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# STRONG CONVERGENCE THEOREMS OF A NEW HYBRID PROJECTION METHOD FOR FINITE FAMILY OF TWO HEMI-RELATIVELY NONEXPANSIVE MAPPINGS IN A BANACH SPACE

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**Abstract.** In this paper, we prove strong convergence theorems of the hybrid projection algorithms for finite family of two hemi-relatively nonexpansive mappings in a Banach space. Using this result, we also discuss the resolvents of two maximal monotone operators in a Banach space. Our results modify and improve the recently ones announced by Plubtieng and Ungchittrakool [*Strong convergence theorems for a common fixed point of two relatively nonexpansive mappings in a Banach space*, J. Approx. Theory 149 (2007), 103–115], Matsushita and Takahashi [*A strong convergence theorem for relatively nonexpansive mappings in a Banach space*, J. Approx. Theory 134 (2005), 257–266] and many others.

**1. Introduction.** Let $E$ be a real Banach space, $C$ be a nonempty closed convex subset of $E$, and $T : C \to C$ be a mapping. Recall that $T$ is nonexpansive if

$$\|Tx - Ty\| \le \|x - y\| \qquad \text{for all } \ x, y \in C. \tag{1}$$

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We denote by $F(T)$ the set of fixed points of $T$, that is $F(T) = \{x \in C : x = Tx\}$. A mapping $T$ is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\|Tx - y\| \leq \|x - y\| \text{ for all } x \in C \text{ and } y \in F(T).$$

It is easy to see that if $T$ is nonexpansive with $F(T) \neq \emptyset$, then it is quasi-nonexpansive. Some iteration processes are often used to approximate a fixed point of a nonexpansive mapping. Mann's iterative algorithm was introduced by Mann [7] in 1953. This iteration process is now known as Mann's iteration process, which is defined as

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0, \tag{2}$$

where the initial guess $x_0$ is taken in $C$ arbitrarily and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is in the interval $[0, 1]$.

In 1967, Halpern [4] first introduced the following iteration scheme:

$$\begin{cases} x_0 = x \in C & \text{chosen arbitrarily,} \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \end{cases} \tag{3}$$

see also Browder [2]. He pointed out that the conditions $\lim_{n\to\infty} \alpha_n = 0$ and $\Sigma_{n=1}^{\infty}\alpha_n = \infty$ are necessary in the sense that, if the iteration (3) converges to a fixed point of $T$, then these conditions must be satisfied.

In 1974, Ishikawa [5] introduced a new iteration scheme, which is defined recursively by

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n)Tx_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Ty_n, \end{cases} \tag{4}$$

where the initial guess $x_0$ is taken in $C$ arbitrarily and the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are in the interval $[0, 1]$.

Many papers have appeared in the literature on Ishikawa's iteration process; see, for example [10, 11] and reference therein.

On the other hand, Matsushita and Takahashi [8] introduced the following iteration: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \Pi_C J^{-1}\big(\alpha_n Jx_n + (1 - \alpha_n)JTx_n\big), \tag{5}$$

where the initial guess element $x_0 \in C$ is arbitrary, $\{\alpha_n\}$ is a real sequence in $[0, 1]$, $T$ is a relatively nonexpansive mapping, $J$ is the duality mapping on $E$ and $\Pi_C$ denotes the generalized projection from $E$ onto a closed convex subset $C$ of $E$. They proved that the sequence $\{x_n\}$ converges weakly to a fixed point of $T$. Moreover, Matsushita and Takahashi [9] proposed the following modification of iteration (5):

$$\begin{cases} x_0 \in C & \text{chosen arbitrarily,} \\ y_n = J^{-1}\big(\alpha_n Jx_n + (1 - \alpha_n)JTx_n\big), \\ C_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0), & n = 0, 1, 2, \ldots. \end{cases} \tag{6}$$

and proved that the sequence $\{x_n\}$ converges strongly to $\Pi_{F(T)}(x_0)$.

In 2007, Plubtieng and Ungchittrakool [11] proved the following iteration for two relatively nonexpansive mappings $T$ in a Banach space $E$:

$$\begin{cases} x_0 = x \in C \qquad \text{chosen arbitrarily}, \\ z_n = J^{-1}\big(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JTx_n + \beta_n^{(3)} JSx_n\big), \\ y_n = J^{-1}\big(\alpha_n Jx_n + (1-\alpha_n) JTz_n\big), \\ H_n = \{v \in C : \phi(v, y_n) \le \phi(v, x_n)\}, \\ W_n = \{v \in C : \langle x_n - v, Jx - Jx_n\rangle \ge 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x, \end{cases} \tag{7}$$

and

$$\begin{cases} x_0 = x \in C \qquad \text{chosen arbitrarily}, \\ z_n = J^{-1}\big(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JTx_n + \beta_n^{(3)} JSx_n\big), \\ y_n = J^{-1}\big(\alpha_n Jx_0 + (1-\alpha_n) JTz_n\big), \\ H_n = \big\{v \in C : \phi(v, y_n) \le \phi(v, x_n) + \alpha_n(\|x_0\|^2 + 2\langle v, Jx_n - Jx\rangle)\big\}, \\ W_n = \big\{v \in C : \langle x_n - v, Jx - Jx_n\rangle \ge 0\big\}, \\ x_{n+1} = P_{H_n \cap W_n} x, \end{cases} \tag{8}$$

the sequences $\{x_n\}$ generated by (7) and (8) converge to $P_F x$, $F := F(T) \cap F(S)$, where $P_F$ is the generalized projection from $C$ onto $F$.

