CORE EQUIVALENCE IN ECONOMY UNDER AWARENESS

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Abstract. The extended notion of pure exchange economy under uncertainty, called an economy with awareness structure, is presented, where each trader having strictly monotone preferences makes decision under his/her awareness and belief. We show an extension of the core equivalence theorem: The ex-post core coincides with the set of all generalized expectations equilibria in awareness for the economy.

1. Introduction. This article relates economies and distributed belief. The purposes are: First, to present an extended notion of economy under uncertainty, called an *economy with awareness structure*, where each trader makes decision in his/her awareness and belief under incomplete information. Secondly, to show the 'core equivalence theorem' under generalized expectations equilibrium in the extended economy:

MAIN THEOREM. *In a pure exchange economy under uncertainty, the traders are assumed to have an awareness structure and and they are risk averse. Then the ex-post core coincides with the set of all expectations equilibria in awareness for the economy.*

Recently researchers in such fields as economics, AI, and computer science have become interested in reasoning of belief and knowledge. There are pragmatic concerns about the relationship between knowledge (belief) and actions. Of most interest to us is the emphasis on situations involving the distributed knowledge (belief) of a group of agents rather than that of a single agent. At the heart of any analysis of such situations as a conversation, a bargaining session or a protocol run by processes is the interaction between

agents. An agent in a group must take into account not only events that have occurred in the world but also the knowledge of the other agents in the group.

Many authors have investigated several notions of core in an economy under asymmetric information (e.g., Wilson [14], Volij [13], Einy et al [6] and Forges et al [9]). The serious limitations of their analysis is the assumption of the 'partition' structure of information the traders receive, together with the assumption of common-knowledge on traders' willingness to trade. From the epistemic point of view, the partition structure represents the traders' knowledge satisfying the postulates: 'Truth' **T** (what is known is true), the 'positive introspection' **4** (that we know what we do) and the 'negative introspection' **5** (that we know what we do not know). The postulate **5** is indeed so strong that it describes the hyper-rationality of traders, and thus it is particularly objectionable. So is the common knowledge assumption because the notion of common knowledge is defined by an infinite recursion of all of individual knowledge.

This article explores the extent to which two assumptions, partition structure and common-knowledge, are gneralized in economies under uncertainty. As has already been pointed out in Geanakoplos [8], this relaxation can potentially yield important results in a world with imperfectly Bayesian agents.

The idea has been performed in different settings. Among other things Geanakoplos [8] showed the no speculation theorem in the extended rational expectations equilibrium under the assumption that the information structure is reflexive, transitive and *nested* (Corollary 3.2 in Geanakoplos [8]). The condition 'nestedness' is interpreted as a requisite on the 'memory' of the trader. Recently, Matsuhisa, Ishikawa and Hoshino [11] extended the core equivalence theorem in an economy under generalized information structure without the nestedness condition.

This article is organized as follows: In Section 2 we begin with illustrating our model by a simple example. In Section 3 we propose the model: An economy with awareness structure. In Section 4 we present the notions of expectations equilibrium in awareness and ex-post core in the economy. We state explicitly the core equivalence theorem and sketch the proof. Finally we conclude by giving some remarks.

2. Illustrative example. Let us consider the exchange economy under uncertainty as follows: There are two traders 1 and 2, and two commodities x and y. We let x_t and y_t respectively mean the amounts of commodities x and y that the trader t possesses. Let Ω be the state space $\{\omega_1, \omega_2, \omega_3\}$. The traders' information structures are given as follows: Trader 1 has the partition information $P_1(\omega) = \{\omega\}$ for all $\omega \in \Omega$, but trader 2 has the non-partition information $P_2(\omega_1) = \Omega, P_2(\omega_2) = \{\omega_1, \omega_2\}$ and $P_2(\omega_3) = \{\omega_1, \omega_3\}$. Trader 1 has her initial endowment $e_1(\omega) = (\frac{3}{2}, 3)$ for every $\omega \in \Omega$, and trader 2 has his initial endowment $e_2(\omega) = (1, 1)$ for every $\omega \in \Omega$. Furthermore they have the utilities U_1 and U_2 as follows: $U_1(x_1, y_1; \omega) = x_1^{\frac{1}{3}} y_1^{\frac{2}{3}}$ for every state $\omega \in \Omega$, $U_2(x_2, y_2; \omega_1) = U_2(x_2, y_2; \omega_2) = x_2 y_2$, $U_2(x_2, y_2; \omega_3) = x_2^{\frac{1}{2}} y_2^{\frac{1}{2}}$, and they have different priors: $\pi_1(\omega) = 1/3$ for every state, and $\pi_2(\omega_1) = 1/2$ and $\pi_2(\omega) = 1/4$ for $\omega \neq \omega_1$.

