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# ON THE COLORED JONES POLYNOMIALS OF RIBBON LINKS, BOUNDARY LINKS AND BRUNNIAN LINKS

SAKIE SUZUKI
*Research Institute for Mathematical Sciences, Kyoto University*
*Kyoto, 606-8502, Japan*
*E-mail: sakie@kurims.kyoto-u.ac.jp*

**Abstract.** Habiro gave principal ideals of $\mathbb{Z}[q, q^{-1}]$ in which certain linear combinations of the colored Jones polynomials of algebraically-split links take values. The author proved that the same linear combinations for ribbon links, boundary links and Brunnian links are contained in smaller ideals of $\mathbb{Z}[q, q^{-1}]$ generated by several elements. In this paper, we prove that these ideals also are principal, each generated by a product of cyclotomic polynomials.

**1. Introduction.** After the discovery of the Jones polynomial, Reshetikhin and Turaev [7] defined an invariant of framed links whose components are colored by finite-dimensional representations of a ribbon Hopf algebra. The *colored Jones polynomial* can be defined as the Reshetikhin–Turaev invariant of links whose components are colored by finite-dimensional representations of the quantized enveloping algebra $U_h(sl_2)$.

We are interested in the relationship between *algebraic properties* of the colored Jones polynomial and *topological properties* of links.

In this paper, we consider the following three types of links.

A link is called a *ribbon link* if it bounds the image of an immersion from a disjoint union of disks into $S^3$ with only ribbon singularities.

An $n$-component link $L = L_1 \cup \ldots \cup L_n$ is called a *boundary link* if it bounds a disjoint union of $n$ Seifert surfaces $F_1, \ldots, F_n$ in $S^3$ such that $L_i$ bounds $F_i$ for $i = 1, \ldots, n$.

A link $L$ is called a *Brunnian link* if every proper sublink of $L$ is trivial.

In [4], Habiro used certain linear combinations $J_{L;\tilde{P}'_{l_1},\ldots,\tilde{P}'_{l_n}}$, $l_1, \ldots, l_n \geq 0$, of the colored Jones polynomials of a link $L$ to construct the unified Witten–Reshetikhin–Turaev invariants for integral homology spheres. He proved that $J_{L;\tilde{P}'_{l_1},\ldots,\tilde{P}'_{l_n}}$ for an algebraically-split, 0-framed link $L$ is contained in a certain principal ideal of $\mathbb{Z}[q, q^{-1}]$ (Theorem 2.1).

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This result was improved by the present author [8, 9, 10, 11] in the special case of ribbon links, boundary links (Theorem 2.2) and Brunnian links (Theorem 2.4) by using ideals $I_{l_1}, \dots, I_{l_n}$ of $\mathbb{Z}[q, q^{-1}]$, where Theorem 2.2 for boundary links had been conjectured by Habiro [4]. Here, in [8], we gave an alternative proof of the fact that the Jones polynomial of an $n$-component ribbon link is divisible by the Jones polynomial of the $n$-component trivial link, which was proved first by Eisermann [1]. The results in [4, 8, 9, 10, 11] are proved by using the *universal $sl_2$ invariant of bottom tangles* (cf. [3, 4]), which has the universality property for the colored Jones polynomial of links.

In this paper, we prove that the ideal $I_l$, $l \geq 0$, is a principal ideal generated by a product of cyclotomic polynomials (Theorem 3.1), and rewrite Theorems 2.1, 2.2 and 2.4 by using these generators (Proposition 3.3).

**2. Results for the colored Jones polynomial.** In this section, we recall results in [4, 9, 10, 11] for the colored Jones polynomial. In what follows, we assume that links are 0-framed. For the definition of the quantized enveloping algebra $U_h(sl_2)$ see, e.g., [6, 4, 9]. We set $q = \exp h$.

For $m \geq 1$, let $V_m$ denote the $m$-dimensional irreducible representation of $U_h(sl_2)$. Let $\mathcal{R}$ denote the representation ring of $U_h(sl_2)$ over $\mathbb{Q}(q^{1/2})$, i.e., $\mathcal{R}$ is the $\mathbb{Q}(q^{1/2})$-algebra

$$\mathcal{R} = \mathrm{Span}_{\mathbb{Q}(q^{1/2})}\{V_m \mid m \geq 1\}$$

with the multiplication induced by the tensor product. It is well known that $\mathcal{R} = \mathbb{Q}(q^{1/2})[V_2]$.

