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THE WAVELET TYPE SYSTEMS

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Abstract. We consider biorthogonal systems of functions on the interval $[0, 1]$ or $\mathbb{T}$ which have the same dyadic scaled estimates as wavelets. We present properties and examples of these systems.

1. Introduction. In recent years more and more attention has been paid in mathematical papers to bases on a bounded set, the interval $[0, 1]$ or the circle $\mathbb{T}$ (see for instance [1], [14], [15], [24]). Sometimes these systems are some modifications of wavelets on $\mathbb{R}$ ([13], [23]). Unfortunately, even in this case, they do not have the structure of wavelets (i.e. they do not come from one fixed function), therefore they are more difficult to study.

In this paper we consider properties of orthogonal or biorthogonal systems on $[0, 1]$ or $\mathbb{T}$ consisting of functions which have dyadic scaled estimates (see (I), (II), (III) below). All the systems from the papers mentioned above satisfy this kind of estimate. Moreover, many of classical systems also fulfil conditions (I), (II) and (III). This list includes e.g. the Haar system, the Franklin system, both orthonormal and biorthonormal spline systems ([9], [11]), the orthogonal system of trigonometric conjugates to the Franklin function (see [4], and Theorem 3.3 for estimates) or periodic wavelets.

2. Systems with dyadic scaled estimates

2.1. *Preliminaries and notation.* We will consider systems on the interval $[0, 1]$ as well as on the one-dimensional circle $\mathbb{T}$, so let $(\mathbb{I}, d)$ denote either the metric space $([0, 1], d_1)$ or $(\mathbb{T}, d_2)$, where

$$d_1(x, y) = |x - y|, \quad x, y \in [0, 1], \quad d_2(x, y) = \min(|x - y|, 1 - |x - y|), \quad x, y \in \mathbb{T}.$$

If $j \in \mathbb{N}$ and $k \in \{1, 2, \dots, 2^j\}$ then by $I_{j,k}$ we denote the interval $\left[\frac{k-1}{2^j}, \frac{k}{2^j}\right]$ and for $n \in \mathbb{N}$ we define $n * I_{j,k}$ as the set $\{x \in \mathbb{I} : \ d(x, \frac{k}{2^j}) \leq \frac{n}{2^j}\}$.

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By *a biorthogonal wavelet type system* on $\mathbb{I}$ we mean a biorthogonal system $\{\psi_n, \phi_n\}_{n=-N}^{\infty}$, where natural $N \geq -1$ is given, consisting of functions on $\mathbb{I}$ satisfing the following conditions:

(I) There is a constant $M > 0$ such that for any $n \in \{-N, \ldots, 0, 1\}$ and $x \in \mathbb{I}$

$$|\psi_n(x)| \leq M \quad \text{and} \quad |\phi_n(x)| \leq M.$$

(II) For $j \geq 0$, $k \in \{1, 2, \ldots, 2^j\}$ and $x \in \mathbb{I}$

$$|\psi_{2^j+k}(x)| \leq 2^{\frac{j}{2}} S\Big(2^j d\Big(x, \frac{k}{2^j}\Big)\Big), \quad |\phi_{2^j+k}(x)| \leq 2^{\frac{j}{2}} S\Big(2^j d\Big(x, \frac{k}{2^j}\Big)\Big),$$

where the function S satisfies some kind of integral condition. In this paper we assume that

(III) $S : [0, \infty) \to \mathbb{R}$ is a nonincreasing function such that

$$\int_0^{\infty} \ln(1 + x) S(x) dx < +\infty.$$

2.2. *Properties of the function S.* As a consequence of the monotonicity of S and of condition (III) one can get

LEMMA 2.1. *There is a constant C such that*

$$\sum_{j=0}^{\infty} \sum_{k=1}^{\infty} 2^j S(k 2^j) < C$$

and for $j \geq 0$ and $x, y \in \mathbb{I}$

$$\sum_{k=1}^{2^j} S\Big(2^j d\Big(x, \frac{k}{2^j}\Big)\Big) S\Big(2^j d\Big((y, \frac{k}{2^j}\Big)\Big) \leq C S(2^{j-1} d(x, y)). \ \blacksquare$$

