

EQUIVARIANT MORITA EQUIVALENCES BETWEEN PODLEŚ SPHERES

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Abstract. We show that the family of Podleś spheres is complete under equivariant Morita equivalence (with respect to the action of quantum $SU(2)$), and determine the associated orbits. We also give explicit formulas for the actions which are equivariantly Morita equivalent with the quantum projective plane. In both cases, the computations are made by examining the localized spectral decomposition of a generalized Casimir element.

Introduction. This paper is concerned with $SU_q(2)$, the quantum $SU(2)$ group, at real values $0 < q < 1$ ([18]). In [14], the $SU_q(2)$ -homogeneous spaces were classified which have the same spectral decomposition as the ordinary action of $SU(2)$ on the 2-sphere (and whose spin 1-part generates the algebra). They form a continuous one-parameter-family S_{qc}^2 , called *Podleś spheres*, and are indexed by a number $c \in [0, +\infty]$. In this paper, we give a classification with respect to a weaker equivalence relation, namely *equivariant Morita equivalence*. The notation we follow in the Introduction will be the one of [14]. (In the paper itself we will use a different notational convention which is more convenient for our purposes).

THEOREM 0.1. *Write*

$$c : [0, +\infty] \rightarrow [0, +\infty] : x \mapsto (q^{-x} - q^x)^{-2}.$$

Then

$$S_{qc(x)}^2 \underset{SU_q(2)\text{-Morita equiv.}}{\cong} S_{qc(y)}^2 \Leftrightarrow \exists m \in \mathbb{Z} \text{ with } y = |x + m|.$$

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Moreover, any quantum homogeneous space \mathbb{X} of $SU_q(2)$ which is equivariantly Morita equivalent with a Podleś sphere is itself a Podleś sphere.

REMARK. The moreover-part follows from the results of [17] and the classification in [14], but we will give an independent proof.

For the equatorial Podleś sphere $S_{q\infty}^2$, there is an $SU_q(2)$ -equivariant \mathbb{Z}_2 -symmetry, which allows us to form the quantum projective plane $\mathbb{R}P_q^2$ as an $SU_q(2)$ -homogeneous space (see e.g. [6]). The following theorem provides the classification of quantum homogeneous spaces which are $SU_q(2)$ -Morita equivalent with $\mathbb{R}P_q^2$.

THEOREM 0.2. For $l \in \frac{1}{2}\mathbb{N}_0$, let B_l be the unital $*$ -algebra generated by elements X, Z, Y and $4l + 1$ elements A_s , where

$$s \in \{-2l, -2l + 1, \dots, 2l - 1, 2l\},$$

satisfying the following relations:

$$\begin{cases} Y^* = X, \\ Z^* = Z, \\ A_s^* = (-1)^s A_{-s}, \\ \\ \begin{cases} XZ = q^2 ZX, \\ A_s Z = -q^{-2s} Z A_s, \end{cases} \\ \\ \begin{cases} X A_s = -A_{s-1}(1 + q^{2s+2l-1}Z) \text{ for } s > -2l, \\ X A_{-2l} = -A_{-2l} X, \\ X^* A_s = -A_{s+1}(1 - q^{2s-2l+1}Z) \text{ for } s < 2l, \\ X^* A_{2l} = -A_{2l} X^* \end{cases} \end{cases}$$

and

$$\left\{ \begin{array}{l} \begin{cases} X^* X = (1 - q^{2l-1}Z)(1 + q^{-2l-1}Z), \\ X X^* = (1 - q^{2l+1}Z)(1 + q^{-2l+1}Z), \end{cases} \\ \\ \begin{cases} A_s A_{s'} = (-1)^s X^{-(s+s')} (q^{2s'-2l+1}Z; q^2)_{s+2l} \\ \quad \times (-q^{-2l+1}Z; q^2)_{s'+2l} \quad \text{for } s + s' \leq 0, \\ = (-1)^s (X^*)^{s+s'} (q^{2s'-2l+1}Z; q^2)_{2l-s'} \\ \quad \times (-q^{2s+2s'-2l+1}Z; q^2)_{2l-s} \quad \text{for } s + s' \geq 0. \end{cases} \end{array} \right.$$

(In particular, the unital $*$ -algebra $\text{Alg}(X, Z, Y)$ generated by X, Z, Y is an isomorphic copy of the Podleś sphere at parameter $c(2l)$.)

Then we can define on B_l an ergodic action of $SU_q(2)$ which agrees with the usual action on the copy $\text{Alg}(X, Z, Y)$ of the Podleś sphere, and such that

$$\theta_{2l} = (q^{\frac{1}{2}s(s-1)} \frac{(q^{4l-2s+2}; q^2)_{s+2l}^{1/2}}{(q^2; q^2)_{s+2l}^{1/2}} A_s)_s \in B_l \otimes \mathbb{C}^{4l+1}$$

is a π_{2l} -eigenvector (where π_r for $r \in \frac{1}{2}\mathbb{N}$ denotes the spin r -representation of $SU_q(2)$, whose precise (infinitesimal) form can be found in the proof of Proposition 3.2).

If we denote by the formal symbol \mathbb{X}_l the quantum homogeneous space associated with the action on B_l , then a quantum homogeneous space \mathbb{X} of $SU_q(2)$ is equivariantly Morita equivalent with $\mathbb{R}P_q^2$ iff it is isomorphic to $\mathbb{R}P_q^2$ or one of the \mathbb{X}_l .

Classically (i.e. for $q = 1$), the B_l correspond to the inductions to $SU(2)$ of the actions $\text{Ad}(\gamma_l)$ of D_∞^* , where $D_\infty^* \subseteq SU(2)$ is the double cover of the infinite dihedral group $S^1 \rtimes \mathbb{Z}_2$, and with the γ_l denoting its 2-dimensional irreducible representations. Note that the B_l have a natural equivariant \mathbb{Z}_2 -gradation with X, Y, Z even and the A_s odd.

To prove these theorems, we will proceed as follows. In [2], we introduced a $*$ -algebra $U_q(-, +)$, equipped with a right $U_q(su(2))$ -module $*$ -algebra structure, where $U_q(su(2))$ is the quantum universal enveloping $*$ -algebra of $su(2)$. We showed that the Podleś spheres (for $c \neq 0$) can be realized as equivariant sub-quotients of this $*$ -algebra, by evaluation of a certain central and self-adjoint Casimir element. But as $U_q(-, +)$ also has a compatible $U_q(su(2))$ -co-module $*$ -algebra structure (more precisely, an $U_q(su(2))$ -Yetter-Drinfel'd $*$ -algebra structure), one can compose representations of $U_q(-, +)$ with ordinary representations of $U_q(su(2))$, and split these up into irreducibles (a classical method). From applying such a composition to the mentioned Casimir element, its decomposition can easily be deduced by a (trivial) spectral decomposition. From such a procedure we will then be able to prove Theorem 0.1. Also Theorem 0.2 will be proved in a similar fashion.

Let us remark that by [17], the computation of the orbit under \mathbb{G} -equivariant Morita equivalence for an ergodic action α on a unital C^* -algebra B can be found by studying the representation theory of $B \rtimes \mathbb{G}$. From this observation, it follows that our work will be directly connected with [16], where the infinitesimal version $\text{Pol}(S_{qc}^2) \rtimes U_q(su(2))$ is studied from a representation theoretic viewpoint. We will at the appropriate places remark where we make contact with [16], but on the whole our approach is a little different as we tend to work locally.

The contents of this paper are as follows.

After a section containing notational conventions, our *first section* introduces those quantum group concepts we will need in the paper. All the statements in this section are either well-known or easy to prove. In the *second section*, we prove Theorem 0.1, and as a corollary compute the equivariant Picard group for the Podleś spheres. In the *third section*, we prove Theorem 0.2.

Notations. In the remainder of the article, q will denote a real number strictly between 0 and 1. We then write

$$\lambda = (q - q^{-1})^{-1}.$$

We will also use a different parametrization τ of $[-\infty, +\infty]$, namely

$$\tau(x) = q^{-x} - q^x \quad \text{for } x \in [-\infty, +\infty].$$

All our vector spaces will be over the ground field \mathbb{C} . For V a vector space, we denote by $L(V)$ the space of linear endomorphisms of V , and by V° the space of linear functionals. If V is endowed with a Hilbert space structure \mathcal{H} , we denote by $B(\mathcal{H})$ the $*$ -algebra

of bounded operators. When we have a basis e_i of a vector space V , parametrized by a set I , then e_i is interpreted to be zero if $i \notin I$.

By \odot , we will denote the algebraic tensor product of two vector spaces or algebras over \mathbb{C} . By \otimes , we will denote the tensor product between Hilbert spaces, or the minimal tensor product between C^* -algebras. We will also use the leg notation for tensor products: for example, if we have spaces V_1, V_2, V_3 , and X an operator in $L(V_1 \odot V_3)$, we denote by X_{13} the operator on $V_1 \odot V_2 \odot V_3$ acting as X on the first and third component, and as the identity on the second component.

For $r \in \mathbb{N} \cup \{\infty\}$ and $a \in \mathbb{C}$, we denote by $(a; q)_r$ the q -factorial

$$(a; q)_r = \prod_{k=0}^{r-1} (1 - q^k a).$$

1. Preliminaries

1.1. Quantum groups. We will freely use the language of Hopf algebras, Hopf $*$ -algebras, and C^* -algebraic compact quantum groups (see e.g. [9]). For a Hopf algebra (H, Δ) , we will use Sweedler notation in the form

$$\Delta(h) = h_{(1)} \otimes h_{(2)} \quad \text{for } h \in H.$$

A C^* -algebraic compact quantum group will always be written in the form $(C(\mathbb{G}), \Delta)$, and we then refer to the symbol \mathbb{G} as ‘the compact quantum group’. The associated Hopf $*$ -algebra is written $\text{Pol}(\mathbb{G})$. Except for the preliminary section, we will only be interested in these objects for one particular quantum group, namely $\mathbb{G} = SU_q(2)$.