Set

$$\{i\}_q = q^i - 1, \quad \{i\}_{q,n} = \{i\}_q\{i-1\}_q \cdots \{i-n+1\}_q, \quad \{n\}_q! = \{n\}_{q,n},$$

for $i \in \mathbb{Z}$, $n \geq 0$.

Habiro [4] studied the following elements in $\mathcal{R}$

$$\tilde{P}'_l = \frac{q^{l/2}}{\{l\}_q!} \prod_{i=0}^{l-1} (V_2 - q^{i+1/2} - q^{-i-1/2}),$$

for $l \geq 0$, which are used in an important technical step in his construction of the unified Witten–Reshetikhin–Turaev invariants for integral homology spheres.

For the definition of the colored Jones polynomial $J_{L;X_1,\dots,X_n}$ of $L$ with $i$th component $L_i$ colored by $X_i \in \mathcal{R}$ see, e.g., [5, 4, 9].

Habiro [4] proved the following.

THEOREM 2.1 (Habiro [4]). *Let $L$ be an $n$-component algebraically-split link. We have*

$$J_{L;\tilde{P}'_{l_1},\dots,\tilde{P}'_{l_n}} \in \frac{\{2l_{\max}+1\}_{q,l_{\max}+1}}{\{1\}_q} \mathbb{Z}[q, q^{-1}], \tag{1}$$

*for $l_1, \dots, l_n \geq 0$, where $l_{\max} = \max(l_1, \dots, l_n)$.*

Set

$$f_{l,k} = \{l-k\}_q!\{k\}_q!, \tag{2}$$

for $0 \leq k \leq l$. For $l \geq 0$, let $I_l$ be the ideal of $\mathbb{Z}[q, q^{-1}]$ generated by $f_{l,0}, \dots, f_{l,l}$.

In [9, 10], we proved the following.

THEOREM 2.2 ([9, 10]). *Let $L$ be an $n$-component ribbon or boundary link. For $l_1, \dots, l_n$ nonnegative, we have*

$$J_{L;\tilde{P}'_{l_1},\dots,\tilde{P}'_{l_n}} \in \frac{\{2l_{\max}+1\}_{q,l_{\max}+1}}{\{1\}_q} \prod_{1\le i\le n,\, i\ne i_M} I_{l_i}, \tag{3}$$

*where $l_{\max} = \max(l_1, \dots, l_n)$ and $i_M$ is an integer such that $l_{i_M} = l_{\max}$.*

REMARK 2.3. Theorem 2.2 for boundary links had been conjectured by Habiro [4].

In [11], we proved the following.

THEOREM 2.4 ([11]). *Let $L$ be an $n$-component algebraically-split Brunnian link with $n \ge 2$. We have*

$$J_{L;\tilde{P}'_{l_1},\dots,\tilde{P}'_{l_n}} \in \frac{\{2l_{\max}+1\}_{q,l_{\max}+1}}{\{1\}_q\{l_{\min}\}_q!} \prod_{1\le i\le n,\, i\ne i_M,i_m} I_{l_i}, \tag{4}$$

*for $l_1, \dots, l_n \ge 0$, where $l_{\max} = \max(l_1, \dots, l_n)$, $l_{\min} = \min(l_1, \dots, l_n)$ and $i_M, i_m$, $i_M \ne i_m$, are integers such that $l_{i_M} = l_{\max}$, $l_{i_m} = l_{\min}$, respectively.*

Note that the condition "algebraically-split" in Theorem 2.4 is not necessary when $n \ge 3$.

Let us compare Theorems 2.1, 2.2 and 2.4. For $l_1, \dots, l_n \ge 0$, $n \ge 2$, let $Z_a^{(l_1,\dots,l_n)}$, $Z_{r,b}^{(l_1,\dots,l_n)}$ and $Z_{\mathrm{Br}}^{(l_1,\dots,l_n)}$ denote the ideals of $\mathbb{Z}[q,q^{-1}]$ at the right hand sides of (1), (3) and (4), respectively, i.e., we set

$$Z_a^{(l_1,\dots,l_n)} = \frac{\{2l_{\max}+1\}_{q,l_{\max}+1}}{\{1\}_q}\mathbb{Z}[q,q^{-1}],$$
$$Z_{r,b}^{(l_1,\dots,l_n)} = \frac{\{2l_{\max}+1\}_{q,l_{\max}+1}}{\{1\}_q} \prod_{1\le i\le n,\, i\ne i_M} I_{l_i},$$
$$Z_{\mathrm{Br}}^{(l_1,\dots,l_n)} = \frac{\{2l_{\max}+1\}_{q,l_{\max}+1}}{\{1\}_q\{l_{\min}\}_q!} \prod_{1\le i\le n,\, i\ne i_M,i_m} I_{l_i}.$$