In 2008, Takahashi et al. [13] proved the following theorem by a new hybrid method. We call such a method the shrinking projection method.

Theorem 1.1 (Takahashi et al. [13]). *Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $T$ be a nonexpansive mapping of $C$ into $H$ such that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1} x_0$, define a sequence $\{u_n\}$ of $C$ as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n) T u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}} x_0, \qquad n \in \mathbb{N}, \end{cases} \tag{9}$$

*where $0 \le \alpha_n \le a < 1$ for all $n \in \mathbb{N}$. Then, $\{u_n\}$ converges strongly to $z_0 = P_{F(T)} x_0$.*

This paper considers the following explicit cyclic algorithm:

$$\begin{aligned} x_1 &= \alpha_0 x_0 + (1-\alpha_0) T_0 x_0, \\ x_2 &= \alpha_1 x_1 + (1-\alpha_1) T_1 x_1, \\ &\ \vdots \\ x_n &= \alpha_{n-1} x_{n-1} + (1-\alpha_{n-1}) T_{n-1} x_{n-1}, \\ x_{n+1} &= \alpha_n x_n + (1-\alpha_n) T_0 x_n. \end{aligned} \tag{10}$$

It can rewritten into compact form as follows

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) T_{[n]} x_n, \tag{11}$$

where $T_{[n]} = T_i$, $i = n \,(\mathrm{mod}\, N)$.

In this paper, motivated by Plubtieng and Ungchittrakool's result [11], we use an idea to modify (7) and (8) for finite family of two hemi-relatively nonexpansive mappings to have strong convergence theorems in a Banach space by using the shrinking projection method. Our result extends and improves the recent results by Plubtieng and Ungchittrakool [11] and many authors.

**2. Preliminaries.** Let $E$ be a real Banach space with dual $E^*$. Denote by $\langle \cdot, \cdot \rangle$ the duality product. The normalized duality mapping $J$ from $E$ to $E^*$ is defined by

$$Jx = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}, \tag{12}$$

for all $x \in E$.

A Banach space $E$ is said to have the Kadec-Klee property if for every sequence $\{x_n\}$ of $E$ satisfying $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$ we have $x_n \to x$. It is known that if $E$ is uniformly convex, then $E$ has the Kadec-Klee property.

If $C$ is a nonempty closed convex subset of real Hilbert space $H$ and $P_C : H \to C$ is the metric projection, then $P_C$ is nonexpansive. Alber [1] has recently introduced a generalized projection operator $\Pi_C$ in a Banach space $E$ which is an analogue representation of the metric projection in Hilbert spaces. We denote by $\omega_w(\{z_n\})$ the set of all weak subsequential limits of a bounded sequence $\{z_n\}$ in $C$.

Let $E$ be a smooth Banach space. The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \qquad \text{for all } x, y \in E. \tag{13}$$

The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x)$, that is, $\Pi_C x = x^*$, where $x^*$ is the solution to the minimization problem

$$\phi(x^*, x) = \min_{y \in C} \phi(y, x),$$

existence and uniqueness of the operator $\Pi_C$ follow from the properties of the functional $\phi(y, x)$ and strict monotonicity of the mapping $J$. In Hilbert space, $\Pi_C = P_C$. It is obvious from the definition of the function $\phi$ that

$$(\|y\| - \|x\|)^2 \le \phi(y, x) \le (\|y\| + \|x\|)^2 \qquad \text{for all } x, y \in E. \tag{14}$$

Remark 2.1 ([12]). If $E$ is a strictly convex and smooth Banach space, then for all $x, y \in E$, $\phi(y, x) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(y, x) = 0$ then $x = y$. From (14), we have $\|x\| = \|y\|$. This implies $\langle y, Jx \rangle = \|y\|^2 = \|Jx\|^2$. From the definition of $J$, we have $Jx = Jy$. Since $J$ is one-to-one, we have $x = y$.

Lemma 2.2 (Kamimura and Takahashi [6]). *Let $E$ be a smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to R$ such that $g(0) = 0$ and $g(\|x - y\|) \le \phi(x, y)$ for all $x, y \in B_r$.*

Lemma 2.3 (Kamimura and Takahashi [6]). *Let $E$ be a uniformly convex and smooth real Banach space and let $\{x_n\}, \{y_n\}$ be two sequences of $E$. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

LEMMA 2.4 (Alber [1]). *Let $E$ be a reflexive, strictly convex, and smooth real Banach space, let $C$ be a nonempty closed convex subset of $E$ and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \qquad \forall y \in C. \tag{15}$$

LEMMA 2.5 (Cho et al. [3]). *Let $X$ be a uniformly convex Banach space and $B_r(0)$ be a closed ball of $X$. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu g(\|x - y\|)$$