LEMMA 2.2. *Let B be an arbitrary family of dyadic intervals with disjoint interiors. For each interval $I_{j_0,m}$ and $t \notin I_{j_0,m}$ we have*

$$\sum_{\substack{I_{j,k} \in B \\ I_{j,k} \subset I_{j_0,m}}} \sum_{i=0}^{\infty} 2^i S(2^{j+i-1} d(t, I_{j,k})) \leq \sum_{i=0}^{\infty} 2^i S(2^{j_0+i-1} d(t, I_{j_0,m})). \tag{1}$$

Proof. If we denote the left side of inequality (1) by L, then

$$L = \lim_{n \to \infty} \sum_{j=j_0}^{j_0+n} \sum_{\substack{I_{j,k} \in B \\ I_{j,k} \subset I_{j_0,m}}} \sum_{i=0}^{\infty} 2^i S(2^{j+i-1} d(t, I_{j,k})) =: \lim_{n \to \infty} a_n(B).$$

Let $b_n = \sup a_n(B)$, where the supremum is taken over all the families B. We will show that the sequence $\{b_n\}_{n \geq 0}$ is constant. Obviously $b_n \leq b_{n+1}$. On the other hand

$$a_{n+1}(B) = a_n(B_n) + \sum_{\substack{k' : I_{j_0+n+1,k'} \in B \\ I_{j_0+n+1,k'} \subset I_{j_0,m}}} \sum_{i=0}^{\infty} 2^i S(2^{j_0+n+i} d(t, I_{j_0+n+1,k'})),$$

where B_n comes from B after excluding all the intervals shorter than $1/2^{j_0+n}$.

Let $I_{j_0+n+1,k'} \in B$ and $I_{j_0+n+1,k'} \subset I_{j_0,m}$. The interval $I_{j_0+n+1,k'}$ is a half of some interval $I_{j_0+n,k}$. Let us denote the second half of $I_{j_0+n,k}$ (which may also belong to B) by $I_{j_0+n+1,k''}$. The interiors of two dyadic intervals are either disjoint or one of them contains the other, thus $I_{j_0+n,k} \subset I_{j_0,m}$. But $d(t, I_{j_0+n+1,k'}) \geq d(t, I_{j_0+n,k})$ and $d(t, I_{j_0+n+1,k''}) \geq d(t, I_{j_0+n,k})$, therefore from the monotonicity of S we have

$$\sum_{i=0}^{\infty} 2^i S(2^{j_0+n+i} d(t, I_{j_0+n+1,k'})) + \sum_{i=0}^{\infty} 2^i S(2^{j_0+n+i} d(t, I_{j_0+n+1,k''}))$$
$$\leq \sum_{i=0}^{\infty} 2^{i+1} S(2^{j_0+n+i} d(t, I_{j_0+n,k})) = \sum_{i=1}^{\infty} 2^i S(2^{j_0+n+i-1} d(t, I_{j_0+n,k})).$$

Since $I_{j_0+n+1,k'} \in B$, it follows that $I_{j_0+n,k} \notin B$ and the interior of $I_{j_0+n,k}$ is disjoint from intervals of B_n. From this we get

$$a_{n+1}(B) \leq a_n(B_n \cup A),$$

where A is a family of dyadic intervals of length equal to $1/2^{j_0+n}$ which are disjoint from elements of B_n. Thus

$$b_{n+1} \leq b_n.$$

Since the sequence $\{b_n\}_{b \geq 0}$ is constant, we have

$$L \leq b_0 = \sum_{i=0}^{\infty} 2^i S(2^{j_0+i-1} d(t, I_{j_0,m})). \ \blacksquare$$

2.3. *Properties of wavelet type systems.* Below we list some properties of wavelet type systems which have been proved by the author in [30]

