac{\xi'}{|\xi|} & -\frac{\xi'^T \xi'}{|\xi|^2} & \frac{\xi' \xi_n}{|\xi|^2} \\ \mp \frac{\xi_n}{|\xi|} & -\frac{\xi_n^T \xi'}{|\xi|^2} & \frac{\xi_n^2}{|\xi|^2} \end{pmatrix},$$

$$\widehat{G}_{DW,\pm}^{(3)}(\xi', \xi_n) = \frac{\gamma}{a_2^{(\pm)} \alpha} \frac{\xi_n}{|\xi|^{1/2}} \frac{1}{ia_2^{(\pm)} |\xi|^{1/2} \xi_n + |\xi'|^2} \begin{pmatrix} 0 & 0 & 0 \\ -\xi' & \mp \frac{\xi'^T \xi'}{|\xi|} & \pm \frac{\xi' \xi_n}{|\xi|} \\ 0 & 0 & 0 \end{pmatrix};$$

and

$$\begin{aligned} \widehat{G}_{DW,\pm}^{(Res)}(\xi', x_n, y_n) &= \{ \widehat{G}_{DW,\pm}^{(Res,1)}(\xi') e^{-a_2^{(\pm)} |\xi'|^{1/2} x_n} + \widehat{G}_{DW,\pm}^{(Res,2)}(\xi') e^{-\frac{1}{a_2^{(\pm)}} |\xi'|^{3/2} x_n} \} \\ &\quad \times \{ e^{-\frac{1}{a_2^{(\pm)}} |\xi'|^{3/2} y_n} Q_0 + (e^{-a_2^{(\pm)} |\xi'|^{1/2} y_n} - e^{-\frac{1}{a_2^{(\pm)}} |\xi'|^{3/2} y_n}) (I - Q_0) \} \end{aligned}$$

with

$$\begin{aligned} \widehat{G}_{DW,\pm}^{(Res,1)}(\xi') &= \frac{1}{a_2^{(\pm)}} \begin{pmatrix} 0 & 0 & 0 \\ \mp \xi' |\xi'|^{1/2} & -\frac{\xi'^T \xi'}{|\xi'|^{1/2}} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \widehat{G}_{DW,\pm}^{(Res,2)}(\xi') &= i \frac{a_2^{(\pm)} \alpha}{\gamma} \begin{pmatrix} \mp |\xi'|^{3/2} & -|\xi'|^{1/2} \xi' & 0 \\ \xi' |\xi'|^{3/2} & \pm \frac{\xi'^T \xi'}{|\xi'|^{1/2}} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

(iii) Let ℓ be a multi-index and let $2 \leq p \leq \infty$. Then there exist constants $c_0 > 0$ and $C > 0$ such that

$$\begin{aligned} \|\partial_x^\ell \mathcal{U}_D(t) u_0\|_p &\leq C t^{-\frac{n}{2}(1-\frac{1}{p}) - \frac{|\ell|}{2}} \|u_0\|_1, \\ \|\partial_x^\ell \mathcal{U}_{DW}(t) u_0\|_p &\leq C t^{-\frac{n}{2}(1-\frac{1}{p}) - \frac{n-1}{4}(1-\frac{2}{p}) - \frac{2}{p} q(|\ell|) - (1-\frac{2}{p}) \frac{|\ell|}{4}} \|u_0\|_1, \\ \|\partial_x^\ell \mathcal{U}_\infty(t) u_0\|_p &\leq C e^{-c_0 t} (\|m_0\|_{H^{k_p(|\ell|-1)}} + \|\rho_0\|_{H^{k_p(|\ell|)}}). \end{aligned}$$

for all $t \geq 1$. Here $q(|\ell|)$ and $k_p(|\ell|)$ are defined in (0.4) and (0.5), respectively.

COROLLARY 1.2. *Assume that $n \geq 2$. Let ℓ be a multi-index and let $2 \leq p \leq \infty$. Then, there exist positive constants c_0 and C such that the following estimates in (i) and (ii) hold for all $t \geq 1$.*

(i) *If $|\ell| \leq n - 1$, then*

$$\|\partial_x^\ell \mathcal{U}(t)u_0\|_p \leq C\{t^{-\frac{n}{2}(1-\frac{1}{p})-\frac{2}{p}q(|\ell|)-(1-\frac{2}{p})\frac{|\ell|}{2}}\|u_0\|_1 + e^{-c_0t}(\|m_0\|_{H^{k_p(|\ell|-1)}} + \|\rho_0\|_{H^{k_p(|\ell|)}})\}.$$