*for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$.*

## 3. Main results

THEOREM 3.1. *Let $E$ be a uniformly convex and uniformly smooth Banach space and let $C$ be a nonempty closed convex subset of $E$. Given an integer $N \geq 1$, let, for each $0 \leq i \leq N-1$, $T_i$ and $S_i$ be two hemi-relatively nonexpansive mappings from $C$ into itself with $\mathcal{T} = \bigcap_{i=0}^{N-1} F(T_i)$, $\mathcal{S} = \bigcap_{i=0}^{N-1} F(S_i)$ and $F := \mathcal{T} \cap \mathcal{S}$ nonempty, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ be sequences of real numbers such that $0 \leq \alpha_n < 1$ for all $n \in \mathbb{N} \cup \{0\}$, $\limsup_{n\to\infty} \alpha_n < 1$, $0 \leq \beta_n, \gamma_n, \delta_n \leq 1$, $\beta_n + \gamma_n + \delta_n = 1$ for all $n \in \mathbb{N} \cup \{0\}$, $\liminf_{n\to\infty} \beta_n\gamma_n > 0$ and $\liminf_{n\to\infty} \beta_n\delta_n > 0$. Define a sequence $\{x_n\}$ by the following algorithm:*

$$\begin{cases} x_0 \in C \qquad \text{chosen arbitrarily,} \\ z_n = J^{-1}\big(\beta_n J x_n + \gamma_n J T_{[n]} x_n + \delta_n J S_{[n]} x_n\big), \\ y_n = J^{-1}\big(\alpha_n J x_n + (1 - \alpha_n) J z_n\big), \\ C_{n+1} = \{v \in C : \phi(v, y_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \qquad \forall n \geq 0, \end{cases} \tag{16}$$

*where $J$ is the duality mapping on $E$ and $T_{[n]} = T_i$, $S_{[n]} = S_i$, $i = n \,(\mathrm{mod}\, N)$. Then $\{x_n\}$ converges strongly to $\Pi_F x_0$, where $\Pi_F$ is the generalized projection from $C$ onto $F$.*

*Proof.* We first show that $C_{n+1}$ is closed and convex for each $n \in \mathbb{N} \cup \{0\}$. From the definition of $C_{n+1}$ it is obvious that $C_{n+1}$ is closed for each $n \in \mathbb{N} \cup \{0\}$. For any $n \in \mathbb{N} \cup \{0\}$,

$$\phi(v, y_n) \leq \phi(v, x_n) \Longleftrightarrow 2\langle v, Jx_n - Jy_n\rangle + \|y_n\|^2 - \|x_n\|^2 \leq 0,$$

and hence $C_{n+1}$ is convex. Next, we show that $F \subset C_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$. Let $p \in F$ and let $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} \phi(p, z_n) &= \phi\big(p, J^{-1}(\beta_n J x_n + \gamma_n J T_{[n]} x_n + \delta_n J S_{[n]} x_n)\big) \\ &= \|p\|^2 - 2\langle p, \beta_n J x_n + \gamma_n J T_{[n]} x_n + \delta_n J S_{[n]} x_n\rangle \\ &\qquad + \|\beta_n J x_n + \gamma_n J T_{[n]} x_n + \delta_n J S_{[n]} x_n\|^2 \\ &\leq \|p\|^2 - 2\beta_n\langle p, Jx_n\rangle - 2\gamma_n\langle p, JT_{[n]}x_n\rangle - 2\delta_n\langle p, JS_{[n]}x_n\rangle \\ &\qquad + \beta_n\|x_n\|^2 + \gamma_n\|T_{[n]}x_n\|^2 + \delta_n\|S_{[n]}x_n\|^2 \\ &\leq \beta_n\phi(p, x_n) + \gamma_n\phi(p, T_{[n]}x_n) + \delta_n\phi(p, S_{[n]}x_n) \\ &\leq \beta_n\phi(p, x_n) + \gamma_n\phi(p, x_n) + \delta_n\phi(p, x_n) = \phi(p, x_n), \end{aligned} \tag{17}$$

and

$$\begin{aligned}
\phi(p, y_n) &= \phi\big(p, J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Jz_n)\big) \\
&= \|p\|^2 - 2\langle p, \alpha_n Jx_n + (1-\alpha_n)Jz_n\rangle + \|\alpha_n Jx_n + (1-\alpha_n)Jz_n\|^2 \\
&\le \|p\|^2 - 2\alpha_n\langle p, Jx_n\rangle - 2(1-\alpha_n)\langle p, Jz_n\rangle + \alpha_n\|x_n\|^2 + (1-\alpha_n)\|z_n\|^2 \\
&\le \alpha_n\phi(p, x_n) + (1-\alpha_n)\phi(p, z_n) \\
&\le \alpha_n\phi(p, x_n) + (1-\alpha_n)\phi(p, x_n) \\
&= \phi(p, x_n).
\end{aligned} \tag{18}$$

So, $p \in C_n$ for all $n \in \mathbb{N} \cup \{0\}$, and we have $F \subset C_n$. This implies that $\{x_n\}$ is well defined. Since $x_{n+1} = \Pi_{C_{n+1}}x_0$ and $x_{n+1} \in C_{n+1} \subset C_n$, we get

$$\phi(x_n, x_0) \le \phi(x_{n+1}, x_0),$$

for all $n \ge 0$. Therefore, $\{\phi(x_n, x_0)\}$ is nondecreasing.