For $l_1, \dots, l_n \ge 0$, we have

$$Z_{r,b}^{(l_1,\dots,l_n)} \subset Z_a^{(l_1,\dots,l_n)}, \quad Z_{r,b}^{(l_1,\dots,l_n)} \subset Z_{\mathrm{Br}}^{(l_1,\dots,l_n)},$$

since we have

$$Z_{r,b}^{(l_1,\dots,l_n)} = \Big(\prod_{1\le i\le n, i\ne i_M} I_{l_i}\Big)\cdot Z_a^{(l_1,\dots,l_n)} = \big(\{l_{\min}\}_q!\, I_{l_{\min}}\big)\cdot Z_{\mathrm{Br}}^{(l_1,\dots,l_n)}.$$

On the other hand, there is no inclusion which satisfies for all $l_1, \dots, l_n \ge 0$ between $Z_a^{(l_1,\dots,l_n)}$ and $Z_{\mathrm{Br}}^{(l_1,\dots,l_n)}$. For example, we have $Z_a^{(2,2,2,2)} \not\subset Z_{\mathrm{Br}}^{(2,2,2,2)}$ and $Z_{\mathrm{Br}}^{(2,2,2,2)} \not\subset Z_a^{(2,2,2,2)}$ since

$$\begin{aligned} Z_a^{(2,2,2,2)} &= \frac{\{5\}_{q,3}}{\{1\}_q}\mathbb{Z}[q,q^{-1}] \\ &= (q-1)^2(q+1)(q^2+q+1)(q^2+1)(q^4+q^3+q^2+q^1+1)\mathbb{Z}[q,q^{-1}], \\ Z_{\mathrm{Br}}^{(2,2,2,2)} &= \frac{\{5\}_{q,3}}{\{1\}_q\{2\}_q!}\{1\}_q^4\mathbb{Z}[q,q^{-1}] \\ &= (q-1)^4(q^2+q+1)(q^2+1)(q^4+q^3+q^2+q^1+1)\mathbb{Z}[q,q^{-1}]. \end{aligned}$$

Thus we have the following refinement of Theorem 2.4.

THEOREM 2.5. *Let $L$ be an $n$-component algebraically-split Brunnian link with $n \geq 2$. We have*

$$J_{L;\tilde{P}'_{l_1},\ldots,\tilde{P}'_{l_n}} \in Z_a^{(l_1,\ldots,l_n)} \cap Z_{\mathrm{Br}}^{(l_1,\ldots,l_n)},$$

*for $l_1,\ldots,l_n \geq 0$.*

## 3. Main result for the ideal $I_l$.

In this section, we state the main result of this paper.

For $l \geq 0$, recall from (2) the generators $f_{l,0},\ldots,f_{l,l}$ of the ideal $I_l$. Set

$$g_l = GCD(f_{l,0},\ldots,f_{l,l}).$$

It is clear that $I_l \subset g_l\mathbb{Z}[q,q^{-1}]$. The opposite inclusion follows if and only if $I_l$ is principal. Since $\mathbb{Z}[q,q^{-1}]$ is *not* a principal ideal domain, there is a problem if $I_l$ is principal or not. The main result in this paper (Theorem 3.1) is that $I_l$ is principal, where we determine $g_l$ explicitly. The proof is in Section 4.

For $m \geq 1$, let $\Phi_m = \prod_{d|m}(q^d-1)^{\mu(m/d)} \in \mathbb{Z}[q]$ denote the $m$th cyclotomic polynomial, where $\prod_{d|m}$ denotes the product over all positive divisors $d$ of $m$, and $\mu$ is the Möbius function. For $r \in \mathbb{Q}$, we denote by $\lfloor r \rfloor$ the largest integer smaller than or equal to $r$.

