Contents

Abstract

An *ideal* of N-tuples of operators is a class invariant with respect to unitary equivalence which contains direct sums of arbitrary collections of its members as well as their (reduced) parts. New decomposition theorems (with respect to ideals) for N-tuples of closed densely defined linear operators acting in a common (arbitrary) Hilbert space are presented. Algebraic and order (with respect to containment) properties of the class $\mathcal{CDD}_N$ of all unitary equivalence classes of such N-tuples are established and certain ideals in $\mathcal{CDD}_N$ are distinguished. It is proved that infinite operations in $\mathcal{CDD}_N$ may be reconstructed from the direct sum operation of a pair. *Prime decomposition* in $\mathcal{CDD}_N$ is proposed and its uniqueness (in a certain sense) is established. The issue of classification of ideals in $\mathcal{CDD}_N$ (up to isomorphism) is discussed. A model for $\mathcal{CDD}_N$ is described and its concrete realization is presented. A new partial order of N-tuples of operators is introduced and its fundamental properties are established. The importance of unitary disjointness of N-tuples and the way how it 'tidies up' the structure of $\mathcal{CDD}_N$ are emphasized.

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1. INTRODUCTION

1.1. Preface

Criterions for unitary equivalence of two (bounded linear) operators (acting on Hilbert spaces) and the classification of operators up to unitary equivalence are subjects which fascinated many mathematicians inspired by methods and ideas from the quite well explored area of normal operators. The literature dealing with these and related topics is still growing, let us mention here only a few: Brown [2] classified quasi-normal operators; Halmos and McLaughlin [17] reduced the issue of unitary equivalence of arbitrary bounded operators to partial isometries; Ernest [9], Hadwin [15, 16] and others (e.g. [21]) investigated operator-valued spectra which generalized standard (scalar) spectrum of a normal operator; Ernest [9], Brown, Fong and Hadwin [3] and Loebl [23] studied parts (that is, suboperators) of operators. It was Ernest [9] who first showed that—in a sense—the classification of all operators up to unitary equivalence is an essentially unattainable objective, although he gave an equivalent condition for two (totally arbitrary) bounded operators to be unitarily equivalent. It was formulated by means of certain (operator-valued) spectra of operators and multiplicity theory extended from normal to all bounded operators (roughly speaking, he adapted and generalized the classical Hahn–Hellinger theorem).

The present paper is motivated by his approach to this subject. One of our aims is to finish Ernest's programme of exploring the realm of unitary equivalence classes of closed densely defined operators by making no assumptions either on the dimension of Hilbert spaces or on boundedness of operators (this solves the problem posed by Ernest in point c of §7 of Chapter 5 of [9]). Even more, we study the class $\mathcal{CDD}_N$ of finite systems (N-tuples) of closed densely defined operators acting in (totally arbitrary) common Hilbert spaces. Surprisingly, such general considerations lead to more elegant results and reveal features which become invisible when one restricts only to separable spaces. Although $\mathcal{CDD}_N$ is not a set but a class, we shall show that it is 'controlled' by a single N-tuple (acting in a nonseparable space; cf. Proposition 3.4.8) and this observation will enable us to find an (algebraic as well as order) model for $\mathcal{CDD}_N$ (Theorem 4.4.2). An elementary form of the model will enable us to establish several interesting properties of $\mathcal{CDD}_N$ (e.g. (AO13)–(AO14), page 34). Also the central decomposition (of an operator acting in a separable space) introduced by Ernest may be extended to a general context and translated into a more attractive (at least for us) form of a 'prime decomposition' similar to the one for natural numbers (Theorem 5.6.14).

Another aspect discussed in this work concerns various (known) results on decompositions of operators. There are many results stating that a certain operator may be uniquely decomposed into two (or more) parts, the first of which is of a special type and the second has no nontrivial part of this type. The latter part is often named 'completely non-*sth*' or 'purely *sth*'. Let us mention only a few such results:

(DC1) a contraction operator may be decomposed into a unitary part and a completely non-unitary one,
(DC2) a bounded operator may be decomposed into a normal (respectively selfadjoint) part and a completely non-normal (resp. non-selfadjoint) one,
(DC3) a closed densely defined operator admits a unique decomposition into a normal, a purely formally normal and a completely non-formally normal part ([33])

(other results in this fashion are included e.g. in [34], [10], [32], [5]). There is a striking resemblance in the above statements. And this is not a coincidence. In this paper we put **all** results of this type in *one* general frame. To be more precise, let us introduce the notion of an *ideal*. It is any nonempty class $\mathcal{A}$ of closed densely defined operators which satisfies the following three axioms:

- if A and B are unitarily equivalent, then $A \in \mathcal{A} \Leftrightarrow B \in \mathcal{A}$,
- every part (including the trivial one acting on a zero-dimensional Hilbert space) of a member of $\mathcal{A}$ belongs to $\mathcal{A}$,
- $\bigoplus_{s\in S} A_s \in \mathcal{A}$ whenever $\{A_s\}_{s\in S} \subset \mathcal{A}$ (and S is a nonempty set).

For every ideal $\mathcal{A}$ we denote by $\mathcal{A}^\perp$ the class of all operators A none of whose nontrivial parts belongs to $\mathcal{A}$. In Theorem 2.4.2 we show that whenever $\mathcal{A}$ and $\mathcal{B}$ are ideals, so is $\mathcal{A}^\perp$, and every (closed densely defined) operator T acting in a (completely arbitrary) Hilbert space $\mathcal{H}$ induces a unique decomposition $\mathcal{H} = \mathcal{H}_{11} \oplus \mathcal{H}_{10} \oplus \mathcal{H}_{01} \oplus \mathcal{H}_{00}$ such that $\mathcal{H}_{jk}$ are reducing subspaces for T and $T|_{\mathcal{H}_{11}} \in \mathcal{A} \cap \mathcal{B}$, $T|_{\mathcal{H}_{10}} \in \mathcal{A} \cap \mathcal{B}^\perp$, $T|_{\mathcal{H}_{01}} \in \mathcal{A}^\perp \cap \mathcal{B}$ and $T|_{\mathcal{H}_{00}} \in \mathcal{A}^\perp \cap \mathcal{B}^\perp$. This result covers (DC1)–(DC3) and all the above-mentioned theorems on decompositions.

Ernest [9, Definition 1.7] introduced the notion of *disjoint* operators, say A and B. In this paper we denote it by writing '$A \perp_u B$' and call A and B *unitarily disjoint*. (Unitary disjointness, as a relation, behaves as singularity of measures or orthogonality in Hilbert spaces. Moreover, unitary disjointness is formulated in order-theoretic terms in the same way as disjointness in Banach lattices, where the disjointness of two vectors x and y is indicated by writing $x \perp y$. This is why we prefer using '$\perp_u$' rather than Ernest's original notation.) For Ernest the disjointness was only one of possible relations between operators. His *Lebesgue decomposition* of one operator with respect to another (Proposition 2.12 and Definition 2.13 in [9]) is merely one of many interesting results. Another aim of our work is to emphasize the importance of (unitary) disjointness (for example, we demonstrate how Ernest's central decomposition, or our prime one, may be translated into the 'intrinsic' language of operators, with the use of unitary disjointness; also the proof of our Theorem 2.4.2 depends on the properties of unitary disjointness). Roughly speaking, composing direct sums of arbitrary collections of operators is very *chaotic*, while the direct sum of a family of mutually unitarily disjoint operators is well 'arranged'. We may compare this with representing a simple Borel function (i.e. one whose range is finite) as a linear combination of the characteristic functions of Borel sets—this may be done in infinitely many ways; there is however only one such representation in which all the sets appearing form a partition of the domain of the function. This form of a simple function tells us *everything* about the function. The same occurs in the class $\mathcal{CDD}_N$ (see e.g. Theorem 3.6.1) when an N-tuple is written as the direct sum of a collection of mutually unitarily disjoint N-tuples. To distinguish between these specific decompositions and 'chaotic' ones, we call every direct sum (as well as any collection) of pairwise unitarily disjoint N-tuples *regular*. The notion of regularity may easily be adapted to 'continuous' versions of direct sums (defined in Chapter 5.5 by means of direct integrals). This generalization turns out to be crucial for formulating our Prime Decomposition Theorem (Theorem 5.6.14).

The main tools we use are, as in Ernest's work [9], techniques of von Neumann algebras. In Chapters 2.1–5.1 and 6.1–6.2 we involve the dimension theory of $\mathcal{W}^*$-algebras, especially a property recently discovered by Sherman [31]. All results of these chapters may be formulated and proved in the language of a 'semigroup' $\mathcal{CDD}_N$ with the direct sum of a pair as the only available operation (cf. Chapter 4.2). The remainder (Chapters 5.2–5.6) depends on the reduction theory due to von Neumann [25]. This deals with topological and measure-theoretic aspects which are introduced in Chapters 5.2–5.4. It is assumed that the reader is familiar with basics of von Neumann algebras (it is enough to know the material of [29], [18, 19] and [35]).

The main results of the paper are Theorems 2.4.2 (page 14), 3.6.1 (page 28), 4.4.2 (page 47), 5.6.14 (page 96) and 6.1.4 (page 101).

1.2. Basic notation and terminology

In this paper $\mathbb{R}_+ = [0,\infty)$ and all Hilbert spaces are over the complex field. $\mathcal{H}$ and $\mathcal{K}$ denote (possibly trivial) Hilbert spaces. By an *operator* we mean a linear function between linear subspaces of Hilbert spaces. The Hilbert space dimension of $\mathcal{H}$ is denoted by $\dim \mathcal{H}$. $\mathcal{B}(\mathcal{H},\mathcal{K})$ and $\mathcal{U}(\mathcal{H},\mathcal{K})$ denote, respectively, the Banach space of all bounded operators from $\mathcal{H}$ into $\mathcal{K}$ and the set of all unitary operators from $\mathcal{H}$ onto $\mathcal{K}$, and $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H},\mathcal{H})$ and $\mathcal{U}(\mathcal{H}) = \mathcal{U}(\mathcal{H},\mathcal{H})$. Whenever A is an operator, $\mathcal{D}(A)$, $\mathcal{R}(A)$, $\overline{\mathcal{D}}(A)$ and $\overline{\mathcal{R}}(A)$ stand for, respectively, the domain and the range of A and their closures. Additionally, $\mathcal{N}(A)$ denotes the kernel of A. The direct sum of a collection of Hilbert spaces $\{\mathcal{H}_s\}_{s\in S}$ is denoted by $\bigoplus_{s\in S}\mathcal{H}_s$ and $\oplus_s x_s$ is the member of $\bigoplus_{s\in S}\mathcal{H}_s$ corresponding to a family $\{x_s\}_{s\in S}$ of vectors such that $x_s \in \mathcal{H}_s$ and $\sum_{s\in S}\|x_s\|^2 < \infty$. The same notation is used for direct sums of operators: if $\{A_s\}_{s\in S}$ is a family of operators, $A = \bigoplus_{s\in S} A_s$ is an operator with

$$\mathcal{D}(A) = \Big\{\oplus_s x_s \in \bigoplus_{s\in S}\overline{\mathcal{D}}(A_s)\colon x_s \in \mathcal{D}(A_s)\ (s\in S),\ \sum_{s\in S}\|A_s x_s\|^2 < \infty\Big\}$$

and for $x = \oplus_s x_s \in \mathcal{D}(A)$, $Ax = \oplus_s(A_s x_s) \in \bigoplus_{s\in S}\overline{\mathcal{R}}(A_s)$.

For two operators A and B acting in a common Hilbert space we write $A \subset B$ provided $\mathcal{D}(A) \subset \mathcal{D}(B)$ and $Bx = Ax$ for $x \in \mathcal{D}(A)$.

Let A be a closed densely defined operator in $\mathcal{H}$. A closed linear subspace E of $\mathcal{H}$ is said to be *reducing* for A if $P_E A \subset A P_E$ where P_E is the orthogonal projection onto E and $\mathcal{D}(AP_E) = P_E^{-1}(\mathcal{D}(A))$. The *reduced part* of A to E is denoted by $A|_E$ and it is the restriction of A to $\mathcal{D}(A) \cap E$. The set of all reducing subspaces for A is denoted by $\operatorname{red}(A)$. A subspace $E \in \operatorname{red}(A)$ is *centrally* reducing if $P_E P_K = P_K P_E$ for any $K \in \operatorname{red}(A)$. The collection of all centrally reducing subspaces is denoted by $\operatorname{cred}(A)$. The $*$-*commutant* of A is the set $\mathcal{W}'(A)$ consisting of all $T \in \mathcal{B}(\mathcal{H})$ such that $TA \subset AT$ and $T^*A \subset AT^*$; and $\mathcal{W}''(A) = (\mathcal{W}'(A))'$ is the $*$-*bicommutant* of A. When A is bounded, we may also use $\mathcal{W}(A)$ to denote the smallest von Neumann algebra containing A; in that case $\mathcal{W}(A) = \mathcal{W}''(A)$ (thanks to von Neumann's bicommutant theorem). The *polar decomposition* of A has the form $A = Q|A|$ where $|A|$ is the square root of A^*A (obtained e.g. by the functional calculus for unbounded selfadjoint operators) and Q is a partial isometry with $\mathcal{N}(Q) = \mathcal{N}(A)$. Whenever we use the notation 'Q_T' with T being a closed densely defined operator, this denotes the partial isometry appearing in the polar decomposition of T.

2. GENERAL DECOMPOSITION THEOREM

2.1. Preliminaries

In the whole paper, N is a fixed positive integer corresponding to the length of tuples of operators. Whenever $\mathcal{H}$ is a Hilbert space, $\mathrm{CDD}(\mathcal{H})$ is the collection of all closed densely defined linear operators acting in $\mathcal{H}$ and $\mathrm{CDD}_N(\mathcal{H}) = [\mathrm{CDD}(\mathcal{H})]^N$. That is, $\mathrm{CDD}_N(\mathcal{H})$ consists of all N-tuples of members of $\mathrm{CDD}(\mathcal{H})$. Further, we put

$$\mathrm{CDD}_N = \bigcup_{\mathcal{H}} \mathrm{CDD}_N(\mathcal{H})$$

where $\mathcal{H}$ runs over all Hilbert spaces (including zero-dimensional). For simplicity, we shall write CDD in place of CDD_1. For every $\boldsymbol{A} = (A_1, \ldots, A_N) \in \mathrm{CDD}_N$ there is a unique Hilbert space, denoted by $\overline{\mathfrak{D}}(\boldsymbol{A})$, such that $\boldsymbol{A} \in \mathrm{CDD}_N(\overline{\mathfrak{D}}(\boldsymbol{A}))$. In particular, $\overline{\mathfrak{D}}(\boldsymbol{A}) = \overline{\mathfrak{D}}(A_j)$ for $j = 1, \ldots, N$.

Suppose $\boldsymbol{A} = (A_1, \ldots, A_N) \in \mathrm{CDD}_N$. We define $\boldsymbol{A}^*$, $|\boldsymbol{A}|$ and $\boldsymbol{Q}_{\boldsymbol{A}}$ (as members of CDD_N) in a coordinatewise manner: $\boldsymbol{A}^* = (A_1^*, \ldots, A_N^*)$, $|\boldsymbol{A}| = (|A_1|, \ldots, |A_N|)$ and $\boldsymbol{Q}_{\boldsymbol{A}} = (Q_{A_1}, \ldots, Q_{A_N})$. In the same way we may define other operations on N-tuples, if only they can be made on each of their entries. For example, if each of A_j's is one-to-one and has dense image, we may define $\boldsymbol{A}^{-1}$ as $(A_1^{-1}, \ldots, A_N^{-1})$.

Everywhere below in items (DF1)–(DF11), $\boldsymbol{A} = (A_1, \ldots, A_N)$, $\boldsymbol{B} = (B_1, \ldots, B_N)$ and $\boldsymbol{A}^{(s)} = (A_1^{(s)}, \ldots, A_N^{(s)})$ represent arbitrary members of CDD_N. For a single operator, some of the notions stated below are well-known and some of them were introduced in [9] (with different notation). Probably the only new notion is the *strong* order '$\leqslant^s$' defined in (DF8) below.

(DF1) Let $\bigoplus_{s \in S} \boldsymbol{A}^{(s)} = (\bigoplus_{s \in S} A_1^{(s)}, \ldots, \bigoplus_{s \in S} A_N^{(s)})$. For a positive cardinal α define $\alpha \odot \boldsymbol{A} = \bigoplus_{\xi < \xi_\alpha} \boldsymbol{A}^{(\xi)}$ where ξ_α is the first ordinal of cardinality α and $\boldsymbol{A}^{(\xi)} = \boldsymbol{A}$ for any $\xi < \xi_\alpha$.

(DF2) $\boldsymbol{A}$ is *trivial* provided $\overline{\mathfrak{D}}(\boldsymbol{A})$ is zero-dimensional; otherwise $\boldsymbol{A}$ is *nontrivial.*

(DF3) $\boldsymbol{A}$ is *bounded* iff each of $A_1, \ldots, A_N$ is a bounded operator; for bounded $\boldsymbol{A}$ let $\|\boldsymbol{A}\| := \max(\|A_1\|, \ldots, \|A_N\|)$, otherwise $\|\boldsymbol{A}\| := \infty$. We say a bounded N-tuple $\boldsymbol{A}$ *assumes its norm* provided there is $x \in \overline{\mathfrak{D}}(\boldsymbol{A})$ of norm 1 with

$$\max(\|A_1 x\|, \ldots, \|A_N x\|) = \|\boldsymbol{A}\|.$$

(DF4) Let $\mathrm{red}(\boldsymbol{A}) = \bigcap_{j=1}^N \mathrm{red}(A_j)$ and for $E \in \mathrm{red}(\boldsymbol{A})$,

$$\boldsymbol{A}|_E = (A_1|_E, \ldots, A_N|_E);$$

$\mathrm{cred}(\boldsymbol{A})$ consists of all $E \in \mathrm{red}(\boldsymbol{A})$ such that $P_E P_K = P_K P_E$ for every $K \in \mathrm{red}(\boldsymbol{A})$.

(DF5) The $*$-*commutant* of $\boldsymbol{A}$ is the set $\mathcal{W}'(\boldsymbol{A}) = \bigcap_{j=1}^{N} \mathcal{W}'(A_j) \subset \mathcal{B}(\overline{\mathcal{D}}(\boldsymbol{A}))$ and $\mathcal{W}''(\boldsymbol{A}) = (\mathcal{W}'(\boldsymbol{A}))'$ is the $*$-*bicommutant* of $\boldsymbol{A}$. When $\boldsymbol{A}$ is bounded, we may also use $\mathcal{W}(\boldsymbol{A})$ to denote the smallest von Neumann algebra including $\{A_1, \ldots, A_N\}$; in that case $\mathcal{W}(\boldsymbol{A}) = \mathcal{W}''(\boldsymbol{A})$.

(DF6) $\boldsymbol{A} \equiv \boldsymbol{B}$ ($\boldsymbol{A}$ and $\boldsymbol{B}$ are *unitarily equivalent*) iff there is $U \in \mathcal{U}(\overline{\mathcal{D}}(\boldsymbol{A}), \overline{\mathcal{D}}(\boldsymbol{B}))$ such that $A_j = U^{-1} B_j U$ for $j = 1, \ldots, N$.

(DF7) $\boldsymbol{A} \leqslant \boldsymbol{B}$ iff $\boldsymbol{A} \equiv \boldsymbol{B}|_E$ for some $E \in \operatorname{red}(\boldsymbol{B})$.

(DF8) $\boldsymbol{A} \leqslant^s \boldsymbol{B}$ iff $\boldsymbol{A} \equiv \boldsymbol{B}|_E$ for some $E \in \operatorname{cred}(\boldsymbol{B})$.

(DF9) $\boldsymbol{A}$ and $\boldsymbol{B}$ are *unitarily disjoint*, in symbols $\boldsymbol{A} \perp_u \boldsymbol{B}$, if there is no nontrivial N-tuple $\boldsymbol{X} \in \mathrm{CDD}_N$ with $\boldsymbol{X} \leqslant \boldsymbol{A}$ and $\boldsymbol{X} \leqslant \boldsymbol{B}$.

(DF10) $\boldsymbol{A}$ is *covered* by $\boldsymbol{B}$, in symbols $\boldsymbol{A} \ll \boldsymbol{B}$, if $\boldsymbol{A} \leqslant \alpha \odot \boldsymbol{B}$ for some cardinal α.

(DF11) The symbols '$\boxplus$' and '$\bigboxplus$' will often be used instead of '$\oplus$' and '$\bigoplus$' in situations when all summands are mutually unitarily disjoint. So, whenever the notation $\boldsymbol{A} \boxplus \boldsymbol{B}$ or $\bigboxplus_{s\in S} \boldsymbol{A}^{(s)}$ appears, this will always imply that $\boldsymbol{A} \perp_u \boldsymbol{B}$ or, respectively, $\boldsymbol{A}^{(s')} \perp_u \boldsymbol{A}^{(s'')}$ for any distinct $s', s'' \in S$. The direct sum (a collection) is called *regular* provided all its summands (elements) are mutually unitarily disjoint.

The reader should notice that a function $\operatorname{red}(\boldsymbol{A}) \ni E \mapsto P_E \in \mathcal{W}'(\boldsymbol{A})$ establishes a one-to-one correspondence between $\operatorname{red}(\boldsymbol{A})$ and the set $E(\mathcal{W}'(\boldsymbol{A}))$ of all orthogonal projections belonging to $\mathcal{W}'(\boldsymbol{A})$. What is more, this map sends $\operatorname{cred}(\boldsymbol{A})$ onto $E(\mathcal{W}'(\boldsymbol{A})) \cap \mathcal{Z}(\mathcal{W}'(\boldsymbol{A}))$ where $\mathcal{Z}(\mathcal{W}'(\boldsymbol{A}))$ is the center of $\mathcal{W}'(\boldsymbol{A})$.

It is quite easy to check that '$\equiv$' is an equivalence relation on CDD_N and thus for each $\boldsymbol{A} \in \mathrm{CDD}_N$ we may consider the equivalence class of $\boldsymbol{A}$ with respect to '$\equiv$', which we shall denote by A. Let $\mathcal{CDD}_N$ be the class of (all) equivalence classes of all members of CDD_N and let $\mathcal{CDD} = \mathcal{CDD}_1$. Elements of $\mathcal{CDD}_N$ will be denoted by $\mathsf{A}, \mathsf{B}, \mathsf{X}, \mathsf{Y}$ and so on, and their corresponding representatives by $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{X}, \boldsymbol{Y}$. The symbol O is reserved to denote the equivalence class of a trivial element of CDD_N. O is the unique member of $\mathcal{CDD}_N$ whose representatives act on zero-dimensional Hilbert spaces. (It is also the neutral element for $\oplus$'.) For every $\mathsf{A} \in \mathcal{CDD}_N$, the following are well defined, in an obvious manner: A^*, $|\mathsf{A}|$, Q_{A} (corresponding to $\boldsymbol{Q}_{\boldsymbol{A}}$) and $\dim(\mathsf{A}) = \dim \overline{\mathcal{D}}(\boldsymbol{A})$. For simplicity, we shall use the term 'N-tuple' for members of CDD_N as well as of $\mathcal{CDD}_N$.

Some of the notions in (DF1)–(DF11) may be adapted to members of $\mathcal{CDD}_N$ as follows:

(UE1) Let $\bigoplus_{s\in S} \mathsf{A}^{(s)} = \mathsf{X}$ where $\boldsymbol{X} = \bigoplus_{s\in S} \boldsymbol{A}^{(s)}$. For any cardinal $\mathfrak{m} > 0$, put $\mathfrak{m} \odot \mathsf{A} = \mathsf{Y}$ where $\boldsymbol{Y} = \mathfrak{m} \odot \boldsymbol{A}$. Additionally, let $0 \odot \mathsf{A} = \mathsf{O}$.

(UE2) A is bounded, nontrivial, trivial iff so is $\boldsymbol{A}$. $\|\mathsf{A}\| = \|\boldsymbol{A}\|$; A assumes its norm iff so does $\boldsymbol{A}$.

(UE3) $\mathsf{A} \leqslant \mathsf{B}$, $\mathsf{A} \leqslant^s \mathsf{B}$, $\mathsf{A} \perp_u \mathsf{B}$, $\mathsf{A} \ll \mathsf{B}$ iff the corresponding relation holds for $\boldsymbol{A}$ and $\boldsymbol{B}$. Note that $\mathsf{A} \leqslant^s \mathsf{B} \Rightarrow \mathsf{A} \leqslant \mathsf{B} \Rightarrow \mathsf{A} \ll \mathsf{B}$.

(UE4) Notation $\mathsf{A} \boxplus \mathsf{B}$ or $\bigboxplus_{s\in S} \mathsf{A}^{(s)}$ includes information that $\mathsf{A} \perp_u \mathsf{B}$ or, respectively, $\mathsf{A}^{(s')} \perp_u \mathsf{A}^{(s'')}$ for any distinct indices $s', s'' \in S$. The direct sum of (a family of) members of $\mathcal{CDD}_N$ is *regular* iff all its summands (elements) are pairwise unitarily disjoint.

A starting point for all of our investigations is the following classical result (see e.g. [9, Theorem 1.3]).

PROPOSITION 2.1.1. *'$\leqslant$' and '$\leqslant^s$' are partial orders on $\mathcal{CDD}_N$. More precisely, if* $\mathsf{A} \leqslant \mathsf{B}$ *and* $\mathsf{B} \leqslant \mathsf{A}$, *then* $\mathsf{A} = \mathsf{B}$.

2.2. The $\mathfrak{b}$-transform

This chapter is mainly devoted to single operators. We fix a Hilbert space $\mathcal{H}$ and an operator $T \in \mathrm{CDD}(\mathcal{H})$. Let I be the identity operator on $\mathcal{H}$.

DEFINITION 2.2.1. The $\mathfrak{b}$-*transform* of T is the operator

$$\mathfrak{b}(T) = T(I + |T|)^{-1} \in \mathcal{B}(\mathcal{H}).$$

The reader should verify with no difficulties

PROPOSITION 2.2.2. *Let* $S = \mathfrak{b}(T)$.

(A) $\mathfrak{b}(|T|) = |S| = |T|(I + |T|)^{-1}$ *and* $Q_T = Q_S$.
(B) $\|Sx\| < \|x\|$ *for each* $x \in \mathcal{H} \setminus \{0\}$.
(C) $T = S(I - |S|)^{-1} =: \mathfrak{ub}(S)$.
(D) $\mathcal{W}'(T) = \mathcal{W}'(S)$. *Consequently,* $\mathrm{red}(T) = \mathrm{red}(S)$ *and* $\mathrm{cred}(T) = \mathrm{cred}(S)$. *For every* $E \in \mathrm{red}(T)$, $\mathfrak{b}(T|_E) = S|_E$.
(E) *The $\mathfrak{b}$-transform establishes a one-to-one correspondence between closed densely defined operators in $\mathcal{H}$ and operators $S \in \mathcal{B}(\mathcal{H})$ satisfying* (B).
(F) $\mathfrak{b}(\bigoplus_{s\in S} T_s) = \bigoplus_{s\in S} \mathfrak{b}(T_s)$ *for an arbitrary family* $\{T_s\}_{s\in S} \subset \mathrm{CDD}$.

The following result is slightly surprising.

THEOREM 2.2.3. *For every* $T \in \mathrm{CDD}$, $\mathfrak{b}(T^*) = [\mathfrak{b}(T)]^*$.

Proof. Let $T = Q|T|$ be the polar decomposition of T. Then $T^* = Q^*|T^*|$ is the polar decomposition of T^*. Put $\mathcal{H} = \overline{\mathcal{D}}(T)$, $S = \mathfrak{b}(T)$ and $S' = \mathfrak{b}(T^*)$. Fix $x, y \in \mathcal{H}$, put $u = (I + |T|)^{-1}x \in \mathcal{D}(T)$ and $v = (I + |T^*|)^{-1}y \in \mathcal{D}(T^*)$ and observe that

$$\begin{aligned}\langle Sx, y\rangle = \langle Tu, (I + |T^*|)v\rangle &= \langle Tu, v\rangle + \langle Q|T|u, |T^*|v\rangle \\ &= \langle u, T^*v\rangle + \langle |T|u, T^*v\rangle = \langle (I + |T|)u, T^*v\rangle = \langle x, S'y\rangle,\end{aligned}$$

which finishes the proof. ∎

Involving the $\mathfrak{b}$-transform we now easily prove

THEOREM 2.2.4. *Let $\mathcal{H}$ be a nonseparable Hilbert space and $\{T_s\}_{s\in S} \subset \mathrm{CDD}(\mathcal{H})$ be a* ***countable*** *family of operators. For every nonzero $x \in \mathcal{H}$ there is a separable space $E \subset \mathcal{H}$ containing x such that $E \in \mathrm{red}(T_s)$ for each $s \in S$.*

Proof. By Proposition 2.2.2(D), we may assume each T_s is bounded (because we may replace T_s by $\mathfrak{b}(T_s)$). Now it suffices to put $E = \overline{\mathrm{lin}}\{S_1 \cdot \ldots \cdot S_n x\colon n \geqslant 1,\ S_1, \ldots, S_n \in \{T_s\colon s \in S\} \cup \{T_s^*\colon s \in S\} \cup \{I\}\}$ where I is the identity operator on $\mathcal{H}$. ∎

Now for $\boldsymbol{A} = (A_1, \ldots, A_N) \in \mathrm{CDD}_N$ put $\mathfrak{b}(\boldsymbol{A}) = (\mathfrak{b}(A_1), \ldots, \mathfrak{b}(A_N))$ and $\mathfrak{b}(\mathsf{A}) = \mathsf{X}$ where $\boldsymbol{X} = \mathfrak{b}(\boldsymbol{A})$. Below we list the most important properties of the $\mathfrak{b}$-transform on $\mathcal{CDD}_N$ and CDD_N.

(BT1) $\mathfrak{b}(\mathsf{A}) = \mathsf{O} \Leftrightarrow \mathsf{A} = \mathsf{O}$.
(BT2) $\mathfrak{b}(\mathsf{A})$ is bounded, $\mathfrak{b}(\mathsf{A}^*) = [\mathfrak{b}(\mathsf{A})]^*$, $|\mathfrak{b}(\mathsf{A})| = \mathfrak{b}(|\mathsf{A}|)$ and $\mathsf{Q}_{\mathfrak{b}(\mathsf{A})} = \mathsf{Q}_{\mathsf{A}}$.
(BT3) $\mathcal{W}'(\boldsymbol{A}) = \mathcal{W}'(\mathfrak{b}(\boldsymbol{A}))$, $\mathcal{W}''(\boldsymbol{A}) = \mathcal{W}(\mathfrak{b}(\boldsymbol{A}))$; $\mathrm{red}(\boldsymbol{A}) = \mathrm{red}(\mathfrak{b}(\boldsymbol{A}))$ and $\mathrm{cred}(\boldsymbol{A}) = \mathrm{cred}(\mathfrak{b}(\boldsymbol{A}))$; for every $E \in \mathrm{red}(\boldsymbol{A})$, $\mathfrak{b}(\boldsymbol{A}|_E) = \mathfrak{b}(\boldsymbol{A})|_E$.
(BT4) $\mathfrak{b}(\bigoplus_{s\in S} \mathsf{A}^{(s)}) = \bigoplus_{s\in S} \mathfrak{b}(\mathsf{A}^{(s)})$.
(BT5) If '$\sim$' denotes one of the relations $=, \leqslant, \leqslant^s, \ll, \perp_u$, then $\mathsf{A} \sim \mathsf{B} \Leftrightarrow \mathfrak{b}(\mathsf{A}) \sim \mathfrak{b}(\mathsf{B})$.

2.3. Background on von Neumann algebras

Let $\mathcal{M}$ be a von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$. Denote by $E(\mathcal{M})$ the set of all orthogonal projections in $\mathcal{M}$ and by $\mathcal{Z}(\mathcal{M})$ the center of $\mathcal{M}$. By '$\sim$' we shall denote the Murray–von Neumann equivalence on $E(\mathcal{M})$. Further, put $\mathcal{E}(\mathcal{M}) = E(\mathcal{M})/\sim$ and let '$\preccurlyeq$' denote the Murray–von Neumann order on $\mathcal{E}(\mathcal{M})$. Finally, for each $p \in E(\mathcal{M})$, $c_p \in E(\mathcal{Z}(\mathcal{M}))$ stands for the central support of p.

It was observed by several mathematicians that the order '$\leqslant$' on $\mathcal{CDD}$ translates into the Murray–von Neumann order between (equivalence classes of) projections in a suitable von Neumann algebra. This was explicitly stated and proved in [9, Proposition 1.35]. It is nothing new that the same idea works for tuples of operators. We formulate this precisely in the next result which is the main tool of the paper.

Proposition 2.3.1. *Let $\boldsymbol{T} \in \mathrm{CDD}_N(\mathcal{H})$, $E, F \in \mathrm{red}(\boldsymbol{T})$, $\boldsymbol{A} = \boldsymbol{T}|_E$ and $\boldsymbol{B} = \boldsymbol{T}|_F$. Further, let $\mathcal{M} = \mathcal{W}'(\boldsymbol{T})$, $p = P_E$ and $q = P_F$ $(p, q \in E(\mathcal{M}))$. Then*

(a) $\boldsymbol{A} \equiv \boldsymbol{B} \Leftrightarrow p \sim q$,
(b) $\boldsymbol{A} \leqslant \boldsymbol{B} \Leftrightarrow p \preccurlyeq q$,
(c) $\boldsymbol{A} \leqslant^s \boldsymbol{B} \Leftrightarrow p \sim c_p q$,
(d) $\boldsymbol{A} \perp_u \boldsymbol{B} \Leftrightarrow c_p c_q = 0$,
(e) $\boldsymbol{A} \ll \boldsymbol{B} \Leftrightarrow p \leqslant c_q$.

Proof. We shall only prove (c), since the other points are covered by [9, Proposition 1.35] ((d) is stated there in another form; its present form may be deduced e.g. from [35, Lemma 1.7]). For this purpose put $\mathcal{M}_0 = q\mathcal{M}q$, $z_0 = c_p q \in E(\mathcal{Z}(\mathcal{M}_0))$ and let $K \in \mathrm{cred}(\boldsymbol{B})$ be the range of z_0. If $z_0 \sim p$, then by (a), $\boldsymbol{A} \equiv \boldsymbol{B}|_K$ and thus $\boldsymbol{A} \leqslant^s \boldsymbol{B}$. Conversely, if the last inequality is satisfied, there is $z_0 \in E(\mathcal{Z}(\mathcal{M}_0))$ such that $p \sim z_0$ (again by (a)). But $\mathcal{Z}(\mathcal{M}_0) = \mathcal{Z}(\mathcal{M})q$ and hence $z_0 = zq$ for some $z \in E(\mathcal{Z}(\mathcal{M}))$. Finally, note that $c_p = c_{zq}$ (since $p \sim zq$) and $c_{zq} = zc_q$ and therefore $zq = zc_q q = c_p q$. ■

Some consequences of Proposition 2.3.1 are formulated below (these are adaptations of suitable results of [9]).

(PR1) $\mathsf{A} \sim \mathsf{X} \oplus \mathsf{Y}$ and $\mathsf{A} \perp_u \mathsf{Y}$ imply $\mathsf{A} \sim \mathsf{X}$ when '$\sim$' is replaced by one of $\leqslant, \leqslant^s, \ll$.
(PR2) If $\mathsf{A}^{(s)} \perp_u \mathsf{B}^{(t)}$ for all $s \in S$ and $t \in T$, then $\bigoplus_{s\in S} \mathsf{A}^{(s)} \perp_u \bigoplus_{t\in T} \mathsf{B}^{(t)}$.

(PR3) The function $\mathrm{cred}(\boldsymbol{A}) \ni E \mapsto \mathsf{X}(E) \in \{\mathsf{B} \in \mathcal{CDD}_N : \mathsf{B} \leqslant^s \mathsf{A}\}$ where $\boldsymbol{X}(E) = \boldsymbol{A}|_E$ is a (well defined) bijection.

(PR4) For every $E \in \mathrm{red}(\boldsymbol{A})$, $\boldsymbol{A}|_E \perp_u \boldsymbol{A}|_{E^\perp} \Leftrightarrow E \in \mathrm{cred}(\boldsymbol{A})$.

(PR5) For every pair (A, B) such that $\mathsf{A} \leqslant^s \mathsf{B}$ there is a unique $\mathsf{X} \in \mathcal{CDD}_N$ such that $\mathsf{B} = \mathsf{A} \boxplus \mathsf{X}$. Notation: $\mathsf{B} \boxminus \mathsf{A} := \mathsf{X}$. (So, $\mathsf{B} \boxminus \mathsf{A}$ makes sense iff $\mathsf{A} \leqslant^s \mathsf{B}$.)

(PR6) For every $\mathsf{X} \in \mathcal{CDD}_N$ and a cardinal α, $\{\mathsf{Y} \in \mathcal{CDD}_N : \mathsf{Y} \leqslant^s \alpha \odot \mathsf{X}\} = \{\alpha \odot \mathsf{Y} : \mathsf{Y} \leqslant^s \mathsf{X}\}$.

Following (PR5), let us adopt the following convention: whenever for a pair (A, B) there is a unique X for which $\mathsf{B} = \mathsf{A} \oplus \mathsf{X}$, we shall denote X by $\mathsf{B} \ominus \mathsf{A}$. Observe that $\mathsf{A} \leqslant \mathsf{B}$ provided $\mathsf{B} \ominus \mathsf{A}$ makes sense.

Combining Proposition 2.3.1 with Sherman's theorem [31], we obtain an interesting

THEOREM 2.3.2. $(\mathcal{CDD}_N, \leqslant)$ *is an order-complete lattice. Precisely, for every nonempty family* (*i.e. a set*) $\{\mathsf{A}^{(s)}\}_{s \in S} \subset \mathcal{CDD}_N$ *there are members* X *and* Y *of* $\mathcal{CDD}_N$ *such that* $\mathsf{X} \leqslant \mathsf{A}^{(s)} \leqslant \mathsf{Y}$ *for each* $s \in S$ *and* $\mathsf{X}' \leqslant \mathsf{X}$ (*respectively* $\mathsf{Y} \leqslant \mathsf{Y}'$) *whenever* $\mathsf{X}' \leqslant \mathsf{A}^{(s)}$ (*respectively* $\mathsf{A}^{(s)} \leqslant \mathsf{Y}'$) *for all* $s \in S$.

Proof. Put $\mathsf{A} = \bigoplus_{s \in S} \mathsf{A}^{(s)}$ and $\mathcal{M} = \mathcal{W}'(\boldsymbol{A})$. By [31], $(\mathcal{E}(\mathcal{M}), \preccurlyeq)$ is an order-complete lattice. So, using Proposition 2.3.1 we see that there are X and Y (both $\leqslant \mathsf{A}$) which correspond to the g.l.b. and l.u.b. (with respect to '$\preccurlyeq$') of the projections corresponding to $\mathsf{A}^{(s)}$'s. Now if X' and Y' are as in the statement of the theorem, consider $\widetilde{\mathsf{A}} = \mathsf{A} \oplus \mathsf{X}' \oplus \mathsf{Y}'$ and $\widetilde{\mathcal{M}} = \mathcal{W}'(\widetilde{\boldsymbol{A}})$ and repeat the above argument. We skip the details. ∎

As is usually done when working with lattices, for every nonempty collection $\mathcal{A} = \{\mathsf{A}^{(s)}\}_{s \in S}$ we shall denote by $\bigvee_{s \in S} \mathsf{A}^{(s)}$ and $\bigwedge_{s \in S} \mathsf{A}^{(s)}$ the l.u.b. and the g.l.b. of $\mathcal{A}$. Observe that $\mathsf{A} \perp_u \mathsf{B}$ iff $\mathsf{A} \wedge \mathsf{B} = \mathsf{O}$.

2.4. Decompositions relative to ideals

Let $\mathcal{A}$ be a subclass of CDD_N. We call $\mathcal{A}$ an *ideal* iff $\mathcal{A}$ satisfies the following four conditions:

(ID1) $\mathcal{A}$ is nonempty,

(ID2) whenever $\boldsymbol{A} \in \mathcal{A}$ and $\boldsymbol{A} \equiv \boldsymbol{B} \in \mathrm{CDD}_N$, then $\boldsymbol{B} \in \mathcal{A}$,

(ID3) for every $\boldsymbol{A} \in \mathcal{A}$ and $E \in \mathrm{red}(\boldsymbol{A})$, $\boldsymbol{A}|_E \in \mathcal{A}$,

(ID4) $\bigoplus_{s \in S} \boldsymbol{A}_s \in \mathcal{A}$ for any nonempty family $\{\boldsymbol{A}_s\}_{s \in S} \subset \mathcal{A}$.

Classical examples of ideals are discussed in Examples 2.4.3 below.

For every subclass $\mathcal{F}$ of CDD_N put

$$\mathcal{F}^\perp = \{\boldsymbol{T} \in \mathrm{CDD}_N : \boldsymbol{T} \perp_u \boldsymbol{F} \text{ for every } \boldsymbol{F} \in \mathcal{F}\}.$$

It is easily seen that $\mathcal{F}^\perp$ is an ideal for any $\mathcal{F} \subset \mathrm{CDD}_N$ (thanks to (PR2)). As we will see later, the 'converse' is also true, that is, $\mathcal{A}$ is an ideal iff $\mathcal{A} = (\mathcal{A}^\perp)^\perp$. This resembles the analogous characterization of closed linear subspaces of Hilbert spaces. However, the above definition of the 'orthogonal complement' is in the spirit of the orthogonality in spaces of measures, and not in Hilbert spaces.

One of the main results of the paper is the following

THEOREM 2.4.1. *Let $\mathcal{A} \subset \mathrm{CDD}_N$ be an ideal. For every $\boldsymbol{T} \in \mathrm{CDD}_N$ there is a unique $E \in \mathrm{red}(\boldsymbol{T})$ such that*

$$\boldsymbol{T}|_E \in \mathcal{A} \quad \text{and} \quad \boldsymbol{T}|_{E^\perp} \in \mathcal{A}^\perp. \tag{2.4.1}$$

Moreover, $E \in \mathrm{cred}(\boldsymbol{T})$ and

$$E = \bigvee \{K \in \mathrm{red}(\boldsymbol{T}) \colon \boldsymbol{T}|_K \in \mathcal{A}\}. \tag{2.4.2}$$

Proof. First we shall show the existence of E satisfying (2.4.1). We may assume that $\boldsymbol{T} \notin \mathcal{A}^\perp$. By Zorn's lemma, there is a maximal family $\{E_s\}_{s\in S}$ of mutually orthogonal nontrivial reducing (for $\boldsymbol{T}$) subspaces with $\boldsymbol{T}|_{E_s} \in \mathcal{A}$ for every $s \in S$. It is clear that (2.4.1) is satisfied with $E = \bigvee_{s\in S} E_s$.

Now assume that $E \in \mathrm{red}(\boldsymbol{T})$ is as in (2.4.1). By (PR4), $E \in \mathrm{cred}(\boldsymbol{T})$. To establish the uniqueness and finish the proof, it is enough to check (2.4.2). But this simply follows from (PR1) and Proposition 2.3.1. (Indeed, if $K \in \mathrm{red}(\boldsymbol{T})$ is such that $\boldsymbol{T}|_K \in \mathcal{A}$, then $\boldsymbol{T}|_K \leqslant \boldsymbol{T}|_E \oplus \boldsymbol{T}|_{E^\perp}$ and $\boldsymbol{T}|_K \perp_u \boldsymbol{T}|_{E^\perp}$. So, we conclude from (PR1) that $\boldsymbol{T}|_K \leqslant \boldsymbol{T}|_E$. Thus, by Proposition 2.3.1, $P_K \preccurlyeq P_E$ in $\mathcal{M} = \mathcal{W}'(\boldsymbol{T})$. But $P_E \in \mathcal{Z}(\mathcal{M})$ and hence $P_K \leqslant P_E$, which means that $K \subset E$.) ∎

For simplicity, let us introduce the following notation. For every ideal $\mathcal{A} \subset \mathrm{CDD}_N$, $\mathcal{A}^{(0)} = \mathcal{A}$ and $\mathcal{A}^{(1)} = \mathcal{A}^\perp$. With this notation, by a simple induction argument we obtain

THEOREM 2.4.2. *Let $\mathcal{A}_1, \ldots, \mathcal{A}_k \subset \mathrm{CDD}_N$ be ideals. For every $\boldsymbol{T} \in \mathrm{CDD}_N(\mathcal{H})$ there is a unique system $\{E_\delta\}_{\delta\in\{0,1\}^k}$ of reducing subspaces for $\boldsymbol{T}$ such that*

(i) *$E_\delta \perp E_{\delta'}$ for distinct $\delta, \delta' \in \{0,1\}^k$; and $\mathcal{H} = \bigoplus_{\delta\in\{0,1\}^k} E_\delta$,*

(ii) *$\boldsymbol{T}|_{E_\delta} \in \bigcap_{j=1}^k \mathcal{A}_j^{(\delta_j)}$ for each $\delta \in \{0,1\}^k$.*

Moreover, $E_\delta \in \mathrm{cred}(\boldsymbol{T})$ and $E_\delta = \bigvee\{K \in \mathrm{red}(\boldsymbol{T}) \colon \boldsymbol{T}|_K \in \bigcap_{j=1}^k \mathcal{A}_j^{(\delta_j)}\}$ for every $\delta \in \{0,1\}^k$.

We leave the proof of Theorem 2.4.2 to the reader.

Theorem 2.4.2 covers any known result on decomposition of a single operator into two parts with one of them of a special class and the other 'completely' (or 'hereditarily') not of this class. Examples are given below.

EXAMPLES 2.4.3.

(A) Let F be a closed subset of the complex plane $\mathbb{C}$. Let $\mathcal{N}(F)$ be the class of all normal operators whose spectrum is contained in F. (Here we assume that operators on zero-dimensional Hilbert spaces are normal and have empty spectra.) It is easily checked that $\mathcal{N}(F)$ is an ideal. Thus, every operator $T \in \mathrm{CDD}$ admits a unique decomposition into a part in $\mathcal{N}(F)$ and the remainder in $\mathcal{N}(F)^\perp$. This means that there is a unique $E \in \mathrm{red}(T)$ such that $T|_E$ is normal, $\sigma(T|_E) \subset F$ and $T|_{E^\perp}$ has no nontrivial reduced part which belongs to $\mathcal{N}(F)$. When $F = \mathbb{C}$, this is the decomposition into the normal part and the completely non-normal part. When $F = \mathbb{R}$, we get the decomposition into the selfadjoint part and the completely non-selfadjoint part. Finally, when $F = \{z \in \mathbb{C} \colon |z| = 1\}$, the operator decomposes into the unitary part and the completely non-unitary part. These three cases are most classical. (Compare with [9, p. 179].)

(B) Single operators of each of the following classes form an ideal: formally normal (for the definition see e.g. [33]); quasinormal; hyponormal; subnormal; contractions. As we will see in Proposition 2.4.4, also the following class $\mathcal{A}$ is an ideal: $T \in \mathcal{A}$ iff T is the direct sum of bounded operators.

(C) Stochel and Szafraniec [33] showed that every operator $T \in \mathrm{CDD}$ admits a unique decomposition of the form $T = T_{\mathrm{nor}} \oplus T_{\mathrm{pfn}} \oplus T_{\mathrm{cnfn}}$ where T_{nor} is normal, T_{pfn} is purely formally normal (here 'purely' means that T_{pfn} is in addition completely non-normal) and T_{cnfn} is completely non-formally normal. Their result is a special case of Theorem 2.4.2.

(D) Ernest [9] distinguished an important class of bounded operators on separable Hilbert spaces, the so-called *smooth* operators (see §6 of Chapter 1 in [9]). Let us say that an operator $T \in \mathrm{CDD}(\mathcal{H})$ where $\mathcal{H}$ is separable is *σ-smooth* iff $\mathfrak{b}(T)$ is the direct sum of countably (finitely or infinitely) many smooth operators. By Proposition 1.52 of [9] and Proposition 2.4.4 below, operators which are direct sums of σ-smooth operators form an ideal. In particular, every closed densely defined operator acting on a separable Hilbert space admits a unique decomposition into a σ-smooth operator and a completely non-smooth one.

(E) Let us give some examples dealing with systems of operators. Let $\mathcal{N}_N$ and $\widetilde{\mathcal{N}}_N$ consist of all N-tuples (belonging to CDD_N) of, respectively, commuting normal and arbitrary normal operators (commutativity may be defined by means of spectral measures or, equivalently, $\mathfrak{b}$-transforms). It is clear that both $\mathcal{N}_N$ and $\widetilde{\mathcal{N}}_N$ are ideals. So, every $\boldsymbol{T} \in \mathrm{CDD}_N$ has a unique decomposition $\boldsymbol{T} = \boldsymbol{T}_{\mathrm{jn}} \oplus \boldsymbol{T}_{\mathrm{psn}} \oplus \boldsymbol{T}_{\mathrm{cnsn}}$ where $\boldsymbol{T}_{\mathrm{jn}} \in \mathcal{N}_N$, $\boldsymbol{T}_{\mathrm{psn}} \in \widetilde{\mathcal{N}}_N$ and no nontrivial reduced part of $\boldsymbol{T}_{\mathrm{psn}}$ is a member of $\mathcal{N}_N$, and no nontrivial reduced part of $\boldsymbol{T}_{\mathrm{cnsn}}$ belongs to $\widetilde{\mathcal{N}}_N$. (The labels 'jn', 'psn' and 'cnsn' appearing here are abbreviations for *jointly normal*, *purely separately normal* and *completely non-separately normal*.) We call an N-tuple A *normal* iff $\boldsymbol{A} \in \mathcal{N}_N$.

