RECENT DEVELOPMENTS
IN FUNCTIONAL EQUATIONS AND INEQUALITIES
BANACH CENTER PUBLICATIONS, VOLUME 99
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2013

ADDITION THEOREMS AND RELATED GEOMETRIC PROBLEMS OF GROUP REPRESENTATION THEORY

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Abstract. The Levi-Civita functional equation $f(gh) = \sum_{k=1}^{n} u_k(g)v_k(h)$ $(g, h \in G)$, for scalar functions on a topological semigroup G, has as the solutions the functions which have finite-dimensional orbits in the right regular representation of G, that is the matrix elements of G. In considerations of some extensions of the L-C equation one encounters with other geometric problems, for example: 1) which vectors x of the space X of a representation $g \mapsto T_g$ have orbits $O(x)$ that are "close" to a fixed finite-dimensional subspace? 2) for which x, $O(x)$ is contained in the sum of a fixed finite-dimensional subspace and a finite-dimensional invariant subspace? 3) what can be said about a pair L, M of finite-dimensional subspaces if $T_gL \cap M \neq \{0\}$ for all $g \in G$? 4) which finite-dimensional subspaces L have the property that for each $g \in G$ there is $0 \neq x \in L$ with $T_gx = x$? The problem 1) arises in the study of the Hyers–Ulam stability of the L-C equation. It leads to the theory of *covariant widths* — the analogues of Kolmogorov widths which measure the distances from a given set to n-dimensional *invariant* subspaces. The problem 2) is related to multivariable extensions of the L-C equation; the study of this problem is based on the theory of subadditive set-valued functions which was developed specially for this aim. To problems 3) and 4) one comes via the study of the equations $\sum_{i=1}^{m} a_i(g)b_i(hg) = \sum_{j=1}^{n} u_j(g)v_j(h)$. We will finish by the consideration of "fractionally-linear version" of the L-C equation which is very important for the theory of integrable dynamical systems.

1. Introduction.

The Levi-Civita functional equation

$$f(x+y) = \sum_{i=1}^{n} a_i(x)b_i(y) \tag{1}$$

2010 *Mathematics Subject Classification*: 20C15, 20M30, 39B22, 39B32, 39B52, 39B62, 39B82.
Key words and phrases: functional equations on semigroups, addition theorems, representations of topological semigroups, stability in the Hyers–Ulam sense, subadditive set-valued functions on groups, elliptic functions.
The paper is in final form and no version of it will be published elsewhere.

was originally considered in the present form by Stéphanos [17], Levi-Civita [5] and Stäkel [16] for differentiable functions on $\mathbb{R}$. Its general solution (even in the class of all measurable functions) is a quasipolynomial of n-th order:

$$f(x) = \sum_{k=1}^{n} P_k(x) e^{\lambda_k x},$$

where P_k are polynomials, $\sum_{k=1}^{n}(\deg P_k + 1) = n$.

These equations, in the form

$$f(gh) = \sum_{i=1}^{n} a_i(g) b_i(h) \tag{2}$$

can be considered for functions on arbitrary semigroup G. For abelian groups this situation was considered by Székelyhidi in 1982 [18]; the general case — by the author in 1994 [10]. Such an extension is quite natural because it relates the problem with the group representation theory, and even the equation (1) on $\mathbb{R}$ is much more transparent from the point of view of this theory. Indeed, if we are looking for the solutions of (2) in some functional space X on G we just need to consider the representation of G by the right shifts on X and note that f satisfies (2) if and only if its orbit $O(f) = \{f(gh) : h \in G\}$ is contained in a subspace of dimension $\leq n$. So the study of equations (2) is related to the study of the vectors with finite-dimensional orbits in the representation spaces (*finite vectors*). To formulate the general result and for further discussion we need to recall the following notion.

Definition 1.1. A function f on a (topological) semigroup G is called a *matrix element* if there is a (continuous) representation T of G on a finite-dimensional space X, a vector $\xi \in X$ and a functional $\eta \in X^*$ such that $f(g) = \langle T(g)\xi, \eta\rangle$. The dimension of X is called the order of the matrix element f.

The set of all matrix elements of order n is denoted by $\mathfrak{M}_n(G)$.

The following result is based on the fact that matrix elements are precisely the finite vectors of regular representations.

Theorem 1.2 ([10, 11]). *Let G be a unital topological semigroup. A continuous function $f : G \to \mathbb{C}$ satisfies* (2) *if and only if it is a matrix element of order n. If f is bounded then the corresponding representation can be chosen to be bounded.*

For many classical groups all finite-dimensional representations are described and their matrix elements are well known. For instance, for the group of circle $\mathbb{T}$ the matrix elements are trigonometrical polynomials. The matrix elements of $\mathbb{R}^n$ are quasi-polynomials in n variables, that is the functions of the form

$$\sum_{k=1}^{N} e^{(\lambda_k, x)} P_k(x),$$

where all P_k are polynomials, $\lambda_k \in \mathbb{C}^n$. This implies the classical result of Levi-Civita.

The theory developed for Levi-Civita equation extends naturally in many directions. Dealing with such extensions we will always try to formulate and discuss the corresponding problems in the setting of the representation theory.

Before presenting the main results of the paper, we note that apart from the general equations (2) it is important to study its *specifications*, that is the equations that arise from (2) when some additional conditions on the functions f, a_i, b_i are imposed. Among them are the equations of Cauchy, Pexider, Lobachevsky, Gauss, Jensen, Thielman and many others (see [1]). The solutions of particular specifications are of course the matrix elements of G, but it needs sometimes considerable efforts to handle them precisely even if the (semi)group is such that its matrix elements are transparently described.

In this paper we gather some results on the representation theory approach to the functional equations (2) and related topics. Most of them were obtained in papers of the author; we present also some theorems of Székelyhidi, Buchstaber, Braden and other mathematicians; a few new results are also included (with full proofs).

2. Covariant n-widths and stability of the Levi-Civita equation. The stability of an equation in the Ulam–Hyers sense means, roughly speaking, that each function which "almost satisfies" the equation is "close" to a precise solution of the equation. It turns out that the Levi-Civita equation is stable when considered in various functional classes on amenable groups [11]. The proof of such results uses some approximation technique for subsets of a Banach space in which a representation of a topological group acts.