THEOREM 3.1. *For $l \geq 0$, the ideal $I_l$ is the principal ideal generated by $g_l$. Moreover, we have*

$$g_l = \prod_{m\geq 1} \Phi_m^{t_{l,m}}, \tag{5}$$

*where*

$$t_{l,m} = \begin{cases} \lfloor \frac{l+1}{m} \rfloor - 1 & \text{for } 1 \leq m \leq l, \\ 0 & \text{for } l < m. \end{cases}$$

Here is a table of $t_{l,m}$ for $1 \leq m \leq 4$, $0 \leq l \leq 16$.

| $m \setminus l$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| 2 | 0 | 0 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 6 | 7 | 7 |
| 3 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 4 | 4 | 4 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 3 |

REMARK 3.2. In [11], Theorem 3.1 is used in the proof of Theorem 2.4.

Theorem 3.1 implies that the ideals $Z_{r,b}^{(l_1,\ldots,l_n)}$ and $Z_{\mathrm{Br}}^{(l_1,\ldots,l_n)}$ are principal. Moreover, we can write a generator of each principal ideal $Z_a^{(l_1,\ldots,l_n)}$, $Z_{r,b}^{(l_1,\ldots,l_n)}$ and $Z_{\mathrm{Br}}^{(l_1,\ldots,l_n)}$ as a product of cyclotomic polynomials as follows.

PROPOSITION 3.3. *For $l_1, \dots, l_n \geq 0$, the ideals $Z_a^{(l_1,\dots,l_n)}$, $Z_{r,b}^{(l_1,\dots,l_n)}$ and $Z_{\mathrm{Br}}^{(l_1,\dots,l_n)}$ are principal. Moreover, we have*

$$Z_a^{(l_1,\dots,l_n)} = \prod_{m\geq 1} \Phi_m^{\lfloor(2l_{\max}+1)/m\rfloor - \lfloor l_{\max}/m\rfloor - \lfloor 1/m\rfloor}\mathbb{Z}[q,q^{-1}],$$

$$Z_{r,b}^{(l_1,\dots,l_n)} = \prod_{1\leq m\leq 2l_{\max}+1} \Phi_m^{\lfloor(2l_{\max}+1)/m\rfloor - \lfloor l_{\max}/m\rfloor - \lfloor 1/m\rfloor + \sum_{1\leq i\leq n,\, i\neq i_M} t_{l_i,m}}\mathbb{Z}[q,q^{-1}],$$

$$Z_{\mathrm{Br}}^{(l_1,\dots,l_n)} = \prod_{1\leq m\leq 2l_{\max}+1} \Phi_m^{\lfloor(2l_{\max}+1)/m\rfloor - \lfloor l_{\max}/m\rfloor - \lfloor 1/m\rfloor - \lfloor l_{\min}/m\rfloor + \sum_{1\leq i\leq n,\, i\neq i_M, i_m} t_{l_i,m}}\mathbb{Z}[q,q^{-1}].$$

*Proof.* The assertion for $Z_a^{(l_1,\dots,l_n)}$ follows from

$$\{l\}_{q,i} = \prod_{m\geq 1} \Phi_m^{\lfloor l/m\rfloor - \lfloor (l-i)/m\rfloor}, \tag{6}$$

for $0 \leq i \leq l$. The assertion for $Z_{r,b}^{(l_1,\dots,l_n)}$ and $Z_{\mathrm{Br}}^{(l_1,\dots,l_n)}$ follows from (6) and Theorem 3.1. ■

COROLLARY 3.4. *For $l_1, \dots, l_n \geq 0$, we have*

$$Z_a^{(l_1,\dots,l_n)} \cap Z_{\mathrm{Br}}^{(l_1,\dots,l_n)} = \prod_{m\geq 1} \Phi_m^{\lfloor(2l_{\max}+1)/m\rfloor - \lfloor l_{\max}/m\rfloor - \lfloor 1/m\rfloor + \max(0, \sum_{1\leq i\leq n,\, i\neq i_M, i_m} t_{l_i,m} - \lfloor l_{\min}/m\rfloor)}\mathbb{Z}[q,q^{-1}].$$