(F) If $\mathcal{A} \subset \mathrm{CDD}$ is an ideal, so are $\Delta_N(\mathcal{A}) \subset \mathrm{CDD}_N$ and $\mathcal{A}^{[N]} \subset \mathrm{CDD}_N$ where $\mathcal{A}^{[N]}$ consists of all N-tuples $(A_1, \ldots, A_N)$ with $A_1, \ldots, A_N \in \mathcal{A}$ acting in a common Hilbert space, and

$$\Delta_N(\mathcal{A}) = \{(A_1, \ldots, A_N)\colon A_1 = \cdots = A_N \in \mathcal{A}\}.$$

(G) Theorem 2.4.1 may be briefly reformulated in the following way: $\mathrm{CDD}_N = \mathcal{A} \oplus \mathcal{A}^{\perp}$ for every ideal $\mathcal{A} \subset \mathrm{CDD}_N$. Using this notation, Theorem 2.4.2 with $k = 2$ asserts that

$$\mathrm{CDD}_N = (\mathcal{A} \cap \mathcal{B}) \oplus (\mathcal{A} \cap \mathcal{B}^{\perp}) \oplus (\mathcal{A}^{\perp} \cap \mathcal{B}) \oplus (\mathcal{A}^{\perp} \cap \mathcal{B}^{\perp}) \tag{2.4.3}$$

for any two ideals $\mathcal{A}$ and $\mathcal{B}$ in CDD_N. The counterpart of (2.4.3) for linear subspaces K and L of a Hilbert space $\mathcal{H}$ is satisfied only when P_K and P_L commute. Thus, as we have said earlier, the 'orthogonal complement' for ideals behaves in a similar manner to the orthogonal complement in lattices of measures (or in more general structures such as abstract L-spaces).

The next result is useful for producing ideals.

Proposition 2.4.4. *Let $\mathcal{A}$ be a subclass of* CDD_N *and* Θ_N *be the class of all trivial members of* CDD_N*.*

(a) *The class*

$$J(\mathcal{A}) = \Big\{\boldsymbol{T} \in \mathrm{CDD}_N\colon \textit{for some set } S,\ \boldsymbol{T} = \bigoplus_{s\in S} \boldsymbol{X}^{(s)} \textit{ with } \boldsymbol{X}^{(s)} \leqslant \boldsymbol{Y}^{(s)} \in \mathcal{A} \cup \Theta_N\Big\}$$

is an ideal and it is the smallest ideal which contains $\mathcal{A}$.

(b) *$\mathcal{A}$ is an ideal iff $\mathcal{A} = (\mathcal{A}^{\perp})^{\perp}$.*

Proof. To show (a), we only need to check that $\boldsymbol{A} \in J(\mathcal{A})$ provided $\boldsymbol{A} \leqslant \bigoplus_{s\in S} \boldsymbol{Y}^{(s)}$ with $\boldsymbol{Y}^{(s)} \in \mathcal{A}$. Assuming $\boldsymbol{A}$ is nontrivial, take a maximal family $\mathcal{E} = \{E_\gamma\}_{\gamma\in\Gamma}$ of mutually orthogonal nontrivial reducing subspaces for $\boldsymbol{A}$ such that $\boldsymbol{A}|_{E_\gamma} \leqslant \boldsymbol{X}^{(\gamma)}$ for some $\boldsymbol{X}^{(\gamma)} \in \mathcal{A}$ ($\gamma \in \Gamma$). Let F be the orthogonal complement of $\bigoplus_{\gamma\in\Gamma} E_\gamma$ (in $\overline{\mathcal{D}}(\boldsymbol{A})$). We only need to check that F is trivial. We infer from the maximality of $\mathcal{E}$ that $\boldsymbol{A}|_F \in \mathcal{A}^{\perp}$. Thus, thanks to (PR2), $\boldsymbol{A}|_F \perp_u \bigoplus_{s\in S} \boldsymbol{Y}^{(s)}$ and hence, by (PR1), $\boldsymbol{A}|_F$ is trivial and we are done.

The 'if' part of (b) is immediate, while 'only if' follows from Theorem 2.4.1. ∎

Remark 2.4.5. In Proposition 3.5.1 we shall show that for every ideal $\mathcal{A}$ there is a (unique up to unitary equivalence under some additional assumptions on $\boldsymbol{A}$) N-tuple $\boldsymbol{A}$ such that $\mathcal{A} = \{\boldsymbol{B}\colon \boldsymbol{B} \ll \boldsymbol{A}\}$. Thus, our Theorem 2.4.1 is a generalization of Ernest's Proposition 2.12 in [9].

The rest of the paper is devoted to the class $\mathcal{CDD}_N$.

3. STRUCTURAL DECOMPOSITION

3.1. Strong order

Everywhere below the prefix '$\leqslant^s$' says that the relevant term is understood with respect to this order. The aim of this chapter is to prove

THEOREM 3.1.1. *Let $\mathfrak{B}$ be a nonempty set of members of $\mathcal{CDD}_N$ and let $\mathsf{A}, \mathsf{B} \in \mathcal{CDD}_N$.*

(A) *$\mathfrak{B}$ has the $\leqslant^s$-g.l.b.*
(B) *$\mathfrak{B}$ has the $\leqslant^s$-l.u.b. if and only if every two-point subset of $\mathfrak{B}$ is $\leqslant^s$-upper bounded. In that case, $\inf_{\leqslant^s} \mathfrak{B} = \bigwedge \mathfrak{B}$ and $\sup_{\leqslant^s} \mathfrak{B} = \bigvee \mathfrak{B}$.*
(C) *The following conditions are equivalent:*
  (i) *the set $\{\mathsf{A}, \mathsf{B}\}$ is $\leqslant^s$-upper bounded,*
  (ii) *$\mathsf{A} \leqslant^s \mathsf{A} \vee \mathsf{B}$ and $\mathsf{B} \leqslant^s \mathsf{A} \vee \mathsf{B}$,*
  (iii) *A and B may be written in the forms $\mathsf{A} = \mathsf{E} \boxplus \mathsf{X}$ and $\mathsf{B} = \mathsf{E} \boxplus \mathsf{Y}$ for some $\mathsf{E}, \mathsf{X}, \mathsf{Y} \in \mathcal{CDD}_N$ such that $\mathsf{X} \perp_u \mathsf{Y}$.*
(D) *If $\{\mathsf{A}, \mathsf{B}\}$ is $\leqslant^s$-upper bounded, then $\mathsf{A} \leqslant \mathsf{B} \Leftrightarrow \mathsf{A} \leqslant^s \mathsf{B}$.*

Proof. We begin with (C). The implications (iii)$\Rightarrow$(ii)$\Rightarrow$(i) are immediate (indeed, if (iii) is fulfilled, $\mathsf{A} \vee \mathsf{B} = \mathsf{E} \boxplus \mathsf{X} \boxplus \mathsf{Y}$). To see that (iii) follows from (i), let $\mathsf{F} \in \mathcal{CDD}_N$ $\leqslant^s$-majorize A and B. This means that $\boldsymbol{A} \equiv \boldsymbol{F}|_K$ and $\boldsymbol{B} \equiv \boldsymbol{F}|_L$ for some $K, L \in \operatorname{cred}(\boldsymbol{F})$. Then P_K and P_L commute and therefore $K = M \oplus K'$ and $L = M \oplus L'$ where $M = K \cap L$, $K' = M^\perp \cap K$ and $L' = M^\perp \cap L$. Note that then $\boldsymbol{E} = \boldsymbol{F}|_M$, $\boldsymbol{X} = \boldsymbol{F}|_{K'}$ and $\boldsymbol{Y} = \boldsymbol{F}|_{L'}$ are pairwise unitarily disjoint and $\mathsf{A} = \mathsf{E} \boxplus \mathsf{X}$ and $\mathsf{B} = \mathsf{E} \boxplus \mathsf{Y}$.

Now we turn to (B). Suppose every two-point subset of $\mathfrak{B}$ is $\leqslant^s$-upper bounded. Let M be such that $\mathsf{B} \leqslant \mathsf{M}$ for every $\mathsf{B} \in \mathfrak{B}$. Put $\mathcal{M} = \mathcal{W}'(\boldsymbol{M})$. For every $\mathsf{B} \in \mathfrak{B}$ take $K(\mathsf{B}) \in \operatorname{red}(\boldsymbol{M})$ such that $\boldsymbol{B} \equiv \boldsymbol{M}|_{K(\mathsf{B})}$ and put $p_{\mathsf{B}} = P_{K(\mathsf{B})} \in \mathcal{M}$.

For a moment fix $\mathsf{A}, \mathsf{B} \in \mathfrak{B}$. By (C), there is $\mathsf{F} \leqslant \mathsf{M}$ such that $\mathsf{A} \leqslant^s \mathsf{F}$ and $\mathsf{B} \leqslant^s \mathsf{F}$. We infer, involving Proposition 2.3.1, that there is a projection $q \in E(\mathcal{M})$ for which $p_{\mathsf{A}} \sim c_{p_{\mathsf{A}}} q$ and $p_{\mathsf{B}} \sim c_{p_{\mathsf{B}}} q$. Then $c_{p_{\mathsf{B}}} p_{\mathsf{A}} \sim c_{p_{\mathsf{B}}} c_{p_{\mathsf{A}}} q$ and $c_{p_{\mathsf{A}}} p_{\mathsf{B}} \sim c_{p_{\mathsf{A}}} c_{p_{\mathsf{B}}} q$. This proves that

$$c_{p_{\mathsf{B}}} p_{\mathsf{A}} \sim c_{p_{\mathsf{A}}} p_{\mathsf{B}} \tag{3.1.1}$$

for all $\mathsf{A}, \mathsf{B} \in \mathfrak{B}$. Now put $w = \bigvee \{c_{p_{\mathsf{A}}} \colon \mathsf{A} \in \mathfrak{B}\} \in \mathcal{Z}(\mathcal{M})$. There is a family $\{z_{\mathsf{A}}\}_{\mathsf{A} \in \mathfrak{B}}$ of mutually orthogonal central projections in $\mathcal{M}$ such that $z_{\mathsf{A}} \leqslant c_{p_{\mathsf{A}}}$ for every $\mathsf{A} \in \mathfrak{B}$ and $\sum_{\mathsf{A} \in \mathfrak{B}} z_{\mathsf{A}} = w$. Put

$$q = \sum_{\mathsf{A} \in \mathfrak{B}} z_{\mathsf{A}} p_{\mathsf{A}} \in E(\mathcal{M}).$$

For $\mathsf{A}, \mathsf{B} \in \mathfrak{B}$ we have, by (3.1.1), $z_{\mathsf{B}} c_{p_{\mathsf{A}}} q = z_{\mathsf{B}} c_{p_{\mathsf{A}}} p_{\mathsf{B}} \sim z_{\mathsf{B}} c_{p_{\mathsf{B}}} p_{\mathsf{A}} = z_{\mathsf{B}} p_{\mathsf{A}}$ and consequently (since $w \geqslant c_{p_{\mathsf{A}}}$),

$$p_{\mathsf{A}} = \sum_{\mathsf{B} \in \mathfrak{B}} z_{\mathsf{B}} p_{\mathsf{A}} \sim \sum_{\mathsf{B} \in \mathfrak{B}} z_{\mathsf{B}} c_{p_{\mathsf{A}}} q = c_{p_{\mathsf{A}}} q.$$

Now if $E \in \operatorname{red}(\boldsymbol{M})$ is the range of q and $\boldsymbol{M}' = \boldsymbol{M}|_E$, Proposition 2.3.1 shows that $\mathsf{A} \leqslant^s \mathsf{M}'$ for every $\mathsf{A} \in \mathfrak{B}$. Hence, replacing M by M', we may assume that $p_{\mathsf{A}} \in \mathcal{Z}(\mathcal{M})$. It is known that in that case $\bigvee_{\mathsf{A} \in \mathfrak{B}} p_{\mathsf{A}}$ and $\bigwedge_{\mathsf{A} \in \mathfrak{B}} p_{\mathsf{A}}$ are, respectively, the l.u.b. and the g.l.b. with respect to '$\preccurlyeq$' in $E(\mathcal{M})$. It is left as an exercise that (B) now follows.

Finally, (A) follows from (B), and (D) is left to the reader. ∎

As a very special case of Theorem 3.1.1 we get

Corollary 3.1.2. *If $\{\mathsf{A}^{(s)}\}_{s \in S}$ is a nonempty family of mutually unitarily disjoint N-tuples, then $\bigvee_{s \in S} \mathsf{A}^{(s)} = \boxplus_{s \in S} \mathsf{A}^{(s)}$.*

Proof. One easily checks that $\boxplus_{s \in S} \mathsf{A}^{(s)}$ is the $\leqslant^s$-l.u.b. of $\{\mathsf{A}^{(s)}\}_{s \in S}$. Thus the assertion follows from Theorem 3.1.1. ∎

Example 3.1.3. $[N = 1]$ Let I_j for $j = 1, 2$ be the identity operator on a j-dimensional Hilbert space. It is clear that $\mathsf{I}_1 \leqslant \mathsf{I}_2$, $\mathsf{I}_1 \wedge \mathsf{I}_2 = \mathsf{I}_1$ and $\mathsf{I}_1 \vee \mathsf{I}_2 = \mathsf{I}_2$, while $\inf_{\leqslant^s}\{\mathsf{I}_1, \mathsf{I}_2\} = \mathsf{O}$ and $\{\mathsf{I}_1, \mathsf{I}_2\}$ is not $\leqslant^s$-upper bounded. This shows that $\leqslant^s$-g.l.b. in general differs from $\leqslant$-g.l.b. (although both always exist).

Proposition 3.1.4.

(A) *If $\mathsf{A} \leqslant \boxplus_{s \in S} \mathsf{B}^{(s)}$, then $\mathsf{A} = \boxplus_{s \in S} (\mathsf{A} \wedge \mathsf{B}^{(s)})$.*
(B) *Suppose $\mathsf{A}^{(s)} \leqslant \mathsf{X}$ $(s \in S \neq \emptyset)$ and $\mathsf{B} \leqslant^s \mathsf{X}$. Then*

$$\Big[\bigvee_{s \in S} \mathsf{A}^{(s)}\Big] \wedge \mathsf{B} = \bigvee_{s \in S} [\mathsf{A}^{(s)} \wedge \mathsf{B}].$$

If in addition $\bigoplus_{s \in S} \mathsf{A}^{(s)} \leqslant \mathsf{X}$, then

$$\Big[\bigoplus_{s \in S} \mathsf{A}^{(s)}\Big] \wedge \mathsf{B} = \bigoplus_{s \in S} [\mathsf{A}^{(s)} \wedge \mathsf{B}].$$

Proof. To prove (A), put $\mathsf{B} = \boxplus_{s \in S} \mathsf{B}^{(s)}$. Since each $\mathsf{B}^{(s)}$ corresponds to a central projection in $\mathcal{W}'(\boldsymbol{B})$, the assertion easily follows. The same argument works for (B)—here B corresponds to a central projection in $\mathcal{W}'(\boldsymbol{X})$. ∎

A counterpart of a part of Proposition 3.1.4 for the order '$\leqslant$' will be proved in Theorem 4.4.10. However, this will be much more complicated.

3.2. Steering projections in $\mathcal{W}^*$-algebras

We would like to propose a slightly modified approach to the so-called *dimension theory* of $\mathcal{W}^*$-algebras (see e.g. [18, Chapter 5, §5] and [19, Chapter 6]; [35, Chapter 5, §1]; [13, 14]; [37]; [31]). Usually one decomposes a projection in a $\mathcal{W}^*$-algebra into (in a sense) 'homogeneous' parts, as done by Griffin [13, 14], Tomiyama [37] and Sherman [31]. In the next chapter we will do essentially the same but in a different manner, convenient for applications to the class $\mathcal{CDD}_N$. In every $\mathcal{W}^*$-algebra $\mathcal{M}$ we shall distinguish a projection, called *steering*, and we shall show how it 'controls' the Murray–von Neumann order on $\mathcal{E}(\mathcal{M})$. As we will see, the steering projection is defined in different ways for type II_1; type II_∞; and type I or III algebras. Therefore we shall divide our investigations into these three cases.

3.2.1. Type II_1. When $\mathcal{M}$ is a type II_1 $\mathcal{W}^*$-algebra, it seems to be most appropriate to call the unit of $\mathcal{M}$ the steering projection.

3.2.2. Types I and III. We assume that $\mathcal{M}$ is a type I or III $\mathcal{W}^*$-algebra. We say $\mathcal{M}$ is *quasi-commutative* iff $p \sim c_p$ for every $p \in E(\mathcal{M})$. A projection $p \in E(\mathcal{M})$ is *quasi-abelian* iff $p = 0$ or $p\mathcal{M}p$ is quasi-commutative.

LEMMA 3.2.1. *For $p \in E(\mathcal{M})$ the following conditions are equivalent:*

(i) *p is quasi-abelian,*
(ii) *for every $q \in E(\mathcal{M})$ with $q \leqslant p$, $q \sim c_q p$,*
(iii) *for every $q \in E(\mathcal{M})$, $p \preccurlyeq q \Leftrightarrow p \leqslant c_q$.*

Proof. The equivalence of (i) and (ii) follows from the fact that the central support of $q \in E(p\mathcal{M}p)$ with respect to $p\mathcal{M}p$ coincides with $c_q p$ (where c_q is the central support of q in $\mathcal{M}$).

To show that (iii) follows from (ii), assume that $0 \neq p \leqslant c_q$ and take a maximal family $\{p_s\}_{s\in S} \subset E(\mathcal{M})$ of nonzero projections such that $p_s \leqslant p$, $p_s \preccurlyeq q$ for $s \in S$ and $c_{p_s} c_{p_t} p = 0$ for distinct $s, t \in S$. Notice that then $p = \sum_{s\in S} c_{p_s} p$ and $c_{p_s} c_{p_t} c_p = 0$ for different $s, t \in S$. Now we infer from (ii) that $c_{p_s} p \preccurlyeq q$ and consequently $c_{p_s} p \preccurlyeq c_{p_s} c_p q$. So, $p = \sum_{s\in S} c_{p_s} p \preccurlyeq (\sum_{s\in S} c_{p_s} c_p) q \leqslant q$.

Finally, under the assumption of (iii), for $q \leqslant p$ put $r = q + (1 - c_q)p$, notice that $c_r = c_p \geqslant p$ and thus, by (iii), $p \preccurlyeq r$. Consequently, $c_q p \preccurlyeq c_q r = q$ and we are done. ∎

A *steering* projection in (a type I or III $\mathcal{W}^*$-algebra) $\mathcal{M}$ is a quasi-abelian projection $p \in E(\mathcal{M})$ such that $c_p = 1$.

THEOREM 3.2.2.

(I) *Suppose $\mathcal{M}$ is type I. A projection $p \in E(\mathcal{M})$ with $c_p = 1$ is steering iff p is abelian. In particular, $\mathcal{M}$ has a steering projection and any two steering projections are Murray–von Neumann equivalent.*
(II) *Suppose $\mathcal{M}$ is type III. $\mathcal{M}$ has a steering projection and any two steering projections are Murray–von Neumann equivalent.*

Proof. Point (I) is left to the reader. We shall give a sketch of proof of (II). If p and q are steering, then $c_p = c_q = 1$ and thus $p \preccurlyeq q$ and $q \preccurlyeq p$, by Lemma 3.2.1. This establishes uniqueness up to Murray–von Neumann equivalence. To show the existence, take a maximal family $\{p_s\}_{s\in S} \subset E(\mathcal{M})$ of mutually centrally orthogonal nonzero projections each of which is countably decomposable and put $p = \sum_{s\in S} p_s$. Such a projection is steering since each p_s is quasi-abelian, e.g. by [19, Corollary 6.3.5]. ∎

3.2.3. Type II_∞. Finally, assume $\mathcal{M}$ is a type II_∞ $\mathcal{W}^*$-algebra. Since the unit of $\mathcal{M}$ may be written in the form $\sum_{n=1}^{\infty} p_n$ with $p_n \sim 1$ for each $n \geqslant 1$, for every projection $q \in E(\mathcal{M})$ there is a countable infinite family of mutually orthogonal projections each of which is Murray–von Neumann equivalent to q. For each $n \in \{1, 2, \ldots\} \cup \{\omega\}$ we shall write $n \odot q$ to denote any projection (or, a unique member of $\mathcal{E}(\mathcal{M})$) in $\mathcal{M}$ which is the sum of (exactly) n copies of q. (Here by a *copy* we mean any projection which is Murray–von Neumann equivalent to q; $\omega \odot q$ is the sum of $\aleph_0$ copies of q.)

Lemma 3.2.3. *For $p \in E(\mathcal{M})$ the following conditions are equivalent:*

(i) *p is finite,*
(ii) *whenever $p \leqslant c_q$ for $q \in E(\mathcal{M})$, there is a sequence $(z_n)_{n=1}^{\infty}$ of central projections in $\mathcal{M}$ such that $\sum_{n=1}^{\infty} z_n = 1$ and $z_n p \preccurlyeq n \odot q$ for any $n \geqslant 1$.*

Proof. Let $q_0 \in E(\mathcal{M})$ be a finite projection such that $c_{q_0} = 1$. If (ii) is satisfied, then $z_n p \preccurlyeq n \odot q_0$ for a suitable sequence $(z_n)_{n=1}^{\infty}$ of central projections. Then $z_n p$ is finite and thus so is $(\bigvee_{n \geqslant 1} z_n)p = p$.

Conversely, if p is finite and $p \leqslant c_q$, there is a family $\{q_s\}_{s \in S}$ of mutually orthogonal projections such that $p = \sum_{s \in S} q_s$ and $q_s \preccurlyeq q$ for all $s \in S$. Let $\mathrm{tr}\colon p\mathcal{M}p \to \mathcal{Z}(p\mathcal{M}p) = \mathcal{Z}(\mathcal{M})p$ be the trace on $p\mathcal{M}p$. There are central (in $\mathcal{M}$) projections $z_{n,k}^{(s)}$ with $1 \leqslant k \leqslant 2^n$ and $n \geqslant 1$ such that

$$\mathrm{tr}(q_s) = \sum_{n=1}^{\infty} \Big(\sum_{k=1}^{2^n} \frac{k}{2^n} z_{n,k}^{(s)} p \Big).$$

Since $\mathrm{tr}(z_{n,k}^{(s)} p) \leqslant 2^n \,\mathrm{tr}(q_s)$, we have $z_{n,k}^{(s)} p \preccurlyeq 2^n \odot q_s \preccurlyeq 2^n \odot q$. Moreover, we infer from the relation $p = \mathrm{tr}(p) = \sum_{s \in S} \mathrm{tr}(q_s)$ that $\bigvee_{s,n,k} z_{n,k}^{(s)} \geqslant p$. Reindexing the family $\{z_{n,k}^{(s)}\}_{s,n,k}$ we obtain a collection $\{w_t\}_{t \in T} \subset E(\mathcal{Z}(\mathcal{M}))$ such that

$$w_t p \preccurlyeq m(t) \odot q \quad \text{and} \quad w := \bigvee_{t \in T} w_t \geqslant p$$

where $m(t)$ is some positive integer. Now let $\{v_t\}_{t \in T}$ be a family of mutually orthogonal central projections such that $v_t \leqslant w_t$ $(t \in T)$ and $\sum_{t \in T} v_t = w$. Let $* \notin T$, $v_* = 1 - w$ and $m(*) = 1$. Observe that $v_t p \preccurlyeq m(t) \odot q$ for every $t \in T_* := T \cup \{*\}$, and $\sum_{t \in T_*} v_t = 1$. Finally, define z_n for $n > 0$ by $z_n = \sum \{v_t \colon t \in T_*,\ m(t) = n\}$. ■

Let $E_\omega(\mathcal{M}) = \{q \in E(\mathcal{M}) \colon q \sim \omega \odot p$ for some finite projection $p\}$.

Lemma 3.2.4.

(a) *For every $p \in E_\omega(\mathcal{M})$ and a properly infinite projection $q \in E(\mathcal{M})$, $p \preccurlyeq q \Leftrightarrow p \leqslant c_q$.*
(b) *If $p \in E_\omega(\mathcal{M})$ is such that $c_p = 1$, then $q \sim c_q p$ for every $q \in E_\omega(\mathcal{M})$.*
(c) *If $p \in E_\omega(\mathcal{M})$ and $z \in E(\mathcal{Z}(\mathcal{M}))$, then $zp \in E_\omega(\mathcal{M})$.*

Proof. Point (c) is immediate and (b) follows from (a) and (c). So, it suffices to check (a). Assume p and q are as there and $p \leqslant c_q$. Take a finite projection p_0 such that $p \sim \omega \odot p_0$. By Lemma 3.2.3, $z_n p_0 \preccurlyeq n \odot q$ for a suitable sequence $(z_n)_{n=1}^{\infty}$ of central projections. Since q is properly infinite, $q \sim \omega \odot q$ and hence $z_n p_0 \preccurlyeq z_n q$, which gives $p_0 \preccurlyeq q$. Consequently, $p \sim \omega \odot p_0 \preccurlyeq \omega \odot q \sim q$ and we are done. ■

A *steering* projection in (a type II_∞ $\mathcal{W}^*$-algebra) $\mathcal{M}$ is a projection $p \in E_\omega(\mathcal{M})$ with $c_p = 1$. Since $E_\omega(\mathcal{M})$ consists of properly infinite projections, Lemma 3.2.4 ensures that any two steering projections in $\mathcal{M}$ are Murray–von Neumann equivalent.

Now if $\mathcal{M}$ is an arbitrary $\mathcal{W}^*$-algebra, the steering projection of $\mathcal{M}$ is defined as the sum of the steering projections of type I, II_1, II_∞ and III parts of $\mathcal{M}$. It is clear that any two steering projections in $\mathcal{M}$ are Murray–von Neumann equivalent. The reader should also verify that if $p \in E(\mathcal{M})$ is a steering projection, then $c_p = 1$, and zp is a steering projection of $\mathcal{M}z$ for every central projection z in $\mathcal{M}$.

3.3. Decomposition relative to a steering projection

Let us first generalize the idea of the previous chapter. Whenever α is an (arbitrary) cardinal number and p and q are projections in a $\mathcal{W}^*$-algebra $\mathcal{M}$, p is said to be a *copy* of q provided $p \sim q$; and $p \sim \alpha \odot q$ iff p is a sum of α copies of q. In particular, $p \sim 0 \odot q$ is equivalent to $p = 0$. When $\mathcal{M}$ contains α mutually orthogonal copies of q, we shall also write $p \preccurlyeq \alpha \odot q$ with the obvious meaning. Similarly, we shall say that p contains α orthogonal copies of q iff $q' \sim \alpha \odot q$ for some projection $q' \leqslant p$.

Using standard methods (such as Lemma 6.3.9 and Theorem 6.3.11 of [19]; cf. [1, Proposition III.1.7.1]), similar to those in [13, 14], [37] or [31], one shows the next result (we skip its proof). To simplify its statement, let us define $\Lambda_I = \mathrm{Card}$ (the class of all cardinals), $\Lambda_{II} = \mathrm{Card}_\infty \cup \{0, 1\}$ and $\Lambda_{III} = \mathrm{Card}_\infty \cup \{0\}$ where Card_∞ is the class of all infinite cardinals. For any cardinal α, α^+ is the immediate successor of α, that is, $\alpha^+ = \min\{\beta \in \mathrm{Card}\colon \beta > \alpha\}$. Below '$\sim$' refers to the Murray–von Neumann equivalence in $E(\mathcal{M})$.

THEOREM 3.3.1. *Let $\mathcal{M}$ be a properly infinite $\mathcal{W}^*$-algebra, p a steering projection of $\mathcal{M}$ and let $\mathcal{A} = p\mathcal{M}p$. Let $z^I, z^{II}, z^{III} \in \mathcal{Z}(\mathcal{A})$ be projections such that $z^I + z^{II} + z^{III} = p$ and $\mathcal{A}z^i$ is of type i for $i = I, II, III$. For every $q \in E(\mathcal{M})$ there is a unique system $\{z^I_\alpha(q)\}_{\alpha\in\Lambda_I} \cup \{z^{II}_\alpha(q)\}_{\alpha\in\Lambda_{II}} \cup \{z^{III}_\alpha(q)\}_{\alpha\in\Lambda_{III}} \subset \mathcal{Z}(\mathcal{A})$ of mutually orthogonal projections such that for $i = I, II, III$, $\sum_{\alpha\in\Lambda_i} z^i_\alpha(q) = z^i$ and $c_{z^i_\alpha(q)}q \sim \alpha \odot z^i_\alpha(q)$ if only $\alpha \in \Lambda_i$ and $(i, \alpha) \neq (II, 1)$, while $c_{z^{II}_1(q)}q$ is finite and $z^{II}_1(q) \sim \omega \odot c_{z^{II}_1(q)}q$.*

What is more, $z^i_\alpha(q)$ may be characterized as follows:

$$z^{II}_1(q) = \bigvee\{w \in E(\mathcal{A}) |\ w \leqslant z^{II},\ \forall v \in E(\mathcal{A}),\ 0 \neq v \leqslant w\colon\ c_v q \neq 0 \text{ and } q \text{ contains no copy of } \omega \odot v\}$$

and when $(i, \alpha) \neq (II, 1)$,

$$z^i_\alpha(q) = \bigvee\{w \in E(\mathcal{A}) |\ w \leqslant z^i,\ c_w q \sim \alpha \odot w,\ \forall v \in E(\mathcal{A})\colon\ 0 \neq v \leqslant w \Rightarrow q \text{ does not contain } \alpha^+ \text{ orthogonal copies of } v\}.$$

The statement of the above theorem is complicated. We have formulated it in this way, having in mind further applications to the class $\mathcal{CDD}_N$.

For the purpose of this paper, let us introduce the following

Definition 3.3.2. Let $i \in \{I, II, III\}$ and $\alpha \in \mathrm{Card}_\infty$. A $\mathcal{W}^*$-algebra $\mathcal{M}$ is said to be of *(pure) type* i_α iff $\mathcal{M}$ is of pure type i and $1 \sim \alpha \odot p$ where p is the steering projection.

Recall that the above definition of type I_α $\mathcal{W}^*$-algebras is equivalent to the classical definition of this type, and that below types I_n for finite n and II_1 are understood in the usual sense.

Proposition 3.3.3. *For every $\mathcal{W}^*$-algebra $\mathcal{M}$ there is a unique system*

$$\{z_\alpha^i \colon i \in \{I, II, III\},\ \alpha \in \Lambda_i \setminus \{0\}\} \subset E(\mathcal{Z}(\mathcal{M}))$$

such that $1 = \sum_{i,\alpha} z_\alpha^i$ and for each i and α either $z_\alpha^i = 0$ or $\mathcal{M} z_\alpha^i$ is of pure type i_α.

To simplify the statements of next results, we fix $i \in \{I, II, III\}$, $\gamma \in \mathrm{Card}_\infty$, a type i_γ $\mathcal{W}^*$-algebra $\mathcal{M}$ and a steering projection p of $\mathcal{M}$. Additionally, we put $\mathcal{A} = p\mathcal{M}p$ and $\Lambda = \{\alpha \in \Lambda_i \colon \alpha \leqslant \gamma\}$. For every $q \in E(\mathcal{M})$ let $z_\alpha(q) = z_\alpha^i(q)$ where $z_\alpha^i(q)$ is as in Theorem 3.3.1. It is easily seen that $z_\alpha(q) = 0$ for $\alpha > \gamma$ and $\sum_{\alpha\in\Lambda} z_\alpha(q) = p$. Therefore for every $q \in E(\mathcal{M})$ we shall deal with a <u>set</u> $\{z_\alpha(q)\}_{\alpha\in\Lambda}$ of projections.

We skip the proof of the next result (cf. [31]).

Proposition 3.3.4. *For $q, q' \in E(\mathcal{M})$ the following conditions are equivalent:*

(i) $q \preccurlyeq q'$,
(ii) $z_\beta(q) z_\alpha(q') = 0$ *whenever $\alpha, \beta \in \Lambda$ and $\beta > \alpha$; and $c_{z_1(q)} c_{z_1(q')} q \preccurlyeq c_{z_1(q)} c_{z_1(q')} q'$ provided $i = II$.*

The following result explains the terminology proposed by us.

Proposition 3.3.5. *Let $q \in E(\mathcal{M})$ be nonzero. Then $c_q \sim \gamma \odot q$ and $\mathcal{M}$ does not contain γ^+ orthogonal copies of q.*

Proof. The second claim is left to the reader. For every positive cardinal $\beta \in \Lambda$ let S_β be a set such that $\mathrm{card}(S_\beta) = \beta$ and let $\kappa_\beta \colon S_\gamma \times S_\beta \to S_\gamma$ be a bijection. Since $\mathcal{M}$ is of type i_γ, there is a collection $\{p_s\}_{s\in S_\gamma}$ of mutually orthogonal projections which are Murray–von Neumann equivalent to p and sum up to 1. For $s \in S_\gamma$ let

$$q_s = \sum_{\beta\in\Lambda\setminus\{0\}} c_{z_\beta(q)} \sum_{t\in S_\beta} p_{\kappa_\beta(s,t)}.$$

Since $c_{z_\beta(q)} p_s \sim z_\beta(q)$ and $\sum_{\beta\in\Lambda\setminus\{0\}} c_{z_\beta(q)} = c_q$, we have $q_s \sim q$ for $s \in S_\gamma$. Finally,

$$\sum_{s\in S_\gamma} q_s = \sum_{\beta\in\Lambda\setminus\{0\}} c_{z_\beta(q)} \sum_{(s,t)\in S_\gamma\times S_\beta} p_{\kappa_\beta(s,t)} = \sum_{\beta\in\Lambda\setminus\{0\}} c_{z_\beta(q)} = c_q. \ \blacksquare$$

Proposition 3.3.6. *For every $q \in E(\mathcal{M})$ there are projections $q_\#, q^\# \in E(\mathcal{M})$ such that $1 - q_\# \sim q \sim 1 - q^\#$ and $q_\# \preccurlyeq q' \preccurlyeq q^\#$ for every $q' \in E(\mathcal{M})$ with $1 - q' \sim q$. Moreover, $q^\# \sim 1$ and $q_\# \sim 1 - c_{z_\gamma(q)}$.*

Proof. Since for $i = I, III$ arguments are similar, we shall only sketch the proof for $i = II$ (which is most complicated). Since $1 \sim 2 \odot 1$, it is clear that there is $q^\# \in E(\mathcal{M})$ such that $q^\# \sim 1$ and $1 - q^\# \sim q$. Thus we only need to find $q_\#$. For each $\beta \in \Lambda$ let S_β be a set of cardinality β and $\{p_s\}_{s\in S_\gamma}$ be a collection of mutually orthogonal projections which are Murray–von Neumann equivalent to p and sum to 1. We assume that $S_\beta \subset S_\gamma$ for each $\beta \in \Lambda$. Let $s_1 \in S_1$. Take $v \in E(\mathcal{M})$ with $v \leqslant p_{s_1}$ and $v \sim c_{z_1(q)}q$, and put

$$q_\# = c_{z_1(q)}(p_{s_1} - v) + \sum_{\beta\in\Lambda} c_{z_\beta(q)} \sum_{s\in S_\gamma\setminus S_\beta} p_s.$$

Since $\sum_{\beta\in\Lambda} c_{z_\beta(q)} = 1$ and $\operatorname{card}(S_\gamma \setminus S_\beta) = \gamma$ if only $\beta < \gamma$, we infer that $q_\# \sim 1 - c_{z_\gamma(q)}$. This implies that $z_\gamma(q_\#) = (1-c_{z_\gamma(q)})p = p - z_\gamma(q) = \sum_{\beta\in\Lambda\setminus\{\gamma\}} z_\beta(q)$, $z_0(q_\#) = c_{z_\gamma(q)}p = z_\gamma(q)$ and $z_\beta(q_\#) = 0$ for each $\beta \in \Lambda \setminus \{0, \gamma\}$ (in particular, $z_1(q_\#) = 0$). Further, observe that $c_{z_1(q)}v = v$ and thus

$$1 - q_\# = v + \sum_{\beta\in\Lambda\setminus\{1\}} c_{z_\beta(q)} \sum_{s\in S_\beta} p_s,$$

which yields $1 - q_\# \sim q$. Now let $q' \in E(\mathcal{M})$ be such that $1 - q' \sim q$. Thanks to Proposition 3.3.4, $q_\# \preccurlyeq q'$ iff $z_\beta(q_\#)z_\alpha(q') = 0$ whenever $\alpha, \beta \in \Lambda$ and $\alpha < \beta$ (because $z_1(q_\#) = 0$). In our situation this is equivalent to $z_\beta(q)z_\alpha(q') = 0$ for all $\alpha, \beta \in \Lambda \setminus \{\gamma\}$. For such α and β we have

$$w := c_{z_\beta(q)}c_{z_\alpha(q')} = wq' + w(1 - q')$$

and $w(1 - q') \sim wq$. But

$$\begin{cases} wq' \sim \alpha \odot (wp) & \text{if } \alpha \neq 1, \\ wq' \quad \text{is finite} & \text{if } \alpha = 1, \end{cases} \quad \text{and} \quad \begin{cases} wq \sim \beta \odot (wp) & \text{if } \beta \neq 1, \\ wq \quad \text{is finite} & \text{if } \beta = 1. \end{cases}$$

We conclude that either w is finite (and hence $w = 0$) or $w \sim \max(\alpha, \beta) \odot wp$. At the same time, thanks to e.g. Proposition 3.3.5, $w \sim \gamma \odot wp$, which implies that $w = 0$ and we are done. ∎

Since in every finite $\mathcal{W}^*$-algebra $\mathcal{W}$, $1 - q' \sim q$ iff $q' \sim 1 - q$ for any $q, q' \in E(\mathcal{W})$, Proposition 3.3.6 gives

THEOREM 3.3.7. *Let $\mathcal{W}$ be a $\mathcal{W}^*$-algebra and $q \in E(\mathcal{W})$. There are projections $q^\#$ and $q_\#$ such that $1 - q_\# \sim q \sim 1 - q^\#$ and $q_\# \preccurlyeq q' \preccurlyeq q^\#$ whenever $q' \in E(\mathcal{W})$ is such that $1 - q' \sim q$. What is more, if $\mathcal{W}$ is properly infinite, then $q^\# \sim 1$ and $q_\#$ is Murray–von Neumann equivalent to a central projection.*

Our last aim of this chapter is

PROPOSITION 3.3.8. *Let S be an (infinite) set whose size is a limit cardinal. Let $\{p_s\}_{s\in S}$ be a family of mutually orthogonal projections in a $\mathcal{W}^*$-algebra $\mathcal{W}$ which sum to 1. For a nonempty set $A \subset S$ put $q_A = \sum_{s\in A} p_s$. Then 1 is the l.u.b. of the family $\{q_A\colon A \subset S,\ 0 < \operatorname{card}(A) < \operatorname{card}(S)\}$ with respect to the Murray–von Neumann order.*

Proof. Thanks to Proposition 3.3.3, we may and do assume that $\mathcal{W}$ is of pure type i_γ. Since the assertion is known to be true for finite algebras $\mathcal{W}$, we assume in addition that $\mathcal{W}$ is properly infinite—that is, that γ is infinite. Finally, we reduce our considerations to the case when the steering projection p of $\mathcal{W}$ is countably decomposable.

Let $q \in E(\mathcal{W})$ be such that $q_A \preccurlyeq q$ for each $A \in \mathcal{S} := \{A \subset S\colon 0 < \operatorname{card}(A) < \operatorname{card}(S)\}$. We need to show that $q \sim 1$. Equivalently, we have to prove that $z^i_\alpha(q) = 0$ provided $\alpha < \gamma$. When $i = \mathit{II}$, $c_{z^{\mathit{II}}_1(q)}q$ is finite and $c_{z^{\mathit{II}}_1(q)} \sum_{s\in A} p_s \preccurlyeq c_{z^{\mathit{II}}_1(q)}q$ for each

$A \in \mathcal{S}$, which implies that $c_{z_1^{II}(q)} \preccurlyeq c_{z_1^{II}(q)}q$. Consequently, $c_{z_1^{II}(q)}$ is finite and thus $z_1^{II}(q) = 0$. Also when $i = I$ and α is finite, $z_\alpha^i(q) = 0$, because then $\alpha \odot p$ is finite.

Now let α be infinite. Then $c_{z_\alpha^i(q)}q \sim \alpha \odot z_\alpha^i(q) \sim \alpha \odot (c_{z_\alpha^i(q)}p)$ and $c_{z_\alpha^i(q)}q_A \preccurlyeq c_{z_\alpha^i(q)}q$ for any $A \in \mathcal{S}$. Towards a contradiction, assume $z_\alpha^i(q) \neq 0$. Replacing $\mathcal{W}$ by $\mathcal{W}c_{z_\alpha^i(q)}$, we may assume $c_{z_\alpha^i(q)} = 1$, that is, $z_\alpha^i(q) = p$. We then have $q \sim \alpha \odot p$, $1 \sim \gamma \odot p$ and $q_A \preccurlyeq q$ ($A \in \mathcal{S}$). We consider two cases. When $\operatorname{card}(S) \leqslant \alpha$, we easily get $p_s \preccurlyeq q$ and thus $1 = \sum_{s \in S} p_s \preccurlyeq \alpha \odot q \sim \alpha^2 \odot p$, which contradicts the facts that $\alpha^2 < \gamma$ and $1 \sim \gamma \odot p$.

Finally, assume that $\operatorname{card}(S) > \alpha$. Since p is countably decomposable and $q \sim \alpha \odot p$, $\operatorname{card}(\{s \in A\colon p_s \neq 0\}) \leqslant \alpha$ for any $A \in \mathcal{S}$ (because $q_A \preccurlyeq q$). We conclude that $A := \{s \in S\colon p_s \neq 0\} \in \mathcal{S}$ (because $\operatorname{card}(S) > \alpha^+$). But then $1 = q_A \preccurlyeq q$ and we are done. ∎

Example 3.3.9. As the following example shows (compare with [37, Example 3]), the assumption in Proposition 3.3.8 that the size of S is a limit cardinal is essential. Let $\mathcal{H}$ be a Hilbert space of dimension $\aleph_1$, S be a set of cardinality $\aleph_1$ and let $\{e_s\}_{s \in S}$ be an orthonormal basis of $\mathcal{H}$. Further, let $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and for $s \in S$ let $p_s \in E(\mathcal{M})$ be the orthogonal rank-one projection onto the linear span of e_s. Now if q_A's are defined as in Proposition 3.3.8, then $q_A \preccurlyeq q_J$ for every nonempty set $A \subset S$ of size less than $\aleph_1$ where J is a countable infinite subset of S and hence 1 is not equivalent to the l.u.b. (which is q_J).

3.4. Minimal and semiminimal tuples

The idea of steering projections will now be adapted to the class $\mathcal{CDD}_N$. Following Ernest [9], we say a nontrivial N-tuple $\mathsf{A} \in \mathcal{CDD}_N$ is (*of*) *type* I, II, III iff $\mathcal{W}'(\boldsymbol{A})$ is such. Additionally, we let the trivial N-tuple be of each of these types.

We begin with a result which will find many applications.

Lemma 3.4.1. *Every collection of mutually unitarily disjoint nontrivial N-tuples has cardinality not greater than $2^{\aleph_0}$.*

Proof. Suppose
$$\mathsf{A}^{(s)} \perp_u \mathsf{A}^{(s')} \tag{3.4.1}$$
(and $\mathsf{A}^{(s)} \neq \mathsf{O}$) for distinct $s, s' \in S$. For $n \in J = \{1, 2, \ldots\} \cup \{\aleph_0\}$ let $\mathcal{H}_n$ be a fixed Hilbert space of dimension n. By Theorem 2.2.4, for each $s \in S$ there are $n(s) \in J$ and $\boldsymbol{B}^{(s)} \in \mathrm{CDD}_N(\mathcal{H}_{n(s)})$ such that $\mathsf{B}^{(s)} \leqslant \mathsf{A}^{(s)}$. We infer from (3.4.1) that $\boldsymbol{B}^{(s)} \neq \boldsymbol{B}^{(s')}$ for distinct $s, s' \in S$. Now the assertion easily follows from the fact that $\operatorname{card}(\mathrm{CDD}_N(\mathcal{H}_n)) \leqslant 2^{\aleph_0}$ for every $n \in J$. ∎

Definition 3.4.2. $\mathsf{A} \in \mathcal{CDD}_N$ is said to be *minimal* iff for every $\mathsf{B} \in \mathcal{CDD}_N$,
$$\mathsf{A} \ll \mathsf{B} \;\Rightarrow\; \mathsf{A} \leqslant \mathsf{B}.$$
A is said to be *multiplicity free* ($\mathsf{A} \in \mathcal{MF}_N$) iff there is no nontrivial $\mathsf{B} \in \mathcal{CDD}_N$ for which $2 \odot \mathsf{B} \leqslant \mathsf{A}$, and is a *hereditary idempotent* ($\mathsf{A} \in \mathcal{HI}_N$) iff $\mathsf{B} = 2 \odot \mathsf{B}$ for every $\mathsf{B} \leqslant \mathsf{A}$. We shall write $\mathsf{A} \in \mathcal{HIM}_N$ whenever A is both a hereditary idempotent and minimal.

Minimal members of $\mathcal{CDD}_N$ correspond to quasi-abelian projections.

REMARK 3.4.3. The work of Ernest [9] deals with (single) bounded operators. In this context, our definition of a multiplicity free operator is equivalent to Ernest's (Definition 1.21 in [9]).

THEOREM 3.4.4.

(I) *For every* $\mathsf{A} \in \mathcal{CDD}_N$,
$$\mathsf{A} = 2 \odot \mathsf{A} \Leftrightarrow \mathsf{A} = \aleph_0 \odot \mathsf{A}.$$
(II) *For* $\mathsf{A} \in \mathcal{CDD}_N$ *the following conditions are equivalent:*

(i) A *is minimal,*
(ii) *for each* $\mathsf{B} \in \mathcal{CDD}_N$, $\mathsf{B} \leqslant \mathsf{A} \Rightarrow \mathsf{B} \leqslant^s \mathsf{A}$.

If A *is minimal and* $\mathsf{B} \leqslant \mathsf{A}$, *then* B *is minimal as well.*

(III) *For* $\mathsf{A} \in \mathcal{CDD}_N$ *the following conditions are equivalent:*

(i) $\mathsf{A} \in \mathcal{MF}_N$,
(ii) $\mathsf{A} = \mathsf{O}$ *or* $\mathcal{W}'(\boldsymbol{A})$ *is commutative.*

In particular, if $\mathsf{A} \in \mathcal{MF}_N$ *and* $\mathsf{B} \leqslant \mathsf{A}$, *then* $\mathsf{B} \in \mathcal{MF}_N$ *as well.*

(IV) *Every multiplicity free N-tuple is minimal and unitarily disjoint from any hereditary idempotent.*

(V) *If* $\mathsf{A} \in \mathcal{HI}_N$ *and* $\mathsf{B} \ll \mathsf{A}$, *then* $\mathsf{B} \in \mathcal{HI}_N$ *as well.*

(VI) *There exist unique* $\mathsf{J}_I, \mathsf{J}_{III} \in \mathcal{CDD}_N$ *such that* $\mathsf{J}_I \in \mathcal{MF}_N$, $\mathsf{J}_{III} \in \mathcal{HIM}_N$, $\mathsf{J}_I \boxplus \mathsf{J}_{III}$ *is minimal and for every* $\mathsf{A} \in \mathcal{CDD}_N$:

(a) $\mathsf{A} \in \mathcal{MF}_N$ *iff* $\mathsf{A} \leqslant \mathsf{J}_I$,
(b) $\mathsf{A} \ll \mathsf{J}_I$ *iff* $\mathsf{A} = \mathsf{O}$ *or* $\mathcal{W}'(\boldsymbol{A})$ *is type I,*
(c) $\mathsf{A} \in \mathcal{HI}_N$ *iff* $\mathsf{A} \ll \mathsf{J}_{III}$, *iff* $\mathsf{A} = \mathsf{O}$ *or* $\mathcal{W}'(\boldsymbol{A})$ *is type III,*
(d) $\mathsf{A} \in \mathcal{HIM}_N$ *iff* $\mathsf{A} \leqslant \mathsf{J}_{III}$,
(e) A *is minimal iff* $\mathsf{A} \leqslant \mathsf{J}_I \boxplus \mathsf{J}_{III}$.

What is more, $\dim(\mathsf{J}_I) + \dim(\mathsf{J}_{III}) \leqslant 2^{\aleph_0}$.

Proof. In all points of the theorem we make use of Proposition 2.3.1. The counterparts of points (I) and (V) are well known for projections in $\mathcal{W}^*$-algebras, (II) follows from Lemma 3.2.1, (III) is immediate, (IV) is implied by (III) and the relevant definitions. To prove (VI), take a maximal collection (cf. Lemma 3.4.1) of nontrivial mutually unitarily disjoint multiplicity free N-tuples (respectively hereditary idempotents) whose representatives act in separable spaces and define J_I (J_{III}) as the direct sum of this family. One may check that the N-tuple obtained in this way belongs to $\mathcal{MF}_N$ ($\mathcal{HIM}_N$) and—since J_I and J_{III} are unitarily disjoint—that $\mathsf{J}_I \boxplus \mathsf{J}_{III}$ is minimal. It follows from the maximality of the family considered and Theorem 2.2.4 that J_I and J_{III} are the greatest members of $\mathcal{MF}_N$ and $\mathcal{HIM}_N$. The details are left to the reader (cf. Propositions 2.12, 1.27 and 1.29 and Corollary 1.37 in [9]). (For the proof of (b) and (c) see also Theorem 3.6.1.) ■

Theorem 3.4.4 shows that there is a greatest minimal N-tuple in $\mathcal{CDD}_N$, namely $\mathsf{J}_I \boxplus \mathsf{J}_{III}$, and that it covers all type I and III N-tuples. Since there are also type II ones, we need to introduce one more notion.

Definition 3.4.5. $\mathsf{A} \in \mathcal{CDD}_N$ is said to be *semiminimal* ($\mathsf{A} \in \mathcal{SM}_N$) iff it is unitarily disjoint from every minimal N-tuple and satisfies the following condition. Whenever $\mathsf{B} \in \mathcal{CDD}_N$ is such that $\mathsf{A} \ll \mathsf{B}$, A may be written in the form $\mathsf{A} = \boxplus_{n=1}^{\infty} \mathsf{A}_n$ where $\mathsf{A}_n \leqslant n \odot \mathsf{B}$ for each $n \geqslant 1$.

Before stating the next result, we underline that there is no greatest semiminimal member of $\mathcal{CDD}_N$.

Theorem 3.4.6.

