KNOTS IN POLAND III
BANACH CENTER PUBLICATIONS, VOLUME 100
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2014

ABOUT PRESENTATIONS OF BRAID GROUPS AND THEIR GENERALIZATIONS

V. V. VERSHININ

Département des Sciences Mathématiques, Université Montpellier II
Place Eugéne Bataillon, 34095 Montpellier cedex 5, France
and Sobolev Institute of Mathematics, Novosibirsk, 630090, Russia
E-mail: vershini@math.univ-montp2.fr, versh@math.nsc.ru

Abstract. In the paper we give a survey of rather new notions and results which generalize classical ones in the theory of braids. Among such notions are various inverse monoids of partial braids. We also observe presentations different from standard Artin presentation for generalizations of braids. Namely, we consider presentations with small number of generators, Sergiescu graph-presentations and Birman–Ko–Lee presentation. The work of V. V. Chaynikov on the word and conjugacy problems for the singular braid monoid in Birman–Ko–Lee generators is described as well.

Contents

2010 *Mathematics Subject Classification*: Primary 20F36; Secondary 20M18, 57M.
Key words and phrases: Braid, presentation, inverse braid monoid, Artin–Brieskorn group, singular braid monoid, word problem.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc100-0-14

1. Introduction. The purpose of this paper is to give a survey on some recent notions and results concerning generalizations of the braids.

Classical braid groups Br_n can be defined in several ways. Either as a set of isotopy classes of system of n curves in a three-dimensional space (what is the same as the fundamental group of the configuration space of n points on a plane) or as the mapping class group of a disc with n points deleted D_n with its boundary fixed, what is equivalent to the subgroup of the braid automorphisms of the automorphism group of a free group $\operatorname{Aut} F_n$. For the exact definitions we make a reference here to a monograph on braid, for example the book of C. Kassel and V. Turaev [46] or to the previous surveys of the author [79, 81, 84].

The *pure* braid group P_n is defined as the kernel of the canonical epimorphism τ_n from braids to the symmetric group Σ_n:

$$1 \to P_n \to Br_n \xrightarrow{\tau_n} \Sigma_n \to 1.$$

We fix the canonical Artin presentation [2] of the braid group Br_n. It has generators σ_i, $i = 1, \dots, n-1$, and two types of relations:

$$\begin{cases} \sigma_i\sigma_j = \sigma_j\sigma_i, & \text{if } \ |i-j| > 1, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}. \end{cases} \tag{1.1}$$

The generators σ_i correspond to the following automorphisms of F_n:

$$\begin{cases} x_i \mapsto x_{i+1}, \\ x_{i+1} \mapsto x_{i+1}^{-1}x_ix_{i+1}, \\ x_j \mapsto x_j, \qquad j \neq i, i+1. \end{cases} \tag{1.2}$$

Of course, there exist other presentations of the braid group. Let

$$\sigma = \sigma_1\sigma_2\dots\sigma_{n-1}, \tag{1.3}$$

then the group Br_n is generated by σ_1 and σ because

$$\sigma_{i+1} = \sigma^i\sigma_1\sigma^{-i}, \quad i = 1, \dots, n-2. \tag{1.4}$$

The relations for the generators σ_1 and σ are the following

$$\begin{cases} \sigma_1\sigma^i\sigma_1\sigma^{-i} = \sigma^i\sigma_1\sigma^{-i}\sigma_1 & \text{for } \ 2 \le i \le n/2, \\ \sigma^n = (\sigma\sigma_1)^{n-1}. \end{cases} \tag{1.5}$$

The presentation (1.5) was given by Artin in the initial paper [2]. It was also mentioned in the books by F. Klein [48] and by H. S. M. Coxeter and W. O. J. Moser [23].

V. Ya. Lin in [55] gives a slightly different form of this presentation. Let $\beta \in Br_n$ be defined by the formula

$$\beta = \sigma\sigma_1.$$

Then there is the presentation of the group Br_n with generators σ_1 and β and relations:

$$\begin{cases} \beta\sigma^{i-1}\beta = \sigma^i\beta\sigma^{-i-1}\beta\sigma^i & \text{for } 2 \le i \le n/2, \\ \sigma^n = \beta^{n-1}. \end{cases}$$

This presentation is called *special* in [55].

An interesting series of presentations was given by V. Sergiescu [72]. For every planar graph he constructed a presentation of the group Br_n, where n is the number of vertices of the graph, with generators corresponding to edges and relations reflecting the geometry of the graph. To each edge e of the graph he associates the braid β_e which is a clockwise half-twist along e (see Figure 1.1). Artin's classical presentation (1.1) in this context corresponds to the graph consisting of the interval from 1 to n with the natural numbers (from 1 to n) as vertices and with segments between them as edges.

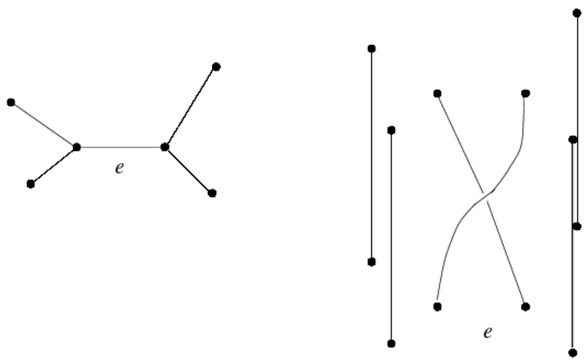


Fig. 1.1. Edges and geometric braids

To be precise, let Γ be a planar graph. We call it *normal* if Γ is connected, and it has no loops or intersections. Let $S(\Gamma)$ be the set of vertices of Γ. If Γ is not a tree then we define next what is a *pseudocycle* on it. The bounded part of the complement of Γ in the plane is the disjoint union of a finite number of open disks $D_1, \dots, D_m$, $m > 1$. The boundary of D_j on the plane is a subgraph $\Gamma(D_j)$ of Γ. We choose a point O in the interior of D_j, and an edge σ of $\Gamma(D_j)$ with vertices v_1, v_2. We suppose that the triangle Ov_1v_2 is oriented anticlockwise. We denote σ by $\sigma(e_1)$. We define the *pseudocycle associated to* D_j to be the sequence of edges $\sigma(e_1)\dots\sigma(e_p)$ such that:

— if the vertex v_{j+1} is not univalent, then $\sigma(e_{j+1})$ is the first edge on the left of $\sigma(e_j)$ (we consider $\sigma(e_j)$ going from v_j to v_{j+1}) and the vertex v_{j+2} is the other vertex adjacent to $\sigma(e_{j+1})$;
— if the vertex v_{j+1} is univalent, then $\sigma(e_{j+1}) = \sigma(e_j)$ and $v_{j+2} = v_j$;
— the vertex v_{p+1} is the vertex v_1.

Let $\gamma = \sigma(e_1)\dots\sigma(e_p)$ be a pseudocycle of Γ. Let $i = 1,\dots,p$. If $\sigma(e_i) = \sigma(e_j)$ for some $j \neq i$, then we say that

- $\sigma(e_i)$ is the *start edge* of a reverse if $j = i+1$ (we set $e_{p+1} = e_1$),
- $\sigma(e_i)$ is the *end edge* of a reverse if $j = i-1$ (we set $e_0 = e_p$).

In the following we set $\sigma_1\dots\sigma_p$ for the pseudocycle $\sigma(e_1)\dots\sigma(e_p)$.

THEOREM 1.1 (V. Sergiescu [72]). *Let Γ be a normal planar graph with n vertices. The braid group Br_n admits a presentation $\langle X_\Gamma \,|\, R_\Gamma\rangle$, where $X_\Gamma = \{\sigma \,|\, \sigma$ is an edge of $\Gamma\}$ and R_Γ is the set of following relations*:

- *Disjointedness relations* (*DR*): *if σ_i and σ_j are disjoint, then $\sigma_i\sigma_j = \sigma_j\sigma_i$.*
- *Adjacency relations* (*AR*): *if σ_i, σ_j have a common vertex, then $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$.*
- *Nodal relations* (*NR*): *if $\{\sigma_1, \sigma_2, \sigma_3\}$ have only one common vertex and they are clockwise oriented* (*Figure* 1.2), *then*

$$\sigma_1\sigma_2\sigma_3\sigma_1 = \sigma_2\sigma_3\sigma_1\sigma_2.$$

- *Pseudocycle relations* (*PR*): *if $\sigma_1\dots\sigma_m$ is a pseudocycle and σ_1 is not the start edge or σ_m the end edge of a reverse* (*Figure* 1.3), *then*

$$\sigma_1\sigma_2\dots\sigma_{m-1} = \sigma_2\sigma_3\dots\sigma_m.$$

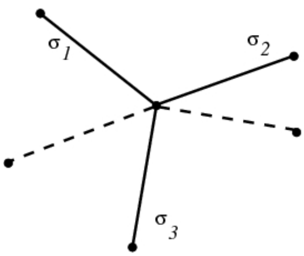


Fig. 1.2. Nodal relation

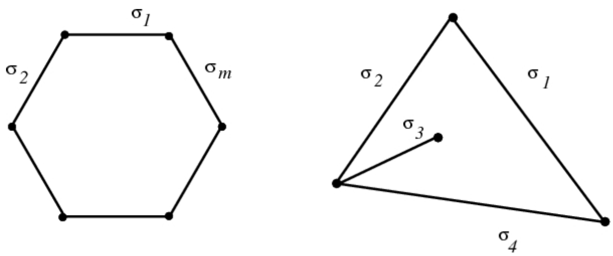


Fig. 1.3. Pseudocycle relation; on the left $\sigma_1\sigma_2\dots\sigma_{m-1} = \sigma_2\dots\sigma_m = \dots = \sigma_m\dots\sigma_{m-2}$. On the right $\sigma_1\sigma_2\sigma_3^2 = \sigma_2\sigma_3^2\sigma_4 = \sigma_3^2\sigma_4\sigma_1$ and $\sigma_3\sigma_4\sigma_1\sigma_2 = \sigma_4\sigma_1\sigma_2\sigma_3$

REMARK 1.1. Theorem 1.1 is true for infinite graphs. Let Γ be the direct limit of its finite subgraphs Γ_i, then the braid group Br_Γ is the direct limit of the subgroups Br_{Γ_i}.

The graph presentation of Sergiescu underlines the geometric character of braids, its connection with configuration spaces. In this survey we confirm this proposing a statement: for every generalization of braids of *geometric* character there exists a graph presentation.

Birman, Ko and Lee [14] introduced the presentation with the generators a_{ts} with $1 \le s < t \le n$ and relations

$$\begin{cases} a_{ts}a_{rq} = a_{rq}a_{ts} & \text{for } (t-r)(t-q)(s-r)(s-q) > 0, \\ a_{ts}a_{sr} = a_{tr}a_{ts} = a_{sr}a_{tr} & \text{for } 1 \le r < s < t \le n. \end{cases}$$

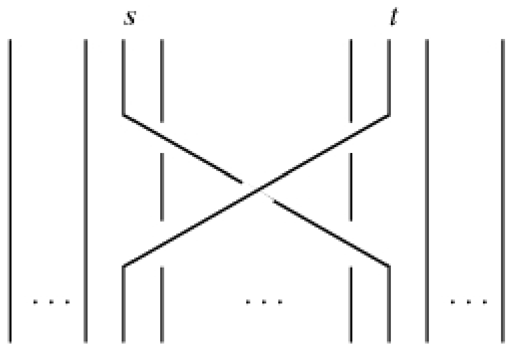


Fig. 1.4.

The generators a_{ts} are expressed by the canonical generators σ_i in the following form:

$$a_{ts} = (\sigma_{t-1}\sigma_{t-2}\cdots\sigma_{s+1})\sigma_s(\sigma_{s+1}^{-1}\cdots\sigma_{t-2}^{-1}\sigma_{t-1}^{-1}) \quad \text{for} \quad 1 \leq s < t \leq n.$$

Geometrically the generators $a_{s,t}$ are depicted in Figure 1.4. These generators are very natural and for this presentation Birman, Ko and Lee proposed an algorithm which solves the word problem with the speed $\mathcal{O}(m^2n)$ while Garside algorithm [37] improved by W. Thurston has a speed $\mathcal{O}(m^2n\log n)$, where m is the length of a word and n is the number of strands (see [30], Corollary 9.5.3). The question of generalization of this presentation for other types of braids was raised in [14].

In Section 2 we describe generalizations of braids that will be involved. In Section 3 we give the presentations with few generators, in Section 4 we study graph-presentations in the sense of V. Sergiescu and in Section 5 we give the Birman–Ko–Lee presentation for the singular braid monoid. In Section 6 we describe the work of V. V. Chaynikov [20] on the word and conjugacy problems for the singular braid monoid in Birman–Ko–Lee generators. In Sections 7–9 we study inverse monoids of partial braids.

The author is thankful to the organizers of Knots in Poland III, Józef Przytycki and Paweł Traczyk for the excellent conference.

2. Generalizations of braids. It is interesting to obtain the analogues of the presentations mentioned in the Introduction for various generalizations of braids [3], [13], [16], [27], [35], [80].

2.1. Artin–Brieskorn braid groups. Let I be a set and $M = (m_{i,j})$ be a matrix, $m_{i,j} \in \mathbb{N}^+ \cup \{\infty\}$, $i, j \in I$, with the following conditions: $m_{i,i} = 1$ and $m_{i,j} > 1$ for $i \neq j$. J. Tits in [74] defines the *Coxeter group of type M* as a group with generators w_i, $i \in I$, and relations

$$(w_i w_j)^{m_{i,j}} = e, \quad i, j \in I.$$

The corresponding braid groups, which are called *Artin–Tits groups*, have the elements s_i, $i \in I$, as the generators and the following set of defining relations:

$$\operatorname{prod}(m_{i,j}; s_i, s_j) = \operatorname{prod}(m_{j,i}; s_j, s_i),$$

where $\operatorname{prod}(m; x, y)$ denotes the product $xyxy\ldots$ (m factors).

Classification of irreducible finite Coxeter groups is well known (see for example Theorem 1, Chapter VI, §4 of [15]). It consists of the three infinite series: A, B and D as well as the exceptional groups E_6, E_7, E_8, F_4, G_2, H_3, H_4 and $I_2(p)$.

Let N be a finite set of cardinality n, say $N = \{v_1, \dots, v_n\}$. Let us equip elements of N with the signs, i.e. let $SN = \{\delta_1 v_1, \dots, \delta_n v_n\}$, where $\delta_i = \pm 1$. The Coxeter group $W(B_n)$ of type B can be interpreted as a group of signed permutations of the set SN:

$$W(B_n) = \{\sigma\text{—bijection of } SN : (-x)\sigma = -(x)\sigma \text{ for } x \in SN\}. \tag{2.1}$$

The *generalized braid group* (or *Artin–Brieskorn group*) $Br(W)$ *of* W [16], [27] corresponds to the case of finite Coxeter group W. The classical braids on k strings Br_k are obtained by this construction if W is the symmetric group on k symbols. In this case $m_{i,i+1} = 3$, and $m_{i,j} = 2$ if $j \neq i, i+1$.