EXAMPLE 3.5. Let $L$ be an $n$-component *algebraically-split* link. By Theorem 2.1 and Proposition 3.3, we have

$$J_{L;\tilde{P}_1',\dots,\tilde{P}_1'} \in \Phi_1\Phi_2\Phi_3\mathbb{Z}[q,q^{-1}],$$
$$J_{L;\tilde{P}_2',\dots,\tilde{P}_2'} \in \Phi_1^2\Phi_2\Phi_3\Phi_4\Phi_5\mathbb{Z}[q,q^{-1}],$$
$$J_{L;\tilde{P}_3',\dots,\tilde{P}_3'} \in \Phi_1^3\Phi_2^2\Phi_3\Phi_4\Phi_5\Phi_6\Phi_7\mathbb{Z}[q,q^{-1}].$$

Let $L$ be an $n$-component algebraically-split *Brunnian* link with $n \geq 2$. By Theorem 2.5 and Corollary 3.4, we have

$$J_{L;\tilde{P}_1',\dots,\tilde{P}_1'} \in \Phi_1^{\delta_{n,2}+n-2}\Phi_2\Phi_3\mathbb{Z}[q,q^{-1}],$$
$$J_{L;\tilde{P}_2',\dots,\tilde{P}_2'} \in \Phi_1^{2(\delta_{n,2}+n-2)}\Phi_2\Phi_3\Phi_4\Phi_5\mathbb{Z}[q,q^{-1}],$$
$$J_{L;\tilde{P}_3',\dots,\tilde{P}_3'} \in \Phi_1^{3(\delta_{n,2}+n-2)}\Phi_2^{\delta_{n,2}+n-1}\Phi_3\Phi_4\Phi_5\Phi_6\Phi_7\mathbb{Z}[q,q^{-1}].$$

Let $L$ be an $n$-component *ribbon* or *boundary* link. By Theorem 2.2 and Proposition 3.3, we have

$$J_{L;\tilde{P}_1',\dots,\tilde{P}_1'} \in \Phi_1^n\Phi_2\Phi_3\mathbb{Z}[q,q^{-1}],$$
$$J_{L;\tilde{P}_2',\dots,\tilde{P}_2'} \in \Phi_1^{2n}\Phi_2\Phi_3\Phi_4\Phi_5\mathbb{Z}[q,q^{-1}],$$
$$J_{L;\tilde{P}_3',\dots,\tilde{P}_3'} \in \Phi_1^{3n}\Phi_2^{n+1}\Phi_3\Phi_4\Phi_5\Phi_6\Phi_7\mathbb{Z}[q,q^{-1}].$$

EXAMPLE 3.6. For $n \geq 3$, let $M_n$ be Milnor's $n$-component Brunnian link depicted in Figure 1. Note that $M_3$ is the Borromean ring. We have

$$J_{M_n;\tilde{P}'_1,\dots,\tilde{P}'_1} = (-1)^n q^{-2n+4} \Phi_1^{n-2} \Phi_2^{n-2} \Phi_3 \Phi_4^{n-3},$$

which we will prove in a forthcoming paper [12]. This implies that Theorem 2.4 is best possible for the divisibility by $\Phi_1$ and $\Phi_3$ of $J_{L;\tilde{P}'_1,\dots,\tilde{P}'_1}$ with $L$ Brunnian. By Theorem 2.2, this also implies that each $M_n$ is not ribbon or boundary.

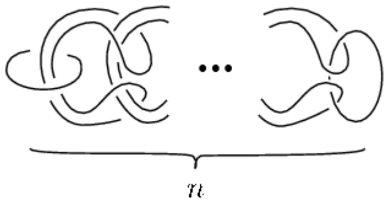


Fig. 1. Milnor's link $M_n$

**4. Proof of Theorem 3.1.** For $a_1, \dots, a_m \in \mathbb{Z}[q, q^{-1}]$, let $(a_1, \dots, a_m)$ denote the ideal in $\mathbb{Z}[q, q^{-1}]$ generated by $a_1, \dots, a_m \in \mathbb{Z}[q, q^{-1}]$.

For $l \geq 0$, recall that $I_l = (f_{l,0}, f_{l,1}, \dots, f_{l,l})$ with $f_{l,i} = \{l-i\}_q!\{i\}_q!$ for $0 \leq i \leq l$.