In the economy, we can see that traders' initial endowments (e_1, e_2) are ex-ante Pareto optimal. and we can observe that equilibrium price $p = (p_1, p_2)$ is the same between

traders 1, 2 ($p_1 = p_2$), and the endowments give a generalized notion of rational expectations equilibrium for the price p. It is noted that trader 2's information $\sigma(p) \cap P_2$ given by the price may be still a non-partition.

This situation is very interesting: Even though traders have different priors and they process different information, they do not trade at all. It should be noted that P_2 satisfies the reflexivity: For any $\omega \in \Omega$, $\omega \in P_2(\omega)$, however it does not satisfy the transitivity: $P_2(\xi) \subseteq P_2(\omega)$ whenever $\xi \in P_2(\omega)$. Moreover P_2 is not *nested*: An information structure $(P_i)_{i \in N}$ is said to be *nested* if for each $i \in N$ and for all states ω and ξ in Ω, either $P_i(\omega) \cap P_i(\xi) = \emptyset$, or else $P_i(\omega) \subseteq P_i(\xi)$ or $P_i(\omega) \supseteq P_i(\xi)$. In this article we shall investigate the pure exchange economies under generalized information structure as like this example.

3. The model. Let Ω be a non-empty *finite* set called a *state space* and 2^Ω the field consisting of all subsets of Ω. Each member of 2^Ω is called an *event* and each element of Ω called a *state*. The space of the *traders* is a measure space (T, Σ, μ) in which T is a set of *traders*, Σ is a σ-field of subsets of T whose elements are called *coalitions*, and μ is a measure on Σ. We present a model of awareness according to E. Dekel, B. L. Lipman and A. Rustichini [5]. A different approach of awareness models is discussed in R. Fagin, J. Y. Halpern, Y. Moses and M. Y. Vardi [7].

3.1. *Awareness and belief.* A *belief structure* is a tuple $\langle \Omega, (B_t)_{t \in N} \rangle$ in which $B_t : 2^\Omega \to 2^\Omega$ is trader t's *belief* operator. The interpretation of the event $B_t E$ is that 't believes E.' An *awareness structure* is a tuple $\langle \Omega, (A_t)_{t \in T}, (B_t)_{t \in T} \rangle$ in which $\langle \Omega, (B_t)_{t \in T} \rangle$ is a belief structure and A_t is t's *awareness* operator on 2^Ω defined by

PL $\qquad A_t E = B_t E \cup B_t(\Omega \setminus B_t E)$ for every E in 2^Ω.

This is called the *plausibility property* **PL**. The interpretation of $A_t E$ is that 't is aware of E.' The property **PL** says that t is aware of E if he believes it or if he believes that he does not believe it. An event E is said to be t's *evident belief* if $T \subseteq B_t T$. We can think of it as embodying the essence of what is involved in t making his/her direct observations.

3.2. *Associated information structure.* We shall give the generalized notion of information partition in the line of Bacharach [3].

DEFINITION 1. The *associated information structure* $(P_t)_{t \in T}$ with an awareness structure $\langle \Omega, (A_t), (B_t) \rangle$ is the class of t's *associated information function* P_t of Ω into 2^Ω defined by $P_t(\omega) = \bigcap_{E \in 2^\Omega} \{E \mid \omega \in E \subseteq B_t E\}$. (If there is no event E for which $\omega \in E \subseteq B_t E$ then we take $P_t(\omega)$ to be undefined.) The *domain of P_t* denoted by $\mathrm{Dom}(P_t)$ is the set of all the states at which P_t is defined.

The mapping P_t is *reflexive* on its domain in the sense below:

Ref $\quad \omega \in P_t(\omega)$ for every $\omega \in \mathrm{Dom}(P_t)$,

and it is said to be *transitive* if:

Trn $\quad \xi \in P_t(\omega)$ implies $P_t(\xi) \subseteq P_t(\omega)$ for any $\xi, \omega \in \mathrm{Dom}(P_t)$.