The braid group of type B_n has the canonical presentation with generators σ_i, $i = 1, \dots, n-1$, and τ, and relations:

$$\begin{cases} \sigma_i\sigma_j = \sigma_j\sigma_i, & \text{if } \ |i-j| > 1, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, & \\ \tau\sigma_i = \sigma_i\tau, & \text{if } \ i \geq 2, \\ \tau\sigma_1\tau\sigma_1 = \sigma_1\tau\sigma_1\tau. & \end{cases} \tag{2.2}$$

This group can be identified with the fundamental group of the configuration space of distinct points on the plane with one point deleted [52], [76], what is the same as the braid group on n strands on the annulus, $Br_n(Ann)$. A geometric interpretation of generators $\tau, \sigma_1, \dots, \sigma_{n-1}$ is given in Figure 2.1.

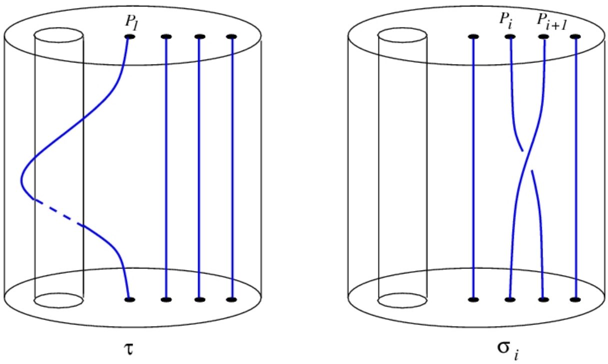


Fig. 2.1. Geometric interpretation of generators $\tau, \sigma_1, \dots, \sigma_{n-1}$ of $Br_n(Ann)$

The braid groups of the type D_n has the canonical presentation with generators σ_i and ρ, and relations:

$$\begin{cases} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{if } \ |i-j| > 1, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, & \\ \rho\sigma_i = \sigma_i\rho & \text{if } \ i = 1, 3, \dots, n-1, \\ \rho\sigma_2\rho = \sigma_2\rho\sigma_2. & \end{cases} \tag{2.3}$$

Let V be a complex finite-dimensional vector space. A *pseudo-reflection* of $GL(V)$ is a non-trivial element s of $GL(V)$ which acts trivially on a hyperplane, called the reflecting hyperplane of s. Suppose that W is a finite subgroup of $GL(V)$ generated by pseudo-reflections; the corresponding braid groups were studied by M. Broué, G. Malle and R. Rouquier [18] and also by D. Bessis and J. Michel [12]. As in the classical case these groups can be defined as fundamental groups of complement in V of the reflecting hyperplanes. The following classical conjecture generalizes the case of braid groups:

The universal cover of complement in V of the reflecting hyperplane is contractible.

(See for example the book by Orlik and Terao [63], p. 163 and p. 259.)

This conjecture was proved by David Bessis [11]. It means that these groups have naturally defined finite-dimensional manifold as $K(\pi, 1)$-spaces.

2.2. Braid groups on surfaces. Let Σ be a surface. The nth braid group of Σ can be defined as the fundamental group of configuration space of n points on Σ. Let S^2 be a sphere. The corresponding braid group $Br_n(S^2)$ has simple geometric interpretation as a group of isotopy classes of braids lying in a layer between two concentric spheres. It has the presentation with generators δ_i, $i = 1, \dots, n-1$, which satisfy the braid relations (1.1) and the following sphere relation:

$$\delta_1\delta_2 \dots \delta_{n-2}\delta_{n-1}^2\delta_{n-2} \dots \delta_2\delta_1 = 1. \tag{2.4}$$

This presentation was found by O. Zariski [88] in 1936 and then rediscovered by E. Fadell and J. Van Buskirk [32] in 1961.

Presentations of braid groups on all closed surfaces were obtained by G. P. Scott [71] and others.

2.3. Braid-permutation group. Let BP_n be the subgroup of $\operatorname{Aut} F_n$, generated by both sets of the automorphisms σ_i of (1.2) and ξ_i of the following form:

$$\begin{cases} x_i \mapsto x_{i+1}, \\ x_{i+1} \mapsto x_i, \\ x_j \mapsto x_j, \qquad j \neq i, i+1, \end{cases} \tag{2.5}$$

This is the nth *braid-permutation group* introduced by R. Fenn, R. Rimányi and C. Rourke [35] who gave a presentation of this group: it consists of the set of generators: $\{\xi_i, \sigma_i : i = 1, 2, \dots, n-1\}$ such that σ_i satisfy the braid relations, ξ_i satisfy the symmetric group relations and both of them satisfy the following mixed relations:

$$\begin{cases} \sigma_i\xi_j = \xi_j\sigma_i, & \text{if } |i-j| > 1, \\ \xi_i\xi_{i+1}\sigma_i = \sigma_{i+1}\xi_i\xi_{i+1}, \\ \sigma_i\sigma_{i+1}\xi_i = \xi_{i+1}\sigma_i\sigma_{i+1}. \end{cases} \tag{2.6}$$

R. Fenn, R. Rimányi and C. Rourke gave a geometric interpretation of BP_n as a group of *welded braids.*

This group was also studied by A. G. Savushkina [70] under the name of *group of conjugating automorphisms* and notation $\mathbf{C}_n$.

Braid-permutation group has an interesting geometric interpretation as a motion group. This group was introduced in the Ph.D. thesis of David Dahm, a student of

Ralph Fox. It appeared in literature in the paper of Deborah Goldsmith [41] and then has been studied by various authors, see [44], for instance. This is an analogue of the interpretation of the classical braid group as a mapping class group of a punctured disc. Instead of n points in a disc we consider n unlinked unknotted circles in a 3-ball. The fundamental group of the complement of n circles is also the free group F_n. Interchanging two neighbour points in the case of the braid group corresponds to an automorphism (1.2) of the free group. In the case of circles this automorphism corresponds to a motion of two neighbour circles when one of the circles is passing inside the other. Simple interchange of two neighbour circles corresponds to the automorphism (2.5).

Another motivation for studying braid-permutation groups is given by the *pure braid-permutation group* $P\Sigma_n$, the kernel of the canonical epimorphism $BP_n \to \Sigma_n$. In the context of the motion group it is called the *group of loops*, but it has even a longer history and is connected with classical works of J. Nielsen [62] and W. Magnus [56] (see also [57]), as follows. Let us denote the kernel of the natural map

$$\operatorname{Aut} F_n \to GL(n, \mathbb{Z})$$

by IA_n. These groups are similar to the Torelli subgroups of the mapping class groups. Nielsen and Magnus gave automorphisms which generate IA_n as a group. These automorphisms are named as follows:

- $\chi_{k,i}$ for $i \neq k$ with $1 \leq i, k \leq n$, and
- $\theta(k; [s,t])$ for k, s, t distinct integers with $1 \leq k, s, t \leq n$ and $s < t$.

The definition of the map $\chi_{k,i}$ is given by the formula

$$\chi_{k,i}(x_j) = \begin{cases} x_j & \text{if } k \neq j, \\ (x_i^{-1})(x_k)(x_i) & \text{if } k = j. \end{cases}$$

The map $\theta(k; [s,t])$ is defined by the formula

$$\theta(k; [s,t])(x_j) = \begin{cases} x_j & \text{if } k \neq j, \\ (x_k) \cdot ([x_s, x_t]) & \text{if } k = j. \end{cases}$$

for which the commutator is given by $[a,b] = a^{-1} \cdot b^{-1} \cdot a \cdot b$.

The group IA_2 is isomorphic to the group of inner automorphisms $\operatorname{Inn}(F_2)$, which is isomorphic to the free group F_2. The group IA_3 is not finitely presented [51].

Consider the subgroup of IA_n generated by the $\chi_{k,i}$, the *group of basis conjugating automorphisms* of a free group. This is exactly $P\Sigma_n$. McCool gave a presentation for it [59].

The cohomology of $P\Sigma_n$ was computed by C. Jensen, J. McCammond, and J. Meier in [44]. N. Kawazumi [47], T. Sakasai [68], T. Satoh [69] and A. Pettet [66] have given related cohomological information for IA_n. The integral cohomology of the natural direct limit of the groups $\operatorname{Aut} F_n$ is given in work of S. Galatius [36].

THEOREM 2.1 (A. G. Savushkina [70]). *The group* BP_n *is the semi-direct product of the symmetric group on n-letters* Σ_n *and the group* $P\Sigma_n$ *with a split extension*

$$1 \longrightarrow P\Sigma_n \longrightarrow \mathrm{BP}_n \longrightarrow \Sigma_n \longrightarrow 1.$$

The Lie algebra structure obtained from the descending central series of the group $P\Sigma_n$ was studied by F. R. Cohen, J. Pakianathan, V. V. Vershinin and J. Wu [21] and by B. Berceanu and S. Papadima [9]. Certain subgroups of $P\Sigma_n$ were studied by V. Bardakov and R. Mikhailov [6].

2.4. Singular braid monoid. The set of singular braids on n strands, up to isotopy, forms a monoid. This is the *singular braid monoid* or *Baez–Birman monoid* SB_n [3], [13]. It can be presented as the monoid with generators g_i, g_i^{-1}, x_i, $i = 1, \dots, n-1$, and relations

$$\begin{aligned}
\sigma_i\sigma_j &= \sigma_j\sigma_i, \quad \text{if } |i-j|>1,\\
x_ix_j &= x_jx_i, \quad \text{if } |i-j|>1,\\
x_i\sigma_j &= \sigma_jx_i, \quad \text{if } |i-j|\neq 1,\\
\sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1},\\
\sigma_i\sigma_{i+1}x_i &= x_{i+1}\sigma_i\sigma_{i+1},\\
\sigma_{i+1}\sigma_ix_{i+1} &= x_i\sigma_{i+1}\sigma_i,\\
\sigma_i\sigma_i^{-1} &= \sigma_i^{-1}\sigma_i = 1.
\end{aligned}$$

In pictures σ_i corresponds to canonical generator of the braid group and x_i represents an intersection of the ith and $(i+1)$th strand as in Figure 2.2. The singular braid monoid on two strings is isomorphic to $\mathbb{Z}\oplus\mathbb{Z}^+$. This monoid embeds in a group SG_n [34] which

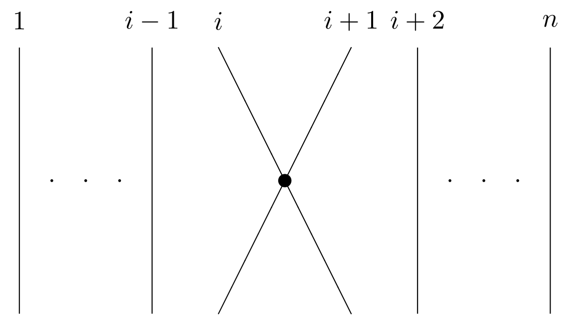


Fig. 2.2.

is called the *singular braid group*:

$$SB_n \to SG_n.$$

So, in SG_n the elements x_i become invertible and all relations of SB_n remain true.

Principal motivations for study of the singular braid monoid lie in the Vassiliev theory of finite type invariants [75]. Essential step in this theory is that a link invariant is extended from usual links to singular ones. Singular links and singular braids are connected via singular versions of Alexander theorem proved by Birman [13] and Markov theorem proved by B. Gemein [38], so that as well as in the classical case a singular link is an equivalence class (by conjugation and stabilization) of singular braids. Therefore the study of singular braid monoid, especially such questions as conjugation problem, is interesting not only because of its general importance in Algebra but because of the connections with Knot Theory.

2.5. Other generalizations of braids that are not considered in the paper. Garside's solution of the word and conjugacy problems for braids had a great influence for the subsequent research on braids. Tools developed by Garside were put as the definitions for Gaussian and Garside groups [26], [24] or even Garside groupoids [50]. The latter notion is connected also with the mapping class groups.

Another direction of generalizations are the parenthesized braids [43], [17], [25]. Motivations for these studies are in D. Bar-Natan's works on noncommutative tangles [4], [5] and, on the other hand, in connections with Thompson's group [19].

3. Presentations of generalizations of braids with few generators. The presentation with two generators gives an economic way (from the point of view of generators) to have a vision of the braid group. We give here the extension of this presentation for the natural generalizations of braids. The results of this section were obtained in [83].

3.1. Artin–Brieskorn groups and complex reflection groups. For the braid groups of type B_n from the canonical presentation (2.2) we obtain the presentation with three generators σ_1, σ and τ and the following relations:

$$\begin{cases} \sigma_1\sigma^i\sigma_1\sigma^{-i} = \sigma^i\sigma_1\sigma^{-i}\sigma_1 & \text{for } \ 2 \le i \le n/2, \\ \sigma^n = (\sigma\sigma_1)^{n-1}, & \\ \tau\sigma^i\sigma_1\sigma^{-i} = \sigma^i\sigma_1\sigma^{-i}\tau & \text{for } \ 2 \le i \le n-2, \\ \tau\sigma_1\tau\sigma_1 = \sigma_1\tau\sigma_1\tau. & \end{cases} \tag{3.1}$$

If we add the relations

$$\begin{cases} \sigma_1^2 = 1, \\ \tau^2 = 1 \end{cases}$$

to (3.1), we arrive at a presentation of the Coxeter group of type B_n.

Similarly, for the braid groups of the type D_n from the canonical presentation (2.3) we can obtain the presentation with three generators σ_1, σ and ρ and the relations:

$$\begin{cases} \sigma_1\sigma^i\sigma_1\sigma^{-i} = \sigma^i\sigma_1\sigma^{-i}\sigma_1 & \text{for } \ 2 \le i \le n/2, \\ \sigma^n = (\sigma\sigma_1)^{n-1}, & \\ \rho\sigma^i\sigma_1\sigma^{-i} = \sigma^i\sigma_1\sigma^{-i}\rho & \text{for } \ i = 0, 2, \dots, n-2, \\ \rho\sigma\sigma_1\sigma^{-1}\rho = \sigma\sigma_1\sigma^{-1}\rho\sigma\sigma_1\sigma^{-1}. & \end{cases} \tag{3.2}$$

If we add the relations

$$\begin{cases} \sigma_1^2 = 1, \\ \rho^2 = 1 \end{cases}$$

to (3.2) we come to a presentation of the Coxeter group of type D_n.

For the exceptional braid groups of types E_6–E_8 our presentations look similar to the presentation for the groups of type D (3.2). We give it here for E_8: it has three generators σ_1, σ and ω and the following relations:

$$\begin{cases} \sigma_1\sigma^i\sigma_1\sigma^{-i} = \sigma^i\sigma_1\sigma^{-i}\sigma_1 \quad \text{for} \quad i = 2,3,4, \\ \sigma^8 = (\sigma\sigma_1)^7, \\ \omega\sigma^i\sigma_1\sigma^{-i} = \sigma^i\sigma_1\sigma^{-i}\omega \quad \text{for} \quad i = 0,1,3,4,5,6, \\ \omega\sigma^2\sigma_1\sigma^{-2}\omega = \sigma^2\sigma_1\sigma^{-2}\omega\sigma^2\sigma_1\sigma^{-2}. \end{cases} \tag{3.3}$$

Similarly, if we add the relations

$$\begin{cases} \sigma_1^2 = 1, \\ \omega^2 = 1 \end{cases}$$

to (3.3) we arrive at a presentation of the Coxeter group of type E_8.

As for the other exceptional braid groups, F_4 has four generators and it follows from its Coxeter diagram that there is no sense to speak about analogues of the Artin presentation (1.5), G_2 and $I_2(p)$ already have two generators and H_3 has three generators. For H_4 it is possible to diminish the number of generators from four to three and the presentation will be similar to that of B_4.