Indeed, suppose that a bounded function f on G almost satisfies (2) in the sense that

$$\Big|f(gh) - \sum_{i=1}^{n} a_i(g)b_i(h)\Big| < \delta, \tag{3}$$

for all $g, h \in G$, where $\delta > 0$. This means that all functions $g \mapsto f(gh)$ are on the distance $< \delta$ (in the uniform metric) from the subspace L, spanned by the functions $a_i(g)$, $i = 1, \dots, n$. In other words, the orbit $O(f)$ of f is on the distance $\leq \delta$ from L. On the other hand, let f be on the distance $< \varepsilon$ from some function ϕ, satisfying the equation

$$\phi(gh) = \sum_{i=1}^{n} u_i(g)v_i(h).$$

Then the linear span L' of $O(\phi)$ is contained in the linear span of $\{u_1, \dots, u_n\}$ and so $\dim L' \leq n$. Thus $O(f)$ is on the distance $\leq \varepsilon$ from an *invariant* subspace of dimension $\leq n$. This leads to a general problem from the representation theory: given a representation $g \mapsto T_g$ of a group G on a Banach space X, for any invariant subset $K \subset X$ to estimate its distance to n-dimensional invariant subspaces of X via the distances to arbitrary n-dimensional subspaces.

The minimal distance of a set from n-dimensional subspaces is called its n-width. More precisely, for each $n \in \mathbb{N}$, let $\mathcal{L}_n(X)$ be the set of all subspaces $L \subset X$ with $\dim L \leq n$. The *n-width* $p_n(K)$ of a bounded subset $K \subset X$ is defined by the condition

$$p_n(K) = \inf_{L \in \mathcal{L}_n} \sup_{x \in K} \operatorname{dist}(x, L). \tag{4}$$

The technique of n-widths is a useful tool of the approximation theory; for $K = AB_1(X)$ where A is a bounded linear operator, $B_1(X)$ the unit ball of X, the n-widths $p_n(K)$ are called singular numbers of A, they play an outstanding role in the operator theory.

So let T be a representation of a group G on X. We introduce a covariant version $p_n^G(K)$ of n-widths replacing the set $\mathcal{L}_n$ in (4) by the subset $\mathcal{L}_n^G \subset \mathcal{L}_n$ of all G-invariant subspaces:

$$p_n^G(K) = \inf_{L \in \mathcal{L}_n^G} \sup_{x \in K} \operatorname{dist}(x, L). \tag{5}$$

Clearly, $p_n(K) \le p_n^G(K)$. The following two statements show that sometimes an opposite type estimate holds.

Theorem 2.1 ([11]). *Let T_g be a unitary representation of a group G on a Hilbert space H, and $K \subset H$ be an invariant subset. Then*

$$p_n^G(K) < \sqrt{n+1}\, p_n(K).$$

For the general Banach spaces we have to restrict the class of groups and our estimates are more complicated.

Theorem 2.2 ([11]). *Let G be an amenable group. For any $n \in \mathbb{N}$, $C > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that, if T is an isometric dual representation and K is a G-invariant subset with $p_n(K) < \delta$ and $\sup_{x \in K} \|x\| \le C$, then $p_n^G(K) < \epsilon$.*

This implies stability of the Levi-Civita equation for bounded functions on amenable groups:

Theorem 2.3 ([11]). *Let G be an amenable locally compact group. Then for any $\epsilon > 0$, $C > 0$ and $n \in \mathbb{N}$, there is $\delta > 0$ such that, if $f \in L^\infty(G)$ with $\|f\| < C$ satisfies the condition*

$$\Big| f(gh) - \sum_{i=1}^n a_i(g) b_i(h) \Big| < \delta, \tag{6}$$

for all $g, h \in G$, where a_i, b_i are some functions on G, then there exists $\phi \in \mathfrak{M}_n(G)$ with $\|f - \phi\| < \epsilon$.

Using Theorem 2.1 we obtain more precise estimates for the case of compact groups.

Theorem 2.4 ([11]). *Let G be a compact group. If $f \in L^2(G)$ satisfies (6) for almost all g, h, then there is $\phi \in \mathfrak{M}_n(G)$ with $\|f - \phi\|_2 < (n+1)^{1/2}\delta$.*

Now we turn to unbounded solutions of (2). For the general geometric background we should consider a representation T of a group G on a topological linear space X that contains an invariant subspace Y which is supplied with a T-invariant complete norm $\|\cdot\|_Y$, such that the inclusion $(Y, \|\cdot\|_Y) \to X$ is continuous.

Let us say that a set $E \subset X$ is on the finite $\|\cdot\|_Y$-distance from a subset $F \subset X$ if there is $C > 0$ such that for any $x \in E$, there is $z \in F$ with $x - z \in Y$ and $\|x - z\|_Y < C$.

Theorem 2.5 ([11]). *Suppose that G is amenable. For a vector $x \in X$, the following conditions are equivalent*:

(i) *$x = y + z$, where $y \in Y$, z is contained in an invariant subspace of dimension $\le n$*;

(ii) *there is an n-dimensional subspace L of X such that the orbit of x is on the finite $\|\cdot\|_Y$-distance from L.*

Applying this result to the regular representation of G on the space X of Haar-measurable functions on G and the subspace $Y = L^\infty(G)$ of X, we obtain

COROLLARY 2.6 ([11]). *A measurable function f on an amenable group G satisfies the condition*

$$\sup_{g,h\in G}\Big|f(gh) - \sum_{i=1}^{n} u_i(g)v_i(h)\Big| < \infty,$$

for some measurable functions u_i, v_i, if and only if $f = \phi + f_1$, where $f_1 \in \mathfrak{M}_n(G)$, $\phi \in L^\infty(G)$.

As with the Levi-Civita equation itself, one can obtain some consequences for specifications of (2). For example, the following result follows from Corollary 2.6.

COROLLARY 2.7 ([11]). *A measurable function f on a locally compact group G satisfies the condition*

$$\sup_{g,h\in G} |f(gh) - f(g)f(h)| < \infty$$

if and only if it is either bounded or multiplicative.

3. A generalization of the Levi-Civita equation related to richly periodic spaces of functions

3.1. Special subspaces in G-spaces. In this section we study the functional equation

$$\sum_{i=1}^{m} a_i(g)b_i(hg) = \sum_{j=1}^{n} u_j(g)v_j(h), \qquad g, h \in G; \tag{7}$$

it reduces to (2) when $m = 1$ and $a_1(g) = 1$.