REMARK. To avoid overloading certain statements, we will in the remainder of this section always assume that \mathbb{G} is *co-amenable*, so that $C(\mathbb{G})$ is uniquely determined by $\text{Pol}(\mathbb{G})$.

The following easy lemma will be needed at a certain point. Let H be an algebra, and V a right H -module. We then denote by $V_{\text{fin}} \subseteq V$ the submodule of all *locally finite elements*, i.e.

$$V_{\text{fin}} = \{v \in V \mid \{v \cdot h \mid h \in H\} \text{ is finite-dimensional}\}.$$

LEMMA 1.1. *Let (H, Δ) be a Hopf algebra with invertible antipode, and let V and W be two right H -modules. Then*

$$(V \odot W)_{\text{fin}} = V_{\text{fin}} \odot W_{\text{fin}}.$$

We also make the following remark. Let (H, Δ) be a Hopf $(*)$ -algebra, and let A be a right module $(*)$ -algebra for (H, Δ) . (The compatibility with the $*$ -structure means that $(a \cdot h)^* = a^* \cdot S(h)^*$). Let V be a finite-dimensional vector space (resp. Hilbert space) with a left H -module structure by a $(*)$ -preserving unital homomorphism $\pi : H \rightarrow L(V)$. Then $A \odot L(V)$ can be made into a right module $(*)$ -algebra by the formula

$$(a \otimes x) \cdot h := (a \cdot h_{(2)}) \otimes \pi(S(h_{(1)}))x\pi(h_{(3)}).$$

If we are furthermore in the following situation:

- (K, Δ) is Hopf $(*)$ -algebra paired with (H, Δ) by a map $\iota_H : H \rightarrow K^\circ$ (with the compatibility

$$\iota_H(h^*)(k) = \overline{\iota_H(h)(S(k)^*)}$$

in the $*$ -case),

- the module $(*)$ -algebra structure on A is induced from a left comodule $(*)$ -algebra structure α of K on A , and
- π is induced from a (unitary) corepresentation $U \in K \odot L(V)$,

then

$$(a \otimes x) \cdot h = (\iota_H(h) \otimes \iota \otimes \iota)(U_{13}^{-1}(\alpha(a) \otimes x)U_{13}).$$

Also, in the general case, the module $A \odot L(V)$ is isomorphic to $V \odot A \odot V^\circ$ with the tensor module structure (where V now carries the right H -module structure $v \cdot h := \pi(S(h))v$, and with V° endowed with the right module structure $\omega \cdot h := \omega(\pi(h) \cdot)$).

1.2. Coactions. We begin with the following remark on terminology. We will use the equivalent notions of *(co)module algebra* and *(co)action*, whenever one of them is more convenient. In the C^* -algebra context, we will always assume that the co-unit condition is satisfied.

Our next remarks concern *ergodic* coactions. We call a left coaction α of a Hopf algebra on a unital algebra B *ergodic* if the identity $\alpha(b) = 1 \otimes b$ for some $b \in B$ implies that $b \in \mathbb{C}1$. If α is an ergodic coaction of a C^* -algebraic compact quantum group $(C(\mathbb{G}), \Delta)$ on a unital C^* -algebra B , we will write $B = C(\mathbb{X})$ for some formal symbol \mathbb{X} , and call it a ‘ \mathbb{G} -homogeneous space’. We then denote by $\text{Pol}(\mathbb{X})$ the linear span of the finite-dimensional spectral subspaces of $C(\mathbb{X})$. It is a $*$ -algebra carrying a natural coaction of $\text{Pol}(\mathbb{G})$ by restricting α . One also has a (unique) invariant (and faithful) state $\varphi_{\mathbb{X}}$ on $C(\mathbb{X})$, obtained by integrating out the coaction (so $\varphi_{\mathbb{X}}(x)1_{C(\mathbb{X})} = (\iota \otimes \varphi_{\mathbb{G}})\alpha(x)$ for all $x \in C(\mathbb{X})$, where $\varphi_{\mathbb{G}}$ is the invariant state on $C(\mathbb{G})$). Note that $C(\mathbb{X})$ is completely determined by $\text{Pol}(\mathbb{X})$, by our co-amenability assumption on \mathbb{G} (see [11], Proposition 3.8).

The following result by F. Boca ([1]) is fundamental.

THEOREM 1.2. *Let \mathbb{X} be a homogeneous space for a compact quantum group \mathbb{G} . Then any irreducible representation of \mathbb{G} appears in $C(\mathbb{X})$ with only finite multiplicity.*

The following lemma will also be used at some point.

LEMMA 1.3. *Let \mathbb{G} be a compact quantum group, \mathcal{H} a Hilbert space, and let $B \subseteq B(\mathcal{H})$ be a (not necessarily closed) unital sub- $*$ -algebra with a coaction α_B by $\text{Pol}(\mathbb{G})$. Assume that there exists a normal state ω in $B(\mathcal{H})_*$ whose restriction to B is faithful and α_B -invariant. Then if $A \subseteq B$ is a unital sub- $*$ -algebra for which*

- α_B restricts to an ergodic coaction of $\text{Pol}(\mathbb{G})$ on A , and
- the weak closures of A and B coincide,

then $A = B$.

Proof. Suppose that $B \neq A$. We may then take an irreducible representation π of \mathbb{G} and a non-zero element $x \in B_\pi$, the spectral subspace for π in B , such that $x \notin A$. As A_π is finite-dimensional by Boca’s theorem, we may moreover assume that x is orthogonal to

A_π , and hence to A (where A is equipped with the pre-Hilbert space structure $\langle a', a \rangle := \omega(a^*a')$). But as ω is normal, we would then get $\omega(xy) = 0$ for all $y \in A'' = B''$. Clearly this gives a contradiction with the faithfulness of ω . ■

1.3. Morita equivalence for coactions. Let α_i be left coactions of $C(\mathbb{G})$ on unital C^* -algebras B_i . One says the B_i are \mathbb{G} -Morita equivalent if there exists a unital C^* -algebra E with a left coaction α , together with a \mathbb{G} -invariant self-adjoint projection e , such that, denoting $e_1 = e$ and $e_2 = 1 - e$, we have that Ee_1E and Ee_2E are norm-dense in E , and $e_i E e_i \cong B_i$ by a \mathbb{G} -covariant isomorphism. Alternatively, it is more common to define the B_i to be \mathbb{G} -Morita equivalent if there exists an equivariant B_1 - B_2 -equivalence Hilbert bimodule (see e.g. the remark after Theorem 2.5 in [13]). The equivalence of the latter definition with the above ‘linking algebra’ picture is well-known and easily proven. It is also easily shown that \mathbb{G} -Morita equivalence is indeed an equivalence relation.

If the α_i are ergodic, and we write $B_i = C(\mathbb{X}_i)$, we will also call the \mathbb{X}_i themselves \mathbb{G} -Morita equivalent.

The following results can be deduced from the ones in section 4 of [17].

PROPOSITION 1.4. *Let the $\mathbb{X}_1, \mathbb{X}_2$ be two \mathbb{G} -homogeneous quantum spaces. Then the following are equivalent.*

- *The \mathbb{X}_i are \mathbb{G} -Morita equivalent.*
- *There exists a finite-dimensional unitary corepresentation U of $C(\mathbb{G})$ on a Hilbert space \mathcal{H} and a \mathbb{G} -invariant projection $p \in C(\mathbb{X}_2) \otimes B(\mathcal{H})$ such that*

$$C(\mathbb{X}_1) \cong p(C(\mathbb{X}_2) \otimes B(\mathcal{H}))p$$

by a \mathbb{G} -equivariant isomorphism.

Here $C(\mathbb{X}_2) \otimes B(\mathcal{H})$ is, as before, equipped with the coaction

$$x \mapsto U_{13}^*(\alpha \otimes \iota)(x)U_{13}.$$

To prove \Rightarrow , take an equivariant equivalence Hilbert bimodule $(\mathcal{E}, \alpha_{\mathcal{E}})$ between $C(\mathbb{X}_1)$ and $C(\mathbb{X}_2)$, a suitable unitary left corepresentation U of $C(\mathbb{G})$ on a Hilbert space \mathcal{H} , and any finite set of non-zero elements $\xi_i \in \mathcal{E}$ such that $\alpha_{\mathcal{E}}(\xi_i) = \sum_j U_{ij}^* \otimes \xi_j$. Then using the ergodicity of the action of \mathbb{X}_1 , one shows that, possibly up to a non-zero scalar, the map

$$\mathcal{E} \rightarrow C(\mathbb{X}_2) \otimes \mathcal{H}_U : \xi \mapsto \sum_i \langle \xi, \xi_i \rangle_{C(\mathbb{X}_2) \otimes e_i}$$

is a \mathbb{G} -equivariant isometry between $C(\mathbb{X}_2)$ -Hilbert modules, where the range is equipped with the coaction $x \mapsto U_{13}^*(\alpha \otimes \iota)(x)$, and where the e_i form a basis of \mathcal{H} which gives rise to the matrix coefficients U_{ij} of U .

To prove \Leftarrow , the essential point to prove is that for any \mathbb{G} -invariant projection p , the Hilbert module $p(C(\mathbb{X}_2) \otimes \mathcal{H})$ is still full (cf. [17], Lemma 4.5). This will follow from the fact that $(\iota \otimes \omega)(p) \in C(\mathbb{X}_2)$ is invariant for a well-chosen faithful state $\omega \in B(\mathcal{H})_*$ (namely an invariant functional for the action $x \mapsto U(1 \otimes x)U^*$ by $(\text{Pol}(\mathbb{G}), \Delta^{\text{op}})$).

