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A NOTE ON THE EXPONENTIAL DIOPHANTINE EQUATION $(4m^2+1)^x + (5m^2-1)^y = (3m)^z$

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Abstract. Let m be a positive integer. Using an upper bound for the solutions of generalized Ramanujan–Nagell equations given by Y. Bugeaud and T. N. Shorey, we prove that if $3 \nmid m$, then the equation $(4m^2+1)^x + (5m^2-1)^y = (3m)^z$ has only the positive integer solution $(x,y,z) = (1,1,2)$.

1. Introduction. Let $\mathbb{Z}, \mathbb{N}$ be the sets of all integers and positive integers respectively. Let a, b, c be fixed coprime positive integers with $\min\{a,b,c\} > 1$. Recently, many papers investigated the exponential Diophantine equation

$$a^x + b^y = c^z, \quad x, y, z \in \mathbb{N}. \tag{1.1}$$

(see [FM, HTY, L, M1, M2, M3, M4, MT]). In this connection, N. Terai [T] proved that if a, b, c satisfy

$$a = 4m^2+1, \quad b = 5m^2-1, \quad c = 3m, \quad m \in \mathbb{N}, \tag{1.2}$$

then (1.1) has only the solution $(x,y,z) = (1,1,2)$ under some conditions. The proof of this result is based on elementary methods and Baker's method.

In the present paper, using an upper bound for the solutions of generalized Ramanujan–Nagell equations given by Y. Bugeaud and T. N. Shorey [BS], we prove a general result:

THEOREM 1.1. *Let a, b, c be positive integers satisfying* (1.2)*. If $3 \nmid m$, then* (1.1) *has only the solution $(x,y,z) = (1,1,2)$.*

2. Preliminaries. In this section, we assume that a, b, c are positive integer satisfying (1.2). Then, (1.1) can be written as

$$(4m^2+1)^x + (5m^2-1)^y = (3m)^z, \quad x, y, z \in \mathbb{N}. \tag{2.1}$$

We further assume that (x,y,z) is a solution of (2.1) with $(x,y,z) \neq (1,1,2)$.

LEMMA 2.1. *If $m > 1$ and $3 \nmid m$, then $2 \nmid m$, $2 \nmid x$, $y = 1$ and $2 \mid z$.*

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Proof. Since $m > 1$ and $\min\{x, y\} \geq 1$, we see from (2.1) that $z \geq 2$ and $1 + (-1)^y \equiv (4m^2 + 1)^x + (5m^2 - 1)^y \equiv (3m)^z \equiv 0 \pmod{m^2}$, whence

$$2 \nmid y. \tag{2.2}$$

Since $(x, y, z) \neq (1, 1, 2)$, we have $z \geq 3$. Hence, by (2.1) and (2.2), we get $(4x + 5y)m^2 \equiv (4m^2 + 1)^x + (5m^2 - 1)^y \equiv (3m)^z \equiv 0 \pmod{m^3}$, so

$$4x + 5y \equiv 0 \pmod{m}. \tag{2.3}$$

If $2 \mid m$, then from (2.3) we get $2 \mid y$, which contradicts (2.2). So we have

$$2 \nmid m. \tag{2.4}$$

If $2 \mid x$, then from (2.1), (2.2) and (2.4) we get

$$1 = \left(\frac{-(5m^2 - 1)}{3m}\right), \tag{2.5}$$

where $\left(\frac{*}{*}\right)$ is the Jacobi symbol. Since $3 \nmid m$, we have $m^2 \equiv 1 \pmod{3}$, $-(5m^2 - 1) \equiv -1 \pmod{3}$ and $\left(\frac{-(5m^2-1)}{3}\right) = -1$. Hence, by (2.5), we get

$$1 = \left(\frac{-(5m^2 - 1)}{3}\right)\left(\frac{-(5m^2 - 1)}{m}\right) = -\left(\frac{-(5m^2 - 1)}{m}\right) = -\left(\frac{1}{m}\right) = -1, \tag{2.6}$$

a contradiction. This implies that

$$2 \nmid x. \tag{2.7}$$

If $2 \nmid z$, then from (2.1), (2.2) and (2.7) we get

$$1 = \left(\frac{3m(5m^2 - 1)}{4m^2 + 1}\right) = \left(\frac{3}{4m^2 + 1}\right)\left(\frac{m}{4m^2 + 1}\right)\left(\frac{5m^2 - 1}{4m^2 + 1}\right). \tag{2.8}$$