(ii) *If $|\ell| \geq n$, then*

$$\|\partial_x^\ell \mathcal{U}(t)u_0\|_p \leq \begin{cases} C\{t^{-\frac{n}{2}(1-\frac{1}{p})-\frac{n-1}{4}(1-\frac{2}{p})-\frac{1}{4p}-\frac{|\ell|}{4}}\|u_0\|_1 \\ \quad + e^{-c_0t}(\|m_0\|_{H^{k_p(|\ell|-1)}} + \|\rho_0\|_{H^{k_p(|\ell|)}})\} \quad (n = 2), \\ C\{t^{-\frac{n}{2}(1-\frac{1}{p})-\frac{n-1}{4}(1-\frac{2}{p})-\frac{3}{4p}-\frac{|\ell|}{4}}\|u_0\|_1 \\ \quad + e^{-c_0t}(\|m_0\|_{H^{k_p(|\ell|-1)}} + \|\rho_0\|_{H^{k_p(|\ell|)}})\} \quad (n \geq 3). \end{cases}$$

REMARK. The diffusive part $\mathcal{U}_D(t)u_0$ is approximated by the incompressible flow in large time, i.e.,

$$\mathcal{U}_D(t)u_0 \sim \begin{pmatrix} \tilde{p}(t) \\ \tilde{v}(t) \end{pmatrix} \quad \text{as } t \rightarrow \infty,$$

where $\begin{pmatrix} \tilde{p}(t) \\ \tilde{v}(t) \end{pmatrix}$ is the solution of the nonstationary Stokes problem

$$\begin{aligned} \operatorname{div} \tilde{v} &= 0, \\ \partial_t \tilde{v} - \alpha \Delta \tilde{v} + \gamma \nabla \tilde{p} &= 0, \\ \tilde{v}|_{x_n=0} &= 0, \quad \tilde{v}|_{t=0} = \mathbb{P}m_0 \end{aligned}$$

with $\tilde{p}(t) \in L^{\tilde{r}}(\mathbf{R}_+^n)$ for all $t > 0$ and some fixed $2 < \tilde{r} < \infty$. Here \mathbb{P} denotes the orthogonal projector from $\{L^2(\mathbf{R}_+^n)\}^n$ onto the solenoidal subspace associated with the Helmholtz decomposition of $\{L^2(\mathbf{R}_+^n)\}^n$.

To analyze the nonlinear problem (0.1)–(0.2), we will make use of estimate for $\mathcal{U}_D(t)[\partial_x u_0]$:

THEOREM 1.5. *Let $u_0 = \begin{pmatrix} 0 \\ m_0 \end{pmatrix}$ and let $2 \leq p \leq \infty$. Then there exist constants $c_0 > 0$ and $C > 0$ such that*

$$\begin{aligned} \|\mathcal{U}_D(t)[\partial_x u_0]\|_p &\leq Ct^{-\frac{n}{2}(1-\frac{1}{p})-\frac{1}{2}}\|m_0\|_1, \\ \|\mathcal{U}_{DW}(t)[\partial_x u_0]\|_p &\leq Ct^{-\frac{n}{2}(1-\frac{1}{p})-\frac{n-1}{4}(1-\frac{2}{p})-\frac{1}{4}}\|m_0\|_1, \\ \|\mathcal{U}_\infty(t)[\partial_x u_0]\|_p &\leq Ce^{-c_0t}\|m_0\|_{H^{k_p(0)}}. \end{aligned}$$

for all $t \geq 1$. As a result,

$$\|\mathcal{U}(t)[\partial_x u_0]\|_p \leq Ct^{-\frac{n}{2}(1-\frac{1}{p})-\frac{1}{2}(1-\frac{1}{p})}(\|m_0\|_1 + \|m_0\|_{H^{k_p(0)}}).$$

REMARKS. (i) In the case of the Cauchy problem, $\|\mathcal{E}(t)[\partial_x u_0]\|_{L^p(\mathbf{R}^n)} = O(t^{-\frac{n}{2}(1-\frac{1}{p})-\frac{1}{2}})$.

(ii) The estimate $\|\mathcal{U}(t)[\partial_x u_0]\|_2 = O(t^{-\frac{n}{4}-\frac{1}{4}})$ is optimal for all $n \geq 2$.

(iii) The slower decay rate arises from the terms with $e^{-a_2^{(\pm)}|\xi|^{1/2}y_n}$ in the representation of $\mathcal{U}_{DW}(t)[\partial_x u_0]$. In particular, the term with

$$e^{-a_1|\xi|^2 t \pm i\gamma|\xi|t} \widehat{G}_{DW,\pm}^{(1)}(\xi', \xi_n) e^{i\xi_n x_n} (-a_2^{(\pm)}|\xi|^{1/2}) e^{-a_2^{(\pm)}|\xi|^{1/2} y_n} (I - Q_0)$$

decays in L^2 norm as $O(t^{-\frac{n}{4}-\frac{1}{4}})$.

2. Nonlinear problem. We next present our results on the large time behavior of solutions to the nonlinear problem (0.1)–(0.2). In what follows we write s_0 for $[\frac{n}{2}] + 1$. We first state the assumptions on the data.

ASSUMPTIONS.

