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REMARK ON THE INEQUALITY OF F. RIESZ

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Abstract. We prove F. Riesz' inequality assuming the boundedness of the norm of the first arithmetic mean of the functions $|\varphi_n|^p$ with $p \geq 2$ instead of boundedness of the functions φ_n of an orthonormal system.

1. Inequality of F. Riesz. Let (φ_k) be an orthonormal system in $[a, b]$, that is,

$$\int_a^b \varphi_k(t)\varphi_n(t)dt = 0 \qquad (k, n = 1, 2, 3, \dots),$$
$$\int_a^b |\varphi_k(t)|^2 \, dt = 1 \qquad (k = 1, 2, 3, \dots),$$

where $\varphi_k \in L^2_{[a,b]}$ $(k = 1, 2, 3, \dots)$, and let

$$a_k(f) = \int_a^b f(t)\varphi_k(t)dt \qquad (k = 1, 2, 3, \dots)$$

be the sequence of Fourier coefficients of a function $f \in L^2_{[a,b]}$ with respect to the system (φ_n). The well known result of F. Riesz states

THEOREM 1 (F. Riesz [4]). *Let*

$$|\varphi_k(t)| \leq M \quad \text{for almost all } t \in [a, b] \quad \text{and} \quad k = 1, 2, 3, \dots$$

with M independent of k, and let $p \in (1, 2]$ and p' be such that $1/p + 1/p' = 1$.

1° If $f \in L^p_{[a,b]}$ then

$$\Big(\sum_{k=1}^{\infty} |a_k(f)|^{p'}\Big)^{1/p'} \leq M^{\frac{2-p}{p}} \Big(\int_a^b |f(t)|^p \, dt\Big)^{1/p} .$$

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2° If $(a_k) \in l^p$ *then there exists* $f \in L^{p'}_{[a,b]}$ *such that* $a_k = a_k(f)$ *and*

$$\Big(\int_a^b |f(t)|^{p'} dt\Big)^{1/p'} \leq M^{\frac{2-p}{p}} \Big(\sum_{k=1}^{\infty} |a_k(f)|^p\Big)^{1/p}.$$

A generalization of this result was obtained by J. Marcinkiewicz and A. Zygmund [2], where a condition on the $L^q_{[a,b]}$ $(2 < q \leq \infty)$ norm of the functions φ_k was used, and the constant M was replaced by a sequence of constants M_k (see also [3] p. 166).

In the present note we consider another slightly more general condition on the system (φ_k). We suppose the boundedness of the first arithmetic mean of the $L^q_{[a,b]}$ $(2 \leq q \leq \infty)$ norms of the functions φ_k instead of their boundedness.

THEOREM 2. *Let*

$$\sum_{k=1}^{n} \Big(\int_a^b |\varphi_k(t)|^q \, dt\Big)^{p'/q} \leq M^{p'} n (b-a)^{p'/q} \quad \text{when} \;\; q < \infty$$

and

$$\sum_{k=1}^{n} |\varphi_k(t)|^{p'} \leq M^{p'} n \quad \text{for almost all} \;\; t \in [a,b] \quad \text{when} \;\; q = \infty$$

with M *independent of* n, $q \geq p'$, $\varphi_k \in L^q_{[a,b]}$ *for every* $k = 1, 2, 3, \ldots$ *with* $p' \geq 2$ *and let* $p \in (1, 2]$ *such that* $1/p + 1/p' = 1$.

1° If $f \in L^p_{[a,b]}$ *then*

$$\Big(\sum_{k=1}^{\infty} |a_k(f)|^{p'}\Big)^{1/p'} \leq M^{\frac{2-p}{p}} \Big(\int_a^b |f(t)|^p \, dt\Big)^{1/p}.$$

2° If $(a_k) \in l^p$ *then there exists* $f \in L^{p'}_{[a,b]}$ *such that* $a_k = a_k(f)$ *and*

$$\Big(\int_a^b |f(t)|^{p'} \, dt\Big)^{1/p'} \leq M^{\frac{2-p}{p}} \Big(\sum_{k=1}^{\infty} |a_k(f)|^p\Big)^{1/p}.$$

2. Proof of Theorem 2. We will use the same notation as in the book [1]. The main part of the proof is the same as in [1, Theorem 6.3.1], therefore we will give only the part which is essentially different. The modification is based on an estimation of the coefficients $\overline{a}_k$ which also leads to the inequality (14) from the proof of Theorem 6.3.1 of [1].

So, since

$$\overline{a}_k = \int_a^b \overline{f}(t) \varphi_k(t) dt$$

then, by the Hölder inequality,

$$\overline{a}_k \leq \Big(\int_a^b |\overline{f}(t)|^p \, dt\Big)^{1/p} \Big(\int_a^b |\varphi_k(t)|^{p'} \, dt\Big)^{1/p'}$$

and consequently, by our assumption,

$$\begin{aligned}1 = \sum_{k=1}^{r} |\overline{a}_k|^{p'} &\le \sum_{k=1}^{r} \int_a^b |\varphi_k(t)|^{p'}\,dt \Big(\int_a^b \big|\overline{f}(t)\big|^p\,dt \Big)^{p'/p} \\ &\le (b-a) \sum_{k=1}^{r} \Big(\frac{1}{b-a} \int_a^b |\varphi_k(t)|^q\,dt \Big)^{p'/q} \Big(\int_a^b \big|\overline{f}(t)\big|^p\,dt \Big)^{p'/p} \\ &\le M^{p'} r(b-a)(\Delta(p))^{p'/p}\,.\end{aligned}$$

Hence

$$\Delta(p) \ge \frac{1}{r^{p-1}(b-a)^{p-1}M^p},$$

which is the above mentioned inequality (14).

This modification completes the proof of 1°.

The proof of 2° is based on the proof of 1°, so it is exactly the same as that in [1].

3. Remark. This version of the assumption in the theorem of F. Riesz is sometimes more useful in applications, e.g. in investigation of strong summability of orthogonal expansions.

References

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