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SOME HOMOLOGICAL PROPERTIES OF BANACH ALGEBRAS ASSOCIATED WITH LOCALLY COMPACT GROUPS

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Abstract. We investigate some homological notions of Banach algebras. In particular, for a locally compact group G we characterize the most important properties of G in terms of some homological properties of certain Banach algebras related to this group. Finally, we use these results to study generalized biflatness and biprojectivity of certain products of Segal algebras on G.

1. Introduction. There are some important homological notions such as biflatness and biprojectivity that Helemskii investigated in the category of Banach algebras. Namely, a Banach algebra $\mathcal{A}$ is called *biflat* [*biprojective*] if there exists a bounded $\mathcal{A}$-bimodule morphism $\theta : (\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A})^* \to \mathcal{A}^*$ [$\rho : \mathcal{A} \to \mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A}$] such that

$$\theta \circ \pi_{\mathcal{A}}^* = \mathrm{id}_{\mathcal{A}^*} \quad [\pi_{\mathcal{A}} \circ \rho = \mathrm{id}_{\mathcal{A}}],$$

where $\pi_{\mathcal{A}} : \mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A} \to \mathcal{A}$ is the product morphism, $\widehat{\otimes}$ is the projective tensor product of Banach algebras and $\mathcal{A}^*$ is the continuous dual of $\mathcal{A}$. For more details about homology of Banach algebras we refer the reader to [H1, H2]; see also [D, P, RU]. For a locally compact group G, the group algebra $L^1(G)$ is biflat if and only if G is amenable [H2]. Also, $L^1(G)$ is biprojective if and only if G is compact [H1]. These show that some important properties of G are equivalent to certain homological properties of $L^1(G)$.

The author of [Z] introduced and studied the approximate version of biprojectivity which is based on a property of bimodule morphisms from algebra. In fact, the Banach algebra $\mathcal{A}$ is called *approximately biprojective* if there is a net (ρ_γ) of bounded $\mathcal{A}$-bimodule morphisms from $\mathcal{A}$ into $\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A}$ such that $\pi_{\mathcal{A}} \circ \rho_\gamma(a) \to a$ for all $a \in \mathcal{A}$. We also recall from [SSS] that $\mathcal{A}$ is *approximately biflat* if there is a bounded net (θ_γ) of bounded $\mathcal{A}$-bimodule morphisms from $(\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A})^*$ into $\mathcal{A}^*$ such that $\mathrm{W^*OT}\text{-}\lim_\gamma \theta_\gamma \circ \pi_{\mathcal{A}}^* = \mathrm{id}_{\mathcal{A}^*}$, where W*OT is the weak* operator topology on $B(\mathcal{A}^*)$. For a locally compact group G, it is of interest to determine when a Segal algebra $S^1(G)$

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is approximately biflat or approximately biprojective. For example in [SSS], for a [SIN]-group G, the authors proved that $S^1(G)$ is approximately biflat if and only if G is amenable. Further, they showed that if $S^1(G)$ is symmetric, then G is compact if and only if $S^1(G)$ is a projective $L^1(G)$-bimodule.

Let $\Delta(\mathcal{A})$ be the *spectrum* of $\mathcal{A}$, consisting of all non-zero characters on $\mathcal{A}$, and let $\phi \in \Delta(\mathcal{A})$. More recently, Pourabbas and Shahami [SP] introduced and studied homological concepts of ϕ-biflatness and ϕ-biprojectivity of Banach algebras. Precisely, $\mathcal{A}$ is called *ϕ-biflat* [*ϕ-biprojective*] if there exists a bounded $\mathcal{A}$-bimodule morphism $\rho : \mathcal{A} \to (\mathcal{A} \widehat{\otimes} \mathcal{A})^{**}$ [$\rho : \mathcal{A} \to \mathcal{A} \widehat{\otimes} \mathcal{A}$] such that

$$\widetilde{\phi} \circ \pi_{\mathcal{A}}^{**} \circ \rho = \phi \quad [\phi \circ \pi_{\mathcal{A}} \circ \rho = \phi],$$

where $\widetilde{\phi}$ is the unique extension of ϕ on $\mathcal{A}^{**}$. These concepts are closely related to some cohomological properties, namely, ϕ-amenability and ϕ-contractibility. Recall from [KLP1] and [KLP2] that $\mathcal{A}$ is called *left ϕ-amenable* if for all Banach $\mathcal{A}$-bimodules X for which the left module action is given by $a \cdot x = \phi(a)x$ ($a \in \mathcal{A}$, $x \in X$), every continuous derivation from $\mathcal{A}$ into X^* is inner.