(I) *For* $\mathsf{A} \in \mathcal{CDD}_N$, $\mathsf{A} \in \mathcal{SM}_N$ *iff* $\mathsf{A} = \mathsf{O}$ *or* $\mathcal{W}'(\boldsymbol{A})$ *is type* II_1. *In particular, if* $\mathsf{A} \in \mathcal{SM}_N$ *and* $\mathsf{B} \leqslant \mathsf{A}$, *then* $\mathsf{B} \in \mathcal{SM}_N$ *as well; the direct sum of finitely many semiminimal* N*-tuples belongs to* $\mathcal{SM}_N$.

(II) *There is a unique* $\mathsf{J}_{II} \in \mathcal{CDD}_N$ *such that for every* $\mathsf{A} \in \mathcal{SM}_N$ *there is* $\mathsf{B} \in \mathcal{SM}_N$ *for which* $\mathsf{J}_{II} = \aleph_0 \odot (\mathsf{A} \boxplus \mathsf{B})$. *Moreover,* $\dim(\mathsf{J}_{II}) \leqslant 2^{\aleph_0}$ *and*

(a) *for* $\mathsf{E}, \mathsf{F} \in \mathcal{CDD}_N$ *with* $\mathsf{E} \leqslant \mathsf{F} \leqslant \mathsf{J}_{II}$,

$$\mathsf{E} \leqslant^s \mathsf{F} \leqslant^s \mathsf{J}_{II} \iff \mathsf{E} = 2 \odot \mathsf{E} \text{ and } \mathsf{F} = 2 \odot \mathsf{F}, \tag{3.4.2}$$

(b) $\mathsf{A} \ll \mathsf{J}_{II}$ *iff* $\mathsf{A} = \mathsf{O}$ *or* $\mathcal{W}'(\boldsymbol{A})$ *is type II.*

Proof. Point (I) follows from Lemma 3.2.3 and Theorem 3.4.4 from which we infer that $\mathcal{W}'(\boldsymbol{A})$ is type II for every $\mathsf{A} \in \mathcal{SM}_N$ (because every semiminimal N-tuple is unitarily disjoint from $\mathsf{J}_I \boxplus \mathsf{J}_{III}$). To prove (II), we proceed similarly to the proof of Theorem 3.4.4. Take a maximal family $\mathcal{A}$ of mutually unitarily disjoint nontrivial members of $\mathcal{SM}_N$ whose representatives act in separable spaces and denote by $\mathsf{S}(\mathcal{A})$ its direct sum. Next put $\mathsf{J}_{II} = \aleph_0 \odot \mathsf{S}(\mathcal{A})$. We check that $\mathsf{S}(\mathcal{A}) \in \mathcal{SM}_N$ for every such $\mathcal{A}$. Further, we note that for two maximal families $\mathcal{A}$ and $\mathcal{A}'$ one has $\mathsf{S}(\mathcal{A}) \ll \mathsf{S}(\mathcal{A}') \ll \mathsf{S}(\mathcal{A})$ and consequently, by the definition of semiminimality, $\aleph_0 \odot \mathsf{S}(\mathcal{A}') = \aleph_0 \odot \mathsf{S}(\mathcal{A})$. Having this, one easily shows the uniqueness of J_{II} and all suitable properties of it. (For example, if $\mathsf{E} = 2 \odot \mathsf{E}$, then $\mathsf{E} = \aleph_0 \odot \mathsf{E}$ and it suffices to apply Lemma 3.2.4.) ■

The reader should notice that J_{II} corresponds to the steering projection of a type II_∞ $\mathcal{W}^*$-algebra.

Remark 3.4.7. Point (II) of Theorem 3.4.6 implies that J_{II} is the greatest element of the class $\mathcal{SM}_N^\infty = \{\aleph_0 \odot \mathsf{A} \colon \mathsf{A} \in \mathcal{SM}_N\}$ (and hence $\mathcal{SM}_N^\infty$ is a set) and that for any $\mathsf{A}, \mathsf{B} \in \mathcal{SM}_N^\infty$, $\mathsf{A} \leqslant \mathsf{B} \Leftrightarrow \mathsf{A} \leqslant^s \mathsf{B}$.

Let us denote by J the N-tuple $\mathsf{J}_I \boxplus \mathsf{J}_{II} \boxplus \mathsf{J}_{III}$. We call J the *unity* of $\mathcal{CDD}_N$. Since every $\mathcal{W}^*$-algebra admits a decomposition into type I, II and III parts, we have

Proposition 3.4.8. *For every* $\mathsf{A} \in \mathcal{CDD}_N$, $\mathsf{A} \ll \mathsf{J}$.

Remark 3.4.9. It is worth noting that $\dim(\mathsf{J}_i) = 2^{\aleph_0}$ for $i = I, II, III$. We shall prove this later (see Corollary 5.1.9). We conclude from this and Proposition 3.4.8 that for an infinite cardinal α there exists $\mathsf{A} \in \mathcal{CDD}_N$ such that $\dim(\mathsf{A}) = \alpha$ and $\mathsf{X} \leqslant \mathsf{A}$ whenever $\dim(\mathsf{X}) \leqslant \alpha$ iff $\alpha \geqslant 2^{\aleph_0}$. If this happens, such an A is of course unique and one may check that $\mathsf{A} = \alpha \odot \mathsf{J}$.

We shall also need

Proposition 3.4.10. *For every nontrivial* $\mathsf{A} \leqslant \mathsf{J}$ *there is* $\mathsf{B} \leqslant^s \mathsf{A}$ *such that* $0 < \dim(\mathsf{B}) \leqslant \aleph_0$.

Proof. By Theorem 2.2.4, there is a nontrivial $\mathsf{B}_0 \leqslant \mathsf{A}$ such that $\boldsymbol{B}_0$ acts in a separable Hilbert space. We may assume that $\mathsf{B}_0 \leqslant \mathsf{J}_i$ for some $i \in \{I, II, III\}$. If $i \neq II$, we automatically have $\mathsf{B}_0 \leqslant^s \mathsf{A}$; while when $i = II$, it suffices to notice that $\aleph_0 \odot \mathsf{B}_0 \leqslant^s \aleph_0 \odot \mathsf{A}$ (by (3.4.2)) and to apply (PR6) (page 13) to find $\mathsf{B} \leqslant^s \mathsf{A}$ with $\aleph_0 \odot \mathsf{B} = \aleph_0 \odot \mathsf{B}_0$. ∎

Example 3.4.11. When $N = 1$, one may check that a bounded normal operator on a separable Hilbert space is multiplicity free iff it is $*$-cyclic (an operator $T \in \mathcal{B}(\mathcal{H})$ is $*$-*cyclic* iff there is $x \in \mathcal{H}$ for which the linear span of $\{x\} \cup \{S_1 \dots S_m x \colon m \geqslant 1,\ S_1, \dots, S_m \in \{T, T^*\}\}$ is dense in $\mathcal{H}$). Taking this into account, one may ask whether every $*$-cyclic type I operator is multiplicity free. As this simple example shows, this fails to be true. Let $T = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and $S = T \oplus T$. Of course, $S \notin \mathcal{MF}_1$. However, S is $*$-cyclic. (For $u = (1,0,0,1)$, $Su = (0,1,0,0)$, $S^*u = (0,0,1,0)$ and $S^*Su = (1,0,0,0)$.)

3.5. Unities of ideals

Adapting conditions (ID1)–(ID4) (page 13) to the realm $\mathcal{CDD}_N$, we obtain the notion of an ideal in $\mathcal{CDD}_N$. Equivalently, a nonempty class $\mathcal{A} \subset \mathcal{CDD}_N$ is an ideal provided $\mathcal{A}$ is order-complete (i.e. $\bigvee \mathcal{F} \in \mathcal{A}$ for every nonempty set $\mathcal{F} \subset \mathcal{A}$) and $\mathfrak{m} \odot \mathsf{A} \in \mathcal{A}$ whenever $\mathfrak{m}$ is a cardinal and $\mathsf{A} \leqslant \mathsf{B}$ for some $\mathsf{B} \in \mathcal{A}$.

Theorem 2.4.1 asserts that for every ideal $\mathcal{A} \subset \mathcal{CDD}_N$ and $\mathsf{X} \in \mathcal{CDD}_N$ there is a unique $\mathsf{Y} \in \mathcal{A}$ such that $\mathsf{Y} \leqslant^s \mathsf{X}$ and $\mathsf{X} \boxminus \mathsf{Y} \in \mathcal{A}^\perp$. We shall denote this unique Y by $\mathsf{E}(\mathsf{X}|\mathcal{A})$. Similarly, if A is any member of $\mathcal{CDD}_N$, $\mathsf{E}(\mathsf{X}|\mathsf{A}) := \mathsf{E}(\mathsf{X}|\{\mathsf{B} \colon \mathsf{B} \ll \mathsf{A}\})$. $\mathsf{E}(\mathsf{X}|\mathsf{A})$ is Ernest's A-shadow of X (see [9, Definition 2.13]).

One may easily verify that

$$\mathsf{E}\Big(\bigoplus_{s \in S} \mathsf{X}^{(s)} | \mathcal{A}\Big) = \bigoplus_{s \in S} \mathsf{E}(\mathsf{X}^{(s)} | \mathcal{A})$$

for every ideal $\mathcal{A} \subset \mathcal{CDD}_N$ and any family $\{\mathsf{X}^{(s)}\}_{s \in S} \subset \mathcal{CDD}_N$. We shall use the above property repeatedly.

Let $\mathcal{A}$ be an ideal in $\mathcal{CDD}_N$. The N-tuple $\mathsf{J}(\mathcal{A}) := \mathsf{E}(\mathsf{J}|\mathcal{A})$ is uniquely determined by $\mathcal{A}$ and is called the *unity* of $\mathcal{A}$. Proposition 3.4.8 implies

Proposition 3.5.1. *For every ideal* $\mathcal{A}$ *in* $\mathcal{CDD}_N$,

$$\mathcal{A} = \{\mathsf{X} \in \mathcal{CDD}_N \colon \mathsf{X} \ll \mathsf{J}(\mathcal{A})\},$$

$\mathsf{J}(\mathcal{A}) \leqslant^s \mathsf{J}$ *and* $\mathsf{J}(\mathcal{A}) = \bigvee \{\mathsf{A} \leqslant \mathsf{J} \colon \mathsf{A} \in \mathcal{A}\}$.

Corollary 3.5.2. *There is a one-to-one correspondence between ideals in* $\mathcal{CDD}_N$ *and members* A *of* $\mathcal{CDD}_N$ *such that* $\mathsf{A} \leqslant^s \mathsf{J}$. *The correspondence is established by the assignments* $\mathcal{A} \mapsto \mathsf{J}(\mathcal{A})$ *and* $\mathsf{A} \mapsto \{\mathsf{B} \colon \mathsf{B} \ll \mathsf{A}\}$. *In particular, there are at most* $2^{2^{\aleph_0}}$ *ideals in* $\mathcal{CDD}_N$.

Example 3.5.3. Let $\mathcal{N}_N \subset \mathcal{CDD}_N$ be the ideal of all normal N-tuples (see Examples 2.4.3(E)). Since $\mathcal{W}''(\boldsymbol{M})$ is commutative for every $\mathsf{M} \in \mathcal{N}_N$, $\mathcal{W}'(\boldsymbol{M})$ is type I and hence $\mathsf{M} \ll \mathsf{J}_I$. Here we shall give a description of $\mathsf{J}(\mathcal{N}_N)$. First of all, $\mathsf{M} \leqslant \mathsf{J}$ iff $\mathcal{W}'(\boldsymbol{M})$ is commutative (provided $\mathsf{M} \neq \mathsf{O}$ and $\mathsf{M} \in \mathcal{N}_N$). When $\boldsymbol{M}$ acts in a separable Hilbert space, this is equivalent to the fact that $\boldsymbol{M}$ is $*$-cyclic. That is, there has to exist $x \in \overline{\mathcal{D}}(\boldsymbol{M})$ such that the smallest reducing subspace for $\boldsymbol{M}$ which contains x coincides with $\overline{\mathcal{D}}(\boldsymbol{M})$. (Indeed, if $\mathsf{M} \in \mathcal{MF}_N \cap \mathcal{N}_N$ is such that $0 < \dim(\mathsf{M}) \leqslant \aleph_0$, then both $\mathcal{W}'(\boldsymbol{M})$ and $\mathcal{W}''(\boldsymbol{M})$ are commutative, which means that $\mathcal{W}''(\boldsymbol{M})$ is a MASA and consequently $\mathcal{W}''(\boldsymbol{M})$ is cyclic or, equivalently, $\boldsymbol{M}$ is $*$-cyclic. Conversely, if $\mathsf{M} \in \mathcal{N}_N$ and $\boldsymbol{M}$ is $*$-cyclic, then $\boldsymbol{M}$ is unitarily equivalent to $\boldsymbol{M}_\mu$ for some probability Borel measure μ on $\mathbb{C}^N$ where $\boldsymbol{M}_\mu = (M_{z_1}, \ldots, M_{z_N})$ and M_{z_j} is the multiplication operator by z_j in $L^2(\mu)$. One may show that $\mathcal{W}'(\boldsymbol{M}_\mu)$ coincides with the algebra of all multiplication operators by members of $L^\infty(\mu)$ and hence $\mathsf{M} \in \mathcal{MF}_N$.) Having this, one shows that $\mathsf{J}(\mathcal{N}_N)$ may be represented as follows. Take a maximal family $\{\mu_s\}_{s\in S}$ of mutually orthogonal probability Borel measures on $\mathbb{C}^N$. For each $s \in S$ let $\boldsymbol{M}^{(s)} = \boldsymbol{M}_{\mu_s}$ (defined as above). One may check that $\mathsf{J}(\mathcal{N}_N) = \boxplus_{s\in S} \mathsf{M}^{(s)}$. Moreover, for two probability Borel measures μ and λ on $\mathbb{C}^N$: (a) $\boldsymbol{M}_\mu \leqslant \boldsymbol{M}_\lambda \Leftrightarrow \mu \ll \lambda$; (b) $\boldsymbol{M}_\mu \equiv \boldsymbol{M}_\lambda \Leftrightarrow \mu \ll \lambda \ll \mu$; (c) $\boldsymbol{M}_\mu \perp_u \boldsymbol{M}_\lambda \Leftrightarrow \mu \perp \lambda$. A similar (and more detailed) construction will appear in Chapter 5.6.

Theorems 3.4.4 and 3.4.6 show that for $i \in \{I, II, III\}$ the ideal

$$\mathcal{I}_i = \{\mathsf{X} \in \mathcal{CDD}_N \colon \mathsf{X} \ll \mathsf{J}_i\}$$

consists of all N-tuples of type i.

3.6. Decomposition relative to the unity

Recall that $\Lambda_I = \mathrm{Card}$, $\Lambda_{II} = \mathrm{Card}_\infty \cup \{0, 1\}$ and $\Lambda_{III} = \mathrm{Card}_\infty \cup \{0\}$. For simplicity, let $\Upsilon = \{(i, \alpha) \colon i \in \{I, II, III\},\ \alpha \in \Lambda_i\}$ and $\Upsilon_* = \Upsilon \setminus \{(II, 1)\}$.

Theorem 3.6.1. *For every* $\mathsf{A} \in \mathcal{CDD}_N$ *there are a unique regular collection*

$$\{\mathsf{E}^i_\alpha(\mathsf{A}) \colon (i, \alpha) \in \Upsilon\}$$

and a unique $\mathsf{E}_{sm}(\mathsf{A}) \in \mathcal{CDD}_N$ *such that for* $i \in \{I, II, III\}$,

$$\mathsf{J}_i = \boxplus_{\alpha \in \Lambda_i} \mathsf{E}^i_\alpha(\mathsf{A}),$$

$\mathsf{E}_{sm}(\mathsf{A})$ *is semiminimal and* $\mathsf{E}^{II}_1(\mathsf{A}) = \aleph_0 \odot \mathsf{E}_{sm}(\mathsf{A})$, *and*

$$\mathsf{A} = \mathsf{E}_{sm}(\mathsf{A}) \boxplus \boxplus_{(i,\alpha)\in\Upsilon_*} \alpha \odot \mathsf{E}^i_\alpha(\mathsf{A}). \tag{3.6.1}$$

What is more, $\mathsf{E}_{sm}(\mathsf{A}) = \mathsf{A} \wedge \mathsf{E}^{II}_1(\mathsf{A})$ *and* $\mathsf{E}^i_\alpha(\mathsf{A})$*'s may be characterized as follows:*

$$\mathsf{E}^{II}_1(\mathsf{A}) = \bigvee \{\mathsf{E} \leqslant \mathsf{J}_{II} |\ \mathsf{E} \ll \mathsf{A},\ \forall\, \mathsf{F} \leqslant \mathsf{E},\ \mathsf{F} \neq \mathsf{O} \colon \aleph_0 \odot \mathsf{F} \nleqslant \mathsf{A}\} \tag{3.6.2}$$

and for $(i, \alpha) \in \Upsilon_*$,

$$\mathsf{E}^i_\alpha(\mathsf{A}) = \bigvee \{\mathsf{E} \leqslant \mathsf{J}_i |\ \alpha \odot \mathsf{E} \leqslant \mathsf{A},\ \forall\, \mathsf{F} \leqslant \mathsf{E},\ \mathsf{F} \neq \mathsf{O} \colon \alpha^+ \odot \mathsf{F} \nleqslant \mathsf{A}\}.$$

Proof. By Proposition 3.4.8, there is an infinite cardinal γ such that $\mathsf{A} \leqslant \gamma \odot \mathsf{J} =: \mathsf{B}$. Put $\mathcal{M} = \mathcal{W}'(\boldsymbol{B})$, observe that $\boldsymbol{J}$ corresponds (by Proposition 2.3.1) to a steering projection of $\mathcal{M}$ and apply Theorem 3.3.1. (Use Theorem 3.4.6 to deduce that a suitable $\mathsf{E}_{sm}(\mathsf{A})$ is semiminimal. Note that if X and Y correspond, by Proposition 2.3.1, to projections p and q, then $p \sim \alpha \odot q$ is equivalent to $\mathsf{X} = \alpha \odot \mathsf{Y}$.) ∎

The system $\{\mathsf{E}^i_\alpha(\mathsf{A})\colon (i,\alpha) \in \Upsilon\}$ appearing in Theorem 3.6.1 is said to be the *partition of unity induced by* A. (In general, a *partition of unity* is any regular collection $\{\mathsf{E}^{(j)}\}_{j\in I}$ such that $\mathsf{J} = \boxplus_{j\in I} \mathsf{E}^{(j)}$. Note that in that case $\mathsf{E}^{(j)} \leqslant^s \mathsf{J}$ for each $j \in I$.)

Remark 3.6.2. Theorem 3.6.1 may be equivalently formulated as follows: after fixing a representative $\boldsymbol{J}$ for J for every $\boldsymbol{A} \in \mathrm{CDD}_N$ there are unique systems $\{H^i_\alpha\colon (i,\alpha) \in \Upsilon\} \subset \mathrm{cred}(\boldsymbol{J})$ and $\{K^i_\alpha\colon (i,\alpha) \in \Upsilon\} \subset \mathrm{cred}(\boldsymbol{A})$ such that $\overline{\mathcal{D}}(\boldsymbol{J}_i) = \bigoplus_{\alpha\in\Lambda_i} H^i_\alpha$ for $i \in \{I, II, III\}$; $\overline{\mathcal{D}}(\boldsymbol{A}) = \bigoplus_{(i,\alpha)\in\Upsilon} K^i_\alpha$; $\mathcal{W}'(\boldsymbol{A}|_{K^{II}_1})$ is type II_1, $\aleph_0 \odot \boldsymbol{A}|_{K^{II}_1} \equiv \boldsymbol{J}|_{H^{II}_1}$ and for every $(i,\alpha) \in \Upsilon_*$,

$$\boldsymbol{A}|_{K^i_\alpha} \equiv \alpha \odot \boldsymbol{J}|_{H^i_\alpha}.$$

(In particular, K^I_0, K^{II}_0 and K^{III}_0 are trivial.)

As an immediate consequence of Proposition 3.3.4 we obtain

Proposition 3.6.3. *For any* $\mathsf{A}, \mathsf{B} \in \mathcal{CDD}_N$, $\mathsf{A} \leqslant \mathsf{B}$ *iff* $\mathsf{E}^i_\alpha(\mathsf{A}) \perp_u \mathsf{E}^i_\beta(\mathsf{B})$ *whenever* $(i,\alpha), (i,\beta) \in \Upsilon$ *and* $\alpha > \beta$; *and* $\mathsf{E}_{sm}(\mathsf{A}) \wedge \mathsf{E}^{II}_1(\mathsf{B}) \leqslant \mathsf{E}_{sm}(\mathsf{B})$.

One may also show

Proposition 3.6.4. *For any* $\mathsf{A}, \mathsf{B} \in \mathcal{CDD}_N$, $\mathsf{A} \leqslant^s \mathsf{B}$ *iff* $\mathsf{E}^i_\alpha(\mathsf{A}) \leqslant \mathsf{E}^i_\alpha(\mathsf{B})$ *whenever* $(i,\alpha) \in \Upsilon$ *is such that* $\alpha \neq 0$, *and* $\mathsf{E}_{sm}(\mathsf{A}) \leqslant^s \mathsf{E}_{sm}(\mathsf{B})$.

The proofs of Propositions 3.6.3 and 3.6.4 are skipped.

Other interesting consequences of Theorem 3.6.1 are stated below.

Corollary 3.6.5. *Let* $\mathsf{A}, \mathsf{B} \in \mathcal{CDD}_N$ *and let* α *be an arbitrary infinite cardinal number such that* $\alpha \geqslant \max(\dim(\mathsf{A}), \dim(\mathsf{B}))$.

(I) $\mathsf{A} \ll \mathsf{B} \Leftrightarrow \alpha \odot \mathsf{A} \leqslant^s \alpha \odot \mathsf{B}$.
(II) $\mathsf{A} \ll \mathsf{B} \ll \mathsf{A} \Leftrightarrow \alpha \odot \mathsf{A} = \alpha \odot \mathsf{B}$.

Proof. In both items the implication '$\Leftarrow$' is immediate. Conversely, observe that for each $\mathsf{X} \in \mathcal{CDD}_N$ and $(i,\beta) \in \Upsilon$, $\mathsf{E}^i_\beta(\mathsf{X}) = \mathsf{O}$ provided $\beta > \dim(\mathsf{X})$. This implies that if $\beta \geqslant \max(\aleph_0, \dim(\mathsf{X}))$, then $\beta \odot \mathsf{X} = \beta \odot \mathsf{E}$ for some $\mathsf{E} \leqslant^s \mathsf{J}$. This yields the direct implication in both (I) and (II). (Observe that if $\mathsf{E}' \leqslant^s \mathsf{E}''$, then $\gamma \odot \mathsf{E}' \leqslant^s \gamma \odot \mathsf{E}''$ for every cardinal γ.) ∎

Corollary 3.6.6. *A nonempty class* $\mathcal{A}$ *is an ideal iff* $\mathcal{A}$ *satisfies the following three conditions:*

(a) *for every* $\mathsf{A} \in \mathcal{CDD}_N$ *and* $\alpha \in \mathrm{Card}_\infty$, $\mathsf{A} \in \mathcal{A} \Leftrightarrow \alpha \odot \mathsf{A} \in \mathcal{A}$,
(b) *whenever* $\{\mathsf{A}^{(s)}\}_{s\in S} \subset \mathcal{A}$ *is a regular family of* N*-tuples such that* $0 < \dim(\mathsf{A}) \leqslant \aleph_0$, *then* $\boxplus_{s\in S} \mathsf{A}^{(s)} \in \mathcal{A}$,
(c) $\mathsf{A} \leqslant^s \mathsf{B}$ *and* $\mathsf{B} \in \mathcal{A}$ *imply* $\mathsf{A} \in \mathcal{A}$.

Proof. The necessity is clear. The sufficiency is in fact a consequence of Corollary 3.6.5. Indeed, if $\mathsf{A} \leqslant \mathsf{B}$ and $\mathsf{B} \in \mathcal{A}$, then $\alpha \odot \mathsf{A} \leqslant^s \alpha \odot \mathsf{B}$ for some infinite cardinal α (by Corollary 3.6.5). It follows from (a) that $\alpha \odot \mathsf{B} \in \mathcal{A}$ and so $\alpha \odot \mathsf{A} \in \mathcal{A}$ (by (c)) and $\mathsf{A} \in \mathcal{A}$, again by (a). Finally, if $\{\mathsf{A}^{(j)}\}_{j\in I} \subset \mathcal{A}$ and $\mathsf{A} = \bigoplus_{j\in I} \mathsf{A}^{(j)}$, then for large enough $\alpha \in \mathrm{Card}_\infty$ one has $\alpha \odot \mathsf{A}^{(j)} = \alpha \odot \mathsf{E}^{(j)}$ with $\mathsf{E}^{(j)} \leqslant^s \mathsf{J}$ ($j \in I$) and $\alpha \odot \mathsf{A} = \alpha \odot \mathsf{E}$ for some $\mathsf{E} \leqslant^s \mathsf{J}$ (see the proof of Corollary 3.6.5). Thanks to (a), $\mathsf{E}^{(j)} \in \mathcal{A}$ and it is enough to show that $\mathsf{E} \in \mathcal{A}$. We see that $\mathsf{E}^{(j)} \leqslant^s \mathsf{E}$ and $\mathsf{E} = \bigvee_{j\in I} \mathsf{E}^{(j)}$. These imply (cf. the proof of Theorem 3.1.1) that there is a regular family $\{\mathsf{B}^{(j)}\}_{j\in I}$ such that $\boxplus_{j\in I} \mathsf{B}^{(j)} = \mathsf{E}$ and $\mathsf{B}^{(j)} \leqslant^s \mathsf{E}^{(j)}$ ($j \in I$). We infer from (c) that $\mathsf{B}^{(j)} \in \mathcal{A}$ for all $j \in I$. Now thanks to Proposition 3.4.10, each $\mathsf{B}^{(j)}$ may be written in the form $\boxplus_{s\in S_j} \mathsf{A}^{(s,j)}$ with $0 < \dim(\mathsf{A}^{(s,j)}) \leqslant \aleph_0$. Consequently, (c) yields $\mathsf{A}^{(s,j)} \in \mathcal{A}$ and hence $\mathsf{E} \in \mathcal{A}$ as well, by (b). ∎

Example 3.6.7. Sometimes it may be useful to consider the *common* partition of unity for several members of $\mathcal{CDD}_N$ (in particular, to find the partition of unity induced by their direct sum). It may be understood as follows. For simplicity, we shall describe this idea only for two N-tuples. Below we involve Proposition 3.1.4 several times, with no comment.

Let $\mathsf{A}, \mathsf{B} \in \mathcal{CDD}_N$. Let

$$\Upsilon^2 = \{(i,\alpha,\beta)\colon (i,\alpha),(i,\beta) \in \Upsilon\} \quad \text{and} \quad \Upsilon^2_* = \{(i,\alpha,\beta)\colon (i,\alpha),(i,\beta) \in \Upsilon_*\}.$$

For $(i,\alpha,\beta) \in \Upsilon^2$ let $\mathsf{E}^i_{\alpha,\beta} = \mathsf{E}^i_\alpha(\mathsf{A}) \wedge \mathsf{E}^i_\beta(\mathsf{B})$. Additionally, we put

$$\mathsf{E}_{sm,\alpha} = \mathsf{E}_{sm}(\mathsf{A}) \wedge \mathsf{E}^{II}_\alpha(\mathsf{B}) \quad \text{and} \quad \mathsf{E}_{\alpha,sm} = \mathsf{E}^{II}_\alpha(\mathsf{A}) \wedge \mathsf{E}_{sm}(\mathsf{B})$$

for $\alpha \in \Lambda_{II}$. One may check that then $\mathsf{J}_i = \boxplus_{\alpha,\beta\in\Lambda_i} \mathsf{E}^i_{\alpha,\beta}$ for $i \in \{I, II, III\}$; $\mathsf{E}_{\alpha,sm}$ and $\mathsf{E}_{sm,\alpha}$ are semiminimal and

$$\mathsf{E}^{II}_{1,\alpha} = \aleph_0 \odot \mathsf{E}_{sm,\alpha} \quad \text{and} \quad \mathsf{E}^{II}_{\alpha,1} = \aleph_0 \odot \mathsf{E}_{\alpha,sm} \tag{3.6.3}$$

for each $\alpha \in \Lambda_{II}$. Further,

$$\mathsf{A} = \Big(\boxplus_{\alpha\in\mathrm{Card}_\infty} \mathsf{E}_{sm,\alpha}\Big) \boxplus \Big(\boxplus_{\alpha\in\mathrm{Card}_\infty} \alpha \odot \mathsf{E}^{II}_{\alpha,1}\Big) \boxplus \Big(\boxplus_{(i,\alpha,\beta)\in\Upsilon^2_*} \alpha \odot \mathsf{E}^i_{\alpha,\beta}\Big) \boxplus (\mathsf{E}_{sm,1} \boxplus \mathsf{E}_{sm,0}) \tag{3.6.4}$$

and correspondingly

$$\mathsf{B} = \Big(\boxplus_{\alpha\in\mathrm{Card}_\infty} \alpha \odot \mathsf{E}^{II}_{1,\alpha}\Big) \boxplus \Big(\boxplus_{\alpha\in\mathrm{Card}_\infty} \mathsf{E}_{\alpha,sm}\Big) \boxplus \Big(\boxplus_{(i,\alpha,\beta)\in\Upsilon^2_*} \beta \odot \mathsf{E}^i_{\alpha,\beta}\Big) \boxplus (\mathsf{E}_{1,sm} \boxplus \mathsf{E}_{0,sm}). \tag{3.6.5}$$

In particular, thanks to (3.6.3),

$$\mathsf{A} \oplus \mathsf{B} = [\mathsf{E}_{sm,0} \boxplus \mathsf{E}_{0,sm} \boxplus (\mathsf{E}_{sm,1} \oplus \mathsf{E}_{1,sm})] \boxplus \boxplus_{(i,\alpha,\beta)\in\Upsilon^2_\#} (\alpha+\beta) \odot \mathsf{E}^i_{\alpha,\beta}$$

where $\Upsilon^2_{\#} = \Upsilon^2 \setminus \{(\mathit{II}, \alpha, \beta)\colon (\alpha, \beta) = (0, 1), (1, 0), (1, 1)\}$. So (below $(i, \gamma) \in \Upsilon_*$),

$$\begin{cases} \mathsf{E}_{sm}(\mathsf{A} \oplus \mathsf{B}) = \mathsf{E}_{sm,0} \boxplus \mathsf{E}_{0,sm} \boxplus [\mathsf{E}_{sm,1} \oplus \mathsf{E}_{1,sm}], \\ \mathsf{E}^{\mathit{II}}_1(\mathsf{A} \oplus \mathsf{B}) = \mathsf{E}^{\mathit{II}}_{0,1} \boxplus \mathsf{E}^{\mathit{II}}_{1,0} \boxplus \mathsf{E}^{\mathit{II}}_{1,1}, \\ \mathsf{E}^i_\gamma(\mathsf{A} \oplus \mathsf{B}) = \bigboxplus \{\mathsf{E}^i_{\alpha,\beta}\colon (i, \alpha, \beta) \in \Upsilon^2_{\#},\ \alpha + \beta = \gamma\}. \end{cases} \tag{3.6.6}$$

In a similar manner one may find formulas for $\mathsf{A} \vee \mathsf{B}$ and $\mathsf{A} \wedge \mathsf{B}$ and the partitions of unity induced by them.

4. TOPOLOGICAL MODEL

4.1. Algebraic and order properties

The following is folklore (see e.g. [19, Exercise 6.9.14]): if p and q are two projections in a von Neumann algebra $\mathcal{M}$ such that $n \odot p \sim n \odot q$ for some $n \geqslant 1$, then $p \sim q$. This has an interesting consequence for the class $\mathcal{CDD}_N$:

(AO1) $n \odot \mathsf{A} = n \odot \mathsf{B} \Rightarrow \mathsf{A} = \mathsf{B}$

provided n is positive and finite. Further properties in this style are listed below.

(AO2) For finite positive n and m: $n \odot \mathsf{A} = m \odot \mathsf{B} \Leftrightarrow \mathsf{A} = k \odot \mathsf{X}$ and $\mathsf{B} = l \odot \mathsf{X}$ for some $\mathsf{X} \in \mathcal{CDD}_N$ with $k = m/\operatorname{GCD}(n,m)$ and $l = n/\operatorname{GCD}(n,m)$ ('GCD' stands for the greatest common divisor). If $n \neq m$, then $n \odot \mathsf{A} = m \odot \mathsf{A} \Leftrightarrow \mathsf{A} = \aleph_0 \odot \mathsf{A}$.

(AO3) If α and β are cardinals such that $\alpha < \beta$ and β is infinite, then

$$\alpha \odot \mathsf{A} = \beta \odot \mathsf{B} \;\Leftrightarrow\; \mathsf{A} = \beta \odot \mathsf{B}.$$

((AO2) and (AO3) follow from (3.6.1); cf. also the beginning of Chapter 4.3.)

(AO4) For a nontrivial $\mathsf{A} \in \mathcal{CDD}_N$ the following conditions are equivalent:

(i) for any $\mathsf{X}, \mathsf{Y} \in \mathcal{CDD}_N$, $\mathsf{A} \oplus \mathsf{X} = \mathsf{A} \oplus \mathsf{Y} \Leftrightarrow \mathsf{X} = \mathsf{Y}$,
(ii) $\mathsf{B} \leqslant^s \mathsf{A}$ and $\mathsf{A} \oplus \mathsf{B} = \mathsf{A}$ imply $\mathsf{B} = \mathsf{O}$,
(iii) $\mathcal{W}'(\boldsymbol{A})$ is finite,
(iv) $\mathsf{E}^i_\alpha(\mathsf{A}) = \mathsf{O}$ for each $i \in \{I, II, III\}$ and infinite α.

All N-tuples A satisfying (i) form a **set**, denoted by $\mathcal{FIN}_N$. $(\mathcal{FIN}_N, \oplus)$ is a semigroup which may be enlarged to an Abelian group (by (i)).

(AO5) For any $\mathsf{A} \in \mathcal{FIN}_N$ and $\mathsf{B} \geqslant \mathsf{A}$ there is a unique X such that $\mathsf{A} \oplus \mathsf{X} = \mathsf{B}$. Thus, $\mathsf{B} \ominus \mathsf{A}$ is well defined in that case.

(AO6) Let S be an infinite set whose size is a limit cardinal. For every family $\{\mathsf{A}^{(s)}\}_{s \in S} \subset \mathcal{CDD}_N$,

$$\bigoplus_{s \in S} \mathsf{A}^{(s)} = \bigvee \Big\{ \bigoplus_{s \in S'} \mathsf{A}^{(s)} \colon S' \subset S,\ 0 < \operatorname{card}(S') < \operatorname{card}(S) \Big\}. \tag{4.1.1}$$

In particular, for every sequence $(\mathsf{B}^{(n)})_{n=1}^\infty \subset \mathcal{CDD}_N$,

$$\bigoplus_{n=1}^\infty \mathsf{B}^{(n)} = \bigvee_{n=1}^\infty \mathsf{B}^{(1)} \oplus \cdots \oplus \mathsf{B}^{(n)},$$

and for each $\mathsf{A} \in \mathcal{CDD}_N$ and an infinite limit cardinal γ,

$$\gamma \odot \mathsf{A} = \bigvee_{\alpha<\gamma} \alpha \odot \mathsf{A}.$$

(By Proposition 3.3.8.)

(AO7) Whenever $\mathsf{A} \leqslant \mathsf{B}$, there are $(\mathsf{B} \ominus \mathsf{A})^{\triangledown}, (\mathsf{B} \ominus \mathsf{A})_{\vartriangle} \in \mathcal{CDD}_N$ such that $\mathsf{A} \oplus \mathsf{X} = \mathsf{B}$ iff $(\mathsf{B}\ominus\mathsf{A})_{\vartriangle} \leqslant \mathsf{X} \leqslant (\mathsf{B}\ominus\mathsf{A})^{\triangledown}$. Moreover, if $\mathsf{B} = 2\odot\mathsf{B}$, then $(\mathsf{B}\ominus\mathsf{A})_{\vartriangle} \leqslant^s \mathsf{B} = (\mathsf{B}\ominus\mathsf{A})^{\triangledown}$. $\mathsf{B} \ominus \mathsf{A}$ is well defined iff $(\mathsf{B} \ominus \mathsf{A})^{\triangledown} = (\mathsf{B} \ominus \mathsf{A})_{\vartriangle}$. (See Theorem 3.3.7.)

(AO8) If $\mathsf{A} \leqslant^s \mathsf{B}$, then $(\mathsf{B} \ominus \mathsf{A})_{\vartriangle} = \mathsf{B} \boxminus \mathsf{A}$. (Thanks to (PR1), page 12.)

(AO9) $(\mathsf{B} \ominus \mathsf{A})_{\vartriangle} \leqslant (\mathsf{B} \ominus \mathsf{X})_{\vartriangle} \oplus (\mathsf{X} \ominus \mathsf{A})_{\vartriangle} \leqslant (\mathsf{B} \ominus \mathsf{X})^{\triangledown} \oplus (\mathsf{X} \ominus \mathsf{A})^{\triangledown} \leqslant (\mathsf{B} \ominus \mathsf{A})^{\triangledown}$ whenever $\mathsf{A} \leqslant \mathsf{X} \leqslant \mathsf{B}$.

(AO10) $(\mathsf{B} \ominus \mathsf{A})_{\vartriangle} \leqslant^s (\mathsf{B} \ominus \mathsf{A})^{\triangledown}$ provided $\mathsf{A} \leqslant \mathsf{B}$.

Let us prove (AO10). We use the notation of Example 3.6.7. We infer from (3.6.4) and (3.6.5) that $\mathsf{A} \leqslant \mathsf{B}$ iff $\mathsf{E}_{sm,1} \leqslant \mathsf{E}_{1,sm}$, $\mathsf{E}_{sm,0} = \mathsf{O}$ and for any $\gamma \in \mathrm{Card}_\infty$ and $(i,\alpha,\beta) \in \Upsilon^2_*$ with $\alpha > \beta$,

$$\mathsf{E}^{II}_{\gamma,1} = \mathsf{E}^{i}_{\alpha,\beta} = \mathsf{O}.$$

In that case (3.6.4) reduces to

$$\mathsf{A} = \Big(\boxplus_{\alpha\in\mathrm{Card}_\infty} \mathsf{E}_{sm,\alpha}\Big) \boxplus \Big(\boxplus_{\substack{(i,\alpha,\beta)\in\Upsilon^2_*\\ \alpha\leqslant\beta}} \alpha \odot \mathsf{E}^i_{\alpha,\beta}\Big) \boxplus \mathsf{E}_{sm,1},$$

while (3.6.5) is equivalent to

$$\mathsf{B} = \Big(\boxplus_{\alpha\in\mathrm{Card}_\infty} \alpha \odot \mathsf{E}^{II}_{1,\alpha}\Big) \boxplus \Big(\boxplus_{\substack{(i,\alpha,\beta)\in\Upsilon^2_*\\ \alpha\leqslant\beta}} \beta \odot \mathsf{E}^i_{\alpha,\beta}\Big) \boxplus \mathsf{E}_{sm}(\mathsf{B}).$$

Now we infer from the above formulas and (AO5) that

$$(\mathsf{B} \ominus \mathsf{A})_{\vartriangle} = [\mathsf{E}_{sm}(\mathsf{B}) \ominus \mathsf{E}_{sm,1}] \boxplus \Big(\boxplus_{\alpha\in\mathrm{Card}_\infty} \alpha \odot \mathsf{E}^{II}_{1,\alpha}\Big)$$

$$\boxplus \Big(\boxplus\{(\beta-\alpha) \odot \mathsf{E}^i_{\alpha,\beta}\colon (i,\alpha,\beta) \in \Upsilon^2_*,\ \alpha < \beta\}\Big)$$

where $\beta - \alpha = \beta$ provided β is infinite (and $\beta > \alpha$). The above formula may be written in the following concise form:

$$(\mathsf{B}\ominus\mathsf{A})_{\vartriangle} = [\mathsf{E}_{sm}(\mathsf{B})\ominus(\mathsf{E}_{sm}(\mathsf{A})\wedge\mathsf{E}^{II}_1(\mathsf{B}))]\boxplus\Big[\boxplus_{(i,\alpha,\beta)\in\Upsilon^2_+} (\beta-\alpha)\odot(\mathsf{E}^i_\alpha(\mathsf{A})\wedge\mathsf{E}^i_\beta(\mathsf{B}))\Big] \tag{4.1.2}$$

where $\Upsilon^2_+ = \{(i,\alpha,\beta) \in \Upsilon^2\colon \alpha < \beta,\ (i,\alpha,\beta) \neq (II,0,1)\}$. It is also easy to verify that $(\mathsf{B} \ominus \mathsf{A})^{\triangledown} = (\mathsf{B} \ominus \mathsf{A})_{\vartriangle} \oplus \mathsf{X}$ where $\mathsf{X} = \boxplus_{\alpha\in\mathrm{Card}_\infty} \alpha \odot [\mathsf{E}^I_{\alpha,\alpha} \boxplus \mathsf{E}^{II}_{\alpha,\alpha} \boxplus \mathsf{E}^{III}_{\alpha,\alpha}]$. Since $\mathsf{X} \perp_u (\mathsf{B} \ominus \mathsf{A})_{\vartriangle}$, the proof of (AO10) is finished. Recall that we have shown that

$$(\mathsf{B} \ominus \mathsf{A})^{\triangledown} \boxminus (\mathsf{B} \ominus \mathsf{A})_{\vartriangle} = \boxplus_{\substack{\alpha\in\mathrm{Card}_\infty\\ i\in\{I,II,III\}}} \alpha \odot (\mathsf{E}^i_\alpha(\mathsf{A}) \wedge \mathsf{E}^i_\alpha(\mathsf{B})). \tag{4.1.3}$$

In particular, $(\mathsf{B}\ominus\mathsf{A})^{\nabla}=(\mathsf{B}\ominus\mathsf{A})_{\Delta}$ if and only if $\mathsf{E}^i_\alpha(\mathsf{A})\perp_u \mathsf{E}^i_\alpha(\mathsf{B})$ for every infinite α. This proves

(AO11) Whenever $\mathsf{A}\leqslant\mathsf{B}$, $B\ominus\mathsf{A}$ is well defined iff $\mathsf{E}^i_\alpha(\mathsf{A})\perp_u \mathsf{E}^i_\alpha(\mathsf{B})$ for any $\alpha\in\mathrm{Card}_\infty$ and $i\in\{I,II,III\}$.

(AO12) $(\mathsf{B}\ominus\mathsf{X})_\Delta\vee(\mathsf{X}\ominus\mathsf{A})_\Delta\leqslant(\mathsf{B}\ominus\mathsf{A})_\Delta$ whenever $\mathsf{A}\leqslant\mathsf{X}\leqslant\mathsf{B}$.

(AO13) For any nonempty set $\{\mathsf{A}^{(s)}\}_{s\in S}\subset\mathcal{CDD}_N$ and $\mathsf{B}\in\mathcal{CDD}_N$, $\mathsf{B}\vee(\bigwedge_{s\in S}\mathsf{A}^{(s)})=\bigwedge_{s\in S}(\mathsf{B}\vee\mathsf{A}^{(s)})$ and $\mathsf{B}\wedge(\bigvee_{s\in S}\mathsf{A}^{(s)})=\bigvee_{s\in S}(\mathsf{B}\wedge\mathsf{A}^{(s)})$.

(AO14) For any nonempty set $\{\mathsf{A}^{(s)}\}_{s\in S}$ of N-tuples, any $\mathsf{A},\mathsf{B}\in\mathcal{CDD}_N$ and each $\alpha\in\mathrm{Card}$,

$$\begin{aligned}&\alpha\odot(\mathsf{A}\wedge\mathsf{B})=(\alpha\odot\mathsf{A})\wedge(\alpha\odot\mathsf{B}),\\ &\alpha\odot(\textstyle\bigwedge_{s\in S}\mathsf{A}^{(s)})=\bigwedge_{s\in S}(\alpha\odot\mathsf{A}^{(s)})\quad\text{if}\quad \begin{array}{l}\alpha\text{ is finite or}\\ \forall s\in S\colon \mathsf{E}_{sm}(\mathsf{A}^{(s)})=\mathsf{O},\end{array}\\ &\alpha\odot(\textstyle\bigvee_{s\in S}\mathsf{A}^{(s)})=\bigvee_{s\in S}(\alpha\odot\mathsf{A}^{(s)}).\end{aligned}$$

For the proofs of (AO12)–(AO14) see Corollary 4.4.3, Theorem 4.4.10 and Proposition 4.4.11.

Example 4.1.1. Taking into account (AO14), it seems to be surprising that in general $\alpha\odot(\bigwedge_{s\in S}\mathsf{A}^{(s)})$ differs from $\bigwedge_{s\in S}(\alpha\odot\mathsf{A}^{(s)})$ for infinite cardinals α, even if S is countable. Let us give a counterexample. Let $\alpha\geqslant\aleph_0$ and $\mathsf{X}\in\mathcal{SM}_N$ be nontrivial. There is a sequence $(\mathsf{A}^{(n)})_{n=1}^\infty$ such that $n\odot\mathsf{A}^{(n)}=\mathsf{X}$ (see the beginning of Chapter 4.3). Then $\alpha\odot\mathsf{A}^{(n)}=\alpha\odot\mathsf{X}\neq\mathsf{O}$, while $\bigwedge_{n=1}^\infty\mathsf{A}^{(n)}=\mathsf{O}$.

(AO12) has an interesting consequence.

Proposition 4.1.2. *Let $\mathcal{A},\mathcal{B}\subset\mathcal{CDD}_N$ be nonempty sets. Then $\bigvee(\mathcal{A}\oplus\mathcal{B})=(\bigvee\mathcal{A})\oplus(\bigvee\mathcal{B})$ and $\bigwedge(\mathcal{A}\oplus\mathcal{B})=(\bigwedge\mathcal{A})\oplus(\bigwedge\mathcal{B})$ where $\mathcal{A}\oplus\mathcal{B}=\{\mathsf{A}\oplus\mathsf{B}\colon\ \mathsf{A}\in\mathcal{A},\ \mathsf{B}\in\mathcal{B}\}$.*

Proof. Since the case of l.u.b.'s is much simpler, we prove only the g.l.b. part. It is clear that $(\bigwedge\mathcal{A})\oplus(\bigwedge\mathcal{B})\leqslant\bigwedge(\mathcal{A}\oplus\mathcal{B})$. To see the converse inequality, assume that $\mathsf{X}\leqslant\mathsf{A}\oplus\mathsf{B}$ for any $\mathsf{A}\in\mathcal{A}$ and $\mathsf{B}\in\mathcal{B}$. Fix $\mathsf{B}\in\mathcal{B}$ and put $\mathsf{E}=\bigwedge(\mathcal{A}\oplus\{\mathsf{B}\})$. For each $\mathsf{A}\in\mathcal{A}$ we clearly have $\mathsf{A}\oplus\mathsf{B}\geqslant\mathsf{E}\geqslant\mathsf{B}$ and consequently, thanks to (AO12), $(\mathsf{E}\ominus\mathsf{B})_\Delta\leqslant[(\mathsf{A}\oplus\mathsf{B})\ominus\mathsf{B}]_\Delta\leqslant\mathsf{A}$ where the last inequality follows from the definition of $[\ldots]_\Delta$. So, $(\mathsf{E}\ominus\mathsf{B})_\Delta\leqslant\bigwedge\mathcal{A}$ and therefore $\mathsf{E}=(\mathsf{E}\ominus\mathsf{B})_\Delta\oplus\mathsf{B}\leqslant(\bigwedge\mathcal{A})\oplus\mathsf{B}$. This shows that

$$\bigwedge(\mathcal{A}\oplus\{\mathsf{B}\})\leqslant\Big(\bigwedge\mathcal{A}\Big)\oplus\mathsf{B},$$

which yields

$$\begin{aligned}\bigwedge(\mathcal{A}\oplus\mathcal{B})&=\bigwedge_{\mathsf{B}\in\mathcal{B}}\Big[\bigwedge(\mathcal{A}\oplus\{\mathsf{B}\})\Big]\leqslant\bigwedge_{\mathsf{B}\in\mathcal{B}}\Big[\Big(\bigwedge\mathcal{A}\Big)\oplus\mathsf{B}\Big]\\ &=\bigwedge\Big(\mathcal{B}\oplus\Big\{\bigwedge\mathcal{A}\Big\}\Big)\leqslant\Big(\bigwedge\mathcal{B}\Big)\oplus\Big(\bigwedge\mathcal{A}\Big),\end{aligned}$$

and we are done. ∎

Corollary 4.1.3. *Let $\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3,\ldots$ be nonempty sets of members of $\mathcal{CDD}_N$ and let $\mathcal{A}=\{\bigoplus_{n=1}^\infty\mathsf{A}^{(n)}\colon\mathsf{A}^{(n)}\in\mathcal{A}_n\ (n\geqslant1)\}$. Then $\bigvee\mathcal{A}=\bigoplus_{n=1}^\infty(\bigvee\mathcal{A}_n)$.*

Proof. It is clear that $\bigvee\mathcal{A} \leqslant \bigoplus_{n=1}^{\infty}(\bigvee\mathcal{A}_n)$. Conversely, by (AO6), $\bigoplus_{n=1}^{\infty}(\bigvee\mathcal{A}_n) = \bigvee_{n\geqslant 1}\big[(\bigvee\mathcal{A}_1)\oplus\cdots\oplus(\bigvee\mathcal{A}_n)\big]$. Now by induction and Proposition 4.1.2, $(\bigvee\mathcal{A}_1)\oplus\cdots\oplus(\bigvee\mathcal{A}_n) = \bigvee(\mathcal{A}_1\oplus\cdots\oplus\mathcal{A}_n) \leqslant \bigvee\mathcal{A}$. ∎

In the next chapter we shall prove a counterpart of Corollary 4.1.3 for uncountable collections of sets of N-tuples (see Theorem 4.2.2).

Example 4.1.4. It may be surprising that the counterpart of Corollary 4.1.3 for infima fails to be true, even if each $\mathcal{A}_n$ is a finite collection of minimal normal N-tuples. That is, in general $\bigwedge(\bigoplus_{n=1}^{\infty}\mathcal{A}_n)$ differs from $\bigoplus_{n=1}^{\infty}(\bigwedge\mathcal{A}_n)$ where $\bigoplus_{n=1}^{\infty}\mathcal{A}_n = \{\bigoplus_{n=1}^{\infty}\mathsf{A}^{(n)}\colon \mathsf{A}^{(n)}\in\mathcal{A}_n\}$. Let us justify this claim.