Considering both parts of (7) as functions of h, one can look at the equation in a more geometrical way. Indeed, it says that for each $g \in G$ the element $f_g = \sum_{i=1}^m a_i(g)b_i$ has the property that its shift $T_g f_g$ belongs to the span of $\{v_i\}$. If we assume that the m-tuple $(a_i)_{i=1}^m$ is linearly independent (this clearly does not reduce generality) and not all functions b_i are zero, then f_g is non-zero for g in a non-void open subset of G. It is not difficult to translate this into the terms of representations.

Let T be a representation of a group G on a linear space X. We will call a finite-dimensional subspace L of X *special* (*locally special*), if there is a finite-dimensional subspace $S \subset X$ such that for each $g \in G$ (respectively, for each g in a non-empty open subset of G), there is a non-zero vector $x \in L$ with $T_g x \in S$.

Clearly, a finite-dimensional subspace containing a finite vector is special. Our result on (locally) special subspaces says that the converse is true:

THEOREM 3.1 ([14]). *Let G be a topological group and T be a continuous representation of G on a linear topological space X. Then each special subspace of X contains a finite vector. If G is connected then the same is true for locally special subspaces.*

In other words, a minimal (locally) special subspace is one-dimensional.

This has a flavor of stability. Let us say that two subspaces L_1, L_2 of X are "close", if their intersection is non-zero. Then the assertion is: if each element of the orbit of a finite-dimensional subspace L is "close" to a finite-dimensional subspace S then L is "close" to a finite-dimensional invariant subspace.

To obtain a description of continuous solutions of equation (7) we apply the previous result to the right regular representation of G on the space $C(G)$ of all continuous functions on G. As we know, in this case finite vectors are just the matrix elements of G.

Corollary 3.2 ([14]). *Let continuous functions a_i, b_i $(i = 1, \dots, m)$ and u_j, v_j $(j = 1, \dots, n)$ on a connected topological group G satisfy the equality* (7). *If the functions a_i are linearly independent, then all b_i are matrix elements.*

Remark 3.3. The statement of Corollary 3.2 remains true if we assume that the condition (7) holds for all $h \in G$ and all $g \in U$, where U is a non-void open subset of G such that the restrictions of $a_1, a_2, \dots, a_m$ to U form a linearly independent set.

As an example of an application of Corollary 3.2 let us consider the equation

$$f(z) = \sum_{k=0}^{n} (z-y)^k g_k(y) + (z-y)^n h(z), \qquad z, y \in \mathbb{R},\ z \neq y,\ n \geq 1. \tag{8}$$

It was shown by Cross and Kannappan [4] that all solutions of (8) are polynomials. This result can be deduced from Corollary 3.2 because the change of variables $x = z - y$ reduces this equation to the form

$$f(x+y) - x^n h(x+y) = \sum_{k=0}^{n} x^k g_k(y) \tag{9}$$

which is a special case of (7) with $G = \mathbb{R}$. By Corollary 3.2, functions f and h (and, therefore, all g_k) are quasipolynomials. The simple analysis of coefficients of the dominant exponent shows that all these quasipolynomials are, as a matter of fact, polynomials.

3.2. Singular spaces. Let us study the special case of equation (7)

$$\sum_{k=1}^{m} a_k(g) b_k(hg) = \sum_{k=1}^{m} a_k(g) b_k(h). \tag{10}$$

Under some additional assumptions (for example, if all a_i are linearly independent and not all b_i are zero) the equality means that the linear hull of $\{b_1, \dots, b_m\}$ in $C(G)$ contains non-zero functions f with any period $g \in U$, where U is a non-empty open subset of G: $f(hg) = f(h)$ for all $h \in G$.

In terms of representations the situation can be described as follows. Let as above T be a continuous representation of a connected topological group G on a linear topological space X. Let us call a finite-dimensional subspace $L \subset X$ *singular* (*locally singular*) if for each $g \in G$ (respectively, for each $g \in U$, where U is a non-void open subset of G), there is a non-zero $x \in L$ with

$$T_g x = x. \tag{11}$$

Clearly, each locally singular subspace is locally special and, therefore, it contains a finite vector, by Theorem 3.1. Furthermore, since each locally singular subspace contains a minimal locally singular subspace, and each subspace containing a locally singular subspace is locally singular, it suffices to study the structure of minimal locally singular subspaces.

LEMMA 3.4 ([14]). *Let L be a locally singular subspace. Then the subspace L_f consisting of all finite vectors in L is also locally singular.*

COROLLARY 3.5. *Each minimal locally singular subspace consists of finite vectors and, as a consequence, is contained in a finite-dimensional invariant subspace.*

Further analysis shows that the structure of minimal locally singular spaces depends on the group. Restricting ourselves to the case of a commutative group G let us consider two principal cases: the class of compact groups and the groups $\mathbb{R}^n$.

Let us say that a vector x is *fixed* for a representation T of G if $x \neq 0$ and $T_g x = x$ for all $g \in G$. In this case the subspace $\mathbb{C}x$ is singular. The following theorem shows that, in the case of compact group, all minimal singular subspaces are of this form, that is, have dimension 1.

THEOREM 3.6 ([14]). *If G is a compact connected abelian group, then each locally singular subspace of X contains a fixed vector.*

Note that it can be reformulated as a somewhat unusual result on fixed points.

COROLLARY 3.7. *Let G be a compact connected abelian group of linear maps of a real Banach space X and let W be a finite-dimensional subset of X. If each operator $g \in G$ has a fixed point in W then G has a common fixed point in W.*

Proof. Let us denote by L the linear span of W and by P the projection of L onto the subspace of all common fixed points for G,

$$P = \frac{1}{\mu(G)} \int_G g \, d\mu,$$

where μ is the Haar measure on G. Let $W_1 = (1 - P)W$. If $0 \in W_1$ then $W \cap PX \neq \emptyset$ and we are done.

Suppose that $0 \notin W_1$ and let us prove that $\operatorname{span}(W_1)$ is a singular subspace. Indeed, any $g \in G$ has a fixed vector $x \in W$, therefore $(1 - P)x \in W_1$ is also fixed for g since it is non-zero by our assumption and $g(1 - P)x = (1 - P)gx = (1 - P)x$.