For $0 \leq k \leq l$, we have

$$(f_{l,0}, f_{l,1}, \dots, f_{l,k}) = \{l-k\}_q!(h_{l,k,0}, h_{l,k,1}, \dots, h_{l,k,k})$$

with

$$h_{l,k,i} = f_{l,i}/\{l-k\}_q! = \{l-i\}_{q,k-i}\{i\}_q!,$$

for $1 \leq i \leq k$. Set

$$I_{l,k} = (h_{l,k,0}, h_{l,k,1}, \dots, h_{l,k,k}),$$
$$g_{l,k} = GCD(h_{l,k,0}, h_{l,k,1}, \dots, h_{l,k,k}).$$

Note that $I_{l,l} = I_l$ and $g_{l,l} = g_l$.

In what follows, for $a \in \mathbb{Z}[q, q^{-1}] \setminus \{0\}$ and $m \geq 1$, let $d_m(a)$ denote the largest integer $i$ such that $a \in \Phi_m^i \mathbb{Z}[q, q^{-1}]$. For $0 \leq k \leq l$, we can write

$$g_{l,k} = \prod_{m \geq 1} \Phi_m^{d_m(g_{l,k})},$$

since each $h_{l,k,i}$ is a product of cyclotomic polynomials.

LEMMA 4.1. *For $0 \leq k \leq l$, we have*

$$d_m(g_{l,k}) = \begin{cases} \lfloor \frac{l+1}{m} \rfloor - 1 - \lfloor \frac{l-k}{m} \rfloor & \text{for } 1 \leq m \leq k, \\ 0 & \text{for } k < m. \end{cases}$$

*Proof.* We have

$$\begin{aligned} d_m(g_{l,k}) &= \min\{d_m\bigl(h_{l,k,i}\bigr) \,|\, 0 \le i \le k\} \\ &= \min\{d_m\bigl(\{l-i\}_{q,k-i}\{i\}_q!\bigr) \,|\, 0 \le i \le k\} \\ &= \min\Bigl\{\Bigl\lfloor\frac{l-i}{m}\Bigr\rfloor - \Bigl\lfloor\frac{l-k}{m}\Bigr\rfloor + \Bigl\lfloor\frac{i}{m}\Bigr\rfloor \,|\, 0 \le i \le k\Bigr\} \\ &= \min\Bigl\{\Bigl\lfloor\frac{l-i}{m}\Bigr\rfloor + \Bigl\lfloor\frac{i}{m}\Bigr\rfloor \,|\, 0 \le i \le k\Bigr\} - \Bigl\lfloor\frac{l-k}{m}\Bigr\rfloor. \end{aligned}$$

If $k < m$, then we have $d_m(g_{l,k}) = 0$ since $d_m\bigl(h_{l,k,k}\bigr) = d_m\bigl(\{k\}_q!\bigr) = 0$.

Let $1 \le m \le k$. Since we have

$$\Bigl\lfloor\frac{l-(i+am)}{m}\Bigr\rfloor + \Bigl\lfloor\frac{i+am}{m}\Bigr\rfloor = \Bigl\lfloor\frac{l-i}{m}\Bigr\rfloor + \Bigl\lfloor\frac{i}{m}\Bigr\rfloor,$$

for $0 \le i \le k$ and $a \in \mathbb{Z}$, we have

$$\min\Bigl\{\Bigl\lfloor\frac{l-i}{m}\Bigr\rfloor + \Bigl\lfloor\frac{i}{m}\Bigr\rfloor \,|\, 0 \le i \le k\Bigr\} = \min\Bigl\{\Bigl\lfloor\frac{l-i}{m}\Bigr\rfloor + \Bigl\lfloor\frac{i}{m}\Bigr\rfloor \,|\, 0 \le i \le m-1\Bigr\}.$$

Here, for $0 \le i \le m-1$, we have $\lfloor\frac{i}{m}\rfloor = 0$ and $\lfloor\frac{l-i}{m}\rfloor$ takes the minimum with $i = m-1$. Thus we have

$$\min\Bigl\{\Bigl\lfloor\frac{l-i}{m}\Bigr\rfloor + \Bigl\lfloor\frac{i}{m}\Bigr\rfloor \,|\, 0 \le i \le m-1\Bigr\} = \Bigl\lfloor\frac{l-(m-1)}{m}\Bigr\rfloor = \Bigl\lfloor\frac{l+1}{m}\Bigr\rfloor - 1.$$

This implies

$$d_m(g_{l,k}) = \Bigl\lfloor\frac{l+1}{m}\Bigr\rfloor - 1 - \Bigl\lfloor\frac{l-k}{m}\Bigr\rfloor.$$

Hence we have the assertion. ■

Note that we have the latter part (5) of Theorem 3.1 as follows.