Also, $\mathcal{A}$ is called *left character amenable* if it is left ϕ-amenable for all $\phi \in \Delta(\mathcal{A})$ and it has a bounded right approximate identity. Similarly, one defines right ϕ-amenable and right character amenable Banach algebras. If a Banach algebra is both left and right ϕ-amenable [character amenable], it is called *ϕ-amenable* [*character amenable*]. It was shown in [KLP1] that the left ϕ-amenability of $\mathcal{A}$ is equivalent to the existence of a *left ϕ-mean* in $\mathcal{A}^{**}$, that is, a functional m in $\mathcal{A}^{**}$ satisfying

$$m(\phi) = 1, \quad a \odot m = \phi(a)m \quad (a \in \mathcal{A}).$$

Here $\odot$ denotes the left Arens product on $\mathcal{A}^{**}$. The notion of left character amenability for Banach algebras was introduced by Monfared [M3]. Also, the concept of left ϕ-contractibility of $\mathcal{A}$ was introduced and studied in [HMT] by Hu, Monfared and Traynor. An algebra $\mathcal{A}$ is called *left ϕ-contractible* if for all Banach $\mathcal{A}$-bimodules X for which the right module action is given by $x \cdot a = \phi(a)x$ ($a \in \mathcal{A}$, $x \in X$), every continuous derivation from $\mathcal{A}$ into X is inner. Similarly, one defines right ϕ-contractibility for $\mathcal{A}$. Moreover, if $\mathcal{A}$ is both left and right ϕ-contractible it is called *ϕ-contractible*.

These notions of amenability and contractibility have been investigated for various classes of Banach algebras associated with locally compact groups. For example, in [ANN1] and [ANN2], the authors characterized left ϕ-amenability and left ϕ-contractibility of abstract Segal algebras.

A detailed plan of the paper is as follows. In Section 2, we write I_ϕ for the closed two-sided ideal $\ker\phi$ in $\mathcal{A}$ and assume that $\mathcal{A}I_\phi$ is dense in I_ϕ. Then we show that the approximate biflatness and ϕ-biflatness of $\mathcal{A}$ each imply the left ϕ-amenability of $\mathcal{A}$. A similar result is valid for generalized biprojective and ϕ-biprojective versions. We also introduce the notions of character biprojectivity and character biflatness of Banach algebras. Special cases regarding Banach algebras related to a locally compact group and a certain product of Banach algebras are also investigated there.

In Section 3, we examine some homological properties of abstract Segal algebras. In particular, for a Segal algebra $S^1(G)$ of a locally compact group G we characterize compactness of G in terms of approximate biprojectivity and character biprojectivity of $S^1(G)$. We also prove that if either G is an [IN]-group or $S^1(G)$ is symmetric, then $S^1(G)$ is character biflat if and only if G is amenable. Applying these results, we study generalized biflatness and biprojectivity of the ϕ_1-Lau product $S^1(G) \times_{\phi_1} S^1(G)$.

2. Some homological properties of Banach algebras. Let $\mathcal{A}$ be a Banach algebra. The left Arens product on $\mathcal{A}^{**}$ is denoted by "$\odot$". This product extends the given product on $\mathcal{A}$ to the entire $\mathcal{A}^{**}$ in such a way that for every $m \in \mathcal{A}^{**}$ the map $n \mapsto n \odot m$ is weak*-weak* continuous on $\mathcal{A}^{**}$. Moreover, for each $m, n \in \mathcal{A}^{**}$ and $\phi \in \Delta(\mathcal{A})$ we have $(m \odot n)(\phi) = m(\phi)n(\phi)$. Therefore, the character ϕ extends uniquely to some character $\widetilde{\phi}$ on $\mathcal{A}^{**}$. We notice that the Banach algebra $\mathcal{A} \widehat{\otimes} \mathcal{A}$ is a Banach $\mathcal{A}$-bimodule via the following module actions:

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca \quad (a, b, c \in \mathcal{A}).$$

The biflatness version of the result below on Banach algebras has been proved by Helemskii [H2]. The arguments used in the proof for the classical case can be applied for the following result. Hence, the proof is omitted.

PROPOSITION 2.1. *Let $\mathcal{A}$ be a Banach algebra with $\phi \in \Delta(\mathcal{A})$. Then the following statements are equivalent:*

(i) *$\mathcal{A}$ is ϕ-biflat.*
(ii) *There is an $\mathcal{A}$-bimodule morphism $\theta : (\mathcal{A} \widehat{\otimes} \mathcal{A})^* \to \mathcal{A}^*$ such that $\theta \circ \pi_{\mathcal{A}}^*(\phi) = \phi$.*

Following [SP], a Banach algebra $\mathcal{A}$ is called *ϕ-Johnson amenable* [*ϕ-Johnson contractible*] if there exists an element u in $(\mathcal{A} \widehat{\otimes} \mathcal{A})^{**}$ $[\mathcal{A} \widehat{\otimes} \mathcal{A}]$ for which

$$\widetilde{\phi}(\pi_{\mathcal{A}}^{**}(u)) = 1 \quad [\phi(\pi_{\mathcal{A}}(u)) = 1], \quad a \cdot u = u \cdot a \quad (a \in \mathcal{A}).$$

In [SP], the authors studied the relations between ϕ-Johnson amenability and ϕ-biflatness, and the relations between ϕ-Johnson contractibility and ϕ-biprojectivity.

REMARK 2.2. Let $\mathcal{A}$ be a Banach algebra and let $\phi \in \Delta(\mathcal{A})$. Following [HMT], $\mathcal{A}$ is called *ϕ-amenable* if it is both left and right ϕ-amenable.

Moreover, it was shown in [SP] that $\mathcal{A}$ is ϕ-Johnson amenable if and only if it is both left and right ϕ-amenable. Thus, ϕ-amenability and ϕ-Johnson amenability of $\mathcal{A}$ are equivalent. Similarly, ϕ-contractibility and ϕ-Johnson contractibility of $\mathcal{A}$ are equivalent.