We call $P_t(\omega)$ t's *evidence set* at ω, which is interpreted as the basis for all t's evident beliefs. This is because each t's evident belief E is decomposed into a union of all evidence sets contained in E. Furthermore P_t is said to be *symmetric* if:

Sym $\xi \in P_t(\omega)$ implies $P_t(\xi) \ni \omega$ for any ω and $\xi \in \mathrm{Dom}(P_t)$.

REMARK 1. M. Bacharach [3] introduces the *strong epistemic model* equivalent to the Kripke semantics of the modal logic **S5**. The model is a tuple $\langle \Omega, (K_t)_{t\in T} \rangle$, in which t's *knowledge operator* $K_t : 2^\Omega \to 2^\Omega$ satisfies the five postulates: For every E, F of 2^Ω,

N $K_t\Omega = \Omega$;   **K** $K_t(E \cap F) = K_tE \cap K_tF$;   **T** $K_tF \subseteq F$;
4 $K_tF \subseteq K_tK_tF$;   **5** $\Omega \setminus K_tF \subseteq K_t(\Omega \setminus K_tF)$.

t's associated *information partition* P_t with K_t is defined by $P_t(\omega) = \bigcap_{T \in 2^\Omega}\{T \in 2^\Omega \,|\, \omega \in K_tT\}$; it satisfies the postulates **Ref**, **Trn** and **Sym**. This is uniquely determined by (Ω, K_t). The strong epistemic model can be interpreted as the awareness structure $\langle \Omega, (A_t), (B_t) \rangle$ such that B_t is the knowledge operator. In this situation it is easily verified that A_t must be the *trivial* operator; i.e. $A_t(F) = \Omega$ for every $F \in 2^\Omega$, and that $\mathrm{Dom}(P_t) = \Omega$.

3.3. *Economy with awareness structure.* A pure exchange economy *under uncertainty* is a structure $\mathcal{E} = \langle T, \Sigma, \mu, \Omega, \mathbf{e}, (U_t)_{t\in T}, (\pi_t)_{t\in T} \rangle$ consisting of the following structure and interpretations: There are l commodities in each state of the state space Ω; the consumption set of trader t is $\mathbf{R}^l_+$; (T, Σ, μ) is the measure space of the *traders*; $\mathbf{e}(t, \cdot) : T \times \Omega \to \mathbf{R}^l_+$ is t's *initial endowment*; $U_t : \mathbf{R}^l_+ \times \Omega \to \mathbf{R}$ is t's von-Neumann and Morgenstern utility function; π_t is a subjective prior on Ω for $t \in T$. For simplicity it is assumed that (Ω, π_t) is a *finite* probability space with π_t *full support* for all $t \in T$; i.e., $\pi_t(\omega) \neq 0$ for every $\omega \in \Omega$.

DEFINITION 2. A pure exchange economy *with awareness structure* is a structure $\mathcal{E}^A = \langle \mathcal{E}, (A_t)_{t\in T}, (B_t)_{t\in T}, (P_t)_{t\in T} \rangle$, in which $\mathcal{E}$ is a pure exchange economy under uncertainty, and $\langle \Omega, (A_t)_{t\in T}, (B_t)_{t\in T}, (P_t)_{t\in T} \rangle$ is an awareness structure with $(P_t)_{t\in T}$ the associated information structure. By the *domain* of the economy $\mathcal{E}^A$ we mean $\mathrm{Dom}(\mathcal{E}^A) = \cap_{t\in T}\mathrm{Dom}(P_t)$. The economy is called *atomless* if (T, Σ, μ) is a non-atomic measure space.

We always assume the condition below:

A-0 $\mathrm{Dom}(\mathcal{E}^A) \neq \emptyset$.

REMARK 2. We can plainly observe that an economy under asymmetric information coincides with an economy $\mathcal{E}^A$ with the awareness structure $\langle \Omega, (A_t)_{t\in T}, (B_t)_{t\in T} \rangle$ given by the strong epistemic model, and that $\mathrm{Dom}(\mathcal{E}^A) = \Omega$.