We can summarize informally what we were doing. Let a group have a presentation which can be expressed by a "Coxeter-like" graph. If there exists a linear subgraph corresponding to the standard presentation of the classical braid group, then in the "braid-like" presentation of our group the part that corresponds to the linear subgraph can be replaced by two generators and relations (1.5). This recipe can be applied to the complex reflection groups [73] whose "Coxeter-like" presentations is obtained in [18], [12]. For the series of the complex braid groups $B(2e,e,r)$, $e \geq 2$, $r \geq 2$, which correspond to the complex reflection groups $G(de,e,r)$, $d \geq 2$ [18], we take the linear subgraph with nodes $\tau_2,\dots,\tau_r$, and put as above $\tau = \tau_2\dots\tau_r$. The group $B(2e,e,r)$ have presentation with generators $\tau_2, \tau, \sigma, \tau_2'$ and relations

$$\begin{cases} \tau_2\tau^i\tau_2\tau^{-i} = \tau^i\tau_2\tau^{-i}\tau_2 \quad \text{for} \quad 2 \leq i \leq r/2, \\ \tau^r = (\tau\tau_2)^{r-1}, \\ \sigma\tau^i\tau_2\tau^{-i} = \tau^i\tau_2\tau^{-i}\sigma \quad \text{for} \quad 1 \leq i \leq r-2, \\ \sigma\tau_2'\tau_2 = \tau_2'\tau_2\sigma, \\ \tau_2'\tau\tau_2\tau^{-1}\tau_2' = \tau\tau_2\tau^{-1}\tau_2'\tau\tau_2\tau^{-1}, \\ \tau\tau_2\tau^{-1}\tau_2'\tau_2\tau\tau_2\tau^{-1}\tau_2'\tau_2 = \tau_2'\tau_2\tau\tau_2\tau^{-1}\tau_2'\tau_2\tau\tau_2\tau^{-1}, \\ \underbrace{\tau_2\sigma\tau_2'\tau_2\tau_2'\tau_2\tau_2'\dots}_{e+1 \text{ factors}} = \underbrace{\sigma\tau_2'\tau_2\tau_2'\tau_2\tau_2'\tau_2\dots}_{e+1 \text{ factors}}. \end{cases} \tag{3.4}$$

If we add the relations

$$\begin{cases} \sigma^d = 1, \\ \tau_2^2 = 1, \\ \tau_2'^2 = 1 \end{cases}$$

to (3.4) we come to a presentation of the complex reflection group $G(de,e,r)$.

The braid group $B(d,1,n)$, $d>1$, has the same presentation as the Artin–Brieskorn group of type B_n, but if we add the relations

$$\begin{cases} \sigma_1^2 = 1, \\ \tau^d = 1 \end{cases}$$

to (3.1) then we arrive at a presentation of the complex reflection group $G(d,1,n)$, $d \geq 2$.

For the series of braid groups $B(e,e,r)$, $e \geq 2$, $r \geq 3$, which correspond to the complex reflection groups $G(e,e,r)$, $e \geq 2$, $r \geq 3$, we take again the linear subgraph with the nodes $\tau_2, \dots, \tau_r$, and put as above $\tau = \tau_2 \dots \tau_r$. The group $B(e,e,r)$ may have the presentation with generators τ_2, τ, τ_2' and relations

$$\text{(3.5)} \qquad \begin{cases} \tau_2\tau^i\tau_2\tau^{-i} = \tau^i\tau_2\tau^{-i}\tau_2 \quad \text{for} \quad 2 \leq i \leq r/2, \\ \tau^r = (\tau\tau_2)^{r-1}, \\ \tau_2'\tau\tau_2\tau^{-1}\tau_2' = \tau\tau_2\tau^{-1}\tau_2'\tau\tau_2\tau^{-1}, \\ \tau\tau_2\tau^{-1}\tau_2'\tau_2\tau\tau_2\tau^{-1}\tau_2'\tau_2 = \tau_2'\tau_2\tau\tau_2\tau^{-1}\tau_2'\tau_2\tau\tau_2\tau^{-1}, \\ \underbrace{\tau_2\tau_2'\tau_2\tau_2'\tau_2\tau_2'\dots}_{e \text{ factors}} = \underbrace{\tau_2'\tau_2\tau_2'\tau_2\tau_2'\tau_2\dots}_{e \text{ factors}}. \end{cases}$$

If $e = 2$ then this is precisely the presentation for the Artin–Brieskorn group of type D_r (3.2). If we add the relations

$$\begin{cases} \tau_2^2 = 1, \\ \tau_2'^2 = 1 \end{cases}$$

to (3.5), then we obtain a presentation of the complex reflection group $G(e,e,r)$, $e \geq 2$, $r \geq 3$.

As for the exceptional (complex) braid groups, it is reasonable to consider the groups $Br(G_{30})$, $Br(G_{33})$ and $Br(G_{34})$ which correspond to the complex reflection groups G_{30}, G_{33} and G_{34}.

The presentation for $Br(G_{30})$ is similar to the presentation (3.1) of $Br(B_4)$ with the last relation replaced by the relation of length 5: the three generators σ_1, σ and τ and the following relations:

$$\text{(3.6)} \qquad \begin{cases} \sigma_1\sigma^2\sigma_1\sigma^{-2} = \sigma^2\sigma_1\sigma^{-2}\sigma_1, \\ \sigma^4 = (\sigma\sigma_1)^3, \\ \tau\sigma^i\sigma_1\sigma^{-i} = \sigma^i\sigma_1\sigma^{-i}\tau \qquad \text{for} \quad i = 2,3, \\ \tau\sigma_1\tau\sigma_1\tau = \sigma_1\tau\sigma_1\tau\sigma_1. \end{cases}$$

If we add the relations

$$\begin{cases} \sigma_1^2 = 1, \\ \tau^2 = 1 \end{cases}$$

to (3.6), then we obtain a presentation of complex reflection group G_{30}.

As for the groups $Br(G_{33})$ and $Br(G_{34})$, we give here the presentation for the latter one because the "Coxeter-like" graph for $Br(G_{33})$ has one node less in the linear subgraph (discussed earlier) than that of $Br(G_{34})$. This presentation has the three generators s, z

($z = stuvx$ in the reflection generators) and w and the following relations:

$$\begin{cases} sz^isz^{-i} = z^isz^{-i}s & \text{for} \quad i = 2, 3, \\ z^6 = (zs)^5, \\ wz^isw^{-i} = z^isz^{-i}w & \text{for} \quad i = 0, 3, 4, \\ wz^isz^{-i}w = z^isz^{-i}wz^isz^{-i} & \text{for} \quad i = 1, 2, \\ wz^2sz^{-2}wzsz^{-1}wz^2sz^{-2} = zsz^{-1}wz^2sz^{-2}wzsz^{-1}w. \end{cases} \tag{3.7}$$

In the same way if we add the relations

$$\begin{cases} s^2 = 1, \\ w^2 = 1 \end{cases}$$

to (3.7), then we come to a presentation of the complex reflection group G_{34}.

We can obtain presentations with few generators for the other complex reflection groups using the already observed presentations of the braid groups. For G_{25} and G_{32} we can use the presentations (1.5) for the classical braid groups Br_4 and Br_5 with the only additional relation

$$\sigma_1^3 = 1.$$

3.2. Sphere braid groups: few generators. The presentation has two generators δ_1, δ which satisfy relations (1.5) (where σ_1 is replaced by δ_1, and σ is replaced by δ) and the following sphere relation:

$$\delta^n(\delta_1\delta^{-1})^{n-1} = 1.$$

3.3. Braid-permutation groups. For the case of the braid-permutation group BP_n we add the new generator σ, defined by (1.3) to the set of standard generators of BP_n; then relations (1.4) and the following relations hold

$$\xi_{i+1} = \sigma^i\xi_1\sigma^{-i}, \quad i = 1, \dots, n-2.$$

This gives a possibility to get rid of ξ_i as well as of σ_i for $i \geq 2$.

THEOREM 3.1. *The braid-permutation group BP_n has a presentation with generators σ_1, σ, and ξ_1 and relations*

$$\begin{cases} \sigma_1\sigma^i\sigma_1\sigma^{-i} = \sigma^i\sigma_1\sigma^{-i}\sigma_1 & \textit{for} \quad 2 \leq i \leq n/2, \\ \sigma^n = (\sigma\sigma_1)^{n-1}, \\ \xi_1\sigma^i\sigma_1\sigma^{-i} = \sigma^i\sigma_1\sigma^{-i}\xi_1 & \textit{for} \quad i = 2, \dots, n-2, \\ \xi_1\sigma^i\xi_1\sigma^{-i} = \sigma^i\xi_1\sigma^{-i}\xi_1 & \textit{for} \quad i = 2, \dots, n-2, \\ \xi_1\sigma\xi_1\sigma^{-1}\sigma_1 = \sigma\sigma_1\sigma^{-1}\xi_1\sigma\xi_1\sigma^{-1}, \\ \xi_1\sigma\xi_1\sigma^{-1}\xi_1 = \sigma\xi_1\sigma^{-1}\xi_1\sigma\xi_1\sigma^{-1}, \\ \xi^2 = 1. \end{cases}$$

3.4. Few generators for the singular braid monoid. If we add the new generator σ, defined by (1.3) to the set of generators of SB_n then the following relations hold

$$x_{i+1} = \sigma^ix_1\sigma^{-i}, \quad i = 1, \dots, n-2. \tag{3.8}$$

This gives a possibility to get rid of x_i, $i \geq 2$.

THEOREM 3.2. *The singular braid monoid SB_n has a presentation with generators σ_1, σ_1^{-1}, σ, σ^{-1} and x_1 and relations*

$$\begin{cases} \sigma_1\sigma^i\sigma_1\sigma^{-i} = \sigma^i\sigma_1\sigma^{-i}\sigma_1 \quad \textit{for} \quad 2 \leq i \leq n/2, \\ \sigma^n = (\sigma\sigma_1)^{n-1}, \\ x_1\sigma^i\sigma_1\sigma^{-i} = \sigma^i\sigma_1\sigma^{-i}x_1 \quad \textit{for} \quad i = 0, 2, \dots, n-2, \\ x_1\sigma^i x_1\sigma^{-i} = \sigma^i x_1\sigma^{-i}x_1 \quad \textit{for} \quad 2 \leq i \leq n/2, \\ \sigma^n x_1 = x_1\sigma^n, \\ x_1\sigma\sigma_1\sigma^{-1}\sigma_1 = \sigma\sigma_1\sigma^{-1}\sigma_1\sigma x_1\sigma^{-1}, \\ \sigma_1\sigma_1^{-1} = \sigma_1^{-1}\sigma_1 = 1, \\ \sigma\sigma^{-1} = \sigma^{-1}\sigma = 1. \end{cases} \tag{3.9}$$

4. Graph-presentations

4.1. Braid groups of type B via graphs. Graph presentations for the braid groups of the type B and for the singular braid monoid were studied by the author. We recall that the group $Br_n(Ann)$ embeds in the braid group Br_{n+1} as the subgroup of braids with the first strand fixed.

In the following we consider a normal planar graph Γ such that there exists a distinguished vertex v and such that the graph Γ minus the vertex v and all the edges adjacent to v is connected also. We call such Γ a 1-*punctured* graph.

THEOREM 4.1. *Let Γ be a 1-punctured graph with $n+1$ vertices. The braid group $Br_n(Ann)$ admits the presentation $\langle X_\Gamma \,|\, R_\Gamma\rangle$, where $X_\Gamma = \{\sigma_a, \tau_b \,|\, a$ is an edge of Γ not adjacent to the distinguished vertex v and b is an edge adjacent to $v\}$ and R_Γ is the following set of relations*:

- *Disjointedness relations* (*DR*): *if the edges a and c* (*respectively b and c*) *are disjoint, then $\sigma_a\sigma_c = \sigma_c\sigma_a$* (*respectively $\tau_b\sigma_c = \sigma_c\tau_b$*).
- *Adjacency relations* (*AR*): *if the edges a and c* (*respectively b and c*) *have a common vertex, then $\sigma_a\sigma_c\sigma_a = \sigma_c\sigma_a\sigma_c$* ($\tau_b\sigma_c\tau_b\sigma_c = \sigma_c\tau_b\sigma_c\tau_b$).
- *Nodal relations* (*NR*): *Let a, b, c be three edges which have only one common vertex and are clockwise ordered. If the edges a, b, c are not adjacent to v, then*

  $$\sigma_a\sigma_b\sigma_c\sigma_a = \sigma_b\sigma_c\sigma_a\sigma_b;$$

  if the edges a, c are not adjacent to v and b is adjacent to v, then

  $$\sigma_a\sigma_b\tau_c\sigma_a = \sigma_b\tau_c\sigma_a\sigma_b,$$
  $$\tau_b\sigma_c\sigma_a\tau_b\sigma_c = \sigma_a\tau_b\sigma_c\sigma_a\tau_b.$$

- *Pseudocycle relations* (*PR*): *if the edges $a_1, \dots, a_m$ form a pseudocycle, a_1 is not the start edge or a_m the end edge of a reverse and all a_i are not adjacent to v, then*

  $$\sigma_{a_1}\sigma_{a_2}\dots\sigma_{a_{m-1}} = \sigma_{a_2}\sigma_{a_3}\dots\sigma_{a_m}.$$

  If a_1, a_m are adjacent to v, then

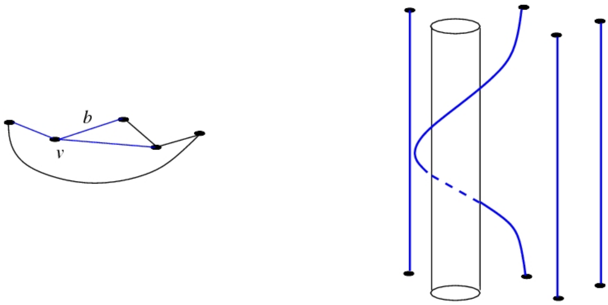
  $$\tau_{a_1}\sigma_{a_2}\dots\sigma_{a_{m-1}} = \sigma_{a_2}\sigma_{a_3}\dots\tau_{a_m}.$$

Remark 4.1. As in Theorem 1.1, the nodal relation (NR) implies also the equality

$$\sigma_a\sigma_b\sigma_c\sigma_a = \sigma_b\sigma_c\sigma_a\sigma_b = \sigma_c\sigma_a\sigma_b\sigma_c.$$

The geometric interpretation of generators is the following. The distinguished vertex corresponds to the deleted point of the plane. To any edge a that is not adjacent to v we associate the corresponding positive half-twist. To any edge b adjacent to v we associate the braid τ_b as in Figure 4.1.

Remark 4.2. This Theorem as well as Theorem 1.1 is true for infinite graphs via the direct limit arguments.