THEOREM 2.3. *Let $\{\psi_n, \phi_n\}_{n=-N}^{\infty}$ be a wavelet type system. If the system $\{\psi_n\}_{n=-N}^{\infty}$ is a basis in $L_2(\mathbb{I})$, then it is also a basis in $L_p(\mathbb{I})$, $1 \leq p < \infty$.*

The proof is an adaptation of the argument for orthogonal wavelet on $\mathbb{R}^d$ from the paper by S. E. Kelly, M. A. Kon, L. A. Raphael [22]. Moreover, the following theorem is true:

THEOREM 2.4. *Let $\{\psi_n, \phi_n\}_{n=-N}^{\infty}$ be a wavelet type system and let $\{\psi_n\}_{n=-N}^{\infty}$ be a basis in $L_2(\mathbb{I})$. If the set of linear combinations of elements of $\{\psi_n\}_{n=-N}^{\infty}$ is dense in $C(\mathbb{I})$, then $\{\psi_n\}_{n=-N}^{\infty}$ is a basis in $C(\mathbb{I})$.*

REMARK 2.5. It is not hard to prove that the system $\{\psi_n\}_{n=-N}^{\infty} \subset C(\mathbb{I})$ is linearly dense in $C(\mathbb{I})$ if and only if

$$P_n 1 \to 1 \quad \text{uniformly} \quad \text{on} \quad \mathbb{I},$$

where P_n are the partial sum operators, i.e.

$$P_n g = \sum_{i=-N}^{n} (g, \phi_i) \psi_i.$$

Using Lemma 2.2 we can prove

THEOREM 2.6. *Let $\{\psi_n, \phi_n\}_{n=-N}^{\infty}$ be a wavelet type system. If $\{\psi_n\}_{n=-N}^{\infty}$ is a Riesz basis in $L_2(\mathbb{I})$, then $\{\psi_n\}_{n=-N}^{\infty}$ is an unconditional basis in $L_p(\mathbb{I})$ for $1 < p < \infty$.*

The idea of the proof comes from the papers [21], [29], [16], where the case of wavelets on $\mathbb{R}$ is considered.

It turns out that conditions (I), (II), (III) ensure that each function ψ_n $(n \geq 1)$ can be estimated by the maximal function of the Haar function χ_n and vice versa:

LEMMA 2.7. *Let $\{\psi_n, \phi_n\}_{n=-N}^{\infty}$ be a wavelet type system. Then there is a constant C such that for all $j \geq 0$, $k \in \{1, 2, \ldots, 2^j\}$ and $t \in \mathbb{I}$ we have*

$$|\chi_{2^j+k}(t)| \leq C \cdot M\psi_{2^j+k}(t),$$

$$|\psi_{2^j+k}(t)| \leq C \cdot M\chi_{2^j+k}(t),$$

where Mf denotes the Hardy-Littewood maximal function of f.

From this fact and C. Feffermann's and E. Stein's theorem (see for instance [28], Theorem II.1) we get

THEOREM 2.8. *Let $\{\psi_n, \phi_n\}_{n=-N}^{\infty}$ be a wavelet type system. Then for each $1 < p < \infty$ and an arbitrary sequence of real numbers $\{a_n\}_{n\geq 1}$,*

$$\Big\| \Big\{ \sum_{n=1}^{\infty} a_n^2 \chi_n^2 \Big\}^{\frac{1}{2}} \Big\|_p \sim \Big\| \Big\{ \sum_{n=1}^{\infty} a_n^2 \psi_n^2 \Big\}^{\frac{1}{2}} \Big\|_p, \tag{2}$$

where the implied constants depend only on p and S.

REMARK 2.9. The above theorem is also true when $p = 1$ or when $p \in (0, 1)$ and the function S satisfies an additional assumption, namely

$$\int_0^{\infty} S^p(x)dx < +\infty,$$

but the proof in these cases is much more complicated. The method of the proof is based on the papers [19] and [20].

Let us note that if $\{f_n\}$ is an unconditional basis in $L_p(\mathbb{I})$, $1 < p < \infty$, then from Khintchin's ineqality we have

$$\Big\| \sum_n a_n f_n \Big\|_p \sim \Big\| \Big\{ \sum_n a_n^2 f_n^2 \Big\}^{\frac{1}{2}} \Big\|_p.$$

Therefore, as a consequence of Theorem 2.8, we get the boundedness of the shift operators.

