RECENT DEVELOPMENTS
IN FUNCTIONAL EQUATIONS AND INEQUALITIES
BANACH CENTER PUBLICATIONS, VOLUME 99
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2013

ON THE INVERSE STABILITY OF FUNCTIONAL EQUATIONS

ZENON MOSZNER

Department of Mathematics, Pedagogical University
Podchorążych 2, 30-084 Kraków, Poland
E-mail: zmoszner@up.krakow.pl

Abstract. The inverse stability of functional equations is considered, i.e. when the function, approximating a solution of the equation, is an approximate solution of this equation.

1. Introduction. The theory of stability of functional equations, initiated by Ulam and Hyers (see [6]), is inspired by the problem: *for the fixed functional equation, is the approximate solution of this equation the approximation of a solution of this equation*? It is known by the results of Forti (see [4]) that the answer is no for the functional equation of homomorphism of the free group, generated by two elements, to the additive group of reals.

It is possible to consider the inverse problem: for the fixed functional equation, is the approximation of a solution of this equation the approximate solution of this equation? More exactly:

— the functional equation $L(f) = R(f)$ is said to be *inversely stable* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every function g if there exists a solution f of this equation such that $\rho(g, f) \leq \delta$ (ρ is a metric) for all arguments in the functions g and f, then $\rho(L(g), R(g)) \leq \varepsilon$ for all variables in the equation.

This stability is called the inverse stability since the stability in the sense of Ulam–Hyers is defined as follows:

— the functional equation $L(f) = R(f)$ is *stable* (in the sense of Ulam–Hyers) if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every function g for which $\rho(L(g), R(g)) \leq \delta$ for all variables in the equation there exists a solution f of the equation such that $\rho(g, f) \leq \varepsilon$ for all arguments in the functions g and f.

2010 *Mathematics Subject Classification*: Primary 39B82; Secondary 39B62.

Key words and phrases: Stability, inverse stability, superstability, b-stability, uniform b-stability, inverse b-stability, inverse uniform b-stability, functional equations, difference equations, differential equations, integral linear equation.

The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc99-0-8

2. Inverse stability

Example 2.1. The equation of homomorphism

$$f(x\cdot y)=f(x)\cdot f(y)$$

for f from a groupoid $(G_1,\cdot)$ to a groupoid $(G_2,\cdot)$ with a metric ρ invariant with respect to the operation "$\cdot$" in G_2 is inversely stable. In fact, for $f,g:G_1\to G_2$ we have

$$\begin{aligned}\rho\big(g(x\cdot y),g(x)\cdot g(y)\big)&\le\rho\big(g(x\cdot y),f(x\cdot y)\big)+\rho\big(f(x\cdot y),f(x)\cdot f(y)\big)\\&\quad+\rho\big(f(x)\cdot f(y),g(x)\cdot f(y)\big)+\rho\big(g(x)\cdot f(y),g(x)\cdot g(y)\big)\\&=\rho\big(g(x\cdot y),f(x\cdot y)\big)+\rho\big(f(x\cdot y),f(x)\cdot f(y)\big)+\rho\big(f(x),g(x)\big)+\rho\big(f(y),g(y)\big)\end{aligned}$$

and if f is a solution of the equation and $\rho(g(x),f(x))\le\frac{\varepsilon}{3}$ for $x\in G_1$, then

$$\rho(g(x\cdot y),g(x)\cdot g(y))\le\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon,\qquad x,y\in G_1.$$

Example 2.2. The functional equation of isometry

$$\rho_2\big(f(x),f(y)\big)=\rho_1(x,y),$$

where $f:X_1\to X_2$ and X_i is a space with the metric ρ_i for $i=1,2$, is inversely stable. Indeed, since the metric ρ_2 is the uniformly continuous function for arbitrary $\varepsilon>0$ there exists a $\delta>0$ ($\delta=\frac{\varepsilon}{2}$) such that $|\rho_2(u_1,v_1)-\rho_2(u_2,v_2)|\le\varepsilon$ for $\rho_2(u_1,u_2)\le\delta$ and $\rho_2(v_1,v_2)\le\delta$. If for $g:X_1\to X_2$ there exists an isometric mapping f such that $\rho_2(g(u),f(u))\le\delta$ for $u\in X_1$, then

$$\big|\rho_2(g(x),g(y))-\rho_1(x,y)\big|=\big|\rho_2(g(x),g(y))-\rho_2(f(x),f(y))\big|\le\varepsilon.$$