For every $u\in L^{\infty}([0,1])$ we shall write, for simplicity, $\boldsymbol{X}_u$ to denote the N-tuple $(M_u,\ldots,M_u)$ where M_u is the multiplication operator by u on $L^2([0,1])$. For each pair (n,m) of naturals with $1\leqslant m\leqslant n$ let $j_{n,m}$ be the characteristic function of $[0,1]\setminus[(m-1)/n, m/n]$. Additionally, let $\mathrm{id}\in L^{\infty}([0,1])$ be the identity map on $[0,1]$. Put $\mathsf{A}_{n,m} = \mathsf{X}_{j_{n,m}\,\mathrm{id}}$ and $\mathcal{A}_n = \{\mathsf{A}_{n,j}\colon j=1,\ldots,n\}$. Then $\mathcal{A}_n\subset\mathcal{MF}_N$ (because $\mathcal{W}'(\boldsymbol{X}_{\mathrm{id}}) = \{M_u\colon u\in L^{\infty}([0,1])\}$) and $\bigwedge\mathcal{A}_n = \mathsf{O}$ for every $n\geqslant 1$. However, if $(m_n)_{n=1}^{\infty}$ is any sequence of natural numbers such that $1\leqslant m_n\leqslant n$, then $\bigoplus_{n=1}^{\infty}\mathsf{A}_{n,m_n}\geqslant\bigvee_{n\geqslant 1}\mathsf{A}_{n,m_n} = \mathsf{X}_{\mathrm{id}}$ (the last equality holds since $\bigcup_{n=1}^{\infty}([0,1]\setminus[(m_n-1)/n, m_n/n])$ is of full Lebesgue measure in $[0,1]$). Consequently, $\bigwedge(\bigoplus_{n=1}^{\infty}\mathcal{A}_n)\geqslant\mathsf{X}_{\mathrm{id}}\neq\mathsf{O} = \bigoplus_{n=1}^{\infty}(\bigwedge\mathcal{A}_n)$.

One may deduce from Example 3.3.9 that the assumption in (AO6) that the size of S is a limit cardinal is essential (in the next chapter we shall discuss (4.1.1) in detail for sets S whose cardinality is not limit). However, for semiminimal parts of N-tuples a stronger property (than (AO6)) holds in general (see below). For simplicity, for every set S let us denote by $\mathcal{P}_f(S)$ and $\mathcal{P}_{\omega}(S)$ the families of all finite and, respectively, countable (finite or infinite) subsets of S.

Proposition 4.1.5. *Let S be an infinite set and $\{\mathsf{A}^{(s)}\}_{s\in S}$ be an arbitrary collection of N-tuples, $\mathsf{A} = \bigoplus_{s\in S}\mathsf{A}^{(s)}$ and*

$$\mathsf{A}' = \bigvee\Big\{\bigoplus_{s\in S_0}\mathsf{A}^{(s)}\colon S_0\in\mathcal{P}_f(S)\Big\}.$$

Then $\mathsf{E}_{sm}(\mathsf{A}) = \mathsf{E}_{sm}(\mathsf{A}')$ and $\mathsf{E}^i_{\alpha}(\mathsf{A}) = \mathsf{E}^i_{\alpha}(\mathsf{A}')$ for each $(i,\alpha)\in\Upsilon$ with finite α.

Proof. It is clear that $\mathsf{E}^i_0(\mathsf{A}) = \mathsf{E}^i_0(\mathsf{A}')$ for $i\in\{\mathit{I},\mathit{II},\mathit{III}\}$. Further, let us prove that

$$\mathsf{E}^{\mathit{II}}_1(\mathsf{A}) = \mathsf{E}^{\mathit{II}}_1(\mathsf{A}'). \tag{4.1.4}$$

Since $\mathsf{A}'\leqslant\mathsf{A}\ll\mathsf{A}'$, (3.6.2) (page 28) shows that $\mathsf{E}^{\mathit{II}}_1(\mathsf{A})\leqslant\mathsf{E}^{\mathit{II}}_1(\mathsf{A}')$. Conversely, if $\mathsf{X}_a = \mathsf{E}(\mathsf{X}|\mathsf{E}^{\mathit{II}}_1(\mathsf{A}'))$ for every $\mathsf{X}\in\mathcal{CDD}_N$, then $(\mathsf{A}')_a = \bigvee\big\{\bigoplus_{s\in S_0}(\mathsf{A}^{(s)})_a\colon S_0\in\mathcal{P}_f(S)\big\}$ and $\mathsf{A}_a = \bigoplus_{s\in S}(\mathsf{A}^{(s)})_a$. But $(\mathsf{A}')_a = \mathsf{E}_{sm}(\mathsf{A}')\in\mathcal{SM}_N$ and hence $(\mathsf{A}')_a = \mathsf{A}_a$, thanks to Proposition 4.1.6 (see below). So, $\mathsf{A}_a\in\mathcal{SM}_N$ and consequently, again by (3.6.2), $\mathsf{E}^{\mathit{II}}_1(\mathsf{A}')\leqslant\mathsf{E}^{\mathit{II}}_1(\mathsf{A})$. This proves (4.1.4).

Now we have $\mathsf{E}_{sm}(\mathsf{A}) = \mathsf{E}(\mathsf{A}|\mathsf{E}^{\mathit{II}}_1(\mathsf{A})) = \mathsf{A}_a = (\mathsf{A}')_a = \mathsf{E}_{sm}(\mathsf{A}')$.

It remains to check that $\mathsf{E}_n^I(\mathsf{A}) = \mathsf{E}_n^I(\mathsf{A}')$ for natural n. Let $\mathsf{F} = \boxplus_{n=1}^{\infty} \mathsf{E}_n^I(\mathsf{A})$ and $\mathsf{F}' = \boxplus_{n=1}^{\infty} \mathsf{E}_n^I(\mathsf{A}')$. It is enough to show that $\mathsf{F} = \mathsf{F}'$, which we leave to the reader (for it is similar to the proof of (4.1.4)). ∎

The following result is in the same spirit.

PROPOSITION 4.1.6. *Let S be an infinite set, $\{\mathsf{A}^{(s)}\}_{s\in S} \subset \mathcal{CDD}_N$ and let*
$$\mathsf{A} = \bigvee\Big\{\bigoplus_{s\in S_0} \mathsf{A}^{(s)} \colon S_0 \in \mathcal{P}_f(S)\Big\}.$$
If $\mathsf{A} \in \mathcal{FIN}_N$, then $\mathsf{A} = \bigoplus_{s\in S} \mathsf{A}^{(s)}$.

Proof. Let $\mathcal{M} = \mathcal{W}'(\boldsymbol{A})$ and let $p_s \in E(\mathcal{M})$ correspond (by Proposition 2.3.1) to $\mathsf{A}^{(s)}$ ($s \in S$). Further, let $\mathrm{tr}\colon \mathcal{M} \to \mathcal{Z}(\mathcal{M})$ be the trace on $\mathcal{M}$. For every $s \in S$ put $w_s = \mathrm{tr}(p_s)$. Since $\bigoplus_{s\in S_0} \mathsf{A}^{(s)} \leqslant \mathsf{A}$ where $S_0 \in \mathcal{P}_f(S)$, $\sum_{s\in S_0} w_s \leqslant 1$ and consequently $\sum_{s\in S} w_s$ is convergent and the sum is not greater than 1. Recall that for any $q, q' \in E(\mathcal{M})$, $q \preccurlyeq q' \Leftrightarrow \mathrm{tr}(q) \leqslant \mathrm{tr}(q')$ (see e.g. [35, Corollary 5.2.8] or [19, Theorem 8.4.3]). This implies that it is possible, well ordering the set S and using transfinite induction, to construct a family $\{q_s\}_{s\in S}$ of mutually orthogonal projections in $\mathcal{M}$ such that $p_s \sim q_s$ for any $s \in S$. Hence $\sum_{s\in S} q_s \leqslant 1$, which yields $\bigoplus_{s\in S} \mathsf{A}^{(s)} \leqslant \mathsf{A}$ and we are done. ∎

4.2. Reconstructing infinite operations

Classical algebraic structures deal with operations on pairs (such as the action of a semigroup). However, some operations naturally make sense also for infinitely (possibly uncountably) many arguments (e.g. unions of sets) and sometimes it is necessary to use these extended 'infinite' operations in order to understand, formulate or prove some statements. Unless infinite operations can be 'defined' (or characterized) in terms of their finite versions, every such theorem may be seen as unformulable or unprovable in the language of the original algebraic structure. The most typical example of an infinite operation is the union of a family of sets. However, it may be characterized by means of the union of two sets. Namely, for any family $\mathcal{A}$ put $\mathcal{A}^{\Delta} = \{B \colon A \cup B = B$ for each $A \in \mathcal{A}\}$ and then $\bigcup \mathcal{A}$ is the **unique** set $B \in \mathcal{A}^{\Delta}$ such that $B \cup C = C$ for any $C \in \mathcal{A}^{\Delta}$. This characterization is possible for one simple reason: the union coincides with the l.u.b. of the family with respect to the inclusion order which may be defined in terms of the union of a pair. When we turn to the class $\mathcal{CDD}_N$, the direct sum operation cannot be characterized in a similar manner, because $\bigoplus_{s\in S} \mathsf{A}^{(s)}$ differs, in general, from $\bigvee\{\bigoplus_{s\in S_0} \mathsf{A}^{(s)} \colon S_0$ a finite subset of $S\}$. Nevertheless, infinite direct sums may be reconstructed from finite ones, and this is the subject of this chapter. Thus, every result of the paper concerning unitary equivalence classes of N-tuples is a part of the theory which starts with the class $\mathcal{CDD}_N$ and the operation $\mathcal{CDD}_N \times \mathcal{CDD}_N \ni (\mathsf{A}, \mathsf{B}) \mapsto \mathsf{A} \oplus \mathsf{B} \in \mathcal{CDD}_N$. (This refers to the material of Chapters 2.1–5.1, but **not** to the rest.)

Our aim is to show that $\bigoplus_{s\in S} \mathsf{A}^{(s)}$ may be 'recognized' if the only admissible 'tool' in the class $\mathcal{CDD}_N$ is the direct sum of a pair. Below we show step by step how to do this. Each of the steps listed begins with a tool which may be defined.

(ST1) 'O': It is the unique member A of $\mathcal{CDD}_N$ such that $\mathsf{A}\oplus\mathsf{X} = \mathsf{X}$ for every $\mathsf{X} \in \mathcal{CDD}_N$.

(ST2) '$\leqslant$': $\mathsf{A} \leqslant \mathsf{B}$ iff $\mathsf{B} = \mathsf{A}\oplus\mathsf{X}$ for some $\mathsf{X} \in \mathcal{CDD}_N$. Accordingly, the l.u.b.'s and g.l.b.'s are well defined.

(ST3) '$\perp_u$': $\mathsf{A} \perp_u \mathsf{B} \Leftrightarrow \mathsf{A} \wedge \mathsf{B} = \mathsf{O}$.

(ST4) '$\leqslant^s$': $\mathsf{A} \leqslant^s \mathsf{B}$ iff $\mathsf{B} = \mathsf{A} \oplus \mathsf{X}$ for some X such that $\mathsf{X} \perp_u \mathsf{A}$.

(ST5) '$\boxplus$' and '$\boxminus$': $\mathsf{A} = \boxplus_{s\in S} \mathsf{A}^{(s)}$ (S any set) iff $\mathsf{A}^{(s)} \perp_u \mathsf{A}^{(s')}$ for distinct $s, s' \in S$ and $\mathsf{A} = \bigvee_{s\in S} \mathsf{A}^{(s)}$; if $\mathsf{A} \leqslant^s \mathsf{B}$, $\mathsf{B}\boxminus\mathsf{A}$ is the unique X such that $\mathsf{X} \perp_u \mathsf{A}$ and $\mathsf{B} = \mathsf{X}\oplus\mathsf{A}$.

(ST6) '$\bigoplus_{n=1}^{\infty}$': $\bigoplus_{n=1}^{\infty} \mathsf{A}^{(n)} = \bigvee_{n\geqslant 1} \mathsf{A}^{(1)} \oplus \cdots \oplus \mathsf{A}^{(n)}$. In particular, $\aleph_0 \odot \mathsf{A}$ is well defined for each A.

(ST7) '$\ll$': $\mathsf{A} \ll \mathsf{B}$ iff there is no $\mathsf{X} \neq \mathsf{O}$ such that $\mathsf{X} \leqslant \mathsf{A}$ and $\mathsf{X} \perp_u \mathsf{B}$.

(ST8) '$\mathsf{E}(\mathsf{A}|\mathsf{B})$': $\mathsf{X} = \mathsf{E}(\mathsf{A}|\mathsf{B}) \Leftrightarrow \exists\,\mathsf{Y}\colon \mathsf{A} = \mathsf{X} \oplus \mathsf{Y},\ \mathsf{X} \ll \mathsf{B}$ and $\mathsf{Y} \perp_u \mathsf{B}$.

(ST9) 'Multiplicity free N-tuples': $\mathsf{A} \in \mathcal{MF}_N$ if and only if there is no $\mathsf{X} \neq \mathsf{O}$ such that $\mathsf{X} \oplus \mathsf{X} \leqslant \mathsf{A}$. Accordingly, J_I is well defined.

(ST10) 'Minimal N-tuples': A is minimal iff $\mathsf{A} \leqslant \mathsf{X}$ whenever $\mathsf{A} \ll \mathsf{X}$.

(ST11) 'Hereditary idempotents': $\mathsf{A} \in \mathcal{HI}_N \Leftrightarrow \mathsf{B} = \mathsf{B} \oplus \mathsf{B}$ for each $\mathsf{B} \leqslant \mathsf{A}$. Accordingly, the class $\mathcal{HIM}_N$ and J_{III} are well defined, thanks to (ST10).

(ST12) 'Semiminimal N-tuples': Use (ST5), (ST7) and the definition of semiminimality.

(ST13) 'J_{II}': $\mathsf{J}_{II} = \bigvee\{\aleph_0 \odot \mathsf{X}\colon \mathsf{X} \in \mathcal{SM}_N\}$ (use (ST6) and (ST12)).

(ST14) '$\alpha \odot \mathsf{A}$ for $\mathsf{A} \leqslant \mathsf{J}_i$' with $i \in \{I, II, III\}$: Thanks to (ST6), we may assume that $\alpha > \aleph_0$. If $i = II$, $\alpha \odot \mathsf{A} = \alpha \odot (\aleph_0 \odot \mathsf{A})$ and $\aleph_0 \odot \mathsf{A} \leqslant^s \mathsf{J}_{II}$; while for $i \neq II$ one has $\mathsf{A} \leqslant^s \mathsf{J}_i$. These remarks show that we may assume that $\mathsf{A} \leqslant^s \mathsf{J}_i$. Under this assumption, $\alpha \odot \mathsf{A}$ may be characterized by transfinite induction with respect to α. When α is limit, it suffices to apply (AO6) (page 32). On the other hand, if $\alpha = \beta^+$ and $\mathsf{A} \neq \mathsf{O}$,

$$\alpha \odot \mathsf{A} = \bigwedge\{\mathsf{X}|\ \forall \mathsf{B} \leqslant^s \mathsf{A},\ \mathsf{B} \neq \mathsf{O}\colon \beta \odot \mathsf{B} \lneqq \mathsf{E}(\mathsf{X}|\mathsf{B})\}.$$

(This formula may be deduced from Theorem 3.6.1.)

(ST15) '$\mathsf{E}^i_\alpha(\mathsf{A})$ and $\mathsf{E}_{sm}(\mathsf{A})$': Use Theorem 3.6.1 and previous steps.

(ST16) '$\alpha \odot \mathsf{A}$' (arbitrary A): Use (ST15), Theorem 3.6.1, (ST14) and (ST5).

(ST17) '$\aleph_0 \cdot \dim(\mathsf{A})$': Since

$$\aleph_0 \cdot \dim(\mathsf{A}) = \sum_{n=1}^{\infty} \aleph_0 \cdot \dim(\mathsf{E}^I_n(\mathsf{A})) + \aleph_0 \cdot \dim(\mathsf{E}^{II}_1(\mathsf{A})) + \sum_{\substack{\alpha\in\mathrm{Card}_\infty \\ i\in\{I,II,III\}}} \alpha[\aleph_0 \cdot \dim(\mathsf{E}^i_\alpha(\mathsf{A}))]$$

(cf. Theorem 3.6.1; $\dim(\mathsf{E}_{sm}(\mathsf{A})) = \dim(\mathsf{E}^{II}_1(\mathsf{A}))$), it suffices to characterize the cardinal number $\aleph_0 \cdot \dim(\mathsf{A})$ for $\mathsf{A} \leqslant^s \mathsf{J}$. But in that case this is quite easy, thanks to Proposition 3.4.10: $\aleph_0 \cdot \dim(\mathsf{A})$ (provided $\mathsf{A} \leqslant^s \mathsf{J}$ is nontrivial) is the least infinite cardinal α such that each regular subfamily of $\{\mathsf{X}\colon \mathsf{O} \neq \mathsf{X} \leqslant^s \mathsf{A}\}$ has size not greater than α.

Now we are ready to characterize infinite direct sums. For simplicity, let us put $\mathrm{Dim}(\mathsf{F}) = \aleph_0 \cdot \dim(\mathsf{F})$, $\mathsf{E}_f(\mathsf{F}) = \mathsf{E}^{II}_1(\mathsf{F}) \boxplus \boxplus_{n=1}^{\infty} \mathsf{E}^I_n(\mathsf{F})$ and $\mathsf{E}_\alpha(\mathsf{F}) = \mathsf{E}^I_\alpha(\mathsf{F}) \boxplus \mathsf{E}^{II}_\alpha(\mathsf{F}) \boxplus \mathsf{E}^{III}_\alpha(\mathsf{F})$ for $\alpha \in \{0\} \cup \mathrm{Card}_\infty$ and any $\mathsf{F} \in \mathcal{CDD}_N$. By (ST17) and (ST15), 'Dim', 'E_f' and 'E_α' are well defined. Fix a collection $\{\mathsf{A}^{(s)}\}_{s\in S}$ of N-tuples and put $\mathsf{A} = \bigoplus_{s\in S} \mathsf{A}^{(s)}$ and

$\mathsf{A}_f = \bigvee\{\bigoplus_{s\in S'} \mathsf{A}^{(s)}\colon S' \in \mathcal{P}_f(S)\}$. A_f is 'known' by (ST2). Thanks to (ST16) and (ST5), it suffices to find $\mathsf{E}_{sm}(\mathsf{A})$ and $\mathsf{E}^i_\alpha(\mathsf{A})$. By Proposition 4.1.5, $\mathsf{E}^i_\alpha(\mathsf{A}) = \mathsf{E}^i_\alpha(\mathsf{A}')$ for $(i,\alpha) \in \Upsilon$ with finite α and $\mathsf{E}_{sm}(\mathsf{A}) = \mathsf{E}_{sm}(\mathsf{A}')$. Since $\mathsf{E}^i_\alpha(\mathsf{A}) = \mathsf{E}_\alpha(\mathsf{A}) \wedge \mathsf{J}_i$, we see that it remains to find $\mathsf{E}_\alpha(\mathsf{A})$ for infinite α (thanks to (ST9), (ST11) and (ST13)). Let us show that

$$\mathsf{E}_\alpha(\mathsf{A}) = \bigvee \mathfrak{X} \tag{4.2.1}$$

where

$$\mathfrak{X} = \Big\{\mathsf{X} \leqslant^s \mathsf{J} \boxminus (\mathsf{E}_0(\mathsf{A}) \boxplus \mathsf{E}_f(\mathsf{A}))\Big|\ \forall \mathsf{Y} \leqslant^s \mathsf{X},\ \mathrm{Dim}(\mathsf{Y}) = \aleph_0\colon \sum_{s\in S} \mathrm{Dim}\, \mathsf{E}(\mathsf{A}^{(s)}|\mathsf{Y}) = \alpha\Big\}.$$

It is clear that $\mathsf{E}_\alpha(\mathsf{A}) \leqslant^s \mathsf{J} \boxminus (\mathsf{E}_0(\mathsf{A}) \boxplus \mathsf{E}_f(\mathsf{A}))$. Furthermore, if $\mathsf{Y} \leqslant^s \mathsf{E}_\alpha(\mathsf{A})$, then $\bigoplus_{s\in S} \mathsf{E}(\mathsf{A}^{(s)}|\mathsf{Y}) = \mathsf{E}(\mathsf{A}|\mathsf{Y}) = \alpha \odot \mathsf{Y}$ and thus

$$\aleph_0 \cdot \sum_{s\in S} \dim(\mathsf{E}(\mathsf{A}^{(s)}|\mathsf{Y})) = \alpha$$

provided $\mathrm{Dim}(\mathsf{Y}) = \aleph_0$. This yields $\mathsf{E}_\alpha(\mathsf{A}) \in \mathfrak{X}$. Consequently, the proof of (4.2.1) will be complete if we show that $\mathsf{X} \leqslant \mathsf{E}_\alpha(\mathsf{A})$ for every $\mathsf{X} \in \mathfrak{X}$. To get this inequality, it is enough to check that $\mathsf{X} \perp_u \mathsf{E}_\beta(\mathsf{A})$ for any infinite $\beta \neq \alpha$. Suppose $\mathsf{Y}' = \mathsf{X} \wedge \mathsf{E}_\beta(\mathsf{A})$ is nontrivial. Since $\mathsf{Y}' \leqslant^s \mathsf{J}$, we infer from Proposition 3.4.10 that there is $\mathsf{Y} \leqslant^s \mathsf{Y}'$ with $\mathrm{Dim}(\mathsf{Y}) = \aleph_0$. But then $\mathsf{Y} \leqslant^s \mathsf{X}$ and $\mathsf{E}(\mathsf{A}|\mathsf{Y}) = \beta \odot \mathsf{Y}$ (because $\mathsf{Y} \leqslant^s \mathsf{E}_\beta(\mathsf{A})$). Consequently, $\aleph_0 \cdot \sum_{s\in S} \dim(\mathsf{E}(\mathsf{A}^{(s)}|\mathsf{Y})) = \beta$, which contradicts the fact that $\mathsf{X} \in \mathfrak{X}$ and finishes the proof of (4.2.1).

The arguments of this chapter prove

Proposition 4.2.1. *If $\Phi\colon \mathcal{CDD}_N \to \mathcal{CDD}_N$ is a bijective assignment such that $\Phi(\mathsf{A} \oplus \mathsf{B}) = \Phi(\mathsf{A}) \oplus \Phi(\mathsf{B})$ for any $\mathsf{A}, \mathsf{B} \in \mathcal{CDD}_N$, then Φ preserves all notions, features and operations appearing in* (ST1)–(ST17) *and*

$$\Phi\Big(\bigoplus_{s\in S} \mathsf{A}^{(s)}\Big) = \bigoplus_{s\in S} \Phi(\mathsf{A}^{(s)})$$

for any set $\{\mathsf{A}^{(s)}\}_{s\in S} \subset \mathcal{CDD}_N$.

Let us now discuss the relation between both sides of (4.1.1) for a set S whose cardinality is not limit, i.e. $\mathrm{card}(S) = \gamma^+$ for some infinite cardinal γ. Let

$$\mathsf{A}' = \bigvee\Big\{\bigoplus_{s\in S'} \mathsf{A}^{(s)}\colon S' \subset S,\ 0 < \mathrm{card}(S') \leqslant \gamma\Big\}.$$

By Proposition 4.1.5, $\mathsf{E}_{sm}(\mathsf{A}) = \mathsf{E}_{sm}(\mathsf{A}')$ and $\mathsf{E}^i_\alpha(\mathsf{A}) = \mathsf{E}^i_\alpha(\mathsf{A}')$ for every $(i,\alpha) \in \Upsilon$ with finite α. These equalities are more general: we claim that

$$\mathsf{E}^i_\alpha(\mathsf{A}) = \mathsf{E}^i_\alpha(\mathsf{A}') \tag{4.2.2}$$

provided $\alpha \neq \gamma, \gamma^+$. To show this, it suffices to check that $\mathsf{E}^i_\alpha(\mathsf{A}') \leqslant \mathsf{E}^i_\alpha(\mathsf{A})$ for $\alpha \neq \gamma, \gamma^+$ and $\mathsf{E}^i_\gamma(\mathsf{A}') \boxplus \mathsf{E}^i_{\gamma^+}(\mathsf{A}') \leqslant \mathsf{E}^i_\gamma(\mathsf{A}) \boxplus \mathsf{E}^i_{\gamma^+}(\mathsf{A})$ (since we deal with partitions of unity). We need to check only infinite α's.

First assume $\aleph_0 \leqslant \alpha < \gamma$. Fix $\mathsf{X} \leqslant^s \mathsf{E}^i_\alpha(\mathsf{A}')$ with $\mathrm{Dim}(\mathsf{X}) = \aleph_0$. Let

$$S' = \{s \in S\colon \mathsf{E}(\mathsf{A}^{(s)}|\mathsf{X}) \neq \mathsf{O}\}.$$

If the size of S' were greater than α, there would exist a subset S'' of S' with $\operatorname{card}(S'') = \alpha^+ \leqslant \gamma$. Then we would have $\bigoplus_{s\in S''} \mathsf{A}^{(s)} \leqslant \mathsf{A}'$ and $\alpha \odot \mathsf{X} = \mathsf{E}(\mathsf{A}'|\mathsf{X}) \geqslant \bigoplus_{s\in S''} \mathsf{E}(\mathsf{A}^{(s)}|\mathsf{X})$, which contradicts the fact that $\operatorname{Dim}(\alpha\odot\mathsf{X}) = \alpha < \operatorname{card}(S'') \leqslant \operatorname{Dim}(\bigoplus_{s\in S''} \mathsf{E}(\mathsf{A}^{(s)}|\mathsf{X}))$. We infer that $\operatorname{card}(S') \leqslant \alpha$. So, $\mathsf{E}(\mathsf{A}|\mathsf{X}) = \mathsf{E}(\mathsf{A}'|\mathsf{X})$. Consequently, $\mathsf{E}(\mathsf{A}|\mathsf{E}^i_\alpha(\mathsf{A}')) = \mathsf{E}(\mathsf{A}'|\mathsf{E}^i_\alpha(\mathsf{A}'))$ $= \alpha \odot \mathsf{E}^i_\alpha(\mathsf{A}')$ (thanks to Proposition 3.4.10). This means that $\mathsf{E}^i_\alpha(\mathsf{A}') \leqslant \mathsf{E}^i_\alpha(\mathsf{A})$.

Now assume that $\alpha \geqslant \gamma^+ = \operatorname{card}(S)$. As before, take $\mathsf{X} \leqslant^s \mathsf{E}^i_\alpha(\mathsf{A}')$ with $\operatorname{Dim}(\mathsf{X}) = \aleph_0$. Then $\alpha \odot \mathsf{X} = \mathsf{E}(\mathsf{A}'|\mathsf{X}) \geqslant \mathsf{E}(\mathsf{A}^{(s)}|\mathsf{X})$ for every $s \in S$ and thus $\mathsf{E}(\mathsf{A}|\mathsf{X}) = \bigoplus_{s\in S} \mathsf{E}(\mathsf{A}^{(s)}|\mathsf{X}) \leqslant \operatorname{card}(S) \odot (\alpha \odot \mathsf{X}) = \alpha \odot \mathsf{X} = \mathsf{E}(\mathsf{A}'|\mathsf{X}) \leqslant \mathsf{E}(\mathsf{A}|\mathsf{X})$. So, $\mathsf{E}(\mathsf{A}|\mathsf{X}) = \mathsf{E}(\mathsf{A}'|\mathsf{X})$ and consequently $\mathsf{E}^i_\alpha(\mathsf{A}') \leqslant \mathsf{E}^i_\alpha(\mathsf{A})$.

To finish the proof of (4.2.2), it remains to consider $\alpha = \gamma$. For $\mathsf{X} = \mathsf{E}^i_\gamma(\mathsf{A}')$ we readily have $\gamma \odot \mathsf{X} = \mathsf{E}(\mathsf{A}'|\mathsf{X}) \leqslant \mathsf{E}(\mathsf{A}|\mathsf{X}) = \bigoplus_{s\in S} \mathsf{E}(\mathsf{A}^{(s)}|\mathsf{X}) \leqslant \operatorname{card}(S) \odot \mathsf{E}(\mathsf{A}'|\mathsf{X}) = \gamma^+ \odot \mathsf{X}$. These inequalities imply that $\mathsf{X} \leqslant \mathsf{E}^i_\gamma(\mathsf{A}) \boxplus \mathsf{E}^i_{\gamma^+}(\mathsf{A})$ and we are done.

Having (4.2.2), we obtain $\mathsf{E}^i_\gamma(\mathsf{A}') \boxplus \mathsf{E}^i_{\gamma^+}(\mathsf{A}') = \mathsf{E}^i_\gamma(\mathsf{A}) \boxplus \mathsf{E}^i_{\gamma^+}(\mathsf{A})$. We also know that $\mathsf{E}^i_{\gamma^+}(\mathsf{A}') \leqslant^s \mathsf{E}^i_{\gamma^+}(\mathsf{A})$ (by the above argument and Theorem 3.1.1). So, $\mathsf{E}^i_{\gamma^+}(\mathsf{A}) \boxminus \mathsf{E}^i_{\gamma^+}(\mathsf{A}')$ gives full information about the difference between A and A'. We have

$$\mathsf{E}^i_{\gamma^+}(\mathsf{A}) \boxminus \mathsf{E}^i_{\gamma^+}(\mathsf{A}') = \bigvee\{\mathsf{Y} \leqslant^s \mathsf{E}^i_\gamma(\mathsf{A}')|\ \forall \mathsf{X} \leqslant^s \mathsf{Y},\ \mathsf{X} \neq \mathsf{O}\colon \operatorname{card}(\{s \in S\colon \mathsf{E}(\mathsf{A}^{(s)}|\mathsf{X}) \neq \mathsf{O}\}) = \gamma^+\}. \quad (4.2.3)$$

Indeed, $\mathsf{E}^i_{\gamma^+}(\mathsf{A}) \boxminus \mathsf{E}^i_{\gamma^+}(\mathsf{A}') \leqslant^s \mathsf{E}^i_\gamma(\mathsf{A}')$ and if $\mathsf{X} \leqslant^s \mathsf{E}^i_{\gamma^+}(\mathsf{A}) \boxminus \mathsf{E}^i_{\gamma^+}(\mathsf{A}')$ is nontrivial, then there is $\mathsf{X}' \leqslant^s \mathsf{X}$ with $\operatorname{Dim}(\mathsf{X}') = \aleph_0$ (by Proposition 3.4.10). Then $\bigoplus_{s\in S} \mathsf{E}(\mathsf{A}^{(s)}|\mathsf{X}') = \mathsf{E}(\mathsf{A}|\mathsf{X}') = \gamma^+ \odot \mathsf{X}'$ and

$$\operatorname{Dim}(\mathsf{E}(\mathsf{A}^{(s)}|\mathsf{X}')) \leqslant \operatorname{Dim}(\mathsf{E}(\mathsf{A}'|\mathsf{X}')) = \operatorname{Dim}(\gamma \odot \mathsf{X}') = \gamma,$$

which implies that the set $\{s \in S\colon \mathsf{E}(\mathsf{A}^{(s)}|\mathsf{X}') \neq \mathsf{O}\}$ has cardinality γ^+. Conversely, if Y is a member of the set appearing on the right-hand side of (4.2.3), then necessarily $\mathsf{Y} \perp_u \mathsf{E}^i_{\gamma^+}(\mathsf{A}')$ and $\mathsf{Y} \leqslant \mathsf{E}^i_\gamma(\mathsf{A}) \boxplus \mathsf{E}^i_{\gamma^+}(\mathsf{A})$. So, we only need to show that $\mathsf{Y} \perp_u \mathsf{E}^i_\gamma(\mathsf{A})$. Suppose, on the contrary, that $\mathsf{Y}' = \mathsf{Y} \wedge \mathsf{E}^i_\gamma(\mathsf{A})$ ($\leqslant^s \mathsf{Y}$) is nontrivial. Then there is $\mathsf{X} \leqslant^s \mathsf{Y}'$ such that $\operatorname{Dim}(\mathsf{X}) = \aleph_0$. Observe that $\bigoplus_{s\in S} \mathsf{E}(\mathsf{A}^{(s)}|\mathsf{X}) = \mathsf{E}(\mathsf{A}|\mathsf{X}) = \gamma \odot \mathsf{X}$, which contradicts the fact that $\operatorname{card}(\{s \in S\colon \mathsf{E}(\mathsf{A}^{(s)}|\mathsf{X}) \neq \mathsf{O}\}) = \gamma^+$.

One may deduce from (4.2.2) and (4.2.3) that (below we use the notation introduced in this chapter)

$$\mathsf{A} = \mathsf{A}' \vee [\gamma^+ \odot (\mathsf{E}_{\gamma^+}(\mathsf{A}) \boxminus \mathsf{E}_{\gamma^+}(\mathsf{A}'))]. \quad (4.2.4)$$

The above remarks show that Example 3.3.9 demonstrates all reasons for which it may happen that (4.1.1) is false. We end the chapter with the announced

Theorem 4.2.2. *Let S be a nonempty set and $\{\mathcal{A}_s\}_{s\in S}$ be a collection of nonempty subsets of $\mathcal{CDD}_N$. Then*

$$\bigvee\Big(\bigoplus_{s\in S} \mathcal{A}_s\Big) = \bigoplus_{s\in S}\Big(\bigvee \mathcal{A}_s\Big) \quad (4.2.5)$$

where $\bigoplus_{s\in S} \mathcal{A}_s = \{\bigoplus_{s\in S} \mathsf{X}^{(s)}\colon \mathsf{X}^{(s)} \in \mathcal{A}_s\ (s \in S)\}$.

Proof. The inequality '$\leqslant$' in (4.2.5) is clear. We shall prove the converse by transfinite induction on $\operatorname{card}(S)$. The cases when $\operatorname{card}(S) < \aleph_0$ or $\operatorname{card}(S) = \aleph_0$ are included in

Proposition 4.1.2 and Corollary 4.1.3, respectively. Assume β is an uncountable cardinal such that (4.2.5) is satisfied provided $\operatorname{card}(S) < \beta$. Now suppose $\operatorname{card}(S) = \beta$. If β is limit, the assertion (i.e. the inequality '$\geqslant$' in (4.2.5)) follows from (AO6) (page 32) and the transfinite induction hypothesis. Thus we may assume that $\beta = \gamma^+$. Put $\mathcal{A} = \bigoplus_{s\in S} \mathcal{A}_s$, $\mathsf{A}^{(s)} = \bigvee \mathcal{A}_s$ $(s \in S)$, $\mathsf{A} = \bigoplus_{s\in S} \mathsf{A}^{(s)}$ and $\mathsf{A}' = \bigvee\{\bigoplus_{s\in S'} \mathsf{A}^{(s)}\colon S' \subset S,\ 0 < \operatorname{card}(S') \leqslant \gamma\}$. From the transfinite induction hypothesis, $\mathsf{A}' \leqslant \bigvee \mathcal{A}$. Hence, according to (4.2.4), we only need to show that $\gamma^+ \odot (\mathsf{E}_{\gamma^+}(\mathsf{A}) \boxminus \mathsf{E}_{\gamma^+}(\mathsf{A}')) \leqslant \bigvee \mathcal{A}$. Having in mind the partition of unity induced by $\bigvee \mathcal{A}$, we see that the last inequality will be satisfied if only

$$\mathsf{E}_{\gamma^+}(\mathsf{A}) \boxminus \mathsf{E}_{\gamma^+}(\mathsf{A}') \perp_u \mathsf{E}^i_\alpha\Big(\bigvee \mathcal{A}\Big) \tag{4.2.6}$$

for every $(i, \alpha) \in \Upsilon$ with $\alpha \leqslant \gamma$. Suppose (4.2.6) is false for some $\alpha \leqslant \gamma$. Then Proposition 3.4.10 implies that there is $\mathsf{X} \in \mathcal{CDD}_N$ such that $0 < \dim(\mathsf{X}) \leqslant \aleph_0$,

$$\mathsf{X} \leqslant^s \mathsf{E}^i_\alpha\Big(\bigvee \mathcal{A}\Big) \quad \text{and} \quad \mathsf{X} \leqslant^s \mathsf{E}_{\gamma^+}(\mathsf{A}) \boxminus \mathsf{E}_{\gamma^+}(\mathsf{A}'). \tag{4.2.7}$$

The first relation of (4.2.7) yields $\mathsf{E}(\bigvee \mathcal{A}|\mathsf{X}) \leqslant \alpha \odot \mathsf{X}$ (by the characterization of $\mathsf{E}^i_\alpha(\bigvee \mathcal{A})$ given in Theorem 3.6.1). Consequently,

$$\dim\Big(\mathsf{E}\Big(\bigoplus_{s\in S} \mathsf{B}^{(s)}|\mathsf{X}\Big)\Big) \leqslant \alpha \tag{4.2.8}$$

whenever $\mathsf{B}^{(s)} \in \mathcal{A}_s$ $(s \in S)$. But $\mathsf{E}(\bigoplus_{s\in S} \mathsf{B}^{(s)}|\mathsf{X}) = \bigoplus_{s\in S} \mathsf{E}(\mathsf{B}^{(s)}|\mathsf{X})$ and thus (4.2.8) changes into $\sum_{s\in S} \dim(\mathsf{E}(\mathsf{B}^{(s)}|\mathsf{X})) \leqslant \alpha$. So, whatever $\mathsf{B}^{(s)} \in \mathcal{A}_s$ we choose,

$$\operatorname{card}(\{s \in S\colon \mathsf{B}^{(s)} \not\perp_u \mathsf{X}\}) \leqslant \gamma. \tag{4.2.9}$$

However, the second relation of (4.2.7) combined with (4.2.3) implies that the set $S' = \{s \in S\colon \mathsf{A}^{(s)} \not\perp_u \mathsf{X}\}$ has size γ^+. Observe that for $s \in S$, if $\mathsf{Y} \perp_u \mathsf{X}$ for every $\mathsf{Y} \in \mathcal{A}_s$, then necessarily $\mathsf{A}^{(s)} = \bigvee \mathcal{A}_s \perp_u \mathsf{X}$ and hence $s \notin S'$. We conclude that for every $s \in S'$ there is $\mathsf{B}^{(s)} \in \mathcal{A}_s$ such that $\mathsf{B}^{(s)} \not\perp_u \mathsf{X}$. Now (4.2.9) contradicts the fact that $\operatorname{card}(S') = \gamma^+$. Consequently, (4.2.6) is satisfied and we are done. ∎

4.3. Semigroup of semiminimal tuples

This chapter is devoted to a deeper study of $\mathcal{SM}_N$. Thanks to (AO4) (page 32), $\mathcal{SM}_N$ is a **set** and $(\mathcal{SM}_N, \oplus)$ is a semigroup which may be enlarged to an Abelian group.

A similar construction to the following may be found in [9, Proposition 1.41]. Fix a nontrivial $\mathsf{A} \in \mathcal{SM}_N$. Since $\mathcal{W}'(\boldsymbol{A})$ is type II_1, for every $n \geqslant 1$ there is a unique (by (AO1), page 32) $\mathsf{A}^{(n)} \in \mathcal{SM}_N$ such that $\mathsf{A} = n \odot \mathsf{A}^{(n)}$. We denote it by $\frac{1}{n} \odot \mathsf{A}$. Now if w is a positive rational number and $w = p/q$ with natural coprime p and q, we define $w \odot \mathsf{A}$ as $p \odot (\frac{1}{q} \odot \mathsf{A})$. Finally, for a positive real number t let

$$t \odot \mathsf{A} = \bigvee\{w \odot \mathsf{A}\colon w \in \mathbb{Q},\ w \leqslant t\}.$$

Additionally, put $t \odot \mathsf{O} = \mathsf{O}$ for each $t \in \mathbb{R}_+$. Using traces on $*$-commutants of semiminimal N-tuples (i.e. on $\mathcal{W}'(\boldsymbol{A})$ for $\mathsf{A} \in \mathcal{SM}_N$), one shows that for any $s, t \in \mathbb{R}_+$ and any $\mathsf{A}, \mathsf{B} \in \mathcal{SM}_N$,

(VS1) $0 \odot \mathsf{A} = \mathsf{O}$; $1 \odot \mathsf{A} = \mathsf{A}$,
(VS2) $s \odot \mathsf{A} \leqslant t \odot \mathsf{A}$ provided $s \leqslant t$,
(VS3) $t \odot \mathsf{A} = \bigwedge\{x \odot \mathsf{A}\colon x > t\}$ and for $t > 0$, $t \odot \mathsf{A} = \bigvee\{x \odot \mathsf{A}\colon 0 \leqslant x < t\}$,
(VS4) $(st) \odot \mathsf{A} = s \odot (t \odot \mathsf{A})$; $(s+t) \odot \mathsf{A} = (s \odot \mathsf{A}) \oplus (t \odot \mathsf{A})$,
(VS5) $t \odot (\mathsf{A} \oplus \mathsf{B}) = (t \odot \mathsf{A}) \oplus (t \odot \mathsf{B})$,
(VS6) if '$\sim$' denotes one of '$\leqslant$', '$\leqslant^s$', '$\ll$', '$\perp_u$' and $t > 0$, then $t \odot \mathsf{A} \sim t \odot \mathsf{B} \Leftrightarrow \mathsf{A} \sim \mathsf{B}$,
(VS7) if $\mathsf{A} = \bigoplus_{s\in S} \mathsf{A}^{(s)}$, then $t \odot \mathsf{A} = \bigoplus_{s\in S}(t \odot \mathsf{A}^{(s)})$ (this follows from (VS5), (VS6) and Proposition 4.1.6),
(VS8) $\mathfrak{b}(\mathsf{A}) \in \mathcal{SM}_N$ and $\mathfrak{b}(t \odot \mathsf{A}) = t \odot \mathfrak{b}(\mathsf{A})$,
(VS9) for every sequence $(t_n)_{n=1}^{\infty}$ of nonnegative reals, $(\sum_{n=1}^{\infty} t_n) \odot \mathsf{A} = \bigoplus_{n=1}^{\infty} t_n \odot \mathsf{A}$ (where ∞ is identified with $\aleph_0$, if applicable).

Now by (VS1), (VS4), (VS5) and (AO4) (page 32), there is a real vector space

$$(\mathcal{E}_N, +, \cdot) \supset (\mathcal{SM}_N, \oplus, \odot).$$

The above inclusion means that addition and multiplication by reals in $\mathcal{E}_N$ extend, respectively, '$\oplus$' and '$\odot$' defined above. $\mathcal{SM}_N$ as a subset of $\mathcal{E}_N$ is a cone (that is, $\mathcal{SM}_N + \mathcal{SM}_N \subset \mathcal{SM}_N$, $\mathbb{R}_+ \cdot \mathcal{SM}_N \subset \mathcal{SM}_N$ and $\mathcal{SM}_N \cap (-\mathcal{SM}_N) = \{0\} = \{\mathsf{O}\}$). We may assume that $\mathcal{E}_N = \mathcal{SM}_N - \mathcal{SM}_N$. Under this assumption, we may consider the partial order on $\mathcal{E}_N$ induced by $\mathcal{SM}_N$: $\xi_1 \leqslant_{\mathcal{E}} \xi_2 \Leftrightarrow \xi_2 - \xi_1 \in \mathcal{SM}_N$ ($\xi_1, \xi_2 \in \mathcal{E}_N$). It may be checked that for $\mathsf{A}, \mathsf{B} \in \mathcal{SM}_N$, $\mathsf{A} \leqslant_{\mathcal{E}} \mathsf{B} \Leftrightarrow \mathsf{A} \leqslant \mathsf{B}$. So, '$\leqslant_{\mathcal{E}}$' extends '$\leqslant$' and therefore we shall omit the subscript '$\mathcal{E}$' in '$\leqslant_{\mathcal{E}}$'. Since every nonempty subset of $\mathcal{SM}_N$ which is upper bounded in $\mathcal{SM}_N$ has the l.u.b. (in $\mathcal{SM}_N$), $\mathcal{E}_N$ is a conditionally complete lattice (which means that every nonempty upper bounded subset of $\mathcal{E}_N$ has the l.u.b. in $\mathcal{E}_N$). Our aim is to find a 'model' for the lattice $\mathcal{E}_N$.

Until the end of the chapter we fix a representative $\boldsymbol{J}_{II}$ of J_{II}, a compact Hausdorff space Ω_{II} homeomorphic to the Gelfand spectrum of $\mathcal{Z}(\mathcal{W}'(\boldsymbol{J}_{II}))$ and an isomorphism $\Psi\colon \mathcal{Z}(\mathcal{W}'(\boldsymbol{J}_{II})) \to \mathcal{C}(\Omega_{II})$ of $*$-algebras where $\mathcal{C}(\Omega_{II})$ is the algebra of all continuous complex-valued functions on Ω_{II}. Every $\mathsf{A} \leqslant^s \mathsf{J}_{II}$ corresponds to a unique central projection z_{A} in $\mathcal{W}'(\boldsymbol{J}_{II})$. Let U_{A} be a clopen (i.e. simultaneously closed and open) subset of Ω_{II} whose characteristic function coincides with $\Psi(z_{\mathsf{A}})$. Ω_{II} is extremely disconnected (that is, the closure of every open subset of Ω_{II} is open as well; see [18, Theorem 5.2.1]) and the assignment $\mathsf{A} \mapsto U_{\mathsf{A}}$ establishes a one-to-one correspondence between all N-tuples $\mathsf{X} \in \mathcal{SM}_N^{\infty}$ (where $\mathcal{SM}_N^{\infty} = \{\mathsf{X} \in \mathcal{CDD}_N\colon \mathsf{X} \leqslant^s \mathsf{J}_{II}\} = \{\aleph_0 \odot \mathsf{A}\colon \mathsf{A} \in \mathcal{SM}_N\}$) and all clopen subsets of Ω_{II}. Moreover, for $\mathsf{A}, \mathsf{B} \in \mathcal{SM}_N^{\infty}$, $\mathsf{A} \leqslant^s \mathsf{B} \Leftrightarrow U_{\mathsf{A}} \subset U_{\mathsf{B}}$.

For every $\mathsf{A} \in \mathcal{SM}_N$, $\widetilde{\mathsf{A}} = \aleph_0 \odot \mathsf{A} \leqslant^s \mathsf{J}_{II}$ and thus $U_{\widetilde{\mathsf{A}}}$ makes sense. This set is said to be the *support* of A and denoted by $\operatorname{supp}_{\Omega_{II}} \mathsf{A}$. There is no difficulty in verifying that $\operatorname{supp}_{\Omega_{II}} \mathsf{A} \subset \operatorname{supp}_{\Omega_{II}} \mathsf{B}$ (respectively $\operatorname{supp}_{\Omega_{II}} \mathsf{A} \cap \operatorname{supp}_{\Omega_{II}} \mathsf{B} = \emptyset$) provided $\mathsf{A} \ll \mathsf{B}$ (respectively $\mathsf{A} \perp_u \mathsf{B}$) and $\mathsf{A}, \mathsf{B} \in \mathcal{SM}_N$.

The following idea comes from the theory of $\mathcal{W}^*$-algebras ([18, Definition 5.6.5]) especially when working with the so-called extended center valued traces (see the notes on page 329 of [35] and Definition V.2.33 there). We consider the set

$$\mathcal{M}(\Omega_{II}) = \{f \in \mathcal{C}(\Omega_{II}, [-\infty, +\infty])\colon f^{-1}(\mathbb{R}) \text{ is dense in } \Omega_{II}\}.$$

To make the space $\mathcal{M}(\Omega_{II})$ a real vector space, we need the following well-known result (it follows from [18, Corollary 5.2.11] or [35, Corollary III.1.8]; see also [12] for more general results in this direction).

Lemma 4.3.1. *If $\mathcal{X}$ and K are compact Hausdorff spaces and $\mathcal{X}$ is extremely disconnected, then every continuous function of an arbitrary open dense subset of $\mathcal{X}$ into K is extendable to a continuous function of $\mathcal{X}$ into K.*

Now if $f, g \in \mathcal{M}(\Omega_{II})$, the set $D = f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R})$ is open and dense in Ω_{II} and the function $f|_D + g|_D$ is well defined and continuous. Consequently, thanks to Lemma 4.3.1, there is a unique member of $\mathcal{M}(\Omega_{II})$, which we shall denote by $f + g$, which coincides with the usual sum on D. Similarly one defines $f \cdot g$ and $t \cdot f$ for $t \in \mathbb{R}$. We leave it as an exercise that $\mathcal{M}(\Omega_{II})$ is a real vector space with these operations. Further, we equip $\mathcal{M}(\Omega_{II})$ with the pointwise order. One may easily check that $\mathcal{M}(\Omega_{II})$ is a lattice (i.e. every finite nonempty subset of $\mathcal{M}(\Omega_{II})$ has the l.u.b. and the g.l.b.). What is more, $\mathcal{M}(\Omega_{II})$ is conditionally complete, since Ω_{II} is extremely disconnected (this follows from [35, Proposition III.1.7]). We shall show that $\mathcal{E}_N$ and $\mathcal{M}(\Omega_{II})$ are lattice-isomorphic. For every $f \in \mathcal{M}(\Omega_{II})$ let $\operatorname{supp} f$ be the closure of the set $\{x \in \Omega_{II} \colon f(x) \neq 0\}$. Since Ω_{II} is extremely disconnected, $\operatorname{supp} f$ is clopen.

When $\mathcal{X}$ is a clopen subset of Ω_{II}, let $\mathcal{M}(\Omega_{II}|\mathcal{X})$ be the set of all $f \in \mathcal{M}(\Omega_{II})$ for which $\operatorname{supp} f \subset \mathcal{X}$. Then $\mathcal{M}(\Omega_{II}|\mathcal{X})$ is a sublattice of $\mathcal{M}(\Omega_{II})$. By $\mathcal{M}_+(\Omega_{II})$ and $\mathcal{M}_+(\Omega_{II}|\mathcal{X})$ we denote the cones of nonnegative elements of the respective lattices.