In [NS], the authors showed that the left ϕ-contractibility of $\mathcal{A}$ is equivalent to the existence of a *left ϕ-mean* in $\mathcal{A}$, that is, an element m in $\mathcal{A}$ for which $\phi(m) = 1$ and $am = \phi(a)m$ for all $a \in \mathcal{A}$. Below, I_ϕ denotes the closed two-sided ideal $\ker\phi$ in $\mathcal{A}$. The following argument is inspired by the proof of the part (iv)⇒(i) of [SSS, Theorem 3.5].

PROPOSITION 2.3. *Let $\mathcal{A}$ be a Banach algebra with $\phi \in \Delta(\mathcal{A})$ such that $\mathcal{A}I_\phi$ $[I_\phi\mathcal{A}]$ is dense in I_ϕ. Then under each of the following conditions, $\mathcal{A}$ is left [right] ϕ-contractible:*

(i) *$\mathcal{A}$ is ϕ-biprojective.*
(ii) *$\mathcal{A}$ is approximately biprojective.*

Proof. We give the proof for (i). The argument for (ii) is similar. Suppose that ρ is an $\mathcal{A}$-bimodule morphism from $\mathcal{A}$ into $\mathcal{A}\,\widehat{\otimes}\,\mathcal{A}$ such that

$$\phi \circ \pi_{\mathcal{A}} \circ \rho = \phi.$$

Now, consider the continuous linear map $T : \mathcal{A}\,\widehat{\otimes}\,\mathcal{A} \to \mathcal{A}$ defined by

$$T(a \otimes b) = \phi(b)a \qquad (a, b \in \mathcal{A}).$$

Thus, for any $a \in \mathcal{A}$ and $w \in \mathcal{A}\,\widehat{\otimes}\,\mathcal{A}$ we have

$$T(a \cdot w) = aT(w), \qquad T(w \cdot a) = \phi(a)T(w).$$

Therefore, the map $\widetilde{\rho} : \mathcal{A} \to \mathcal{A}$ defined by $\widetilde{\rho} := T \circ \rho$ is a left $\mathcal{A}$-module morphism and $\widetilde{\rho}(ab) = \widetilde{\rho}(a)\phi(b)$ for all $a, b \in \mathcal{A}$. Hence, $\widetilde{\rho}(ab) = 0$ for all $a \in \mathcal{A}$ and $b \in I_\phi$. As $\mathcal{A}I_\phi$ is dense in I_ϕ, we conclude that $\widetilde{\rho} = 0$ on I_ϕ. Given $b_0 \in \mathcal{A}$ with $\phi(b_0) = 1$, we see that $\phi \circ \pi_{\mathcal{A}} \circ \rho(b_0) = 1$. Now, consider $a_0 := \widetilde{\rho}(b_0)$ in $\mathcal{A}$. Since $\phi\circ T = \phi\circ\pi_{\mathcal{A}}$, it follows that $\phi(a_0) = \phi\circ\pi_{\mathcal{A}}\circ\rho(b_0) = 1$. Moreover, for each $a \in \mathcal{A}$ we have $ab_0 - \phi(a)b_0 \in I_\phi$ which implies that

$$\begin{aligned} aa_0 - \phi(a)a_0 &= a\widetilde{\rho}(b_0) - \phi(a)\widetilde{\rho}(b_0) = \widetilde{\rho}(ab_0) - \phi(a)\widetilde{\rho}(b_0) \\ &= \widetilde{\rho}(ab_0 - \phi(a)b_0) = 0. \end{aligned}$$

This shows that a_0 is a left ϕ-mean in $\mathcal{A}$. Hence, $\mathcal{A}$ is left ϕ-contractible. ∎

PROPOSITION 2.4. *Let $\mathcal{A}$ be a Banach algebra with $\phi \in \Delta(\mathcal{A})$ such that $\mathcal{A}I_\phi$ $[I_\phi\mathcal{A}]$ is dense in I_ϕ. Then under each of the following conditions, $\mathcal{A}$ is left [right] ϕ-amenable:*

(i) *$\mathcal{A}$ is ϕ-biflat.*
(ii) *$\mathcal{A}$ is approximately biflat.*

Proof. We only prove (ii). The assertion (i) can be proved by a simple modification of the same argument. Suppose that there is a net $\theta_\gamma : (\mathcal{A} \widehat{\otimes} \mathcal{A})^* \to \mathcal{A}^*$ of bounded $\mathcal{A}$-bimodule morphisms such that

$$\mathrm{w}^*\text{-}\lim_\gamma \theta_\gamma \circ \pi_{\mathcal{A}}^*(f) = f \qquad (f \in \mathcal{A}^*).$$

Given $b_0 \in \mathcal{A}$ with $\phi(b_0) = 1$, there is a γ_0 such that

$$\langle \theta_{\gamma_0} \circ \pi_{\mathcal{A}}^*(\phi), b_0 \rangle \neq 0.$$