By the definition of $x_n$ and Lemma 2.4, we have

$$\phi(x_n, x_0) = \phi(\Pi_{C_n}x_0, x_0) \le \phi(p, x_0) - \phi(p, \Pi_{C_n}x_0) \le \phi(p, x_0), \tag{19}$$

for all $p \in F \subset C_n$. Thus, $\phi(x_n, x_0)$ is bounded. Moreover, by (14), we have that $\{x_n\}$ is bounded. So, $\lim_{n\to\infty}\phi(x_n, x_0)$ exists. Again by Lemma 2.4, we have

$$\begin{aligned}
\phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n}x_0) \\
&\le \phi(x_{n+1}, x_0) - \phi(\Pi_{C_n}x_0, x_0) \\
&= \phi(x_{n+1}, x_0) - \phi(x_n, x_0),
\end{aligned}$$

for all $n \ge 0$. Thus, $\phi(x_{n+1}, x_n) \to 0$ as $n \to \infty$. Since $x_{n+1} = \Pi_{C_n}x_0 \in C_n$, from the definition of $C_{n+1}$, we also have

$$\phi(x_{n+1}, y_n) \le \phi(x_{n+1}, x_n), \tag{20}$$

for all $n \in \mathbb{N} \cup \{0\}$. So, we have $\lim_{n\to\infty}\phi(x_{n+1}, y_n) = 0$. Using Lemma 2.3, we obtain

$$\lim_{n\to\infty}\|x_{n+1} - y_n\| = \lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{21}$$

Since $J$ is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty}\|Jx_{n+1} - Jy_n\| = \lim_{n\to\infty}\|Jx_{n+1} - Jx_n\| = 0. \tag{22}$$

For each $n \in \mathbb{N} \cup \{0\}$, we observe that

$$\begin{aligned}
\|Jx_{n+1} - Jy_n\| &= \|Jx_{n+1} - (\alpha_n Jx_n + (1-\alpha_n)Jz_n)\| \\
&= \|\alpha_n(Jx_{n+1} - Jx_n) + (1-\alpha_n)(Jx_{n+1} - Jz_n)\| \\
&= \|(1-\alpha_n)(Jx_{n+1} - Jz_n) - \alpha_n(Jx_n - Jx_{n+1})\| \\
&\ge (1-\alpha_n)\|Jx_{n+1} - Jz_n\| - \alpha_n\|Jx_n - Jx_{n+1}\|.
\end{aligned}$$

It follows that

$$\|Jx_{n+1} - Jz_n\| \le \frac{1}{1-\alpha_n}\big(\|Jx_{n+1} - Jy_n\| + \alpha_n\|Jx_n - Jx_{n+1}\|\big).$$

By (22) and $\limsup_{n\to\infty}\alpha_n < 1$, we obtain

$$\lim_{n\to\infty}\|Jx_{n+1} - Jz_n\| = 0.$$

Since $J^{-1}$ is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|x_{n+1} - z_n\| = 0. \tag{23}$$

It follows from (21) that

$$\|x_n - z_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \to 0,$$

Since $J$ is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\to\infty} \|Jx_n - Jz_n\| = 0.$$

Next, we show that $\|x_n - T_{[n]}x_n\| \to 0$ and $\|x_n - S_{[n]}x_n\| \to 0$. Since $\{x_n\}$ is bounded, $\phi(p, T_{[n]}x_n) \le \phi(p, x_n)$ and $\phi(p, S_{[n]}x_n) \le \phi(p, x_n)$ where $p \in F$. We also obtain that $\{Jx_n\}$, $\{JT_{[n]}x_n\}$ and $\{JS_{[n]}x_n\}$ are bounded, then there exists $r > 0$ such that $\{Jx_n\}$, $\{JT_{[n]}x_n\}$, $\{JS_{[n]}x_n\} \subset B_r(0)$. From Lemma 2.5, we have

$$\begin{aligned}
\phi(p, z_n) &= \phi\big(p, J^{-1}(\beta_n Jx_n + \gamma_n JT_{[n]}x_n + \delta_n JS_{[n]}x_n)\big)\\
&= \|p\|^2 - 2\beta_n \langle p, Jx_n\rangle - 2\gamma_n \left\langle p, JT_{[n]}x_n\right\rangle - 2\delta_n \left\langle p, JS_{[n]}x_n\right\rangle\\
&\qquad + \|\beta_n Jx_n + \gamma_n JT_{[n]}x_n + \delta_n JS_{[n]}x_n\|^2\\
&\le \|p\|^2 - 2\beta_n \langle p, Jx_n\rangle - 2\gamma_n \left\langle p, JT_{[n]}x_n\right\rangle - 2\delta_n \left\langle p, JS_{[n]}x_n\right\rangle \qquad (24)\\
&\qquad + \beta_n\|x_n\|^2 + \gamma_n\|T_{[n]}x_n\|^2 + \delta_n\|S_{[n]}x_n\|^2 - \beta_n\gamma_n g(\|Jx_n - JT_{[n]}x_n\|)\\
&= \beta_n\phi(p, x_n) + \gamma_n\phi(p, T_{[n]}x_n) + \delta_n\phi(p, S_{[n]}x_n) - \beta_n\gamma_n g(\|Jx_n - JT_{[n]}x_n\|)\\
&\le \phi(p, x_n) - \beta_n\gamma_n g(\|Jx_n - JT_{[n]}x_n\|)).
\end{aligned}$$