For the next step of our considerations we need

Lemma 4.3.2. *Let Ω be the Gelfand spectrum of a commutative $\mathcal{W}^*$-algebra. Every dense $\mathcal{G}_\delta$ subset of Ω has dense interior. What is more, for each Borel function $f \colon \Omega \to \mathbb{R}$ there is an open dense set $D \subset \Omega$ such that $f|_D$ is continuous.*

Lemma 4.3.2 follows from [18, Lemma 5.2.10] combined with Proposition III.1.15 and Theorem III.1.17 of [35] (see also the note preceding Corollary III.1.16 there).

Let $\{f_n\}_{n=1}^{\infty} \subset \mathcal{M}_+(\Omega_{II})$ be such that $\sum_{k=1}^{n} f_k \leqslant g$ for some $g \in \mathcal{M}_+(\Omega_{II})$ and each n. We define $\sum_{n=1}^{\infty} f_n \in \mathcal{M}_+(\Omega_{II})$ as follows. Let $f \colon \Omega \to \mathbb{R}$ be given by $f(x) = \sum_{n=1}^{\infty} f_n(x)$ provided the series is convergent, and $f(x) = 0$ otherwise. By Lemma 4.3.2, there is an open dense subset D of Ω_{II} such that $f|_D$ is continuous. We define $\sum_{n=1}^{\infty} f_n \in \mathcal{M}_+(\Omega_{II})$ as the unique continuous extension of $f|_D$. Since $f \leqslant g$, $(\sum_{n=1}^{\infty} f_n)(x) = \sum_{n=1}^{\infty} f_n(x)$ for x belonging to an open dense subset of Ω_{II}. One may check that

$$\sum_{n=1}^{\infty} f_n = \sup_{\mathcal{M}(\Omega_{II})} \Big\{ \sum_{k=1}^{n} f_k \colon n \geqslant 1 \Big\}.$$

Fix a nontrivial $\mathsf{X} \in \mathcal{SM}_N$. Let $\mathcal{L}[\mathsf{X}] = \{\mathsf{F} \in \mathcal{SM}_N \colon \mathsf{F} \ll \mathsf{X}\}$ and $\mathcal{X} = \operatorname{supp}_{\Omega_{II}} \mathsf{X}$.

Theorem 4.3.3. *There is a unique operator*

$$\mathcal{L}[\mathsf{X}] \ni \mathsf{F} \mapsto \frac{d\mathsf{F}}{d\mathsf{X}} \in \mathcal{M}_+(\Omega_{II}|\mathcal{X})$$

such that for any $\mathsf{F}, \mathsf{F}^{(n)} \in \mathcal{L}[\mathsf{X}]$ $(n = 1, 2, \ldots)$:

(TR0) $\frac{d\mathsf{X}}{d\mathsf{X}}$ *is the characteristic function of $\mathcal{X}$,*
(TR1) $\operatorname{supp} \frac{d\mathsf{F}}{d\mathsf{X}} \subset \operatorname{supp}_{\Omega_{II}} \mathsf{F}$,

(TR2) $\frac{d(\bigoplus_{n=1}^{\infty} \mathsf{F}^{(n)})}{d\mathsf{X}} = \sum_{n=1}^{\infty} \frac{d\mathsf{F}^{(n)}}{d\mathsf{X}}$ *if* $\bigoplus_{n=1}^{\infty} \mathsf{F}^{(n)} \in \mathcal{SM}_N$ *(see the remarks preceding the theorem).*

Moreover, the above operator has further properties:

(TR1′) $\operatorname{supp} \frac{d\mathsf{F}}{d\mathsf{X}} = \operatorname{supp}_{\Omega_{II}} \mathsf{F}$ *for every* $\mathsf{F} \in \mathcal{L}[\mathsf{X}]$,

(TR2′) *whenever* $\mathsf{A} \in \mathcal{L}[\mathsf{X}]$ *is of the form* $\mathsf{A} = \bigoplus_{s \in S} \mathsf{A}^{(s)}$,

$$\frac{d\mathsf{A}}{d\mathsf{X}} = \sum_{s \in S} \frac{d\mathsf{A}^{(s)}}{d\mathsf{X}} := \sup_{\mathcal{M}(\Omega_{II}|\mathcal{X})} \Big\{ \sum_{s \in S_0} \frac{d\mathsf{A}^{(s)}}{d\mathsf{X}} \colon S_0 \in \mathcal{P}_f(S) \Big\},$$

(TR4′) $\frac{d(t \odot \mathsf{F})}{d\mathsf{X}} = t \frac{d\mathsf{F}}{d\mathsf{X}}$ *for each* $\mathsf{F} \in \mathcal{L}[\mathsf{X}]$,

(TR5′) *for any* $\mathsf{A}, \mathsf{B} \in \mathcal{L}[\mathsf{X}]$, $\mathsf{A} \leqslant \mathsf{B} \Leftrightarrow \frac{d\mathsf{A}}{d\mathsf{X}} \leqslant \frac{d\mathsf{B}}{d\mathsf{X}}$,

(TR6′) *for every* $f \in \mathcal{M}_+(\Omega_{II}|\mathcal{X})$ *there is a unique* $\mathsf{F} \in \mathcal{L}[\mathsf{X}]$ *with* $\frac{d\mathsf{F}}{d\mathsf{X}} = f$.

Proof. The existence of the operator may be deduced from the result on faithful normal extended center valued traces for semifinite $\mathcal{W}^*$-algebras ([35, Theorem V.2.34]) applied to $\mathcal{W}'(\boldsymbol{Y})$ with $\mathsf{Y} = \aleph_0 \odot \mathsf{X}$. The operator may also be constructed as follows. By Theorem 3.4.6, there is $\mathsf{X}' \in \mathcal{SM}_N$ such that $\mathsf{J}_{II} = \aleph_0 \odot (\mathsf{X} \boxplus \mathsf{X}')$. Put $\widetilde{\mathsf{X}} = \mathsf{X} \boxplus \mathsf{X}'$ and let $\mathcal{M} = \mathcal{W}''(\widetilde{\boldsymbol{X}})$. Then $\mathcal{M}' = \mathcal{W}'(\widetilde{\boldsymbol{X}})$. Since $\widetilde{\mathsf{X}} \in \mathcal{SM}_N$, $\mathcal{M}'$ is type II$_1$ and hence there is a trace $\operatorname{tr}\colon \mathcal{M}' \to \mathcal{Z}(\mathcal{M}')$ (with $\operatorname{tr}(1) = 1$). Since $\mathcal{Z}(\mathcal{M}') = \mathcal{Z}(\mathcal{M})$, $\mathcal{Z}(\mathcal{W}''(\boldsymbol{J}_{II})) = \mathcal{Z}(\mathcal{W}'(\boldsymbol{J}_{II}))$ and the function $\mathcal{M} \ni T \mapsto \aleph_0 \odot T \in \mathcal{W}''(\boldsymbol{J}_{II})$ is an isomorphism of $*$-algebras, hence the function $\kappa\colon \mathcal{Z}(\mathcal{M}') \ni T \mapsto \aleph_0 \odot T \in \mathcal{Z}(\mathcal{W}'(\boldsymbol{J}_{II}))$ is a well defined $*$-isomorphism. Define $\operatorname{Tr}\colon \mathcal{M}' \to \mathcal{C}(\Omega_{II})$ by $\operatorname{Tr} = \Psi \circ \kappa \circ \operatorname{tr}$. Now if $\mathsf{A} \ll \mathsf{X}$, by Definition 3.4.5, A may be written in the form $\mathsf{A} = \boxplus_{n=1}^{\infty} \mathsf{A}^{(n)}$ with $\mathsf{A}^{(n)} \leqslant n \odot \mathsf{X}$. This implies that $\frac{1}{n} \odot \mathsf{A}^{(n)} \leqslant \widetilde{\mathsf{X}}$ and thus there is a projection p_n in $\mathcal{M}'(\widetilde{\boldsymbol{X}})$ which corresponds (by Proposition 2.3.1) to $\frac{1}{n} \odot \mathsf{A}^{(n)}$. We put

$$\frac{d\mathsf{A}}{d\mathsf{X}} = \sum_{n=1}^{\infty} n \operatorname{Tr}(p_n). \tag{4.3.1}$$

Since $\mathsf{A}^{(n)} \perp_u \mathsf{A}^{(m)}$ for $n \neq m$, $\operatorname{supp} \operatorname{Tr}(p_n) \cap \operatorname{supp} \operatorname{Tr}(p_m) = \emptyset$ and thus (4.3.1) is well understood, by Lemma 4.3.1. We leave it as an exercise that the definition is independent of the choice of $(\mathsf{A}^{(n)})_{n=1}^{\infty}$ and that all conditions of the theorem are fulfilled (observe that X corresponds to a central projection in $\mathcal{M}'(\widetilde{\boldsymbol{X}})$). Here we focus on the uniqueness of the operator.

If $\mathsf{A} \leqslant^s \mathsf{X}$, then $\mathsf{X} = \mathsf{A} \boxplus \mathsf{B}$ with $\mathsf{B} = \mathsf{X} \boxminus \mathsf{A}$ and $\operatorname{supp}_{\Omega_{II}} \mathsf{B} = \operatorname{supp}_{\Omega_{II}} \mathsf{X} \setminus \operatorname{supp}_{\Omega_{II}} \mathsf{A}$. Consequently, by (TR0)–(TR2), $\frac{d\mathsf{A}}{d\mathsf{X}}$ is the characteristic function of $\operatorname{supp}_{\Omega_{II}} \mathsf{A}$. This shows that $\frac{d\mathsf{F}}{d\mathsf{X}}$ is uniquely determined by (TR0)–(TR2) for $\mathsf{F} \in \mathcal{F}_0 := \{w \odot \mathsf{A}\colon w \in \mathbb{Q}_+,\ \mathsf{A} \leqslant^s \mathsf{X}\}$.

Further, if $\mathsf{A} \leqslant \mathsf{X}$, then A may be written in the form $\mathsf{A} = \bigoplus_{n=1}^{\infty} \mathsf{F}^{(n)}$ with $\mathsf{F}^{(n)} \in \mathcal{F}_0$ (this may be deduced, by means of the trace, from the representation of a continuous function on an extremely disconnected compact Hausdorff space as a series of continuous functions with finite ranges). So, according to (TR2), $\frac{d\mathsf{B}}{d\mathsf{X}}$ is uniquely determined by (TR0)–(TR2) for $\mathsf{B} = w \odot \mathsf{A}$ with rational w and $\mathsf{A} \leqslant \mathsf{X}$. Finally, it suffices to recall that if $\mathsf{A} \in \mathcal{L}[\mathsf{X}]$, then $\mathsf{A} = \boxplus_{n=1}^{\infty} \mathsf{A}^{(n)}$ with $\frac{1}{n} \odot \mathsf{A}^{(n)} \leqslant \mathsf{X}$. ∎

Corollary 4.3.4. *$\mathcal{E}_N$ and $\mathcal{M}(\Omega_{II})$ are isomorphic as ordered vector spaces.*

Proof. Take $\mathsf{X} \in \mathcal{SM}_N$ such that $\aleph_0 \odot \mathsf{X} = \mathsf{J}_{II}$ and define $\Phi_+ \colon \mathcal{SM}_N \ni \mathsf{F} \mapsto \frac{d\mathsf{F}}{d\mathsf{X}} \in \mathcal{M}_+(\Omega_{II})$. By Theorem 4.3.3, Φ_+ is an additive bijection preserving orders. Now it suffices to extend Φ_+ in a standard way: $\Phi(\xi) = \Phi(\xi_+) - \Phi(\xi_-)$. ∎

PROPOSITION 4.3.5. *If* $\mathsf{A}, \mathsf{X}, \mathsf{Y} \in \mathcal{SM}_N$ *are such that* $\mathsf{A} \ll \mathsf{X} \ll \mathsf{Y}$, *then*

$$\frac{d\mathsf{A}}{d\mathsf{Y}} = \frac{d\mathsf{A}}{d\mathsf{X}} \cdot \frac{d\mathsf{X}}{d\mathsf{Y}}. \tag{4.3.2}$$

Proof. Arguing as in the uniqueness part of Theorem 4.3.3, we only need to check that (4.3.2) is satisfied for $\mathsf{A} \leqslant^s \mathsf{X}$. When $\mathsf{A} = \mathsf{X}$, (4.3.2) is clear. So, for arbitrary $\mathsf{A} \leqslant^s \mathsf{X}$, (4.3.2) follows from (TR1) and (TR2). ∎

We end the chapter with the following two remarks.

REMARK 4.3.6. The notation '$\frac{d\mathsf{A}}{d\mathsf{X}}$' suggests denoting the inverse operator, from $\mathcal{M}_+(\Omega_{II}|\mathcal{X})$ onto $\mathcal{L}[\mathsf{X}]$, by $\int f\, d\mathsf{X}$. Thus, for $f \in \mathcal{M}_+(\Omega_{II}|\mathcal{X})$, $\int f\, d\mathsf{X} = \mathsf{B}$ iff $\mathsf{B} \in \mathcal{L}[\mathsf{X}]$ is such that $\frac{d\mathsf{B}}{d\mathsf{X}} = f$. Arguing as in the proof of Theorem 4.3.3, one may show that the operator $\mathcal{M}_+(\Omega_{II}|\mathcal{X}) \ni f \mapsto \int f\, d\mathsf{X} \in \mathcal{L}[\mathsf{X}]$ is uniquely determined by the following three conditions:

(AD1) $\int j_{\mathcal{X}}\, d\mathsf{X} = \mathsf{X}$ where $j_{\mathcal{X}}$ is the characteristic function of $\mathcal{X}$,
(AD2) $\operatorname{supp}_{\Omega_{II}}(\int f\, d\mathsf{X}) \subset \operatorname{supp} f$ for each $f \in \mathcal{M}_+(\Omega_{II}|\mathcal{X})$,
(AD3) if $f \in \mathcal{M}_+(\Omega_{II}|\mathcal{X})$ has the form $f = \sum_{n=1}^{\infty} f_n$ (with $f_n \in \mathcal{M}_+(\Omega_{II}|\mathcal{X})$), then

$$\int f\, d\mathsf{X} = \bigoplus_{n=1}^{\infty} \int f_n\, d\mathsf{X}.$$

Note that (AD3) resembles Lebesgue's classical monotone convergence theorem.

REMARK 4.3.7. Specialists in Hilbert space operators would probably prefer the version of '$\frac{d\mathsf{Y}}{d\mathsf{X}}$' whose values are operators rather than functions. This is possible and may be provided as follows. Since every bounded member of $\mathcal{M}_+(\Omega_{II})$ corresponds, by Ψ, to a nonnegative element of $\mathcal{Z}(\mathcal{W}'(\boldsymbol{J}_{II}))$, each member of $\mathcal{M}_+(\Omega_{II})$ corresponds to a (possibly unbounded) nonnegative selfadjoint operator A such that $\mathfrak{b}(A) \in \mathcal{Z}(\mathcal{W}'(\boldsymbol{J}_{II}))$ (in the theory of von Neumann algebras such an operator A is said to be *affiliated* with $\mathcal{Z}(\mathcal{W}'(\boldsymbol{J}_{II}))$; see e.g. [18, Definition 5.6.2]). Thus, if we let $\boldsymbol{L}[\boldsymbol{X}]$ and $\widehat{\mathcal{Z}}_+(\mathcal{W}'(\boldsymbol{J}_{II}))$ denote, respectively, the classes of all $\boldsymbol{Y} \in \mathrm{CDD}_N$ whose unitary equivalence class is semiminimal and which are covered by $\boldsymbol{X}$ (i.e. $\boldsymbol{Y} \ll \boldsymbol{X}$), and of all the above-mentioned operators A, then Theorem 4.3.3 may be adapted to these settings in such a way that $\frac{d\boldsymbol{Y}}{d\boldsymbol{X}} \in \widehat{\mathcal{Z}}_+(\mathcal{W}'(\boldsymbol{X}))$ for any $\boldsymbol{Y} \in \boldsymbol{L}[\boldsymbol{X}]$ and (here we list only those properties which do not need additional explanations): (a) $\frac{d\boldsymbol{X}}{d\boldsymbol{X}}$ is the unit of $\mathcal{Z}(\mathcal{W}'(\boldsymbol{X}))$ (so, $\frac{d\boldsymbol{X}}{d\boldsymbol{X}}$ is a central projection in $\mathcal{W}'(\boldsymbol{J}_{II})$); (b) $\frac{d\boldsymbol{Y}'}{d\boldsymbol{X}} = \frac{d\boldsymbol{Y}''}{d\boldsymbol{X}}$ iff $\boldsymbol{Y}'$ and $\boldsymbol{Y}''$ are unitarily equivalent; (c) if $\boldsymbol{A} \leqslant m \odot \boldsymbol{X}$ and $\boldsymbol{B} \leqslant n \odot \boldsymbol{X}$ for some natural numbers m and n, then both $\frac{d\boldsymbol{A}}{d\boldsymbol{X}}$ and $\frac{d\boldsymbol{B}}{d\boldsymbol{X}}$ are bounded and $\frac{d(\boldsymbol{A} \oplus \boldsymbol{B})}{d\boldsymbol{X}} = \frac{d\boldsymbol{A}}{d\boldsymbol{X}} + \frac{d\boldsymbol{B}}{d\boldsymbol{X}}$; (d) if $\boldsymbol{Y}_t$ is such that $\mathsf{Y}_t = t \odot \mathsf{Y}$ (for some $\boldsymbol{Y} \in \boldsymbol{L}[\boldsymbol{X}]$ and $t > 0$), then $\frac{d\boldsymbol{Y}_t}{d\boldsymbol{X}} = t \frac{d\boldsymbol{Y}}{d\boldsymbol{X}}$. The reader interested in this approach should consult [18, Theorem 5.6.15].

4.4. Model for the class

Now we shall develop the idea of the previous chapter. This will also be an adaptation of the dimension theory for $\mathcal{W}^*$-algebras. Let $\boldsymbol{J}$ be a representative of J, Ω be a compact Hausdorff space homeomorphic to the Gelfand spectrum of $\mathcal{Z}(\mathcal{W}'(\boldsymbol{J}))$ and let

$$\Psi\colon \mathcal{Z}(\mathcal{W}'(\boldsymbol{J})) \to \mathcal{C}(\Omega)$$

be an isomorphism of $*$-algebras. When the triple $(\boldsymbol{J}, \Omega, \Psi)$ is fixed, J_i for $i = I, II, III$ corresponds to a clopen subset Ω_i of Ω. In what follows, we assume that $\mathrm{Card} \cap \mathbb{R}_+ = \mathbb{Z} \cap \mathbb{R}_+$. We add and multiply two reals and two infinite cardinals in the usual way, and additionally we put $0 \cdot \alpha = \alpha \cdot 0 = 0$ and $t + \alpha = \alpha + t = \alpha + 0 = 0 + \alpha = \alpha = t \cdot \alpha = \alpha \cdot t$ for $t \in \mathbb{R}_+ \setminus \{0\}$ and $\alpha \in \mathrm{Card}_\infty$. We also extend the natural total orders on $\mathbb{R}_+$ and Card_∞ assuming that $t < \alpha$ for every real t and each infinite cardinal α. In this way the order on $\mathbb{R}_+ \cup \mathrm{Card}$ is total and complete. We equip every set $Y \subset \mathbb{R}_+ \cup \mathrm{Card}$ with the topology inherited from the linearly ordered space $I_\alpha := \{\xi \in \mathbb{R}_+ \cup \mathrm{Card}\colon \xi \leqslant \alpha\}$ where $\alpha = \sup(Y \cup \{\aleph_0\})$ (cf. [8, Problem 1.7.4]). Since the topology of the linearly ordered space I_α coincides with the topology inherited from I_β whenever $\aleph_0 \leqslant \alpha < \beta$, this definition of the topology on Y causes no confusion. For every topological space X, we call a function $f\colon X \to \mathbb{R}_+ \cup \mathrm{Card}$ continuous if f is continuous as a function of X into $f(X)$. One may check that for every $\alpha \in \mathrm{Card}_\infty$, I_α is compact, the order is a closed subset of $I_\alpha \times I_\alpha$ and the functions $I_\alpha \times I_\alpha \ni (\xi, \xi') \mapsto \xi + \xi' \in I_\alpha$ and $I_\alpha \times I_\alpha \ni (\xi, \xi') \mapsto \xi \cdot \xi' \in I_\alpha$ are continuous.

Let $\Lambda(\Omega)$ be the class of all continuous functions $u\colon \Omega \to \mathbb{R}_+ \cup \mathrm{Card}$ such that $u(\Omega_I) \subset \mathrm{Card}$ and $u(\Omega_{III}) \subset \{0\} \cup \mathrm{Card}_\infty$. We add and multiply members of $\Lambda(\Omega)$ pointwise. We shall also multiply elements of $\Lambda(\Omega)$ by cardinal numbers pointwise and we equip $\Lambda(\Omega)$ with the pointwise order. For each $f \in \Lambda(\Omega)$, $\operatorname{supp} f$ is the closure of the (open) set $\{x \in \Omega\colon f(x) \neq 0\}$. Observe that $\operatorname{supp} f$ is clopen.

Suppose $\{f_s\}_{s\in S} \subset \Lambda(\Omega)$ is any family such that $\operatorname{supp} f_s \cap \operatorname{supp} f_{s'} = \emptyset$ for distinct $s, s' \in S$. We define $\sum_{s\in S} f_s \in \Lambda(\Omega)$ in the following manner. Let $D_0 = \bigcup_{s\in S} \operatorname{supp} f_s$, $D = D_0 \cup \operatorname{int}(\Omega \setminus D_0)$ ('int' stands for interior) and $u\colon D \to \mathbb{R}_+ \cup \mathrm{Card}$ be given by $u(x) = f_s(x)$ for $x \in \operatorname{supp} f_s$ ($s \in S$) and $u(x) = 0$ for $x \in \operatorname{int}(\Omega \setminus D_0)$. It is clear that D is open and dense in Ω and u is continuous. Now by Lemma 4.3.1, u may be (uniquely) continuously extended to a member of $\Lambda(\Omega)$, denoted by $\sum_{s\in S} f_s$. One may check that in that case $\sum_{s\in S} f_s = \sup_{\Lambda(\Omega)}\{\sum_{s\in S_0} f_s\colon S_0 \in \mathcal{P}_f(S)\}$.

Lemma 4.4.1. *Let $\{f_n\}_{n=1}^\infty \subset \Lambda(\Omega)$ and $u, v\colon \Omega \to \mathbb{R}_+ \cup \mathrm{Card}$ be given by $u(x) = \inf_{n\geqslant 1} f_n(x)$ and $v(x) = \sup_{n\geqslant 1} f_n(x)$ ($x \in \Omega$). There are open dense subsets U and V of Ω such that $u|_U$ and $v|_V$ are continuous.*

Proof. Since the proofs for u and v differ, we shall present both. We start with u for which the proof is simpler. Let $U_0 = u^{-1}(\mathbb{R}_+)$ and for $\alpha \in \mathrm{Card}_\infty$ let $U_\alpha = \operatorname{int} u^{-1}(\{\alpha\})$. Since $U_0 = \bigcup_{n=1}^\infty f_n^{-1}(\mathbb{R}_+)$, U_0 is open. Now the function $u'\colon \Omega \to \mathbb{R}_+$ given by $u'(x) = u(x)$ for $x \in U_0$ and $u'(x) = 0$ otherwise is Borel (because on U_0 it coincides with the infimum of a sequence of continuous functions taking values in $[0, \infty]$, after a suitable change of f_n's). Thus, according to Lemma 4.3.2, there is a dense open subset U' of Ω such that $u'|_{U'}$ is continuous. Consequently, $u|_{U_1}$ is continuous where $U_1 = U_0 \cap U'$ is open and dense in U_0. We see that $U = U_1 \cup \bigcup_{\alpha \in \mathrm{Card}_\infty} U_\alpha$ is open and $u|_U$ is continuous. To show that U is dense in Ω, it remains to check that the set $G = \operatorname{int}[\Omega \setminus (U_0 \cup \bigcup_{\alpha \in \mathrm{Card}_\infty} U_\alpha)]$ is empty. Suppose, for a contradiction, that $G \neq \emptyset$. Note that G is clopen and $u(G) \subset \mathrm{Card}_\infty$. Let $\alpha = \min u(G) \geqslant \aleph_0$. We conclude from the definition of u that $f_n(x) \geqslant \alpha$ for all $x \in G$ and $n \geqslant 1$. What is more, there is $x_0 \in G$ such that $u(x_0) = \alpha$ and there exists $m \geqslant 1$ with $u(x_0) = f_m(x_0)$. Since α is an isolated point of $f_m(G)$, the set $G_0 = f_m^{-1}(\{\alpha\}) \cap G$ is clopen (and nonempty). We see that then $u(x) = \alpha$ for each $x \in G_0$ and hence $G_0 \subset U_\alpha$, which contradicts the definition of G. This finishes the proof for u.

To show the assertion for v, we begin similarly: let $F = v^{-1}(I_{\aleph_0})$ and

$$V_\infty = \bigcup_{\alpha \in \mathrm{Card}_\infty} \operatorname{int} v^{-1}(\{\alpha\}).$$

The set F is closed since $F = \bigcap_{n=1}^\infty f_n^{-1}(I_{\aleph_0})$. We claim that

$$F \cup \operatorname{cl} V_\infty = \Omega \tag{4.4.1}$$

('cl' stands for closure). Again, for contradiction suppose that $D = \Omega \setminus (F \cup \operatorname{cl} V_\infty)$ is nonempty. Since D is open, there is a clopen set $G \neq \emptyset$ such that $G \subset D$. Notice that $v(G) \subset \mathrm{Card}_\infty \setminus \{\aleph_0\}$. Let γ be the first infinite cardinal such that

$$\operatorname{int}[G \cap v^{-1}(I_\gamma)] \neq \emptyset. \tag{4.4.2}$$

Let W be any nonempty clopen subset of $G \cap v^{-1}(I_\gamma)$. Let us show that

$$\gamma = \sup\{\sup f_n(W) \colon n \geqslant 1\} = \sup v(W) > \aleph_0. \tag{4.4.3}$$

Put $\gamma' = \sup\{\sup f_n(W) \colon n \geqslant 1\}$. It is clear that $\gamma' \leqslant \sup v(W) \leqslant \gamma$ (as $v(W) \subset I_\gamma$). On the other hand, by the definition of v, $v(x) \leqslant \gamma'$ for each $x \in W$, which yields $\gamma' > \aleph_0$ (since $W \subset G$) and $W \subset v^{-1}(I_{\gamma'}) \cap G$. We now infer from the definition of γ that $\gamma \leqslant \gamma'$. This proves (4.4.3).

Now let W_0 be an arbitrary nonempty clopen subset of $G \cap v^{-1}(I_\gamma)$ (cf. (4.4.2)). Put $Z = W_0 \cap \bigcup_{n=1}^\infty f_n^{-1}(\{\gamma\})$. Then Z is $\mathcal{F}_\sigma$ and, by Baire's theorem, $\operatorname{int} Z = \emptyset$ (because, thanks to (4.4.3), $\operatorname{int}[W_0 \cap f_n^{-1}(\{\gamma\})] \subset \operatorname{int} v^{-1}(\{\gamma\}) \subset V_\infty$ and $W_0 \cap V_\infty = \emptyset$). An application of Lemma 4.3.2 shows that $\operatorname{int}(\operatorname{cl} Z) = \emptyset$. This implies that there is a nonempty clopen set $W \subset W_0 \setminus Z$. We conclude from the definition of Z that $f_n(x) < \gamma$ for any $x \in W$ and $n \geqslant 1$. But since W is compact, f_n assumes its maximum on W and consequently $\gamma_n := \max(\aleph_0, \sup f_n(W)) < \gamma$. Now by (4.4.3),

$$\sup_{n \geqslant 1} \gamma_n = \gamma. \tag{4.4.4}$$

Further, by the minimality of γ, each of the sets $G_n = G \cap v^{-1}(I_{\gamma_n})$ has empty interior. Moreover, the G_n's are closed ($G_n = G \cap \bigcap_{k=1}^\infty f_k^{-1}(I_{\gamma_n})$). Consequently, another application of Baire's theorem and Lemma 4.3.2 gives $\operatorname{int}[\operatorname{cl}(G_\infty)] = \emptyset$ where $G_\infty = \bigcup_{n=1}^\infty G_n$. But $G_\infty = G \cap v^{-1}(I_\gamma \setminus \{\gamma\})$ (by (4.4.4)). Finally, by (4.4.2), we obtain

$$\operatorname{int}[G \cap v^{-1}(\{\gamma\})] = \operatorname{int}(G \cap v^{-1}(I_\gamma) \setminus G_\infty) \supset \operatorname{int}[G \cap v^{-1}(I_\gamma)] \setminus \operatorname{cl} G_\infty \neq \emptyset,$$

which contradicts the fact that $G \cap V_\infty = \emptyset$. This finishes the proof of (4.4.1).

Relation (4.4.1) means that the set $E = \Omega \setminus \operatorname{cl} V_\infty$ is contained in F and consequently $v(E) \subset I_{\aleph_0}$. Observe that E is clopen and $I_{\aleph_0}$ is both homeomorphic and order-isomorphic to $[0, 1]$. Therefore $v|_E$ is Borel and by Lemma 4.3.2 there is an open dense subset V_0 of E such that $v|_{V_0}$ is continuous. To end the proof, put $V = V_0 \cup V_\infty$. ∎

Now assume $(f_n)_{n=1}^\infty$ is a sequence of members of $\Lambda(\Omega)$. Let $v\colon \Omega \ni x \mapsto \sum_{n=1}^\infty f_n(x) \in \mathbb{R}_+ \cup \mathrm{Card}$. (The series $\sum_{n=1}^\infty f_n(x)$ is understood as the supremum of its partial sums.) It is clear that $v(\Omega_I) \subset \mathrm{Card}$ and $v(\Omega_{III}) \subset \{0\} \cup \mathrm{Card}_\infty$. By Lemma 4.4.1, there is an open dense subset D of Ω such that $v|_D$ is continuous. Consequently, thanks to Lemma 4.3.1, there is a unique $\tilde{v} \in \Lambda(\Omega)$ which extends $v|_D$. This unique extension $\tilde{v}$ will be denoted by $\sum_{n=1}^\infty f_n$. One may check that $\sum_{n=1}^\infty f_n = \sup_{\Lambda(\Omega)} \bigl\{\sum_{k=1}^n f_k \colon n \geqslant 1\bigr\}$.

Now let $\mathsf{A} \in \mathcal{CDD}_N$. Put

$$s(\mathsf{A}) = \mathsf{J} \boxminus (\mathsf{E}_0^{I}(\mathsf{A}) \boxplus \mathsf{E}_0^{II}(\mathsf{A}) \boxplus \mathsf{E}_0^{III}(\mathsf{A})). \tag{4.4.5}$$

Since $s(\mathsf{A}) \leqslant^s \mathsf{J}$, $s(\mathsf{A})$ corresponds to a unique central projection $z_{\mathsf{A}} \in \mathcal{M}'(\boldsymbol{J})$. There is a unique clopen set in Ω, denoted by $\operatorname{supp}_\Omega \mathsf{A}$, whose characteristic function coincides with $\Psi(z_{\mathsf{A}})$. It is clear that for $\mathsf{A}, \mathsf{B} \in \mathcal{CDD}_N$, $\mathsf{A} \ll \mathsf{B} \Leftrightarrow \operatorname{supp}_\Omega \mathsf{A} \subset \operatorname{supp}_\Omega \mathsf{B}$; and $\mathsf{A} \perp_u \mathsf{B} \Leftrightarrow \operatorname{supp}_\Omega \mathsf{A} \cap \operatorname{supp}_\Omega \mathsf{B} = \emptyset$. When $\mathsf{X}, \mathsf{Y} \in \mathcal{SM}_N$ are such that $\mathsf{X} \ll \mathsf{Y}$, $u = d\mathsf{X}/d\mathsf{Y}$ is defined on Ω_{II} and real-valued on an open dense subset D of Ω_{II}. Extending $u|_D$ to a continuous function of Ω into $I_{\aleph_0}$ by putting zero on $\Omega_I \cup \Omega_{III}$ and applying Lemma 4.3.1, we may consider $d\mathsf{X}/d\mathsf{Y}$ as a member of $\Lambda(\Omega)$, as is done in this chapter. With this understanding,

$$\Bigl\{\frac{d\mathsf{X}}{d\mathsf{Y}} \colon \mathsf{X} \in \mathcal{SM}_N,\ \mathsf{X} \ll \mathsf{Y}\Bigr\}$$
$$= \{u \in \Lambda(\Omega) \colon \operatorname{supp} u \subset \operatorname{supp}_\Omega \mathsf{Y},\ u^{-1}(\mathbb{R}_+) \text{ is dense in } \Omega\} \tag{4.4.6}$$

(by Theorem 4.3.3). Since addition is continuous on $I_{\aleph_0}$, $d(\mathsf{X}' \oplus \mathsf{X}'')/d\mathsf{Y} = d\mathsf{X}'/d\mathsf{Y} + d\mathsf{X}''/d\mathsf{Y}$ whenever $\mathsf{X}', \mathsf{X}'' \ll \mathsf{Y}$.

Throughout, j_E denotes the characteristic function of a set $E \subset \Omega$.

Theorem 4.4.2. *Let $\mathsf{T} \in \mathcal{SM}_N$ be such that $\aleph_0 \odot \mathsf{T} = \mathsf{J}_{II}$ (there exists such a T). There is a unique assignment $\Phi_{\mathsf{T}} \colon \mathcal{CDD}_N \to \Lambda(\Omega)$ such that*

(D0) $\Phi_{\mathsf{T}}(\mathsf{T}) = j_{\Omega_{II}}$, $\Phi_{\mathsf{T}}(\mathsf{J}_I) = j_{\Omega_I}$ *and* $\Phi_{\mathsf{T}}(\mathsf{J}_{III}) = \aleph_0 \cdot j_{\Omega_{III}}$,
(D1) $\operatorname{supp} \Phi_{\mathsf{T}}(\mathsf{A}) \subset \operatorname{supp}_\Omega \mathsf{A}$ *for each* $\mathsf{A} \in \mathcal{CDD}_N$,
(D2) $\Phi_{\mathsf{T}}(\alpha \odot \mathsf{A}) = \alpha \cdot \Phi_{\mathsf{T}}(\mathsf{A})$ *for any* $\alpha \in \mathrm{Card}$ *and* $\mathsf{A} \in \mathcal{CDD}_N$,
(D3) *whenever* $\{\mathsf{A}^{(s)}\}_{s \in S} \subset \mathcal{CDD}_N$ *is a regular family (cf.* (D1) *and notes on page* 45*),*
$$\Phi_{\mathsf{T}}\Bigl(\boxplus_{s \in S} \mathsf{A}^{(s)}\Bigr) = \sum_{s \in S} \Phi_{\mathsf{T}}(\mathsf{A}^{(s)}),$$
(D4) *whenever* $(\mathsf{A}^{(n)})_{n=1}^\infty \subset \mathcal{CDD}_N$ *is such that* $\bigoplus_{n=1}^\infty \mathsf{A}^{(n)} \in \mathcal{SM}_N$ *(see notes above),*
$$\Phi_{\mathsf{T}}\Bigl(\bigoplus_{n=1}^\infty \mathsf{A}^{(n)}\Bigr) = \sum_{n=1}^\infty \Phi_{\mathsf{T}}(\mathsf{A}^{(n)}).$$

What is more, $\Lambda(\Omega)$ is order-complete and Φ_{T} has further properties (below, $\mathsf{A}, \mathsf{B} \in \mathcal{CDD}_N$):

(D1′) $\operatorname{supp}\Phi_{\mathsf{T}}(\mathsf{A}) = \operatorname{supp}_\Omega \mathsf{A}$; *in particular,* $\mathsf{A} \ll \mathsf{B}$ *(resp.* $\mathsf{A} \perp_u \mathsf{B}$*) iff* $\operatorname{supp}_\Omega \Phi_{\mathsf{T}}(\mathsf{A}) \subset \operatorname{supp}_\Omega \Phi_{\mathsf{T}}(\mathsf{B})$ (*resp.* $\operatorname{supp}_\Omega \Phi_{\mathsf{T}}(\mathsf{A}) \cap \operatorname{supp}_\Omega \Phi_{\mathsf{T}}(\mathsf{B}) = \emptyset$),

(D4′) *for any sequence* $(\mathsf{A}^{(n)})_{n=1}^\infty \subset \mathcal{CDD}_N$,

$$\Phi_{\mathsf{T}}\Big(\bigoplus_{n=1}^\infty \mathsf{A}^{(n)}\Big) = \sum_{n=1}^\infty \Phi_{\mathsf{T}}(\mathsf{A}^{(n)}),$$

in particular,

$$\Phi_{\mathsf{T}}(\mathsf{A} \oplus \mathsf{B}) = \Phi_{\mathsf{T}}(\mathsf{A}) + \Phi_{\mathsf{T}}(\mathsf{B}), \tag{4.4.7}$$

(D5) $\mathsf{A} \leqslant \mathsf{B} \Leftrightarrow \Phi_{\mathsf{T}}(\mathsf{A}) \leqslant \Phi_{\mathsf{T}}(\mathsf{B})$,

(D6) $\mathsf{A} \leqslant^s \mathsf{B} \Leftrightarrow \Phi_{\mathsf{T}}(\mathsf{A}) = \Phi_{\mathsf{T}}(\mathsf{B}) \cdot j_E$ *for some clopen set* $E \subset \Omega$,

(D7) *for every* $\mathsf{X} \in \mathcal{SM}_N$, $\Phi_{\mathsf{T}}(\mathsf{X}) = d\mathsf{X}/d\mathsf{T}$,

(D8) *for every* $u \in \Lambda(\Omega)$ *there is a unique* $\mathsf{X} \in \mathcal{CDD}_N$ *such that* $\Phi_{\mathsf{T}}(\mathsf{X}) = u$.

Proof. Let us start with the uniqueness of Φ_{T}. First of all, for $\mathsf{A} \in \mathcal{SM}_N$, $s(\mathsf{A}) = \aleph_0 \odot \mathsf{A}$ and hence $\operatorname{supp}_\Omega \mathsf{A}$ coincides with $\operatorname{supp}_{\Omega_{II}} \mathsf{A}$ introduced in the previous chapter. Therefore (D0), (D1) and (D4) combined with Theorem 4.3.3 yield $\Phi_{\mathsf{T}}(\mathsf{A}) = d\mathsf{A}/d\mathsf{T}$ for $\mathsf{A} \in \mathcal{SM}_N$ (notice that $\mathsf{A} \ll \mathsf{T}$ for every such A). Further, we infer from (D0) and (D2) that $\Phi_{\mathsf{T}}(\mathsf{J}_{II}) = \aleph_0 \cdot j_{\Omega_{II}}$ and consequently, by (D3) and (D0),

$$\Phi_{\mathsf{T}}(\mathsf{J}) = j_{\Omega_I} + \aleph_0 \cdot j_{\Omega_{II} \cup \Omega_{III}}. \tag{4.4.8}$$

Now if $\mathsf{X} \leqslant^s \mathsf{J}$, (D3) implies that $\Phi_{\mathsf{T}}(\mathsf{J}) = \Phi_{\mathsf{T}}(\mathsf{X}) + \Phi_{\mathsf{T}}(\mathsf{Y})$ with $\mathsf{Y} = \mathsf{J} \boxminus \mathsf{X}$. What is more, $\operatorname{supp}_\Omega \mathsf{X} \cap \operatorname{supp}_\Omega \mathsf{Y} = \emptyset$, from which we conclude, thanks to (D1), that $\Phi_{\mathsf{T}}(\mathsf{X}) = \Phi_{\mathsf{T}}(\mathsf{J}) \cdot j_{\operatorname{supp}_\Omega \mathsf{X}}$. Finally, if $\mathsf{A} \in \mathcal{CDD}_N$ is arbitrary, the above combined with (D3) and (D2) gives

$$\Phi_{\mathsf{T}}(\mathsf{A}) = \frac{d\mathsf{E}_{sm}(\mathsf{A})}{d\mathsf{T}} + \sum_{(i,\alpha)\in\Upsilon_*} \alpha \cdot \Phi_{\mathsf{T}}(\mathsf{J}) \cdot j_{\operatorname{supp}_\Omega \mathsf{E}^i_\alpha(\mathsf{A})}. \tag{4.4.9}$$

To establish the existence of Φ_{T} together with all suitable properties, define $\Phi_{\mathsf{T}}(\mathsf{A})$ by (4.4.9) with $\Phi_{\mathsf{T}}(\mathsf{J})$ given by (4.4.8). Observe that (D0), (D1′), (D2) and (D7) are satisfied. We now show (4.4.7). We shall apply the calculations in Example 3.6.7. Under the notation of that example, (4.4.9) and (3.6.6) give

$$\Phi_{\mathsf{T}}(\mathsf{A}\oplus\mathsf{B}) = \frac{d\mathsf{E}_{sm,0}}{d\mathsf{T}} + \frac{d\mathsf{E}_{sm,1}}{d\mathsf{T}} + \frac{d\mathsf{E}_{0,sm}}{d\mathsf{T}} + \frac{d\mathsf{E}_{1,sm}}{d\mathsf{T}} + \sum_{(i,\alpha,\beta)\in\Upsilon^2_\#} (\alpha+\beta)\cdot(\Phi_{\mathsf{T}}(\mathsf{J})\cdot j_{\operatorname{supp}_\Omega \mathsf{E}^i_{\alpha,\beta}}).$$

Further, it follows from Theorem 4.3.3 that

$$\frac{d\mathsf{E}_{sm}(\mathsf{A})}{d\mathsf{T}} = \frac{d\mathsf{E}_{sm,0}}{d\mathsf{T}} + \frac{d\mathsf{E}_{sm,1}}{d\mathsf{T}} + \sum_{\alpha\in\mathrm{Card}_\infty} \frac{d\mathsf{E}_{sm,\alpha}}{d\mathsf{T}},$$
$$\frac{d\mathsf{E}_{sm}(\mathsf{B})}{d\mathsf{T}} = \frac{d\mathsf{E}_{0,sm}}{d\mathsf{T}} + \frac{d\mathsf{E}_{1,sm}}{d\mathsf{T}} + \sum_{\alpha\in\mathrm{Card}_\infty} \frac{d\mathsf{E}_{\alpha,sm}}{d\mathsf{T}}.$$

On the other hand, for $(i,\alpha)\in\Upsilon_*$, we have $\mathsf{E}^i_\alpha(\mathsf{A}) = \boxplus_{\beta\in\Lambda_i} \mathsf{E}^i_{\alpha,\beta}$ and $\mathsf{E}^i_\alpha(\mathsf{B}) = \boxplus_{\beta\in\Lambda_i} \mathsf{E}^i_{\beta,\alpha}$, which means that

$$j_{\operatorname{supp}_\Omega \mathsf{E}^i_\alpha(\mathsf{A})} = \sum_{\beta\in\Lambda_i} j_{\operatorname{supp}_\Omega \mathsf{E}^i_{\alpha,\beta}} \quad \text{and} \quad j_{\operatorname{supp}_\Omega \mathsf{E}^i_\alpha(\mathsf{B})} = \sum_{\beta\in\Lambda_i} j_{\operatorname{supp}_\Omega \mathsf{E}^i_{\beta,\alpha}}.$$

Substituting the above in the formulas for $\Phi_{\mathsf{T}}(\mathsf{A})$ and $\Phi_{\mathsf{T}}(\mathsf{B})$, we see that (4.4.7) is satisfied.

Now let g be an arbitrary member of $\Lambda(\Omega)$. For $(i,\alpha) \in \Upsilon_*$ let $U^i_\alpha = \Omega_i \cap \operatorname{int} g^{-1}(\{\alpha\})$ and let U^{II}_1 be the closure of $g^{-1}(\mathbb{R}_+ \setminus \{0\}) \cap \Omega_{II}$. Since Ω is extremely disconnected, the sets U^i_α (with $(i,\alpha) \in \Upsilon$) are clopen and pairwise disjoint. The arguments used in the proof of Lemma 4.4.1 show that their union is dense in Ω. This implies that there is a partition of unity $\{\mathsf{E}^i_\alpha\}_{(i,\alpha)\in\Upsilon} \subset \mathcal{CDD}_N$ such that $\operatorname{supp}_\Omega \mathsf{E}^i_\alpha = U^i_\alpha$ for every $(i,\alpha) \in \Upsilon$. Moreover, thanks to (4.4.6), there is $\mathsf{E}_{sm} \in \mathcal{SM}_N$ such that $d\mathsf{E}_{sm}/d\mathsf{T} = g \cdot j_{U^{II}_1}$. This implies that $\operatorname{supp}_\Omega \mathsf{E}_{sm} = \operatorname{supp}_\Omega \mathsf{E}^{II}_1$ and hence $\mathsf{E}^{II}_1 = \aleph_0 \odot \mathsf{E}_{sm}$. Now the formulas $\mathsf{E}^i_\alpha(\mathsf{A}) := \mathsf{E}^i_\alpha$ and $\mathsf{E}_{sm}(\mathsf{A}) := \mathsf{E}_{sm}$ well define $\mathsf{A} \in \mathcal{CDD}_N$ such that $\Phi_{\mathsf{T}}(\mathsf{A}) = g$. Further, if $\Phi_{\mathsf{T}}(\mathsf{B}) = g$ and $V^i_\alpha = \operatorname{supp}_\Omega \mathsf{E}^i_\alpha(\mathsf{B})$ ($(i,\alpha) \in \Upsilon$), then $V^i_\alpha \subset U^i_\alpha$ for $(i,\alpha) \in \Upsilon$, by (4.4.9). But the union of all V^i_α's is dense in Ω and $U^i_\alpha \setminus V^i_\alpha$ is open. We infer that $V^i_\alpha = U^i_\alpha$ and consequently $d\mathsf{E}_{sm}(\mathsf{B})/d\mathsf{T} = d\mathsf{E}_{sm}(\mathsf{A})/d\mathsf{T}$ and $\mathsf{B} = \mathsf{A}$. This shows (D8).

We are now able to prove (D5). Indeed, if $\mathsf{A} \leqslant \mathsf{B}$, then $\mathsf{B} = \mathsf{A} \oplus \mathsf{X}$ for some X and then, by (4.4.7), $\Phi_{\mathsf{T}}(\mathsf{B}) = \Phi_{\mathsf{T}}(\mathsf{A}) + \Phi_{\mathsf{T}}(\mathsf{X}) \geqslant \Phi_{\mathsf{T}}(\mathsf{A})$. Conversely, if $\Phi_{\mathsf{T}}(\mathsf{A}) \leqslant \Phi_{\mathsf{T}}(\mathsf{B})$, there is $g \in \Lambda(\Omega)$ (see Corollary 4.4.3 below) for which $\Phi_{\mathsf{T}}(\mathsf{B}) = \Phi_{\mathsf{T}}(\mathsf{A}) + g$. We know from the previous argument that $g = \Phi_{\mathsf{T}}(\mathsf{X})$ for some $\mathsf{X} \in \mathcal{CDD}_N$. Consequently, $\Phi_{\mathsf{T}}(\mathsf{B}) = \Phi_{\mathsf{T}}(\mathsf{A} \oplus \mathsf{X})$ and by (D8), $\mathsf{B} = \mathsf{A} \oplus \mathsf{X}$ and we are done.

We have shown that Φ_{T} is a bijective order isomorphism. This implies that $\Lambda(\Omega)$ is order-complete (by Theorem 2.3.2) and for every nonempty set $\{\mathsf{A}^{(s)}\}_{s\in S} \subset \mathcal{CDD}_N$,

$$\Phi_{\mathsf{T}}\Big(\bigvee\Big\{\bigoplus_{s\in S_0} \mathsf{A}^{(s)} \colon S_0 \in \mathcal{P}_f(S)\Big\}\Big) = \sup_{\Lambda(\Omega)}\Big\{\sum_{s\in S_0} \Phi_{\mathsf{T}}(\mathsf{A}^{(s)}) \colon S_0 \in \mathcal{P}_f(S)\Big\}.$$

But this and (AO6) (page 32) imply (D3), (D4) and (D4′). Point (D6) is left to the reader. ∎

Let us call every topological space homeomorphic to Ω an *underlying model space for* $\mathcal{CDD}_N$. We shall show that underlying model spaces for $\mathcal{CDD}_N$ and $\mathcal{CDD}_{N'}$ are homeomorphic for any N and N'. We shall also propose a simplified form of them.

Let us now list a few basic consequences of Theorem 4.4.2. Some of them were announced in Chapter 4.1. For simplicity, we fix $\mathsf{T} \in \mathcal{SM}_N$ such that $\aleph_0 \odot \mathsf{T} = \mathsf{J}_{II}$ and for each $\mathsf{A} \in \mathcal{CDD}_N$, $\widehat{\mathsf{A}}$ will denote $\Phi_{\mathsf{T}}(\mathsf{A})$. Since $\Lambda(\Omega)$ is order-complete, for every nonempty set $\{f_s\}_{s\in S} \subset \Lambda(\Omega)$, $\bigvee_{s\in S} f_s$ and $\bigwedge_{s\in S} f_s$ will stand for, respectively, $\sup_{\Lambda(\Omega)}\{f_s \colon s \in S\}$ and $\inf_{\Lambda(\Omega)}\{f_s \colon s \in S\}$.

Corollary 4.4.3. $(\mathsf{B} \ominus \mathsf{X})_\vartriangle \vee (\mathsf{X} \ominus \mathsf{A})_\vartriangle \leqslant (\mathsf{B} \ominus \mathsf{A})_\vartriangle$ *provided* $\mathsf{A} \leqslant \mathsf{X} \leqslant \mathsf{B}$.