Therefore, the map $\widetilde{\theta} : \mathcal{A}^{**} \to \mathcal{A}^{**}$ defined by $\widetilde{\theta} := T^{**} \circ \theta_{\gamma_0}^*$ is a left $\mathcal{A}$-module morphism, where T is defined as in the proof of Proposition 2.3. Moreover, $\widetilde{\theta}(ab) = \widetilde{\theta}(a)\phi(b)$ for all $a, b \in \mathcal{A}$. It follows that $\widetilde{\theta} = 0$ on I_ϕ. Define the functional m_0 in $\mathcal{A}^{**}$ by

$$m_0 := \widetilde{\theta}(b_0).$$

Hence, the equality $\pi_{\mathcal{A}}^*(\phi) = T^*(\phi)$ implies that

$$m_0(\phi) = \langle \theta \circ \pi_{\mathcal{A}}^*(\phi), b_0 \rangle \neq 0,$$

and for each $a \in \mathcal{A}$ we have

$$\begin{aligned} a \odot m_0 - \phi(a) m_0 &= a \odot \widetilde{\theta}(b_0) - \phi(a)\widetilde{\theta}(b_0) = \widetilde{\theta}(ab_0) - \phi(a)\widetilde{\theta}(b_0) \\ &= \widetilde{\theta}(ab_0 - \phi(a)b_0) = 0. \end{aligned}$$

Therefore, $m_0/m_0(\phi)$ is a left ϕ-mean in $\mathcal{A}^{**}$, and consequently $\mathcal{A}$ is left ϕ-amenable. ■

DEFINITION 2.5. Let $\mathcal{A}$ be a Banach algebra. Then we say that $\mathcal{A}$ is *character biflat* if it is ϕ-biflat for all $\phi \in \Delta(\mathcal{A})$. Similarly, $\mathcal{A}$ is *character biprojective* if it is ϕ-biprojective for all $\phi \in \Delta(\mathcal{A})$.

Let $1 < p < \infty$, and $A_p(G)$ be the generalized Fourier algebra of a locally compact group G. Then $\Delta(A_p(G)) = \{\phi_x : x \in G\}$, where $\phi_x(f) = f(x)$ for all $f \in A_p(G)$. By [M1, Lemma 3.1], $A_p(G)$ is ϕ_x-amenable for all $x \in G$. Therefore, it is character biflat. Furthermore, it follows from [M3, Corollary 2.4] that $A_p(G)$ is character amenable if and only if G is amenable. This shows that character biflatness is weaker than character amenability.

Suppose that G is a locally compact group and let $L^1(G)$ be its group algebra. Suppose that X is a *left introverted* subspace of $L^1(G)^* = L^\infty(G)$, that is, a right $L^1(G)$-submodule of $L^\infty(G)$ such that $F\varphi \in X$ for all $F \in X^*$ and $\varphi \in X$, where $(F\varphi)(f) = F(\varphi f)$ and $(\varphi f)(h) = \varphi(f * h)$ for all $f, h \in L^1(G)$, $F \in X^*$. In this case, the left Arens multiplication is well defined on X^*, and X^* is a Banach algebra with respect to this multiplication. Examples of closed left introverted subspaces of $L^\infty(G)$ include the space $\mathrm{LUC}(G)$ of all left uniformly continuous functions on G, and the space $L_0^\infty(G)$ of all $f \in L^\infty(G)$ which vanish at infinity.

Corollary 2.6. *Let G be a locally compact group and let $\mathcal{A}$ be either of the Banach algebras $L^\infty(G)^*$ or $\mathrm{LUC}(G)^*$. Then the following statements are equivalent:*

(i) *$\mathcal{A}$ is character biflat.*
(ii) *$\mathcal{A}$ is approximately biflat.*
(iii) *G is finite.*

Proof. Suppose that either (i) or (ii) holds. Since $\mathcal{A}$ has a right identity, Proposition 2.4 implies that $\mathcal{A}$ is right ϕ-amenable for all $\phi \in \Delta(\mathcal{A})$. The rest of the proof is similar to that of [HMT, Theorem 3.10] and [NN, Proposition 3.5]. ■

Let us first recall that a locally compact group G is *amenable* if there is a *left invariant mean* on $L^\infty(G)$, that is, a functional $m \in L^\infty(G)^*$ for which $\|m\| = m(1) = 1$ and $m(L_x\varphi) = m(\varphi)$ for all $\varphi \in L^\infty(G)$ and $x \in G$, where $L_x\varphi(y) = \varphi(x^{-1}y)$ for all $y \in G$.

Remark 2.7. The same arguments used in the proofs of Corollary 2.6 and [NN, Proposition 3.6] show that $L_0^\infty(G)^*$ is character biflat if and only if $L_0^\infty(G)^*$ is approximately biflat if and only if G is discrete and amenable.

Before giving the following example we note that a Banach algebra $\mathcal{A}$ is *contractible* if it has a *diagonal*, that is, an element $u \in \mathcal{A} \widehat{\otimes} \mathcal{A}$ for which $a \cdot u = u \cdot a$ and $\pi_{\mathcal{A}}(u)a = a$ for all $a \in \mathcal{A}$. Let us point out that $\mathcal{A}$ is contractible if and only if it is unital and biprojective.