- (A1) The pressure $P = P(\rho)$ is a smooth function of ρ in a neighborhood of ρ^* and $\frac{\partial P}{\partial \rho}(\rho^*) > 0$.
- (A2) The initial perturbation ${}^T(\rho_0 - \rho^*, {}^T m_0)$ belongs to H^s for some $s \in \mathbf{Z}$ with $s \geq s_0 + 1$ and satisfies a suitable compatibility condition.

THEOREM 2.1. (i) *Let $n \geq 2$. Assume that (A1) and (A2) hold for some $s \in \mathbf{Z}$ with $s \geq s_0 + 1$. Then there exists a positive number η_0 such that if the initial perturbation ${}^T(\rho_0 - \rho^*, {}^T m_0) \in H^s \cap L^1$ and if $\|{}^T(\rho_0 - \rho^*, {}^T m_0)\|_{H^s} + \|{}^T(\rho_0 - \rho^*, {}^T m_0)\|_{L^1} \leq \eta_0$, then the perturbation $u(t) \equiv {}^T(\rho(t) - \rho^*, {}^T m(t))$ satisfies*

$$\|u(t)\|_p = O(t^{-\frac{n}{2}(1-\frac{1}{p})})$$

and

$$\|\partial_x u(t)\|_2 = \begin{cases} O(t^{-\frac{n}{4}-\frac{3}{8}}) & (n = 2), \\ O(t^{-\frac{n}{4}-\frac{1}{2}}) & (n \geq 3), \end{cases}$$

for $2 \leq p \leq \infty$ as $t \rightarrow \infty$.

(ii) *For $u_0 = {}^T(\bar{\rho}_0, {}^T \bar{m}_0)$ with $\bar{\rho}_0 \in H^1$ and $\bar{m}_0 = {}^T(\bar{m}_{0,1}, \dots, \bar{m}_{0,n}) \in L^2$, let $\mathcal{U}(t)u_0(x) = {}^T(\bar{\rho}(t, x), {}^T \bar{m}(t, x))$ denote the solution of the linearized problem at ${}^T(\rho^*, 0)$. Then, under the same assumptions on ${}^T(\rho_0 - \rho^*, {}^T m_0)$ in (i),*

$$\|u(t) - \mathcal{U}(t)u_0\|_2 = O(t^{-\frac{n}{4}-\frac{1}{4}} L(t))$$

as $t \rightarrow \infty$, where $u_0 = {}^T(\rho_0 - \rho^*, {}^T m_0)$, and $L(t) = \log(1+t)$ when $n = 2$ and $L(t) = 1$ for $n \geq 3$.

(iii) *Assume, in addition, that (A2) holds for some $s \in \mathbf{Z}$ with $s \geq s_0 + 2$. Then there exists a positive number $\tilde{\eta}_0$ such that If, $\|{}^T(\rho_0 - \rho^*, {}^T m_0)\|_{H^s} + \|{}^T(\rho_0 - \rho^*, {}^T m_0)\|_{L^1} \leq \tilde{\eta}_0$,*

$$\|u(t) - \mathcal{U}(t)u_0\|_\infty = \begin{cases} O(t^{-\frac{n}{2}-\frac{3}{8}} \log t) & (n = 2), \\ O(t^{-\frac{n}{2}-\frac{1}{2}}) & (n \geq 3), \end{cases}$$

as $t \rightarrow \infty$, where $u_0 = {}^T(\rho_0 - \rho^*, {}^T m_0)$.

REMARKS. (i) Since $\|\mathcal{U}(t)u_0\|_p = O(t^{-\frac{n}{2}(1-\frac{1}{p})})$, the estimates in (ii) and (iii) mean that the leading part of $u(t)$ in large time is given by $\mathcal{U}(t)u_0$, similarly to the case of the Cauchy problem [2,6,7].

(ii) The estimates in (ii) and (iii) are obtained via Duhamel's principle

$$u(t) = \mathcal{U}(t)u_0 + \int_0^t \mathcal{U}(t-s) [\operatorname{div} \mathcal{N}(u, \partial_x u)(s)] ds,$$

by applying the estimate for $\mathcal{U}(t)[\partial_x u_0]$ to $\mathcal{U}(t-s) [\operatorname{div} \mathcal{N}(u, \partial_x u)(s)]$. Here $\operatorname{div} \mathcal{N}(u, \partial_x u)$

denotes the nonlinearity. From this identity we may say that the leading part of $u(t) - \mathcal{U}(t)u_0$ is given by the nonlinear interaction of the leading part of $\mathcal{U}(t)u_0$. The decay rate in (ii) is slower than the corresponding one for the Cauchy problem, but, the estimate $\|\mathcal{U}(t)[\partial_x u_0]\|_2 = O(t^{-\frac{3}{4}-\frac{1}{4}})$ is optimal. This would, therefore, indicate that the leading part of the nonlinear interaction in L^2 space may involve some slowly decaying terms not appearing in the Cauchy problem.

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