Therefore, we have

$$\beta_n\gamma_n g(\|Jx_n - JT_{[n]}x_n\|) \le \phi(p, x_n) - \phi(p, z_n).$$

On the other hand, we have

$$\begin{aligned}
\phi(p, x_n) - \phi(p, z_n) &= \|x_n\|^2 - \|z_n\|^2 - 2\langle p, Jx_n - Jz_n\rangle\\
&\le \|x_n - z_n\|(\|x_n\| + \|z_n\|) + 2\|p\|\,\|Jx_n - Jz_n\|.
\end{aligned}$$

It follows from $\|x_n - z_n\| \to 0$ and $\|Jx_n - Jz_n\| \to 0$ that

$$\lim_{n\to\infty} (\phi(p, x_n) - \phi(p, z_n)) = 0. \tag{25}$$

Observing that the assumption $\liminf_{n\to\infty} \beta_n\gamma_n > 0$ and by Lemma 2.2, we also have

$$\lim_{n\to\infty} g\|Jx_n - JT_{[n]}x_n\| = 0.$$

It follows from the property of $g$ that

$$\lim_{n\to\infty} \|Jx_n - JT_{[n]}x_n\| = 0.$$

Since $J^{-1}$ is also uniformly norm-to-norm continuous on bounded sets, we see that

$$\lim_{n\to\infty} \|x_n - T_{[n]}x_n\| = 0.$$

Similarly, one can obtain

$$\lim_{n\to\infty} \|x_n - S_{[n]}x_n\| = 0.$$

Next we show that $\omega_\omega(\{x_n\}) \subset F$, $\omega_\omega(\{x_n\}) = \{x : \exists x_{n_i} \rightharpoonup x\}$. Indeed, we assume that $\bar{x} \in \omega_\omega(\{x_n\})$ and $x_{n_i} \rightharpoonup \bar{x}$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$. We may further assume that $n_i = l \,(\operatorname{mod} N)$ for all $i$. We also have

$$\|x_{n_i+j} - T_{[l+j]}x_{n_i+j}\| = \|x_{n_i+j} - T_{[n_i+j]}x_{n_i+j}\| \to 0,$$

which implies $\bar{x} \in F(T_{[l+j]})$ for all $j \geq 0$. Similarly, we have $\bar{x} \in F(S_{[l+j]})$ for all $j \geq 0$. Therefore, $\bar{x} \in F$.

Finally, we show that $x_n \to \Pi_F x_0$. From $x_{n+1} = \Pi_{C_{n+1}} x_0$ if we take $w \in F \subset C_{n+1}$, we also have $\phi(x_{n+1}, x) \leq \phi(w, x)$. On the other hand, from weak lower semicontinuity of the norm, we have

$$\begin{aligned}
\phi(\bar{x}, x_0) &= \|\bar{x}\|^2 - 2\langle \bar{x}, Jx_0\rangle + \|x_0\|^2 \\
&\leq \liminf_{i\to\infty}(\|x_{n_i}\|^2 - 2\langle x_{n_i}, Jx_0\rangle + \|x_0\|^2) \\
&\leq \liminf_{i\to\infty} \phi(x_{n_i}, x_0) \\
&\leq \limsup_{i\to\infty} \phi(x_{n_i}, x_0) \\
&\leq \phi(w, x_0).
\end{aligned}$$

From the definition of $\Pi_F x_0$, we obtain $\bar{x} = w$ and hence $\lim_{i\to\infty} \phi(x_{n_i}, x_0) = \phi(w, x_0)$. So, we have $\lim_{i\to\infty} \|x_{n_i}\| = \|w\|$. Using the Kadec-Klee property of $E$, we obtain that $\{x_{n_i}\}$ converges strongly to $\Pi_F x_0$. Since $\{x_{n_i}\}$ is an arbitrary weakly convergent subsequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to $\Pi_F x_0$. This completes the proof. ∎

COROLLARY 3.2. *Let $E$ be a uniformly convex and uniformly smooth Banach space and let $C$ be a nonempty closed convex subset of $E$. Let $T$ and $S$ be two hemi-relatively nonexpansive mappings from $C$ into itself with $F := F(T) \cap F(S)$ nonempty and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ for all $n \in \mathbb{N} \cup \{0\}$, $\limsup_{n\to\infty} \alpha_n < 1$, $0 \leq \beta_n, \gamma_n, \delta_n \leq 1$, $\beta_n + \gamma_n + \delta_n = 1$ for all $n \in \mathbb{N} \cup \{0\}$, $\liminf_{n\to\infty} \beta_n\gamma_n > 0$ and $\liminf_{n\to\infty} \beta_n\delta_n > 0$. Define a sequence $\{x_n\}$ by the following algorithm*:

$$\begin{cases}
x_0 \in C & \text{chosen arbitrarily,} \\
z_n = J^{-1}\big(\beta_n Jx_n + \gamma_n JTx_n + \delta_n JSx_n\big), \\
y_n = J^{-1}\big(\alpha_n Jx_n + (1-\alpha_n)Jz_n\big), \\
C_{n+1} = \{v \in C : \phi(v, y_n) \leq \phi(v, x_n)\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_0, & \forall n \geq 0,
\end{cases} \tag{26}$$