Proof. It suffices to prove a counterpart of the corollary in the class $\Lambda(\Omega)$. Let $f, g \in \Lambda(\Omega)$ be such that $f \leqslant g$. The set $D_0 = \{x \in \Omega \colon f(x) < f(y) \text{ or } f(y) \in \mathbb{R}_+\}$ is open in Ω and there is a unique function $u_0 \colon D_0 \to \mathbb{R}_+ \cup \mathrm{Card}$ such that $g(x) = u_0(x) + f(x)$ for every $x \in D_0$. It may be easily seen that u_0 is continuous. Let $D(f,g) = D_0 \cup \operatorname{int}(\Omega \setminus D_0)$ and $u \in \Lambda(\Omega)$ be a unique continuous function (guaranteed by Lemma 4.3.1) such that $u(x) = u_0(x)$ for $x \in D_0$ and $u(x) = 0$ for $x \in D(f,g) \setminus D_0$. We see that $g = f + u$ on $D(f,g)$ and hence $g = f + u$ on Ω. It is easily seen that u is the least member of $(\Lambda(\Omega), \leqslant)$

with this property. We shall denote this u by $(g-f)_\Delta$. It is clear that

$$(\widehat{\mathsf{B}\ominus\mathsf{A}})_\Delta = (\widehat{\mathsf{B}}-\widehat{\mathsf{A}})_\Delta$$

whenever $\mathsf{A}\leqslant\mathsf{B}$. Thus, we need to check that $(h-g)_\Delta \vee (g-f)_\Delta \leqslant (h-f)_\Delta$ if only $f\leqslant g\leqslant h$. It suffices to check a suitable inequality on a dense subset of Ω. We leave it as a simple exercise that it is satisfied for $x\in D(f,g)\cap D(g,h)\cap D(f,h)$. ∎

Remark 4.4.4. Using the same idea as in the proof of Corollary 4.4.3, one may show that whenever $\mathsf{A},\mathsf{B}\in\mathcal{CDD}_N$ are such that $\mathsf{A}\leqslant\mathsf{B}$, then

$$[\mathsf{B}\ominus(\mathsf{B}\ominus\mathsf{A})^\nabla]_\Delta \leqslant^s [\mathsf{B}\ominus(\mathsf{B}\ominus\mathsf{A})_\Delta]_\Delta \leqslant^s \mathsf{A} \lll [\mathsf{B}\ominus(\mathsf{B}\ominus\mathsf{A})_\Delta]^\nabla = [\mathsf{B}\ominus(\mathsf{B}\ominus\mathsf{A})^\nabla]^\nabla.$$

Recall that the *Souslin number* of a topological space X, denoted by $c(X)$ ([8, Problem 1.7.12]), is the least infinite cardinal α such that every family of mutually disjoint nonempty open subsets of X has size not greater than α. Let us modify this by putting $c_*(\emptyset)=0$ and $c_*(X)=c(X)$ for nonempty topological spaces X. It turns out that the modified Souslin numbers of certain clopen subsets of Ω may be used to give the formula for $\dim(\mathsf{A})$ if only this dimension is infinite. Namely,

Proposition 4.4.5. *Let $\mathsf{A}\in\mathcal{CDD}_N$ and $f=\widehat{\mathsf{A}}$. Let U_1^{II} be the closure of the set $f^{-1}(\mathbb{R}_+\setminus\{0\})\cap\Omega_{II}$ and for $(i,\alpha)\in\Upsilon_*$ let $U_\alpha^i=\Omega_i\cap\operatorname{int} f^{-1}(\{\alpha\})$. Then*

$$\aleph_0\cdot\dim(\mathsf{A}) = \sum_{(i,\alpha)\in\Upsilon}\alpha\cdot c_*(U_\alpha^i).$$

Proof. In extremely disconnected spaces, the closures of two disjoint open sets are disjoint as well. Consequently, whenever E is a clopen subset of Ω, $c(E)$ is the least infinite cardinal α such that every family of pairwise disjoint nonempty clopen sets has size not greater than α. Since clopen sets correspond to N-tuples A such that $\mathsf{A}\leqslant^s\mathsf{J}$, the assertion follows from the argument used in (ST17) (page 37). The details are left to the reader (cf. the proof of (D8) in Theorem 4.4.2). ∎

Remark 4.4.6. It is worth mentioning that it is impossible to recognize N-tuples whose representatives act on finite-dimensional spaces by means of corresponding members of $\Lambda(\Omega)$, unless we distinguish some special subsets of Ω, as is done in the next chapter. To see this, it suffices to note that $\widehat{\mathsf{A}}$ is the characteristic function of a one-point subset of Ω_I if e.g. $\boldsymbol{A}=(T,\dots,T)\in\mathrm{CDD}_N$ where T is either the identity operator on $\mathbb{C}$ or a unilateral shift on ℓ_2.

We shall now prove a useful

Lemma 4.4.7.

(A) *For every clopen nonempty set $E\subset\Omega$ there is a family $\{E_s\}_{s\in S}$ of pairwise disjoint clopen nonempty sets such that $c(E_s)=\aleph_0$ for every $s\in S$ and $\bigcup_{s\in S}E_s$ is a dense subset of E.*

(B) *Let $\{f_s\}_{s\in S}$ be a nonempty set of members of $\Lambda(\Omega)$ and let $u=\bigwedge_{s\in S}f_s$ and $v=\bigvee_{s\in S}f_s$. For every clopen nonempty set $E\subset\Omega$ with $c(E)=\aleph_0$ there are a nonempty set $S(E)\in\mathcal{P}_\omega(S)$ and an open dense subset $D(E)$ of E with the following property.*

Whenever $S' \supset S(E)$ $(S' \subset S)$ *and* $x \in D(E)$, *then*

$$u(x) = \inf_{s \in S'} f_s(x) \tag{4.4.10}$$

and if, in addition, $v(E) \subset I_{\aleph_0}$, *then also*

$$v(x) = \sup_{s \in S'} f_s(x).$$

Proof. (A): Let $\mathcal{E} = \{E_s\}_{s \in S}$ be a maximal family of pairwise disjoint nonempty clopen sets such that $c(E_s) = \aleph_0$ and $E_s \subset E$ for every $s \in S$. Let $D = E \setminus \operatorname{cl}(\bigcup_{s \in S} E_s)$. We have to show that D is empty. But this follows from Proposition 3.4.10. Indeed, we infer from that result that every nonempty clopen subset of Ω contains a nonempty clopen set G with $c(G) = \aleph_0$. Consequently, since D is clopen and $\mathcal{E}$ is maximal, $D = \emptyset$.

(B): Let $U_1 = \operatorname{cl} u^{-1}(\mathbb{R}_+) \cap E$ and $U_\alpha = \operatorname{int} u^{-1}(\{\alpha\}) \cap E$ for $\alpha \in \mathrm{Card}_\infty$. We know (cf. the proof of Lemma 4.4.1) that the collection $\mathcal{U} = \{U_\alpha \colon \alpha \in \mathrm{Card}_\infty \cup \{1\}\}$ consists of pairwise disjoint clopen sets whose union is dense in E. Further, for each $\alpha \in \mathrm{Card}_\infty$ and $s \in S$ put $U_{\alpha,s} = U_\alpha \cap f_s^{-1}(\{\alpha\})$. Since $f_s \geqslant \alpha$ on U_α and α is an isolated point of $\mathrm{Card} \setminus \{\beta \in \mathrm{Card} \colon \beta < \alpha\}$, $U_{\alpha,s}$ is clopen. It is clear that $\bigcup_{s \in S} U_{\alpha,s}$ is dense in U_α. (Indeed, the set $G = U_\alpha \setminus \operatorname{cl}(\bigcup_{s \in S} U_{\alpha,s})$ is clopen and $f_s(x) \geqslant \alpha^+$ for any $x \in G$ and $s \in S$ and thus $u' \in \Lambda(\Omega)$ given by $u'|_G \equiv \alpha^+$ and $u' = u$ on $\Omega \setminus G$ is such that $u' \leqslant f_s$ $(s \in S)$, which gives $u' \leqslant u$ and consequently $G = \emptyset$.) Let '$<$' be a well order on S with the first element s_*. We define clopen sets $V_{\alpha,s}$ by transfinite induction as follows. Let $V_{\alpha,s_*} = U_{\alpha,s_*}$ and for any $s \in S \setminus \{s_*\}$,

$$V_{\alpha,s} = U_{\alpha,s} \setminus \operatorname{cl}\Bigl(\bigcup_{s' < s} V_{\alpha,s'}\Bigr).$$

We see that $V_{\alpha,s} \subset U_{\alpha,s}$ and hence

$$u|_{V_{\alpha,s}} = f_s|_{V_{\alpha,s}}. \tag{4.4.11}$$

Further, the sets $V_{\alpha,s}$ $(s \in S)$ are pairwise disjoint. Using transfinite induction one may check that $\operatorname{cl}(\bigcup_{s'<s} V_{\alpha,s'}) = \operatorname{cl}(\bigcup_{s'<s} U_{\alpha,s'})$ for each $s \in S$ and thus

$$\operatorname{cl}\Bigl(\bigcup_{s \in S} V_{\alpha,s}\Bigr) = U_\alpha. \tag{4.4.12}$$

Now we turn to the set U_1. By definition, U_1 is clopen and $u(U_1) \subset I_{\aleph_0}$. In what follows, we assume U_1 is nonempty. Let $g_s = f_s \wedge \aleph_0$. We naturally identify $I_{\aleph_0}$ with $[0, \infty]$. Let $\tau \colon [0, \infty] \ni x \mapsto \frac{x}{x+1} \in [0, 1]$ (with the convention that $\frac{\infty}{\infty+1} = 1$). Put $u' = \tau \circ u|_{U_1} \in \mathcal{C}(U_1, [0, 1])$ and $g'_s = \tau \circ g_s|_{U_1} \in \mathcal{C}(U_1, [0, 1])$. Note that

$$u' = \bigwedge_{s \in S} g'_s. \tag{4.4.13}$$

Since U_1 is clopen in Ω and $\mathcal{C}(\Omega)$ is a $\mathcal{W}^*$-algebra, so is $\mathcal{C}(U_1)$. Further, we conclude from the fact that $c(U_1) = \aleph_0$ that $\mathcal{C}(U_1)$ is countably decomposable. Thus, it may be inferred from [35, Theorem III.1.18] or [29, Proposition 1.18.1] that $\mathcal{C}(U_1)$ is isomorphic to $L^\infty(\mu)$ for some probability space $(X, \mathfrak{M}, \mu)$. Under this isomorphism, g'_s and u' correspond to, respectively, $\xi_s \in L^\infty(\mu)$ and $w \in L^\infty(\mu)$. Consequently, $w = \inf_{L^\infty(\mu)}\{\xi_s \colon s \in S\}$ (by (4.4.13)). For a nonempty set $S_0 \in \mathcal{P}_\omega(S)$ let $w_{S_0} \colon X \ni x \mapsto \inf_{s \in S_0} \xi_s(x) \in [0, 1]$. Since S_0 is countable, w_{S_0} is measurable and hence $w_{S_0} \in L^\infty(\mu)$. Let

$$c = \inf\Big\{ \int_X w_{S_0}\, d\mu\colon S_0 \in \mathcal{P}_\omega(S)\Big\}.$$

It is easily seen that there is $S_1 \in \mathcal{P}_\omega(S)$ for which $c = \int_X w_{S_1}\, d\mu$. Now if s is an arbitrary element of S, then $w_{S_1\cup\{s\}} \leqslant w_{S_1}$ and $\int_X w_{S_1\cup\{s\}}\, d\mu \geqslant c = \int_X w_{S_1}\, d\mu$. These imply that $w_{S_1\cup\{s\}} = w_{S_1}$ (μ-almost everywhere) and consequently $\xi_s \geqslant w_{S_1}$ in $L^\infty(\mu)$. The last inequality gives $w \geqslant w_{S_1} = \inf_{L^\infty(\mu)}\{\xi_s\colon s \in S_1\}$ and therefore $w = w_{S_1}$ (in $L^\infty(\mu)$). In $\mathcal{C}(U_1)$ this is interpreted as $u' = \bigwedge_{s\in S_1} g_s'$, which is equivalent to $u|_{U_1} = \bigwedge_{s\in S_1} g_s|_{U_1}$. Now by Lemma 4.4.1, $u(x) = \inf_{s\in S_1} g_s(x)$ for $x \in D_1$ where D_1 is an open dense subset of U_1. This implies that for each $x \in D_1(E) := D_1 \cap u^{-1}(\mathbb{R}_+) \cap E$ there is $s_x \in S_1$ such that $g_{s_x}(x) \in \mathbb{R}_+$. Consequently, $g_{s_x}(x) = f_{s_x}(x)$ and hence

$$u(x) = \inf_{s\in S_1} f_s(x) \tag{4.4.14}$$

for $x \in D_1(E)$. Notice that $D_1(E)$ is dense in U_1.

Further, observe that the family $\{U_1\} \cup \{V_{\alpha,s}\colon s \in S,\ \alpha \in \mathrm{Card}_\infty\}$ consists of pairwise disjoint clopen subsets of E. Since $c(E) = \aleph_0$, the set $J := \{(\alpha, s)\colon s \in S,\ \alpha \in \mathrm{Card}_\infty,\ U_{\alpha,s} \neq \emptyset\}$ is countable (finite or not). Put $S(E) = S_1 \cup \{s\colon (\alpha, s) \in J\}$ and $D(E) = D_1(E) \cup \bigcup_{(\alpha,s)\in J} V_{\alpha,s}$. We see that $S(E) \in \mathcal{P}_\omega(S)$ and $D(E)$ is open and dense in E (by (4.4.12) and the density of $D_1(E)$ in U_1). Take an arbitrary set S' such that $S(E) \subset S' \subset S$. For each $x \in \Omega$ one has $\inf_{s\in S'} f_s(x) \geqslant u(x)$. On the other hand, if $x \in D(E)$, then either $x \in D_1(E)$ or $x \in V_{\alpha,s}$ for some $(\alpha, s) \in J$. In the first case the inequality $\inf_{s\in S'} f_s(x) \leqslant u(x)$ follows from (4.4.14), and in the second from (4.4.11).

If we additionally assume that $v(E) \subset I_{\aleph_0}$, we have to enlarge the set $S(E)$ defined above and decrease $D(E)$. Arguing as in the paragraph for U_1 (that is, representing $\mathcal{C}(E)$ as $L^\infty(\mu)$ for some probability measure μ), we see that there is $S_2 \in \mathcal{P}_\omega(S)$ such that $v|_E = \bigvee_{s\in S_2} f_s$. By Lemma 4.4.1, there is an open dense subset D_2 of E such that $v(x) = \sup_{s\in S_2} f_s(x)$. Now it suffices to replace $S(E)$ by $S(E) \cup S_2$ and $D(E)$ by $D(E) \cap D_2$. (The details are left to the reader.) ∎

Both points of Lemma 4.4.7 yield

COROLLARY 4.4.8. *Let $\{f_s\}_{s\in S}$ be a nonempty subset of $\Lambda(\Omega)$.*

(A) *There is an open dense subset D of Ω such that for all $x \in D$,*

$$\Big(\bigwedge_{s\in S} f_s\Big)(x) = \inf_{s\in S} f_s(x).$$

(B) *If E is a clopen subset of Ω such that $(\bigvee_{s\in S} f_s)(E) \subset I_{\aleph_0}$, then there exists an open dense subset G of E such that for any $x \in G$, $(\bigvee_{s\in S} f_s)(x) = \sup_{s\in S} f_s(x)$.*

REMARK 4.4.9. We suspect that the counterpart of Corollary 4.4.8(A) for suprema fails to be true in general. However, partial results in this direction may be shown. Let $u = \bigvee_{s\in S} f_s$. Put $U_1 = u^{-1}(I_{\aleph_0})$ and $U_\alpha = \operatorname{int} f^{-1}(\{\alpha\})$ for $\alpha \in \mathrm{Card}_\infty \setminus \{\aleph_0\}$. The argument used in the proof of Lemma 4.4.1 shows that $U_1 \cup \bigcup_{\alpha>\aleph_0} U_\alpha$ is dense in Ω. By Corollary 4.4.8, there is an open dense subset of U_1 such that

$$u(x) = \sup_{s\in S} f_s(x) \tag{4.4.15}$$

for $x \in D_1$. We ask for which $\alpha \in \mathrm{Card}_\infty \setminus \{\aleph_0\}$ there is an open dense subset D_α of U_α such that (4.4.15) is satisfied for all $x \in D_\alpha$. It is quite easy to show that this is so when $\alpha = \beta^+$ for some $\beta \geqslant \aleph_0$ (indeed, it suffices to put $D_\alpha = U_\alpha \cap \bigcup_{s\in S} f_s^{-1}(\{\alpha\})$; since $f_s \leqslant \alpha$ on U_α and α is an isolated point of I_α, the set D_α is open; that $\mathrm{cl}\, D_\alpha = U_\alpha$ may be proved by a standard argument on the difference of these sets). Slightly more difficult is to prove that D_α exists for every limit cardinal α which has countable cofinality, that is, when there is a sequence $(\beta_n)_{n=1}^\infty$ of cardinals such that $\beta_n < \alpha$ for every n, and $\alpha = \sup_{n\geqslant 1} \beta_n$. In that case we put $G = U_\alpha \cap \bigcap_{n=1}^\infty \bigcup_{s\in S} f_s^{-1}(\mathrm{Card} \setminus I_{\beta_n})$ and $D = U_\alpha \setminus \mathrm{cl}\, G$. Our first claim is that D is empty. For if not, there would exist a nonempty clopen set $E \subset D$. Then put $E_n = E \cap \bigcap_{s\in S} f_s^{-1}(I_{\beta_n})$. Noticing that $E = \bigcup_{n=1}^\infty E_n$ (since $E \cap G = \emptyset$) and E_n's are closed, we infer from Baire's theorem that $W = \mathrm{int}\, E_n$ is nonempty for some n and thus $\bigvee_{s\in S}(f_s|_W) \leqslant \beta_n$ (W is clopen), contradicting the fact that $[\bigvee_{s\in S}(f_s|_W)](x) = u(x) = \alpha$ for $x \in W$. So, D is indeed empty and hence G is a dense $\mathcal{G}_\delta$ subset of U_α. Now an application of Lemma 4.3.2 shows that $D_\alpha = \mathrm{int}\, G$ is dense in U_α as well.

The above arguments show that if $(\bigvee_{s\in S} f_s)(\Omega) \cap \mathrm{Card}_\infty$ consists only of cardinals which are nonlimit or have countable cofinality, then $\bigvee_{s\in S} f_s$ may be computed pointwise on an open dense set.

THEOREM 4.4.10. *For every nonempty set* $\{\mathsf{A}^{(s)}\}_{s\in S} \subset \mathcal{CDD}_N$ *and each* $\mathsf{B} \in \mathcal{CDD}_N$*,*

$$\mathsf{B} \wedge \Big(\bigvee_{s\in S} \mathsf{A}^{(s)}\Big) = \bigvee_{s\in S} (\mathsf{B} \wedge \mathsf{A}^{(s)}), \tag{4.4.16}$$

$$\mathsf{B} \vee \Big(\bigwedge_{s\in S} \mathsf{A}^{(s)}\Big) = \bigwedge_{s\in S} (\mathsf{B} \vee \mathsf{A}^{(s)}). \tag{4.4.17}$$

Proof. As usual, we pass to $\Lambda(\Omega)$. Put $f_s = \widehat{\mathsf{A}^{(s)}}$ and $g = \widehat{\mathsf{B}}$. Let $u = \bigwedge_{s\in S} f_s$ and $u' = \bigwedge_{s\in S}(g \vee f_s)$. By Corollary 4.4.8, there are open dense sets D and D' such that $u(x) = \inf_{s\in S} f_s(x)$ for $x \in D$ and $u'(x) = \inf_{s\in S}(g \vee f_s)(x)$ for $x \in D'$. Then for $x \in D \cap D'$,

$$(g \vee u)(x) = \max(g(x), \inf_{s\in S} f_s(x)) = \inf_{s\in S}(\max(g(x), f_s(x)) = u'(x),$$

which gives (4.4.17). Now we turn to (4.4.16).

Let $v = \bigvee_{s\in S} f_s$ and $v' = \bigvee_{s\in S}(g \wedge f_s)$. We only need to show that $v' \geqslant g \wedge v$. As usual, put $U_0 = g^{-1}(I_{\aleph_0}) \cap v^{-1}(I_{\aleph_0})$, $U_1 = g^{-1}(I_{\aleph_0}) \setminus v^{-1}(I_{\aleph_0})$ and $U_\alpha = \mathrm{int}\, g^{-1}(\{\alpha\})$ for $\alpha \in \mathrm{Card}_\infty \setminus \{\aleph_0\}$. We know that each of these sets is clopen and their union is dense in Ω. Hence it suffices to show that $g \wedge v \leqslant v'$ on a dense subset of U_α for any $\alpha \in \{0, 1\} \cup \mathrm{Card}_\infty \setminus \{\aleph_0\}$.

On U_0 it suffices to apply Corollary 4.4.8: if $v'(x) = \sup_{s\in S}(g \wedge f_s)(x)$ for $x \in D'$ and $v(x) = \sup_{s\in S} f_s(x)$ for $x \in D$, then $v' = v \wedge g$ on $D \cap D'$. Further, since $v > \aleph_0$ on U_1, the set $D_1 = U_1 \cap \bigcup_{s\in S} f_s^{-1}(\mathrm{Card} \setminus I_{\aleph_0})$ is dense in U_1. What is more, for every $x \in D_1$ there is $s \in S$ with $f_s(x) > \aleph_0$ and therefore $v'(x) \geqslant (f_s \wedge g)(x) = g(x)$. Consequently, $v' \geqslant g \wedge v$ on D_1 and we are done.

Now fix $\alpha \in \mathrm{Card}_\infty \setminus \{\aleph_0\}$. We divide U_α into two clopen parts: $V_1 = U_\alpha \cap v^{-1}(I_\alpha)$ and $V_2 = U_\alpha \setminus v^{-1}(I_\alpha)$. Let $D_\alpha = V_1 \cup \bigcup_{s\in S}[U_\alpha \setminus f_s^{-1}(I_\alpha)]$. Notice that $f_s \leqslant \alpha$ on V_1

(hence $v' = v$ on V_1) and for every $x \in D_\alpha \setminus V_1$ there is $s \in S$ such that $f_s(x) > \alpha$ (so, $v' = g$ on $D_\alpha \setminus V_1$). This proves that $v' \geqslant v \wedge g$ on D_α. Finally, standard argument shows that $D_\alpha \cap V_2$ is dense in V_2, and this finishes the proof. ■

Proposition 4.4.11. *The assertion of* (AO14) (page 34) *is satisfied.*

Proof. Again, it suffices to prove the counterpart of (AO14) in the realm $\Lambda(\Omega)$. It is clear that $\alpha \cdot (f \vee g) = (\alpha \cdot f) \vee (\alpha \cdot g)$ and $\alpha \cdot (f \wedge g) = (\alpha \cdot f) \wedge (\alpha \cdot g)$ for all $f, g \in \Lambda(\Omega)$ and each $\alpha \in \mathrm{Card}$. Now let $\alpha = k$ be a positive finite cardinal. In order to show that $k \cdot (\bigvee_{s\in S} f_s) = \bigvee_{s\in S}(k \cdot f_s)$ and $k \cdot (\bigwedge_{s\in S} f_s) = \bigwedge_{s\in S}(k \cdot f_s)$, let us consider an 'extended' version of $\Lambda(\Omega)$, namely $\widetilde{\Lambda}(\Omega)$ which is defined in the same way as $\Lambda(\Omega)$ with the only difference that members of $\widetilde{\Lambda}(\Omega)$ send Ω_I into $\mathbb{R}_+ \cup \mathrm{Card}$. We shall prove in Corollary 6.1.2 that Ω_I is homeomorphic to Ω_{II}. Consequently, $\widetilde{\Lambda}(\Omega)$ is order-complete. It is immediate that the assignment $\widetilde{\Lambda}(\Omega) \ni f \mapsto k \cdot f \in \widetilde{\Lambda}(\Omega)$ is a bijective order isomorphism. Hence it preserves g.l.b.'s and l.u.b.'s computed in $\widetilde{\Lambda}(\Omega)$. So, we only need to check that $u := \sup_{\widetilde{\Lambda}(\Omega)} F$ and $v := \inf_{\widetilde{\Lambda}(\Omega)} F$ are in $\Lambda(\Omega)$ for every nonempty set $F \subset \Lambda(\Omega)$. Since the proof for u is similar, we shall only show that $v \in \Lambda(\Omega)$. Let $D_0 = \Omega_I \cap \operatorname{int} v^{-1}(\{0\})$, $B_0 = v^{-1}(\{0\}) \cap \Omega_I \setminus D_0$ and for any positive integer m let $D_m = \Omega_I \cap \operatorname{int} v^{-1}((m-1, m])$ and $B_m = v^{-1}((m-1, m]) \cap \Omega_I \setminus D_m$. We claim that $D = (\Omega_I \cap v^{-1}(\mathrm{Card}_\infty)) \cup \bigcup_{m=0}^\infty D_m$ is dense in Ω_I (D is of course open). Indeed, $\Omega_I \setminus D = \bigcup_{m=0}^\infty B_m$. Since each B_m is nowhere dense (by Lemma 4.3.2), Baire's theorem yields our assertion. Now let $v' \in \Lambda(\Omega)$ be such that $v' = v$ on $(\Omega_I \cap v^{-1}(\mathrm{Card}_\infty)) \cup \Omega_{II} \cup \Omega_{III}$ and $v(D_m) \subset \{m\}$ for every integer $m \geqslant 0$ (see Lemma 4.3.1). We see that $v(x) \leqslant v'(x)$ for $x \in D \cup \Omega_{II} \cup \Omega_{III}$ and consequently $v \leqslant v'$. Moreover, since $v \leqslant f \in \Lambda(\Omega)$ for any $f \in F$, $v' \leqslant f$ as well ($f \in F$) and hence $v = v' \in \Lambda(\Omega)$.

In the second part of the second claim of (AO14) one assumes that $\mathsf{E}_{sm}(\mathsf{A}^{(s)}) = \mathsf{O}$, which corresponds to $f_s(\Omega_{II}) \subset \{0\} \cup \mathrm{Card}_\infty$. Here we shall weaken this, assuming that $f_s(\Omega_{II}) \subset \mathrm{Card}$ for each $s \in S$. It follows from Corollary 4.4.8 that there is an open dense subset D of Ω such that for all $x \in D$, $(\bigwedge_{s\in S} f_s)(x) = \inf_{s\in S} f_s(x)$ as well as $[(\bigwedge_{s\in S}(\alpha \cdot f_s)](x) = \inf_{s\in S}(\alpha \cdot f_s)(x)$. Since all values of (all) f_s's are cardinals, we see that in the last two formulas 'inf' may be replaced by 'min'. But $\alpha \cdot \min_{s\in S} f_s(x) = \min_{s\in S}(\alpha \cdot f_s(x))$ and thus $\alpha \cdot (\bigwedge_{s\in S} f_s)(x) = [\bigwedge_{s\in S}(\alpha \cdot f_s)](x)$ for $x \in D$, and we are done.

We now turn to the last claim: that $\alpha \cdot \bigvee_{f\in F} f = \bigvee_{f\in F}(\alpha \cdot f)$ for every nonempty set $F \subset \Lambda(\Omega)$ and $\alpha \in \mathrm{Card}_\infty$. The inequality '$\geqslant$' is clear. To prove the converse, put $u = \bigvee_{f\in F}(\alpha \cdot f)$. It is enough to show that $\alpha \cdot u = u$. Equivalently, we have to check that for each $x \in \Omega$, $u(x) \geqslant \alpha$ or $u(x) = 0$. Suppose, to the contrary, that $0 < u(x_0) < \alpha$ for some $x_0 \in \Omega$. Take a closed set $B \subset I_\alpha \setminus \{\alpha\}$ such that $u(x_0) \in \operatorname{int} B$ and put $D = \operatorname{int} u^{-1}(B)$. D is clopen and $x_0 \in D$. Now let $u' \in \Lambda(\Omega)$ be given by $u' = u$ on $\Omega \setminus D$ and $u' = 0$ on D. We see that $u'(x_0) < u(x_0)$. However, $\alpha \cdot f \leqslant u'$ for every $f \in F$. Indeed, if $x \in D$, then $\alpha > u(x) \geqslant \alpha \cdot f(x)$, which implies that $f(x) = 0$. Thus, u is not the l.u.b. of $\alpha \cdot F$, and this finishes the proof. ■

Remark 4.4.12. It is natural to ask which function corresponds to $\mathsf{A} = \bigoplus_{s\in S} \mathsf{A}^{(s)}$ for an uncountable set S; in other words, how to express $\sum_{s\in S} f_s := \widehat{\mathsf{A}}$ by means of $f_s = \widehat{\mathsf{A}^{(s)}}$

$(s \in S)$. Lemma 4.4.1 and Theorem 4.4.2 show that for countable S, $\sum_{s\in S} f_s$ may be computed pointwise on an open dense subset of Ω. Let us demonstrate how to find $\sum_{s\in S} f_s$ when S is uncountable. We shall use here the arguments of Chapter 4.2. First of all, let $g = \bigvee\{\sum_{s\in S'} f_s \colon S' \in \mathcal{P}_f(S)\}$ and $U_f = \operatorname{cl} g^{-1}(\mathbb{R}_+)$. It may be deduced from the arguments of Chapter 4.2 that $\sum_{s\in S} f_s = g$ on U_f and the function $f := \sum_{s\in S} f_s$ takes infinite values on $\Omega \setminus U_f$. So, we only need to characterize $U_\alpha = \operatorname{int} f^{-1}(\{\alpha\})$ for $\alpha \in \mathrm{Card}_\infty$ (since we know that $U_f \cup \bigcup_{\alpha\in\mathrm{Card}_\infty} U_\alpha$ is dense in Ω). This is possible thanks to (4.2.1). For this purpose, we define $\dim_E u$ for $u \in \Lambda(\Omega)$ and a nonempty clopen set $E \subset \Omega$ with $c(E) = \aleph_0$ as follows:

$$\dim_E u = \sum\{\alpha \in \mathrm{Card}_\infty \colon E \cap \operatorname{int} u^{-1}(\{\alpha\}) \neq \emptyset\} + c_*(E \cap \operatorname{cl} u^{-1}(\mathbb{R}_+ \setminus \{0\}))$$

(notice that the last summand is either 0 or $\aleph_0$). Now one may conclude from (4.2.1) that U_α is the closure of the union of all clopen sets $V \subset \Omega \setminus U_f$ such that $\sum_{s\in S} \dim_E f_s = \alpha$ for every nonempty clopen set $E \subset V$ with $c(E) = \aleph_0$ (of course, U_α may be empty). We leave the details to the interested readers.

Remark 4.4.13. It is clear that the formula for Φ_{T} essentially depends on T. However, there is a quite simple connection between Φ_{T} and Φ_{S} for any two semiminimal N-tuples T and S such that $\aleph_0 \odot \mathsf{T} = \aleph_0 \odot \mathsf{S} = \mathsf{J}_{II}$. Put $u = j_{\Omega_I \cup \Omega_{III}} + d\mathsf{S}/d\mathsf{T}$ and $D := u^{-1}(\mathbb{R}_+ \setminus \{0\})$. We leave it as an easy exercise that D is dense in Ω and for every $\mathsf{X} \in \mathcal{CDD}_N$, $\Phi_{\mathsf{S}}(\mathsf{X})$ is the unique continuous extension of $(\frac{1}{u}\Phi_{\mathsf{T}}(\mathsf{X}))|_D$.

4.5. Types of tuples

As in the previous chapter, $\widehat{\mathsf{A}} = \Phi_{\mathsf{T}}(\mathsf{A})$ for each $\mathsf{A} \in \mathcal{CDD}_N$ where Φ_{T} is as in Theorem 4.4.2. This notation is in force until the end of the paper.

The following result is an immediate consequence of Proposition 3.5.1.

Proposition 4.5.1. *For every clopen set $E \subset \Omega$ the class*

$$\mathcal{I}[E] := \{\mathsf{A} \in \mathcal{CDD}_N \colon \operatorname{supp} \widehat{\mathsf{A}} \subset E\}$$

is an ideal in $\mathcal{CDD}_N$. Conversely, for every ideal $\mathcal{A} \subset \mathcal{CDD}_N$ there is a (*unique*) *clopen set $K \subset \Omega$ such that $\mathcal{A} = \mathcal{I}[K]$. What is more, $K = \operatorname{supp} \widehat{\mathsf{J}(\mathcal{A})}$.*

For every ideal $\mathcal{A}$, the unique clopen set K such that $\mathcal{A} = \mathcal{I}[K]$ will be denoted by $\operatorname{supp}_\Omega \mathcal{A}$. Below we give some related examples.

Examples 4.5.2. (A) Fix a nonnegative real number r and let $\mathcal{I}(r)$ be the class of all N-tuples X for which $\|\mathsf{X}\| \leqslant r$. It is clear that $\mathcal{I}(r)$ is an ideal. Put $\Omega(r) := \operatorname{supp}_\Omega \mathcal{I}(r)$ and $\Omega(\mathrm{bd}) := \bigcup_{r\geqslant 0} \Omega(r)$. The set $\Omega(\mathrm{bd})$ is open in Ω and for every $\mathsf{X} \in \mathcal{CDD}_N$,

$$\|\mathsf{X}\| < \infty \iff \operatorname{supp} \widehat{\mathsf{X}} \subset \Omega(\mathrm{bd})$$

(indeed, use the fact that $\operatorname{supp} \widehat{\mathsf{X}}$ is compact). What is more, if $\|\mathsf{X}\| < \infty$, then $\|\mathsf{X}\| = \min\{r \geqslant 0 \colon \operatorname{supp} \widehat{\mathsf{X}} \subset \Omega(r)\}$. The ideal $\mathcal{I}[\operatorname{cl} \Omega(\mathrm{bd})]$ consists of all N-tuples which are direct sums of bounded N-tuples. Further, whenever $0 \leqslant s < r$, the ideal $\mathcal{I}[\Omega(r) \setminus \Omega(s)]$ consists of all N-tuples all of whose nontrivial reduced parts have norm greater than s but not greater than r. We conclude that $\Omega(s) = \operatorname{int}(\bigcap_{r>s} \Omega(r))$ for any $s \geqslant 0$. For positive r put $\Omega\{r\} = \Omega(r) \setminus \operatorname{cl}(\bigcup_{s<r} \Omega(s))$ and $\mathcal{I}\{r\} = \mathcal{I}[\Omega\{r\}]$. The ideal $\mathcal{I}\{r\}$ consists of all N-tuples with all nontrivial reduced parts having norm r.

(B) Now let $\mathcal{I}(\mathfrak{b}) := \{\mathfrak{b}(\mathsf{A})\colon \mathsf{A} \in \mathcal{CDD}_N\}$. It follows from the properties of the $\mathfrak{b}$-transform that $\mathcal{I}(\mathfrak{b})$ is an ideal. Let $\Omega(\mathfrak{b}) = \operatorname{supp}_\Omega \mathcal{I}(\mathfrak{b})$. Notice that $\mathcal{I}(\mathfrak{b})$ consists of all N-tuples X such that either $\|\mathsf{X}\| < 1$, or $\|\mathsf{X}\| = 1$ and X does not assume its norm. Consequently, $\Omega(\mathfrak{b}) \subsetneq \Omega(1)$. The ideal $\mathcal{I}[E]$ with $E = \Omega(1) \setminus \Omega(\mathfrak{b})$ consists of all N-tuples each of whose nontrivial reduced part has norm 1 and assumes its norm. In particular, $E \subset \Omega\{1\}$ and the ideal $\mathcal{I}[\Omega\{1\} \setminus E] = \mathcal{I}[\Omega\{1\} \cap \Omega(\mathfrak{b})]$ coincides with the class of all N-tuples each of whose nontrivial reduced parts has norm equal to 1 and does not assume its norm.

As a consequence of Theorem 2.4.1 and Examples 4.5.2 we obtain

Corollary 4.5.3. *Every contraction T acting on a Hilbert space $\mathcal{H}$ induces a unique decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$ such that $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2 \in \operatorname{red}(T)$ and*

(a) *every nontrivial reduced part of $T|_{\mathcal{H}_0}$ admits a nontrivial reduced part of norm less than* 1,
(b) *$T|_{\mathcal{H}_1}$ does not assume its norm (unless $\mathcal{H}_1$ is trivial) and each of its nontrivial reduced parts has norm* 1,
(c) *every nontrivial reduced part of $T|_{\mathcal{H}_2}$ has norm* 1 *and assumes its norm.*

What is more, $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2 \in \operatorname{cred}(T)$.

As done by Ernest [9], the types of $\mathcal{W}''(\boldsymbol{X})$ and $\mathcal{W}'(\boldsymbol{X})$ may be assigned to $\boldsymbol{X}$. It is easily seen (and in fact, already used by us in Theorem 3.6.1) that for every nontrivial $\boldsymbol{X} \in \mathrm{CDD}_N$:

- $\mathcal{W}'(\boldsymbol{X})$ is type I_α ($\alpha \in \mathrm{Card} \setminus \{0\}$) iff $\mathsf{X} = \alpha \odot \mathsf{E}$ for a unique $\mathsf{E} \leqslant^s \mathsf{J}_I$,
- $\mathcal{W}'(\boldsymbol{X})$ is type III_α ($\alpha \in \mathrm{Card}_\infty$) iff $\mathsf{X} = \alpha \odot \mathsf{E}$ for a unique $\mathsf{E} \leqslant^s \mathsf{J}_{III}$,
- $\mathcal{W}'(\boldsymbol{X})$ is type II_1 iff X is semiminimal,
- $\mathcal{W}'(\boldsymbol{X})$ is type II_α ($\alpha \in \mathrm{Card}_\infty$) iff $\mathsf{X} = \alpha \odot \mathsf{E}$ for a unique $\mathsf{E} \leqslant^s \mathsf{J}_{II}$.

Ernest calls a bounded operator T of type i_α provided $\mathcal{W}'(T)$ is of this type (cf. [9, Definition 1.28]). We call a nontrivial N-tuple $\mathsf{X} \in \mathcal{CDD}_N$ *(of) type* I^n (with $n = 1, 2, \ldots, \infty$), II^1, II^∞ or $\mathrm{III}^{(\infty)}$ iff $\mathcal{W}''(\boldsymbol{X})$ is of type I_n, II_1, II_∞ and $\mathrm{III}_{(\infty)}$ (respectively). Additionally, we agree that the trivial N-tuple is of each of these types.

Since a von Neumann algebra is type I, II, III iff so is its commutant, we see that for nontrivial X, $\mathcal{W}''(\boldsymbol{X})$ is type III iff so is $\mathcal{W}'(\boldsymbol{X})$ and thus the above definition causes no confusion. Later we shall see that if a nontrivial X is type i^∞ ($i \in \{I, II, III\}$), then $\mathcal{W}''(\boldsymbol{X})$ is type $i_{\aleph_0}$ and thus there is no need to use uncountable cardinals here.

Fix $i^n \in \{I^1, I^2, \ldots, I^\infty, II^1, II^\infty, III^\infty\}$ and let $\mathcal{I}_{i_n}$ be the class of all N-tuples of type i^n. Our first goal is

Proposition 4.5.4. *$\mathcal{I}_{i_n}$ is an ideal in $\mathcal{CDD}_N$.*

Proof. It suffices to verify all points of Corollary 3.6.6. Point (a) is fulfilled since for any $\alpha \in \mathrm{Card}_\infty$ and nontrivial X, the von Neumann algebras $\mathcal{W}''(\boldsymbol{X})$ and $\mathcal{W}''(\boldsymbol{Y})$ are isomorphic where $\mathsf{Y} = \alpha \odot \mathsf{X}$. Point (b) follows from the following result on $\mathcal{W}^*$-algebras: if $\mathcal{M}$ is a $\mathcal{W}^*$-algebra and $\{z_s\}_{s\in S}$ is a family of mutually orthogonal central projections in $\mathcal{M}$ which sum to 1 and $\mathcal{M}z_s$ is type i_n for each $s \in S$, then $\mathcal{M}$ itself is type i_n. Finally, (c) is a consequence of a similar result: if $\mathcal{M}$ is a type i_n $\mathcal{W}^*$-algebra and z is a (nonzero) central projection in $\mathcal{M}$, then $\mathcal{M}z$ is type i_n as well. ∎

Now put $\Omega_{i_n} = \operatorname{supp}_\Omega \mathcal{J}_{i_n}$. It is clear that the sets $\Omega_{I_1}, \Omega_{I_2}, \ldots, \Omega_{I_\infty}, \Omega_{II_1}, \Omega_{II_\infty}$ and Ω_{III_∞} are pairwise disjoint and their union is dense in Ω. It is obvious that $\Omega_{III_\infty} = \Omega_{III}$. Let us now check that if X is type i^∞ (and nontrivial), then $\mathcal{W}''(\boldsymbol{X})$ is type $i_{\aleph_0}$. Indeed, there is $\mathsf{E} \leqslant^s \mathsf{J}$ (namely, $\mathsf{E} = s(\mathsf{X})$, cf. (4.4.5)) and an infinite cardinal α such that $\alpha \odot \mathsf{X} = \alpha \odot \mathsf{E}$. This implies that $\mathcal{W}''(\boldsymbol{X})$ and $\mathcal{W}''(\boldsymbol{E})$ are isomorphic as $\mathcal{W}^*$-algebras and thus $\mathcal{W}''(\boldsymbol{E})$ is type i_∞. Further, we conclude from Proposition 3.4.10 that $\mathsf{E} = \boxplus_{s\in S} \mathsf{E}^{(s)}$ for a suitable family such that $0 < \dim(\mathsf{E}^{(s)}) \leqslant \aleph_0$. Consequently, $\mathcal{W}''(\boldsymbol{E}^{(s)})$ is type i_∞ for each $s \in S$ and therefore (since $\boldsymbol{E}^{(s)}$ acts in a separable Hilbert space) $\mathcal{W}''(\boldsymbol{E}^{(s)})$ is type $i_{\aleph_0}$. This implies that $\mathcal{W}''(\boldsymbol{E})$ (and hence $\mathcal{W}''(\boldsymbol{X})$) is type $i_{\aleph_0}$ as well.

One may easily check that $\mathcal{J}_{I_1}$ coincides with the ideal $\mathcal{N}_N$ introduced in Examples 2.4.3(E) and studied in Example 3.5.3. Thus Ω_{I_1} corresponds to normal N-tuples.

The sets Ω_{I_n} may be used to compute $\dim(\mathsf{X})$ for every $\mathsf{X} \in \mathcal{CDD}_N$ by means of $\widehat{\mathsf{X}}$. For this, let us introduce the *strict Souslin number*, $c_f(X)$, of a topological space X. Namely, $c_f(X) = c(X)$ iff X is an infinite set and $c_f(X) = \operatorname{card}(X)$ otherwise.

Proposition 4.5.5. *Let* $\mathsf{X} \in \mathcal{CDD}_N$, $f = \widehat{\mathsf{X}}$, $U^i_\alpha = \Omega_i \cap \operatorname{int} f^{-1}(\{\alpha\})$ *for* $(i,\alpha) \in \Upsilon_*$ *and* $U^{II}_1 = \Omega_{II} \cap \operatorname{cl} f^{-1}(\mathbb{R}_+ \setminus \{0\})$. *Then*

$$\begin{aligned}\dim(\mathsf{X}) = &\sum_{n,m=1}^{\infty} nm \cdot c_f(U^I_n \cap \Omega_{I_m}) + \aleph_0 \sum_{n=1}^{\infty} c_f(U^I_n \cap \Omega_{I_\infty})\\ &+ \aleph_0 \cdot c_f(U^{II}_1) + \sum_{\alpha\in\mathrm{Card}_\infty} \alpha[c_f(U^I_\alpha) + c_f(U^{II}_\alpha) + c_f(U^{III}_\alpha)]. \end{aligned} \tag{4.5.1}$$

Proof. As in the proof of Proposition 4.4.5, we see that $\dim(\mathsf{X}) = \sum_{(i,\alpha)\in\Upsilon} \alpha \cdot \dim(\mathsf{E}^i_\alpha(\mathsf{X}))$ and $\aleph_0 \cdot \dim(\mathsf{E}^i_\alpha(\mathsf{X})) = c_*(U^i_\alpha) = \aleph_0 \cdot c_f(U^i_\alpha)$. Moreover, $\dim(\mathsf{E}^{II}_1(\mathsf{X})) \in \mathrm{Card}_\infty \cup \{0\}$. So, to show (4.5.1), it suffices to check $\dim(\mathsf{E}^I_n(\mathsf{X})) = \aleph_0 \cdot c_f(U^I_n \cap \Omega_{I_\infty}) + \sum_{m=1}^\infty m \cdot c_f(U^I_n \cap \Omega_{I_m})$. Write $\mathsf{E}^I_n(\mathsf{X}) = \boxplus_{m=1}^{m=\infty} \mathsf{E}_{n,m}$ with $\mathsf{E}_{n,m} \in \mathcal{J}_{I_m}$ and observe that $\operatorname{supp}_\Omega \mathsf{E}_{n,m} = U^I_n \cap \Omega_{I_m} =: V_{n,m}$. So, it is enough to show that

$$\dim(\mathsf{E}_{n,m}) = m \cdot c_f(V_{n,m}) \tag{4.5.2}$$

(for $m = \infty$ the above means that $\dim(\mathsf{E}_{n,\infty}) = \aleph_0 \cdot c_f(V_{n,\infty})$). If the set $V_{n,m}$ is infinite, then we may decompose it into arbitrarily (finitely) many pairwise disjoint nonempty clopen sets, which shows that representatives of $\mathsf{E}_{n,m}$ act in infinite-dimensional Hilbert spaces and hence (4.5.2) is satisfied in that case (e.g. by Proposition 4.4.5). On the other hand, if $V_{n,m}$ is finite, $\mathsf{E}_{n,m}$ may be decomposed into $\operatorname{card}(V_{n,m})$ irreducible N-tuples of type I^m. Now (4.5.2) easily follows since an irreducible N-tuple of type I^m acts in an m-dimensional Hilbert space. ∎

5. PRIME DECOMPOSITION

5.1. Primes, semiprimes, atoms and fractals

Prime numbers may be defined in two ways (below, n, k and l are positive integers):

- n is prime iff $n \neq 1$, and $n = kl$ implies $k = 1$ or $l = 1$,
- n is prime iff $n \neq 1$, and $n = kl$ implies $k, l \in \{1, n\}$.

These two conditions may naturally be adapted to more general algebraic structures (especially monoids, i.e. semigroups with neutral elements). However, in some structures they may be inequivalent. We will see that this occurs in $\mathcal{CDD}_N$. Therefore we distinguish the following two classes of N-tuples.

Definition 5.1.1. Let $\mathsf{A} \in \mathcal{CDD}_N$ be nontrivial. We say A is a *prime* iff $\mathsf{A} = \mathsf{X} \oplus \mathsf{Y}$ implies $\mathsf{X}, \mathsf{Y} \in \{\mathsf{O}, \mathsf{A}\}$. A is an *atom* iff $\mathsf{A} = \mathsf{X} \oplus \mathsf{Y}$ implies $\mathsf{X} = \mathsf{O}$ or $\mathsf{Y} = \mathsf{O}$.

In case of a single bounded operator, our definition of an atom is equivalent to Ernest's definition of an irreducible operator ([9]). It is clear that every atom is a prime. But not conversely. To see that, let us first prove

Proposition 5.1.2. *For a nontrivial* $\mathsf{A} \in \mathcal{CDD}_N$ *the following conditions are equivalent:*

(i) $\mathcal{W}'(\boldsymbol{A})$ *is a factor,*
(ii) $\mathcal{W}''(\boldsymbol{A})$ *is a factor,*
(iii) $\{\mathsf{X} \in \mathcal{CDD}_N \colon \mathsf{X} \leqslant^s \mathsf{A}\} = \{\mathsf{O}, \mathsf{A}\}$,
(iv) *exactly one of the following three conditions is fulfilled:*

(a) *there are unique* $\mathsf{X} \in \mathcal{MF}_N$ *and a unique positive cardinal* α *such that* $\mathsf{A} = \alpha \odot \mathsf{X}$ *and* $\mathcal{W}'(\boldsymbol{X})$ *constists precisely of the scalar multiples of the identity operator; what is more,* $0 < \dim(\mathsf{X}) \leqslant \aleph_0$,
(b) *there are unique* $\mathsf{X} \in \mathcal{HIM}_N$ *and a unique infinite cardinal* α *such that* $\mathsf{A} = \alpha \odot \mathsf{X}$ *and* $\mathcal{W}'(\boldsymbol{X})$ *is a* (*type III*) *factor; what is more,* $\dim(\mathsf{X}) = \aleph_0$,
(c) *there are* (*nonunique*) $\mathsf{X} \in \mathcal{SM}_N$ *and a unique cardinal* $\alpha \in \{1\} \cup \mathrm{Card}_\infty$ *such that* $\mathsf{A} = \alpha \odot \mathsf{X}$ *and* $\mathcal{W}'(\boldsymbol{X})$ *is a* (*type* II_1) *factor; what is more,* $\dim(\mathsf{X}) = \aleph_0$.

Proof. Points (i) and (ii) are clearly equivalent. Further, it follows from (PR3) (page 13) that (i) is equivalent to (iii). Consequently, we infer from Theorem 3.6.1 that if $\mathcal{W}'(\boldsymbol{A})$ is a factor, then either $\mathsf{A} = \mathsf{E}_{sm}(\mathsf{A})$ or $\mathsf{A} = \beta \odot \mathsf{E}^i_\beta(\mathsf{A})$ for some $(i, \beta) \in \Upsilon_*$. In the first situation put $\mathsf{X} = \mathsf{E}_{sm}$ and $\alpha = 1$; in the second, we consider two cases: if $i \neq II$, put $\mathsf{X} = \mathsf{E}^i_\beta(\mathsf{A})$, otherwise take $\mathsf{X} \in \mathcal{SM}_N$ such that $\aleph_0 \odot \mathsf{X} = \mathsf{E}^{II}_\beta$; in both cases we put

$\alpha = \beta$. Note that $\mathsf{A} = \alpha \odot \mathsf{X}$. Further, we conclude from (PR6) (page 13) that $\mathcal{W}'(\boldsymbol{A})$ is a factor iff so is $\mathcal{W}'(\boldsymbol{X})$. Now Proposition 3.4.10 implies that $\dim(\mathsf{X}) \leqslant \aleph_0$ provided $\{\mathsf{Y} \in \mathcal{CDD}_N \colon \mathsf{Y} \leqslant^s \mathsf{X}\} = \{\mathsf{O}, \mathsf{X}\}$. All the above shows that (i) is equivalent to (iv). ∎

We now have

Proposition 5.1.3. *Let* $\mathsf{A} \in \mathcal{CDD}_N$ *be nontrivial.*

(A) A *is an atom iff* $\mathcal{W}'(\boldsymbol{A})$ *consists precisely of the scalar multiples of the identity operator. If* A *is an atom, then* $\mathsf{A} \leqslant \mathsf{J}_I$ *and* $0 < \dim(\mathsf{A}) \leqslant \aleph_0$.
(B) *Suppose* $\mathsf{A} \in \mathcal{CDD}_N$ *is not an atom. Then* A *is a prime iff* $\dim(\mathsf{A}) = \aleph_0$ *and* $\mathcal{W}'(\boldsymbol{A})$ *is a type III factor.*

Proof. Point (A) is left to the reader. We turn to (B).