τ_b

Fig. 4.1. Geometric interpretation of τ_b

To prove the relation $\tau_b\sigma_c\sigma_a\tau_b\sigma_c = \sigma_a\tau_b\sigma_c\sigma_a\tau_b$ we add two edges d and e, with their corresponding braids τ_d and τ_e as in Figure 4.2. The braid τ_d is equivalent to the braid $\sigma_c^{-1}\tau_b\sigma_c$ and the braid τ_e is equivalent to the braid $\sigma_a\tau_b\sigma_a^{-1}$. Then the braids $\sigma_c^{-1}\tau_b\sigma_c$ and σ_a commute, as well as $\sigma_a\tau_b\sigma_a^{-1}$ and σ_c. So we have the following equalities, that can be easily verified on corresponding braids:

$$\begin{aligned}\tau_b\sigma_c\sigma_a\tau_b\sigma_c &= \sigma_c\sigma_c^{-1}\tau_b\sigma_c\sigma_a\tau_b\sigma_c = \sigma_c\sigma_a\sigma_c^{-1}\tau_b\sigma_c\tau_b\sigma_c \\ &= \sigma_c\sigma_a\sigma_c^{-1}\sigma_c\tau_b\sigma_c\tau_b = \sigma_c\sigma_a\tau_b\sigma_a^{-1}\sigma_a\sigma_c\tau_b = \sigma_a\tau_b\sigma_a^{-1}\sigma_c\sigma_a\sigma_c\tau_b = \sigma_a\tau_b\sigma_c\sigma_a\tau_b.\end{aligned}$$

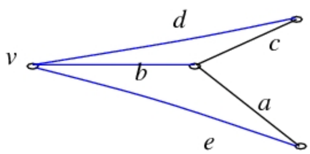


Fig. 4.2. Nodal relation $\tau_b\sigma_c\sigma_a\tau_b\sigma_c = \sigma_a\tau_b\sigma_c\sigma_a\tau_b$ holds in $Br_n(Ann)$

Corollary 4.1. *The automorphism group of $Br_n(Ann)$ contains a group isomorphic to the dihedral group D_{n-1}.*

One can associate to the graph given in Figure 4.3 a presentation for $Br_n(Ann)$.

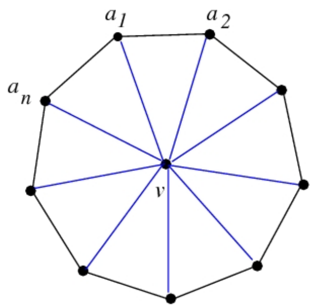


Fig. 4.3. A graph associated to $Br_n(Ann)$

It is possible to generalize such an approach to braid groups on a planar surface, i.e. a surface of genus 0 with $l > 1$ boundary components. In this case one considers a normal planar graph with $k\,(= l-1)$ distinguished vertices $v_1, \dots, v_k$ such that there are no edges connecting distinguished vertices and such that the graph Γ minus the vertices $v_1, \dots, v_k$ and all the edges adjacent to $v_1, \dots, v_k$ is also connected. We label by $\{\tau_{1,j}, \dots, \tau_{m,j}\}$ the edges adjacent to v_j and by $\{\sigma_1, \dots, \sigma_p\}$ the edges disjoint from the set $\{v_1, \dots, v_k\}$. We say that Γ is a k-punctured graph. As in Theorem 4.1 one can associate to any k-punctured graph Γ on n vertices a set of generators for the braid group on n strands on surface of genus 0 with $k+1$ boundary components, with the above geometrical interpretation of generators.

4.2. Graph-presentations for the surface braid groups. These presentations were considered in [8]. Let Γ be a normal graph on an orientable surface Σ and $S(\Gamma)$ denote the set of vertices of Γ. In the same way as earlier we associate to the edges of Γ the corresponding geometric braids on Σ (Figure 1.1) and we define $Br_\Gamma(\Sigma)$ as the subgroup of $Br_{|S(\Gamma)|}(\Sigma)$ generated by these braids.

Proposition 4.1. *Let Σ be an oriented surface such that $\pi_1(\Sigma) \neq 1$ and let Γ be a normal graph on Σ. Then $Br_\Gamma(\Sigma)$ is a proper subgroup of $Br_{|S(\Gamma)|}(\Sigma)$.*

4.3. Sphere braid groups presentations via graphs. Now let the surface Σ be a sphere S^2 and Γ denote a normal finite graph on this sphere. We define a pseudocycle as in Introduction: we consider the set $S^2 \setminus \Gamma$ as the disjoint union of a finite number of open disks $D_1, \dots, D_m$, $m > 1$, and define the *pseudocycle associated to D_j* exactly in the same way.

Let Δ be a maximal tree of a normal graph Γ on $q+1$ vertices. Then Δ has q edges. Let v_1, v_2 be two vertices adjacent to the same edge σ of Δ. Write $\sigma(f_1)$ for σ. We define the *circuit* $\sigma(f_1) \dots \sigma(f_{2q})$ as follows:

— if the vertex v_{j+1} is not univalent, then $\sigma(f_{j+1})$ is the first edge on the left of $\sigma(f_j)$ (we consider $\sigma(f_j)$ going from v_j to v_{j+1}) and the vertex v_{j+2} is the other vertex adjacent to $\sigma(f_{j+1})$;
— if the vertex v_{j+1} is univalent, then $\sigma(f_{j+1}) = \sigma(f_j)$ and $v_{j+2} = v_j$.

This way we come back to v_1 after passing twice through each edge of Δ. Write $\delta_{v_1,v_2}(\Delta)$ for the word in X_Γ corresponding to the circuit $\sigma(f_1)\dots\sigma(f_{2q})$ (Figure 4.4).

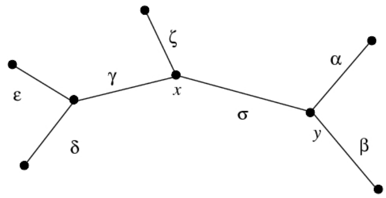


Fig. 4.4. $\delta_{x,y}(\Delta) = \sigma\alpha^2\beta^2\sigma\gamma\delta^2\epsilon^2\gamma\zeta^2$ and $\delta_{y,x}(\Delta) = \sigma\gamma\delta^2\epsilon^2\gamma\zeta^2\sigma\alpha^2\beta^2$

THEOREM 4.2. *Let* Γ *be a normal graph with* n *vertices. The braid group* $Br_n(S^2)$ *admits a presentation* $\langle X_\Gamma \,|\, R_\Gamma\rangle$*, where* $X_\Gamma = \{\sigma \,|\, \sigma \text{ is an edge of } \Gamma\}$ *and* R_Γ *is the set of following relations*: *disjointedness relations* (*DR*); *nodal relations* (*NR, Figure* 1.2); *pseudocycle relations* (*PR, Figure* 1.3), *exactly as in Theorem* 1.1 *and the new tree relations* (*TR*): $\delta_{x,\,y}(\Delta) = 1$*, for every maximal tree* Δ *of* Γ *and every ordered pair of vertices* x, y *such that they are adjacent to the same edge* σ *of* Δ*.*

REMARK 4.3. The statement of Theorem 4.2 is highly redundant. For instance one can show that a relation (TR) on a given maximal tree of Γ, together with the relations (DR), (AR), (NR) and (PR), generate the (TR) relation for any other maximal tree of Γ. Anyway, these presentations are symmetric and one can read off the relations from the geometry of Γ.

REMARK 4.4. Let $\gamma \subseteq \Gamma$ be a star (a graph which consists of several edges joined in one point). For any clockwise ordered subset $\{\sigma_{i_1}, \dots, \sigma_{i_j} \,|\, j \geq 2\}$ of edges of γ the following relation holds in the group $\langle X_\Gamma \,|\, R_\Gamma\rangle$:

$$\sigma_{i_1} \dots \sigma_{i_j}\sigma_{i_1} = \sigma_{i_j}\sigma_{i_1} \dots \sigma_{i_j}.$$

4.3.1. *Geometric interpretation of relations.* It is geometrically evident that the relations (AR) and (DR) hold in $Br_n(S^2)$. Let Γ contain a triangle σ_1, σ_2, τ as in Figure 4.5. Corresponding braids satisfy the relation $\tau = \sigma_1\sigma_2\sigma_1^{-1}$ and thus $\tau\sigma_1 = \sigma_1\sigma_2$ in

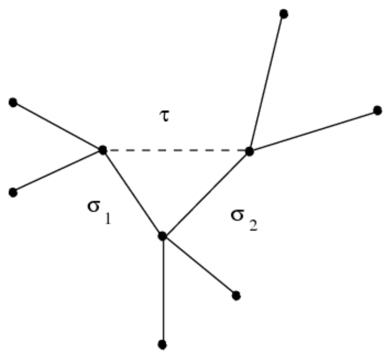


Fig. 4.5. Adding or removing a triangle

$Br_n(S^2)$. The relation $\sigma_1\sigma_2 = \sigma_2\tau$ follows from the braid relation $\sigma_1\sigma_2\sigma_1^{-1} = \sigma_2^{-1}\sigma_1\sigma_2$. Let $\sigma_1, \sigma_2, \sigma_3$ be arranged as in Figure 4.6. We add three edges τ_1, τ_2, τ_3. The nodal relation follows from the pseudocycle relations on triangles $\tau_1\sigma_2\sigma_3$, $\tau_2\sigma_1\sigma_3$ and $\tau_3\sigma_1\sigma_2$. In fact, $\sigma_1\sigma_2\sigma_3\sigma_1 = \sigma_2\tau_3\sigma_3\sigma_1 = \sigma_2\sigma_3\tau_3\sigma_1 = \sigma_2\sigma_3\sigma_1\sigma_2$. All other pseudocycle relations follow from induction on the length of the cycle.

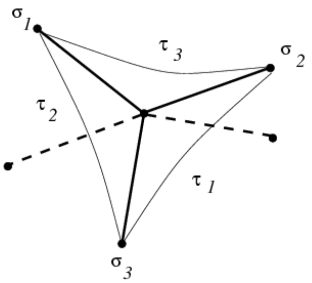


Fig. 4.6. Nodal relation holds in $Br_n(S^2)$

Let Δ be a maximal tree of Γ. Let σ be an edge of Δ and let x, y be the two adjacent vertices. The element $\delta_{x,y}(\Delta)$ corresponds to a (pure) braid such that the braid obtained by removing the string starting from the vertex x is isotopic to the trivial braid. This string goes around (with clockwise orientation) all other vertices (Figure 4.7 on the left). The braid $\delta_{x,y}(\Delta)$ is isotopic to the trivial braid in $Br_n(S^2)$ and so $\delta_{x,y}(\Delta) = 1$ (Figure 4.7). Therefore the natural map $\phi_\Gamma : \langle X_\Gamma \,|\, R_\Gamma \rangle \to Br_n(S^2)$ is a homomorphism.

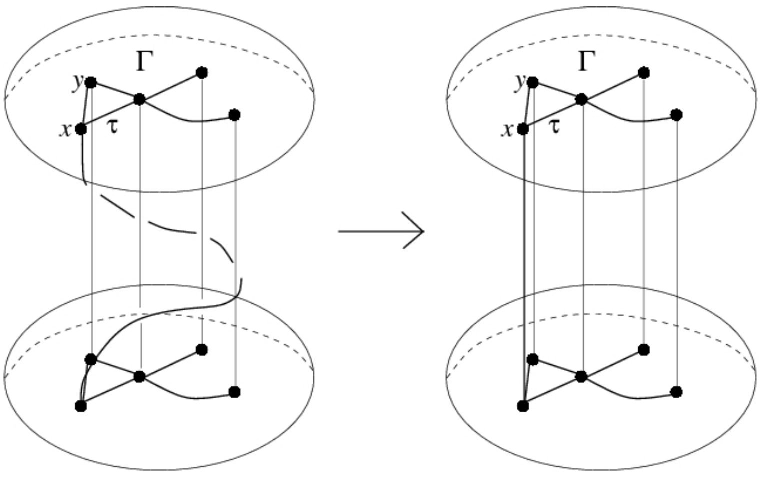


Fig. 4.7. The braid $\delta_{x,\sigma}(\Delta)$ associated to the tree $\Delta = \Gamma \setminus \tau$

4.4. Singular braids and graphs. As in the case of classical braids, one can extend the group $Br_n(\Sigma)$ to the monoid $SB_n(\Sigma)$ of singular braids on n strands on the surface Σ. Presentations for this monoid are given in [7] and [42].

In this section we provide presentations by graphs for the monoid SB_n and for the monoid $SB_n(Ann)$ of singular braids on n strands of the annulus.

Let Γ be a normal planar graph. We associate to any edge a three singular braids: σ_a will denote the positive half-twist associated to a (as in Figure 1.1), σ_a^{-1} will denote the corresponding negative half-twist and x_a the corresponding singular crossing.

THEOREM 4.3. *Let Γ be a normal planar graph with n vertices. The singular braid monoid SB_n has the presentation $\langle X_\Gamma, R_\Gamma\rangle$ where $X_\Gamma = \{\sigma_a, \sigma_a^{-1}, x_a \mid a$ is an edge of $\Gamma\}$ and R_Γ is formed by the following six types of relations*:

- *disjointedness*: *if the edges a and b are disjoint, then*

$$\sigma_a\sigma_b = \sigma_b\sigma_a, \quad x_a x_b = x_b x_a, \quad \sigma_a x_b = x_b\sigma_a,$$

- *commutativity*:

$$\sigma_a x_a = x_a\sigma_a,$$

- *invertibility*:

$$\sigma_a\sigma_a^{-1} = \sigma_a^{-1}\sigma_a = 1,$$

- *adjacency*: *if the edges a and b have a common vertex, then*

$$\sigma_a\sigma_b\sigma_a = \sigma_b\sigma_a\sigma_b,$$
$$x_a\sigma_b\sigma_a = \sigma_b\sigma_a x_b,$$

- *nodal*: *if the edges a, b and c have a common vertex and are placed clockwise, then*

$$\sigma_a\sigma_b\sigma_c\sigma_a = \sigma_b\sigma_c\sigma_a\sigma_b = \sigma_c\sigma_a\sigma_b\sigma_c,$$
$$x_a\sigma_b\sigma_c\sigma_a = \sigma_a\sigma_b\sigma_c x_a,$$
$$\sigma_a\sigma_b x_c\sigma_a = \sigma_b x_c\sigma_a\sigma_b,$$
$$x_a\sigma_b x_c\sigma_a = \sigma_b x_c\sigma_a x_b,$$

- *pseudocycle*: *if the edges $a_1, \ldots, a_n$ form an irreducible pseudocycle and if a_1 is not the starting edge nor a_n is the end edge of a reverse, then*

$$\sigma_{a_1}\ldots\sigma_{a_{n-1}} = \sigma_{a_2}\ldots\sigma_{a_n},$$
$$x_{a_1}\sigma_{a_2}\ldots\sigma_{a_{n-1}} = \sigma_{a_2}\ldots\sigma_{a_{n-1}}x_{a_n}.$$

The last aim of this section is to give graph presentations for the singular braid monoid on n strands of the annulus.