Example 2.8. Let $\mathcal{A}$ and $\mathcal{B}$ be two Banach algebras and $\theta \in \Delta(\mathcal{B})$. The *$\theta$-Lau product* $\mathcal{A} \times_\theta \mathcal{B}$ is the space $\mathcal{A} \times \mathcal{B}$ equipped with the norm $\|(a,b)\| = \|a\| + \|b\|$ and the product

$$(a,b)(a',b') = (aa' + \theta(b)a' + \theta(b')a, bb') \quad (a,a' \in \mathcal{A},\ b,b' \in \mathcal{B}).$$

Then $\mathcal{A} \times_\theta \mathcal{B}$ is a Banach algebra and $\mathcal{A}$ is a closed two-sided ideal of $\mathcal{A} \times_\theta \mathcal{B}$ and $\mathcal{A} \times_\theta \mathcal{B}/\mathcal{A} \cong \mathcal{B}$. Moreover, it was shown in [M2, Proposition 2.4] that

$$\Delta(\mathcal{A} \times_\theta \mathcal{B}) = \Delta(\mathcal{A}) \times \{\theta\} \cup \{0\} \times \Delta(\mathcal{B}).$$

Suppose that $\mathcal{B}$ has an approximate identity.

(i) We show that $\mathcal{A} \times_\theta \mathcal{B}$ is approximately biprojective if and only if $\mathcal{A}$ is contractible and $\mathcal{B}$ is approximately biprojective. In fact, from [NJ, Corollary 3.2] it suffices to show that if $\mathcal{A} \times_\theta \mathcal{B}$ is approximately biprojective, then $\mathcal{A}$ is unital. To see this, we notice that $\mathcal{A} \times_\theta \mathcal{B}$ has an approximate identity by [M2, Proposition 2.3], and therefore it follows from Proposition 2.3 above that $\mathcal{A} \times_\theta \mathcal{B}$ is both left and right $(0,\theta)$-contractible. Thus, $\mathcal{A}$ is unital by an argument similar to that of [M3, Proposition 2.8].

(ii) We show that $\mathcal{A} \times_\theta \mathcal{B}$ is character biflat if and only if $\mathcal{A}$ is character amenable and $\mathcal{B}$ is character biflat. By assumption and [M2, Proposition 2.3],

$\mathcal{A} \times_\theta \mathcal{B}$ has an approximate identity. Therefore, if $\mathcal{A} \times_\theta \mathcal{B}$ is character biflat, then $\mathcal{A} \times_\theta \mathcal{B}$ is Φ-amenable for all $\Phi \in \Delta(\mathcal{A} \times_\theta \mathcal{B})$ by Proposition 2.4. Thus, $\mathcal{A}$ is character amenable and $\mathcal{B}$ is ψ-amenable for all $\psi \in \Delta(\mathcal{B})$ by [M3, Proposition 2.8], and therefore $\mathcal{B}$ is character biflat. The converse follows again from Proposition 2.4 and [M3, Proposition 2.8].

3. Results on Segal algebras. Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a Banach algebra. Then a Banach algebra $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is an *abstract Segal algebra* with respect to $\mathcal{A}$ if $\mathcal{B}$ is a dense left ideal in $\mathcal{A}$ and there exist constants $M > 0$ and $C > 0$ such that for each $a \in \mathcal{A}$ and $b \in \mathcal{B}$ the following hold:

(i) $\|b\|_{\mathcal{A}} \leq M\|b\|_{\mathcal{B}}$.
(ii) $\|ab\|_{\mathcal{B}} \leq C\|a\|_{\mathcal{A}}\|b\|_{\mathcal{B}}$.

We recall from [B, Theorem 2.1] that the restriction map $\phi \mapsto \phi|_{\mathcal{B}}$ from $\Delta(\mathcal{A})$ onto $\Delta(\mathcal{B})$ is a homeomorphism. We further say that $\mathcal{B}$ is *symmetric* if it is also a two-sided dense ideal in $\mathcal{A}$ and for each $a, b \in \mathcal{B}$,

$$\|ba\|_{\mathcal{B}} \leq C\|a\|_{\mathcal{A}}\|b\|_{\mathcal{B}}.$$

We recall from [JAZ] that $\mathcal{A}$ is *ϕ-inner amenable* if there exists a functional $m \in \mathcal{A}^{**}$ such that $m(\phi) = 1$ and $a \odot m = m \odot a$ for all $a \in \mathcal{A}$.

Definition 3.1. Let $\mathcal{A}$ be a Banach algebra and let $\phi \in \Delta(\mathcal{A})$. Then we say that $\mathcal{A}$ is *ϕ-inner contractible* if there is a central element m in $\mathcal{A}$ such that $\phi(m) = 1$.

Lemma 3.2. *Let $\mathcal{A}$ be a Banach algebra with $\phi \in \Delta(\mathcal{A})$ and let $\mathcal{B}$ be an abstract Segal algebra with respect to $\mathcal{A}$. Suppose that $\mathcal{A}$ is ϕ-biflat [ϕ-biprojective]. Then under each of the following statements $\mathcal{B}$ is $\phi|_{\mathcal{B}}$-biflat [$\phi|_{\mathcal{B}}$-biprojective]:*

(i) *$\mathcal{B}$ is $\phi|_{\mathcal{B}}$-inner contractible.*
(ii) *$\mathcal{B}$ is a symmetric abstract Segal algebra.*