We denote by $\mathcal{F}_t$ the field of $\mathrm{Dom}(P_t)$ generated by $\{P_t(\omega)|\ \omega \in \Omega\}$ and denote by $\Pi_t(\omega)$ the atom containing $\omega \in \mathrm{Dom}(P_t)$. We denote by $\mathcal{F}$ the join of all $\mathcal{F}_t(t \in T)$ on $\mathrm{Dom}(\mathcal{E}^A)$; i.e. $\mathcal{F} = \vee_{t\in T}\mathcal{F}_t$, and denote by $\Pi(\omega)$ the atom containing $\omega \in \mathrm{Dom}(\mathcal{E}^A)$ of the field $\mathcal{F} = \vee_{t\in T}\mathcal{F}_t$.

We shall often refer to the following conditions: For every $t \in T$,

A-1 $\int_T \mathbf{e}(t, \omega)d\mu > 0$ for all $\omega \in \Omega$.
A-2 $\mathbf{e}(t, \cdot)$ is $\mathcal{F}_t$-measurable on $\mathrm{Dom}(P_t)$;

A-3   For each $x \in \mathbf{R}_+^l$, the function $U_t(x, \cdot)$ is at least $\mathcal{F}$-measurable on $\text{Dom}(\mathcal{E}^A)$, and the function: $T \times \mathbf{R}_+^l \to \mathbf{R}, (t, x) \mapsto U_t(x, \omega)$ is $\Sigma \times \mathcal{B}$-measurable where $\mathcal{B}$ is the σ-field of all Borel subsets of $\mathbf{R}_+^l$.

A-4   For each $\omega \in \Omega$, the function $U_t(\cdot, \omega)$ is continuous, strictly increasing and quasi-concave on $\mathbf{R}_+^l$.

3.4. *Expectations.* An *assignment* $\mathbf{x}$ is a mapping from $T \times \Omega$ into $\mathbf{R}_+^l$ such that for each $\omega \in \Omega$, the function $\mathbf{x}(\cdot, \omega)$ is μ-measurable, and for each $t \in T$, the function $\mathbf{x}(t, \cdot)$ is at least $\mathcal{F}$-measurable on $\text{Dom}(\mathcal{E}^A)$. We denote by $Ass(\mathcal{E}^A)$ the set of all assignments for the economy $\mathcal{E}^A$. By an *allocation* we mean an assignment $\mathbf{a}$ such that for every $\omega \in \Omega$, $\int_T \mathbf{a}(t, \omega) d\mu \leq \int_T \mathbf{e}(t, \omega) d\mu$. We denote by $Alc(\mathcal{E}^A)$ the set of all allocations, and for each $t \in T$ we denote by $Alc(\mathcal{E}^A)_t$ the set of all the functions $\mathbf{a}(t, \cdot)$ for $\mathbf{a} \in Alc(\mathcal{E}^A)$.

We introduce the revised notion of trader's expectation of utility in $\mathcal{E}^A$. By t's *ex-ante* expectation we mean $\mathbf{E}_t[U_t(\mathbf{x}(t, \cdot)] := \sum_{\omega \in \text{Dom}(P_t)} U_t(\mathbf{x}(t, \omega), \omega)\pi_t(\omega)$ for each $\mathbf{x} \in Ass(\mathcal{E}^A)$. The *interim* expectation $\mathbf{E}_t[U_t(\mathbf{x}(t, \cdot)|P_t]$ is defined by $\mathbf{E}_t[U_t(\mathbf{x}(t, \cdot)|P_t](\omega) := \sum_{\xi \in \text{Dom}(P_t)} U_t(\mathbf{x}(t, \xi), \xi)\pi_t(\{\xi\} \cap A_t(\{\xi\})|P_t(\omega))$.

REMARK 3. It should be noted that we use not the usual notion of posterior $\pi_t(\{\xi\}|P_t(\omega))$ but the revised one $\pi_t(\{\xi\} \cap A_t(\{\xi\})|P_t(\omega))$. For the discussion about this improvement of the notion of posterior, see Section 4 in Matsuhisa and Usami [12].

4. The result

4.1. *Ex-post core.* An assignment $\mathbf{y}$ is called an *ex-post improvement* of a coalition $S \in \Sigma$ on an assignment $\mathbf{x}$ at a state $\omega \in \Omega$ if

Imp1 $\mu(S) > 0$;
Imp2 $\int_S \mathbf{y}(t, \omega) d\mu \leq \int_S \mathbf{e}(t, \omega) d\mu$;   and
Imp3 $U_t(\mathbf{y}(t, \omega), \omega) > U_t(\mathbf{x}(t, \omega), \omega)$ for almost all $t \in S$.