THEOREM 4.4. *The singular braid monoid on n strands of the annulus $SB_n(Ann)$ admits the following presentation*:

— *Generators*: $\sigma_i, \sigma_i^{-1}, x_i$ $(i = 1, \ldots, n-1)$, τ, τ^{-1}.
— *Relations*:

$$\begin{array}{lll} \text{(R1)} & \sigma_i\sigma_j = \sigma_j\sigma_i, & \textit{if} \quad |i-j| > 1; \\ \text{(R2)} & x_i x_j = x_j x_i, & \textit{if} \quad |i-j| > 1; \\ \text{(R3)} & x_i\sigma_j = \sigma_j x_i, & \textit{if} \quad |i-j| \neq 1; \\ \text{(R4)} & \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}; & \\ \text{(R5)} & \sigma_i\sigma_{i+1}x_i = x_{i+1}\sigma_i\sigma_{i+1}; & \\ \text{(R6)} & \sigma_{i+1}\sigma_i x_{i+1} = x_i\sigma_{i+1}\sigma_i; & \end{array}$$

$$\text{(R7)} \quad \tau\sigma_1\tau\sigma_1 = \sigma_1\tau\sigma_1\tau;$$
$$\text{(R8)} \quad \tau\sigma_1\tau x_1 = x_1\tau\sigma_1\tau,$$
$$\text{(R9)} \quad \tau\sigma_i = \sigma_i\tau, \qquad \textit{if} \quad i \geq 2;$$
$$\text{(R10)} \quad \tau x_i = x_i\tau, \qquad \textit{if} \quad i \geq 2;$$
$$\text{(R11)} \quad \sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = \tau\tau^{-1} = \tau^{-1}\tau = 1.$$

The geometric interpretation of σ_i and τ is given in Figure 2.1.

We get the Reidemeister moves for singular knot theory in a solid torus if we add the move depicted in Figure 4.8 to the regular (without singularities) Reidemeister moves of knot theory in a solid torus. This Reidemeister move means how a singular point goes around the axis of the torus (fixed string). The proof that the list (R1)–(R11) is a complete set of relations is standard: every isotopy can be decomposed in a sequence of elementary isotopies which correspond to relations (R1)–(R11) (see also [42]).

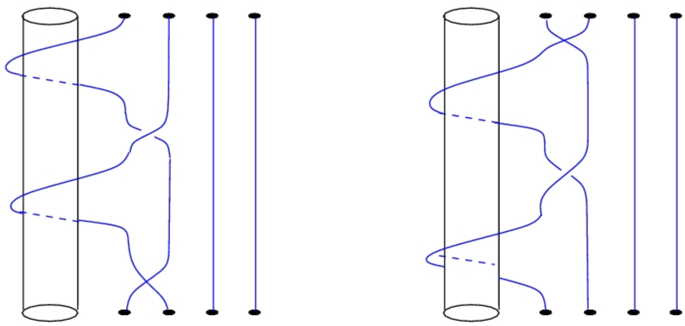


Fig. 4.8. The words $\tau\sigma_1\tau x_1$ and $x_1\tau\sigma_1\tau$ represent the same element in $SB_n(Ann)$

Remark 4.5. The singular braid monoid on n strands of the annulus differs from the singular Artin monoid of type B as defined by R. Corran [22], where the numbers of singular and regular generators are the same. The singular generator associated to τ cannot be interpreted geometrically as above.

As in Section 4.1 we consider 1-punctured graphs. To any edge a disjoint from the distinguished vertex v of Γ we associate three singular braids: σ_a will denote the positive half-twist associated to a, σ_a^{-1} will denote the corresponding negative half-twist and τ_a denotes the corresponding singular crossing.

The graph presentations for the singular braid monoid in the solid torus arise from Theorems 4.3 and 4.4.

Theorem 4.5. *Let Γ be a 1-punctured graph on n vertices. The monoid $SB_n(Ann)$ admits the presentation $\langle X_\Gamma, R_\Gamma\rangle$, where*

— *$X_\Gamma = \{\sigma_a, \sigma_a^{-1}, x_a, \tau_b, \tau_b^{-1}\}$, for any edge a of Γ not incident with the distinguished vertex v, and for any edge b of Γ adjacent to the distinguished vertex v;*
— *R_Γ is formed by the relations given in Theorems 4.1 and 4.3 and the following new nodal and invertibility relations:*

$$\sigma_a\tau_b\sigma_c x_a = x_c\sigma_a\tau_b\sigma_c, \qquad \tau_b\sigma_c\sigma_a\tau_b x_c = x_a\tau_b\sigma_c\sigma_a\tau_b, \qquad \tau_b\tau_b^{-1} = \tau_b^{-1}\tau_b = 1.$$

5. Birman–Ko–Lee presentation for the singular braid monoid. An analogue of the presentation of Birman, Ko and Lee for the singular braid monoid was given in [85]. For $1 \leq s < t \leq n$ and $1 \leq p < q \leq n$ we consider the elements of SB_n which are defined by

$$\begin{cases} a_{ts} = (\sigma_{t-1}\sigma_{t-2}\cdots\sigma_{s+1})\sigma_s(\sigma_{s+1}^{-1}\cdots\sigma_{t-2}^{-1}\sigma_{t-1}^{-1}) & \text{for } 1 \leq s < t \leq n, \\ a_{ts}^{-1} = (\sigma_{t-1}\sigma_{t-2}\cdots\sigma_{s+1})\sigma_s^{-1}(\sigma_{s+1}^{-1}\cdots\sigma_{t-2}^{-1}\sigma_{t-1}^{-1}) & \text{for } 1 \leq s < t \leq n, \\ b_{qp} = (\sigma_{q-1}\sigma_{q-2}\cdots\sigma_{p+1})x_p(\sigma_{p+1}^{-1}\cdots\sigma_{q-2}^{-1}\sigma_{q-1}^{-1}) & \text{for } 1 \leq p < q \leq n. \end{cases}$$

Geometrically the generators $a_{s,t}$ and $b_{s,t}$ are depicted in Figure 5.1.

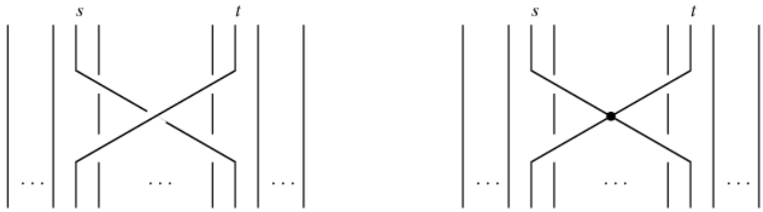


Fig. 5.1.

THEOREM 5.1. *The singular braid monoid SB_n has a presentation with generators a_{ts}, a_{ts}^{-1} for $1 \leq s < t \leq n$ and b_{qp} for $1 \leq p < q \leq n$ and relations*

$$(5.1)\qquad \begin{cases} a_{ts}a_{rq} = a_{rq}a_{ts} & \text{for } (t-r)(t-q)(s-r)(s-q) > 0, \\ a_{ts}a_{sr} = a_{tr}a_{ts} = a_{sr}a_{tr} & \text{for } 1 \leq r < s < t \leq n, \\ a_{ts}a_{ts}^{-1} = a_{ts}^{-1}a_{ts} = 1 & \text{for } 1 \leq s < t \leq n, \\ a_{ts}b_{rq} = b_{rq}a_{ts} & \text{for } (t-r)(t-q)(s-r)(s-q) > 0, \\ a_{ts}b_{ts} = b_{ts}a_{ts} & \text{for } 1 \leq s < t \leq n, \\ a_{ts}b_{sr} = b_{tr}a_{ts} & \text{for } 1 \leq r < s < t \leq n, \\ a_{sr}b_{tr} = b_{ts}a_{sr} & \text{for } 1 \leq r < s < t \leq n, \\ a_{tr}b_{ts} = b_{sr}a_{tr} & \text{for } 1 \leq r < s < t \leq n, \\ b_{ts}b_{rq} = b_{rq}b_{ts} & \text{for } (t-r)(t-q)(s-r)(s-q) > 0. \end{cases}$$

Now we consider the *positive singular braid monoid $SBKL_n^+$* with respect to generators a_{ts} and $b_{t,s}$ for $1 \leq s < t \leq n$. Its relations are (5.1) except the one concerning the invertibility of a_{ts}. Two positive words A and B in the alphabet a_{ts} and $b_{t,s}$ will be said to be *positively equivalent* if they are equal as elements of this monoid. In this case we shall write $A \doteq B$.

The *fundamental word* δ of Birman, Ko and Lee is given by the formula

$$\delta \equiv a_{n(n-1)}a_{(n-1)(n-2)}\dots a_{21} \equiv \sigma_{n-1}\sigma_{n-2}\dots\sigma_2\sigma_1.$$

Its divisibility by any generator a_{ts}, proved in [14], is convenient for us to be expressed in the following form.

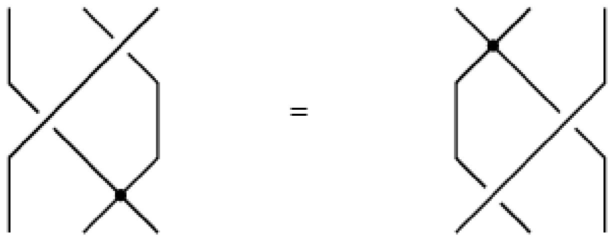


Fig. 5.2.

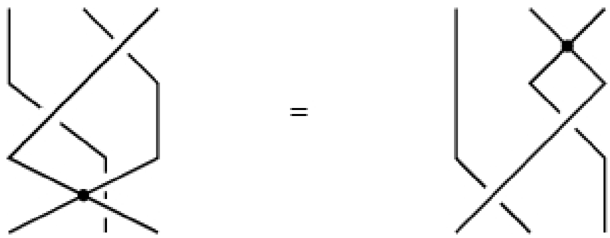


Fig. 5.3.

Proposition 5.1. *The fundamental word δ is positively equivalent to a word that begins or ends with any given generator a_{ts}. The explicit expression for left divisibility is*

$$\delta \doteq a_{ts}a_{n(n-1)}a_{(n-1)(n-2)}\dots a_{(t+1)s}a_{t(t-1)}\dots a_{(s+2)(s+1)}a_{s(s-1)}\dots a_{21}.$$

Proposition 5.2. *For the fundamental word δ the following formulae of commutation are true*

$$\begin{cases} a_{ts}\delta \doteq \delta a_{(t+1)(s+1)} & \text{for } 1 \le s < t < n, \\ a_{ns}\delta \doteq \delta a_{(s+1)1}, \\ b_{ts}\delta \doteq \delta b_{(t+1)(s+1)} & \text{for } 1 \le s < t < n, \\ b_{ns}\delta \doteq \delta b_{(s+1)1}. \end{cases}$$

Geometrically this commutation is shown in Figures 5.2 and 5.3.

The analogues of the other results proved by Birman, Ko and Lee remain valid for the singular braid monoid. They are proved in the work of V. V. Chaynikov [20].

6. The work of V. V. Chaynikov

6.1. Cancellation property. Let $W_1, W_2 \in SBKL_n^+$. By a common multiple of W_1, W_2 (if it exists) we mean a positive word $V \doteq W_1V_1 \doteq W_2V_2$.

Let

$$\delta_{s_k\dots s_1} \equiv a_{s_ks_{(k-1)}}a_{s_{(k-1)}s_{(k-2)}}\dots a_{s_21},$$

where $n \ge s_k > s_{k-1} > \dots > s_1 \ge 1$. The word $\delta_{s_k\dots s_1}$ is the least common multiple (l.c.m.) of the generators a_{ij}, where $i, j \in \{s_k, s_{(k-1)}, \dots s_1\}$, see [14], $\delta \equiv \delta_{n(n-1)\dots 1}$.

We denote the least common multiple of X, Y by $X \vee Y$. Define $(X \vee Y)^*_X$ and $(X \vee Y)^*_Y$ by the equations

$$X \vee Y \doteq X(X \vee Y)^*_X \doteq Y(X \vee Y)^*_Y.$$

Similarly, we denote the greatest common divisor (g.c.d.) of X, Y by $X \wedge Y$. The semigroup BKL_n^+ is a lattice relative to $\vee, \wedge$ [14].

REMARK 6.1. We give the table of l.c.m. for some pairs of generators below. There does not exist $X \vee Y$ for the remaining pairs of $SBKL_n^+$ generators.

X	Y	$X \vee Y$	
$\hat{a}_{ts}$	$\hat{a}_{rq}$	$\hat{a}_{ts}(\hat{a}_{rq}) \doteq \hat{a}_{rq}(\hat{a}_{ts})$	$(t-r)(t-q)(s-r)(s-q) > 0$
		$a_{ts}(a_{sr}) \doteq a_{tr}(a_{ts}) \doteq a_{sr}(a_{tr})$	$t > s > r$
a_{ts}	a_{rq}	$a_{ts}a_{tr}a_{sq} \doteq a_{rq}a_{tq}(a_{rs}) \doteq \delta_{trsq}$	$t > r > s > q$
a_{ts}	b_{ts}	$a_{ts}(b_{ts}) \doteq b_{ts}(a_{ts})$	$t > s > r$
a_{sr}	b_{tr}	$a_{ts}(b_{sr}) \doteq b_{tr}(a_{ts})$	$t > s > r$
a_{sr}	b_{ts}	$a_{sr}(b_{tr}) \doteq b_{ts}(a_{sr})$	$t > s > r$
a_{ts}	b_{sr}	$a_{ts}(a_{sr}b_{ts}) \doteq b_{sr}(\delta_{tsr})$	$t > s > r$
a_{sr}	b_{tr}	$a_{sr}(a_{tr}b_{sr}) \doteq b_{tr}(\delta_{tsr})$	$t > s > r$
a_{tr}	b_{ts}	$a_{tr}(a_{ts}b_{tr}) \doteq b_{ts}(\delta_{tsr})$	$t > s > r$
a_{ts}	b_{rq}	$a_{ts}(\delta_{trq}b_{ts}) \doteq b_{ts}(\delta_{trsq})$	$t > r > s > q$
a_{rq}	b_{ts}	$a_{rq}(a_{tq}a_{rs}b_{rq}) \doteq b_{ts}(\delta_{trsq})$	$t > r > s > q$

Here the symbol $\hat{a}_{ij} \in \{a_{ij},\ b_{ij}\}$ means the same symbol in both parts of one equality.

We call the pairs of generator from the table above *admissible* and all other pairs *inadmissible*. Observe that pairs $\{a_{ij},\ a_{pm}\}$, $\{a_{ij}, b_{pm}\}$ are admissible and $\{b_{ij}, b_{pm}\}$ is admissible if and only if $b_{ij}b_{pm} = b_{pm}b_{ij}$ is the defining relation of SB_n.

THEOREM 6.1 (Left cancellation).

i) *Let $\{x, y\}$ be an admissible pair and $xX \doteq yY$. Then there exists a positive word Z such that $xX \doteq yY \doteq (x \vee y)Z$, where $X \doteq (x \vee y)^*_x Z$ and $Y \doteq (x \vee y)^*_y Z$.*
ii) *If the pair $\{x, y\}$ is inadmissible then the equality $xX \doteq yY$ is impossible (so there does not exist a common multiple for $\{x, y\}$).*

Similarly we can obtain the right cancellation property.

COROLLARY 6.1. *If $A \doteq P$, $B \doteq Q$, $AXB \doteq PYQ$, then the equality $X \doteq Y$ holds in $SBKL_n^+$.*

COROLLARY 6.2. *Suppose that δ is the l.c.m. of the set of generators $\{a_{i_1j_1}, \dots, a_{i_pj_p}\}$ and W is a positive word such that either*

$$W \doteq a_{i_1j_1}A_1 \doteq a_{i_2j_2}A_2 \doteq \dots \doteq a_{i_pj_p}A_p,$$

or

$$W \doteq B_1a_{i_1j_1} \doteq B_2a_{i_2j_2} \doteq \dots \doteq B_pa_{i_pj_p},$$

then $W \doteq \delta Z$ for some positive word Z.