Therefore, by Theorem 3.6, $\operatorname{span}(W_1)$ contains a common fixed vector for G; this contradicts to the fact that all such vectors belong to PX while $\operatorname{span}(W_1) \subset (1 - P)X$. ∎

In the next subsection we will see that in the other special case, $G = \mathbb{R}^n$, the situation is different, so the condition of compactness cannot be dropped in Theorem 3.6. To see, that the commutativity condition is also essential, note that the space $\mathbb{R}^3$ is singular with respect to the proper orthogonal group O_3^+ (each element of O_3^+ is a rotation and therefore has a fixed vector). It is easy to check that it is minimal.

3.3. Richly periodic spaces of functions on $\mathbb{R}^n$. In this subsection we consider the singular subspaces for the group $G = \mathbb{R}^n$. Clearly a finite-dimensional subspace $M \subset C(\mathbb{R}^n)$ is singular (locally singular) if, for each vector $a \in \mathbb{R}^n$ (respectively, for each a in an open subset of $\mathbb{R}^n$), M contains a non-zero function with the period a:

$$f(x + a) = f(x), \qquad x \in \mathbb{R}^n. \tag{12}$$

So the study of locally singular subspaces in $C(\mathbb{R}^n)$ is the study of finite-dimensional subspaces of functions that have a "rich" set of periods.

THEOREM 3.8 ([14]). *Any (locally) singular subspace $L \subset C(\mathbb{R}^n)$ contains a (locally) singular subspace that consists of polynomials.*

Thus each minimal singular space consists of polynomials. To see an example of a minimal singular subspace of dimension > 1 in $C(\mathbb{R}^n)$, for $n > 1$, it suffices to consider the space of all functions of the form $ax_1 + bx_2$, where $a, b \in \mathbb{R}$.

THEOREM 3.9 ([14]). *A minimal locally singular subspace M of polynomials in n variables has dimension $\geq m + 1$ where $m = \max\{\deg(f) : f \in M\}$. This estimate is strict.*

Now we can (partially) describe solutions of functional equation (10) for $G = \mathbb{R}^n$:

COROLLARY 3.10. *Let continuous functions a_k, b_k $(k = 1, \dots, m)$ on $\mathbb{R}^n$ satisfy the equality*

$$\sum_{k=1}^{m} a_k(x)b_k(x + y) = \sum_{k=1}^{m} a_k(x)b_k(y) \tag{13}$$

for all $x, y \in \mathbb{R}^n$. If a_i are linearly independent then all b_k are polynomials.

4. Multivariable versions of the Levi-Civita equation

4.1. Decomposable functions. The solutions of the Levi-Civita equation (1) are those functions f for which $f(x + y)$ belongs to the algebra generated by functions of one variable. It is natural to raise the problem: to describe functions f for which $f(x_1+\ldots+x_n)$ belongs to the algebra generated by functions of fewer variables.

This problem is still not solved in full generality; here we confine ourselves to a more special condition. To formulate it maximally transparently, let us begin with the case of functions of three (real) variables. In this case, instead of the general relation

$$f(x + y + z) = \sum_{i=1}^{N} a_i(x, y)b_i(y, z)c_i(x, z),$$

we study solutions of the following functional equation:

$$f(x + y + z) = \sum_{i=1}^{n} a_i(x)A_i(y, z) + \sum_{i=1}^{m} b_i(y)B_i(x, z) + \sum_{i=1}^{k} c_i(z)C_i(x, y). \tag{14}$$

In general we deal with functions on arbitrary topological semigroup G and impose the condition that $f(g_1g_2 \cdots g_n)$ is a sum of products of functions depending on *disjoint* sets of variables. We call such functions f *decomposable*. Clearly it suffices to consider the products of two factors:

$$f(g_1g_2 \cdots g_n) = \sum_{E} \sum_{j=1}^{N_E} u_j^E v_j^E, \tag{15}$$

where E runs through all proper non-empty subsets of $\{1, 2, \dots, n\}$, $N_E \in \mathbb{N}$ and for each E, the functions u_j^E only depend on variables g_i with $i \in E$, while the v_j^E only depend on g_i with $i \notin E$ (the general case is reduced to this one by joining different groups of variables). All functions $(f,\, u_j^E,\, v_j^E)$ can be considered as unknown, but in fact we obtain only a description of f, which also gives sufficiently rich information about the structure of u_j^E, v_j^E.

Clearly, every matrix element of G admits an addition theorem of the form (15). It turns out that in a quite general situation the equation (15) does not have other solutions apart from the matrix elements.

To workout a more geometric approach to the problem, let us return to equation (14). It is not difficult to deduce from this equation that all functions C_i can be written in the form $C_i(x,y) = D_i(x+y) + \sum_k \alpha_k^i(x)\beta_k^i(y)$. Substituting this into (14) we obtain

$$f(x+y+z) - \sum_{i=1}^{K} c_i(z)D_i(x+y) = \sum_{r=1}^{R} u_r(x,z)v_r(y,z). \tag{16}$$

Let us fix z and denote by w ($= w_z$) the function $t \mapsto f(t+z) - \sum_{i=1}^{K} c_i(z)D_i(t)$. Then (16) shows that w satisfies the Levi-Civita equation. By Theorem 2, w is a matrix element, that is, a finite vector of the right regular representation T. Hence $T_z f - \sum_{i=1}^{K} c_i(z)D_i$ is a finite vector. Denoting by L the linear span of the functions D_i we see that $T_z f$ belongs to the sum of L and a finite-dimensional invariant subspace. So we come to the following problem of the representation theory:

Let T be a representation of G on X, $x \in X$, and suppose that for each $h \in G$,

$$T_h x \in L + V_h,$$

where L is a fixed finite-dimensional subspace and each V_h is a G-invariant subspace of restricted dimension: $\dim V_h \le N$. *Does it imply that G-orbit of x is contained in a finite-dimensional subspace?*

Setting $\tilde{L} = L + \mathbb{C}x$ one can, after some additional simplifications, reformulate the problem as follows:

Let $N \in \mathbb{N}$ and let, for each $h \in G$, there exists an invariant subspace W_h with $\dim W_h \le N$ *such that*

$$T_h\tilde{L} \subset \tilde{L} + W_h. \tag{17}$$

Is G-orbit of $\tilde{L}$ contained in some finite-dimensional subspace L_0?