DEFINITION 3. An allocation $\mathbf{x}$ is said to be an *ex-post core* allocation of an economy with awareness structure $\mathcal{E}^A$ if there is no coalition having an ex-post improvement on $\mathbf{x}$ at any state $\omega \in \text{Dom}(\mathcal{E}^A)$. The *ex-post core* $\mathcal{C}^{ExP}(\mathcal{E}^A)$ is the set of all the ex-post core allocations of $\mathcal{E}^A$.

Let $\mathcal{E}^A$ be the economy with awareness structure and $\mathcal{E}^A(\omega)$ the economy with complete information $\langle T, \Sigma, \mu, \mathbf{e}(\cdot, \omega), (U_t(\cdot, \omega))_{t \in T} \rangle$ for each $\omega \in \Omega$. We denote by $\mathcal{C}(\mathcal{E}^A(\omega))$ the set of all core allocations for $\mathcal{E}^A(\omega)$, and by $\mathcal{W}(\mathcal{E}^A(\omega))$ the set of all competitive equilibria for $\mathcal{E}^A(\omega)$.

PROPOSITION 1. *Let $\mathcal{E}^A$ be a pure exchange economy with awareness structure satisfying the conditions* **A-1**, **A-2** *and* **A-3**. *Suppose that the economy is atomless. The ex-post core of $\mathcal{E}^A$ is non-empty (i.e., $\mathcal{C}^{ExP}(\mathcal{E}^A) \neq \emptyset$). Moreover, $\mathcal{C}^{ExP}(\mathcal{E}^A)$ coincides with the set of all assignments $\mathbf{x}$ such that $\mathbf{x}(\cdot, \omega)$ is a core allocation for the economy $\mathcal{E}^A(\omega)$ for all $\omega \in \text{Dom}(\mathcal{E}^A)$: i.e.,*

$$\mathcal{C}^{ExP}(\mathcal{E}^A) = \{\mathbf{x} \in Alc(\mathcal{E}^A) \mid \mathbf{x}(\cdot, \omega)) \in \mathcal{C}(\mathcal{E}^A(\omega)) \text{ for all } \omega \in \text{Dom}(\mathcal{E}^A)\}.$$

Proof. Modify the proof of Theorem 3.1 in Einy et al [6].

4.2. *Expectations equilibrium in awareness.* A *price system* is a non-zero function $p :$ $\Omega \to \mathbf{R}_+^l$ which is $\mathcal{F}$-measurable on $\mathrm{Dom}(\mathcal{E}^A)$. We denote by $\Delta(p)$ the partition on Ω induced by p, and denote by $\sigma(p)$ the field of Ω generated by $\Delta(p)$. The *budget set* of a trader t at a state ω for a price system p is defined by $B_t(\omega, p) := \{ x \in \mathbf{R}_+^l \mid p(\omega) \cdot x \leq p(\omega) \cdot \mathbf{e}(t, \omega) \}$. Define the mapping $\Delta(p) \cap P_t : \mathrm{Dom}(P_t) \to 2^\Omega$ by $(\Delta(p) \cap P_t)(\omega) := \Delta(p)(\omega) \cap P_t(\omega)$; it is plainly observed that $\Delta(p) \cap P_t$ satisfies **Ref**. We denote by $\sigma(p) \vee \mathcal{F}_t$ the smallest σ-field containing both the fields $\sigma(p)$ and $\mathcal{F}_t$.

DEFINITION 4. An *expectations equilibrium in awareness* for an economy $\mathcal{E}^A$ with awareness structure is a pair $(p, \mathbf{x})$, in which p is a price system and $\mathbf{x}$ is an assignment satisfying the following conditions:

EA1 $\mathbf{x}$ is an allocation;
EA2   For almost all $t \in T$ and for every $\omega \in \Omega$, $\mathbf{x}(t, \omega) \in B_t(\omega, p)$;
EA3   For almost all $t \in T$, if $\mathbf{y}(t, \cdot) : \Omega \to \mathbf{R}_+^l$ is $\mathcal{F}$-measurable on $\mathrm{Dom}(\mathcal{E}^A)$ with $\mathbf{y}(t, \omega) \in B_t(\omega, p)$ for all $\omega \in \Omega$, then $\mathbf{E}_t[U_t(\mathbf{x}(t, \cdot))|\Delta(p) \cap P_t](\omega) \geq \mathbf{E}_t[U_t(\mathbf{y}(t, \cdot))|\Delta(p) \cap P_t](\omega)$ pointwise on $\mathrm{Dom}(P_t)$;
EA4   For every $\omega \in \mathrm{Dom}(\mathcal{E}^A)$, $\int_{t \in T} \mathbf{x}(t, \omega) d\mu = \int_{t \in T} \mathbf{e}(t, \omega) d\mu$.