THEOREM 2.10. *Let $\{\psi_n, \phi_n\}_{n=-N}^{\infty}$ be a wavelet type system. If the system $\{\psi_n\}_{n=-N}^{\infty}$ is a Riesz basis in $L_2(\mathbb{I})$, then the linear shift operators defined by $T_+\psi_n = \psi_{n+1}$ and $T_-\psi_n = \psi_{n-1}$ for $n \geq -N+1$ and $T_-\psi_{-N} = 0$ are bounded in $L_p(\mathbb{I})$, $1 < p < \infty$, and hence $\|f\|_p \sim \|T_+ f\|_p$.*

REMARK 2.11. Under some additional assumptions about the system $\{\psi_n, \phi_n\}_{n=-N}^{\infty}$ we can also prove that the shift operator T_+ is unbounded in $L_1(\mathbb{I})$ (see [30] for details).

Combining Theorems 2.8 and 2.10 we obtain an equivalence of the bases $\{\psi_n\}_{n=-N}^{\infty}$ and $\{\chi_n\}_{n=1}^{\infty}$ in $L_p(\mathbb{I})$ for $1 < p < \infty$:

THEOREM 2.12. *Let $\{\psi_n, \phi_n\}_{n=-N}^{\infty}$ be a wavelet type system. Let $1 < p < \infty$. Let us assume that $\{\psi_n\}_{n=-N}^{\infty}$ is a Riesz basis in $L_2(\mathbb{I})$. Then the series*

$$\sum_{n=1}^{\infty} a_n \psi_{n-N-1} \quad \textit{and} \quad \sum_{n=1}^{\infty} a_n \chi_n$$

are equiconvergent in $L_p(\mathbb{I})$ and their norms are equivalent.

Conditions (I), (II), (III) allow us to prove that the maximal operator

$$P^* f(x) = \sup_{n \geq -N} |P_n f(x)|$$

is of type (p,p) for $1 < p < \infty$ and of weak type $(1,1)$. Thus, using standard methods (see [27], Theorem 3.1.2), we can show the following theorem

THEOREM 2.13. *Let $\{\psi_n, \phi_n\}_{n=-N}^{\infty}$ be a wavelet type system. Let us assume that the system $\{\psi_n\}_{n=-N}^{\infty}$ is a basis in $L_2(\mathbb{I})$. If $f \in L_1(\mathbb{I})$, then $P_{j,k} f(x) \to f(x)$ for a.e. $x \in \mathbb{I}$.*

REMARK 2.14. From Theorem 2.13 we have in particular that

$$P_n 1(x) \to 1 \quad \text{if} \quad n \to \infty \tag{3}$$

for a.e. $x \in \mathbb{I}$. Similarly as in [22] we can prove that if x is Lebesgue's point of f and x satisfies (3) then $P_n f(x) \overset{n \to \infty}{\longrightarrow} f(x)$.

3. Some classical examples

3.1. *The Franklin system.* The Franklin system is the complete orthonormal system obtained from the Schauder system $\{\varphi_n\}_{n=0}^{\infty}$:

$$\varphi_0(t) = 1, \quad \varphi_n(t) = \int_0^t \chi_n(s) ds, \quad n = 1, 2, \ldots$$

by means of Schmidt's orthonormalization procedure ([17], [5]).

If we exclude from the orthonormalization function φ_1 then we get the periodic Franklin system. Both systems satisfy

THEOREM 3.1 (Z. Ciesielski [6]). *There are constants $C > 0$ and $0 < q < 1$ such that for every j, k and $t \in \mathbb{I}$ we have*

$$|f_{2^j+k}(t)| \leq C \cdot 2^{\frac{j}{2}} q^{2^j d(t, k2^{-j})}. \tag{4}$$

Moreover

$$|f'_{2^j+k}(t)| \leq C \cdot 2^{\frac{3j}{2}} q^{2^j d(t, k2^{-j})}. \tag{5}$$

The estimate (4) means that these systems satisfy conditions (I), (II), (III) with the function $S(x) = Cq^x$.