Example 2.3. The equation

$$f(x+y)=f(x)\cdot f(y)$$

for f from the additive reals $(\mathbb{R},+)$ to the multiplicative reals $(\mathbb{R},\cdot)$ with natural metric is not inversely stable. Suppose that this equation is inversely stable. For the function $g(x)=\exp x+\delta$ there exists the solution $f(x)=\exp x$ such that $|g(x)-f(x)|\le\delta$ for $x\in\mathbb{R}$. We have

$$\begin{aligned}|g(x+y)-g(x)\cdot g(y)|&=|\exp(x+y)+\delta-(\exp x+\delta)(\exp y+\delta)|\\&=|\delta-\delta(\exp x+\exp y)-\delta^2|\le\varepsilon,\qquad x,y\in\mathbb{R},\end{aligned}$$

which yields a contradiction.

Example 2.4. The equation

$$f(xy)=f(x)f(y)$$

is not inversely stable in the class of the functions from $(\mathbb{R},\cdot)$ to $(\mathbb{R},\cdot)$. The proof is analogous to the above. This equation is inversely stable in the class of the equi-bounded functions (e.g. by M). In fact, we have

$$\begin{aligned}|g(xy)-g(x)g(y)|&=\big|\big(g(xy)-f(xy)\big)+\big(f(x)(f(y)-g(y))\big)+\big((f(x)-g(x))g(y)\big)\big|\\&\le\delta+M\delta+M\delta=(1+2M)\delta,\qquad x,y\in\mathbb{R}\end{aligned}$$

for a solution f such that $|g(x)-f(x)|\le\delta$ for $x\in\mathbb{R}$.

The same situation is for the equation

$$\big(f(x+y)-f(x)-f(y)\big)\big(f(x+y)+f(x)+f(y)\big)=0$$

and for the equations in Examples 2.17 and 2.19.

Example 2.5. The dilatation equation

$$f(xy)=xf(y)+yf(x)$$

for $f:\mathbb{R}\to\mathbb{R}$ is not inversely stable since

$$|g(xy)-xg(y)-yg(x)|=|\delta(1-x-y)|$$

for $g(x)=0+\delta$ and the function $|\delta(1-x-y)|$ is unbounded. This equation is inversely stable in every class of functions with bounded common domain D (e.g. $|x|\le M$ for $x\in D$) since

$$\begin{aligned}|g(xy)-xg(y)-yg(x)|&=\big|g(xy)-xg(y)-yg(x)-\big(f(xy)-xf(y)-yf(x)\big)\big|\\&\le\delta(1+2M)\end{aligned}$$

for $|g(x)-f(x)|\le\delta$ for $x\in D$, where f is a solution of the equation.

Example 2.6. The equation of the idempotent function

$$f(f(x))=f(x)$$

for $f:\mathbb{R}\to\mathbb{R}$ is not inversely stable. For the proof "ad absurdum" assume that for $\varepsilon=1$ there exists a $\delta>0$ such that for every function $g:\mathbb{R}\to\mathbb{R}$ for which there exists a solution f with $|g(x)-f(x)|\le\delta$, we have $|g(g(x))-g(x)|\le 1$. Therefore for

$$f(x)=\begin{cases}x, & x\in\mathbb{Q},\\ 0, & x\in\mathbb{R}\setminus\mathbb{Q},\end{cases}$$

$g(x)=f(x)+\delta^*$, where $\delta^*\le\delta$ and δ^* is irrational, we have $f(f(x))=f(x)$ and $|g(x)-f(x)|\le\delta$ and

$$|g(g(x))-g(x)|=|\delta^*-(x+\delta^*)|\le 1,\qquad x\in\mathbb{Q},$$

which gives a contradiction.

Conclusion 2.7. The translation equation

$$F(F(x,y),z)=F(x,y+z),$$

for $F:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, is not inversely stable. Assume that this equation is inversely stable and let ε and δ be as in the definition of inverse stability of this equation. Then the equation of the idempotent function has to be inversely stable, too. In fact, let $|g(x)-f(x)|\le\delta$ and $f^2=f$. Putting $G(x,y)=g(x)$ and $F(x,y)=f(x)$ we have $|G(x,y)-F(x,y)|\le\delta$ and F is a solution of translation equation, thus $|G(G(x,y),z)-G(x,y+z)|\le\varepsilon$, also $|g(g(x))-g(x)|\le\varepsilon$.