The allocation $\mathbf{x}$ in $\mathcal{E}^A$ is called an expectations equilibrium *allocation* in awareness for $\mathcal{E}^A$.

We denote by $EA(\mathcal{E}^A)$ the set of all the expectations equilibria of a pure exchange economy $\mathcal{E}^A$, and denote by $\mathcal{A}(\mathcal{E}^A)$ the set of all the expectations equilibrium allocations in awareness for the economy.

THEOREM 2. *Let $\mathcal{E}^A$ be a pure exchange economy with awareness structure satisfying the conditions **A-1**, **A-2**, **A-3** and **A-4**. Suppose that the economy is atomless. Then there exists an expectations equilibrium in awareness for the economy; i.e., $EA(\mathcal{E}^A) \neq \emptyset$.*

Before proceeding with the proof we should note that:

LEMMA 3. *Let $\mathcal{E}^A$ be a pure exchange economy with awareness structure satisfying the condition **A-3**. For every $t \in T$ and for every $\omega \in \mathrm{Dom}(\mathcal{E}^A)$, the event $(\Delta(p) \cap P_t)(\omega)$ can be decomposed into the disjoint union of all the components $\Pi(\xi)$ for $\xi \in (\Delta(p) \cap P_t)(\omega)$; i.e., there exist $\Pi(\xi_1), \ldots, \Pi(\xi_m) \in \mathcal{F}$ such that $(\Delta(p) \cap P_t)(\omega) = \cup_{k=1}^m \Pi(\xi_k)$ as disjoint union. Moreover, if $\mathbf{x}$ is an assignment for $\mathcal{E}^A$ then we obtain that*

$$(1) \qquad \mathbf{E}_t[U_t(\mathbf{x}(t, \cdot))|\Delta(p) \cap P_t](\omega) = \sum_{k=1}^m \frac{\pi_t(\Pi(\xi_k))}{\pi_t((\Delta(p) \cap P_t)(\omega))} U_t(\mathbf{x}(t, \xi_k), \xi_k),$$

where the left hand side is independent of the choices of the representatives $\{\xi_1, \xi_2, \ldots \xi_m\}$ in the partition $\{\Pi(\xi_k) \mid 1 \leq k \leq m\}$ of $(\Delta(p) \cap P_t)(\omega)$. ∎

Proof of Theorem 2. In view of the existence theorem of a competitive equilibrium for an atomless economy with complete information (cf. Theorem 9 in Debreu [4]), it follows that there exists a $(p^*(\omega), \mathbf{x}^*(\cdot, \omega)) \in \mathcal{W}(\mathcal{E}^A(\omega))$ for each $\omega \in \Omega$ by the conditions **A-1**,**A-2**, **A-3** and **A-4**. Define the pair $(p, \mathbf{x})$ as follows: For each $\omega \in \Omega$, denote $\mathbf{x}(t, \xi) := \mathbf{x}^*(t, \omega)$ for all $\xi \in \Pi(\omega)$ and $\omega \in \mathrm{Dom}(\mathcal{E}^A)$, and set $p(\xi) := p^*(\omega)$ for all $\xi \in \Pi(\omega)$ and $\omega \in \mathrm{Dom}(\mathcal{E}^A)$, $p(\xi) := p^*(\omega)$ for $\omega \notin \mathrm{Dom}(\mathcal{E}^A)$. Then we can verify that

$(p, \mathbf{x})$ is an expectations equilibrium in awareness for $\mathcal{E}^A$: For **EA3**. On noting that $\mathcal{E}^A(\xi) = \mathcal{E}^A(\omega)$ for any $\xi \in \Pi(\omega)$, it follows that $(p(\xi), \mathbf{x}(t, \xi)) \in \mathcal{W}(\mathcal{E}^A(\omega))$ for every $\omega \in \Omega$, and thus we can observe **EA3** by Eq (1). The other conditions in Definition 4 are easily verified. ∎