The following lemma will allow us to determine Morita equivalences by an inductive process.

LEMMA 1.5. *Let π_1, \dots, π_n be a generating set of irreducible representations of a compact quantum group \mathbb{G} (i.e. any irreducible representation of \mathbb{G} is contained in some power of $\oplus \pi_i$). Let \mathbb{X}_1 and \mathbb{X}_2 be two \mathbb{G} -homogeneous spaces. Then \mathbb{X}_1 and \mathbb{X}_2 are Morita equivalent iff there exists a finite set of \mathbb{G} -homogeneous spaces $\mathbb{Y}_1, \dots, \mathbb{Y}_m$ with*

- $\mathbb{Y}_1 \cong \mathbb{X}_1$ and $\mathbb{Y}_m \cong \mathbb{X}_2$,
- for each $k \in \{1, 2, \dots, m-1\}$, there exists an $i \in \{1, 2, \dots, n\}$ and a minimal \mathbb{G} -invariant projection $p \in C(\mathbb{Y}_k) \otimes B(\mathcal{H}_{\pi_i})$ such that

$$C(\mathbb{Y}_{k+1}) \cong p(C(\mathbb{Y}_k) \otimes B(\mathcal{H}_{\pi_i}))p.$$

The proof is based on the previous proposition and two basic observations:

- If

$$C(\mathbb{Y}_1) = p_1(C(\mathbb{Y}_2) \otimes B(\mathcal{H}_{\pi_1}))p_1$$

and

$$C(\mathbb{Y}_2) = p_2(C(\mathbb{Y}_3) \otimes B(\mathcal{H}_{\pi_2}))p_2,$$

then with $p_3 = p_1(p_2 \otimes 1) = (p_2 \otimes 1)p_1$ we have

$$C(\mathbb{Y}_1) = p_3(C(\mathbb{Y}_3) \otimes B(\mathcal{H}_{\pi_2 \otimes \pi_1}))p_3.$$

- If $\pi_1 \subseteq \pi_2$ with corresponding projection $p : \mathcal{H}_{\pi_2} \rightarrow \mathcal{H}_{\pi_1}$, then

$$C(\mathbb{Y}) \otimes B(\mathcal{H}_{\pi_1}) = (1 \otimes p)(C(\mathbb{Y}) \otimes B(\mathcal{H}_{\pi_2}))(1 \otimes p).$$

Note that the above two results also (and more naturally) apply to the associated irreducible equivariant $C(\mathbb{X})$ -Hilbert modules, i.e. any irreducible equivariant Hilbert $C(\mathbb{X})$ -module appears as a component in some $C(\mathbb{X}) \otimes \mathcal{H}_\pi$ for π a finite-dimensional representation.

PROPOSITION 1.6. *Let \mathbb{G} be a compact quantum group, and \mathbb{H} a quantum subgroup. Then we can form the \mathbb{G} -homogeneous quantum space $\mathbb{X} = \mathbb{H} \setminus \mathbb{G}$. Any \mathbb{G} -Morita equivalent homogeneous quantum space is then obtained by taking an irreducible unitary corepresentation U of $C(\mathbb{H})$ on a Hilbert space \mathcal{H} , and inducing the associated \mathbb{H} -action on $B(\mathcal{H})$ to \mathbb{G} .*

The proof can be based on the isomorphisms

$$K_0^{\mathbb{G}}(C(\mathbb{H} \setminus \mathbb{G})) \cong K_0(C(\mathbb{H} \setminus \mathbb{G}) \rtimes \mathbb{G}) \cong K_0(C^*(\mathbb{H})),$$

the first isomorphism being the Green-Julg isomorphism which is valid for any action.

1.4. Galois objects

DEFINITION 1.7 ([15]). Let (H, Δ) be a Hopf $(*)$ -algebra, A a unital $(*)$ -algebra, and α a right coaction of (H, Δ) on A . Denote

$$B = \{a \in A \mid \alpha(a) = a \otimes 1\},$$

the fixed point algebra. One says α is Galois if the *Galois map*

$$G : A \underset{B}{\odot} A \rightarrow A \underset{B}{\odot} H : a \otimes a' \mapsto (a \otimes 1)\alpha(a')$$

is bijective.

One says (A, α) is a *Galois object* if α is ergodic (i.e. $B = \mathbb{C}$).

For a Galois object, we write S for the canonical anti-isomorphism $A^{\text{op}} \rightarrow A : a^{\text{op}} \mapsto a$, and denote

$$h_{[1]} \otimes h_{[2]} := (S^{-1} \otimes \iota)(G^{-1}(1 \otimes h)) \in A^{\text{op}} \odot A.$$

The mapping $h \mapsto h_{[1]} \otimes h_{[2]}$ is then a unital homomorphism. As for H itself, one can make A into a right H -module $(*)$ -algebra by means of the *Miyashita-Ulbrich (or adjoint) action*

$$a \triangleleft h := S(h_{[1]})ah_{[2]}.$$

Then $(A, \alpha, \triangleleft)$ is a right Yetter-Drinfel'd module (see [3], or Lemma 2.9 of [15]).

A trivial example of a Galois object is obtained by taking (an isomorphic copy of) the Hopf $(*)$ -algebra itself, with the coaction being the comultiplication. In fact, later on we will be mainly concerned with a particular Galois object for a Hopf $(*)$ -algebra which becomes trivial when forgetting the $(*)$ -structure.

We record the following fact for later use.

LEMMA 1.8. *Let (A, α) be a Galois object for the Hopf $(*)$ -algebra (H, Δ) . Let B be an arbitrary unital $(*)$ -algebra, and let $\pi_B : A \rightarrow B$ be a unital $(*)$ -homomorphism. Then:*

1. *There exists a right H -module $(*)$ -algebra structure on B , determined by*

$$b \triangleleft h = \pi_B(S(h_{[1]}))b\pi_B(h_{[2]}).$$

2. *If (V, π) is a finite dimensional unitary left H -module, then*

$$(\pi_B \otimes \pi)\alpha : A \rightarrow B \odot L(V)$$

is a morphism between right H -module $()$ -algebras. In fact, the module $(*)$ -algebra structure on $B \odot L(V)$ (as at the end of Section 1.1) coincides with the one induced by this $(*)$ -homomorphism as in the first point.*

Proof. The first fact can be proven as in the Hopf $(*)$ -algebra case (see e.g. [9], Lemma 5.5). For the first part of the second fact, use that $(A, \alpha, \triangleleft)$ is a right Yetter-Drinfel'd module. For the second part, the following identity for $h \in H$ will imply the claim:

$$S(h_{(2)[1]}) \otimes h_{(2)[2]} \otimes S(h_{(1)}) \otimes h_{(3)} = S(h_{[1]})_{(0)} \otimes h_{[2](0)} \otimes S(h_{[1]})_{(1)} \otimes h_{[2](1)}.$$

To prove this formula, apply e.g. the identities (2.1.3) and (2.1.2) from Lemma 2.1.7 of [15] to the right hand side. ■

Also the following result will be needed at some point, although only in a very simple situation.

PROPOSITION 1.9. *Let \mathbb{G} be a compact quantum group, B a unital C^* -algebra equipped with an action by \mathbb{G} , and H a finite group (or even quantum group) which has a \mathbb{G} -equivariant Galois action on B . Then B^H and $H \rtimes B$ are \mathbb{G} -equivariantly Morita equivalent.*

Indeed, by a well-known theorem concerning Galois extensions ([10]), we have that $H \rtimes B \cong \text{End}_{B^H}(B)$ by the natural homomorphism (where B is considered just as a right B^H -module on the right hand side). This identification is compatible with the $(*)$ -structure and the \mathbb{G} -action, by assumption, leading to the stated equivariant Morita equivalence.

Restating the proposition in the form we will need it in, the above says that, under the given conditions,

$$p := \frac{1}{|H|} \sum_{h \in H} \lambda_h \in H \rtimes B$$

will be a full \mathbb{G} -invariant projection. If moreover H is abelian, and χ a character, then of course also

$$p_\chi = \frac{1}{|H|} \sum_{h \in H} \chi(h) \lambda_h \in H \rtimes B$$

is full, with

$$p(H \rtimes B)p \cong p_\chi(H \rtimes B)p_\chi \cong B^H$$

equivariantly.

1.5. Quantized universal enveloping algebras

DEFINITION 1.10. We denote by $U_q(su(2))$ the quantized universal enveloping $*$ -algebra of $su(2)$. It is the unital algebra generated by elements E, F, K, K^{-1} , with commutation relations $KE = q^2EK$, $KF = q^{-2}FK$, $KK^{-1} = 1 = K^{-1}K$ and

$$[E, F] = \lambda(K - K^{-1}),$$

where we recall that $\lambda = (q - q^{-1})^{-1}$. The $*$ -operation is determined by $E^* = K^{-1}F$ and $K^* = K$.

We can equip $U_q(su(2))$ with the unital $*$ -homomorphism

$$\Delta : U_q(su(2)) \rightarrow U_q(su(2)) \odot U_q(su(2)),$$

uniquely determined by the fact that $\Delta(K) = K \otimes K$ and

$$\begin{aligned} \Delta(E) &= E \otimes 1 + K^{-1} \otimes E, \\ \Delta(F) &= F \otimes K + 1 \otimes F. \end{aligned}$$

The couple $(U_q(su(2)), \Delta)$ then forms a Hopf $*$ -algebra.