*where $J$ is the duality mapping on $E$. Then $\{x_n\}$ converges strongly to $\Pi_F x_0$, where $\Pi_F$ is the generalized projection from $C$ onto $F$.*

*Proof.* If in Theorem 3.1 we take $T_n = T$ and $S_n = S$ for all $n \in \mathbb{N} \cup \{0\}$, then (16) reduces to (26). ∎

COROLLARY 3.3. *Let $E$ be a uniformly convex and uniformly smooth Banach space and let $C$ be a nonempty closed convex subset of $E$. Let $T$ be a hemi-relatively nonexpansive mapping from $C$ into itself with $F := F(T)$ nonempty and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ be sequences of real numbers such that $0 \leq \alpha_n < 1$ for all $n \in \mathbb{N} \cup \{0\}$, $\limsup_{n\to\infty} \alpha_n < 1$, $0 \leq \beta_n, \gamma_n, \delta_n \leq 1$, $\beta_n + \gamma_n + \delta_n = 1$ for all $n \in \mathbb{N} \cup \{0\}$, $\liminf_{n\to\infty} \beta_n\gamma_n > 0$ and*

$\liminf_{n\to\infty} \beta_n\delta_n > 0$. *Define a sequence* $\{x_n\}$ *by the following algorithm*:

$$\begin{cases} x_0 \in C \qquad \text{chosen arbitrarily,} \\ z_n = J^{-1}\big(\beta_n Jx_n + \gamma_n JTx_n + \delta_n JTx_n\big), \\ y_n = J^{-1}\big(\alpha_n Jx_n + (1-\alpha_n)Jz_n\big), \\ C_{n+1} = \{v \in C : \phi(v, y_n) \le \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \qquad \forall n \ge 0, \end{cases} \tag{27}$$

*where* $J$ *is the duality mapping on* $E$. *Then* $\{x_n\}$ *converges strongly to* $\Pi_F x_0$, *where* $\Pi_F$ *is the generalized projection from* $C$ *onto* $F$.

Corollary 3.4. *Let* $E$ *be a uniformly convex and uniformly smooth Banach space and let* $C$ *be a nonempty closed convex subset of* $E$. *Given an integer* $N \ge 1$, *let, for each* $0 \le i \le N-1$, $T_i$ *be a hemi-relatively nonexpansive mapping from* $C$ *into itself with* $F := \bigcap_{i=0}^{N-1} F(T_i)$ *nonempty and* $\{\alpha_n\}$ *be a sequence of real numbers such that* $0 \le \alpha_n < 1$ *for all* $n \in \mathbb{N}\cup\{0\}$, $\limsup_{n\to\infty} \alpha_n < 1$. *Define a sequence* $\{x_n\}$ *by the following algorithm*:

$$\begin{cases} x_0 \in C \qquad \text{chosen arbitrarily,} \\ y_n = J^{-1}\big(\alpha_n Jx_n + (1-\alpha_n)JT_{[n]}x_n\big), \\ C_{n+1} = \{v \in C : \phi(v, y_n) \le \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \qquad \forall n \ge 0, \end{cases} \tag{28}$$

*where* $J$ *is the duality mapping on* $E$ *and* $T_{[n]} = T_i$, $i = n \pmod N$. *Then* $\{x_n\}$ *converges strongly to* $\Pi_F x_0$, *where* $\Pi_F$ *is the generalized projection from* $C$ *onto* $F$.

*Proof.* If in Theorem 3.1 we put $\beta_n = \delta_n = 0$ and $\gamma_n = 1$ for all $n \in \mathbb{N}\cup\{0\}$, then (16) reduces to (28). ∎

Corollary 3.5. *Let* $E$ *be a uniformly convex and uniformly smooth Banach space and let* $C$ *be a nonempty closed convex subset of* $E$. *Let* $T$ *be a hemi-relatively nonexpansive mapping from* $C$ *into itself with* $F := F(T)$ *nonempty and* $\{\alpha_n\}$ *be a sequence of real numbers such that* $0 \le \alpha_n < 1$ *for all* $n \in \mathbb{N}\cup\{0\}$, $\limsup_{n\to\infty} \alpha_n < 1$. *Define a sequence* $\{x_n\}$ *by the following algorithm*:

$$\begin{cases} x_0 \in C \qquad \text{chosen arbitrarily,} \\ y_n = J^{-1}\big(\alpha_n Jx_n + (1-\alpha_n)JTx_n\big), \\ C_{n+1} = \{v \in C : \phi(v, y_n) \le \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \qquad \forall n \ge 0, \end{cases} \tag{29}$$

*where* $J$ *is the duality mapping on* $E$. *Then* $\{x_n\}$ *converges strongly to* $\Pi_F x_0$, *where* $\Pi_F$ *is the generalized projection from* $C$ *onto* $F$.