One can show that there is a *smallest* W_h satisfying (17).

It is natural to try to construct the subspace L_0 in the form

$$L_0 = \tilde{L} + \sum_{h\in G} W_h.$$

We have $T_{gh}\tilde{L} \subset \tilde{L} + W_{gh}$ and, on the other hand,

$$T_{gh}\tilde{L} \subset T_g\big(\tilde{L} + W_h\big) \subset \tilde{L} + W_h + W_g.$$

By the minimality of W_h, we obtain that

$$W_{gh} \subset W_g + W_h. \tag{18}$$

This condition of *subadditivity* of a subspace-valued function on a semigroup plays a key role in what follows. We have to deduce from (18) and the assumption

$$\sup_{h\in G} \dim W_h < \infty \tag{19}$$

that all W_h are contained in a common finite-dimensional invariant subspace, or, equivalently,

$$\dim\Big(\sum_{h\in G} W_h\Big) < \infty. \tag{20}$$

4.2. Topologically finitely generated semigroups. If G is finitely generated, then the implication

$$(18) \wedge (19) \Longrightarrow (20)$$

is straightforward. The condition of finite generateness allows also to overcome further obstacles and to describe decomposable functions on such semigroups as matrix elements. We will obtain more general results related to this approach. To formulate them, let us call a continuous function f on a topological semigroup *a locally matrix element* if its restriction to any finitely generated subsemigroup $H \subset G$ is a matrix element of H.

To distinguish the notions of a matrix element and a locally matrix element let us consider the following example.

Let $G = \mathbb{Z}^\infty$, the group of all sequences $x = (x_1, x_2, \dots)$ of integers with only finite numbers of non-zero components. Let us set $f(x) = \sum_{k=1}^{\infty} x_k^k$. It is not difficult to check that the functions $f(x) - f(x+e_j)$, where $\{e_j\}_{j=1}^{\infty}$ is the standard basis in $\mathbb{Z}^\infty$, depend on different components of x and therefore are linearly independent. Thus the orbit of $f(x)$ in the regular representation of G is infinite-dimensional, f is not a matrix element of G. On the other hand, f is a locally matrix function. To see this it suffices to consider its restrictions to subgroups $\mathbb{Z}^k$, because each finitely generated subgroup of G is contained in some $\mathbb{Z}^k$. Clearly $f|_{\mathbb{Z}^k}$ satisfies an equation of the form (1) whence it is a matrix element of $\mathbb{Z}^k$.

Theorem 4.1 ([13]). *Let f be a continuous function on a unital topological semigroup G. If, for some $n \geq 2$, f satisfies* (15)*, then f is a locally matrix element of G.*

Of course for topologically finitely generated semigroups the theorem establishes that the solutions of (15) are matrix elements. This is true for a more wide class of topological semigroups.

Let us call a topological semigroup G *approximately finitely generated* if there is $k \in \mathbb{N}$ such that G is the closure of an increasing sequence of subsemigroups with k generators.

For example the semigroup $\mathbb{R}_+^k$ is approximately k-generated, non-being topologically finitely generated.

Corollary 4.2. *If, in the assumptions of Theorem* 4.1*, G is approximately finitely generated, then f is a matrix element.*

As a consequence we have the following result:

Corollary 4.3. *All continuous functions on $\mathbb{R}^n$ or $\mathbb{R}_+^n$ satisfying the equation* (15) *are quasipolynomials.*

4.3. Subadditive set-valued functions on groups. Here we describe an approach that enables us to show that *bounded* decomposable functions on *groups* are matrix elements. It is based on the consideration of special set-valued maps.

Given a semigroup G and a set Ω let us call a map $F : G \mapsto 2^\Omega$ *subadditive* if

$$F(gh) \subset F(g) \cup F(h) \qquad \text{for all } g, h \in G. \tag{21}$$

We study the following problem: suppose that each $F(g)$ contains $\le n$ elements, does it imply that all $F(g)$ are contained in a finite set? If yes, what can one say about its cardinality?

We will denote the cardinality of a set A by $|A|$.

It is not difficult to check that for each semigroup with finite number k of generators, all $F(g)$ are contained in a set of the cardinality not larger than kn.

One can also show that the same is true if G is approximately k-generated semigroup.

On the other hand, the free abelian semigroup $\mathcal{F}_\infty$ with generators $\{e_i : i \in \mathbb{N}\}$ does not have this property. Indeed, set $\Omega = \mathbb{N}$. Each $g \in \mathcal{F}_\infty$ can be uniquely written in the form $g = \sum_{i\in K} n_i e_i$, for some finite subset K of $\mathbb{N}$, with $n_i \ge 1$, $i \in K$. Let us define $F(g)$ as the one-element set $\{i_g\}$ where i_g is the maximal element in K. Clearly F is subadditive, $|F(g)| = 1$ for every $g \in G$ and $\bigcup_{g\in G} F(g) = \mathbb{N}$.

The following result says that for each *group*, the answer is positive.

THEOREM 4.4 ([15]). *Let F be a subadditive set-valued function on a group G and let $\sup_{g\in G} |F(g)| = n \in \mathbb{N}$. Then $|\bigcup_{g\in G} F(g)| \le 4n$. If, in addition, $F(g^{-1}) = F(g)$, for all $g \in G$, then $|\bigcup_{g\in G} F(g)| \le 2n$.*

It can be shown that the constant 4 in Theorem 4.4 cannot be replaced by any $C < 2$.

There is an analogue of the theorem for the maps to the measure spaces.

Let $\mathcal{M}(\Omega, \mu)$ be the σ-algebra of measurable subsets of a measure space (Ω, μ). For $A, B \in \mathcal{M}(\Omega, \mu)$, we write $A \subset B$ if $\mu(B \setminus A) = 0$. In this sense we understand the subadditivity condition (21).

THEOREM 4.5 ([15]). *Let G be a group, $F : G \to \mathcal{M}(\Omega, \mu)$ a subadditive map. If $\mu(F(g)) \le a$, for some $a > 0$ and all $g \in G$, then there is an $A \in \mathcal{M}(\Omega, \mu)$ such that $\mu(A) \le 4a$ and $F(g) \subset A$ for all $g \in G$.*

Now we will formulate a result on the Hyers–Ulam stability of the condition (21).