COROLLARY 4.2. *For $l \ge 0$, we have*

$$g_l = g_{l,l} = \prod_{m \ge 1} \Phi_m^{t_{l,m}}.$$

From now, we prove the following generalization of Theorem 3.1.

PROPOSITION 4.3. *For $0 \le k \le l$, the ideal $I_{l,k}$ is the principal ideal generated by $g_{l,k}$.*

For $1 \le k \le l$, set

$$\tilde{g}_{l,k} = g_{l,k}/g_{l,k-1}.$$

We have

$$\begin{aligned} \tilde{g}_{l,k} &= \prod_{1 \le m \le k} \Phi_m^{\lfloor(l+1)/m\rfloor - 1 - \lfloor(l-k)/m\rfloor - (\lfloor(l+1)/m\rfloor - 1 - \lfloor(l-k+1)/m\rfloor)} \\ &= \prod_{1 \le m \le k} \Phi_m^{\lfloor(l-k+1)/m\rfloor - \lfloor(l-k)/m\rfloor} = \prod_{\substack{m|l-k+1 \\ 1 \le m \le k}} \Phi_m. \end{aligned}$$

We use the following technical lemma.

LEMMA 4.4. *For $1 \le k \le l$, we have*

$$\Bigl(\{l-k+1\}_q, \{k\}_q \frac{\{k-1\}_q!}{g_{l,k-1}}\Bigr) = (\tilde{g}_{l,k}).$$

*(Note that $g_{l,k-1} = GCD(\{l\}_{q,k-1}, \{l-1\}_{q,k-2}\{1\}_q, \ldots, \{k-1\}_q!)$ divides $\{k-1\}_q!$.)*

*Proof of Proposition 4.3 by assuming Lemma 4.4* . We use induction on $k$. For $k = 0$, clearly we have

$$I_{l,0} = (g_{l,0}) = (1).$$

For $k \geq 1$, we have

$$\begin{aligned} I_{l,k} &= (h_{l,k,0}, h_{l,k,1}, \ldots, h_{l,k,k}) \\ &= \big(\{l\}_{q,k}, \{l-1\}_{q,k-1}\{1\}_q, \ldots, \{l-k+1\}_q\{k-1\}_q!, \{k\}_q!\big) \\ &= \big(\{l-k+1\}_q(\{l\}_{q,k-1}, \{l-1\}_{q,k-2}\{1\}_q, \ldots, \{k-1\}_q!), \{k\}_q!\big) \\ &= (\{l-k+1\}_q g_{l,k-1}, \{k\}_q!) \\ &= g_{l,k-1}\Big(\{l-k+1\}_q, \{k\}_q \frac{\{k-1\}_q!}{g_{l,k-1}}\Big) \\ &= (g_{l,k-1}\tilde{g}_{l,k}) = (g_{l,k}), \end{aligned}$$

where the third equality is given by

$$h_{l,k,i} = \{l-k+1\}_q \cdot \{l-i\}_{q,k-i-1}\{i\}_q!,$$

for $0 \leq i \leq k-1$, and the fourth equality is given by the assumption of the induction. Hence we have the assertion. ■

In what follows, we prove Lemma 4.4. We use the following two lemmas, which are well-known.

Lemma 4.5 (cf. Habiro [2, Lemma 4.1]). *For $a, b \geq 0$, the following conditions are equivalent*:

(i) $(\Phi_a, \Phi_b) = \mathbb{Z}[q, q^{-1}]$,
(ii) $\frac{a}{b} \neq p^i$ *for any prime number $p \geq 0$ and $i \in \mathbb{Z}$.*

Lemma 4.6. *Let $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbb{Z}[q, q^{-1}]$ such that $(a_i, b_j) = \mathbb{Z}[q, q^{-1}]$ for all $1 \leq i \leq m$, $1 \leq j \leq n$. We have*

$$(a_1 a_2 \cdots a_m, b_1 b_2 \cdots b_n) = \mathbb{Z}[q, q^{-1}].$$

*Proof of Lemma 4.4.* It is enough to prove the following two equalities.

$$GCD\Big(\{l-k+1\}_q, \{k\}_q \frac{\{k-1\}_q!}{g_{l,k-1}}\Big) = \tilde{g}_{l,k}, \tag{7}$$

$$\Big(\{l-k+1\}_q/\tilde{g}_{l,k}, \{k\}_q \frac{\{k-1\}_q!}{g_{l,k-1}}/\tilde{g}_{l,k}\Big) = \mathbb{Z}[q, q^{-1}], \tag{8}$$

for $1 \leq k \leq l$.