The problem of the Ulam–Hyers stability of translation equation is still open. This equation is Ulam–Hyers stable in the class of functions continuous with respect to each variable (see [11]).

EXAMPLE 2.8. The dynamical system is determined by the function $F : I \times \mathbb{R} \to I$ (I is an interval) which satisfies the translation equation and the identity condition $F(x, 0) = x$ for $x \in I$. This identity condition is evidently inversely stable (the condition $|G(x, y) - F(x, y)| \le \delta$ for $F(x, 0) = x$ implies $|G(x, 0) - x| \le \delta$). It is also stable since $|G(x, 0) - x| \le \delta$ implies $|G(x, y) - F^*(x, y)| \le \delta$ for $F^*(x, y) = G(x, y)$ if $y \ne 0$ and $F^*(x, 0) = x$.

The function F in the dynamical system is supposed to be continuous, thus it is reasonable to consider the stability of this system in the class of continuous functions. The function F^* above may be discontinuous even if the function G is continuous (e.g. for $G(x, y) = x + \delta$ for $I = [a, +\infty)$). The identity condition is also stable in the class of continuous functions; it is sufficient to put for the proof:

$$F_1(x, y) = G(x, y) - G(x, 0) + x \quad \text{if} \quad I = \mathbb{R},$$

$$F_2(x, y) = |G(x, y) - G(x, 0) + x - a| + a \quad \text{if} \quad I = [a, +\infty),$$

$$F_3(x, y) = F_2(x, y) + \alpha(y) \quad \text{if} \quad I = (a, +\infty),$$

where $\alpha : \mathbb{R} \to \mathbb{R}$ is continuous, $\alpha(0) = 0$, $0 < \alpha(y) \le \delta$ for $y \ne 0$ (e.g. $\alpha(y) = |\frac{2\delta}{\pi} \arctan y|$),

$$F_4(x, y) = -|G(x, y) - G(x, 0) + x - b| + b \quad \text{if} \quad I = (-\infty, b],$$

$$F_5(x, y) = F_4(x, y) + \alpha(y) \quad \text{if} \quad I = (-\infty, b),$$

where $\alpha : \mathbb{R} \to \mathbb{R}$ is continuous, $\alpha(0) = 0$, $-\delta < \alpha(y) < 0$ for $y \ne 0$,

$$F_6(x, y) = -\big||G(x, y) - G(x, 0) + x - a| + a - b\big| + b \quad \text{if} \quad I = [a, b],$$

$$F_7(x, y) = F_6(x, y) + \beta(x, y) \quad \text{if} \quad I = [a, b) \text{ or } I = (a, b] \text{ or } I = (a, b),$$

where $\beta : I \times \mathbb{R} \to \mathbb{R}$ is continuous, $\beta(x, 0) = 0$, $|\beta(x, y)| \le \delta$ and

$$a - F_6(x, y) \le \beta(x, y) \le b - F_6(x, y) \text{ for } y \ne 0.$$

We have $|G(x, y) - F_i(x, y)| \le k_i\delta$, where

$$k_i = \begin{cases} 1, & i = 1, \\ 3, & i = 2, 4, 6, \\ 4, & i = 3, 5, 7, \end{cases}$$

if $|G(x, 0) - x| \le \delta$.

The function F_2 may be not good for $I = (a, +\infty)$. In fact, e.g. for $I = (0, +\infty)$ and a continuous mapping $G : I \times \mathbb{R} \to I$ satisfying $G(x, 0) = x + \delta$, $G(1, 1) = \delta$ we have $F_2(1, 1) = 0 \notin I$. The function α is also essential in the definition of the function F_3. Similar situation is for the functions F_5 and F_7.

Notice that in the class of continuous functions the translation equation and the identity condition are Ulam–Hyers stable and the system (the dynamical system) is not stable in this sense if $I \ne \mathbb{R}$ (see [11]). This system is not inversely stable, too. In fact, for $F(x, y) = (\sqrt[3]{x} + y)^3$ and $G(x, y) = F(x, y) + \delta$, F is a solution of translation equation, $F(x, 0) = x$ for $x \in \mathbb{R}$ and we have $|G(x, y) - F(x, y)| \le \delta$ for $x, y \in \mathbb{R}$ and

$$|G(G(0, 0), z) - G(0, 0 + z)| = |G(\delta, z) - G(0, z)| = |\delta + 3z(\sqrt[3]{\delta})^2 + 3z^2\sqrt[3]{\delta}|, \qquad z \in \mathbb{R},$$

which yields a contradiction.