Notice that $4m^2+1 \equiv 1 \pmod{4}$, $4m^2+1 \equiv -1 \pmod{3}$ and $5m^2-1 \equiv (3m)^2 \pmod{4m^2 + 1}$. We have

$$\left(\frac{3}{4m^2 + 1}\right) = -1, \quad \left(\frac{m}{4m^2 + 1}\right) = \left(\frac{5m^2 - 1}{4m^2 + 1}\right) = 1. \tag{2.9}$$

Therefore, (2.8) is false. Consequently,

$$2 \mid z. \tag{2.10}$$

Finally, since $4m^2 + 1 \equiv 5 \pmod{8}$, $5m^2 - 1 \equiv 4 \pmod{8}$ and $(3m)^2 \equiv 1 \pmod{8}$, we deduce from (2.1), (2.7) and (2.10) that $5 + 4^y \equiv 5^x + 4^y \equiv (4m^2 + 1)^x + (5m^2 - 1)^y \equiv (3m)^z \equiv ((3m)^2)^{z/2} \equiv 1 \pmod{8}$, whence

$$y = 1. \tag{2.11}$$

Thus, by (2.4), (2.7), (2.10) and (2.11), the lemma is proved. ∎

LEMMA 2.2. *If* $m \geq 7$ *and* $3 \nmid m$, *then*

$$x \geq \frac{1}{8}(7m^4 - 45m^2 - 10). \tag{2.12}$$

Proof. By Lemma 2.1, we have $2 \nmid m$, $x \geq 3$, $y = 1$ and $2 \mid z$. Since $m \geq 7$, by (2.1), we get $(3m)^z > (4m^2+1)^x \geq (4m^2+1)^3 > 64m^6 > (3m)^4$. Thus $z \geq 6$. Further, since $y = 1$ and $z \geq 6$, we see from (2.1) that

$$(4x+5) + \binom{x}{2} 16m^2 \equiv 0 \pmod{m^4}. \tag{2.13}$$

By (2.13), we get $4x + 5 \equiv 0 \pmod{m^2}$ and

$$4x + 5 = m^2 r, \qquad r \in \mathbb{N}. \tag{2.14}$$

Substituting (2.14) into (2.13), we have

$$2r + 4x(4x-4) \equiv 2r + 45 \equiv 0 \pmod{m^2}. \tag{2.15}$$

Since $2 \nmid m$, we find from (2.15) that

$$r \equiv \frac{m^2 - 45}{2} \pmod{m^2}. \tag{2.16}$$

Further, since r and $\frac{m^2-45}{2}$ are positive integers with $\frac{m^2-45}{2} < m^2$, we have

$$r = \frac{m^2-45}{2} + m^2 s, \qquad s \in \mathbb{N},\ s \geq 0, \tag{2.17}$$

by (2.16). Substituting (2.17) into (2.14), we get

$$x = \frac{1}{4}(m^2 r - 5) = \frac{1}{8}(m^4(2s+1) - 45m^2 - 10). \tag{2.18}$$

Furthermore, since $m^4 \equiv m^2 \equiv 1 \pmod 8$ and x is a positive integer, we see from (2.18) that $m^4(2s+1) - 45m^2 - 10 \equiv (2s+1) - 55 \equiv (2s+1) - 7 \equiv 0 \pmod 8$. This implies that $2s+1 \geq 7$ since $2s+1$ is a positive integer. Thus, by (2.18), x satisfies (2.12). ∎

For any nonnegative integer s, let F_s and L_s denote the sth Fibonacci number and sth Lucas number respectively. Then

$$F_0 = 0, \quad F_1 = 1, \quad F_{s+2} = F_{s+1} + F_s, \quad s \geq 0. \tag{2.19}$$

LEMMA 2.3 ([BS, Theorem 2]). *Let* D_1, D_2, k *be positive integers such that* $\min\{D_1, D_2, k\} > 1$ *and* $\gcd(D_1, D_2) = \gcd(k, 2D_1D_2) = 1$. *If* (X, n) *is a solution of the equation*

$$D_1 X^2 + D_2 = k^n, \qquad X, n \in \mathbb{N}, \tag{2.20}$$

then

$$n < \frac{4}{\pi}\sqrt{D_1 D_2}\log(2e\sqrt{D_1 D_2}), \tag{2.21}$$

except possibly for the following cases:

(i) $D_1 f^2 + D_2 = k^g$ *and* $3D_1 f^2 - D_2 = \pm 1$, *where* f, g *are positive integers.*

(ii) $(D_1, D_2, k) = (F_{t-2\varepsilon}, L_{t+\varepsilon}, F_t)$, *where* t *is a positive integer with* $t \geq 2$, $\varepsilon \in \{\pm 1\}$.