DEFINITION 1.11. We denote by $U_q(-, +)$ the $*$ -algebra which, as an algebra, is generated by elements X, Y, Z, Z^{-1}, T with commutation relations $XZ = q^2ZX$, $YZ = q^{-2}ZY$, $Z^{-1}Z = 1 = ZZ^{-1}$ and

$$\begin{cases} YX = 1 + q^{-1}TZ - q^{-2}Z^2, \\ XY = 1 + q TZ - q^2 Z^2. \end{cases}$$

The $*$ -structure is uniquely determined by the formulas $X^* = Y$, $Z^* = Z$.

Note that T can be expressed in terms of X, Y and $Z^{\pm 1}$. Then T can be shown to be central and self-adjoint. It is interpreted as the *Casimir element* of $U_q(-, +)$.

REMARK. It is easily shown that $U_q(-, +)$ coincides (after introducing a square root of Z) with the $*$ -algebra $\hat{\mathcal{Y}}_c$ from [16], section 5 for $c \neq 0$ (the c can then be removed by rescaling the parameters). It is also the same $*$ -algebra (again after adjoining a square root of Z) as the one denoted by the corresponding symbol in [2].

PROPOSITION 1.12. *The $*$ -algebra $U_q(-, +)$ can be made into a $(U_q(\mathfrak{su}(2)), \Delta)$ -Galois object by the coaction α , defined on the generators X, Y, Z, T by*

$$\begin{aligned}\alpha(Z) &= Z \otimes K^{-1}, \\ \alpha(X) &= X \otimes 1 + Z \otimes (q^{-1/2}\lambda^{-1}E), \\ \alpha(Y) &= Y \otimes 1 + Z \otimes (q^{-1/2}\lambda^{-1}K^{-1}F), \\ \alpha(T) &= T \otimes K + Z \otimes (\lambda^{-2}FE - q^{-1}(K - K^{-1})) \\ &\quad + X \otimes (q^{1/2}\lambda^{-1}F) + Y \otimes (q^{1/2}\lambda^{-1}EK).\end{aligned}$$

This fact can be shown as follows: if we forget the $*$ -structure, then $U_q(-, +)$ is an isomorphic copy of $U_q(\mathfrak{su}(2))$ by the following identifications:

$$\begin{aligned}X &\leftrightarrow iq^{-\frac{1}{2}}\lambda^{-1}E, \\ Y &\leftrightarrow iq^{-\frac{1}{2}}\lambda^{-1}K^{-1}F, \\ Z &\leftrightarrow iK^{-1}, \\ T &\leftrightarrow i(\lambda^{-2}EF + qK^{-1} + q^{-1}K).\end{aligned}$$

The above coaction is then just the comultiplication of $U_q(\mathfrak{su}(2))$, which shows the Galois map is an isomorphism. One should then check the compatibility with the $*$ -operation separately, but this is clear on sight. One could say that we have made a ‘Wick rotation’ for one Borel subalgebra (generated by E and K), but left the remaining part (generated by F) unaltered. This will explain why we will get unilaterally infinite-dimensional representations of our $*$ -algebra later on. Also note that via the above isomorphism, T is identified with an imaginary scalar multiple of the Casimir element of $U_q(\mathfrak{su}(2))$, but for the new $*$ -structure it is self-adjoint.

Hence, as we have already mentioned, the above Galois object is thus trivial when forgetting the $*$ -structure. Alternatively, one can consider the real version of $U_q^{\mathbb{R}}(\mathfrak{su}(2))$, i.e. the Hopf \mathbb{R} -algebra generated by E, F and K , and similarly consider the real version $U_q^{\mathbb{R}}(-, +)$. Then $U_q^{\mathbb{R}}(-, +)$ is a non-trivial Galois object for $U_q^{\mathbb{R}}(\mathfrak{su}(2))$, but becomes trivialized after complexification.

Let us also remark how the way in which $\hat{\mathcal{Y}}_c$ appears in [16] might lead one to think it could be a Galois object. For $\hat{\mathcal{Y}}_c$ can be seen as the relative commutant (or centralizer) of $\text{Pol}(S_{qc}^2)$ inside $\text{Pol}(S_{qc}^2) \rtimes U_q(\mathfrak{su}(2))$. It is easy to check that a dual coaction on a smash (or crossed) product always restricts to the centralizer of the copy of the original algebra, so that we deduce from the above that $\hat{\mathcal{Y}}_c$ will indeed be a right $U_q(\mathfrak{su}(2))$ -comodule $*$ -algebra. Now the dual coaction on a smash product is always a Galois coaction. So one might naively believe that the restriction to the centralizer will then also be Galois, but this is not true in general. However, in the present case $\hat{\mathcal{Y}}_c$ splits of as a tensor product ([4]), and by this fortuitous instance the restricted coaction *does* become a Galois object. To illustrate the subtleness of this situation, we mention that the associated analytic result is *not* true: the relative commutant of $\mathcal{L}^\infty(S_{qc}^2)$ inside $\mathcal{L}^\infty(S_{qc}^2) \rtimes SU_q(2)$ *does not* become a Galois object (or even a Galois action) for $\mathcal{L}(SU_q(2))$ (the analytic version of $U_q(\mathfrak{su}(2))$). However, one can remedy this situation in another way, and we will come back to this in future work.

As $U_q(-, +) \cong U_q(su(2))$ when the $*$ -structure is ignored, we can deduce the following result from [8]. Recall the terminology ‘locally finite’ introduced above Lemma 1.1.

PROPOSITION 1.13. *The locally finite elements of $U_q(-, +)$ w.r.t. the adjoint action form a unital $*$ -algebra $U_q^{\text{fin}}(-, +)$ which is generated by X, Y, Z and T .*

Indeed, it follows from [8] that $U_q^{\text{fin}} = \{Z^m \mid m \in \mathbb{N}\} \triangleleft U_q(su(2))$, so the foregoing proposition follows from the easily verified fact that $\text{span}\{X, Y, Z, T\}$ is closed under the action by $U_q(su(2))$ (using for example the explicit formula for this action below). One may also easily verify that $U_q^{\text{fin}}(-, +)$ can be represented by generators and relations in the same way as $U_q(-, +)$.

Let us present the concrete formulas for the adjoint action on $U_q(-, +)$ by $U_q(su(2))$. If $b \in U_q(-, +)$, then

$$\begin{aligned} b \triangleleft K &= ZbZ^{-1}, \\ b \triangleleft (q^{-1/2}\lambda^{-1}E) &= Z^{-1}[b, X], \\ b \triangleleft (q^{3/2}\lambda^{-1}F) &= [b, Y]Z^{-1}. \end{aligned}$$

It is also apparent from the formula for α that there exists an α -equivariant (and hence \triangleleft -equivariant) $*$ -automorphism σ of $U_q(-, +)$, determined by

$$\sigma : U_q(-, +) \rightarrow U_q(-, +) : b \mapsto -b \quad \text{for } b \in \{X, Z, Y, T\}.$$

1.6. The compact quantum group $SU_q(2)$. We will not need to know the explicit form of $C(SU_q(2))$ or even $\text{Pol}(SU_q(2))$, and therefore simply recall from e.g. [9], section 4.4 that there exists a non-degenerate pairing between $\text{Pol}(SU_q(2))$ and $U_q(su(2))$. One then has the following result.

PROPOSITION 1.14. *There is a one-to-one-correspondence between the following structures:*

- *Left coactions of $(C(SU_q(2)), \Delta)$ with a finite-dimensional space of invariant elements.*
- *Right module $*$ -algebras A for $U_q(su(2))$ such that*
 - *$A_{\text{fin}} = A$.*
 - *All eigenvalues for the action of K are positive.*
 - *The space of $a \in A$ with $a \cdot g = \varepsilon(g)a$ for all $g \in U_q(su(2))$ is finite-dimensional.*
 - *There exists a faithful unital $*$ -homomorphism of A into a unital C^* -algebra.*

1.7. Podleś spheres. *Warning:* For notational reasons, we will follow a slightly different convention than the more common one used in the Introduction: we will use the index $\tau(x) = q^{-x} - q^x$ instead of $c(x) = \tau(x)^{-2}$. Note that τ is an odd function.

DEFINITION 1.15. Let $x \in (-\infty, +\infty)$, and denote $\tau = \tau(x)$. The $*$ -algebra $\text{Pol}(S_{q\tau}^2)$ is generated by three elements X_τ, Z_τ, Y_τ with $X_\tau^* = Y_\tau$, $Z_\tau^* = Z_\tau$, $X_\tau Z_\tau = q^2 Z_\tau X_\tau$, $Y_\tau Z_\tau = q^{-2} Z_\tau Y_\tau$, and with

$$\begin{cases} Y_\tau X_\tau = (1 - q^{x-1} Z_\tau)(1 + q^{-x-1} Z_\tau), \\ X_\tau Y_\tau = (1 - q^{x+1} Z_\tau)(1 + q^{-x+1} Z_\tau). \end{cases}$$

It carries a right $U_q(\mathfrak{su}(2))$ -module $*$ -algebra structure, induced from a left $\text{Pol}(SU_q(2))$ -coaction, for which the span of the $1, X_\tau, Z_\tau, Y_\tau$ is a direct sum of the trivial and the spin 1-representation of $SU_q(2)$. The corresponding action of $SU_q(2)$ is then ergodic.

We call the symbol $S_{q\tau}^2$ the *Podleš sphere at parameter τ* . When $\tau = 0$, we call it the *equatorial Podleš sphere*.

REMARKS. 1. One also has the *standard Podleš sphere* $S_{q\infty}^2 \cong SU_q(2)/S^1$. As it is degenerate from our point of view, we will treat it separately later on.