Let $\delta > 0$; for $A, B \in \mathcal{M}(\Omega, \mu)$, we write $A \subset_\delta B$ if $\mu(A \setminus B) < \delta$.

A map $F : G \to \mathcal{M}(\Omega, \mu)$ is called *δ-subadditive* if

$$F(gh) \subset_\delta F(g) \cup F(h) \qquad \text{for all } g, h \in G. \tag{22}$$

It is natural to consider such maps for sufficiently small δ; in the following theorem we impose the corresponding restriction.

THEOREM 4.6 ([15]). *Let $F : G \to \mathcal{M}(\Omega, \mu)$ be a δ-subadditive function on a group G and $\mu(F(g)) < a$ for all $g \in G$. Assume that $\delta < a/3$. Then there is a set $K \subset \Omega$ such that $\mu(K) \le 6a$ and $F(g) \subset_{8\delta} K$.*

4.4. Subadditive subspace-valued maps. Let $\mathcal{S}(X)$ be the lattice of all closed subspaces of a Banach space X; by the sum of a finite or infinite family of subspaces we mean the closure of the linear span of their union. The subspace-valued map on a group G is just a map $F : G \to \mathcal{S}(X)$. A subspace-valued map F is called subadditive if

$$F(gh) \subset F(g) + F(h) \qquad \text{for all } g, h \in G. \tag{23}$$

Clearly (18) means that the map $g \mapsto W_g$ is subadditive; this shows that subadditive subspace-valued maps naturally arise in the theory of functional equations.

Subadditive subspace-valued maps can be considered as "noncommutative analogue" of subadditive set-valued maps; the counterpart of the cardinality of the set $F(g)$ is the dimension $\dim F(g)$ of the subspace $F(g)$. Our aim is to establish a non-commutative analogue of Theorem 4.4, that is to prove that the image of F is contained in a subspace of restricted dimension if dimensions of all $F(g)$ are bounded by a common constant.

Assume that all subspaces $F(g)$, for $g \in G$, are finite-dimensional, and set

$$n(F) = \sup_{g \in G} \dim F(g), \qquad N(F) = \dim \sum_{g \in G} F(g). \tag{24}$$

In general, the inequality $N(F) < Cn(F)$ and even the implication $n(F) < \infty \Longrightarrow N(F) < \infty$ do not hold, as the following example shows.

EXAMPLE 4.7. Let $G = X$ be considered as a group with respect to the addition. For each $0 \neq x \in G$, let $F(x) = \mathbb{C}x$, the one-dimensional subspace of X containing x. Let also $F(0) = \{0\}$. Then clearly F is a subadditive map of G to $\mathcal{S}(X)$, $n(F) = 1$, $N(F) = \infty$.

We obtain a positive result assuming that all $F(g)$ are invariant subspaces of a *uniformly bounded representation of finite multiplicity* in a Banach space.

Let T be a representation of a group Γ on X, and let π be an irreducible representation of Γ. Let us say that π *occurs* in T if there is an invariant subspace $Y \subset X$ such that the restriction $T|_Y$ of T to Y is equivalent to π. In this case we call Y *a π-subspace.* Furthermore π *occurs with finite multiplicity in* T if the linear span Y_π of all π-subspaces in X is finite-dimensional (equivalently, the number of elements in any family of linearly independent π-subspaces does not exceed some number $m = m(\pi, T)$).

Let us say that T is an *fm-representation* if each irreducible representation of Γ which occurs in T occurs with finite multiplicity. Moreover if the multiplicity $m(\pi, T)$ of each occurring in T irreducible representation π can be evaluated via the dimension of this representation:

$$m(\pi, T) \leq \phi(\dim \pi),$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing function, then we say that T *has uniform multiplicity* ϕ.

We will consider representations by bounded operators on Banach spaces; a representation T is *uniformly bounded* if $\sup_{g \in G} \|T_g\| < \infty$.

THEOREM 4.8 ([15]). *Let T be a uniformly bounded fm-representation of a group Γ and $\mathcal{E}$ be the structure of all finite-dimensional T-invariant subspaces. Let G be a group and $F : G \to \mathcal{E}$ be a map such that*

$$F(g_1 g_2) \subset F(g_1) + F(g_2)$$

and $\dim F(g) \leq n$ for each $g \in G$. Then $\sum_{g \in G} F(g)$ is finite-dimensional. Moreover if T has uniform multiplicity ϕ then

$$\dim\Big(\sum_{g \in G} F(g)\Big) \leq (4n - 1)n\phi(n).$$

The main idea of the proof is to relate to F the set-valued function $f : G \to 2^{\Omega}$, where Ω is the set of all (equivalence classes of) bounded irreducible representations of G, by setting

$$f(g) = \{\pi \in \Omega : \pi \text{ occurs in } T|_{F(g)}\}.$$

The condition of finite multiplicity provides that Theorem 4.4 can be applied.

If T is the left regular representation of Γ on the space $\mathcal{C}_b(\Gamma)$ of all bounded continuous functions on Γ, then each Y_{π_i} is the space of all matrix elements of π_i, therefore

$$\phi(\dim(\pi)) = \dim(\pi) \leq n.$$

Thus T is a bounded fm-representation of Γ of uniform multiplicity $\phi(t) = t$, and we get the estimate

$$\dim \sum_{g \in G} F(g) \leq (4n-1)n^2.$$

Arguing as in Section 4.1 we deduce from Theorem 4.8 the following result which is also in spirit of stability, because it establishes that, to prove that a finite-dimensional subspace L is contained in a finite-dimensional invariant subspace, it is sometimes sufficient to know that the image of L under each operator T_g is close to L in some sense.

Theorem 4.9 ([15]). *Let T be the left regular representation of a group G on the space $X = \mathcal{C}_b(G)$, and let $n \in \mathbb{N}$. Suppose that a finite-dimensional subspace $L \subset X$ has the property that for any $g \in G$, there is an invariant subspace $L(g) \subset X$ with $\dim L(g) \leq n$ and the condition*

$$T_g L \subset L + L(g)$$

holds. Then L is contained in a finite-dimensional invariant subspace $\widetilde{L}$ of X such that

$$\dim \widetilde{L} \leq \dim L + (4n-1)n^2.$$

This gives us the possibility to obtain the following description of bounded decomposable functions.