The unconditionality of the Franklin system in $L_p(\mathbb{I})$ for $1 < p < \infty$ was proved by S. V. Bočkariev in [2]. A stronger fact, i.e. the equivalence in these spaces with the Haar basis was shown in [12].

The boundedness of the shift operator for the Franklin system in $L_p(\mathbb{I})$ $(1 < p < \infty)$ arises from [12], while the unboundedness in $L_1(\mathbb{I})$ is proved in [18].

3.2. *Splines of higher orders.* If we apply the Schmidt orthonormalization procedure to the set
$$\{1, t, \dots, t^{m+1}, G^{m+1}\chi_n(t),\ n \geq 2\}$$
($m \geq -1$, $Gf(t) = \int_0^t f(s)ds$) then we get a complete orthonormal system $\{f_n^{(m)}\}_{n=-m}^{\infty}$ of splines of order m. (For $m = -1$ this is the Haar system, for $m = 0$ the Franklin system.)

From the system $\{f_n^{(m)}\}_{n=-m}^{\infty}$ we can obtain new systems:
$$f_n^{(m,k)} = \begin{cases} D^k f_n^{(m)} & \text{for } 0 < k \leq m+1, \quad n \geq k-m, \\ H^{-k} f_n^{(m)} & \text{for } -m-2 \leq k < 0, \quad n \geq -k-m, \end{cases}$$
where D is the differentiation operator and $Hf(t) = \int_t^1 f(s)ds$. It is known that
$$|f_{j,l}^{(m,k)}(x)| \leq C 2^{\frac{j}{2}+kj} q^{2^j d(x, \frac{l}{2^j})}, \qquad \|f_{j,l}^{(m,k)}\|_p \sim 2^{j(\frac{1}{2}-\frac{1}{p})+kj}$$
and $(f_{n_1}^{(m,k)}, f_{n_2}^{(m,-k)}) = \delta_{n_1,n_2}$.

Now we consider the biorthogonal systems $\{h_n^{(m,k)}, h_n^{(m,-k)}\}_{n=|k|-m}^{\infty}$, where
$$h_n^{(m,k)} = \begin{cases} f_n^{(m,k)} \|f_n^{(m,k)}\|_2^{-1} & \text{for } 0 \leq k \leq m+1, \\ f_n^{(m,k)} \|f_n^{(m,-k)}\|_2 & \text{for } 0 \leq -k \leq m+1, \end{cases} \qquad m \geq -1, \quad |k| \leq m+1.$$
The orthogonal and biorthogonal spline systems were considered in [25], [9] and [7]. In [26] it was proved that these systems are Riesz bases while the case of $L_p(\mathbb{I})$ for $1 < p < \infty$ (unconditionality, equivalence with the Haar basis and convergence almost everywhere of Fourier expansion) was considered in [8].

Let us note that from the above mentioned properties of the functions $f_n^{(m,k)}$ we have that for $0 \leq k \leq m+1$ the biorthogonal system $\{h_n^{(m,k)}, h_n^{(m,-k)}\}_{n=k-m}^{\infty}$ satisfies (I) and (II) with the function $S(x) = Cq^x$ $(0 < q < 1)$, which of course satisfies (III).

The next examples of wavelet type systems are spline systems with boundary conditions constructed by Z. Ciesielski and T. Figiel in [10] and [11]. For these systems we have similar estimates.

3.3. *The conjugate Franklin system.* In this subsection let f_n denote the periodic Franklin functions. The conjugate Franklin system is the system
$$\{1\} \cup \{\tilde{f}_n :\ n \geq 2\} \tag{6}$$
where
$$\tilde{f}_n(x) = -\int_0^{\frac{1}{2}} (f_n(x+t) - f_n(x-t)) \cot \pi t \, dt.$$
The system (6) is an orthonormal basis in $L_2(\mathbb{I})$, because the map $f \to \tilde{f}$ is an isometry on $\{f \in L_2(\mathbb{I}) :\ \int_{\mathbb{I}} f(x) = 0\}$. Moreover, this map is bounded in $L_p(\mathbb{I})$ for $1 < p < \infty$ and the Franklin system is an unconditional basis in these spaces, hence the system (6) is an unconditional basis, too.