REMARK 4. As for the notion of expected equilibrium in awareness, we impose the condition that the assignment is $\mathcal{F}$-measurable under the budget constraints in **EA3**. However, in the usual notion of rational expectations equilibrium for an economy under asymmetric information, each rational expectations equilibrium allocation maximizes trader t's interim expectations on all of the $\sigma(p) \vee F_t$-*measurable* assignments under t's budget constraint (cf. Einy et al. [6]). It seems problematic that a trader might not maximize their expected utilities on the assignments under the finest common information.

4.3. *Proof of the main theorem.* We can now state explicitly the main theorem:

THEOREM 4. *Let $\mathcal{E}^A$ be a pure exchange economy with awareness structure satisfying the conditions **A-1**, **A-2**, **A-3** and **A-4**. Suppose that the economy is atomless. Then the ex-post core coincides with the set of all expectations equilibrium allocations in awareness; i.e., $\mathcal{C}^{ExP}(\mathcal{E}^A) = \mathcal{A}(\mathcal{E}^A)$.*

Because $\mathcal{E}^A(\omega)$ is an atomless economy for each $\omega \in \Omega$, we can observe that $\mathcal{C}(\mathcal{E}^A(\omega)) = \mathcal{W}(\mathcal{E}^A(\omega))$ for any $\omega \in \Omega$ by the core equivalence theorem of Aumann ([1], [2]). The main theorem immediately follows from Proposition 1 together with Proposition 5 as below.

PROPOSITION 5. *Let $\mathcal{E}^A$ be an economy with awareness structure satisfying the conditions **A-1**, **A-2**, **A-3** and **A-4**. Then the set of all expectations equilibrium allocations in awareness $\mathcal{A}(\mathcal{E}^A)$ coincides with the set of all the assignments $\mathbf{x}$ such that $\mathbf{x}(\cdot, \omega)$ is a competitive equilibrium allocation for the economy with complete information $\mathcal{E}^A(\omega)$ for all $\omega \in \mathrm{Dom}(\mathcal{E}^A)$; i.e.,*

$$\mathcal{A}(\mathcal{E}^A) = \{\mathbf{x} \in Alc(\mathcal{E}^A) \mid \text{There is a price system } p \text{ such that}$$
$$(p(\omega), \mathbf{x}(\cdot, \omega)) \in \mathcal{W}(\mathcal{E}^A(\omega)) \text{ for all } \omega \in \mathrm{Dom}(\mathcal{E}^A)\}.$$

Proof of Theorem 4. Let $\mathbf{x} \in \mathcal{A}(\mathcal{E}^A)$. By Proposition 5, $(p(\omega), \mathbf{x}(\cdot, \omega)) \in \mathcal{W}(\mathcal{E}^A(\omega))$ for each $\omega \in \mathrm{Dom}(\mathcal{E}^A)$, and so $\mathbf{x}(\cdot, \omega) \in \mathcal{C}(\mathcal{E}^A(\omega))$ by the theorem of Aumann ([1], [2]). By Proposition 1, $\mathcal{C}^{ExP}(\mathcal{E}^A) \supseteq \mathcal{A}(\mathcal{E}^A)$. The converse shall be shown as follows: Let $\mathbf{x} \in \mathcal{C}^{ExP}(\mathcal{E}^A)$. By Proposition 1, $\mathbf{x}(\cdot, \omega) \in \mathcal{C}(\mathcal{E}^A(\omega))$ for every $\omega \in \mathrm{Dom}(\mathcal{E}^A)$. By the above theorem of Aumann, there is $p^*(\omega) \in \mathbf{R}_+^l$ such that $(p^*(\omega), \mathbf{x}(\cdot, \omega)) \in \mathcal{W}(\mathcal{E}^A(\omega))$. Defining the price system p by the same way in the proof of Theorem 2, we can obtain that $(p(\omega), \mathbf{x}(\cdot, \omega)) \in \mathcal{W}(\mathcal{E}^A(\omega))$ for every $\omega \in \mathrm{Dom}(\mathcal{E}^A)$. By Proposition 5, $\mathcal{C}^{ExP}(\mathcal{E}^A) \subseteq \mathcal{A}(\mathcal{E}^A)$. ∎