Proof. We only prove (i). The proof of (ii) is similar. Suppose that b_0 is a central element in $\mathcal{B}$ with $\phi(b_0) = 1$, and consider the map $S : \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{B}\widehat{\otimes}\mathcal{B}$ defined by

$$S(a \otimes a') = ab_0 \otimes b_0a'.$$

Since b_0 is central in $\mathcal{B}$, the map S is a $\mathcal{B}$-bimodule morphism. Now, let $\theta_{\mathcal{A}}$ from $(\mathcal{A} \widehat{\otimes} \mathcal{A})^*$ into $\mathcal{A}^*$ be an $\mathcal{A}$-bimodule morphism such that

$$\theta_{\mathcal{A}} \circ \pi^*_{\mathcal{A}}(\phi) = \phi.$$

It is clear that the restriction map $\mathcal{P} : \mathcal{A}^* \to \mathcal{B}^*$ is a $\mathcal{B}$-bimodule morphism. Thus the map $\theta_{\mathcal{B}} : (\mathcal{B}\widehat{\otimes}\mathcal{B})^* \to \mathcal{B}^*$ defined by $\theta_{\mathcal{B}} = \mathcal{P} \circ \theta_{\mathcal{A}} \circ S^*$ is a $\mathcal{B}$-bimodule morphism. On the other hand, since $\phi(b_0) = 1$, it follows that

$$S^* \circ \pi^*_{\mathcal{B}}(\phi|_{\mathcal{B}}) = S^*(\phi|_{\mathcal{B}} \otimes \phi|_{\mathcal{B}}) = \phi \otimes \phi = \pi^*_{\mathcal{A}}(\phi).$$

Thus

$$\theta_{\mathcal{B}} \circ \pi^*_{\mathcal{B}}(\phi|_{\mathcal{B}}) = \mathcal{P} \circ \theta_{\mathcal{A}} \circ S^* \circ \pi^*_{\mathcal{B}}(\phi|_{\mathcal{B}}) = \mathcal{P} \circ \theta_{\mathcal{A}} \circ \pi^*_{\mathcal{A}}(\phi) = \mathcal{P}(\phi) = \phi|_{\mathcal{B}},$$

and this shows that $\mathcal{B}$ is $\phi|_{\mathcal{B}}$-biflat. ∎

The following result is obtained by similar arguments to those given in [ANN1, Propositions 2.3 and 2.5] and so the proof will be omitted.

PROPOSITION 3.3. *Let $\mathcal{A}$ be a Banach algebra with $\phi \in \Delta(\mathcal{A})$ and let $\mathcal{B}$ be a symmetric abstract Segal algebra with respect to $\mathcal{A}$. Then $\mathcal{A}$ is ϕ-amenable [ϕ-contractible] if and only if $\mathcal{B}$ is $\phi|_{\mathcal{B}}$-amenable [$\phi|_{\mathcal{B}}$ contractible].*

COROLLARY 3.4. *Let $\mathcal{A}$ be a Banach algebra with $\phi \in \Delta(\mathcal{A})$ and let $\mathcal{B}$ be a symmetric abstract Segal algebra with respect to $\mathcal{A}$. Suppose that $\mathcal{B}$ is $\phi|_{\mathcal{B}}$-inner amenable [$\phi|_{\mathcal{B}}$-inner contractible]. Then the following statements are equivalent:*

(i) *$\mathcal{A}$ is ϕ-biflat [ϕ-biprojective].*
(ii) *$\mathcal{B}$ is $\phi|_{\mathcal{B}}$-biflat [$\phi|_{\mathcal{B}}$-biprojective].*
(iii) *$\mathcal{B}$ is $\phi|_{\mathcal{B}}$-amenable [$\phi|_{\mathcal{B}}$-contractible].*
(iv) *$\mathcal{A}$ is ϕ-amenable [ϕ-contractible].*

Proof. That (i) implies (ii) follows from Lemma 3.2, and that (ii) implies (iii) follows from Remark 2.2 and [SP, Proposition 3.3]. Moreover, the implication (iii)⇒(iv) is given by Proposition 3.3. Finally, (iv)⇒(i) can be deduced from Remark 2.2 and [SP, Lemma 3.1]. ∎

LEMMA 3.5. *Let $\mathcal{A}$ be a ϕ-inner contractible Banach algebra for some $\phi \in \Delta(\mathcal{A})$. If $\mathcal{A}$ is either approximately biprojective or ϕ-biprojective, then it is ϕ-contractible.*

Proof. We only prove the approximately biprojective case. The other case is similar. Suppose that (ρ_γ) is a net of $\mathcal{A}$-bimodule morphisms from $\mathcal{A}$ into $\mathcal{A}\widehat{\otimes}\mathcal{A}$ such that $\pi_{\mathcal{A}}\circ\rho_\gamma(a) \to a$ for all $a \in \mathcal{A}$. Suppose that b_0 is a central element in $\mathcal{A}$ with $\phi(b_0) = 1$. Then there is a γ_0 such that $\phi\circ\pi_{\mathcal{A}}\circ\rho_{\gamma_0}(b_0) \neq 0$. Let the map $T : \mathcal{A} \widehat{\otimes} \mathcal{A} \to \mathcal{A}$ be as in the proof of Proposition 2.3 and let $\widetilde{\rho} : \mathcal{A} \to \mathcal{A}$ be the map $\widetilde{\rho} := T \circ \rho_{\gamma_0}$. If we set $a_0 := \widetilde{\rho}(b_0)$, then $\phi(a_0) \neq 0$ and for each $a \in \mathcal{A}$ we have

$$aa_0 = a\widetilde{\rho}(b_0) = \widetilde{\rho}(ab_0) = \widetilde{\rho}(b_0a) = \widetilde{\rho}(b_0)\phi(a) = \phi(a)a_0.$$