Theorem 4.10 ([15]). *A bounded continuous function f on a topological group G satisfies* (15) *if and only if it is a matrix element of G.*

5. Addition theorems of rational type

5.1. Functional equations and integrable Hamiltonian systems. Here we discuss functional equations of the type

$$f(t+s) = \frac{\sum_{i=1}^{n} y_i(t)u_i(s)}{\sum_{j=1}^{m} z_j(t)v_j(s)}\,. \tag{25}$$

Again, by a solution of (25) we mean a function f for which there exist functions y_i, u_i, z_j, v_j satisfying (25); thus we actually speak about functions that admit an *addition theorem of rational type*. Obviously, we may assume that the functions y_i (as well as u_i, z_j, v_j) are linearly independent.

It should be noted that equations (25) arise in a wide variety of situations. Let us consider how they occur in the context of integrable systems of particles on the line (see, e.g., [2]). Let $q_1(t), q_2(t), \dots, q_n(t)$ be the coordinates of N particles on the line, interacting with the integrable potential $\sum_{k=1}^{N} U(q_j - q_k)$. Then the dynamics of the system is described by the system of ODE

$$\ddot{q} = \sum_{k=1}^{N} U(q_j - q_k), \qquad j = 1, 2, \dots, n. \tag{26}$$

Such a dynamical system is said to admit a Lax representation if it is equivalent to the matrix equation $\dot{L} = [L, M]$, where L and M are matrix-valued functions of q_i and p_i. It follows from this representation that the functions $J_k = \frac{1}{k}\operatorname{tr}\{L_k\}$ $(k = 1, 2, \dots, n)$ are integrals of the system (26). If it is proved that they are independent and in involution, then the system is completely integrable.

Since it is difficult to find a Lax pair for a given equation, usually one operates in reverse order: postulates a form of matrices L and M and then seeks restrictions necessary to obtain a given equations (26). These restrictions typically involve the study of functional equations.

For example, starting with the ansatz

$$\begin{cases} L_{jk} = p_j \delta_{jk} + g(1 - \delta_{jk}) A(q_j - q_k), \\ M_{jk} = g\big[\delta_{jk} \sum_{l \neq k} B(q_j - q_l) - (1 - \delta_{jk}) C(q_j - q_k)\big] \end{cases}$$

one finds that $\dot{L} = [L, M]$ yields the equations of motion (26) for the Hamiltonian system

$$H = \frac{1}{2} \sum_{j=1}^{N} p_j^2 + g^2 \sum_{j<k} U(q_j - q_k), \qquad U(x) = A(x)A(-x) + \text{const},$$

provided that $C(x) = -A'(x)$ and the functions $A(x)$ and $B(x)$ satisfy the functional equation

$$A(x+y) = \frac{A(x)A'(y) - A'(x)A(y)}{B(x) - B(y)}\,. \tag{27}$$

In the same sense the functional equation

$$\phi(t+s) = \frac{\alpha(t)\alpha'(s) - \alpha'(t)\alpha(s)}{\beta(t)\beta'(s) - \beta'(t)\beta(s)} \tag{28}$$

is associated with the relativistic Calogero–Moser systems.

We will not concentrate on other examples, but note that in all works only analytic solutions of (25) were sought.

5.2. Reduction to the system of ODEs. Our approach to analysis of such functional equations is based on the reduction of (25) to an overdetermined system of ordinary differential equations. We assume that all functions y_i and z_j are continuously differentiable on some interval $I_1 \in \mathbb{R}$, all functions u_i and v_j are continuously differentiable on some interval $I_2 \in \mathbb{R}$, and the function f is continuously differentiable on some interval $I \supset (I_1 + I_2)$. Let us introduce two notions.

DEFINITION 5.1. We say that the families $\{y_i\}_{i=1}^{n}$ and $\{z_j\}_{j=1}^{m}$ are *jointly linearly independent* if the family $\{y_i z_j\}_{i,j}$ is linearly independent.

The failure of the joint linear independence for families $\{y_i\}_{i=1}^n$ and $\{z_j\}_{j=1}^m$ participating in (25) often leads to an excessive simplification of the equation; for example (27) in this case reduces to the Levi-Civita equation and therefore its solutions, satisfying this conditions are quasipolynomials. So we restrict ourselves to the study of solutions with jointly linearly independent $\{y_i\}_{i=1}^n$ and $\{z_j\}_{j=1}^m$ as well as $\{u_i\}_{i=1}^n$ and $\{v_j\}_{j=1}^m$.

THEOREM 5.2 ([12]). *In the assumption of the joint linear independence, the functional equation* (25) *holds for some differentiable function f if and only if there exist constants C_{lk}^{ij} such that*

$$\begin{cases} y_i' z_j - y_i z_j' = \sum_{l,k} C_{lk}^{ij} y_l z_k, & i \le n;\ j \le m, \\ u_i' v_j - u_i v_j' = \sum_{l,k} C_{ij}^{lk} u_l v_k, & i \le n;\ j \le m. \end{cases} \tag{29}$$

Basing on Theorem 5.2 one can obtain a description of a "generic" class of functions admitting addition theorem (25).

DEFINITION 5.3. We say that the families $\{y_i\}_{i=1}^n$ and $\{z_j\}_{j=1}^m$ are *jointly quadratically dependent* if they satisfy the nontrivial relation

$$\sum_{i,l=1}^{n} \sum_{j,k=1}^{m} C_{lk}^{ij} y_i(t) z_j(t) y_l(t) z_k(t) = 0$$

where C_{lk}^{ij} are constants.

THEOREM 5.4 ([12]). *Let f be a function satisfying* (25). *Then, either the families $\{y_i\}$ and $\{z_j\}$ are jointly quadratically dependent, or f is a ratio of quasipolynomials.*

5.3. The case of quadratically dependent families. Theorem 5.4 shows that the "degenerated" cases are the most interesting ones. Moreover, in specifications of (25) that arise in the theory of Hamiltonian systems, the quadratic dependence of families $\{y_i\}_{i=1}^n$ and $\{z_j\}_{j=1}^m$ (as well as $\{u_i\}_{i=1}^n$ and $\{v_j\}_{j=1}^m$) holds automatically. Here we discuss the solutions in this case, restricting ourselves to the very important for applications case $m = n = 2$.