COROLLARY 6.3 (Embedding theorem). *The canonical homomorphism*

$$SBKL_n^+ \to SB_n$$

is injective.

6.2. Word and conjugacy problems in SB_n. The word problem in SB_n (in classical generators) was solved by R. Corran [22], see also [85]. Let us fix an arbitrary linear order on the set of generators of $SBKL_n^+$ and extend it to the deg-lex order on words of the generators of $SBKL_n^+$. With this order, we first order words by total degree (the length of the word on given generators) and we break ties by the lex order. By the *base* of the positive word W we mean the least (relative to the deg-lex order on the words on the generators of $SBKL_n^+$) word which represents the same element as W in $SBKL_n^+$. Observe that this word is unique. If the positive word A is not divisible by δ we denote its base by $\overline{A}$.

THEOREM 6.2. *Every word W in SB_n has a unique representation of shape $\delta^m\overline{A}$, where m is an integer and A is not divisible by δ.*

This gives a normal form for SB_n in Birman–Ko–Lee generators. The process of computation of this normal form is the same as given by Garside [37]. First, suppose that P is any positive word in the generators $SBKL_n^+$. Among all positive words positively equivalent to P choose a word in the form $\delta^t A$ with t maximal. Then A is prime to δ and we have

$$P \doteq \delta^t\overline{A}.$$

Now, let W be an arbitrary word in SB_n. Then we may put

$$W \equiv W_1(c_1)^{-1}W_2(c_2)^{-1}\dots(c_k)^{-1}W_{k+1},$$

where each W_j is a positive word of length ≥ 0, and c_l are generators $a_{t,s}$, the only possible invertible generators. For each c_l there exists a positive word D_l such that $c_lD_l \doteq \delta$, so that $(c_l)^{-1} = D_l\delta^{-1}$, and hence

$$W = W_1D_1\delta^{-1}W_2D_2\delta^{-1}\dots W_kD_k\delta^{-1}W_{k+1}.$$

Moving the factors δ^{-1} to the left, we obtain $W = \delta^kP$, where P is positive, so we can express it in the form $\delta^t\overline{A}$ and finally we obtain the normal form

$$W = \delta^m\overline{A}.$$

Let us consider the conjugacy problem. We say that two elements $u, v \in SB_n$ are conjugated if there exists $g \in B_n$ such that $g^{-1}ug = v$. We denote this by $u \sim v$.

Let u be a positive word. Define the set of all positive elements conjugated with u as follows: $C^+(u) = \{v \,|\, v \sim u, v \in SBKL_n^+\}$.

The following properties are obvious and very close to the ones proved in [29], [14]:

i) *The set of all positive words of limited length is finite.*
ii) *The set $C^+(u)$ is finite.*
iii) *The element δ^n generates the center of SB_n.*

Now fix two words $u, v \in SB_n$. We can assume that they are positive (otherwise we multiply them by the element δ^{nk}, where k is big enough to cancel all negative letters).

THEOREM 6.3. *The elements u, v are conjugated if and only if the sets $C^+(u)$ and $C^+(v)$ contain the same elements.*

There exists the following algorithm for constructing $C^+(u)$. Define $C_0^+(u) := \{u\}$. If the set $C_i^+(u)$ is already constructed define

$$C_{i+1}^+(u) := \{v^g \mid g \text{ divides } \delta;\ v \in C_i^+\} \cap SBKL_n^+.$$

The set $C_k^+(u)$ stabilizes on the finite step, so we put

$$C^+(u) := \bigcup_{k\geq 0} C_k^+(u).$$

7. Inverse monoids. The notion of *inverse semigroup* was introduced by V. V. Wagner in 1952 [87]. By definition it means that for any element a of a semigroup (monoid) M there exists a unique element b (which is called *inverse*) with the following two conditions:

$$a = aba \tag{7.1}$$

$$b = bab. \tag{7.2}$$

Roots of this notion can be seen in the von Neumann regular rings [61] where only one condition (7.1) holds for non-necessary unique b, or in the Moore–Penrose pseudoinverse for matrices [60], [64] where both conditions (7.1) and (7.2) hold (and certain supplementary conditions also). See the books [65] and [53] as general references for inverse semigroups.

The typical example of an inverse monoid is a monoid of partial (defined on a subset) injections of a set. For a finite set this gives us the notion of a *symmetric inverse monoid* I_n which generalizes and includes the classical symmetric group Σ_n. A presentation of symmetric inverse monoid was obtained by L. M. Popova [67], see also formulae (7.3)–(7.4) below.

Recently the *inverse braid monoid* IB_n was constructed in [28] by D. Easdown and T. G. Lavers. It arises from a very natural operation on braids: deleting one or several strands. By the application of this procedure to braids in Br_n we get *partial* braids [28]. The multiplication of partial braids is shown in Figure 7.1. At the last stage it is necessary

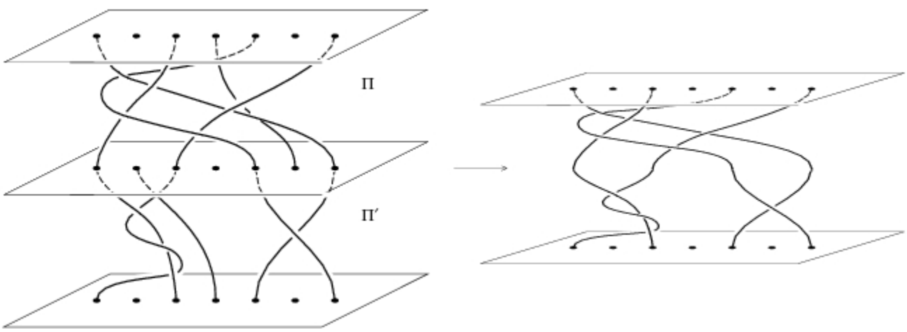


Fig. 7.1.

to remove any arc that does not join the upper or lower planes. The set of all partial braids with this operation forms an inverse braid monoid IB_n.

One of the motivations for studying IB_n is that it is a natural setting for the *Brunnian* (or *Makanin*) braids, which were also called *smooth* braids by G. S. Makanin, who first mentioned them in [49] (page 78, question 6.23), and D. L. Johnson [45]. By the usual definition a braid is Brunnian if it becomes trivial after deleting any strand, see formulae (8.9)–(8.13). According to the work of Fred Cohen, Jon Berrick, Wu Jie, Yang Loi Wong [10], Brunnian braids are connected with homotopy groups of spheres.

The following presentation for the inverse braid monoid was obtained in [28]. It has the generators σ_i, σ_i^{-1}, $i = 1, \dots, n-1$, ϵ, which satisfy the braid relations (1.1) and the following relations:

$$\begin{cases} \sigma_i \sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1 & \text{for all } i, \\ \epsilon\sigma_i = \sigma_i\epsilon & \text{for } i \geq 2, \\ \epsilon\sigma_1\epsilon = \sigma_1\epsilon\sigma_1\epsilon = \epsilon\sigma_1\epsilon\sigma_1, \\ \epsilon = \epsilon^2 = \epsilon\sigma_1^2 = \sigma_1^2\epsilon. \end{cases} \tag{7.3}$$

Geometrically the generator ϵ means that the first strand in the trivial braid is absent.

If we replace the first relation in (7.3) by the following set of relations

$$\sigma_i^2 = 1 \text{ for } \text{ all } i, \tag{7.4}$$

and delete the superfluous relations

$$\epsilon = \epsilon\sigma_1^2 = \sigma_1^2\epsilon,$$

we get a presentation of the symmetric inverse monoid I_n [67]. We also can simply add the relations (7.4) if we do not worry about redundant relations. We get a canonical map [28]

$$\tau_n : IB_n \to I_n, \tag{7.5}$$

which is a natural extension of the corresponding map for the braid and symmetric groups.

More balanced relations for the inverse braid monoid were obtained in [40]. Let ϵ_i denote the braid which is obtained from the trivial by deleting of the ith strand, formally:

$$\begin{cases} \epsilon_1 = \epsilon, \\ \epsilon_{i+1} = \sigma_i^{\pm 1}\epsilon_i\sigma_i^{\pm 1}. \end{cases}$$

So, the generators are: σ_i, σ_i^{-1}, $i = 1, \dots, n-1$, ϵ_i, $i = 1, \dots, n$, and relations are the following:

$$\begin{cases} \sigma_i \sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1 & \text{for all } i, \\ \epsilon_j\sigma_i = \sigma_i\epsilon_j & \text{for } \ j \neq i, i+1, \\ \epsilon_i\sigma_i = \sigma_i\epsilon_{i+1}, \\ \epsilon_{i+1}\sigma_i = \sigma_i\epsilon_i, \\ \epsilon_i = \epsilon_i^2, \\ \epsilon_{i+1}\sigma_i^2 = \sigma_i^2\epsilon_{i+1} = \epsilon_{i+1}, \\ \epsilon_i\epsilon_{i+1}\sigma_i = \sigma_i\epsilon_i\epsilon_{i+1} = \epsilon_i\epsilon_{i+1}, \end{cases} \tag{7.6}$$

plus the braid relations (1.1).

7.1. Inverse reflection monoid of type B. It can be defined in the same way as the corresponding Coxeter group (2.1) as the *monoid of partial signed permutations* $I(B_n)$:

$$I(B_n) = \{\sigma \text{ is a partial bijection of } SN : (-x)\sigma = -(x)\sigma \text{ for } x \in SN$$
$$\text{and } x \in \operatorname{dom}\sigma \text{ if and only if } -x \in \operatorname{dom}\sigma\},$$

where $\operatorname{dom}\sigma$ means domain of definition of the monomorphism σ. This monoid was studied in [31].

8. Properties of inverse braid monoid. In relations (7.3) we have one generator for the idempotent part and $n-1$ generators for the group part. If we minimize the number of generators of the group part and take the presentation (1.5) for the braid group we get a presentation of the inverse braid monoid with generators σ_1, σ, ϵ, and relations:

$$\begin{cases} \sigma_1\sigma_1^{-1} = \sigma_1^{-1}\sigma_1 = 1, \\ \sigma\sigma^{-1} = \sigma^{-1}\sigma = 1, \\ \epsilon\sigma^i\sigma_1\sigma^{-i} = \sigma^i\sigma_1\sigma^{-i}\epsilon \ \text{ for } 1 \le i \le n-2, \\ \epsilon\sigma_1\epsilon = \sigma_1\epsilon\sigma_1\epsilon = \epsilon\sigma_1\epsilon\sigma_1, \\ \epsilon = \epsilon^2 = \epsilon\sigma_1^2 = \sigma_1^2\epsilon, \end{cases}$$

plus (1.5).

Let Γ be a normal planar graph (see Introduction). Let us add new generators ϵ_v which correspond to each vertex of the graph Γ. Geometrically it means the absence in the trivial braid of one strand corresponding to the vertex v. We orient the graph Γ arbitrarily and so we get a starting $v_0 = v_0(e)$ and a terminal $v_1 = v_1(e)$ vertex for each edge e. Consider the following relations

$$(8.1) \qquad \begin{cases} \sigma_e\sigma_e^{-1} = \sigma_e^{-1}\sigma_e = 1, & \text{for all edges of } \Gamma, \\ \epsilon_v\sigma_e = \sigma_e\epsilon_v, & \text{if the vertex } v \text{ and the edge } e \text{ do not intersect}, \\ \epsilon_{v_0}\sigma_e = \sigma_e\epsilon_{v_1}, & \text{where } v_0 = v_0(e),\ v_1 = v_1(e), \\ \epsilon_{v_1}\sigma_e = \sigma_e\epsilon_{v_0}, & \\ \epsilon_v = \epsilon_\nu^2, & \\ \epsilon_{v_i}\sigma_e^2 = \sigma_e^2\epsilon_{v_i} = \epsilon_{v_i}, & i = 0,1, \\ \epsilon_{v_0}\epsilon_{v_1}\sigma_e = \sigma_e\epsilon_{v_0}\epsilon_{v_1} = \epsilon_{v_0}\epsilon_{v_1}. & \end{cases}$$

THEOREM 8.1. *We get a Sergiescu graph presentation of the inverse braid monoid IB_n if we add to the graph presentation of the braid group Br_n relations* (8.1).

Let EF_n be a monoid of partial isomorphisms of a free group F_n defined as follows. Let a be an element of the symmetric inverse monoid I_n, $a \in I_n$, $J_k = \{j_1, \dots, j_k\}$ is the image of a, and elements $i_1, \dots, i_k$ belong to domain of the definition of a. The monoid EF_n consists of isomorphisms of free subgroups

$$\langle x_{i_1}, \dots, x_{i_k}\rangle \to \langle x_{j_1}, \dots, x_{j_k}\rangle$$

such that

$$f_a : x_i \mapsto w_i^{-1} x_{a(i)} w_i,$$

if i is among $i_1, \dots, i_k$ and not defined otherwise and w_i is a word on $x_{j_1}, \dots, x_{j_k}$. The composition of f_a and g_b, $a, b \in I_n$, is defined for x_i belonging to the domain of $a \circ b$. We put $x_{j_m} = 1$ in a word w_i if x_{j_m} does not belong to the domain of definition of g. We define a map ϕ_n from IB_n to EF_n expanding the canonical inclusion

$$Br_n \to \operatorname{Aut} F_n$$

by the condition that $\phi_n(\epsilon)$ as a partial isomorphism of F_n is given by the formula

$$\phi_n(\epsilon)(x_i) = \begin{cases} x_i & \text{if } i \geq 2, \\ \text{not defined}, & \text{if } i = 1. \end{cases} \tag{8.2}$$

Using the presentation (7.3) we see that ϕ_n is a correctly defined homomorphism of monoids

$$\phi_n : IB_n \to EF_n.$$

THEOREM 8.2. *The homomorphism ϕ_n is a monomorphism.*

Theorem 8.2 gives also a possibility to interpret the inverse braid monoid as a monoid of isotopy classes of maps. As usual consider a disc D^2 with n fixed points. Denote the set of these points by Q_n. The fundamental group of D^2 with these points deleted is isomorphic to F_n. Consider homeomorphisms of D^2 onto a copy of the same disc with the condition that only k points of Q_n, $k \leq n$ (say $i_1, \dots, i_k$) are mapped bijectively onto the k points (say $j_1, \dots, j_k$) of the second copy of D^2. Consider the isotopy classes of such homeomorphisms and denote such set by $IM_n(D^2)$. Evidently it is a monoid.

THEOREM 8.3. *The monoids IB_n and $IM_n(D^2)$ are isomorphic.*

These considerations can be generalized to the following definition. Consider a surface $S_{g,b,n}$ of the genus g, b boundary components and with a chosen set Q_n of n fixed interior points. Let f be a homeomorphism of $S_{g,b,n}$ which maps k points, $k \leq n$, from Q_n: $\{i_1, \dots, i_k\}$ to k points $\{j_1, \dots, j_k\}$ also from Q_n. In the same way let h be a homeomorphism of $S_{g,b,n}$ which maps l points, $l \leq n$, from Q_n, say $\{s_1, \dots, s_l\}$ to l points $\{t_1, \dots, t_l\}$ again from Q_n. Consider the intersection of the sets $\{j_1, \dots, j_k\}$ and $\{s_1, \dots, s_l\}$, let it be the set of cardinality m, it may be empty. Then the composition of f and h maps m points of Q_n to m points (may be different) of Q_n. If $m = 0$ then the composition does not take into account the set Q_n. Denote the set of isotopy classes of such maps by $\mathcal{IM}_{g,b,n}$. This standard composition of f and g as maps defines a structure of monoid on $\mathcal{IM}_{g,b,n}$.