2. There is an equivariant $*$ -isomorphism σ_τ from $\text{Pol}(S_{q\tau}^2)$ to $\text{Pol}(S_{q,-\tau}^2)$ sending b_τ to $-b_{-\tau}$ for $b \in \{X, Z, Y\}$. Hence up to isomorphism, $\text{Pol}(S_{q,\tau}^2)$ only depends on $|\tau|$, and we could parametrize Podleš spheres by $c = \frac{1}{\tau^2}$, the convention we used in the Introduction. For the purposes of the article, it will be more convenient *not* to identify such Podleš spheres immediately. For example, on the equatorial Podleš sphere we get in particular an involutive equivariant automorphism σ_0 , which plays an important rôle in the theory.

The following was proven in [2], and can be verified directly. We will denote by $\text{Pol}^{\text{ext}}(S_{q\tau}^2)$ the $*$ -algebra which is obtained by adjoining to $\text{Pol}(S_{q\tau}^2)$ an inverse of Z (which clearly does not introduce additional relations).

PROPOSITION 1.16. *There is a $U_q(\mathfrak{su}(2))$ -equivariant unital $*$ -homomorphism*

$$\pi_\tau : U_q(-, +) \rightarrow \text{Pol}^{\text{ext}}(S_{q\tau}^2),$$

induced by sending a generator $b \in \{X, Z, Y\}$ to the corresponding element b_τ . The kernel of this homomorphism is generated by the element $T - \tau$. Under this morphism, $U_q^{\text{fin}}(-, +)$ is sent to $\text{Pol}(S_{q\tau}^2)$.

REMARKS. • We note that also the standard Podleš sphere can be obtained in a similar manner, using instead the $*$ -algebra $U_q(0, +)$ from [2].

- From the observations in Section 1.5, it follows that the action of $U_q(\mathfrak{su}(2))$ on the (localized) Podleš sphere is inner. This was also observed in [16].
- The isomorphisms σ_τ mentioned before the proposition are seen to be induced from the automorphism σ at the end of Section 1.5.

The following result gives a classification of all irreducible $*$ -representations of $\text{Pol}(S_{q\tau}^2)$ (see [14]).

PROPOSITION 1.17. *Any irreducible $*$ -representation of $\text{Pol}(S_{q\tau(x)}^2)$ on a Hilbert space is either*

- *faithful, in which case it is isomorphic to one of the following two $*$ -representations π_\pm on $l^2(\mathbb{N})$:*

$$\begin{cases} Z_{\tau(x)} \mapsto Z_{\tau(x), \pm} : & e_k \mapsto \pm q^{2k \mp x + 1} e_k, \\ X_{\tau(x)} \mapsto X_{\tau(x), \pm} : & e_k \mapsto \pm (1 - q^{2k})^{1/2} (1 + q^{2k \mp 2x})^{1/2} e_{k-1}. \end{cases}$$

- *one-dimensional, by sending Z_τ to zero and X_τ to a complex number of modulus 1.*

Note that the above also classifies all irreducible representations of $U_q^{\text{fin}}(-, +)$, which were computed in [16]. It is obvious what is meant then by the *-representation

$$\pi_{\tau, \pm} : U_q^{\text{fin}}(-, +) \rightarrow B(l^2(\mathbb{N})).$$

The equality

$$\pi_{\tau, -} = \pi_{-\tau, +} \circ \sigma$$

is easily observed. If we consider the pre-Hilbert space $V = \mathbb{C}[\mathbb{N}]$ with its natural orthonormal basis e_k , we can represent $\text{Pol}^{\text{ext}}(S_{q\tau}^2)$ (and $U_q(-, +)$) as a *-algebra of adjointable endomorphisms of V (i.e. banded operators) by the same formulas as the one in the foregoing proposition. To avoid overloading the notation, we will make no distinction between an element in $\pi_{\pm}(\text{Pol}(S_{q\tau}^2))$ seen as an operator on $l^2(\mathbb{N})$ or its restriction to V .

We end by introducing some extra notational conventions.

NOTATION 1.18. For $b \in \{X, Y, Z\}$, we will identify b_{τ} with the operator

$$b_{\tau} := b_{\tau, -} \oplus b_{\tau, +} \in L(V \oplus V),$$

and we write \mathbf{b}_{τ} for the operator

$$\mathbf{b}_{\tau} := b_{-\tau, +} \oplus b_{\tau, +} \in L(V \oplus V).$$

We write

$$\text{Pol}_+(S_{q\tau}^2), \text{Pol}_-(S_{q\tau}^2), \text{Pol}(S_{q\tau}^2), \text{Pol}_a(S_{q\tau}^2)$$

for the images of $U_q^{\text{fin}}(-, +)$ under the respective *-representations $\pi_{\tau, \pm}, \pi_{\tau} = \pi_{\tau, -} \oplus \pi_{\tau, +}$ and $\pi_{\tau, a} = \pi_{-\tau, +} \oplus \pi_{\tau, +}$, all of whose images are isomorphic.

2. Equivariant Morita equivalences for the Podleś spheres. In this section, we will prove Theorem 0.1. We will fix $x \in (-\infty, +\infty)$, and continue to write $\tau := \tau(x) = q^{-x} - q^x$.

NOTATION 2.1. For $w \in \{+, -, , a\}$, we consider $\text{Pol}_w^{\text{ext}}(S_{q\tau}^2) \odot M_2(\mathbb{C})$ with the right $U_q(\mathfrak{su}(2))$ -module *-algebra structure as at the end of Section 1.1, using on \mathbb{C}^2 the spin 1/2-representation $\pi_{1/2}$. We then denote by $\pi_{\tau, w}^{(2)}$ the morphism from $U_q(-, +)$ to $\text{Pol}_w^{\text{ext}}(S_{q\tau}^2) \odot M_2(\mathbb{C})$ as in Lemma 1.8.2.

We denote by $\{e_+, e_-\}$ the canonical basis of \mathbb{C}^2 , so that $U_q(\mathfrak{su}(2))$ acts by

$$\begin{cases} Ke_{\pm} = q^{\mp 1}e_{\pm}, \\ Ee_{\pm} = q^{1/2}\delta_{\pm, +}e_{-}, \\ Fe_{\pm} = q^{-1/2}\delta_{\pm, -}e_{+}. \end{cases}$$

We further denote the product basis elements $e_k \otimes e_{\pm}$ of $l^2(\mathbb{N}) \otimes \mathbb{C}^2$ as $e_{k, \pm}$.

PROPOSITION 2.2. *The self-adjoint operator $T_{\tau, +}^{(2)} := \pi_{\tau, +}^{(2)}(T)$ is bounded, its spectrum consisting of two eigenvalues. Moreover, it is an invariant element in $\text{Pol}_+(S_{q\tau}^2) \odot M_2(\mathbb{C})$.*

Proof. First of all, it is clear that $T_{\tau, +}^{(2)}$ will be invariant, as T is invariant for the adjoint action \triangleleft (it is a central element of $U_q(-, +)$), and $\pi_{\tau, +}^{(2)}$ is equivariant. Then $T_{\tau, +}^{(2)} \in \text{Pol}_+(S_{q\tau}^2) \odot M_2(\mathbb{C})$ by Lemma 1.1, the remark after it and Proposition 1.16, using the expression for $\alpha(T)$ given in Proposition 1.12.

Next, a straightforward computation shows that $T_{\tau,+}^{(2)}$ preserves the span of the 2-element set $\{e_{k,+}, e_{k+1,-}\}$ for $k \geq 0$, with the resulting 2-by-2-matrix being given by

$$\begin{aligned} (T_{\tau,+}^{(2)})_{+,+} &= q^{-1}\tau - (q^{-1} - q)q^{2k-x+2}, \\ (T_{\tau,+}^{(2)})_{+,-} &= \lambda^{-1}(1 - q^{2k+2})^{1/2}(1 + q^{2k-2x+2})^{1/2}, \\ (T_{\tau,+}^{(2)})_{-,+} &= \lambda^{-1}(1 - q^{2k+2})^{1/2}(1 + q^{2k-2x+2})^{1/2}, \\ (T_{\tau,+}^{(2)})_{-,-} &= q\tau + (q^{-1} - q)q^{2k-x+2}. \end{aligned}$$

(The remaining vector $e_{0,-}$ is an eigenvector, with eigenvalue the right lower corner of the above matrix with $k = -1$).

We find that the eigenvalues of these matrices are $\tau(x+1) = q^{-(x+1)} - q^{x+1}$ and $\tau(x-1) = q^{-(x-1)} - q^{x-1}$, and in particular are independent of k . This proves that $T_{\tau,+}^{(2)}$ has precisely two eigenvalues. ■

REMARK. Note that the eigenvalues of $T_{\tau,+}^{(2)}$ naturally appear as differences of q -powers, in contrast with the classical Casimir element of $U_q(\mathfrak{su}(2))$ whose eigenvalues are *sums* of q -powers.

As we will need it in the next section, we write down a basis of orthogonal eigenvectors for $T_{\tau,+}^{(2)}$.