Proof of Proposition 5. Let $\mathbf{x} \in \mathcal{A}(\mathcal{E}^A)$ and $(p, \mathbf{x}) \in EA(\mathcal{E}^A)$. We shall show that $(p(\omega), \mathbf{x}(\cdot, \omega)) \in \mathcal{W}(\mathcal{E}^A(\omega))$ for any $\omega \in \Omega$: Suppose to the contrary that there exist a state $\omega_0 \in \mathrm{Dom}(\mathcal{E}^A)$ and non-null set $S \subseteq T$ with the property: For each $s \in S$ there is an $\mathbf{a}(s, \omega_0) \in B_s(\omega_0, p)$ such that $U_s(\mathbf{a}(s, \omega_0), \omega_0) > U_s(\mathbf{x}(s, \omega_0), \omega_0)$. Define the function $\mathbf{y} : T \times \Omega \to \mathbf{R}_+^l$ by: $\mathbf{y}(t, \xi) := \mathbf{a}(t, \omega_0)$ for $\xi \in \Pi(\omega_0)$, $\mathbf{y}(t, \xi) := \mathbf{x}(t, \xi)$ otherwise. On

noting that $\mathbf{y}(t, \cdot)$ is $\mathcal{F}$-measurable on $\mathrm{Dom}(\mathcal{E}^A)$ and π_t is full support, we can obtain by Eq (1) that for all $s \in S$, $\mathbf{E}_s[U_s(\mathbf{x}(s, \cdot))|\Delta(p) \cap P_s](\omega_0) < \mathbf{E}_s[U_s(\mathbf{y}(s, \cdot))|\Delta(p) \cap P_s](\omega_0)$, contrary to $(p, \mathbf{x}) \in EA(\mathcal{E}^A)$. The converse will be shown as follows: Let $\mathbf{x} \in \mathcal{A}ss(\mathcal{E}^A)$ with $(p(\omega), \mathbf{x}(\cdot, \omega)) \in \mathcal{W}(\mathcal{E}^A(\omega))$ for all $\omega \in \mathrm{Dom}(\mathcal{E}^A)$. Define the price system $p^* : \Omega \to \mathbf{R}^l_+$ by $p^*(\xi) := p(\omega)$ for all $\xi \in \Pi(\omega)$ and $\omega \in \mathrm{Dom}(\mathcal{E}^A)$, $p^*(\xi) := p(\omega)$ for $\omega \notin \mathrm{Dom}(\mathcal{E}^A)$. We can observe that $(p^*, \mathbf{x}) \in EA(\mathcal{E}^A)$: For **EA3**. Let $\mathbf{y}(t, \cdot) : \Omega \to \mathbf{R}^l_+$ be an $\mathcal{F}$-measurable function with $\mathbf{y}(t, \omega) \in B_t(\omega, p^*)$ for all $\omega \in \mathrm{Dom}(\mathcal{E}^A)$. Since $(p^*(\omega), \mathbf{x}(\cdot, \omega)) \in \mathcal{W}(\mathcal{E}^A(\omega))$ it follows that $U_t(\mathbf{x}(t, \omega), \omega) \geq U_t(\mathbf{y}(t, \omega), \omega)$ for almost all $t \in T$. Therefore by Eq (1), $\mathbf{E}_t[U_t(\mathbf{x}(t, \cdot))|\Delta(p^*) \cap P_t](\omega) \geq \mathbf{E}_t[U_t(\mathbf{y}(t, \cdot))|\Delta(p^*) \cap P_t](\omega)$, as required. The other conditions in Definition 4 can be easily verified. ∎

5. Concluding remarks. Our real concern in this article is about relationship between players' beliefs and their decision making, especially when and how the players take corporate actions under their decisions. We focus on extending the core equivalence theorem of Aumann ([1], [2]) to an economy with traders having 'awareness and belief' model. We have shown that the nature of the equivalence between core and expected equilibrium allocations is dependent not on common-belief assumption nor on the partition structure of traders' information, but on the structure of awareness and belief when each player receives information. It is appropriate to comment on the assumptions in our results. The suppression of any of **A-1** to **A-4** renders Theorem 2 vulnerable to the discussion and the example proposed in Remarks 4.6 of Matsuhisa and Ishikawa [10].

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