EXAMPLE 2.9. If the linear functional equation

$$f(\alpha(x)) = \beta(x)f(x) + \gamma(x),$$

where $f, \alpha, \beta, \gamma : \mathbb{R} \to \mathbb{R}$, has a solution, then it is inversely stable if and only if the function β is bounded. In fact, if $|\beta(x)| \le M$ for an $M \in \mathbb{R}$ and all $x \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is such that $|g(x) - f(x)| \le \frac{\varepsilon}{1+M}$ for a solution f of the equation and all $x \in \mathbb{R}$, then

$$\begin{aligned}
&|g(\alpha(x)) - \beta(x)g(x) - \gamma(x)| \\
&\qquad = |g(\alpha(x)) - \beta(x)g(x) - \gamma(x) - \big(f(\alpha(x)) - \beta(x)f(x) - \gamma(x)\big)| \\
&\qquad\quad \le |g(\alpha(x)) - f(\alpha(x))| + |\beta(x)||g(x) - f(x)| \le \frac{\varepsilon}{1+M} + \frac{M\varepsilon}{1+M} = \varepsilon.
\end{aligned}$$

Suppose that β is unbounded and the equation is inversely stable. Then for 1 there exists a $\delta > 0$ such that for every function $g : \mathbb{R} \to \mathbb{R}$ satisfying $|g(x) - f(x)| \le \delta$ for a solution f of the equation and all $x \in \mathbb{R}$, we have $|g(\alpha(x)) - \beta(x)g(x) - \gamma(x)| \le 1$. Putting $g(x) = f(x) + \delta$ we have $|g(x) - f(x)| \le \delta$ and

$$|f(\alpha(x)) + \delta - \beta(x)(f(x) + \delta) - \gamma(x)| = |\delta - \beta(x)\delta| \le 1,$$

which gives a contradiction.

If the equation in consideration has no solutions, then it is inversely stable.

EXAMPLE 2.10. The functional equation of periodic function

$$f(x + c) = f(x),$$

where f is a function from a groupoid $(G, +)$ to a metric space (S, ρ), is inversely stable. In fact, if $\rho(g(x), f(x)) \le \frac{\varepsilon}{2}$ for a $g : G \to S$ and a periodic function $f : G \to S$, then

$$\rho(g(x + c), g(x)) \le \rho(g(x + c), f(x + c)) + \rho(f(x + c), f(x)) + \rho(f(x), g(x)) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This equation is not Ulam–Hyers stable for $f : \mathbb{R} \to \mathbb{R}$ and $c \ne 0$ since, for the indirect proof, it is sufficient to put $g(x) = \frac{\delta x}{c}$ in the definition of Ulam–Hyers stability. We also see that there exists a periodic function f such that $|g(x) - f(x)| \le \varepsilon$. Putting here $x = nc$ we have $|\frac{\delta nc}{c} - f(0)| \le \varepsilon$, which yields a contradiction.

EXAMPLE 2.11. The Gołąb–Schinzel equation

$$f(x + yf(x)) = f(x)f(y),$$

where $f : \mathbb{R} \to \mathbb{R}$, is not inversely stable. For the indirect proof, let f be an unbounded solution of this equation (thus $f(0) = 1$), e.g. $f(x) = 1 + x$, ε and δ be as in the definition of inverse stability of this equation and $g(x) = f(x) + \delta$. Since we have $|g(x) - f(x)| \le \delta$, thus

$$\varepsilon \ge |g(x + yg(x)) - g(x)g(y)| = |f\big(x + y(f(x) + \delta)\big) + \delta - (f(x) + \delta)(f(y) + \delta)|$$

and for $y = 0$

$$|f(x) + \delta - (f(x) + \delta)(1 + \delta)| = |-\delta(\delta + f(x))| \le \varepsilon,$$

which gives a contradiction (f is unbounded).