PROPOSITION 8.1. *The monoid $\mathcal{IM}_{g,b,n}$ is inverse.*

We call the monoid $\mathcal{IM}_{g,b,n}$ the *inverse mapping class monoid.* If $g = 0$ and $b = 1$ we get the inverse braid monoid. In the general case $\mathcal{IM}_{g,b,n}$ the role of the empty braid plays the mapping class group $\mathcal{M}_{g,b}$ (without fixed points).

We remind that a monoid M is *factorisable* if $M = EG$ where E is a set of idempotents of M and G is a subgroup of M.

PROPOSITION 8.2. *The monoid $\mathcal{IM}_{g,b,n}$ can be written in the form*

$$\mathcal{IM}_{g,b,n} = E\mathcal{M}_{g,b,n},$$

where E is a set of idempotents of $\mathcal{IM}_{g,b,n}$ and $\mathcal{M}_{g,b,n}$ is the corresponding mapping class group. So this monoid is factorisable.

Let Δ be the Garside's fundamental word in the braid group Br_n [37]. It can be defined by the formula

$$\Delta = \sigma_1 \dots \sigma_{n-1}\sigma_1 \dots \sigma_{n-2} \dots \sigma_1\sigma_2\sigma_1.$$

PROPOSITION 8.3. *The generators ϵ_i commute with Δ in the following way:*

$$\epsilon_i\Delta = \Delta\epsilon_{n+1-i}.$$

PROPOSITION 8.4. *The center of IB_n consists of the union of the center of the braid group Br_n (generated by Δ^2) and the empty braid $\varnothing = \epsilon_1 \dots \epsilon_n$.*

Let $\mathcal{E}$ be the monoid generated by one idempotent generator ϵ.

PROPOSITION 8.5. *The abelianization of IB_n is isomorphic to an abelian monoid AB generated (as an abelian monoid) by elements ϵ, α and $-\alpha$, subject to the following relations*

$$\begin{cases} \alpha + (-\alpha) = 0, \\ 2\epsilon = \epsilon, \\ \epsilon + \alpha = \epsilon. \end{cases}$$

So, it is isomorphic to the quotient-monoid of $\mathcal{E} \oplus \mathbb{Z}$ by the relation $\epsilon + 1 = \epsilon$. The canonical map of abelianization

$$a : IB_n \to AB$$

is given by the formula

$$\begin{cases} a(\epsilon_i) = \epsilon, \\ a(\sigma_i) = \alpha. \end{cases}$$

Let $\epsilon_{k+1,n}$ denote the partial braid with the trivial first k strands and the absent rest $n-k$ strands. It can be expressed using the generator ϵ or the generators ϵ_i as follows

$$\epsilon_{k+1,n} = \epsilon\sigma_{n-1} \dots \sigma_{k+1}\epsilon\sigma_{n-1} \dots \sigma_{k+2}\epsilon \dots \epsilon\sigma_{n-1}\sigma_{n-2}\epsilon\sigma_{n-1}\epsilon, \tag{8.3}$$

$$\epsilon_{k+1,n} = \epsilon_{k+1}\epsilon_{k+2} \dots \epsilon_n, \tag{8.4}$$

It was proved in [28] that every partial braid has a representative of the form

$$\sigma_{i_1} \dots \sigma_1 \dots \sigma_{i_k} \dots \sigma_k\epsilon_{k+1,n}x\epsilon_{k+1,n}\sigma_k \dots \sigma_{j_k} \dots \sigma_1 \dots \sigma_{j_1}, \tag{8.5}$$

$$k \in \{0, \dots, n\},\ x \in Br_k,\ 0 \le i_1 < \dots < i_k \le n-1,\ 0 \le j_1 < \dots < j_k \le n-1. \tag{8.6}$$

Note that in the formula (8.5) we can delete one of the $\epsilon_{k+1,n}$, but we shall use the form (8.5) because of convenience: two symbols $\epsilon_{k+1,n}$ serve as markers to distinguish the elements of Br_k. We can put the element $x \in Br_k$ in the Markov normal form [58] and get the corresponding *Markov normal form for the inverse braid monoid IB_n.*

Among positive words on the alphabet $\{\sigma_1 \dots \sigma_n\}$ let us introduce a lexicographical ordering with the condition that $\sigma_1 < \sigma_2 < \dots < \sigma_n$. For a positive word V the *base* of V is the smallest positive word which is positively equal to V. The base is uniquely determined. If a positive word V is prime to Δ, then for the base of V the notation $\overline{V}$ will be used (compare with Section 6.2).

Theorem 8.4. *Every word W in IBr_n can be uniquely written in the form*

$$\sigma_{i_1} \dots \sigma_1 \dots \sigma_{i_k} \dots \sigma_k \epsilon_{k+1,n} x \epsilon_{k+1,n} \sigma_k \dots \sigma_{j_k} \dots \sigma_1 \dots \sigma_{j_1}, \tag{8.7}$$

$$k \in \{0, \dots, n\},\ x \in Br_k,\ 0 \le i_1 < \dots < i_k \le n-1,\ 0 \le j_1 < \dots < j_k \le n-1, \tag{8.8}$$

where x is written in the Garside normal form for Br_k

$$\Delta^m \overline{V},$$

where m is an integer.

Theorem 8.4 is evidently true also for the presentation with ϵ_i, $i = 1, \dots, n$. In this case the elements $\epsilon_{k+1,n}$ are expressed by (8.4).

We call the form of a word W established in Theorem 8.4 the *Garside left normal form for the inverse braid monoid IB_n* and the index m—the *power* of W. In the same way we can define the *Garside right normal form for the inverse braid monoid* and the corresponding variant of Theorem 8.4 is true.

Theorem 8.5. *The necessary and sufficient condition for two words in IB_n to be equal is that their Garside normal forms are identical. The Garside normal form gives a solution to the word problem in the braid group.*

Garside normal form for the braid groups was detailed in the subsequent works of S. I. Adyan [1], W. Thurston [30], E. El-Rifai and H. R. Morton [29]. Namely, there was introduced the *left-greedy form* (in the terminology of W. Thurston [30])

$$\Delta^t A_1 \dots A_k,$$

where A_i are the successive possible longest *fragments of the word* Δ (in the terminology of S. I. Adyan [1]) or *positive permutation braids* (in the terminology of E. El-Rifai and H. R. Morton [29]). In the same way the *right-greedy form* is defined. These greedy forms are defined for the inverse braid monoid in the same way.

Let us consider the elements $m \in IB_n$ satisfying the equation:

$$\epsilon_i m = \epsilon_i. \tag{8.9}$$

Geometrically this means that removing the strand (if it exists) that starts at the point with the number i we get a trivial braid on the remaining $n-1$ strands. It is equivalent to the condition

$$m \epsilon_{\tau(m)(i)} = \epsilon_{\tau(m)(i)}, \tag{8.10}$$

where τ is the canonical map to the symmetric monoid (7.5). With the exception of ϵ_i itself all such elements belong to Br_n. We call such braids as *i-Brunnian* and denote the subgroup of i-Brunnian braids by A_i. The subgroups A_i, $i = 1, \ldots, n$, are conjugate

$$A_i = \sigma_{i-1}^{-1} \ldots \sigma_1^{-1} A_1 \sigma_1 \ldots \sigma_{i-1} \tag{8.11}$$

free subgroups. The group A_1 is freely generated by the set $\{x_1, \ldots, x_{n-1}\}$ [45], where

$$x_i = \sigma_{i-1}^{-1} \ldots \sigma_1^{-1} \sigma_1^2 \sigma_1 \ldots \sigma_{i-1}. \tag{8.12}$$

The intersection of all subgroups of i-Brunnian braids is the group of Brunnian braids

$$Brunn_n = \bigcap_{i=1}^{n} A_i. \tag{8.13}$$

That is the same as $m \in Brunn_n$ if and only if the equation (8.9) holds for all i.

9. Monoids of partial generalized braids. Construction of partial braids can be applied to various generalizations of braids, namely to those where geometric or diagrammatic construction of braids takes place. Let Σ_g be a surface of genus g possibly with boundary components and punctures. We consider partial braids lying in a layer between two such surfaces: $\Sigma_g \times I$ and take a set of isotopy classes of such braids. We get a monoid of partial braid on a surface Σ_g, denote it by $IB_n(\Sigma_g)$. An interesting case is when the surface is a sphere S^2. So our partial braids are lying in a layer between two concentric spheres.

THEOREM 9.1. *We get a presentation of the monoid $IB_n(S^2)$ if we add to the presentation* (7.3) *or to the presentation* (7.6) *of IB_n the sphere relation* (2.4)*. It is a factorisable inverse monoid.*

The monoid $IB(B_n)$ of partial braids of the type B can be considered also as a submonoid of IB_{n+1} consisting of partial braids with the first strand fixed. An interpretation as a monoid of isotopy classes of homeomorphisms is possible as well. Consider a disc D^2 with given $n+1$ points. Denote the set of these points by Q_{n+1}. Consider homeomorphisms of the disc D^2 onto a copy of the same disc with the condition that the first point is always mapped into itself and among the other n points only k points, $k \leq n$ (say $i_1, \ldots, i_k$) are mapped bijectively onto the k points (say $j_1, \ldots, j_k$) of the set Q_{n+1} (without the first point) of second copy of the disc D^2. The isotopy classes of such homeomorphisms form the monoid $IB(B_n)$.

THEOREM 9.2. *We get a presentation of the monoid $IB(B_n)$ if we add to the presentation* (7.3) *or the presentation* (7.6) *of IB_n one generator τ, the type B relation* (2.2) *and the following relations*

$$\begin{cases} \tau\tau^{-1} = \tau^{-1}\tau = 1, \\ \epsilon_1\tau = \tau\epsilon_1 = \epsilon_1. \end{cases} \tag{9.1}$$

It is a factorisable inverse monoid.

REMARK 9.1. Theorem 9.2 can be naturally generalized for partial braids in handlebodies [77].

We define an action of the monoid $IB(B_n)$ on the set SN (see Section 2.1) by partial isomorphisms as follows

$$\sigma_i(\delta_j v_j) = \begin{cases} \delta_i v_{i+1}, & \text{if } j = i, \\ \delta_{i+1} v_i, & \text{if } j = i+1, \\ \delta_j v_j, & \text{if } j \neq i, i+1, \end{cases} \tag{9.2}$$

$$\tau(\delta_j v_j) = \begin{cases} -\delta_1 v_1, & \text{if } j = 1, \\ \delta_j v_j, & \text{if } j \neq 1, \end{cases} \tag{9.3}$$

$$\operatorname{dom} \epsilon = \{\delta_2 v_2, \dots, \delta_n v_n\}, \tag{9.4}$$

$$\epsilon(\delta_j v_j) = \delta_j v_j, \qquad \text{if } j = 2, \dots, n, \tag{9.5}$$

$$\operatorname{dom} \epsilon_i = \{\delta_1 v_1, \dots, \widehat{\delta_i v_i}, \dots, \delta_n v_n\}, \tag{9.6}$$

$$\epsilon_i(\delta_j v_j) = \delta_j v_j, \qquad \text{if } j = 1, \dots, \widehat{i}, \dots, n. \tag{9.7}$$

Direct checking shows that the relations of the inverse braid monoid of type B are satisfied by the corresponding compositions of partial isomorphisms defined by σ_i, τ and ϵ_i.

THEOREM 9.3. *The action given by the formulae* (9.2)–(9.7) *defines a homomorphism of inverse monoids* $\rho_B : IB(B_n) \to I(B_n)$ *such that the following diagram commutes*

$$\begin{array}{ccc} Br(B_n) & \longrightarrow & W(B_n) \\ \downarrow & & \downarrow \\ IB(B_n) & \xrightarrow{\rho_B} & I(B_n) \end{array} \tag{9.8}$$

(*where the vertical arrows mean inclusion of the group of invertible elements into a monoid*).

THEOREM 9.4. *The homomorphism* $\rho_B : IB(B_n) \to I(B_n)$ *is an epimorphism. We get a presentation of the monoid* $I(B_n)$ *if in the presentation of* $IB(B_n)$ *we replace the first relation in* (7.3) *by the following set of relations*

$$\sigma_i^2 = 1 \quad \textit{for all } i,$$

and delete the superfluous relations

$$\epsilon = \epsilon\sigma_1^2 = \sigma_1^2\epsilon,$$

and we replace the first relation in (9.1) *by the relation*

$$\tau^2 = 1.$$

We remind that $\mathcal{E}$ denotes the monoid generated by one idempotent generator ϵ.

PROPOSITION 9.1. *The abelianization* $Ab(IB(B_n))$ *of the monoid* $IB(B_n)$ *is isomorphic to the monoid* $\mathcal{E} \oplus \mathbb{Z}^2$, *factorized by the relations*

$$\begin{cases} \epsilon + \tau = \epsilon, \\ \epsilon + \sigma = \epsilon, \end{cases}$$

where τ and σ are generators of $\mathbb{Z}^2$. The canonical map of abelianization

$$a : IB(B_n) \to Ab(IB(B_n))$$

is given by the formulae

$$\begin{cases} a(\epsilon_i) = \epsilon, \\ a(\tau) = \tau, \\ a(\sigma_i) = \sigma. \end{cases}$$

The canonical map from $Ab(IB(B_n))$ to $Ab(I(B_n))$ consists of factorizing $\mathbb{Z}^2$ modulo 2.

Let BP_n be the braid-permutation group (see Section 2.3). Consider the image of monoid I_n in EF_n by the map defined by the formulae (2.5), (8.2). Take also the monoid IB_n lying in EF_n under the map ϕ_n of Theorem 8.2. We define the *braid-permutation* monoid as a submonoid of EF_n generated by both images of IB_n and I_n and denote it by IBP_n. It can be also defined by the diagrams of partial welded braids.

THEOREM 9.5. *We get a presentation of the monoid IBP_n if we add to the presentation of BP_n the generator ϵ, relations* (7.3) *and the analogous relations between ξ_i and ϵ, or generators ϵ_i, $1 \leq i \leq n$, relations* (7.6) *and the analogous relations between ξ_i and ϵ_i. It is a factorisable inverse monoid. Monoid IBP_n is isomorphic to the monoid EF_n of partial isomorphisms of braid-conjugation type.*

The virtual braids [82] can be defined by the plane diagrams with real and virtual crossings. The corresponding Reidemeister moves are the same as for the welded braids of the braid-permutation group with one exception. The forbidden move corresponds to the last mixed relation for the braid-permutation group (2.6). This allows to define the partial virtual braids and the corresponding monoid IVB_n. So the mixed relation for IVB_n have the form

$$\begin{cases} \sigma_i\xi_j = \xi_j\sigma_i, & \text{if } |i-j| > 1, \\ \xi_i\xi_{i+1}\sigma_i = \sigma_{i+1}\xi_i\xi_{i+1}. \end{cases} \tag{9.9}$$

THEOREM 9.6. *We get a presentation of the monoid IVB_n if we delete the last mixed relation in the presentation of IBP_n, that is, replace the relations* (2.6) *by* (9.9). *It is a factorisable inverse monoid. The canonical epimorphism*

$$IVB_n \to IBP_n$$

is evidently defined.