LEMMA 2.3. *An orthonormal set of eigenvectors for $T_{\tau,+}^{(2)}$ at eigenvalue $\tau(x \pm 1)$ is given by the $\xi_{k,+}^{\tau(x \pm 1)}$, where $k \in \mathbb{N}$ and*

$$\begin{aligned} (1 + q^{2x})^{1/2} \cdot \xi_{k,+}^{\tau(x+1)} &= (e_{k-1,+} \quad e_{k,-}) \cdot \begin{pmatrix} -(1 - q^{2k})^{1/2} \\ q^x(1 + q^{2k-2x})^{1/2} \end{pmatrix}, \\ (1 + q^{2x})^{1/2} \cdot \xi_{k,+}^{\tau(x-1)} &= (e_{k,+} \quad e_{k+1,-}) \cdot \begin{pmatrix} q^x(1 + q^{2k-2x+2})^{1/2} \\ (1 - q^{2k+2})^{1/2} \end{pmatrix}. \end{aligned}$$

We can also introduce an operator $T_{\tau,-}^{(2)}$ w.r.t. $\pi_{\tau,-}$ in a similar way, and the relation $\pi_{\tau,-} = \pi_{-\tau,+} \circ \sigma$ then immediately gives that $T_{\tau,-}^{(2)} = -T_{-\tau,+}^{(2)}$. We denote the respective eigenvectors for the eigenvalues $\tau(x \pm 1)$ of $T_{\tau,-}^{(2)}$ as

$$\begin{aligned} (1 + q^{2x})^{1/2} \cdot \xi_{k,-}^{\tau(x+1)} &= (e_{k,+} \quad e_{k+1,-}) \cdot \begin{pmatrix} (1 + q^{2k+2x+2})^{1/2} \\ q^x(1 - q^{2k+2})^{1/2} \end{pmatrix}, \\ (1 + q^{2x})^{1/2} \cdot \xi_{k,-}^{\tau(x-1)} &= (e_{k-1,+} \quad e_{k,-}) \cdot \begin{pmatrix} -q^x(1 - q^{2k})^{1/2} \\ (1 + q^{2k+2x})^{1/2} \end{pmatrix}. \end{aligned}$$

We will also need to know the invariant functional on $\text{Pol}(S_{q\tau}^2)$. The following result was proven in [12] (see also [16]). We use Notation 1.18.

PROPOSITION 2.4. *Let φ_τ be the faithful normal positive functional on $B(l^2(\mathbb{N}) \oplus l^2(\mathbb{N}))$ which has \mathbf{Z}_τ as its associated trace class operator. Then the restriction of φ_τ to $\text{Pol}(S_{q\tau}^2)$ is $U_q(\mathfrak{su}(2))$ -invariant.*

One way to prove this is as follows: we want to show $\varphi_\tau(x \triangleleft b) = \varphi_\tau(x)\varepsilon(b)$ for $x \in \text{Pol}(S_{q\tau}^2)$ and $b \in \{E, F, K^{\pm 1}\}$. First show invariance for elements in $\text{Pol}(S_{q\tau}^2) \cdot Z$, which are trace class operators. One can use here the formulas in terms of the inner

action without worrying about the unboundedness (of Z^{-1} and the trace Tr). One is left with showing invariance for elements of the form X^n or $(X^*)^n$ with $n \in \mathbb{N}$. But the only non-trivial case to consider is $n = 1$, for which we can simply compute the values.

We now give a proof of Theorem 0.1.

Proof of Theorem 0.1. Let us first note that we can apply Proposition 1.14 to $\text{Pol}(S_{q\tau}^2) \odot M_2(\mathbb{C})$, so that we can work on the level of $U_q(\mathfrak{su}(2))$.

We use the notation of Proposition 2.2. Write

$$p_+ \in \text{Pol}_+(S_{q\tau}^2) \odot M_2(\mathbb{C})$$

for the eigenprojection of $T_{\tau,+}^{(2)}$ which corresponds to the eigenvalue $\tau(x+1)$. Then by Proposition 1.16, the restriction of $\pi_{\tau,+}^{(2)}$ to $p_+(V \odot \mathbb{C}^2)$ factors through $\text{Pol}^{\text{ext}}(S_{q\tau(x+1)}^2)$. As the image of $Z_{\tau(x+1)}$ is easily seen to have distinct non-zero positive eigenvalues, it follows from the classification of *-representations of the $\text{Pol}(S_{q\tau}^2)$ that this representation of $\text{Pol}(S_{q\tau(x+1)}^2)$ on $p_+(l^2(\mathbb{N}) \otimes \mathbb{C}^2)$ is a copy of $\pi_{\tau(x+1),+}$. (In fact, one may check directly that the isomorphism is simply given by sending $\xi_{k,+}^{\tau(x+1)}$ to e_k .) The similar statements hold for the eigenspace of $\tau(x-1)$, as well as for the π_- -representations. We denote p_- for the eigenprojection of $T_{\tau,-}^{(2)}$ at eigenvalue $\tau(x+1)$, and

$$p = p_- \oplus p_+ \in B((l^2(\mathbb{N}) \oplus l^2(\mathbb{N})) \otimes \mathbb{C}^2).$$

Let us denote $B = p(\text{Pol}(S_{q\tau}^2) \odot M_2(\mathbb{C}))p$. To see that

$$B = \text{Pol}(S_{q\tau(x+1)}^2), \quad (*)$$

let us first remark that, by the preceding paragraph, the restriction of $\pi_{\tau(x)}^{(2)}$ to $p((V \oplus V) \otimes \mathbb{C}^2)$ is precisely $\pi_{\tau(x+1)}$. Hence $\text{Pol}(S_{q\tau(x+1)}^2) \subseteq B$ equivariantly. Further, if $\varphi_{1/2}$ is the invariant state on $M_2(\mathbb{C})$ for the adjoint spin 1/2-action of $SU_q(2)$, and φ_τ the invariant functional of the previous proposition, then $\varphi_\tau \otimes \varphi_{1/2}$ is invariant on $\text{Pol}(S_{q\tau}^2) \odot M_2(\mathbb{C})$. It follows that there exists a faithful normal functional on $B(p((l^2(\mathbb{N}) \oplus l^2(\mathbb{N})) \otimes \mathbb{C}^2)) \cong B(l^2(\mathbb{N}) \oplus l^2(\mathbb{N}))$ which restricts to an invariant functional on B .

Now we remark that $B \subseteq B(l^2(\mathbb{N})) \oplus B(l^2(\mathbb{N}))$. As $B(l^2(\mathbb{N})) \oplus B(l^2(\mathbb{N})) = \text{Pol}(S_{q\tau(x+1)}^2)''$, we can conclude (*) by Lemma 1.3.

Hence $S_{q\tau(x)}^2$ and $S_{q\tau(x+1)}^2$ are $SU_q(2)$ -Morita equivalent by Proposition 1.4. A similar analysis with the eigenvalue $\tau(x-1)$ shows $p^\perp(\text{Pol}(S_{q\tau}^2) \odot M_2(\mathbb{C}^2))p^\perp = \text{Pol}(S_{q\tau(x-1)}^2)$. This proves that all Podleś spheres $S_{q\tau(x)}^2$ and $S_{q\tau(y)}^2$ with $x, y \in \mathbb{R}$ and $x - y \in \mathbb{Z}$ are $SU_q(2)$ -Morita equivalent.

As the spin 1/2 representation is generating, it follows from Lemma 1.5 that $S_{q\tau(x)}$ is equivariantly Morita equivalent with some \mathbb{X} iff $\mathbb{X} \cong S_{q\tau(y)}$ for some y with $x - y \in \mathbb{Z}$. The statement of Theorem 0.1 now follows for $x \neq \infty$ by observing that $\text{Pol}(S_{q\tau(x)}) \cong \text{Pol}(S_{q\tau(-x)})$.

Finally, the standard Podleś sphere S_{q0}^2 is only equivariantly Morita equivalent with itself by the remark after Proposition 1.6. Indeed, it is the quotient space of $SU_q(2)$ by S^1 , but the latter only has one-dimensional irreducible representations, so any induced coaction is isomorphic to the original one. ■

We end this section with the following observation.

COROLLARY 2.5 (of the proof of Theorem 0.1). *With $\mathbb{G} = SU_q(2)$, the equivariant Picard group $\text{Pic}_{\mathbb{G}}(S_{q\tau}^2)$ of the Podleś spheres is determined as follows.*

- $\text{Pic}_{\mathbb{G}}(S_{q\infty}^2) \cong \mathbb{Z}$,
- $\text{Pic}_{\mathbb{G}}(S_{q\tau(x)}^2) \cong \mathbb{Z}_2$ for $x \in \mathbb{Z}$,
- $\text{Pic}_{\mathbb{G}}(S_{q\tau(x)}^2)$ is the trivial group in the remaining cases.

By the equivariant Picard group for a \mathbb{G} -homogeneous space \mathbb{X} , we mean the equivalence classes of equivariant equivalence $C(\mathbb{X})$ -Hilbert bimodules, with composition given by the balanced $C(\mathbb{X})$ -product.

Proof. For $S_{q\infty}^2$ the result follows as the equivalence classes of irreducible imprimitivity Hilbert modules are labeled by $\mathbb{Z} = \text{Irrep}(S^1)$, and $\text{Pol}(S_{q\infty}^2)$ has no outer equivariant automorphisms. It is easily verified that the resulting group structure is also \mathbb{Z} .

For $S_{q\tau(x)}^2$ with $x \notin \mathbb{Z}$, we have computed that any irreducible imprimitivity Hilbert module has some $\text{Pol}(S_{q\tau(y)}^2)$ as its endomorphism algebra, where $y \in x + \mathbb{Z}$. As $S_{q\tau(x)} \cong S_{q\tau(y)}$ equivariantly iff $x = \pm y$, and as $S_{q\tau(x)}^2$ has no outer equivariant automorphisms, the result for this case also follows.

Finally, for S_{q0}^2 , the first part of the previous argument still applies, but now we will have $\text{Out}_{\mathbb{G}}(\text{Pol}(S_{q0}^2)) = \mathbb{Z}_2$. Hence $\text{Pic}_{\mathbb{G}}(S_{q0}^2) \cong \mathbb{Z}_2$. As the $S_{q\tau(x)}$ with $x \in \mathbb{Z}$ are \mathbb{G} -Morita equivalent with S_{q0}^2 , the result follows also for these cases. (In fact, observe that the $\text{Pol}(S_{q\tau(2l)}^2)$ -linear span of the A_s inside the $*$ -algebra B_l of Theorem 0.2 give a concrete equivariant equivalence (pre-)Hilbert C^* -bimodule between $\text{Pol}(S_{q\tau(2l)})$ and $\text{Pol}(S_{q\tau(-2l)}) \cong \text{Pol}(S_{q\tau(2l)})$.) ■

REMARK. A similar distinction between equilateral Podleś spheres (i.e. $\tau \neq 0$ or ∞), and the only further one that we are aware of, appears in Proposition 9 of [7], where the spectral decomposition of a certain subspace of the restricted duals of the Podleś spheres is computed. However, now the set of exceptional cases is slightly larger, as they are given by the $\tau(x)$ with $x \in \frac{1}{2}\mathbb{N}$. We have not examined in detail whether there is any direct connection with the above result.