This shows that $\mathcal{A}$ is left ϕ-contractible. Similarly, we can show that it is also right ϕ-contractible. Hence, $\mathcal{A}$ is ϕ-contractible. ∎

Let G be a locally compact group with left Haar measure λ_G and let $L^1(G)$ be the group algebra of G endowed with the norm $\|\cdot\|_1$ and the convolution product $*$. Let $\widehat{G}$ be the set of all continuous homomorphisms from G into the circle group $\mathbb{T}$. Then $\Delta(L^1(G)) = \{\phi_\alpha : \alpha \in \widehat{G}\}$, where $\phi_\alpha(f) = \int_G \overline{\alpha(x)} f(x)\, d\lambda_G(x)$ for all $f \in L^1(G)$ (see [HR, Theorem 23.7] for details). It is well known that G is amenable if and only if $L^1(G)$ is left ϕ_1-amenable.

A dense linear subspace $S^1(G)$ of $L^1(G)$, which is a Banach space under some norm $\|\cdot\|_{S^1}$, is said to be a *Segal algebra* on G if it satisfies the following conditions:

(i) there exists $M > 0$ such that $\|f\|_1 \le M\|f\|_{S^1}$ for all $f \in S^1(G)$;
(ii) $S^1(G)$ is left translation invariant, $\|L_x f\|_{S^1} = \|f\|_{S^1}$ and the map $x \mapsto L_x f$ from G into $S^1(G)$ is continuous for all $f \in S^1(G)$ and $x \in G$, where $L_x f \in S^1(G)$ is defined by $(L_x f)(y) = f(x^{-1}y)$ for all $y \in G$.

Note that a Segal algebra $S^1(G)$ is an abstract Segal algebra with respect to $L^1(G)$ and it is well known that $S^1(G)$ has a left approximate identity which is bounded in L^1-norm. For each $\alpha \in \widehat{G}$, regarding ϕ_α as an element of $\Delta(S^1(G))$ we have $\Delta(S^1(G)) = \{\phi_\alpha : \alpha \in \widehat{G}\}$.

A Segal algebra $S^1(G)$ on G is *symmetric* if it is right translation invariant, $\|R_x f\|_{S^1} = \|f\|_{S^1}$ and the map $x \mapsto R_x f$ from G into $S^1(G)$ is continuous for all $f \in S^1(G)$ and $x \in G$, where $R_x f \in S^1(G)$ is defined by $(R_x f)(y) = f(yx)$ for all $y \in G$. Note that every symmetric Segal algebra is a two-sided ideal in $L^1(G)$ and has an approximate identity which is bounded in L^1-norm (we refer to [R] and [RS]).

Recall that a locally compact group G is called an [IN]-*group* if G contains a compact invariant neighborhood of the identity. Thus all compact groups are [IN]-groups.

REMARK 3.6. Let $S^1(G)$ be a Segal algebra on a locally compact group G. It follows from [MO] that G is an [IN]-group if and only if $L^1(G)$ is ϕ_1-inner contractible. On the one hand, for each $\alpha \in \widehat{G}$ and $f, g \in L^1(G)$ the equality $f * \alpha g = \alpha(\overline{\alpha} f * g)$ implies that $L^1(G)$ is ϕ_1-inner contractible if and only if it is ϕ_α-inner contractible. On the other hand, the center $ZS^1(G)$ of $S^1(G)$ is dense in the center $ZL^1(G)$ of $L^1(G)$ (see [KR, Theorem 2]). It follows that G is an [IN]-group if and only if $S^1(G)$ is ϕ_α-inner contractible for some $\alpha \in \widehat{G}$.

We recall from [GZ] that a Banach algebra $\mathcal{A}$ is *pseudo-contractible* if it has a *central approximate diagonal*, that is, a net (u_γ) in $\mathcal{A} \widehat{\otimes} \mathcal{A}$ such that $a \cdot u_\gamma = u_\gamma \cdot a$ for all γ and $\pi_{\mathcal{A}}(u_\gamma)a \to a$ for all $a \in \mathcal{A}$. We also recall from [GZ, Proposition 3.8] that $\mathcal{A}$ is pseudo-contractible if and only if it is approximately biprojective and has a central approximate identity. It was shown in [CGZ, Theorem 5.3] that $S^1(G)$ is pseudo-contractible if and only if G is compact. This is also contained in [SSS, Theorem 3.5]. In the following theorem we show that compactness of G is equivalent to some homological properties of $S^1(G)$.