LEMMA 2.4 ([W]). *The largest Fibonacci number of the form* $4m^2 + 1$ *is* 5.

LEMMA 2.5 ([N]). *The equation*

$$5^x + 4^y = 3^z, \quad x, y, z \in \mathbb{N}, \tag{2.22}$$

has only the solution $(x, y, z) = (1, 1, 2)$.

LEMMA 2.6. *The equation*

$$101^x + 124^y = 15^z, \quad x, y, z \in \mathbb{N}, \tag{2.23}$$

has only the solution $(x, y, z) = (1, 1, 2)$.

Proof. Since (2.23) is the special case of (2.1) for $m = 5$, by Lemma 2.1, if (x, y, z) is a solution of (2.23) with $(x, y, z) \neq (1, 1, 2)$, then $2 \nmid x$, $x \geq 3, y = 1$, $2 \mid z$ and $z \geq 4$. Hence, by (2.23), $0 \equiv 15^z \equiv 101^x + 124^y \equiv (1 + 100)^x + 124 \equiv (1 + 100x) + (-1) \equiv 100x \pmod{5^3}$. It implies that

$$5 \mid x. \tag{2.24}$$

On the other hand, since $101 \equiv 2 \pmod{11}$, $124 \equiv 3 \pmod{11}$, $2 \nmid x$, $y = 1$ and $2 \mid z$, we deduce from (2.23) and (2.24) that

$$\begin{aligned} 15^z &\equiv (15^{z/2})^2 \equiv 101^x + 124^y \equiv 2^x + 3 \equiv (2^5)^{x/5} + 3 \\ &\equiv (-1)^{x/5} + 3 \equiv (-1) + 3 \equiv 2 \pmod{11}. \end{aligned} \tag{2.25}$$

But, since $11 \equiv 3 \pmod 8$ and 2 is not a quadratic residue modulo 11, (2.25) is false. Thus, (2.23) has only the solution $(x, y, z) = (1, 1, 2)$. ∎

3. Proof of Theorem 1.1. Assume that (x, y, z) is a solution of (2.1) with $(x, y, z) \neq (1, 1, 2)$. By Lemmas 2.5 and 2.6, the conclusion of the theorem holds for $m \in \{1, 5\}$. Since $3 \nmid m$, by Lemma 2.1, we have $2 \nmid m$, $m \geq 7$, $2 \nmid x$ and $y = 1$. This implies that the equation

$$(4m^2 + 1)X^2 + (5m^2 - 1) = (3m)^n, \quad X, n \in \mathbb{N}, \tag{3.1}$$

has a solution

$$(X, n) = ((4m^2 + 1)^{(x-1)/2}, z). \tag{3.2}$$

Since $(5m^2 - 1) \pm 1 \leq 5m^2 < 3(4m^2 + 1)$, we have

$$3(4m^2 + 1)f^2 - (5m^2 - 1) \neq \pm 1, \quad f \in \mathbb{N}. \tag{3.3}$$

On the other hand, by Lemma 2.4,

$$(4m^2 + 1, 5m^2 - 1, 3m) \neq (F_{t-2\varepsilon}, L_{t+\varepsilon}, F_t), \quad t \in \mathbb{N}, \varepsilon \in \{\pm 1\}. \tag{3.4}$$

Therefore, by (3.3) and (3.4), applying Lemma 2.3 to (3.1) and (3.2), we get

$$z < \frac{4}{\pi}\sqrt{(4m^2+1)(5m^2-1)}\log(2e\sqrt{(4m^2+1)(5m^2-1)}). \tag{3.5}$$

Since $m \geq 7$, by Lemma 2.2, x satisfies (2.12). Hence, by (2.1) and (2.12), we have $(3m)^z > (4m^2+1)^x > (2m)^{2x}$ and

$$z > 2x\frac{\log(2m)}{\log(3m)} > \frac{1}{4}(7m^4 - 45m^2 - 10)\frac{\log(2m)}{\log(3m)}. \tag{3.6}$$

The combination of (3.5) and (3.6) yields

$$\begin{aligned}(7m^4 - 45m^2 - 10)&\frac{\log(2m)}{\log(3m)} \\ &< \frac{16}{\pi}\sqrt{(4m^2+1)(5m^2-1)}\log(2e\sqrt{(4m^2+1)(5m^2-1)}).\end{aligned} \tag{3.7}$$

But, since $m \geq 7$, (3.7) is false. Thus, (2.1) has only the solution $(x, y, z) = (1, 1, 2)$.

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