PROPOSITION 2.12. *The stability and the inverse stability are independent.*

Proof. The equation of homomorphism if G_1 is a free group, generated by two elements and G_2 is the additive group of reals, is not stable (see [4]) and it is inversely stable (see Example 2.1). Conversely, the equation of the idempotent function (see Example 2.3) is not inversely stable and it is stable (it is sufficient to put, for the function $g : \mathbb{R} \to \mathbb{R}$ such that $|g(g(x)) - g(x)| \le \varepsilon$, $f(x) = x$ for $x \in g(\mathbb{R})$ and $f(x) = g(x)$ for $x \in \mathbb{R} \setminus g(\mathbb{R})$). ■

PROPOSITION 2.13. *It is possible that a functional equation is inversely stable and an equivalent equation is not. If for two equivalent equations $L_i(f) = R_i(f)$ $(i = 1, 2)$ there exists a sequence $\varepsilon_n \to 0$, $\varepsilon_n > 0$ such that*

$$\{g : \rho(L_1(g), R_1(g)) \le \varepsilon_n\} \subset \{g : \rho(L_2(g), R_2(g)) \le \varepsilon_n\}, \qquad n \in \mathbb{N},$$

and the first equation is inversely stable, then the second equation is inversely stable, too.

Proof. The equations

$$f(x + y) = f(x) + f(y)$$

and

$$(f(x + y))^2 = (f(x) + f(y))^2$$

are equivalent (see [7], p. 380) and the first of them is inversely stable (see Example 2.1), while the second one is not. For the indirect proof it is sufficient to put $g(x) = x + \delta$ and proceed analogously as in Example 2.4.

Fix an $\varepsilon > 0$ and for an $\varepsilon_n \le \varepsilon$ let δ be such as in the inverse stability of the first equation. Assume also that $\rho(g, f) \le \delta$ for a solution f of the second equation. Then $\rho(g, f) \le \delta$ for f as the solution of the first equation, and therefore $\rho(L_1(g), R_1(g)) \le \varepsilon_n$ as well as $\rho(L_2(g), R_2(g)) \le \varepsilon_n \le \varepsilon$. ■

An analogous result for the stability is the following proposition.

PROPOSITION 2.14. *If for two equivalent equations $L_i(f) = R_i(f)$ $(i = 1, 2)$ there exists a sequence $\delta_n \to 0$, $\delta_n > 0$, such that*

$$\{g : \rho(L_2(g), R_2(g)) \le \delta_n\} \subset \{g : \rho(L_1(g), R_1(g)) \le \delta_n\}, \qquad n \in \mathbb{N},$$

and the first equation is stable, then the second equation is stable, too.

REMARK 2.15. A system of two inversely stable functional equations is evidently inversely stable. This statement is not true in the case of stability in Ulam–Hyers sense. On the other hand, a system of two functional equations may be inversely stable although the equations in this system are not. In fact, the equations in the system

$$f(x + y) = f(x)f(y)$$

and

$$f(x + y) = -f(x)f(y),$$

for $f : \mathbb{R} \to \mathbb{R}$, are not inversely stable (see Example 2.3 for the first equation; similarly one can deal with the second equation), while the system is. This system has only the solution $f \equiv 0$. If

$$|g(x) - 0| \le \delta = \frac{\sqrt{1 + 4\varepsilon} - 1}{2},$$

then

$$|g(x+y) - g(x)g(y)| \leq \delta + \delta^2 = \varepsilon$$

and

$$|g(x+y) + g(x)g(y)| \leq \delta + \delta^2 = \varepsilon.$$

Let us next recall that the equation $L(f) = R(f)$ is said to be *superstable* if for every function g such that $\rho(L(g), R(g))$ is bounded, the function g is bounded or it is a solution of the equation.

PROPOSITION 2.16. *If the functional equation $L(f) = R(f)$, where $f : \mathbb{R} \to \mathbb{R}$, is superstable in some class of functions and it has an unbounded solution f in this class such that for every $\delta > 0$ the function $f+\delta$ is not a solution of this equation, then this equation is not inversely stable.*

Proof. Suppose that the equation is inversely stable, ε and δ are as in the definition of inverse stability of this equation and $g = f + \delta$. Thus $|g - f| \leq \delta$ and $|L(g) - R(g)| \leq \varepsilon$. By the superstability of the equation, the function g has to be a solution of the equation or it has to be bounded, which is impossible since $g = f + \delta$. ∎