First note that if A is type III, then $\mathsf{A} \ll \mathsf{J}_{III}$. Consequently, if in addition $\dim(\mathsf{A}) = \aleph_0$, then $\mathsf{A} = \mathsf{E}^{III}_{\aleph_0}(\mathsf{A})$ and thus A is minimal. But then $\{\mathsf{X} \in \mathcal{CDD}_N \colon \mathsf{X} \leqslant \mathsf{A}\} = \{\mathsf{X} \in \mathcal{CDD}_N \colon \mathsf{X} \leqslant^s \mathsf{A}\}$. So, the sufficiency of the conditions formulated in the proposition for A to be a prime follows from Proposition 5.1.2. Conversely, if A is a prime but not an atom, an application of Proposition 5.1.2 shows that $\mathsf{A} = \alpha \odot \mathsf{X}$ for suitable α and X. Since $\mathsf{X} \leqslant \mathsf{A}$, we infer that $\mathsf{A} = \mathsf{X}$. So, $\mathsf{X} \notin \mathcal{MF}_N$ (because A is not an atom) and X is not semiminimal since $\mathsf{O} \neq \frac{1}{2} \odot \mathsf{Y} \lneqq \mathsf{Y}$ for every nontrivial $\mathsf{Y} \in \mathcal{SM}_N$. We infer that $\mathsf{X} \in \mathcal{HIM}_N$. Thus, $\mathcal{W}'(\boldsymbol{A})$ is type III and, of course, it is a factor. ∎

Let A be a prime which is not an atom. It follows from Proposition 5.1.3 that $\mathsf{A} = \aleph_0 \odot \mathsf{A}$. Consequently, $\operatorname{red}(\boldsymbol{A})$ is an infinite set. However, for every $E \in \operatorname{red}(\boldsymbol{A})$, $\boldsymbol{A}|_E \equiv \boldsymbol{A}$ (because A is prime). Conversely, if $\boldsymbol{B} \in \mathrm{CDD}_N$ is such that $\operatorname{card}(\operatorname{red}(\boldsymbol{B})) > 2$ and $\boldsymbol{B}|_E \equiv \boldsymbol{B}$ for any $E \in \operatorname{red}(\boldsymbol{B})$, then B is a prime and not an atom. This observation leads us to

Definition 5.1.4. A *fractal* is a prime which is not an atom.

We see that every prime A is either an atom (if $\mathsf{A} \neq 2 \odot \mathsf{A}$) or a fractal (if $\mathsf{A} = 2 \odot \mathsf{A}$) and that A is type I or type III. It is immediate that two different primes are unitarily disjoint.

A counterpart of primes for type II N-tuples are semiprimes.

Definition 5.1.5. A nontrivial N-tuple A is said to be a *semiprime* iff A is not of the form $n \odot \mathsf{B}$ where n is a natural number and B is a prime, and the following condition is fulfilled: whenever $\mathsf{O} \neq \mathsf{X} \leqslant \mathsf{A}$, there is a natural number m such that $\mathsf{A} \leqslant m \odot \mathsf{X}$.

Semiprimes may be characterized as follows.

Proposition 5.1.6.

(I) *A nontrivial* N*-tuple* A *is a semiprime iff* $\mathcal{W}'(\boldsymbol{A})$ *is a type* II_1 *factor.*
(II) *Let* A *be a semiprime. Then* A *is semiminimal and* $\dim(\mathsf{A}) = \aleph_0$. *If* $\mathsf{B} \ll \mathsf{A}$*, then* B *is a semiprime iff* $\mathsf{B} = t \odot \mathsf{A}$ *for some* $t \in \mathbb{R}_+ \setminus \{0\}$.

Proof. First assume that $\mathcal{W}'(\boldsymbol{A})$ is a type II_1 factor. Then necessarily $\mathsf{A} \neq n \odot \mathsf{B}$ for any prime B, and $\mathsf{A} \in \mathcal{SM}_N$. Moreover, $\mathcal{W}'(\aleph_0 \odot \boldsymbol{A})$ is a factor as well. We conclude that $\operatorname{supp}_{\Omega_{II}} \mathsf{A}$ consists of a single point (see Chapter 4.3). This implies that if $\mathsf{O} \neq \mathsf{X} \leqslant \mathsf{A}$, then $\frac{d\mathsf{X}}{d\mathsf{A}} = \lambda \cdot \frac{d\mathsf{A}}{d\mathsf{A}}$ for some real number $\lambda > 0$. But $\lambda \cdot \frac{d\mathsf{A}}{d\mathsf{A}} = \frac{d(\lambda \odot \mathsf{A})}{d\mathsf{A}}$ and therefore $\mathsf{X} = \lambda \odot \mathsf{A}$. Now it suffices to take a natural number m such that $m\lambda \geqslant 1$ to see that $\mathsf{A} \leqslant m \odot \mathsf{X}$. Consequently, A is a semiprime.

We now assume that A is a semiprime. Observe that then $\mathsf{A} = \mathsf{X} \boxplus \mathsf{Y}$ implies $\mathsf{X} = \mathsf{O}$ or $\mathsf{Y} = \mathsf{O}$. We infer that $\mathcal{W}'(\boldsymbol{A})$ is a factor. So, according to Proposition 5.1.2, $\mathsf{A} = \alpha \odot \mathsf{X}$ for suitable α and X. Since A is a semiprime and $\mathsf{O} \neq \mathsf{X} \leqslant \mathsf{A}$, $\alpha \odot \mathsf{X} \leqslant m \odot \mathsf{X}$ for some natural number m. This implies that either $\mathsf{A} = \mathsf{X} \in \mathcal{HIM}_N$ or $\alpha \leqslant m$. Again taking into account that A is a semiprime, we see that $\mathsf{A} = \mathsf{X} \in \mathcal{SM}_N$ and hence $\mathcal{W}'(\boldsymbol{A})$ is type II_1 and $\dim(\mathsf{A}) = \aleph_0$. Further, if $\mathsf{B} = t \odot \mathsf{A}$, then B is semiminimal (hence $\mathcal{W}'(\boldsymbol{B})$ is type II_1) and the $\mathcal{W}^*$-algebras $\mathcal{Z}(\mathcal{W}'(\boldsymbol{B}))$, $\mathcal{Z}(\mathcal{W}'(\aleph_0 \odot \boldsymbol{B}))$, $\mathcal{Z}(\mathcal{W}'(\aleph_0 \odot \boldsymbol{A}))$ and $\mathcal{Z}(\mathcal{W}'(\boldsymbol{A}))$ are isomorphic (since $\aleph_0 \odot \mathsf{B} = \aleph_0 \odot \mathsf{A}$), which implies that $\mathcal{W}'(\boldsymbol{B})$ is a factor. Consequently, B is a semiprime. Finally, if B is a semiprime such that $\mathsf{B} \ll \mathsf{A}$, then from the semiminimality of B it follows that $\frac{1}{n} \odot \mathsf{B} \leqslant \mathsf{A}$ for some natural number n. Now the first paragraph of the proof shows that then $\frac{1}{n} \odot \mathsf{B} = \lambda \odot \mathsf{A}$ for some $\lambda > 0$, and we are done. ∎

The reader will now easily check that if A is a prime or a semiprime and $\mathsf{X} \in \mathcal{CDD}_N$ is arbitrary, then either $\mathsf{A} \leqslant n \odot \mathsf{X}$ for some natural number n or $\mathsf{A} \perp_u \mathsf{X}$. It turns out that a stronger property may be established, similar to a suitable property of prime numbers. Namely:

Proposition 5.1.7. *Let $\{\mathsf{X}^{(s)}\}_{s \in S} \in \mathcal{CDD}_N$ be a nonempty set and let $\mathsf{A} \leqslant \bigoplus_{s \in S} \mathsf{X}^{(s)}$.*

(I) *If A is a prime, there is $s \in S$ such that $\mathsf{A} \leqslant \mathsf{X}^{(s)}$.*
(II) *Suppose A is a semiprime. For each $s \in S$ let $\lambda_s = \sup\{t \in \mathbb{R}_+ : t \odot \mathsf{A} \leqslant \mathsf{X}^{(s)}\} \in \mathbb{R}_+ \cup \{\aleph_0\}$. Then $\lambda_s \odot \mathsf{A} \leqslant \mathsf{X}^{(s)}$ $(s \in S)$ and $\sum_{s \in S} \lambda_s \geqslant 1$.*

Proof. To prove (I), observe that there is $s \in S$ such that A and $\mathsf{X}^{(s)}$ are not unitarily disjoint. Since A is a prime, this yields $\mathsf{A} \leqslant \mathsf{X}^{(s)}$.

We now turn to (II). By (VS3) (page 41), $\mathsf{A}^{(s)} := \lambda_s \odot \mathsf{A} \leqslant \mathsf{X}^{(s)}$. Assume that $\lambda_s < 1$ for every $s \in S$ and $\lambda = \sum_{s \in S} \lambda_s < \infty$. By the maximality of λ_s, $(1 - \lambda_s) \odot \mathsf{A} = \mathsf{A} \ominus \mathsf{A}^{(s)} \perp_u \mathsf{X}^{(s)} \ominus \mathsf{A}^{(s)} =: \mathsf{Y}^{(s)}$ and consequently $\mathsf{A} \perp_u \mathsf{Y}^{(s)}$. Thus, $\mathsf{A} \perp_u \bigoplus_{s \in S} \mathsf{Y}^{(s)}$. Now since $\bigoplus_{s \in S} \mathsf{X}^{(s)} = (\bigoplus_{s \in S} \mathsf{A}^{(s)}) \oplus (\bigoplus_{s \in S} \mathsf{Y}^{(s)})$, we infer from (PR1) (page 12) that $\mathsf{A} \leqslant \bigoplus_{s \in S} \mathsf{A}^{(s)}$. Further, we see that $\bigvee\{\bigoplus_{s \in S'} \mathsf{A}^{(s)} : S' \in \mathcal{P}_f(S)\} = \lambda \odot \mathsf{A}$. This, combined with Proposition 4.1.6, yields $\lambda \odot \mathsf{A} = \bigoplus_{s \in S} \mathsf{A}^{(s)}$. So, $\mathsf{A} \leqslant \lambda \odot \mathsf{A}$ and hence $\lambda \geqslant 1$. ∎

Denote by $\mathfrak{a}_N$, $\mathfrak{f}_N$ and $\mathfrak{s}_N$ the sets of all, respectively, atoms, fractals and semiprimes in $\mathcal{CDD}_N$. Further, for $n = 1, 2, \ldots, \infty$ let $\mathfrak{a}_N(n)$ be the set of all atoms of type I^n. Similarly, we denote by $\mathfrak{s}_N(1)$ and $\mathfrak{s}_N(\infty)$ the sets of all semiprimes of type II^1 and II^∞, respectively. The reader should notice that an atom A belongs to $\mathfrak{a}_N(n)$ for some finite n iff $\dim(\mathsf{A}) = n$ (and $\mathsf{A} \in \mathfrak{a}_N(\infty)$ iff $\dim(\mathsf{A}) = \aleph_0$). Finally, we put $\mathfrak{p}_N = \mathfrak{a}_N \cup \mathfrak{f}_N \cup \mathfrak{s}_N$.

Proposition 5.1.8. *The sets $\mathfrak{a}_N(n)$ $(n = 1, 2, \ldots, \infty)$, $\mathfrak{f}_N$, $\mathfrak{s}_N(1)$ and $\mathfrak{s}_N(\infty)$ have cardinality $2^{\aleph_0}$. Each of these sets contains a subset of size $2^{\aleph_0}$ consisting of mutually unitarily disjoint N-tuples.*

Proof. Let us first justify that each of the sets $\mathfrak{a}_1(n)$, $\mathfrak{f}_1$, $\mathfrak{s}_1(1)$ and $\mathfrak{s}_1(\infty)$ contains at least one bounded nonzero operator. For $\mathfrak{a}_1(n)$ this is clear, while for $\mathfrak{f}_1$, $\mathfrak{s}_1(1)$ and $\mathfrak{s}_1(\infty)$

it follows from the existence of factors of each type and the results on generators of such factors [38], [11] (the same was in fact observed by Ernest, cf. [9, Proposition 1.30]).

Now let T be a bounded nonzero operator of a suitable type (here by a *type* we mean an atom of type I^n, a fractal or a semiprime of type II^n). Notice that then $\{(rT,\dots,rT)\in \mathrm{CDD}_N\colon r\in(0,\infty)\}$ is a family of mutually unitarily disjoint N-tuples of the same type as T (indeed, if X is a bounded semiprime, then $\|t\odot\mathsf{X}\|=\|\mathsf{X}\|$ for each $t\in\mathbb{R}_+\setminus\{0\}$ and thus $r\boldsymbol{X}\perp_u s\boldsymbol{X}$ for distinct r and s). This proves the second claim of the proposition. To show the first one, it suffices to apply Lemma 3.4.1 and observe that if X is a semiprime, then $\operatorname{card}(\{\mathsf{Y}\in\mathfrak{s}_N\colon \mathsf{Y}\not\perp_u\mathsf{X}\})=\operatorname{card}(\{t\odot\mathsf{X}\colon t\in\mathbb{R}_+\setminus\{0\}\})=2^{\aleph_0}$. ∎

As an immediate consequence of Proposition 5.1.8 we obtain the following result, announced in Remark 3.4.9.

Corollary 5.1.9. *For $i=\{I,II,III\}$, $\dim(\mathsf{J}_i)=2^{\aleph_0}$.*

Denote by $\mathcal{J}^d$ the ideal generated by $\mathfrak{p}_N$ and let $\mathcal{J}^c=(\mathcal{J}^d)^{\perp}$. In other words, $\mathsf{A}\in\mathcal{J}^d$ if $\mathsf{A}=\bigoplus_{\mathsf{X}\in\mathfrak{p}_N}\beta_{\mathsf{X}}\odot\mathsf{X}$ for some family $\{\beta_{\mathsf{X}}\}_{\mathsf{X}\in\mathfrak{p}_N}\subset\mathrm{Card}$; and $\mathsf{A}\in\mathcal{J}^c$ if $\mathsf{P}\leqslant\mathsf{A}$ for no $\mathsf{P}\in\mathfrak{p}_N$. Similarly, whenever $\mathcal{A}$ is an ideal in $\mathcal{CDD}_N$, $\mathcal{A}^d$ and $\mathcal{A}^c$ denote, respectively, the ideals $\mathcal{A}\cap\mathcal{J}^d$ and $\mathcal{A}\cap\mathcal{J}^c$. The ideals $\mathcal{A}^d$ and $\mathcal{A}^c$ are called the *discrete* and *continuous* parts of $\mathcal{A}$. For example, we shall write $\mathcal{J}^c_{III}$, $\mathcal{J}^d_{I_1}$, etc. We also define the discrete and continuous parts of every member of $\mathcal{CDD}_N$ and each clopen set in Ω: $\mathsf{X}^d=\mathsf{E}(\mathsf{X}|\mathcal{J}^d)$ and $\mathsf{X}^c=\mathsf{E}(\mathsf{X}|\mathcal{J}^c)$ for $\mathsf{X}\in\mathcal{CDD}_N$; $\Omega^d=\operatorname{supp}_\Omega\mathcal{J}^d$ and $\Omega^c=\operatorname{supp}_\Omega\mathcal{J}^c$; and $E^d=E\cap\Omega^d$ and $E^c=E\cap\Omega^c$ for a clopen set $E\subset\Omega$. We underline that classically the terms *discrete* and *continuous* as kinds of operators mean *type I* and *without type I parts*, respectively (as used e.g. by Ernest—see [9, Definition 1.22]).

It may be easily checked that $\mathsf{A}\in\mathfrak{p}_N$ iff $\widehat{\mathsf{A}}$ has the form $\widehat{\mathsf{A}}=c\cdot j_{\{x\}}$ where either $c=1$ and $x\in\Omega_I$ or $c\in\mathbb{R}_+\setminus\{0\}$ and $x\in\Omega_{II}$, or $c=\aleph_0$ and $x\in\Omega_{III}$. Therefore Ω^d is the closure of the set of all isolated points of Ω. Consequently, we infer from Lemma 4.3.1 and Proposition 5.1.8 that

Proposition 5.1.10. *Each of the spaces $\Omega^d_{I_n}$ $(n=1,2,\dots,\infty)$, $\Omega^d_{II_1}$, $\Omega^d_{II_\infty}$ and Ω^d_{III} is the Čech–Stone compactification of the discrete space of cardinality $2^{\aleph_0}$.*

Proposition 5.1.10 and the next two results will be used later to classify ideals in $\mathcal{CDD}_N$ up to isomorphism (see Chapter 6.1 for definitions and details).

Proposition 5.1.11. *Every nonempty clopen set $E\subset\Omega^c$ with $c(E)=\aleph_0$ is homeomorphic to the Gelfand spectrum of $L^\infty([0,1])$.*

Proof. There is a (unique) nontrivial $\mathsf{A}\in\mathcal{CDD}_N$ such that $\mathsf{A}\leqslant^s\mathsf{J}$ and $\operatorname{supp}\widehat{\mathsf{A}}=E$. Since $E\subset\Omega^c$ and $c(E)=\aleph_0$,

$$\mathsf{A}\in\mathcal{J}^c\quad\text{and}\quad\dim(\mathsf{A})=\aleph_0.\tag{5.1.1}$$

Further, since $\mathcal{Z}(\mathcal{W}'(\boldsymbol{J}))$ is isomorphic to $\mathcal{C}(\Omega)$, $\mathcal{Z}(\mathcal{W}'(\boldsymbol{A}))$ is isomorphic to $\mathcal{C}(E)$ (because $\mathsf{A}\leqslant^s\mathsf{J}$). This means that E is the Gelfand spectrum of $\mathcal{Z}(\mathcal{W}'(\boldsymbol{A}))$. Now the assertion easily follows from (5.1.1) and Theorem III.1.22 of [35] (which asserts that every commutative von Neumann algebra acting on a separable Hilbert space which has no nonzero minimal projections is isomorphic to $L^\infty([0,1])$). ∎

Now for a clopen set $E \subset \Omega$ let $\kappa_d(E)$ be the size of the set of all isolated points of E and let $\kappa_c(E) = c_*(E^c)$. Additionally, let us denote by $D(\mathfrak{m})$ the discrete space of cardinality $\mathfrak{m}$ and by $\mathfrak{X}$ the Gelfand spectrum of $L^\infty([0,1])$. Recall that for every completely regular topological space X, βX stands for the Čech–Stone compactification of X.

THEOREM 5.1.12. *Any clopen set $E \subset \Omega$ is homeomorphic to the topological disjoint union of $\beta D(\kappa_d(E))$ and $\beta[D(\kappa_c(E)) \times \mathfrak{X}]$.*

Proof. By Lemma 4.4.7 and Proposition 5.1.11, E^c contains an open dense subset homeomorphic to $D(\kappa_c(E)) \times \mathfrak{X}$. Now it suffices to apply Lemma 4.3.1 to infer that E^d and E^c are homeomorphic to, respectively, $\beta D(\kappa_d(E))$ and $\beta[D(\kappa_c(E)) \times \mathfrak{X}]$. ∎

EXAMPLE 5.1.13. It is clear that $\mathfrak{a}_N(1)$ is the collection of all N-tuples acting on a one-dimensional Hilbert space. So, $\mathfrak{a}_N(1)$ may naturally be identified with $\mathbb{C}^N$.

One may also easily check that $\mathfrak{a}_N(2)$ consists of all N-tuples acting on a two-dimensional Hilbert space which are not of type I_1. In other words, if $\boldsymbol{A} = (A_1, \ldots, A_N)$ where $A_1, \ldots, A_N$ are 2 by 2 matrices, then $\mathsf{A} \in \mathfrak{a}_N(2)$ iff $A_j A_k^* \neq A_k^* A_j$ for some $j, k \in \{1, \ldots, N\}$.

For $n \geqslant 3$ the characterization of members of $\mathfrak{a}_N(n)$ is much more complicated.

5.2. Strongly unitarily disjoint families

Thanks to (BT3) (page 12) and suitable characterizations of the kinds of N-tuples appearing below, we see that for every $\mathsf{X} \in \mathcal{CDD}_N$ the following equivalences hold:

> X *is type I, I^n, II, II^1, II^∞, III, minimal, multiplicity free, a hereditary idempotent, semiminimal, a prime, an atom, a fractal or a semiprime iff so is $\mathfrak{b}(\mathsf{X})$.*

However, so far there was no need to use the $\mathfrak{b}$-transform, apart from Theorem 2.2.4. From now on, this transform will be intensively exploited and without it the presentation would be much more complicated.

We say that two classes $\mathcal{A}, \mathcal{B} \subset \mathrm{CDD}_N$ are *unitarily disjoint* iff $\mathcal{A} \perp_u \mathcal{B}$, that is, if $\boldsymbol{A} \perp_u \boldsymbol{B}$ for any $\boldsymbol{A} \in \mathcal{A}$ and $\boldsymbol{B} \in \mathcal{B}$. We begin with a classical

PROPOSITION 5.2.1. *Let $\boldsymbol{A}, \boldsymbol{B} \in \mathrm{CDD}_N$ be nontrivial N-tuples and let $\boldsymbol{X} = \boldsymbol{A} \oplus \boldsymbol{B}$. The following conditions are equivalent:*

(i) $\boldsymbol{A} \perp_u \boldsymbol{B}$,
(ii) $\mathcal{W}'(\boldsymbol{X}) = \{S \oplus T\colon S \in \mathcal{W}'(\boldsymbol{A}), T \in \mathcal{W}'(\boldsymbol{B})\} =: \mathcal{W}'(\boldsymbol{A}) \oplus \mathcal{W}'(\boldsymbol{B})$,
(iii) $I \oplus 0 \in \mathcal{W}''(\boldsymbol{X})$ (*where I is the identity operator on $\overline{\mathcal{D}}(\boldsymbol{A})$ and 0 is the zero operator on $\overline{\mathcal{D}}(\boldsymbol{B})$*).

Proof. Using $\mathfrak{b}$-transform and taking into account properties (BT3)–(BT5) (page 12), we may assume that $\boldsymbol{A}$ and $\boldsymbol{B}$ are bounded. In that case the equivalence of (i) and (ii) follows from Schur's lemma (cf. Theorem 1.5 in [9]; see also Corollary 1.8 there). Further, (ii) easily implies (iii), since $I \oplus 0$ commutes with every member of $\mathcal{W}'(\boldsymbol{A}) \oplus \mathcal{W}'(\boldsymbol{B})$. Finally, if (iii) is satisfied, then all elements of $\mathcal{W}'(\boldsymbol{X})$ commute with $I \oplus 0$ and thus are of the form $S \oplus T$. It is now easily verified that $S \oplus T$ commutes with each entry of $\boldsymbol{X}$ if and only if $S \in \mathcal{W}'(\boldsymbol{A})$ and $T \in \mathcal{W}'(\boldsymbol{B})$. ∎

We are mainly interested in the equivalence of (i) and (iii) in Proposition 5.2.1.

Adapting the concept due to Ernest [9] (see Definition 1.31 and §5.7.f there, especially notes on page 187), let us consider the free complex algebra

$$F = F(z_1, \ldots, z_N; w_1, \ldots, w_N)$$

in $2N$ noncommuting variables $z_1, \ldots, z_N, w_1, \ldots, w_N$. Each member of F may naturally be identified with a polynomial in $2N$ noncommuting variables. Let $*$ be a unique involution on the algebra F such that $z_j^* = w_j$ for $j = 1, \ldots, N$. We denote by $\mathcal{P}(N)$ the $*$-algebra obtained in this way and equip it with the norm given by

$$\|p(z_1, \ldots, z_N; z_1^*, \ldots, z_N^*)\| = \sup_{\|T_j\| \leqslant 1} \|p(T_1, \ldots, T_N; T_1^*, \ldots, T_N^*)\|$$

where the supremum is taken over N-tuples of contractions acting on a (common, arbitrary) Hilbert space. It follows from the definition that for every $p \in \mathcal{P}(N)$ and $\boldsymbol{X} \in \mathrm{CDD}_N$ with $\|\boldsymbol{X}\| \leqslant 1$, $\|p(\boldsymbol{X}, \boldsymbol{X}^*)\| \leqslant \|p\|$. The following is left as an easy exercise (use the separability of $\mathcal{P}(N)$).

LEMMA 5.2.2. *There is a sequence $\{\boldsymbol{M}_n\}_{n=1}^{\infty}$ of atoms in* CDD_N *acting on finite-dimensional Hilbert spaces such that $\|\boldsymbol{M}_n\| \leqslant 1$ $(n \geqslant 1)$ and for every $p \in \mathcal{P}(N)$,*

$$\|p\| = \sup_{n \geqslant 1} \|p(\boldsymbol{M}_n, \boldsymbol{M}_n^*)\|.$$

Making use of the above result and Kaplansky's density theorem [20] (cf. [18, Theorem 5.3.5], [35, Theorem II.4.8], [29, Theorem 1.9.1]) we shall now prove a result which is a starting point for our further investigations. By $\mathcal{P}_1(N)$ we denote the closed unit ball of $\mathcal{P}(N)$. Everywhere below, I and 0 denote the identity and zero operators on suitable Hilbert spaces. Recall that a net $(T_\sigma)_{\sigma \in \Sigma}$ of bounded operators acting on a Hilbert space $\mathcal{H}$ *converges $*$-strongly* to an operator $T \in \mathcal{B}(\mathcal{H})$ iff for any $x \in \mathcal{H}$, $T_\sigma x \to Tx$ $(\sigma \in \Sigma)$ and $T_\sigma^* x \to T^* x$ $(\sigma \in \Sigma)$. We shall denote this by $T_\sigma \xrightarrow{*s} T$.

PROPOSITION 5.2.3.

(I) *Let $\mathcal{A}$ and $\mathcal{B}$ be arbitrary subsets of* CDD_N. *The following conditions are equivalent:*

 (i) *$\mathcal{A}$ and $\mathcal{B}$ are unitarily disjoint,*

 (ii) *there is a net $(p_\sigma)_{\sigma \in \Sigma} \subset \mathcal{P}_1(N)$ such that for any $\boldsymbol{A} \in \mathcal{A}$ and $\boldsymbol{B} \in \mathcal{B}$, $p_\sigma(\mathfrak{b}(\boldsymbol{A}), \mathfrak{b}(\boldsymbol{A})^*) \xrightarrow{*s} I$ and $p_\sigma(\mathfrak{b}(\boldsymbol{B}), \mathfrak{b}(\boldsymbol{B})^*) \xrightarrow{*s} 0$.*

(II) *If $\boldsymbol{A}$ and $\boldsymbol{B}$ are two N-tuples acting in separable Hilbert spaces, then $\boldsymbol{A} \perp_u \boldsymbol{B}$ iff there is a sequence $(p_n)_{n=1}^{\infty} \subset \mathcal{P}_1(N)$ such that $p_n(\mathfrak{b}(\boldsymbol{A}), \mathfrak{b}(\boldsymbol{A})^*) \xrightarrow{*s} I$ and $p_n(\mathfrak{b}(\boldsymbol{B}), \mathfrak{b}(\boldsymbol{B})^*) \xrightarrow{*s} 0$.*

Proof. (I): By (BT5) (page 12), (i) follows from (ii). To prove the converse, assume $\mathcal{A} \perp_u \mathcal{B}$. Let $\boldsymbol{A} = \bigoplus\{\boldsymbol{X} : \boldsymbol{X} \in \mathcal{A}\}$ and $\boldsymbol{B} = \bigoplus\{\boldsymbol{Y} : \boldsymbol{Y} \in \mathcal{B}\}$. By (PR2) (page 12), $\boldsymbol{A} \perp_u \boldsymbol{B}$. Further, let $\{\boldsymbol{M}_n\}_{n=1}^{\infty}$ be as in Lemma 5.2.2. Let $\boldsymbol{M}$ be the direct sum of all $\boldsymbol{M}_n$'s which are unitarily disjoint from $\mathfrak{b}(\boldsymbol{B})$ ($\boldsymbol{M}$ is trivial provided $\boldsymbol{M}_n \leqslant \mathfrak{b}(\boldsymbol{B})$ for each n). Again by

(PR2) and (BT5), $\boldsymbol{M} \oplus \mathfrak{b}(\boldsymbol{A}) \perp_u \mathfrak{b}(\boldsymbol{B})$. Put $\boldsymbol{X} = (\boldsymbol{M} \oplus \mathfrak{b}(\boldsymbol{A})) \boxplus \mathfrak{b}(\boldsymbol{B})$, $\mathcal{H}_1 = \overline{\mathcal{D}}(\boldsymbol{M} \oplus \mathfrak{b}(\boldsymbol{A}))$ and $\mathcal{H}_2 = \overline{\mathcal{D}}(\mathfrak{b}(\boldsymbol{B}))$. It follows from our construction that for each $p \in \mathcal{P}(N)$,

$$\|p\| = \|p(\boldsymbol{X}, \boldsymbol{X}^*)\|. \tag{5.2.1}$$

Let $\mathcal{M} = \{p(\boldsymbol{X}, \boldsymbol{X}^*) \colon p \in \mathcal{P}(N)\}$. It is a unital selfadjoint subalgebra of $\mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$. We infer from von Neumann's double commutant theorem [24] ([18, Theorem 5.3.1], [35, Theorem II.3.9], [29, Theorem 1.20.3]) that the closure of $\mathcal{M}$ in the strong operator topology coincides with $\mathcal{W}''(\boldsymbol{X})$. Further, (5.2.1) implies that the closed unit ball in $\mathcal{M}$ coincides with $\{p(\boldsymbol{X}, \boldsymbol{X}^*) \colon p \in \mathcal{P}_1(N)\}$. An application of Proposition 5.2.1 shows that $I \oplus 0 \in \mathcal{W}''(\boldsymbol{X})$ where $I \in \mathcal{B}(\mathcal{H}_1)$ and $0 \in \mathcal{B}(\mathcal{H}_2)$. Finally, Kaplansky's density theorem asserts that there is a net $(p_\sigma)_{\sigma \in \Sigma} \in \mathcal{P}_1(N)$ such that $p_\sigma(\boldsymbol{X}, \boldsymbol{X}^*) \stackrel{*s}{\to} I \oplus 0$. Since every member of $\mathcal{A}$ and $\mathcal{B}$ is a reduced part of $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively, (ii) holds.

To prove (II), repeat the above argument and observe that in that case both $\mathcal{H}_1$ and $\mathcal{H}_2$ are separable and hence Kaplansky's density theorem asserts the existence of a suitable sequence, since the closed unit ball in $\mathcal{B}(\mathcal{H})$ for separable $\mathcal{H}$ is metrizable in the $*$-strong topology (see e.g. [9, Proposition 2.2]). ∎

Let us now introduce the following

Definition 5.2.4. Let $\mathcal{A}$ and $\mathcal{B}$ be arbitrary collections (sets or classes) of N-tuples. We say that $\mathcal{A}$ and $\mathcal{B}$ are *strongly unitarily disjoint*, in symbols $\mathcal{A} \perp_s \mathcal{B}$, if there is a sequence $(p_n)_{n=1}^\infty \subset \mathcal{P}_1(N)$ such that $p_n(\mathfrak{b}(\boldsymbol{A}), \mathfrak{b}(\boldsymbol{A})^*) \oplus p_n(\mathfrak{b}(\boldsymbol{B}), \mathfrak{b}(\boldsymbol{B})^*) \stackrel{*s}{\to} I \oplus 0$ for any $\boldsymbol{A} \in \mathcal{A}$ and $\boldsymbol{B} \in \mathcal{B}$. Two N-tuples $\boldsymbol{X}, \boldsymbol{Y} \in \mathrm{CDD}_N$ are strongly unitarily disjoint ($\boldsymbol{X} \perp_s \boldsymbol{Y}$) provided so are the sets $\{\boldsymbol{X}\}$ and $\{\boldsymbol{Y}\}$.

The reader should easily notice that for two sets $\mathcal{A}$ and $\mathcal{B}$ of N-tuples, $\mathcal{A} \perp_s \mathcal{B}$ iff $(\bigoplus \mathcal{A}) \perp_s (\bigoplus \mathcal{B})$. It is also clear that if $\mathcal{A}$ and $\mathcal{B}$ are strongly unitarily disjoint, then $\mathcal{A} \perp_u \mathcal{B}$.

Remark 5.2.5. Let $\boldsymbol{A}$ and $\boldsymbol{A}'$ be two unitarily equivalent N-tuples. Observe that then $p(\mathfrak{b}(\boldsymbol{A}), \mathfrak{b}(\boldsymbol{A})^*) \equiv p(\mathfrak{b}(\boldsymbol{A}'), \mathfrak{b}(\boldsymbol{A}')^*)$ for every $p \in \mathcal{P}(N)$. What is more, for every complex number λ and a net $(p_\sigma)_{\sigma \in \Sigma} \subset \mathcal{P}(N)$, $p_\sigma(\mathfrak{b}(\boldsymbol{A}), \mathfrak{b}(\boldsymbol{A})^*) \to \lambda I$ $*$-strongly (strongly, weakly, etc.) iff $p_\sigma(\mathfrak{b}(\boldsymbol{A}'), \mathfrak{b}(\boldsymbol{A}')^*) \to \lambda I$ in the same topology. This means that for any $\mathsf{A} \in \mathcal{CDD}_N$ and $p \in \mathcal{P}(N)$, $p(\mathfrak{b}(\mathsf{A}), \mathfrak{b}(\mathsf{A})^*)$ is a well defined member of $\mathcal{CDD}$ and

$$p_\sigma(\mathfrak{b}(\mathsf{A}), \mathfrak{b}(\mathsf{A})^*) \stackrel{*s}{\to} \lambda I \tag{5.2.2}$$

is well understood. (We do not write in (5.2.2) 'I' instead of 'I' because 'I' represents here the identity operator on a Hilbert space of (arbitrary) suitable dimension. The usage of I may lead to misunderstandings. In fact, (5.2.2) expresses only a property of the net $\{p_\sigma(\mathfrak{b}(\mathsf{A}), \mathfrak{b}(\mathsf{A})^*)\}_{\sigma \in \Sigma}$.) Consequently, in the same way as in Definition 5.2.4 we may define strongly unitarily disjoint subclasses of $\mathcal{CDD}_N$. We use this concept in the next chapters.

Surely the main problem concerning strong unitary disjointness is when two unitarily disjoint families of N-tuples acting in separable Hilbert spaces are strongly unitarily disjoint. We will not answer this question. However, the reader should remember that strong unitary disjointness and unitary disjointness are not equivalent even for families of N-tuples acting on a one-dimensional Hilbert space. Indeed, such N-tuples may be naturally identified with points of $\mathbb{C}^N$. If $p_1, p_2, \dots$ is an arbitrary sequence of members of $\mathcal{P}(N)$ and $\lambda \in \mathbb{C}$, the set $\{z \in \mathbb{C}^N : p_n(\mathfrak{b}(z), \mathfrak{b}(z)^*) \to \lambda\}$ is $\mathcal{F}_{\sigma\delta}$ in $\mathbb{C}^N$. Thus, if $A \subset \mathbb{C}^N$ is not $\mathcal{F}_{\sigma\delta}$, then $A \perp_u \mathbb{C}^N \setminus A$ but A and $\mathbb{C}^N \setminus A$ are not strongly unitarily disjoint.

The next result is a consequence of Proposition 5.2.3. We omit its proof.

Proposition 5.2.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be two* ***countable*** *families of N-tuples acting in separable Hilbert spaces. Then $\mathcal{A} \perp_u \mathcal{B}$ if and only if $\mathcal{A} \perp_s \mathcal{B}$.*

We shall also need the following simple

Lemma 5.2.7. *Let $\boldsymbol{A}$ be a bounded N-tuple acting on a separable Hilbert space such that $\|\boldsymbol{A}\| \leqslant 1$. For every $T \in \mathcal{W}(\boldsymbol{A})$ with $\|T\| \leqslant 1$ there is a sequence $(p_n)_{n=1}^\infty \subset \mathcal{P}_1(N)$ such that $p_n(\boldsymbol{A}, \boldsymbol{A}^*) \stackrel{*s}{\to} T$.*

Proof. We mimic the proof of Proposition 5.2.3. As there, there is a sequence $\{\boldsymbol{M}_n\}_{n=1}^\infty$ of N-tuples of contraction matrices such that $\boldsymbol{M}_n \perp_u \boldsymbol{A}$ for each n and $\|p\| = \|p(\boldsymbol{M} \oplus \boldsymbol{A}, \boldsymbol{M}^* \oplus \boldsymbol{A}^*)\|$ for every $p \in \mathcal{P}(N)$ with $\boldsymbol{M} = \bigoplus_{n=1}^\infty \boldsymbol{M}_n$. Since $\boldsymbol{M} \perp_u \boldsymbol{A}$, $\mathcal{W}'(\boldsymbol{M} \oplus \boldsymbol{A}) = \mathcal{W}'(\boldsymbol{M}) \oplus \mathcal{W}'(\boldsymbol{A})$ (by Proposition 5.2.1). Consequently, $\mathcal{W}''(\boldsymbol{M}) \oplus \mathcal{W}''(\boldsymbol{A}) \subset \mathcal{W}''(\boldsymbol{M} \oplus \boldsymbol{A})$ and thus $0 \oplus T \in \mathcal{W}(\boldsymbol{M} \oplus \boldsymbol{A})$. Finally, since $\boldsymbol{M} \oplus \boldsymbol{A}$ acts on a separable Hilbert space, Kaplansky's density theorem finishes the proof (see the proof of Proposition 5.2.3). ∎

Remark 5.2.8. Let $\mathfrak{P} = \{p_\sigma\}_{\sigma\in\Sigma} \subset \mathcal{P}_1(N)$ be any net and let $\lambda \in \mathbb{C}$. Denote by $\mathcal{I}_{\mathfrak{P}}(\lambda)$ the class of all $\mathsf{X} \in \mathcal{CDD}_N$ for which

$$p_\sigma(\mathfrak{b}(\mathsf{X}), \mathfrak{b}(\mathsf{X})^*) \stackrel{*s}{\to} \lambda I.$$

One easily checks that $\mathcal{I}_{\mathfrak{P}}(\lambda)$ is an ideal and $\mathcal{I}_{\mathfrak{P}}(\lambda) \perp_u \mathcal{I}_{\mathfrak{P}}(\lambda')$ whenever $\lambda' \neq \lambda$. For every subclass $\mathcal{A}$ of $\mathcal{CDD}_N$ let $J(\mathcal{A})$ denote the smallest ideal in $\mathcal{CDD}_N$ which contains $\mathcal{A}$. The above shows that for any two subclasses $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{CDD}_N$, $\mathcal{A} \perp_s \mathcal{B}$ iff $J(\mathcal{A}) \perp_s J(\mathcal{B})$, iff $\mathsf{J}(J(\mathcal{A})) \perp_s \mathsf{J}(J(\mathcal{B}))$. In particular, strong unitary disjointness of sets or classes may always be reduced to strong unitary disjointness of suitable N-tuples X and Y such that $\mathsf{X} \leqslant^s \mathsf{J}$ and $\mathsf{Y} \leqslant^s \mathsf{J}$.

5.3. Measure-theoretic preliminaries

Our next objective is a prime decomposition of N-tuples (Theorem 5.6.14). Essentially this will be based on the same idea (that is, on central decompositions of von Neumann algebras) as Ernest's central decomposition of a bounded operator ([9, Chapter 3]). The difference between his and our approaches (apart from greater generality) is the following. Ernest has focused on a single operator T and studied its (nonscalar) spectrum $\widehat{T}$ and quasi-spectrum $\widetilde{T}$. Central decomposition of the operator T 'takes place' in $\widetilde{T}$. Further the author compares operators (and their central decompositions) which have the same quasi-spectra. It seems to us that Ernest's work was inspired by the spectral theorem for a normal operator. Our work is inspired by the prime decomposition of natural numbers. Our interpretation is therefore in a more algebraic fashion. Also comparing Ernest's work and ours, we may say that his approach is local, while ours is global.

The road to the Prime Decomposition Theorem is long because of measure-theoretic technicalities. First we shall define a Borel structure on the set $\mathcal{SEP}_N \subset \mathcal{CDD}_N$ of all nontrivial N-tuples whose representatives act in separable Hilbert spaces (this is done in this chapter), next we shall generalize the notion of a direct integral to N-tuples (Chapter 5.4) to define 'continuous' direct sums (Chapter 5.5) among which we shall distinguish regular ones (which require unitary disjointness) and finally we shall show that every member of $\mathcal{CDD}_N$ admits a unique (in a sense) regular prime decomposition (Chapter 5.6).

The concept of direct integrals (of Hilbert spaces, operators, von Neumann algebras, etc.) is essentially due to von Neumann and is widely discussed in many classical textbooks on von Neumann algebras. Here we shall focus on main ideas and many proofs will be omitted. The reader interested in details should consult e.g. Chapters 2 and 3 of [9]; [6, 7]; [19, Chapter 14]; §IV.8, §V.6 and Appendix in [35]; [29, Chapter 3]; [30, Chapter I]; or the original paper by von Neumann [25]. It is also assumed that the reader is familiar with basics of measure theory and of reduction theory of von Neumann algebras.

Measurable sets (i.e. elements of a given σ-algebra) will also be called *Borel*. Everywhere below by a *measurable* or *Borel* function from a measurable space $(X, \mathfrak{M})$ into a measurable space $(Y, \mathfrak{N})$ we mean any function $f\colon X \to Y$ such that $f^{-1}(B) \in \mathfrak{M}$ for any $B \in \mathfrak{N}$. The function f is a *Borel isomorphism* if f is a bijection and f and f^{-1} are measurable. For two measures μ and ν defined on a common σ-algebra $\mathfrak{M}$ we shall write $\mu \ll \nu$ iff μ is absolutely continuous with respect to ν, and we call μ and ν (mutually) *singular* iff $\mu \perp \nu$, i.e. μ and ν are concentrated on disjoint measurable sets. If $A \in \mathfrak{M}$, $\mu|_A$ denotes the measure on $\mathfrak{M}$ given by $\mu|_A(B) = \mu(A \cap B)$. For a topological space X, $\mathfrak{B}(X)$ stands for the smallest σ-algebra containing all open subsets of X. Following Takesaki [35, Appendix], we call a measurable space $(X, \mathfrak{M})$ a *standard Borel space* iff $(X, \mathfrak{M})$ is Borel isomorphic to $(Y, \mathfrak{B}(Y))$ where Y is a Borel subset of a separable complete metric space. Equivalently, $(X, \mathfrak{M})$ is standard iff $(X, \mathfrak{M})$ is Borel isomorphic to $(A, \mathfrak{B}(A))$ where A is a countable (finite or not) subset of $[0, 1]$ or $A = [0, 1]$ (cf. [35, Corollary A.11]). If $(X, \mathfrak{M})$ and $(Y, \mathfrak{N})$ are standard Borel spaces and $f\colon X \to Y$ is measurable, then $(X \times Y, \mathfrak{M} \otimes \mathfrak{N})$ is a standard Borel space as well and $\Gamma(f) \in \mathfrak{M} \otimes \mathfrak{N}$ where

$$\Gamma(f) = \{(x, f(x))\colon x \in X\}$$

is the graph of f. The space $(X, \mathfrak{M})$ is *Souslin–Borel* iff it is the image of a standard Borel space under a Borel function and X is *countably separated* (this means that there are sets $E_1, E_2, \ldots \in \mathfrak{M}$ such that for any two distinct points x and y of X there is n with $\operatorname{card}(\{x, y\} \cap E_n) = 1$). In what follows, we shall often identify $I_{\aleph_0}$ with $[0, \infty]$.

Let $(X, \mathfrak{M}, \mu)$ be a measure space (μ need not be σ-finite or complete). We denote by $\mathcal{N}(\mu)$ the null σ-ideal in $\mathfrak{M}$ induced by μ, that is,

$$\mathcal{N}(\mu) = \{A \in \mathfrak{M}\colon \mu(A) = 0\}.$$

$(X, \mathfrak{M}, \mu)$ is said to be a *standard measure* space (or, equivalently, μ is *standard*) iff μ is nonzero σ-finite and $X \setminus Z$ is a standard Borel space for some $Z \in \mathcal{N}(\mu)$. By [35, Corollary A.14], every σ-finite measure on a Souslin–Borel space is standard.

For $n = 1, 2, \dots$ let $\mathcal{H}_n$ be a fixed Hilbert space of dimension n and let $\mathcal{H}_\infty$ be a fixed separable infinite-dimensional Hilbert space (these spaces are fixed for this and the next two chapters). Further, let $\mathcal{H}$ denote one of the spaces $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_\infty$. The norm and the weak topologies of $\mathcal{H}$ induce the same σ-algebra on $\mathcal{H}$ which is for us the default Borel structure of $\mathcal{H}$. Similarly, the $*$-strong, strong and weak operator topologies induce the same Borel structures on $\mathcal{B}(\mathcal{H})$. In other words, the σ-algebra $\mathfrak{W}_\mathcal{H}$ generated by all open sets with respect to any of these topologies is independent of the topology we choose. Moreover, $(\mathcal{B}(\mathcal{H}), \mathfrak{W}_\mathcal{H})$ is a standard Borel space, which means that $(\mathcal{B}(\mathcal{H}), \mathfrak{W}_\mathcal{H})$ is isomorphic as a measurable space to $([0, 1], \mathfrak{B}([0, 1]))$. Addition and multiplication are measurable as functions from $(\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}), \mathfrak{W}_\mathcal{H} \otimes \mathfrak{W}_\mathcal{H})$ into $(\mathcal{B}(\mathcal{H}), \mathfrak{W}_\mathcal{H})$ and the functions $T \mapsto T^*$, $T \mapsto |T|$, $T \mapsto Q_T$ and $T \mapsto T^{-1}$ are measurable as well (the last function is defined on the set of all invertible operators, which is measurable).

The following result will enable us to define a Borel structure on the set $\mathrm{CDD}(\mathcal{H})$.

LEMMA 5.3.1. *The open unit ball B of $\mathcal{B}(\mathcal{H})$ and the set $\mathfrak{b}(\mathcal{H})$ of all $T \in \mathcal{B}(\mathcal{H})$ such that $\|Tx\| < \|x\|$ for any nonzero $x \in \mathcal{H}$ are measurable. The $\mathfrak{b}$-transform is an isomorphism between the measurable spaces $\mathcal{B}(\mathcal{H})$ and B.*

Proof. We shall only explain why $\mathfrak{b}(\mathcal{H})$ is measurable. Notice that $T \in \mathfrak{b}(\mathcal{H})$ iff $\|T\| \leqslant 1$ and $\mathcal{N}(I - T^*T)$ is trivial. Now if P_T denotes the orthogonal projection onto $\mathcal{N}(I - T^*T)$, then the function $T \mapsto P_T$ is measurable, by [9, Proposition 2.4], and we are done. ∎

Since the $\mathfrak{b}$-transform establishes a one-to-one correspondence between members of $\mathrm{CDD}(\mathcal{H})$ and $\mathfrak{b}(\mathcal{H})$, we may introduce

DEFINITION 5.3.2. The *Borel structure* of $\mathrm{CDD}(\mathcal{H})$ is the unique Borel structure which makes the $\mathfrak{b}$-transform an isomorphism. In other words, a set $F \subset \mathrm{CDD}(\mathcal{H})$ is measurable, in symbols $F \in \mathfrak{B}(\mathrm{CDD}(\mathcal{H}))$, iff $\{\mathfrak{b}(X)\colon\ X \in F\} \in \mathfrak{W}_\mathcal{H}$.

Lemma 5.3.1 implies that $\mathrm{CDD}(\mathcal{H})$ is a standard Borel space, that $\mathcal{B}(\mathcal{H})$ is a measurable subset of $\mathrm{CDD}(\mathcal{H})$ and that the original Borel structure of $\mathcal{B}(\mathcal{H})$ coincides with the one inherited from the Borel structure of $\mathrm{CDD}(\mathcal{H})$.

Recall that $\mathrm{CDD}_N(\mathcal{H}) = \mathrm{CDD}(\mathcal{H})^N$. We equip $\mathrm{CDD}_N(\mathcal{H})$ with the product σ-algebra $\mathfrak{B}(\mathrm{CDD}_N(\mathcal{H})) = \mathfrak{B}(\mathrm{CDD}(\mathcal{H})) \otimes \dots \otimes \mathfrak{B}(\mathrm{CDD}(\mathcal{H}))$. Observe that $\mathrm{CDD}_N(\mathcal{H})$ is a standard Borel space and the $\mathfrak{b}$-transform is an isomorphism of the measurable space $\mathrm{CDD}_N(\mathcal{H})$ onto a measurable set $\mathfrak{b}(\mathcal{H})^N$. Moreover, it follows from suitable properties of the $\mathfrak{b}$-transform that each of the functions $\boldsymbol{X} \mapsto \boldsymbol{X}^*$, $\boldsymbol{X} \mapsto |\boldsymbol{X}|$ and $\boldsymbol{X} \mapsto \boldsymbol{Q}_{\boldsymbol{X}}$ (from $\mathrm{CDD}_N(\mathcal{H})$ into itself) is measurable.

Now let $\mathcal{SEP}_N$ be the **set** of all $\mathsf{A} \in \mathcal{CDD}_N$ such that $0 < \dim(\mathsf{A}) \leqslant \aleph_0$. Observe that the function $\Phi\colon \bigcup_{n=1}^{n=\infty} \mathrm{CDD}_N(\mathcal{H}_n) \ni \boldsymbol{X} \mapsto \mathsf{X} \in \mathcal{SEP}_N$ is a surjection. We define a σ-algebra $\mathfrak{B}_N$ on $\mathcal{SEP}_N$ by the rule: $\mathcal{F} \in \mathfrak{B}_N$ iff for every $n \in \{1, 2, \dots, \infty\}$, $\Phi^{-1}(\mathcal{F}) \cap \mathrm{CDD}_N(\mathcal{H}_n) \in \mathfrak{B}(\mathrm{CDD}_N(\mathcal{H}_n))$. It is obvious that the definition of $\mathfrak{B}_N$ is independent of the choice of $\mathcal{H}_n$'s. For every $\mathcal{A} \in \mathfrak{B}_N$ we shall denote by $\mathfrak{B}(\mathcal{A})$ the σ-algebra of all sets $\mathcal{B} \in \mathfrak{B}_N$ contained in $\mathcal{A}$.