In [4] S. V. Bočkariev has proved that the system (6) is a basis in the space $C(\mathbb{T})$. His proof in [4] is based on pointwise estimates for the kernels of the partial sum operators. However, in [3], he has obtained the following pointwise estimates for $\tilde{f}_n$'s:

THEOREM 3.2 (S. V. Bočkariev [3]). *There is a constant B such that for all $j = 0, 1, \dots$ and $k = 1, \dots, 2^j$ we have*

$$|\tilde{f}_{2^j+k}(x)| \le B \min\left(\sqrt{2^j}, \frac{1}{2^{\frac{3}{2}j}\left(d_2(x, \frac{k}{2^j})\right)^2}\right).$$

That is, the conjugate Franklin system is a wavelet type system with $S(x)=C/(1+x)^2$, and the fact that it is a basis in $C(\mathbb{T})$ follows directly from Theorem 2.4.

Here we present more accurate estimates for $\tilde{f}_n$ as a wavelet type system, with $S(x) = C/(1+x)^3$. This more accurate estimate allows us, for example, to get the equivalence (2) for a bigger range of $p \in (0, 1)$ than the estimate from Theorem 3.2, see Remark 2.9.

THEOREM 3.3. *The system (6) fulfils condition (II) with $S(x) = C/(1+x)^3$.*

Proof. It is enough to show condition (II) for $n \ge 2$. Let $j \ge 0$ and $k \in \{1, 2, \dots, 2^j\}$. We can write

$$\begin{aligned}|\tilde{f}_{2^j+k}(x)| \le &\left|\int_0^{\frac{1}{2^j}} (f_{2^j+k}(x+t) - f_{2^j+k}(x-t)) \cot \pi t \, dt\right| \\ &+ \left|\int_{\frac{1}{2^j}}^{\frac{1}{2}} (f_{2^j+k}(x+t) - f_{2^j+k}(x-t)) \cot \pi t \, dt\right| = |V_1| + |V_2|.\end{aligned}$$

Let us estimate the first term. Let $x \in I_{j,l}$. Since $d(x+t, x) \le \frac{1}{2^j}$ and $d(x, x-t) \le \frac{1}{2^j}$, the points $x-t, x, x+t$ belong to $1 * I_{j,l}$. Using (5) we get

$$\begin{aligned}|f_{2^j+k}(x+t) - f_{2^j+k}(x-t)| &\le 2t \sup_{u \in 1*I_{j,l}} |f'_{2^j+k}(u)| \\ &\le C 2^{\frac{3j}{2}} t \sup_{u \in 1*I_{j,l}} q^{2^j d(u, \frac{k}{2^j})} \le C' 2^{\frac{3j}{2}} t q^{2^j d(x, \frac{k}{2^j})}.\end{aligned}$$

Therefore

$$|V_1| \le C 2^{\frac{3j}{2}} \int_0^{\frac{1}{2^j}} t \cot \, \pi t \, dt \cdot q^{2^j d(x, \frac{k}{2^j})} \le C' q^{2^j d(x, \frac{k}{2^j})}.$$

Finally

$$|V_1| \le 2^{\frac{j}{2}} S_1\left(2^j d\left(x, \frac{k}{2^j}\right)\right),$$

where $S_1(x) = C q^x$.