EXAMPLE 2.17. The equations

$f(x+y)f(x-y) = f(x)^2 - f(y)^2$	(sine equation),
$f(xy) = f(x)f(y)$	(homomorphism equation),
$f(\frac{x+y}{2})^2 = f(x)f(y)$	(Lobačevski equation),
$f(x+y) + f(x-y) = 2f(x)f(y)$	(cosine equation),
$\big(f(x) + f(y)\big)\big(f(x+y) - f(x) - f(y)\big) = 0$	(Dhombres equation),
$f(x+y)\big(f(x+y) - f(x) - f(y)\big) = 0$	(Mikusiński equation),

where $f : \mathbb{R} \to \mathbb{R}$, satisfy the assumptions of Proposition 2.16 (see [3, 1, 5, 2, 10] for the superstability of sine, homomorphism, Lobačevski, Dhombres, and Mikusiński equation, respectively, and [1, 5] for the superstability of cosine equation), thus they are not inversely stable (see Example 2.4).

EXAMPLE 2.18. The equation

$$f(x)^2 = a,$$

where $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$, is inversely stable. If $a < 0$, then the equation has no solutions, thus it is inversely stable. If $a \geq 0$, then every solution of this equation has the form $f(x) = \beta(x)\sqrt{a}$, where β is a function from $\mathbb{R}$ to the set $\{-1, 1\}$. Let $\varepsilon, \delta > 0$ be such that $\delta(\delta + 2\sqrt{a}) \leq \varepsilon$ and $g : \mathbb{R} \to \mathbb{R}$ satisfy $|g(x) - \beta(x)\sqrt{a}| \leq \delta$. We have

$$|g(x)| \leq \delta + |\beta(x)\sqrt{a}| = \delta + \sqrt{a},$$

thus

$$\begin{aligned}|g^2(x) - a| &= \big|\big(g(x) - \beta(x)\sqrt{a}\big)\big(g(x) + \beta(x)\sqrt{a}\big)\big| \\ &\leq \delta\big(|g(x)| + |\beta(x)\sqrt{a}|\big) \leq \delta(\delta + 2\sqrt{a}) \leq \varepsilon.\end{aligned}$$

This example is interesting since the function f is in the square in the inversely stable equation.

By an analogous method it is possible to prove that the equation

$$f(x)^n = a,$$

for $f : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$ and $a \in \mathbb{R}$, is inversely stable, too.

Example 2.19. The equation

$$f(x)^2 = |x|,$$

for $f : \mathbb{R} \to \mathbb{R}$, is not inversely stable. The indirect proof is analogous to the previous one. It is enough to consider $g(x) = \sqrt{|x|} + \delta$. The equation

$$f(x)^2 = x$$

is inversely stable since it has no solutions.

Example 2.20. The equations

$$f(f(x)) = 0$$

and

$$f(f(x)) = x \qquad \text{(involution equation)},$$

for $f : \mathbb{R} \to \mathbb{R}$, are not inversely stable. For the indirect proof it is sufficient to take

$$f(x) = \begin{cases} 0, & x \in \mathbb{Z}, \\ E(x), & x \in \mathbb{R} \setminus \mathbb{Z} \end{cases}$$

($E(x)$ denotes the entire part of x), $g(x) = f(x) + \delta$ with an irrational δ for the first equation, and

$$f(x) = \begin{cases} -x, & x \in \mathbb{Q}, \\ x, & x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

$g(x) = f(x) + \delta$ with an irrational δ for the second equation. The involution equation is also not stable (see [8]).

Remark 2.21. The linear difference equation

$$\alpha_k a_{n+k} + \alpha_{k-1} a_{n+k-1} + \ldots + \alpha_1 a_{n+1} + \alpha_0 a_n = 0,$$

where $a_n : \mathbb{N} \to \mathbb{R}$, $\alpha_0, \ldots, \alpha_k \in \mathbb{R}$, is inversely stable ($\delta = \frac{\varepsilon}{\alpha_0 + \ldots + \alpha_k}$ if $\alpha_0 + \ldots + \alpha_k \neq 0$, and $\delta > 0$ is arbitrary if $\alpha_0 + \ldots + \alpha_k = 0$). The difference equation

$$a_{n+1}^2 = a_n a_{n+2}$$

is not inversely stable (it is sufficient to put $b_n = 2^n + \delta$ for the indirect proof).