The system of ODEs (29) in this case can be after some change of variables reduced to a system, containing as a main part the following:

$$\begin{cases} (z' - P_2(z))^2 = P_4(z) \\ \left(y' - \tilde{P}_2(y)\right)^2 = \tilde{P}_4(y) \\ R(y, z) = 0. \end{cases}$$

Here P_2 and $\tilde{P}_2$ are polynomials of degree 2, P_4 and $\tilde{P}_4$ are polynomials of degree 4 while $R(y, z)$ is a polynomial in two variables of degree 2 in each of them; it can written in the form:

$$R(y, z) = (x^2, x, 1) A (y^2, y, 1)^T, \tag{30}$$

where A is an arbitrary 3×3 matrix.

The first two equations in the above system lead to the important conclusion that y and z are the inversion of some elliptic integrals [12]. Their implicit expressions were obtained in the same work for the equations of "symmetric" class

$$f(t+s) = \frac{y(t)u(s) - u(t)y(s)}{z(t)v(s) - v(t)z(s)}\,, \tag{31}$$

which includes many "physical" equations, e.g. (28) and (27). Namely all differentiable solutions of (31) are given by the formula

$$f(x) = Ce^{\lambda x}\,\frac{\Phi(x;\nu_1)}{\Phi(x;\nu_2)}\,, \qquad \text{where } \Phi(x;\nu) = \frac{\sigma(\nu - x)}{\sigma(\nu)\sigma(x)}\,e^{\zeta(\nu)x}.$$

Here $\sigma(x)$ and $\zeta(x)$ are the Weierstrass sigma and zeta functions, $\Phi(x;\nu)$ is called the Baker–Akhiezer function. For the same class of equations, all analytic solutions were earlier found by Braden and Buchstaber [2]. Thus the approach in [12], based on the study of the system (29)), shows that removing the assumption of analyticity does not change the form of solution.

Lundberg [7] have managed to obtain the general solution of (25) (for $m = n = 2$) in the class of continuous *real-valued* functions. This was done in two stages: all meromorphic solutions were found in [6], then all continuous ones — in [7]. The results of [6] were based on the classification of all polynomials $R(y,z)$ defined by (30), up to the special equivalence of matrices A induced by all admissible changes of variables in (29); this classification was obtained in [6] by means of complicated calculations using computer math-programs. For the step "from meromorphic to continuous solutions", a very elegant technique of "sequential derivatives" was developed and applied.

5.4. A related equation. We will finish by the study of the functional equation

$$\frac{f(x+y)}{f(x-y)} = \frac{g(x)+g(y)}{g(x)-g(y)} \tag{32}$$

which was introduced by P. McGill in his work [8] on Brownian motion. In [9] he found all meromorphic solutions (f,g). They are divided into the following six classes (we list only the components g because the corresponding functions f can be found with the aid of the formula $g'(z)/g(z) = 2f'(0)/f(2z)$):

(a) $g(z) = Az$,
(b) $g(z) = A\sin\pi z$,
(c) $g(z) = A\tan\pi z$,
(d) $g(z) = A\,\mathrm{sn}(z;k)$,
(e) $g(z) = A\,\mathrm{sd}(z;k)$,
(f) $g(z) = A\,\mathrm{sc}(z;k)$.

Here and below we use the standard notation for the Jacobi elliptic functions. The functions in the list are normalized by rotation and dilation in such a way that the periods are 2 and $4\omega i$ ($\omega > 0$); the parameter k only depends on ω.

Now we will see that the approach based on the system (29) allows one to find easily all solutions of (32) in a wider class of all functions having two derivatives on some interval of the real axis.

THEOREM 5.5. *All solutions of the equation* (32) *in* $C^2(a,b)$ *are described by the formulas* (a)–(f) *above.*

Proof. By differentiating and some transformations, equation (32) can be reduced to the form (25):

$$\frac{f'(x+y)}{f(x+y)} = \frac{g(x)g'(y) - g'(x)g(y)}{g^2(x) - g^2(y)}\,. \tag{33}$$

It follows from (32) that $g(0) = 0$, which implies, in particular, the joint linear independence for families $\{g, g'\}$ and $\{g^2, 1\}$. Further, since f is not a constant function and $f'(x)/f(x) = g'(0)/g(x)$, we have $g'(0) \neq 0$. Writing the system of differential equations (29) for (33) we conclude that

$$c_{21}^{11} = 1,\ c_{22}^{12} = -1,\ c_{12}^{22} = -c_{11}^{21}$$

and all other coefficients are equal to zero. Thus, the system (29) has the form

$$\begin{cases} g'' = ag + cg^3, \\ g''g^2 - 2g(g')^2 = bg - ag^3, \end{cases}$$

where $a = c_{11}^{21}$, $b = c_{12}^{21}$, $c = c_{11}^{21}$. The last system is equivalent to the equation

$$(g')^2 = \frac{c}{2}\,g^4 + ag^2 - \frac{b}{2}\,.$$

Note that $b \neq 0$ since $g'(0) \neq 0$ and $g(0) = 0$. So, for $y = g\sqrt{-2/b}$ we obtain the differential equation

$$(y')^2 = 1 + ay^2 - \frac{bc}{4}\,y^4. \tag{34}$$

The degenerated case $c = 0$ gives us, respectively, $g = Ax$ when $a = 0$, and $g = \sin Ax$, when $a \neq 0$. (This corresponds to points (a) and (b) from McGill's list of solutions.)

If $c \neq 0$ the solution of (34) is $y(x) = \mathrm{sn}(\varepsilon x; k)/\varepsilon$, with $(1+k^2)\varepsilon^2 = -a$, $k^2\varepsilon^4 = -bc/4$ (see, e.g., [3]). Thus,

$$g(x) = \sqrt{-\frac{b}{2}}\,\frac{\mathrm{sn}(\varepsilon x; k)}{\varepsilon}\,. \tag{35}$$

The parameter k^2 takes values in $\mathbb{C} \setminus \{0, 1\}$. The exceptional case $k^2 = 1$ leads to $g(x) = A\tan x$, while the case $k = 0$ corresponds to $c = 0$ and was already considered. The equalities $i\,\mathrm{sc}(u; k) = \mathrm{sn}(iu; k')$, $k'\,\mathrm{sd}(u; k) = \mathrm{sn}(k'u; -ik/k')$ (see, e.g., [19]) show that we obtain exactly the list (a)–(f). ∎

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