Now we estimate V_2. Let us introduce auxiliary functions

$$B_{j,x,s}(t) = \begin{cases} \cot \pi(t-x) & \text{for } \frac{s}{2^j} \le |t-x| \le \frac{1}{2}, \\ 0 & \text{for } |t-x| < \frac{s}{2^j}. \end{cases}$$

Thus

$$V_2 = \int_{\frac{1}{2^j} \le |t-x| \le \frac{1}{2}} f_{2^j+k}(t) \cot \pi(t-x) dt = \int_{|t-x| \le \frac{1}{2}} f_{2^j+k}(t) B_{j,x,1}(t) dt.$$

Since both functions f_{2^j+k} and $B_{j,x,1}$ are periodic with period 1, we have

$$V_2 = \int_0^1 f_{2^j+k}(t) B_{j,x,1}(t) dt.$$

Let us fix $x \in \mathbb{I}$. Let $W_{j-1}(t)$ denote the piecewise linear periodic function interpolating $B_{j,x,1}$ in the dyadic points from the j-th level (i.e. $0, \frac{1}{2^j}, \frac{2}{2^j}, \ldots, 1$). It may be checked that

$$|B_{j,x,1}(t) - W_{j-1}(t)| \leq C\min\left(2^j, \frac{1}{2^{2j}d(x,t)^3}\right)$$

(see also [4]). Hence

$$V_2 = \int_0^1 f_{2^j+k}(t)(B_{j,x,1}(t) - W_{j-1}(t))dt + \int_0^1 f_{2^j+k}(t)W_{j-1}(t)dt,$$

where the second term is equal to 0 because f_{2^j+k} is orthogonal to $W_{j-1}(t)$. Therefore we can write

$$|V_2| \leq \int_0^1 |f_{2^j+k}(t)|\min\left(2^j, \frac{1}{2^{2j}d(x,t)^3}\right) dt$$

$$\leq \int_{1*I_{j,l}} |f_{2^j+k}(t)|2^j dt + \int_{(1*I_{j,l})^c \cap \{t:\ \frac{1}{2}d(x,\frac{k}{2^j}) \leq d(x,t)\}} |f_{2^j+k}(t)| \frac{1}{2^{2j}(\max(\frac{1}{2^j}, \frac{1}{2}d(x,\frac{k}{2^j})))^3} dt$$

$$+ \int_{(1*I_{j,l})^c \cap \{t:\ \frac{1}{2}d(x,\frac{k}{2^j}) \leq d(\frac{k}{2^j},t)\}} |f_{2^j+k}(t)| \frac{1}{2^{2j}d(x,t)^3} dt.$$

Using the fact that $\|f_{2^j+k}\|_1 \leq C2^{-\frac{j}{2}}$ we get

$$|V_2| \leq C2^{\frac{j}{2}} q^{2^j d(x,\frac{k}{2^j})} + \frac{C}{(1+2^j d(x,\frac{k}{2^j}))^3} 2^{\frac{j}{2}} + C2^{\frac{j}{2}} q^{2^j d(x,\frac{k}{2^j})}.$$

Thus

$$|V_2| \leq 2^{\frac{j}{2}} S_1\left(2^j d\left(x, \frac{k}{2^j}\right)\right) + 2^{\frac{j}{2}} S_2\left(2^j d\left(x, \frac{k}{2^j}\right)\right),$$

where $S_1(x) = Cq^x$ for some $q \in (0,1)$, while $S_2(x) = C/(1+x)^3$, which finishes the proof of Theorem 3.3. ∎

3.4. *Periodic wavelets.* One of the methods of obtaining orthogonal systems on $\mathbb{T}$ is the following: We start from a multiresolution analysis on $\mathbb{R}$, with a scaling function φ and an associated wavelet ψ. Under the assumption $\psi \in L_1(\mathbb{R})$ we can define

$$\psi^{\circ}_{2^j+k}(x) = \sum_{l\in\mathbb{Z}} \psi_{j,k}(x-l).$$

It turns out that the system $\{1\} \cup \{\psi^{\circ}_n\}_{n\geq 1}$ is an orthonormal basis in $L_2(\mathbb{T})$ (see for instance [29]). Moreover, if

$$|\psi(x)| \leq \frac{C}{(1+|x|)^s}$$

for some $s > 1$, then the system $\{1\} \cup \{\psi^{\circ}_n\}_{n\geq 1}$ satisfies (I), (II), (III).

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