Remark 2.22. The inverse stability of differential equations is not interesting since the boundedness of the function does not imply the boundedness of the derivative of this function. For example, for the function $g(x) = x + \alpha(x)$ the inequality $|g(x) - x| \leq \delta$ gives $|\alpha(x)| \leq \delta$ and $|g'(x) - 1| \leq \varepsilon$ gives $|\alpha'(x) - 1| \leq \varepsilon$, thus the differential equation

$$f'(x) = 1$$

is not inversely stable.

REMARK 2.23. The Fredholm integral linear equation

$$f(x) = \int_a^b K(x,t)f(t)\,dt + \alpha(x),$$

where $f:\mathbb{R}\to\mathbb{R}$ is an unknown function and functions $\alpha:\mathbb{R}\to\mathbb{R}$ and $K:\mathbb{R}\times[a,b]\to\mathbb{R}$ are given, is inversely stable if the function $\int_a^b |K(x,t)|\,dt$ is bounded. If $\int_a^b K(x,t)\,dt$ is unbounded, then the equation is not inversely stable. The proof is analogous to the proof in Example 2.6.

3. Inverse b-stability. In the theory of the stability of functional equations we also investigate the notion of the so-called *b-stability*, defined as follows:

— for every function g if $\rho(L(g),R(g))$ is bounded, then there exists a solution f of the equation $L(f)=R(f)$ such that $\rho(g,f)$ is bounded, too.

This definition is not equivalent to the Ulam–Hyers stability.

One can also consider an analogous notion of the *inverse b-stability*:

— for every function g if there exists a solution f of the equation $L(f)=R(f)$ such that $\rho(g,f)$ is bounded, then $\rho(L(g),R(g))$ is bounded, too.

All above-considered equations, which are (not) inversely stable, are (not) inversely b-stable. The inverse stability and the inverse b-stability are not equivalent. For example, the equation

$$f(x)^2 = f(0)^2,$$

where $f:\mathbb{R}\to\mathbb{R}$ with the natural metric, is inversely b-stable and it is not inversely stable. Every solution of this equation is bounded, thus every function $g:\mathbb{R}\to\mathbb{R}$ for which $|g(x)-f(x)|$ is bounded for a solution f, is bounded, and so is $|g(x)^2-g(0)^2|$. Suppose that this equation is inversely stable. Then, for $\varepsilon=1$ there exists a $\delta>0$ such that for every function $g:\mathbb{R}\to\mathbb{R}$ if there exists a solution f for which $|g(x)-f(x)|\le\delta$, then $|g(x)^2-g(0)^2|\le 1$. Putting

$$g(x) = \begin{cases} \frac{1-\delta^2}{\delta}, & x=0,\\ \frac{1+\delta^2}{\delta}, & x\neq 0 \end{cases}$$

we have for the solution $f(x)=\frac{1}{\delta}$ of the equation: $|g(x)-\frac{1}{\delta}|\le\delta$. Therefore we have for $x\neq 0$,

$$|g(x)^2-g(0)^2| = |(g(x)-g(0))(g(x)+g(0))| = 2\delta\,\frac{2}{\delta} = 4 > 1,$$

which yields a contradiction.

This equation is Ulam–Hyers stable. In fact, if $|g(x)^2-g(0)^2|\le\varepsilon^2$ for $x\in\mathbb{R}$, then $|g(x)-g(0)|\le\varepsilon$ or $|g(x)+g(0)|\le\varepsilon$. Putting

$$f(x) = \begin{cases} g(0), & |g(x)-g(0)|\le\varepsilon,\\ -g(0), & |g(x)-g(0)|>\varepsilon, \end{cases}$$

we receive the solution of the equation in consideration for which $|g(x)-f(x)|\le\varepsilon$. This equation is evidently b-stable, too.

The equation

$$f(x)^{-1} = 1,$$

for $f : \mathbb{R} \to \mathbb{R}$, is inversely stable and it is not inversely b-stable, thus the inverse stability does not imply the inverse b-stability. For the proof let $\delta > 0$ be such that $\delta < 1$ and $\frac{\delta}{1-\delta} \leq \varepsilon$ for $\varepsilon > 0$ and $|g(x) - 1| \leq \delta$ for $x \in \mathbb{R}$. We have $1 - |g(x)| \leq \delta$, thus $1 - \delta \leq |g(x)|$ and $|g(x)^{-1}| \leq (1-\delta)^{-1}$. We receive thus

$$|g(x)^{-1} - 1| = \left|\frac{1 - g(x)}{g(x)}\right| \leq \frac{\delta}{1-\delta} \leq \varepsilon.$$