Theorem 3.6. *Let* $E$ *be a uniformly convex and uniformly smooth Banach space and* $C$ *be a nonempty closed convex subset of* $E$. *Given an integer* $N \ge 1$, *let, for each* $0 \le i \le N-1$, $T_i$ *and* $S_i$ *be two hemi-relatively nonexpansive mappings from* $C$ *into itself with* $\mathcal{T} = \bigcap_{i=0}^{N-1} F(T_i)$, $\mathcal{S} = \bigcap_{i=0}^{N-1} F(S_i)$ *and* $F := \mathcal{T}\cap\mathcal{S}$ *nonempty and* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ *be sequences of real numbers such that* $0 \le \alpha_n < 1$ *for all* $n \in \mathbb{N}\cup\{0\}$, $\lim_{n\to\infty} \alpha_n = 0$,

$0 \leq \beta_n, \gamma_n, \delta_n \leq 1$, $\beta_n + \gamma_n + \delta_n = 1$ *for all* $n \in \mathbb{N} \cup \{0\}$, $\liminf_{n\to\infty} \beta_n\gamma_n > 0$ *and* $\liminf_{n\to\infty} \beta_n\delta_n > 0$. *Define a sequence* $\{x_n\}$ *by the following algorithm:*

$$\begin{cases} x_0 \in C & \text{chosen arbitrarily,} \\ z_n = J^{-1}\big(\beta_n Jx_n + \gamma_n JT_{[n]}x_n + \delta_n JS_{[n]}x_n\big), \\ y_n = J^{-1}\big(\alpha_n Jx_0 + (1-\alpha_n)Jz_n\big), \\ C_{n+1} = \big\{v \in C : \phi(v, y_n) \leq \alpha_n\phi(v, x_0) + (1-\alpha_n)\phi(v, x_n)\big\}, \\ x_{n+1} = \Pi_{C_{n+1}}x, & \forall n \geq 0, \end{cases} \tag{30}$$

*where* $J$ *is the duality mapping on* $E$ *and* $T_{[n]} = T_i$, $S_{[n]} = S_i$, $i = n \pmod N$. *Then* $\{x_n\}$ *converges strongly to* $\Pi_F x$, *where* $\Pi_F$ *is the generalized projection from* $C$ *onto* $F$.

*Proof.* We first show that $C_{n+1}$ is closed and convex for each $n \in \mathbb{N} \cup \{0\}$. From the definition of $C_{n+1}$ it is obvious that $C_{n+1}$ is closed for each $n \in \mathbb{N} \cup \{0\}$. By Theorem 3.1, we can prove $C_{n+1}$ is convex.

Next, we show that $F \subset C_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$. Let $p \in F$ and $n \in \mathbb{N} \cup \{0\}$, then we have

$$\begin{aligned} \phi(p, z_n) &= \phi\big(p, J^{-1}(\beta_n Jx_n + \gamma_n JT_{[n]}x_n + \delta_n JS_{[n]}x_n)\big) \\ &= \|p\|^2 - 2\langle p, \beta_n Jx_n + \gamma_n JT_{[n]}x_n + \delta_n JS_{[n]}x_n\rangle \\ &\quad + \|\beta_n Jx_n + \gamma_n JT_{[n]}x_n + \delta_n JS_{[n]}x_n\|^2 \\ &\leq \|p\|^2 - 2\beta_n\langle p, Jx_n\rangle - 2\gamma_n\langle p, JT_{[n]}x_n\rangle - 2\delta_n\langle p, JS_{[n]}x_n\rangle \\ &\quad + \beta_n\|x_n\|^2 + \gamma_n\|T_{[n]}x_n\|^2 + \delta_n\|S_{[n]}x_n\|^2 \\ &\leq \beta_n\phi(p, x_n) + \gamma_n\phi(p, T_{[n]}x_n) + \delta_n\phi(p, S_{[n]}x_n) \\ &\leq \beta_n\phi(p, x_n) + \gamma_n\phi(p, x_n) + \delta_n\phi(p, x_n) \\ &= \phi(p, x_n), \end{aligned} \tag{31}$$

and

$$\begin{aligned} \phi(p, y_n) &= \phi\big(p, J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jz_n)\big) \\ &= \|p\|^2 - 2\langle p, \alpha_n Jx_0 + (1-\alpha_n)Jz_n\rangle + \|\alpha_n Jx_0 + (1-\alpha_n)Jz_n\|^2 \\ &\leq \|p\|^2 - 2\alpha_n\langle p, Jx_0\rangle - 2(1-\alpha_n)\langle p, Jz_n\rangle + \alpha_n\|x_0\|^2 + (1-\alpha_n)\|z_n\|^2 \\ &\leq \alpha_n\phi(p, x_0) + (1-\alpha_n)\phi(p, z_n) \\ &\leq \alpha_n\phi(p, x_0) + (1-\alpha_n)\phi(p, x_n). \end{aligned} \tag{32}$$

So, $p \in C_n$ for all $n \in \mathbb{N} \cup \{0\}$, hence $F \subset C_n$. This implies that $\{x_n\}$ is well defined.