Theorem 3.7. *Let G be a locally compact group and $S^1(G)$ be a Segal algebra on G. Then the following statements are equivalent:*

(i) *G is compact.*
(ii) *$S^1(G)$ is approximately biprojective.*
(iii) *$S^1(G)$ is ϕ_1-contractible.*
(iv) *$S^1(G)$ is ϕ_1-biprojective.*

Proof. Suppose that G is compact. Then $S^1(G)$ is pseudo-contractible by [CGZ, Theorem 5.3]. Thus, (i) implies (ii) by the fact that any pseudo-contractible Banach algebra is approximately biprojective. Similarly, if (i) holds, then $S^1(G)$ is approximately biprojective by the preceding argument. Moreover, compactness of G implies that $S^1(G)$ is ϕ_1-inner contractible by Remark 3.6. Therefore, (iii) holds by Lemma 3.5. That (iii) implies (iv) is trivial. Finally, if either (iv) or (ii) holds, then since $S^1(G)$ has a left approximate identity, it follows from Proposition 2.3 that $S^1(G)$ is left ϕ_1-contractible, that is, there is an element $f_0 \in S^1(G)$ such that $\phi_1(f_0) = 1$ and $g * f_0 = \phi_1(g)f_0$ for all $g \in S^1(G)$. As $S^1(G)$ is dense in $L^1(G)$, we conclude that f_0 is a left ϕ_1-mean in $L^1(G)$. This implies that f_0 is almost everywhere equal to a non-zero constant, and therefore G is compact (see for example [RU, Exercise 1.1.7]). ∎

Corollary 3.8. *Let G be a locally compact group and $S^1(G)$ be a Segal algebra on G. Then G is compact if and only if $S^1(G)$ is character biprojective.*

Proof. Suppose that G is compact. Then $S^1(G)$ is approximately biprojective by Theorem 3.7. Furthermore, Remark 3.6 implies that $S^1(G)$ is ϕ_α-inner contractible for all $\alpha \in \widehat{G}$. It follows from Lemma 3.5 that $S^1(G)$ is character biprojective. The converse is deduced from Theorem 3.7. ∎

Combining Proposition 2.4 and [ANN1, Corollary 3.4], we then have the following result.

Corollary 3.9. *Let G be a locally compact group and $S^1(G)$ be a Segal algebra on G. If $S^1(G)$ is either approximately biflat or ϕ_1-biflat, then G is amenable.*

In general, we do not know whether amenability of G implies character biflatness of $S^1(G)$. However in the case where G is an [IN]-group or $S^1(G)$ is symmetric, we have the following characterization for amenability of G in terms of character biflatness of $S^1(G)$.

Theorem 3.10. *Let G be a locally compact group and $S^1(G)$ be a Segal algebra on G. If either G is an* [IN]*-group or $S^1(G)$ is symmetric, then the following statements are equivalent:*

(i) *G is amenable.*
(ii) *$S^1(G)$ is character biflat.*
(iii) *$S^1(G)$ is ϕ_1-biflat.*
(iv) *$S^1(G)$ is ϕ_1-amenable.*

Proof. Suppose that G is amenable. Then $L^1(G)$ is amenable and consequently it is biflat by [D, Theorem 2.9.65]. Therefore, $S^1(G)$ is character biflat by Lemma 3.2 and Remark 3.6. That (ii) implies (iii) is trivial. When G is an [IN]-group, the implication (iii)⇒(iv) follows from Lemma 3.5, and if $S^1(G)$ is symmetric, this follows from Proposition 2.4 together with the fact that $S^1(G)$ has an approximate identity. The implication (iv)⇒(i) is a direct consequence of Corollary 3.9. ∎

Proposition 3.11. *Let G be a locally compact group and $S^1(G)$ be a Segal algebra on G. Then the following statements are equivalent:*

(i) *$S^1(G) = L^1(G)$ and G is amenable.*
(ii) *$S^1(G) \times_{\phi_1} S^1(G)$ is approximately biflat.*
(iii) *$S^1(G) \times_{\phi_1} S^1(G)$ is character biflat.*
(iv) *$S^1(G) \times_{\phi_1} S^1(G)$ is $(0, \phi_1)$-biflat.*

Proof. (i)⇒(ii). By assumption and Johnson's theorem $S^1(G)$ is amenable and so $S^1(G)\times_{\phi_1}S^1(G)$ is amenable. Hence, $S^1(G)\times_{\phi_1}S^1(G)$ is approximately biflat by [D, Theorem 2.9.65]. Similarly, the implication (i)⇒(iii) holds. Obviously, (iii) implies (iv).

(iv)⇒(i). First note that $S^1(G)$ has a left approximate identity and so $S^1(G) \times_{\phi_1} S^1(G)$ has also a left approximate identity by [M2, Proposition 2.3]. Thus, $S^1(G) \times_{\phi_1} S^1(G)$ is left $(0, \phi_1)$-amenable by Proposition 2.4. Therefore, $S^1(G)$ is left ϕ_1-amenable and it has a bounded right approximate identity by [M3, Proposition 2.8(ii)]. These show that G is amenable and $S^1(G) = L^1(G)$ by [ANN1, Corollary 3.4] and [B, Theorem 1.2], respectively. The implication (ii)⇒(i) can be proved similarly. ∎

The following result can be obtained if we modify the proof of Proposition 3.11.

Proposition 3.12. *Let G be a locally compact group and $S^1(G)$ be a Segal algebra on G. Then the following statements are equivalent:*

(i) *G is finite.*
(ii) *$S^1(G) \times_{\phi_1} S^1(G)$ is approximately biprojective.*
(iii) *$S^1(G) \times_{\phi_1} S^1(G)$ is character biprojective.*
(iv) *$S^1(G) \times_{\phi_1} S^1(G)$ is $(0, \phi_1)$-biprojective.*

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