On the other hand, the boundedness of the function $g(x) - 1$ does not imply the boundedness of the function $g(x)^{-1} - 1$, thus the equation in consideration is not inversely b-stable. This equation is Ulam–Hyers stable. In fact, let $\delta > 0$ be such that $\delta < 1$ and $\frac{\delta}{1-\delta} \leq \varepsilon$ for $\varepsilon > 0$ and assume that $|g(x)^{-1} - 1| \leq \delta$. We have thus

$$|g(x)| - 1 \leq |g(x) - 1| \leq \delta |g(x)|$$

and we receive $|g(x)| \leq \frac{1}{1-\delta}$ and

$$|g(x) - 1| \leq \delta |g(x)| \leq \frac{\delta}{1-\delta} \leq \varepsilon.$$

This equation evidently is not b-stable.

4. Inverse uniform b-stability. We consider the uniform b-stability (see [9]):

— the equation $L(f) = R(f)$ is said to be *uniformly b-stable* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every function g if $\rho(L(g), R(g)) \leq \varepsilon$, then there exists a solution f of the equation such that $\rho(g, f) \leq \delta$.

This b-stability is "uniform" since δ does not depend on the function g.

Per analogiam it is possible to consider the inverse uniform b-stability:

— the equation $L(f) = R(f)$ is said to be *inversely uniformly b-stable* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every function g if there exists a solution f of the equation for which $\rho(g, f) \leq \varepsilon$, then $\rho(L(g), R(g)) \leq \delta$.

This notion of stability is in reality the inverse of the stability in Ulam–Hyers stability (the implication in the inverse uniform b-stability is the inverse of the implication in the Ulam–Hyers stability). The inverse uniform stability implies evidently the inverse b-stability but not inversely. For example, the equation

$$f(x)^2 = f(0)^2,$$

for $f : \mathbb{R} \to \mathbb{R}$, is inversely b-stable (see the remark above) and it is not inversely uniformly b-stable. In fact, for the indirect proof, let $\varepsilon = 1$, the function

$$f(x) = \begin{cases} \delta, & x = 0, \\ -\delta, & x \neq 0, \end{cases}$$

is a solution of the equation and for the function $g(x) = f(x) + 1$ we have

$$|g(x) - f(x)| = 1 = \varepsilon$$

and for $x \neq 0$:

$$|g(x)^2 - g(0)^2| = |\big(f(x)+1\big)^2 - \big(f(0)+1\big)^2| = |(-\delta+1)^2 - (\delta+1)^2| = |-4\delta| > \delta,$$

which gives a contradiction.

The equation in consideration is uniformly b-stable (see Section 3).

Acknowledgments. The author wishes to thank Dr. Krzysztof Ciepliński for the preparation of the manuscript of this paper.

References

[1] J. A. Baker, *The stability of the cosine equation*, Proc. Amer. Math. Soc. 80 (1980), 411–416.

[2] B. Batko, *Stability of Dhombres' equation*, Bull. Austral. Math. Soc. 70 (2004), 499–505.

[3] P. W. Cholewa, *The stability of the sine equation*, Proc. Amer. Math. Soc. 88 (1983), 631–634.

[4] G. L. Forti, *The stability of homomorphisms and amenability, with applications to functional equations*, Abh. Math. Sem. Univ. Hamburg 57 (1987), 215–226.

[5] P. Găvruţa, *On the stability of some functional equations*, in: Stability of Mappings of Hyers–Ulam Type, Hadronic Press Collect. Orig. Artic., Hadronic Press, Palm Harbor, FL, 1994, 93–98.

[6] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. 27 (1941), 222–224.

[7] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality*, Birkhäuser, Basel, 2009.

[8] A. Mach, Z. Moszner, *On some functional equations involving involutions*, Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II 216 (2007), 3–13.

[9] Z. Moszner, *On the stability of functional equations*, Aequationes Math. 77 (2009), 33–88.

[10] Z. Moszner, *On stability of some functional equations and topology of their target spaces*, Ann. Univ. Paedagog. Crac. Stud. Math. 11 (2012), 69–94.

[11] B. Przebieracz, *On the stability of the translation equation and dynamical systems*, Nonlinear Anal. 75 (2012), 1980–1988.