First, we prove (7). Recall that

$$\tilde{g}_{l,k} = \prod_{\substack{m \mid l-k+1 \\ 1 \leq m \leq k}} \Phi_m. \tag{9}$$

Since $\{l-k+1\}_q = \prod_{m \mid l-k+1} \Phi_m$ and $d_m(\{k\}_q \frac{\{k-1\}_q!}{g_{l,k-1}}) = 0$ for $m > k$, it is enough to check that

$$d_m\Big(\{k\}_q \frac{\{k-1\}_q!}{g_{l,k-1}}\Big) \geq 1,$$

for $m|l-k+1$, $1 \leq m \leq k$. Indeed, we have

$$d_m(\{k\}_q) = \begin{cases} 1 & \text{for } m|k, \\ 0 & \text{for } m\nmid k, \end{cases} \tag{10}$$

$$d_m\Big(\frac{\{k-1\}_q!}{g_{l,k-1}}\Big) = \begin{cases} 0 & \text{for } m|k, \\ 1 & \text{for } m\nmid k. \end{cases} \tag{11}$$

Here, (10) is clear and (11) follows from

$$\begin{aligned} d_m\Big(\frac{\{k-1\}_q!}{g_{l,k-1}}\Big) &= \Big\lfloor \frac{k-1}{m} \Big\rfloor - d_m(g_{l,k-1}) \\ &= \Big\lfloor \frac{k-1}{m} \Big\rfloor - \Big\lfloor \frac{l+1}{m} \Big\rfloor + 1 + \Big\lfloor \frac{l-k+1}{m} \Big\rfloor \\ &= \Big\lfloor \frac{um+r-1}{m} \Big\rfloor - \Big\lfloor \frac{(u+u')m+r}{m} \Big\rfloor + 1 + \Big\lfloor \frac{u'm}{m} \Big\rfloor \\ &= \Big\lfloor \frac{r-1}{m} \Big\rfloor + 1 \\ &= \begin{cases} 0 & \text{for } r = 0, \\ 1 & \text{for } 1 \leq r \leq m-1, \end{cases} \end{aligned}$$

where we write $k = um + r$ and $l - k + 1 = u'm$ with $u, u' \geq 1$ and $0 \leq r \leq m-1$.

We prove (8). By Lemmas 4.5 and 4.6, it is enough to prove that there is no pair of integers $m, n \geq 1$ such that

- $\frac{n}{m} = p^i$ for a prime $p$ and $i \in \mathbb{Z}$,
- $d_m(\{l-k+1\}_q/\tilde{g}_{l,k}) \geq 1$, and
- $d_n(\{k\}_q \frac{\{k-1\}_q!}{g_{l,k-1}}/\tilde{g}_{l,k}) \geq 1$.

Note that

$$\{l-k+1\}_q/\tilde{g}_{l,k} = \prod_{\substack{m|l-k+1 \\ m>k}} \Phi_m.$$

Let $m|l-k+1$, $m > k$, i.e., $d_m(\{l-k+1\}_q/\tilde{g}_{l,k}) = 1$. Recall that for $n > k$, we have $d_n(\{k\}_q \frac{\{k-1\}_q!}{g_{l,k-1}}) = 0$. Assume that $1 \leq n \leq k$ and $n|m$, which implies $n|l-k+1$. The conditions $1 \leq n \leq k$ and $n|l-k+1$ imply $d_n(\tilde{g}_{l,k}) = 1$ by (9). By (10) and (11), we have $d_n(\{k\}_q \frac{\{k-1\}_q!}{g_{l,k-1}}) = 1$. Thus we have $d_n(\{k\}_q \frac{\{k-1\}_q!}{g_{l,k-1}}/\tilde{g}_{l,k}) = 0$, which completes the proof. ■

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