The constructions of singular braid monoid SB_n (see Section 2.4) are geometric, so we can easily get the analogous monoid of partial singular braids PSB_n.

THEOREM 9.7. *We get a presentation of the monoid PSB_n if we add to the presentation of SB_n the generators ϵ_i, $1 \leq i \leq n$, relations* (7.6) *and the analogous relations between x_i and ϵ_i.*

REMARK 9.2. The monoid PSB_n is neither factorisable nor inverse.

The construction of braid groups on graphs [39], [33] is geometrical so, in the same way as for the classical braid groups we can define *partial braids on a graph* Γ and the *monoid of partial braids on a graph* Γ which will be evidently inverse, so we call it as *inverse braid monoid on the graph* Γ and we denote it as $IB_n\Gamma$.

References

[1] S. I. Adyan, *Fragments of the word* Δ *in the braid group*, Mat. Zametki 36 (1984), 25–34; English transl.: Math. Notes 36 (1984), 505–520.

[2] E. Artin, *Theorie der Zöpfe*, Abh. Math. Sem. Univ. Hamburg 4 (1925), 47–72.

[3] J. C. Baez, *Link invariants of finite type and perturbation theory*, Lett. Math. Phys. 26 (1992), 43–51.

[4] D. Bar-Natan, *Non-associative tangles*, in: Geometric topology (Athens, GA, 1993), AMS/IP Stud. Adv. Math. 2.1, Amer. Math. Soc., Providence, RI, 1997, 139–183.

[5] D. Bar-Natan, *On associators and the Grothendieck–Teichmuller group. I*, Selecta Math. (N.S.) 4 (1998), 183–212.

[6] V. G. Bardakov, R. Mikhailov, *On certain questions of the free group automorphisms theory*, Comm. Algebra 36 (2008), 1489–1499.

[7] P. Bellingeri, *Surface braid groups and polynomial link invariants*, Thesis, Univ. Grenoble I, 2003.

[8] P. Bellingeri, V. Vershinin, *Presentations of surface braid groups by graphs*, Fund. Math. 188 (2005), 1–20.

[9] B. Berceanu, Ş. Papadima, *Universal representations of braid and braid-permutation groups*, J. Knot Theory Ramifications 18 (2009), 999–1019.

[10] A. J. Berrick, F. R. Cohen, Y. L. Wong, J. Wu, *Configurations, braids, and homotopy groups*, J. Amer. Math. Soc. 19 (2006), 265–326.

[11] D. Bessis, *Finite complex reflection arrangements are* $K(\pi,1)$, arXiv:math.GT/0610777.

[12] D. Bessis, J. Michel, *Explicit presentations for exceptional braid groups*, Experiment. Math. 13 (2004), 257–266.

[13] J. S. Birman, *New points of view in knot theory*, Bull. Amer. Math. Soc. (N.S.) 28 (1993), 253–287.

[14] J. S. Birman, K. H. Ko, S. J. Lee, *A new approach to the word and conjugacy problems in the braid groups*, Adv. Math. 139 (1998), 322–353.

[15] N. Bourbaki, *Groupes et algèbres de Lie*, Chaps. 4–6, Masson, Paris, 1981.

[16] E. Brieskorn, *Sur les groupes de tresses* [*d'après V. I. Arnol'd*], Séminaire Bourbaki, 24ème année (1971/1972), Exp. No. 401, Lecture Notes in Math. 317, Springer, Berlin, 1973, 21–44.

[17] M. Brin, *The algebra of strand splitting. I. A braided version of Thompson's group* V, J. Group Theory 10 (2007), 757–788.

[18] M. Broué, G. Malle, R. Rouquier, *Complex reflection groups, braid groups, Hecke algebras*, J. Reine Angew. Math. 500 (1998), 127–190.

[19] J. W. Cannon, W. J. Floyd, W. R. Parry, *Introductory notes on Richard Thompson's groups*, Enseign. Math. (2) 42 (1996), 215–256.

[20] V. V. Chaynikov, *Garside structure for singular braid monoid in Birman, Ko, Lee generators*, Comm. Algebra 34 (2006), 1981–1995.

[21] F. R. Cohen, J. Pakianathan, V. V. Vershinin, J. Wu, *Basis-conjugating automorphisms of a free group and associated Lie algebras*, in: Groups, Homotopy and Configuration Spaces, Geom. Topol. Monogr. 13, Geom. Topol. Publ., Coventry, 2008, 147–168.

[22] R. Corran, *A normal form for a class of monoids including the singular braid monoids*, J. Algebra 223 (2000), 256–282.

[23] H. S. M. Coxeter, W. O. J. Moser, *Generators and relations for discrete groups*, 3rd ed., Ergeb. Math. Grenzgeb. 14, Springer, Berlin, 1972.

[24] P. Dehornoy, *Groupes de Garside*, Ann. Sci. École Norm. Sup. (4) 35 (2002), 267–306.

[25] P. Dehornoy, *The group of parenthesized braids*, Adv. Math. 205 (2006), 354–409.

[26] P. Dehornoy, L. Paris, *Gaussian groups and Garside groups, two generalisations of Artin groups*, Proc. London Math. Soc. (3) 79 (1999), 569–604.

[27] P. Deligne, *Les immeubles des groupes de tresses généralisés*, Invent. Math. 17 (1972), 273–302.

[28] D. Easdown, T. G. Lavers, *The inverse braid monoid*, Adv. Math. 186 (2004), 438–455.

[29] E. El-Rifai, H. R. Morton, *Algorithms for positive braids*, Quart. J. Math. Oxford Ser. (2) 45 (1994), 479–497.

[30] D. B. A. Epstein, J. W. Cannon, D. F. Holt, S. V. F. Levy, M. S. Paterson, W. P. Thurston, *Word Processing in Groups*, Jones and Bartlett Publ., Boston, MA, 1992.

[31] B. Everitt, J. Fountain, *Partial symmetry, reflection monoids and Coxeter groups*, Adv. Math. 223 (2010), 1782–1814.

[32] E. Fadell, J. Van Buskirk, *The braid groups of E^2 and S^2*, Duke Math. J. 29 (1962), 243–257.

[33] D. Farley, L. Sabalka, *Discrete Morse theory and graph braid groups*, Algebr. Geom. Topol. 5 (2005), 1075–1109.

[34] R. Fenn, E. Keyman, C. Rourke, *The singular braid monoid embeds in a group*, J. Knot Theory Ramifications 7 (1998), 881–892.

[35] R. Fenn, R. Rimányi, C. Rourke, *The braid-permutation group*, Topology 36 (1997), 123–135.

[36] S. Galatius, *Stable homology of automorphism groups of free groups*, Ann. of Math. (2) 173 (2011), 705–768.

[37] F. A. Garside, *The braid group and other groups*, Quart. J. Math. Oxford Ser. (2) 20 (1969), 235–254.

[38] B. Gemein, *Singular braids and Markov's theorem*, J. Knot Theory Ramifications 6 (1997), 441–454.

[39] R. Ghrist, *Configuration spaces and braid groups on graphs in robotics*, in: Knots, braids, and mapping class groups—papers dedicated to Joan S. Birman (New York, 1998), AMS/IP Stud. Adv. Math. 24, Amer. Math. Soc., Providence, RI, 2001, 29–40.

[40] N. D. Gilbert, *Presentations of the inverse braid monoid*, J. Knot Theory Ramifications 15 (2006), 571–588.

[41] D. L. Goldsmith, *The theory of motion groups*, Michigan Math. J. 28 (1981), 3–17.

[42] J. González-Meneses, *Presentations for the monoids of singular braids on closed surfaces*, Comm. Algebra 30 (2002), 2829–2836.

[43] P. Greenberg, V. Sergiescu, *An acyclic extension of the braid group*, Comment. Math. Helv. 66 (1991), 109–138.

[44] C. Jensen, J. McCammond, J. Meier, *The integral cohomology of the group of loops*, Geom. Topol. 10 (2006), 759–784.

[45] D. L. Johnson, *Towards a characterization of smooth braids*, Math. Proc. Cambridge Philos. Soc. 92 (1982), 425–427.

[46] C. Kassel, V. Turaev, *Braid Groups*, Grad. Texts in Math. 247, Springer, New York, 2008.

[47] N. Kawazumi, *Cohomological aspects of Magnus expansions*, arxiv: math.GT/0505497.

[48] F. Klein, *Vorlesungen über höhere Geometrie*, 3. Aufl., bearbeitet und herausgegeben von W. Blaschke, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen 22, J. Springer, Berlin, 1926.

[49] *Kourovka notebook: unsolved problems in group theory*, Seventh edition, Akad. Nauk SSSR Sibirsk. Otdel., Inst. Mat., Novosibirsk, 1980 (Russian).

[50] D. Krammer, *A class of Garside groupoid structures on the pure braid group*, Trans. Amer. Math. Soc. 360 (2008), 4029–4061.

[51] S. Krstić, J. McCool, *The non-finite presentability of $IA(F_3)$ and $GL_2(Z[t,t^{-1}])$*, Invent. Math. 129 (1997), 595–606.

[52] S. Lambropoulou, *Solid torus links and Hecke algebras of B-type*, in: Proceedings of the Conference on Quantum Topology (Manhattan, KS, 1993), World Sci. Publishing, River Edge, NJ, 1994, 225–245.

[53] M. V. Lawson, *Inverse Semigroups. The Theory of Partial Symmetries* World Sci. Publ., River Edge, NJ, 1998.

[54] V. Ya. Lin, *Artin braids and the groups and spaces connected with them*, in: Algebra, Topologiya, Geometriya 17, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, VINITI, Moscow, 1979, 159–227; English transl.: J. Soviet Math. 18 (1982), 736–788.

[55] V. Ya. Lin, *Braids and permutations*, arxiv: math.GR/0404528.

[56] W. Magnus, *Über n-dimensionale Gittertransformationen*, Acta Math. 64 (1935), 353–367.

[57] W. Magnus, A. Karass, D. Solitar, *Combinatorial Group Theory*, Wiley, New York, 1966.

[58] A. A. Markoff, *Foundations of the Algebraic Theory of Tresses*, Trudy Mat. Inst. Steklova, 16 (1945) (Russian, English summary).

[59] J. McCool, *On basis-conjugating automorphisms of free groups*, Canadian J. Math. 38 (1986), 1525–1529.

[60] E. H. Moore, *On the reciprocal of the general algebraic matrix*, Bull. Amer. Math. Soc. 26 (1920), 394–395.

[61] J. von Neumann, *On regular rings*, Proc. Natl. Acad. Sci. USA 22 (1936), 707–713.

[62] J. Nielsen, *Über die Isomorphismen unendlicher Gruppen ohne Relation*, Math. Ann. 79 (1918), 269–272.

[63] P. Orlik, H. Terao, *Arrangements of Hyperplanes*, Grundlehren Math. Wiss. 300, Springer, Berlin, 1992.

[64] R. Penrose, *A generalized inverse for matrices*, Proc. Cambridge Philos. Soc. 51 (1955), 406–413.

[65] M. Petrich, *Inverse Semigroups*, Pure Appl. Math. (N.Y.), John Wiley & Sons, New York, 1984.

[66] A. Pettet, *The Johnson homomorphism and the second cohomology of IA_n*, Algebr. Geom. Topol. 5 (2005), 725–740.

[67] L. M. Popova, *Defining relations of a semigroup of partial endomorphisms of a finite linearly ordered set* (Russian), Leningrad. Gos. Ped. Inst. Učen. Zap. 238 (1962), 78–88.

[68] T. Sakasai, *The Johnson homomorphism and the third rational cohomology group of the Torelli group*, Topology Appl. 148 (2005), 83–111.

[69] T. Satoh, *The abelianization of the congruence IA-automorphism group of a free group*, Math. Proc. Cambridge Philos. Soc. 142 (2007), 239–248.

[70] A. G. Savushkina, *On a group of conjugating automorphisms of a free group*, Mat. Zametki 60 (1996), no. 1, 92–108; English transl.: Math. Notes 60 (1996), 68–80.

[71] G. P. Scott, *Braid groups and the group of homeomorphisms of a surface*, Proc. Cambridge Philos. Soc. 68 (1970), 605–617.

[72] V. Sergiescu, *Graphes planaires et présentations des groupes de tresses*, Math. Z. 214 (1993), 477–490.

[73] G. C. Shephard, J. A. Todd, *Finite unitary reflection groups*, Canadian J. Math. 6 (1954), 274–304.

[74] J. Tits, *Normalisateurs de tores. I. Groupes de Coxeter étendus*, J. Algebra 4 (1966), 96–116.

[75] V. A. Vassiliev, *Complements of Discriminants of Smooth Maps: Topology and Applications*, Transl. Math. Monogr. 98, Amer. Math. Soc., Providence, RI, 1992.

[76] V. V. Vershinin, *Thom spectra of generalized braid groups*, Preprint No 95/02-2, Université de Nantes, 1995.

[77] V. V. Vershinin, *On braid groups in handlebodies*, Sibirsk. Mat. Zh. 39 (1998), 755–764; English transl.: Siberian Math. J. 39 (1998), 645–654.

[78] V. V. Vershinin, *On homological properties of singular braids*, Trans. Amer. Math. Soc. 350 (1998), 2431–2455.

[79] V. V. Vershinin, *Homology of braid groups and their generalizations*, in: Knot Theory (Warsaw, 1995), Banach Center Publ. 42, Polish Acad. Sci., Warsaw, 1998, 421–446.

[80] V. V. Vershinin, *Generalizations of braids from a homological point of view*, Siberian Adv. Math. 9 (1999), no. 2, 109–139.

[81] V. V. Vershinin, *Braid groups and loop spaces*, Uspekhi Mat. Nauk 54 (1999), no. 2, 3–84; English transl.: Russian Math. Surveys 54 (1999), 273–350.

[82] V. V. Vershinin, *On homology of virtual braids and Burau representation*, J. Knot Theory Ramifications 10 (2001), 795–812.

[83] V. V. Vershinin, *On presentations of generalizations of braids with few generators*, Fundam. Prikl. Mat. 11 (2005), no. 4, 23–32.

[84] V. V. Vershinin, *Braids, their properties and generalizations*, in: Handbook of Algebra, Vol. 4, Handb. Algebr. 4, Elsevier/North-Holland, Amsterdam, 2006, 427–465.

[85] V. V. Vershinin, *On the singular braid monoid*, Algebra i Analiz 21 (2009), no. 5, 19–36; English transl.: St. Petersburg Math. J. 21 (2010), 693–704.

[86] V. V. Vershinin, *On inverse braid and reflection monoids of type B*, Sibirsk. Mat. Zh. 50 (2009), 1010–1015; English transl.: Siberian Math. J. 50 (2009), 798–802.

[87] V. V. Wagner, *Generalized groups* (Russian), Doklady Akad. Nauk SSSR (N.S.) 84 (1952), 1119–1122.

[88] O. Zariski, *On the Poincaré group of rational plane curves*, Amer. J. Math. 58 (1936), 607–619.