From the proof of Theorem 3.1, we also obtain $\{x_n\}$ is bounded and $\phi(x_{n+1}, x_n) \to 0$ as $n \to \infty$. Since $x_{n+1} = \Pi_{C_n}x_0 \in C_n$, from the definition of $C_{n+1}$, we also have

$$\phi(x_{n+1}, y_n) \leq \alpha_n\phi(x_{n+1}, x_0) + (1-\alpha_n)\phi(x_{n+1}, x_n). \tag{33}$$

Since $\lim_{n\to\infty} \alpha_n = 0$ and $\phi(x_{n+1}, x_n) \to 0$, we deduce that $\phi(x_{n+1}, y_n) \to 0$. By using Lemma 2.3 we obtain

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{34}$$

Since $J$ is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|Jx_{n+1} - Jy_n\| = \lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = 0. \tag{35}$$

Similarly as in the proof of Theorem 3.1, we obtain

$$\lim_{n\to\infty} \|Jx_{n+1} - Jz_n\| = 0.$$

Since $J^{-1}$ is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|x_{n+1} - z_n\| = 0. \tag{36}$$

By (34) and (36), we have

$$\|x_n - z_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \to 0.$$

Again by Theorem 3.1, we obtain

$$\lim_{n\to\infty} \|x_n - T_{[n]}x_n\| = 0 = \lim_{n\to\infty} \|x_n - S_{[n]}x_n\|.$$

Next we show that $\omega_\omega(\{x_n\}) \subset F$, $\omega_\omega(\{x_n\}) = \{x : \exists x_{n_i} \rightharpoonup x\}$. Indeed, we assume that $\bar{x} \in \omega_\omega(\{x_n\})$ and $x_{n_i} \rightharpoonup \bar{x}$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$. We may further assume that $n_i = l \,(\mathrm{mod}\, N)$ for all $i$. We also have

$$\|x_{n_i+j} - T_{[l+j]}x_{n_i+j}\| = \|x_{n_i+j} - T_{[n_i+j]}x_{n_i+j}\| \to 0,$$

which implies $\bar{x} \in F(T_{[l+j]})$ for all $j \ge 0$. Similarly, we have $\bar{x} \in F(S_{[l+j]})$ for all $j \ge 0$. Therefore, $\bar{x} \in F$.

Finally, we show that $x_n \to \Pi_F x_0$. From $x_{n+1} = \Pi_{C_{n+1}} x_0$ if we take $w \in F \subset C_{n+1}$, we also have $\phi(x_{n+1}, x) \le \phi(w, x)$. On the other hand, from weakly lower semicontinuity of the norm, we have

$$\begin{aligned}
\phi(\bar{x}, x) &= \|\bar{x}\|^2 - 2\langle \bar{x}, Jx\rangle + \|x\|^2 \\
&\le \liminf_{i\to\infty}(\|x_{n_i}\|^2 - 2\langle x_{n_i}, Jx\rangle + \|x\|^2) \\
&\le \liminf_{i\to\infty} \phi(x_{n_i}, x) \\
&\le \limsup_{i\to\infty} \phi(x_{n_i}, x) \\
&\le \phi(w, x).
\end{aligned}$$

From the definition of $\Pi_F x$, we obtain $\bar{x} = w$ and hence $\lim_{i\to\infty} \phi(x_{n_i}, x) = \phi(w, x)$. So, we have $\lim_{i\to\infty} \|x_{n_i}\| = \|w\|$. Using the Kadec-Klee property of $E$, we obtain that $\{x_{n_i}\}$ converges strongly to $\Pi_F x$. Since $\{x_{n_i}\}$ is an arbitrary weakly convergent sequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to $\Pi_F x$. This completes the proof. ■

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## References


[1] Ya. I. Alber, *Metric and generalized projection operators in Banach spaces: properties and applications*, in: Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, ed. by A. G. Kartsatos, Lecture Notes in Pure and Appl. Math. 178, Marcel Dekker, New York, 1996, 15–50.

[2] F. E. Browder, *Fixed points theorems for noncompact mappings in Hilbert spaces*, Proc. Nat. Acad. Sci. U.S.A. 53 (1965), 1272–1276.

[3] Y. J. Cho, H. Zhou, G. Guo, *Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings*, Comput. Math. Appl. 47 (2004), 707–717.

[4] B. Halpern, *Fixed points of nonexpanding maps*, Bull. Amer. Math. Soc. 73 (1976), 957–961.

[5] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. 44 (1974), 147–150.

[6] S. Kamimura, W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space*, SIAM J. Optim. 13 (2002), 938–945.

[7] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. 4 (1953), 506–510.

[8] S. Matsushita, W. Takahashi, *Weak and strong convergence theorems for relatively nonexpansive mappings in a Banach space*, Fixed Point Theory Appl. 2004, 37–47.

[9] S. Matsushita, W. Takahashi, *A strong convergence theorem for relatively nonexpansive mappings in a Banach space*, J. Approx. Theory 134 (2005), 257–266.

[10] S. Plubtieng, K. Ungchittrakool, *Strong convergence of modified Ishikawa iteration for two asymptotically nonexpansive mappings and semigroups*, Nonlinear Anal. 67 (2007), 2306–2315.

[11] S. Plubtieng, K. Ungchittrakool, *Strong convergence theorems for a common fixed point of two relatively nonexpansive mappings in a Banach space*, J. Approx. Theory 149 (2007), 103–115.

[12] Y. Su, D. Wang, M. Shang, *Strong convergence of monotone hybrid algorithm for hemi-relatively nonexpansive mappings*, Fixed Point Theory and Appl. 2008, Article ID 284613, 8 pp.

[13] W. Takahashi, Y. Takeuchi, R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl. 341 (2008), 276–286.