3. Equivariant Morita equivalence for the quantum projective plane. We will first show that the module $*$ -algebra B_l of Theorem 0.2 is well-defined.

We will index the elements $b \in \{X, Y, Z\} \subseteq B_l$ with $2l$ (dropping the τ -symbol w.r.t. previous notation), and denote the A_s as $A_{2l}^{(s)}$. However, when the indices are not crucial in a computation, we will drop them.

We will also use the following orthogonal basis for the pre-Hilbert space $V \oplus V$, where $V = \mathbb{C}[\mathbb{N}]$: for $k \in \mathbb{N} - 2l = \{-2l, -2l + 1, \dots\}$, we denote by $e_{k,+}^{(l)}$ the vector $0 \oplus e_{k+2l}$, and for $k \in \mathbb{N}$, we denote by $e_{k,-}^{(l)}$ the vector $e_k \oplus 0$.

We want to build now on $V \oplus V$ a bounded $*$ -representation of B_l . Namely, we let the generators of B_l correspond to the following banded operators:

$$\begin{aligned} X_{2l}e_{k,\pm}^{(l)} &= \pm(1 \pm q^{2k})^{1/2}(1 \mp q^{2k+4l})^{1/2}e_{k-1,\pm}^{(l)}, \\ Z_{2l}e_{k,\pm}^{(l)} &= \pm q^{2k+2l+1}e_{k,\pm}^{(l)}, \\ X_{2l}^*e_{k,\pm}^{(l)} &= \pm(1 \pm q^{2k+2})^{1/2}(1 \mp q^{2k+4l+2})^{1/2}e_{k+1,\pm}^{(l)}, \\ A_{2l}^{(s)}e_{k,\pm}^{(l)} &= (\pm 1)^s (\pm q^{2k+2s+2}; q^2)_{2l-s}^{1/2} (\mp q^{2k+2}; q^2)_{2l+s}^{1/2} e_{k+s,\mp}^{(l)}. \end{aligned}$$

It is an easy task to check that the commutation relations in Theorem 0.2 are satisfied for these operators. If we restrict to $\text{Pol}(S_{q\tau(2l)}^2) \subseteq B_l$, we see that we get the natural representation $\pi_{\tau(2l)}$.

LEMMA 3.1. *The above representation is faithful.*

Proof. Let us formally write X^{-1} for X^* . Then the commutation relations, together with their adjoints, clearly allow to write any element of B_l as a linear combination of elements of the form

- $X^m Z^n$ with $m \in \mathbb{Z}, n \in \mathbb{N}$,
- $A_s Z^n$ with $s \in \{-2l + 1, -2l + 1, \dots, 2l - 1\}$ and $n \in \mathbb{N}$,
- $A_{-2l} X^m Z^n$ with $m, n \in \mathbb{N}$, and
- $A_{2l} (X^*)^m Z^n$ with $m, n \in \mathbb{N}$.

We will now show that the representations of these monomials are linearly independent. We will in the following already use the same notation for these operator algebraic implementations. Note that in any case none of the above monomials are zero operators.

From looking at the natural $\mathbb{Z} \times \mathbb{Z}_2$ -gradation on B_l by the adjoint action of Z , we see immediately that the above families are linearly independent amongst each other, and that inside each family we can only have linear dependencies of the form $A_s X^m P(Z) = 0$ for some non-zero (Laurent) polynomial P in Z . But it is clear that these do not occur. ■

Using Notation 1.18, consider $\text{Pol}_a^{\text{ext}}(S_{q\tau(2l)}^2) \subseteq L(V \oplus V)$, which induces a module $*$ -algebra structure on the $*$ -algebra of banded operators on $V \oplus V$ by Lemma 1.8.1.

PROPOSITION 3.2. *The above module $*$ -algebra structure restricts to B_l , and coincides with the one described in Theorem 0.2.*

Proof. It is clear that the above module $*$ -algebra structure restricts to $\text{Pol}(S_{q\tau(2l)}^2) \subseteq B_l$, and coincides with the usual one. We are therefore left to show that it behaves in the right way on the operators A_s .

Let us write e for the sign operator

$$e : V \oplus V \rightarrow V \oplus V : e_{k,\pm}^{(l)} \mapsto \pm e_{k,\pm}^{(l)},$$

so that $\mathbf{b}_{2l} = eb_{2l}$ for $b \in \{X, Z, Y\}$. Denote $\theta_{2l}^{(s)} = \lambda_s A_s$ where

$$\lambda_s = q^{\frac{1}{2}s(s-1)} \frac{(q^{4l-2s+2}; q^2)_{s+2l}^{1/2}}{(q^2; q^2)_{s+2l}^{1/2}}.$$

We then have

$$\theta_{2l}^{(s)} \triangleleft K = \mathbf{Z} \theta_{2l}^{(s)} \mathbf{Z}^{-1} = e Z \theta_{2l}^{(s)} Z^{-1} e = -q^{2s} e \theta_{2l}^{(s)} e = q^{2s} \theta_{2l}^{(s)}.$$

This small computation already makes it clear why we cannot use the naive map of $U_q(-, +)$ into $\text{Pol}(S_{q\tau(2l)}^2)$ to define the module $*$ -algebra structure.

Similarly, we compute

$$\begin{aligned}\theta_{2l}^{(-2l)} \triangleleft (q^{-1/2}\lambda^{-1}E) &= Z^{-1}e[\theta_{2l}^{(-2l)}, eX] = Z^{-1}e(\theta_{2l}^{(-2l)}Xe - eX\theta_{2l}^{(-2l)}) \\ &= -Z^{-1}(\theta_{2l}^{(-2l)}X + X\theta_{2l}^{(-2l)}) = -Z^{-1}(\theta_{2l}^{(-2l)}X - \theta_{2l}^{(-2l)}X) = 0,\end{aligned}$$

showing that $\theta_{2l}^{(-2l)}$ is a highest weight vector for the spin $2l$ -representation.

For $s > -2l$, we have

$$X\theta_{2l}^{(s)} = -\frac{\lambda_s}{\lambda_{s-1}}(1 - q^{2l+1}Z)\theta_{2l}^{(s-1)},$$

and, by taking the adjoint of the commutation relations for the X^* , we also have

$$\theta_{2l}^{(s)}X = \frac{\lambda_s}{\lambda_{s-1}}(1 - q^{-2s-2l+1}Z)\theta_{2l}^{(s-1)}.$$

So then we find

$$\theta_{2l}^{(s)} \triangleleft (q^{-1/2}\lambda^{-1}E) = \frac{\lambda_s}{\lambda_{s-1}}(q^{-2s-2l+1} - q^{2l+1})\theta_{2l}^{(s-1)}.$$

Simplifying, this becomes

$$\theta_{2l}^{(s)} \triangleleft E = q^{-s-2l+\frac{1}{2}}\lambda(1 - q^{4l+2s})^{1/2}(1 - q^{4l-2s+2})^{1/2}\theta_{2l}^{(s-1)}.$$

Carrying out a similar calculation for F , or using the compatibility between the module structure and the $*$ -operation, we also find

$$\theta_{2l}^{(s)} \triangleleft F = q^{s-2l-\frac{1}{2}}\lambda(1 - q^{4l+2s+2})^{1/2}(1 - q^{4l-2s})^{1/2}\theta_{2l}^{(s+1)}$$

for $s < 2l$, and $\theta_{2l}^{(2l)} \triangleleft F = 0$. In all, we find that the action of $U_q(\text{su}(2))$ on the elements $\theta_{2l}^{(s)}$ indeed gives a (right) presentation of the spin $2l$ -representation. ■

We now show that the action on B_l is ergodic.

LEMMA 3.3. *The module $*$ -algebra B_l has only the scalar multiples of the unit as its invariant elements.*

Proof. Using the arguments concerning the basis constructed in Lemma 3.1, we see that an invariant element b can be written as $b_1 + b_2$ with $b_1 \in \text{Pol}(S_{q\tau(2l)}^2)$ and b_2 a linear combination of elements of the form A_0Z^n with $n \in \mathbb{N}$. As the natural grading on B_l is $U_q(\text{su}(2))$ -compatible, both b_1 and b_2 have to be invariant. But the action on a Podleś sphere is ergodic, so b_1 reduces to a scalar. On the other hand, set $b_2 = A_0P(Z)$ with $P(Z)$ a polynomial in Z . Then the invariance of b_2 under the adjoint action of E leads to the following functional equation for P :

$$(1 - q^{-2l-1}Z)P(-q^{-2}Z) = (1 + q^{2l-1}Z)P(Z).$$

It is clear that the only solution is $P = 0$. ■

We can thus apply Proposition 1.14 to find that B_l has a well-defined action by $SU_q(2)$. This finishes the existence part of the $SU_q(2)$ -action proposed in Theorem 0.2.

It is also easy to provide the invariant functional on B_l .

PROPOSITION 3.4. *Let φ_l be the normal positive functional on the von Neumann algebra $B(l^2(\mathbb{N}) \oplus l^2(\mathbb{N}))$ which has \mathbf{Z} as its associated trace class operator. Then φ_l is $U_q(su(2))$ -invariant on B_l .*

Proof. Let p_{\pm} be the projections onto the summands of $l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$. We find that the conditional expectation

$$E : B(l^2(\mathbb{N}) \oplus l^2(\mathbb{N})) \rightarrow B(l^2(\mathbb{N})) \oplus B(l^2(\mathbb{N})) : x \mapsto p_-xp_- + p_+xp_+$$

restricts to an equivariant conditional expectation $B_l \rightarrow \text{Pol}(S_{q\tau(2l)}^2)$. Since $\varphi_l = \varphi_{\tau(2l)} \circ E$, the proposition follows from Proposition 2.4. ■

We can now prove Theorem 0.2.

Proof of Theorem 0.2. We first remark that the definition of B_l also makes sense when $l = 0$. In fact, it is easily seen that B_0 is just a copy of $\text{Pol}(S_{q,0}^2) \rtimes \mathbb{Z}_2$ where \mathbb{Z}_2 acts by the automorphism σ_0 (see the remark after Definition 1.15). All results of this section then hold for B_0 , *except* that B_0 is not ergodic: the proof of 3.3 in fact shows that the space of invariants is linearly spanned by 1 and A_0 . Now the ‘antipodal reflection map’ σ_0 on $\text{Pol}(S_{q,0}^2)$ gives a Galois action by \mathbb{Z}_2 (cf. [5], Proposition 2.10). Hence, by the discussion after Proposition 1.9, B_0 is $SU_q(2)$ -equivariantly Morita equivalent with $\text{Pol}(\mathbb{R}P_q^2)$, which is by definition the fixed point algebra under σ_0 . If we denote $p_{\pm} = \frac{1}{2}(1 \pm A_0)$, then $p_{\pm}B_0p_{\pm} \cong \text{Pol}(\mathbb{R}P_q^2)$ equivariantly.

Now for $l \in \frac{1}{2}\mathbb{N}$, let us write $V_{2l,\pm}$ for the space $V = \mathbb{C}[\mathbb{N}]$ considered with the π_{\pm} -action by $\text{Pol}(S_{q\tau(2l)}^2)$, and $V_{2l} = V_{2l,-} \oplus V_{2l,+}$. Consider $B_l \odot M_2(\mathbb{C})$, represented on $V_{2l} \odot \mathbb{C}^2$. Let us write the eigenvectors ξ from Lemma 2.3 and the remark under it as follows:

$$\begin{aligned} e_{k,+}^{(l \pm \frac{1}{2})} &= \xi_{k+2l\pm 1,+}^{\tau(2l\pm 1)}, & k \in \mathbb{N} - (2l \pm 1), \\ e_{k,-}^{(l \pm \frac{1}{2})} &= \xi_{k,-}^{\tau(2l\pm 1)}, & k \in \mathbb{N}. \end{aligned}$$

We may identify the span of the $e_{k,\nu}^{(l \pm \frac{1}{2})}$ over all k with $V_{2l\pm 1,\nu}$, where $\nu \in \{-, +\}$. Then we can write $V_{2l} \odot \mathbb{C}^2$ as $V_{2l-1} \oplus V_{2l+1}$, with corresponding projection maps $p_{2l\pm 1}$. From the results of the previous section, it follows that

$$p_{2l\pm 1}(\text{Pol}(S_{q\tau(2l)}^2) \otimes M_2(\mathbb{C}))p_{2l\pm 1} = \text{Pol}(S_{q\tau(2l\pm 1)}^2),$$

in its natural presentation w.r.t. the basis $e^{(l \pm \frac{1}{2})}$. Now as $\pi_{-\tau,+} = \pi_{\tau,-} \circ \sigma$, we have that in the new basis also

$$\pi_{\tau(2l),a}^{(2)} = \pi_{\tau(2l-1),a} \oplus \pi_{\tau(2l+1),a},$$

where we recall the notations Notation 1.18 and Notation 2.1. By Lemma 1.8.2, the action of $U_q(su(2))$ on $B_l \odot M_2(\mathbb{C})$ will be implemented by this representation.

We want to show now that

$$p_{2l\pm 1}(B_l \otimes M_2(\mathbb{C}))p_{2l\pm 1} = B_{l \pm \frac{1}{2}},$$

where for the moment we assume $l > 0$ in the --case. As the σ -weak closure of B_l is clearly the whole of $B(l^2(\mathbb{N}) \oplus l^2(\mathbb{N}))$, and as the latter has a normal positive functional which restricts to an invariant functional on B_l , by the previous proposition, a similar

argument as in the proof of Theorem 0.1 shows that it is sufficient to prove that the right hand side is *contained* in the left hand side.

We have already remarked above that the copy of the Podleś sphere inside $B_{l \pm \frac{1}{2}}$ will belong to the left hand side. It remains to prove this also for the generators $A_{2l \pm 1}^{(s)}$.

Let us denote $e_{k,\mu,\nu}$ for the vector $e_{k,\mu}^{(l)} \otimes e_\nu$ in $V_{2l} \otimes \mathbb{C}^2$. Then we may write

$$\begin{aligned} \sqrt{1+q^{4l}} \cdot e_{k,\pm}^{(l+\frac{1}{2})} &= (e_{k,\pm,+} \quad e_{k+1,\pm,-}) \cdot \begin{pmatrix} \mp(1 \mp q^{2k+4l+2})^{1/2} \\ q^{2l}(1 \pm q^{2k+2})^{1/2} \end{pmatrix}, \\ \sqrt{1+q^{4l}} \cdot e_{k,\pm}^{(l-\frac{1}{2})} &= (e_{k-1,\pm,+} \quad e_{k,\pm,-}) \cdot \begin{pmatrix} \pm q^{2l}(1 \pm q^{2k})^{1/2} \\ (1 \mp q^{2k+4l})^{1/2} \end{pmatrix}. \end{aligned}$$

Conversely we have

$$\begin{aligned} \frac{1}{\sqrt{1+q^{4l}}} \cdot e_{k,\pm,+} &= (e_{k,\pm}^{(l+\frac{1}{2})} \quad e_{k+1,\pm}^{(l-\frac{1}{2})}) \cdot \begin{pmatrix} \mp(1 \mp q^{2k+4l+2})^{1/2} \\ \pm q^{2l}(1 \pm q^{2k+2})^{1/2} \end{pmatrix}, \\ \frac{1}{\sqrt{1+q^{4l}}} \cdot e_{k,\pm,-} &= (e_{k-1,\pm}^{(l+\frac{1}{2})} \quad e_{k,\pm}^{(l-\frac{1}{2})}) \cdot \begin{pmatrix} q^{2l}(1 \pm q^{2k})^{1/2} \\ (1 \mp q^{2k+4l})^{1/2} \end{pmatrix}. \end{aligned}$$

One computes then that w.r.t. the original basis of $V_{2l} \otimes \mathbb{C}^2 = \begin{pmatrix} V_{2l} \otimes e_+ \\ V_{2l} \otimes e_- \end{pmatrix}$, one has

$$A_{2l+1}^{(0)} = \begin{pmatrix} -A_{2l}^{(0)}(1 - q^{4l+2}Z^2) & q^{2l}A_{2l}^{(-1)}(1 + q^{-2l-1}Z)(1 + q^{2l+1}Z) \\ -q^{2l}A_{2l}^{(1)}(1 - q^{-2l+1}Z)(1 - q^{2l+1}Z) & q^{4l}A_{2l}^{(0)}(1 - q^{-4l-2}Z^2) \end{pmatrix}.$$

A similar computation shows that, for $l > 0$, we can write

$$A_{2l-1}^{(0)} = \begin{pmatrix} -q^{4l}A_{2l}^{(0)} & -q^{2l}A_{2l}^{(-1)} \\ q^{2l}A_{2l}^{(1)} & A_{2l}^{(0)} \end{pmatrix}. \quad (*)$$

We have thus shown that

$$A_{2l \pm 1}^{(0)} \in p_{2l \pm 1}(B_l \otimes M_2(\mathbb{C}))p_{2l \pm 1}.$$

As all other $A_{2l \pm 1}^{(s)}$ lie in $A_{2l \pm 1}^{(0)} \triangleleft U_q(su(2))$, it follows that

$$A_{2l \pm 1}^{(s)} \in p_{2l \pm 1}(B_l \otimes M_2(\mathbb{C}))p_{2l \pm 1}$$

for all s , and so $p_{2l \pm 1}(B_l \otimes M_2(\mathbb{C}))p_{2l \pm 1} = B_{l \pm \frac{1}{2}}$.

Now Theorem 0.2 will follow from Lemma 1.5 and the above discussion, if we can also show that $\text{Pol}(\mathbb{R}P_q^2) \otimes M_2(\mathbb{C}) \cong B_{\frac{1}{2}}$. From the remarks in the first paragraph of this proof, it is sufficient to show that

$$(p_+ \otimes 1)(B_0 \otimes M_2(\mathbb{C}))(p_+ \otimes 1) \cong B_{\frac{1}{2}}.$$

Now an easy computation shows that $(A_0 \otimes 1)p_1(A_0 \otimes 1) = p_{-1}$. As we already know that

$$p_1(B_0 \otimes M_2(\mathbb{C}))p_1 \cong B_{\frac{1}{2}},$$

on which the $SU_q(2)$ -action is ergodic, we must have that p_1 and p_{-1} are minimal projections in the fixed point algebra of $B_0 \otimes M_2(\mathbb{C})$. As also $(A_0 \otimes 1)$ lies in the latter, it follows that the fixed point algebra is in fact $M_2(\mathbb{C})$. Hence

$$(p_+ \otimes 1)(B_0 \otimes M_2(\mathbb{C}))(p_+ \otimes 1) \cong p_1(B_0 \otimes M_2(\mathbb{C}))p_1 \cong B_{\frac{1}{2}},$$

and we are done. ■

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