As shown by Ernest (see [9, Corollary 2.33]), $\mathcal{SEP}_N$ is not countably separated. This makes the investigation of the Borel structure of $\mathcal{SEP}_N$ difficult. The rest of this chapter is devoted to establishing measurability of some (important for us) sets and functions.

For $n = 1, 2, \ldots, \infty$ let $\mathcal{SEP}_N(n)$ consist of all $\mathsf{A} \in \mathcal{SEP}_N$ with $\dim(\mathsf{A}) = n$. It follows from the definition of $\mathfrak{B}_N$ that $\mathcal{SEP}_N(n) \in \mathfrak{B}_N$ for every n. When n is finite, much more can be said (cf. Proposition 2.46 and Corollary 2.47 in [9]):

PROPOSITION 5.3.3. *For every finite n, $\mathcal{SEP}_N(n)$ is a standard Borel space and there are a Borel set $S_n \subset \mathrm{CDD}_N(\mathcal{H}_n)$ and a Borel isomorphism $\chi_n \colon \mathcal{SEP}_N(n) \ni \mathsf{A} \mapsto \boldsymbol{T}_{\mathsf{A}} \in S_n$ such that $\boldsymbol{T}_{\mathsf{A}}$ is a representative of A for every A.*

Proof. It is clear that $\mathcal{CDD}_N(\mathcal{H}_n)$ coincides with the space M_n^N of all N-tuples of $n \times n$ matrices. Let $\pi \colon M_n^N \to \mathcal{SEP}_N(n)$ be the quotient map (i.e. $\pi(\boldsymbol{X}) = \mathsf{X}$). Equip $\mathcal{SEP}_N(n)$ with the quotient topology (induced by π). Since the unitary group of $n \times n$ matrices is compact, $\mathcal{SEP}_N(n)$ is locally compact and π is a proper continuous mapping. Moreover, $\mathcal{SEP}_N(n)$ is separable and metrizable. It is now clear that the σ-algebra generated by all open sets coincides with the one inherited from $\mathfrak{B}_N$. This shows that $\mathcal{SEP}_N(n)$ is a standard Borel space. The existence of S_n and χ_n may easily be deduced e.g. from [22, Corollary XIV.2.1] applied to the partition $\{\pi^{-1}(\{\mathsf{X}\}) \colon \mathsf{X} \in \mathcal{SEP}_N(n)\}$, or from [22, Corollary XIV.1.1] (see also [4]) applied to the multifunction $\mathcal{SEP}_N(n) \ni \mathsf{X} \mapsto \pi^{-1}(\{\mathsf{X}\}) \subset M_n$. ■

Now we are mainly interested in the Borel structure of $\mathcal{SEP}_N(\infty)$. However, in some arguments we shall need to work also with N-tuples acting on finite-dimensional Hilbert spaces and therefore below we explore $\mathrm{CDD}_N(\mathcal{H}_\infty)$ as well as $\mathrm{CDD}_N(\mathcal{H}_n)$ with finite n. Since our main interest is primes and semiprimes, we may restrict our considerations to factor N-tuples defined below. Similar results to those presented below can be found in Chapter 2 of [9].

As before, $\mathcal{H}$ denotes one of the spaces $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_\infty$. The functions $\mathrm{CDD}_N(\mathcal{H}) \ni \boldsymbol{X} \mapsto \mathcal{W}''(\boldsymbol{X}) \in \mathscr{W}(\mathcal{H})$ and $\mathrm{CDD}_N(\mathcal{H}) \ni \boldsymbol{X} \mapsto \mathcal{W}'(\boldsymbol{X}) \in \mathscr{W}(\mathcal{H})$ are measurable when $\mathscr{W}(\mathcal{H})$ denotes the collection of all von Neumann subalgebras of $\mathcal{B}(\mathcal{H})$ and is equipped with the Effros Borel structure [6, 7] (cf. [9, page 54] combined with Theorem IV.8.4 and Corollary IV.8.6 in [35]). Consequently, the following sets are measurable subsets of $\mathrm{CDD}_N(\mathcal{H})$ (compare with notes on page 55 of [9]; [35, Theorem V.6.6] and [26]):

- the set of all atoms, $\mathfrak{a}_N(\mathcal{H}) = \{\boldsymbol{A} \in \mathrm{CDD}_N(\mathcal{H}) \colon \mathsf{A} \in \mathfrak{a}_N\}$,
- the set of all fractals, $\mathfrak{f}_N(\mathcal{H}) = \{\boldsymbol{A} \in \mathrm{CDD}_N(\mathcal{H}) \colon \mathsf{A} \in \mathfrak{f}_N\}$,
- the set of all semiprimes, $\mathfrak{s}_N(\mathcal{H}) = \{\boldsymbol{A} \in \mathrm{CDD}_N(\mathcal{H}) \colon \mathsf{A} \in \mathfrak{s}_N\}$,
- the set of all factor N-tuples,

$$\mathfrak{F}_N(\mathcal{H}) = \{\boldsymbol{A} \in \mathrm{CDD}_N(\mathcal{H}) \colon \mathcal{W}''(\boldsymbol{A}) \text{ is a factor}\},$$

- the sets of all factor N-tuples of type I, I^n, II, II^1, II^∞ and III.

(The above properties imply that $\mathfrak{a}_N, \mathfrak{f}_N, \mathfrak{s}_N$ as well as

$$\mathfrak{F}_N := \{\mathsf{F} \in \mathcal{SEP}_N \colon \mathcal{W}''(\boldsymbol{F}) \text{ is a factor}\}$$

are members of $\mathfrak{B}_N$. When $\mathcal{H}$ is finite-dimensional, $\mathfrak{s}_N(\mathcal{H})$ and $\mathfrak{f}_N(\mathcal{H})$ are of course empty.) We infer from Proposition 5.1.2 that for every $\boldsymbol{F} \in \mathfrak{F}_N(\mathcal{H}) \setminus (\mathfrak{a}_N(\mathcal{H}) \cup \mathfrak{f}_N(\mathcal{H}) \cup \mathfrak{s}_N(\mathcal{H}))$ either there exist a unique $n \in \{2, 3, \ldots, \aleph_0\}$ and a unique $\mathsf{A} \in \mathfrak{a}_N$ such that $\mathsf{F} = n \odot \mathsf{A}$ or there is (nonunique) $\mathsf{A} \in \mathfrak{s}_N$ for which $\mathsf{F} = \aleph_0 \odot \mathsf{A}$. Everywhere below, n and m represent positive integers or ∞.

The following result appears in [9, Corollary 2.11]. Below we give a shorter proof.

LEMMA 5.3.4. *The set*

$$\mathfrak{D}_N(n,m) = \{(\boldsymbol{A},\boldsymbol{B}) \in \mathrm{CDD}_N(\mathcal{H}_n) \times \mathrm{CDD}_N(\mathcal{H}_m) \colon \boldsymbol{A} \perp_u \boldsymbol{B}\}$$

is measurable (i.e. $\mathfrak{D}_N(n,m) \in \mathfrak{B}(\mathrm{CDD}_N(\mathcal{H}_n)) \otimes \mathfrak{B}(\mathrm{CDD}_N(\mathcal{H}_m))$*).*

Proof. Let $\mathcal{K} = \mathcal{H}_\infty$ and let $U_j \colon \aleph_0 \odot \mathcal{H}_j \to \mathcal{K}$ be unitary ($\aleph_0 \odot \mathcal{H}_j$ stands for the Hilbert space in which N-tuples of the form $\aleph_0 \odot \boldsymbol{X}$ with $\boldsymbol{X} \in \mathrm{CDD}_N(\mathcal{H}_j)$ act). Let $\mathcal{Q}$ be the set of all $p \in \mathcal{P}$ such that $\|p\| \leqslant 2$ and all coefficients of p belong to $\mathbb{Q} + i\mathbb{Q}$. It may be deduced from Proposition 5.2.1 and Lemma 5.2.7 that $\boldsymbol{A} \perp_u \boldsymbol{B}$ with $\boldsymbol{A} \in \mathrm{CDD}_N(\mathcal{H}_n)$ and $\boldsymbol{B} \in \mathrm{CDD}_N(\mathcal{H}_m)$ iff there is a sequence $(p_k)_{k=1}^\infty \subset \mathcal{Q}$ such that $U_n p_k(\mathfrak{b}(\aleph_0 \odot \boldsymbol{A}), \mathfrak{b}(\aleph_0 \odot \boldsymbol{A})^*)U_n^{-1} \to I$ and $U_m p_k(\mathfrak{b}(\aleph_0 \odot \boldsymbol{B}), \mathfrak{b}(\aleph_0 \odot \boldsymbol{B})^*)U_m^{-1} \to 0$ strongly as $k \to \infty$. Now if d is a metric on $D = \{T \in \mathcal{B}(\mathcal{K}) \colon \|T\| \leqslant 2\}$ which induces the strong operator topology of D, then for every $p \in \mathcal{Q}$ the function $\psi_p^j \colon \mathrm{CDD}_N(\mathcal{H}_j) \ni \boldsymbol{X} \mapsto U_j p(\mathfrak{b}(\aleph_0 \odot \boldsymbol{X}), \mathfrak{b}(\aleph_0 \odot \boldsymbol{X})^*)U_j^{-1} \in D$ is measurable and thus so is $\theta_p \colon \mathrm{CDD}_N(\mathcal{H}_n) \times \mathrm{CDD}_N(\mathcal{H}_m) \ni (\boldsymbol{X},\boldsymbol{Y}) \mapsto d(\psi_p^j(\boldsymbol{X}), I) + d(\psi_p^m(\boldsymbol{Y}), 0) \in \mathbb{R}_+$. Finally, since $\mathcal{Q}$ is countable, also the function $u \colon \mathrm{CDD}_N(\mathcal{H}_n) \times \mathrm{CDD}_N(\mathcal{H}_m) \ni (\boldsymbol{X},\boldsymbol{Y}) \mapsto \inf_{p\in\mathcal{Q}} \theta_p(\boldsymbol{X},\boldsymbol{Y}) \in \mathbb{R}_+$ is measurable. The observation that $\mathfrak{D}_N(n,m) = u^{-1}(\{0\})$ finishes the proof. ∎

THEOREM 5.3.5. *The sets*

$$\Delta_N(n,m) = \{(\boldsymbol{A},\boldsymbol{B}) \in \mathfrak{F}_N(\mathcal{H}_n) \times \mathfrak{F}_N(\mathcal{H}_m) \colon \boldsymbol{A} \equiv \boldsymbol{B}\}$$

and $\trianglelefteq_N(n,m) = \{(\boldsymbol{A},\boldsymbol{B}) \in \mathfrak{F}_N(\mathcal{H}_n) \times \mathfrak{F}_N(\mathcal{H}_m) \colon \boldsymbol{A} \leqslant \boldsymbol{B}\}$ *are measurable.*

Proof. First of all, note that for $(\boldsymbol{A},\boldsymbol{B}) \in \mathfrak{F}_N(\mathcal{H}_n) \times \mathfrak{F}_N(\mathcal{H}_m)$ we have: $\boldsymbol{A} \not\perp_u \boldsymbol{B} \Leftrightarrow \aleph_0 \odot \mathsf{A} = \aleph_0 \odot \mathsf{B}$. So, Lemma 5.3.4 implies that the set $C(n,m) = \{(\boldsymbol{A},\boldsymbol{B}) \in \mathfrak{F}_N(\mathcal{H}_n) \times \mathfrak{F}_N(\mathcal{H}_m) \colon \aleph_0 \odot \mathsf{A} = \aleph_0 \odot \mathsf{B}\}$ is measurable. Put $L_N(n,m) = \{(\boldsymbol{A},\boldsymbol{B}) \in \mathfrak{F}_N(\mathcal{H}_n) \times \mathfrak{F}_N(\mathcal{H}_m) \colon \boldsymbol{A} \lneqq \boldsymbol{B}\}$ and $R_N(n,m) = \{(\boldsymbol{A},\boldsymbol{B}) \colon (\boldsymbol{B},\boldsymbol{A}) \in L_N(n,m)\}$. Observe that

$$\trianglelefteq_N(n,m) = \Delta_N(n,m) \cup L_N(n,m),$$

$C(n,m) = \Delta_N(n,m) \cup L_N(n,m) \cup R_N(n,m)$ and the sets $\Delta_N(n,m)$, $L_N(n,m)$ and $R_N(n,m)$ are pairwise disjoint. Since $C(n,m)$ is a standard Borel space, it therefore suffices to show that each of these last sets is Souslin (cf. [35, Theorem A.3]). We see that $\Delta(n,m) = \emptyset$ if $n \neq m$ and $\Delta_N(n,n) = \{(\boldsymbol{A}, U\boldsymbol{A}U^{-1}) \colon U \in \mathcal{U}(\mathcal{H}_n),\ \boldsymbol{A} \in \mathfrak{F}_N(\mathcal{H}_n)\}$ (where $U(A_1,\ldots,A_N)U^{-1} = (UA_1U^{-1},\ldots,UA_NU^{-1})$) is the image of a standard Borel space $\mathcal{U}(\mathcal{H}_n) \times \mathfrak{F}_N(\mathcal{H}_n)$ under a Borel function and thus $\Delta_N(n,n)$ is Souslin. Finally, the set $\mathfrak{F}_N^{\mathrm{fin}}(\mathcal{H}_n)$ of all N-tuples $\boldsymbol{X} \in \mathfrak{F}_N(\mathcal{H}_n)$ such that $\mathcal{W}'(\boldsymbol{X})$ is finite is Borel and therefore $L_N(n,m)$ is Souslin, since $L(n,m) = \emptyset$ for $n > m$ or $n = m < \infty$; for $n < m$:

$$\begin{aligned} L_N(n,m) = \{(\boldsymbol{A}, U(\boldsymbol{A} \oplus \boldsymbol{G})U^{-1}) \colon U \in \mathcal{U}(\mathcal{H}_n \oplus \mathcal{H}_{m-n}, \mathcal{H}_m),& \\ \boldsymbol{A} \in \mathfrak{F}_N(\mathcal{H}_n),\ (\boldsymbol{A},\boldsymbol{G}) \in C(n,m-n)\};& \end{aligned}$$

and

$$L_N(\infty,\infty) = \bigcup_{k=1}^{k=\infty} \{(\boldsymbol{A}, U(\boldsymbol{A}\oplus\boldsymbol{G})U^{-1})\colon U \in \mathcal{U}(\mathcal{H}_\infty\oplus\mathcal{H}_k,\mathcal{H}_\infty),$$
$$\boldsymbol{A}\in\mathfrak{F}_N^{\mathrm{fin}}(\mathcal{H}_\infty),\ (\boldsymbol{A},\boldsymbol{G})\in C(\infty,k)\}.$$

The observation that $R_N(n,m)$ is the Borel image of $L_N(n,m)$ finishes the proof. ∎

Corollary 5.3.6. *Let $\mathcal{F}$ be a Borel subset of $\mathfrak{F}_N(\mathcal{H}_n)$ such that the function $\Phi\colon \mathcal{F}\ni \boldsymbol{X}\mapsto \mathsf{X}\in\mathcal{CDD}_N$ is one-to-one. Then $\widehat{\mathcal{F}} = \{\boldsymbol{Y}\in \mathrm{CDD}_N(\mathcal{H}_n)\colon \boldsymbol{Y}\equiv\boldsymbol{X}$ for some $\boldsymbol{X}\in\mathcal{F}\}$ is a Borel subset of $\mathrm{CDD}_N(\mathcal{H}_n)$ and $\mathfrak{F} = \{\mathsf{X}\colon \boldsymbol{X}\in\mathcal{F}\}\subset\mathcal{CDD}_N$ is measurable and it is a standard Borel space.*

Proof. By Theorem 5.3.5, the set $\mathcal{D} = \Delta_N(n,n)\cap(\mathrm{CDD}_N(\mathcal{H}_n)\times\mathcal{F})$ is Borel. What is more, it follows from the assumptions that the function $\mathcal{D}\ni(\boldsymbol{A},\boldsymbol{B})\mapsto\boldsymbol{A}\in\widehat{\mathcal{F}}$ is a bijection. It is also Borel and thus $\widehat{\mathcal{F}}\in\mathfrak{B}(\mathrm{CDD}_N(\mathcal{H}_n))$, by [35, Corollary A.7]. Since $\{\boldsymbol{X}\in\mathrm{CDD}_N(\mathcal{H}_n)\colon \mathsf{X}\in\mathfrak{F}\} = \widehat{\mathcal{F}}$, we find that $\mathfrak{F}\in\mathfrak{B}_N$.

It is clear that Φ is a Borel bijection of $\mathcal{F}$ onto $\mathfrak{F}$. However, if $\mathcal{B}$ is a Borel subset of $\mathcal{F}$, then the above argument shows that $\{\mathsf{X}\colon \boldsymbol{X}\in\mathcal{B}\}\in\mathfrak{B}_N$ and hence Φ is a Borel isomorphism, and the assertion follows. ∎

A variation of Theorem 5.3.5 is contained in

Lemma 5.3.7. *For each $t\in(0,\infty)$ the sets $\Delta_N^t = \{(\boldsymbol{A},\boldsymbol{B})\in\mathfrak{s}_N(\mathcal{H}_\infty)\times\mathfrak{s}_N(\mathcal{H}_\infty)\colon\ \mathsf{A} = t\odot\mathsf{B}\}$ and $\trianglelefteq_N^t = \{(\boldsymbol{A},\boldsymbol{B})\in\mathfrak{s}_N(\mathcal{H}_\infty)\times\mathfrak{s}_N(\mathcal{H}_\infty)\colon\ \mathsf{A}\leqslant t\odot\mathsf{B}\}$ are measurable.*

Proof. Since $\Delta_N^t = \trianglelefteq_N^t\cap\trianglerighteq_N^t$ where $\trianglerighteq_N^t = \{(\boldsymbol{A},\boldsymbol{B})\colon (\boldsymbol{B},\boldsymbol{A})\in\trianglelefteq_N^t\}$, it is enough to prove that $\trianglelefteq_N^t$ is measurable. It is clear that for every $n\geqslant 1$ the function $\mathfrak{s}_N(\mathcal{H}_\infty)\ni\boldsymbol{A}\mapsto n\odot\boldsymbol{A}\in\mathfrak{s}_N(n\odot\mathcal{H}_\infty)$ is measurable. Consequently, thanks to Theorem 5.3.5, the set $D(n,m) = \{(\boldsymbol{A},\boldsymbol{B})\in\mathfrak{s}_N(\mathcal{H}_\infty)\times\mathfrak{s}_N(\mathcal{H}_\infty)\colon n\odot\mathsf{A}\leqslant m\odot\mathsf{B}\}$ is measurable as well. Now if $w_k = m_k/n_k$ are rationals which decrease to t (as k increases to ∞), then $\trianglelefteq_N^t = \bigcap_{k=1}^\infty D(n_k,m_k)$, and we are done. ∎

Whenever $\mathsf{A},\mathsf{B}\in\mathfrak{s}_N$ are such that $\mathsf{A}\ll\mathsf{B}$, there is a unique positive real number denoted by $\mathsf{A}:\mathsf{B}$ such that

$$\mathsf{A} = (\mathsf{A}:\mathsf{B})\odot\mathsf{B}. \tag{5.3.1}$$

Further, we put $\mathsf{O}:\mathsf{X} = 0$ and $(\alpha\odot\mathsf{X}):\mathsf{X} = \alpha$ for any $\mathsf{X}\in\mathfrak{F}_N$ and $\alpha\in\mathrm{Card}_\infty$, and $(n\odot\mathsf{A}):(m\odot\mathsf{A}) = n/m$ for any $\mathsf{A}\in\mathfrak{a}_N$ and positive integers n and m. It is clear that (5.3.1) is satisfied whenever $\mathsf{B}\in\mathfrak{F}_N^{\mathrm{fin}}$ ($=\mathfrak{F}_N\cap\mathcal{FIN}_N$) and $\mathsf{A}\in\mathcal{CDD}_N$ with $\mathsf{A}\ll\mathsf{B}$.

For $n,m\in\{1,2,\ldots,\infty\}$ put $\nabla_N(m,n) = \{(\boldsymbol{A},\boldsymbol{B})\in\mathfrak{F}_N(\mathcal{H}_m)\times\mathfrak{F}_N^{\mathrm{fin}}(\mathcal{H}_n)\colon\ \mathsf{A}\ll\mathsf{B}\}$. It follows from Lemma 5.3.4 that $\nabla_N(m,n)$ is a Borel subset of $\mathrm{CDD}_N(\mathcal{H}_m)\times\mathrm{CDD}_N(\mathcal{H}_n)$. We want to show the measurability of the function

$$\mathrm{Div}\colon \nabla_N(m,n)\ni(\boldsymbol{X},\boldsymbol{Y})\mapsto\mathsf{X}:\mathsf{Y}\in I_{\aleph_0}.$$

It may be easily shown that Div is measurable on $\nabla_N(m,n)$ for finite n and on the set $\nabla_N(\infty,\infty)\setminus(\mathfrak{s}_N(\mathcal{H}_\infty)\times\mathfrak{s}_N(\mathcal{H}_\infty))$ ($\nabla_N(n,\infty)$ is empty if n is finite). On the other hand, $\mathrm{Div}^{-1}((0,t])\cap(\mathfrak{s}_N(\mathcal{H}_\infty)\times\mathfrak{s}_N(\mathcal{H}_\infty)) = \trianglelefteq_N^t$ and therefore Div is measurable on

$\nabla_N(\infty,\infty)\cap(\mathfrak{s}_N(\mathcal{H}_\infty)\times\mathfrak{s}_N(\mathcal{H}_\infty))$ as well. Hence the sets $\{(t,\boldsymbol{A},\boldsymbol{B})\in(0,\infty)\times\mathfrak{s}_N(\mathcal{H}_\infty)\times\mathfrak{s}_N(\mathcal{H}_\infty)\colon \mathsf{A}\leqslant t\odot\mathsf{B}\}$ and

$$\mathscr{R}_N(n,m)=\{(\mathsf{A}:\mathsf{B},\boldsymbol{B},\boldsymbol{A})\in I_{\aleph_0}\times\mathfrak{F}_N^{\mathrm{fin}}(\mathcal{H}_n)\times\mathfrak{F}_N(\mathcal{H}_m)\colon \mathsf{A}\ll\mathsf{B}\}\\ \cup\{(\aleph_0,\boldsymbol{B},\boldsymbol{B})\colon \boldsymbol{B}\in\mathfrak{F}_N(\mathcal{H}_n)\setminus\mathfrak{F}_N^{\mathrm{fin}}(\mathcal{H}_n)\}\quad(5.3.2)$$

are measurable. This fact will be used in the proof of

THEOREM 5.3.8. *Let $(X,\mathfrak{M},\mu)$ be a standard measure space, $\mathcal{F}\subset\mathfrak{F}_N$ be a countably separated measurable set and $\Phi\colon X\ni x\mapsto\mathsf{A}^{(x)}\in\mathcal{F}$ be a measurable function. Further, let $f\colon X\to I_{\aleph_0}\setminus\{0\}$ be a Borel function such that $f(X\setminus\Phi^{-1}(\mathfrak{s}_N))\subset$ Card. Then there are measurable sets $X_1,X_2,\dots,X_\infty\subset X$ and Borel functions $\Phi_n\colon X_n\ni x\mapsto\boldsymbol{B}^{(x)}\in\mathrm{CDD}_N(\mathcal{H}_n)$ $(n=1,2,\dots,\infty)$ such that $\mathsf{B}^{(x)}=f(x)\odot\mathsf{A}^{(x)}$ for each $x\in X':=\bigcup_{n=1}^{n=\infty}X_n$ and $\mu(X\setminus X')=0$. If, in addition,*

$$\mathsf{A}^{(x)}\perp_u\mathsf{A}^{(y)}\tag{5.3.3}$$

for distinct $x,y\in X$, then $\Phi_n(X_n)\in\mathfrak{B}(\mathrm{CDD}_N(\mathcal{H}_n))$ and Φ_n is a Borel isomorphism of X_n onto its range.

Proof. Since $\Phi^{-1}(\mathfrak{F}_N\setminus\mathfrak{F}_N^{\mathrm{fin}})$ is measurable, we may change the function f (with no change of $f(x)\odot\mathsf{A}^{(x)}$) so that $f(x)=\aleph_0$ whenever $\Phi(x)\notin\mathfrak{F}_N^{\mathrm{fin}}$. But then for every $x\in X$,

$$(f(x)\odot\mathsf{A}^{(x)}):\mathsf{A}^{(x)}=f(x).\tag{5.3.4}$$

Further, since μ is σ-finite, we may assume that it is finite. Let $\nu\colon\mathfrak{B}(\mathcal{F})\ni A\mapsto\mu(\Phi^{-1}(A))\in\mathbb{R}_+$. Since $\mathcal{F}$ is the Borel image of a standard Borel space $\bigcup_{n=1}^{n=\infty}\{\boldsymbol{X}\in\mathrm{CDD}_N(\mathcal{H}_n)\colon\mathsf{X}\in\mathcal{F}\}$ (and $\mathcal{F}$ is countably separated), $\mathcal{F}$ is a Souslin-Borel space and therefore ν is a standard measure on $\mathcal{F}$ (cf. [35, Corollary A.14]). So, we may assume (reducing $\mathcal{F}$ and X) that $\mathcal{F}$ and X are standard Borel spaces. For each $n=1,2,\dots,\infty$ let $\mathcal{G}_n$ be the set of all N-tuples $\boldsymbol{X}\in\mathrm{CDD}_N(\mathcal{H}_n)$ such that $\mathsf{X}\in\mathcal{F}(n):=\mathcal{F}\cap\mathcal{SEP}_N(n)$. Note that $\mathcal{G}_n\in\mathfrak{B}(\mathrm{CDD}_N(\mathcal{H}_n))$. Since $\mathcal{F}$ is a standard Borel space, it follows from [35, Theorem A.16] that there are a set $\mathcal{F}_n\in\mathfrak{B}(\mathcal{F}(n))$ and a measurable function $\mathcal{F}_n\ni\mathsf{X}\mapsto\boldsymbol{G}^{(\mathsf{X})}\in\mathcal{G}_n$ such that $\nu(\mathcal{F}(n)\setminus\mathcal{F}_n)=0$ and $\mathsf{G}^{(\mathsf{X})}=\mathsf{X}$ for each $\mathsf{X}\in\mathcal{F}_n$. Again, we may assume that $\mathcal{F}=\bigcup_{n=1}^{n=\infty}\mathcal{F}_n$ (since $\nu(\mathcal{F}\setminus\bigcup_{n=1}^{n=\infty}\mathcal{F}_n)=0$). Put $X(n)=\{x\in X\colon\mathsf{A}^{(x)}\in\mathcal{F}_n\}$ and $\boldsymbol{T}^{(x)}=\boldsymbol{G}^{(\mathsf{A}^{(x)})}$ for $x\in X(n)$. Note that the function $X(n)\ni x\mapsto\boldsymbol{T}^{(x)}\in\mathrm{CDD}_N(\mathcal{H}_n)$ is measurable. This implies that the set $\Gamma_n=\{(x,f(x),\boldsymbol{T}^{(x)})\colon x\in X(n)\}$ is Borel in $X(n)\times I_{\aleph_0}\times\mathrm{CDD}_N(\mathcal{H}_n)$ (as the graph of a Borel function) and consequently for each $m=1,2,\dots,\infty$ the set $\mathscr{B}_{n,nm}=\{(x,\boldsymbol{Y})\colon\mathsf{Y}=f(x)\odot\mathsf{A}^{(x)},\ x\in X(n),\ \boldsymbol{Y}\in\mathfrak{F}_N(\mathcal{H}_{nm})\}$, as the image of $(\Gamma_n\times\mathfrak{F}_N(\mathcal{H}_{nm}))\cap(X(n)\times\mathscr{R}_N(n,nm))$ under the projection map (cf. (5.3.2) and (5.3.4)), which is one-to-one on this set, is Borel as well. Now put $X(n,nm)=\{x\in X(n)\colon f(x)\cdot\dim(\mathsf{A}^{(x)})=nm\}$ and note that $X(n,nm)$'s are measurable sets such that $X(n)=\bigcup_{m=1}^{m=\infty}X(n,nm)$. Since the function $p_{n,nm}\colon\mathscr{B}_{n,nm}\ni(x,\boldsymbol{Y})\mapsto x\in X(n,nm)$ is a Borel surjection, we deduce from [35, Theorem A.16] that there is a Borel function $w_{n,nm}\colon X(n,nm)\to\mathscr{B}_{n,nm}$ such that $(p_{n,nm}\circ w_{n,nm})(x)=x$ for μ-almost all $x\in X(n,nm)$. For $x\in X(n,nm)$ let $\boldsymbol{B}^{(x)}\in\mathfrak{F}_N(\mathcal{H}_{nm})$ be the second coordinate of $w_{n,nm}(x)$. Then the function $\Phi_{n,nm}\colon X(n,nm)\ni x\mapsto\boldsymbol{B}^{(x)}\in\mathrm{CDD}_N(\mathcal{H}_{nm})$ is measurable and for μ-almost all $x \in X$,

$$\mathsf{B}^{(x)} = f(x) \odot \mathsf{A}^{(x)}. \tag{5.3.5}$$

Again, by reducing X, we may assume that (5.3.5) is satisfied for all $x \in X$. Finally, put $X_k = \bigcup\{X(n, nm)\colon nm = k\}$ and let $\Phi_k\colon X_k \to \mathrm{CDD}_N(\mathcal{H}_k)$ be given by $\Phi_k(x) = \Phi_{n,nm}(x)$ provided $nm = k$ and $x \in X(n, nm)$. Since the sets $X(n, nm)$ are pairwise disjoint, Φ_k is well defined and Borel. Thus, if (5.3.3) is satisfied, (5.3.5) implies that Φ_k is one-to-one, and the assertion follows. ∎

5.4. Direct integrals and measurable domains

In this chapter we establish only the most relevant (for our further investigations) properties of direct integrals. The 'continuous' operation in $\mathcal{CDD}_N$ is defined and main results on it appear in the next two chapters.

We now fix a standard measure space $(X, \mathfrak{M}, \mu)$. For a separable Hilbert space $\mathcal{H}$ the Hilbert space $L^2(X, \mathcal{H}) = L^2(\mu, \mathcal{H})$ consists of all (equivalence classes of) measurable functions $\xi\colon X \to \mathcal{H}$ such that $\|\xi\|_2^2 = \int_X \|\xi(x)\|^2\, d\mu(x) < \infty$ ($L^2(\mu, \mathcal{H})$ is separable). Let $X \ni x \mapsto T_x \in \mathrm{CDD}(\mathcal{H})$ be a measurable function. We define an operator $T := \int_X^{\oplus} T_x\, d\mu(x)$ in $L^2(\mu, \mathcal{H})$ by

$$\mathcal{D}(T) = \Big\{\xi \in L^2(\mu, \mathcal{H})\colon \xi(x) \in \mathcal{D}(T_x) \text{ for } \mu\text{-almost all } x \in X \text{ and } \int_X \|T_x\xi(x)\|^2\, d\mu(x) < \infty\Big\}$$

and $(T\xi)(x) = T_x\xi(x)$ for $\xi \in \mathcal{D}(T)$ and (μ-almost all) $x \in X$. It is not obvious that $T\xi$ is measurable (for $\xi \in \mathcal{D}(T)$) and that $T \in \mathrm{CDD}(\mathcal{H})$. These are guaranteed by the next result which may be deduced from [36, Lemma VI.3.3] (cf. [36, Definition VI.3.4]).

Proposition 5.4.1. *For every measurable function $X \ni x \mapsto T_x \in \mathrm{CDD}(\mathcal{H})$ the operator $\int_X^{\oplus} T_x\, d\mu(x)$ is well defined, closed and densely defined. What is more,*

$$\mathfrak{b}\Big(\int_X^{\oplus} T_x\, d\mu(x)\Big) = \int_X^{\oplus} \mathfrak{b}(T_x)\, d\mu(x).$$

Now let $\Phi\colon X' \ni x \mapsto \boldsymbol{T}^{(x)} \in \bigcup_{n=1}^{n=\infty} \mathrm{CDD}_N(\mathcal{H}_n)$, where $X \setminus X' \in \mathcal{N}(\mu)$, be any function and $\boldsymbol{T}^{(x)} = (\boldsymbol{T}_1^{(x)}, \ldots, \boldsymbol{T}_N^{(x)})$ for each $x \in X'$. If there are measurable sets $X_1, X_2, \ldots, X_\infty \subset X'$ such that $\mu(X' \setminus \bigcup_{n=1}^{n=\infty} X_n) = 0$ and $\Phi(X_j) \subset \mathrm{CDD}_N(\mathcal{H}_j)$ (the latter implies that X_j's are pairwise disjoint), and $\Phi|_{X_j}\colon X_j \to \mathrm{CDD}_N(\mathcal{H}_j)$ is measurable for each j, we call Φ *integrable* and define the *direct integral* $\int_X^{\oplus} \boldsymbol{T}^{(x)}\, d\mu(x)$ of the field $\{\boldsymbol{T}^{(x)}\}_{x \in X'}$ by

$$\int_X^{\oplus} \boldsymbol{T}^{(x)}\, d\mu(x) = \bigoplus_{n=1}^{n=\infty} \Big(\int_{X_n}^{\oplus} T_1^{(x)}\, d\mu(x), \ldots, \int_{X_n}^{\oplus} T_N^{(x)}\, d\mu(x)\Big).$$

Below we list the most important (for our investigations) properties of direct integrals of measurable fields of N-tuples.

(di0) $\dim \overline{\mathcal{D}}(\int_X^\oplus \boldsymbol{T}^{(x)}\,d\mu(x)) \leqslant \aleph_0$.
(di1) $\mathfrak{b}(\int_X^\oplus \boldsymbol{T}^{(x)}\,d\mu(x)) = \int_X^\oplus \mathfrak{b}(\boldsymbol{T}^{(x)})\,d\mu(x)$.
(di2) If $X_1, X_2, \ldots$ are pairwise disjoint measurable subsets of X such that $\mu(X_j) > 0$ for each j and $\mu(X \setminus \bigcup_{n=1}^\infty X_n) = 0$, then

$$\int_X^\oplus \boldsymbol{A}^{(x)}\,d\mu(x) \equiv \bigoplus_{n=1}^\infty \int_{X_n}^\oplus \boldsymbol{A}^{(x)}\,d\mu(x).$$

(di3) $\bigoplus_{n=1}^\infty (\int_X^\oplus \boldsymbol{T}_n^{(x)}\,d\mu(x)) \equiv \int_X^\oplus (\bigoplus_{n=1}^\infty \boldsymbol{T}_n^{(x)})\,d\mu(x)$.
(di4) If $\boldsymbol{T}^{(x)} \equiv \boldsymbol{S}^{(x)}$ for μ-almost all $x \in X$, then $\int_X^\oplus \boldsymbol{T}^{(x)}\,d\mu(x) \equiv \int_X^\oplus \boldsymbol{S}^{(x)}\,d\mu(x)$. This follows from (BT5) (page 12), (di1) and the proof of [35, Theorem IV.8.28].
(di5) If ν is a σ-finite measure on $(X, \mathfrak{M})$ such that $\nu \ll \mu \ll \nu$ (that is, $\mathcal{N}(\mu) = \mathcal{N}(\nu)$), then $\int_X^\oplus \boldsymbol{T}^{(x)}\,d\mu(x) \equiv \int_X^\oplus \boldsymbol{T}^{(x)}\,d\nu(x)$.
(di6) If $(Y, \mathfrak{N}, \nu)$ is a standard measure space, $X_0 \in \mathcal{N}(\mu)$, $Y_0 \in \mathcal{N}(\nu)$ and $\psi\colon Y \setminus Y_0 \to X \setminus X_0$ is a Borel isomorphism such that $\mu(\psi(A)) = \nu(A)$ for every $A \in \mathfrak{N}$ disjoint from Y_0, then

$$\int_X^\oplus \boldsymbol{T}^{(x)}\,d\mu(x) \equiv \int_Y^\oplus \boldsymbol{T}^{(\psi(y))}\,d\nu(y).$$

Further, let $X \ni x \mapsto \mathsf{A}^{(x)} \in \mathcal{SEP}_N$ be any function. If there exist Borel sets $X_1, X_2, \ldots, X_\infty \subset X$ and measurable functions

$$X_n \ni x \mapsto \boldsymbol{A}^{(x)} \in \mathrm{CDD}_N(\mathcal{H}_n) \tag{5.4.1}$$

($n = 1, 2, \ldots, \infty$) such that $\mu(X \setminus \bigcup_{n=1}^{n=\infty} X_n) = 0$ and for each $x \in \bigcup_{n=1}^{n=\infty} X_n$, $\boldsymbol{A}^{(x)}$ is a representative of $\mathsf{A}^{(x)}$, we say the field $\{\mathsf{A}^{(x)}\}_{x\in X}$ is *integrable* and we define the *direct integral* $\int_X^\oplus \mathsf{A}^{(x)}\,d\mu(x)$ as the unitary equivalence class of

$$\bigoplus_{n=1}^{n=\infty} \int_{X_n}^\oplus \boldsymbol{A}^{(x)}\,d\mu(x). \tag{5.4.2}$$

Thanks to (di4), $\int_X^\oplus \mathsf{A}^{(x)}\,d\mu(x)$ is well defined, i.e. it is independent of the choice of measurable functions (5.4.1). As is easily seen, in the above situation the function $\bigcup_{n=1}^{n=\infty} X_n \ni x \mapsto \mathsf{A}^{(x)} \in \mathcal{SEP}_N$ is measurable. We call a field $\Psi\colon X \ni x \mapsto \mathsf{B}^{(x)} \in \mathcal{SEP}_N$ *almost measurable* (or *almost Borel*) iff $\Psi|_{X\setminus X_0}$ is Borel for some $X_0 \in \mathcal{N}(\mu)$. Thus, every integrable field is almost measurable.

In our investigations all almost measurable fields are defined on standard measure spaces. Properties (di0)–(di6) may naturally be translated into the realm of unitary equivalence classes of N-tuples:

(DI0) $\int_X^\oplus \mathsf{A}^{(x)}\,d\mu(x) \in \mathcal{SEP}_N$.
(DI1) $\mathfrak{b}(\int_X^\oplus \mathsf{A}^{(x)}\,d\mu(x)) = \int_X^\oplus \mathfrak{b}(\mathsf{A}^{(x)})\,d\mu(x)$.
(DI2) If $X_1, X_2, \ldots$ are pairwise disjoint measurable subsets of X such that $\mu(X_j) > 0$ for each j and $\mu(X \setminus \bigcup_{n=1}^\infty X_n) = 0$, then

$$\int_X^\oplus \mathsf{A}^{(x)}\,d\mu(x) = \bigoplus_{n=1}^\infty \int_{X_n}^\oplus \mathsf{A}^{(x)}\,d\mu(x).$$

(DI3) $\bigoplus_{n=1}^\infty (\int_X^\oplus \mathsf{T}_n^{(x)}\,d\mu(x)) = \int_X^\oplus (\bigoplus_{n=1}^\infty \mathsf{T}_n^{(x)})\,d\mu(x)$.

(DI4) If $(Y, \mathfrak{N}, \nu)$ is a standard measure space, $X_0 \in \mathcal{N}(\mu)$, $Y_0 \in \mathcal{N}(\nu)$, $\psi\colon Y\setminus Y_0 \to X\setminus X_0$ is a Borel isomorphism and $\{\psi(B)\colon B \in \mathcal{N}(\nu),\ B \cap Y_0 = \emptyset\} = \{A \in \mathcal{N}(\nu)\colon A \cap X_0 = \emptyset\}$, then

$$\int_X^{\oplus} \mathsf{A}^{(x)}\, d\mu(x) = \int_Y^{\oplus} \mathsf{A}^{(\psi(y))}\, d\nu(y).$$

A counterpart of regular collections and direct sums ((UE4), page 10) for direct integrals are *regular fields* and *regular direct integrals* '$\int^{\boxplus}$' which we define as follows. Assume $X \ni x \mapsto \mathsf{A}^{(x)} \in \mathcal{SEP}_N$ is an integrable field. If for any two disjoint Borel sets $A, B \subset X$ one has

$$\int_A^{\oplus} \mathsf{A}^{(x)}\, d\mu(x) \perp_u \int_B^{\oplus} \mathsf{A}^{(x)}\, d\mu(x), \tag{5.4.3}$$

we call the field $\{\mathsf{A}^{(x)}\}_{x\in X}$ *regular* and write $\int_X^{\boxplus} \mathsf{A}^{(x)}\, d\mu(x)$ in place of $\int_X^{\oplus} \mathsf{A}^{(x)}\, d\mu(x)$. (Condition (5.4.3) naturally corresponds to (PR2), page 12.) As in the case of direct sums, the notation '$\int^{\boxplus}$' includes the information that the integrable field is regular.

In practice it is quite difficult to verify whether an almost measurable field is integrable. However, as an immediate consequence of Proposition 5.3.3 we obtain

PROPOSITION 5.4.2. *Every almost measurable field of a standard measure space into $\mathcal{SEP}_N \setminus \mathcal{SEP}_N(\infty)$ is integrable.*

Proof. Let $\Phi\colon X \to \mathcal{SEP}_N \setminus \mathcal{SEP}_N(\infty)$ be measurable. The sets $X_n = \Phi^{-1}(\mathcal{SEP}_N(n))$ are Borel and if χ_n's are as in Proposition 5.3.3, then $\chi_n \circ \Phi|_{X_n}$ is a measurable field of representatives for Φ. ∎

In general we are unable to characterize integrable fields taking values in $\mathcal{SEP}_N$. This is in fact not of interest to us. More preferable are regular fields taking values in $\mathfrak{F}_N$. In that case a characterization is possible and we formulate it in the next result. For this purpose we introduce

DEFINITION 5.4.3. A set $\mathcal{F} \in \mathfrak{B}_N$ is said to be a *measurable domain of strong unitary disjointness* iff there is a sequence $(\mathcal{E}_n)_{n=1}^{\infty}$ of subsets of $\mathcal{CDD}_N$ which separates the points of $\mathcal{F}$ and for every $n \geqslant 1$ the families $\mathcal{F} \cap \mathcal{E}_n$ and $\mathcal{F} \setminus \mathcal{E}_n$ are strongly unitarily disjoint (cf. Remark 5.2.5). We shall speak briefly of *measurable domains.*

It follows from the definition that measurable domains consist of pairwise unitarily disjoint N-tuples. It may also be easily verified that the union of a countable family of measurable domains any two of which are strongly unitarily disjoint as well as every measurable subset of a measurable domain are again measurable domains. Another important property of measurable domains is that they are Souslin–Borel. Indeed, when $\mathcal{F}$ is a measurable domain, it is the Borel image of a standard Borel space (by the measurability of $\mathcal{F}$) and $\mathcal{F}$ is countably separated, for if $\mathcal{E} \subset \mathcal{CDD}_N$ is such that $\mathcal{F} \cap \mathcal{E} \perp_s \mathcal{F} \setminus \mathcal{E}$, then $\mathcal{F} \cap \mathcal{E} \in \mathfrak{B}_N$ (because for every sequence $(p_n)_{n=1}^{\infty} \subset \mathcal{P}_1(N)$ and each complex scalar λ the set of all $\boldsymbol{T} \in \mathrm{CDD}_N(\mathcal{H}_k)$ such that $p_n(\mathfrak{b}(\boldsymbol{T}), \mathfrak{b}(\boldsymbol{T})^*)$ converges $*$-strongly to λI is Borel and invariant under unitary equivalence), and thus our claim follows from Definition 5.4.3.

Measurable domains are useful in producing regular fields, as is shown by

PROPOSITION 5.4.4. *Let* $(X, \mathfrak{M}, \mu)$ *be a standard measure space and* $\Phi\colon X \ni x \mapsto \mathsf{A}^{(x)} \in \mathfrak{F}_N$ *be any field. Then the following conditions are equivalent:*

(i) $\{\mathsf{A}^{(x)}\}_{x\in X}$ *is regular,*
(ii) *there is a Borel set* $X' \subset X$ *such that* $X \setminus X' \in \mathcal{N}(\mu)$, $\Phi(X')$ *is a measurable domain and* $\Phi|_{X'}$ *is a Borel isomorphism of* X' *onto its range.*

Proof. First of all, by reducing X, we may assume that X is a standard Borel space. Suppose condition (i) is satisfied. This implies that there is $Z \in \mathcal{N}(\mu)$ and an integrable field $\{\boldsymbol{A}^{(x)}\}_{x\in X\setminus Z} \subset \bigcup_{n=1}^{n=\infty} \mathrm{CDD}_N(\mathcal{H}_n)$ of representatives for Φ. Take a separating sequence $X_1, X_2, \ldots$ of measurable subsets of X. We infer from (di0), (5.4.3) and Proposition 5.2.3 that for each $k \geqslant 1$ there is a sequence $(q_n^{(k)})_{n=1}^{\infty} \subset \mathcal{P}_1(N)$ such that

$$q_n^{(k)}\Big(\mathfrak{b}\Big(\int_{X_k}^{\oplus} \boldsymbol{A}^{(x)}\,d\mu(x)\Big), \mathfrak{b}\Big(\int_{X_k}^{\oplus} \boldsymbol{A}^{(x)}\,d\mu(x)\Big)^*\Big) \stackrel{*s}{\to} I,$$
$$q_n^{(k)}\Big(\mathfrak{b}\Big(\int_{X\setminus X_k}^{\oplus} \boldsymbol{A}^{(x)}\,d\mu(x)\Big), \mathfrak{b}\Big(\int_{X\setminus X_k}^{\oplus} \boldsymbol{A}^{(x)}\,d\mu(x)\Big)^*\Big) \stackrel{*s}{\to} 0.$$

Now taking into account that

$$p\Big(\mathfrak{b}\Big(\int_{D}^{\oplus} \boldsymbol{A}^{(x)}\,d\mu(x)\Big), \mathfrak{b}\Big(\int_{D}^{\oplus} \boldsymbol{A}^{(x)}\,d\mu(x)\Big)^*\Big) = \int_{D}^{\oplus} p(\mathfrak{b}(\boldsymbol{A}^{(x)}), \mathfrak{b}(\boldsymbol{A}^{(x)})^*)\,d\mu(x) \tag{5.4.4}$$

for any measurable set $D \subset X$ and $p \in \mathcal{P}(N)$ (cf. (di1)), we infer from [29, Proposition 3.2.7] that there are a subsequence $(p_n^{(k)})_{n=1}^{\infty}$ of $(q_n^{(k)})_{n=1}^{\infty}$ and a measurable set $X'_k \subset X \setminus Z$ such that $X \setminus X'_k \in \mathcal{N}(\mu)$ and

$$p(\mathfrak{b}(\boldsymbol{A}^{(x)}), \mathfrak{b}(\boldsymbol{A}^{(x)})^*) \stackrel{*s}{\to} j_k(x)I \tag{5.4.5}$$

for any $x \in X'_k$ where j_k is the characteristic function of X_k. Put $X' = \bigcap_{k=1}^{\infty} X'_k$ and note that $\mu(X \setminus X') = 0$. Since $\{X_k\}_{k\geqslant 1}$ is a separating family and thanks to (5.4.5), $\Phi|_{X'}$ is one-to-one. It may also be deduced from Corollary 5.3.6 that $\Phi(X')$ is measurable. Consequently, $\Phi(X')$ is a measurable domain, by (5.4.5). Now it suffices to apply [35, Corollary A.10] to deduce that $\Phi|_{X'}$ is a Borel isomorphism.

We now turn to the converse implication. It follows from Theorem 5.3.8 that Φ is integrable. So, let

$$\{\boldsymbol{A}^{(x)}\}_{x\in X''} \subset \bigcup_{n=1}^{n=\infty} \mathrm{CDD}_N(\mathcal{H}_n)$$

be an integrable field of representatives for Φ where $X'' \subset X'$ and $X \setminus X'' \in \mathcal{N}(\mu)$. Put $\boldsymbol{A} = \int_X^{\oplus} \boldsymbol{A}^{(x)}\,d\mu(x)$. Let $\mathcal{E}_1, \mathcal{E}_2, \ldots$ be a separating family for $\Phi(X')$ such that

$$\Phi(X') \cap \mathcal{E}_k \perp_s \Phi(X') \setminus \mathcal{E}_k \tag{5.4.6}$$

for every k. It follows from the observation preceding the proposition that $\mathcal{E}_k \cap \Phi(X') \in \mathfrak{B}_N$. Consequently, the sets $X_k = \Phi^{-1}(\mathcal{E}_k) \cap X''$ $(k = 1, 2, \ldots)$ are measurable and separate the points of X'' (because Φ is one-to-one on $X' \supset X''$). We infer, by [35, Corollary A.12], that the σ-algebra of subsets of X'' generated by the X_k's coincides with $\mathfrak{M}'' := \{A \subset X''\colon A \in \mathfrak{M}\}$. Further, the space $\overline{\mathcal{D}}(\boldsymbol{A})$ has the form $\bigoplus_{n=1}^{n=\infty} L^2(X''_n, \mathcal{H}_n)$

where $X_1'', X_2'', \ldots$ are pairwise disjoint members of $\mathfrak{M}''$ whose union is X''. For each k let M_k be multiplication by the characteristic function j_k of X_k'' on $\overline{\mathcal{D}}(\boldsymbol{A})$. Fix for a moment k. By (5.4.6), there is a sequence $(p_n)_{n=1}^{\infty} \subset \mathcal{P}_1(N)$ such that $p_n(\mathfrak{b}(\boldsymbol{A}^{(x)}), \mathfrak{b}(\boldsymbol{A}^{(x)})^*)$ converges $*$-strongly to $j_k(x)I$ for every $x \in X''$. Since in addition $\|p_n(\mathfrak{b}(\boldsymbol{A}^{(x)}), \mathfrak{b}(\boldsymbol{A}^{(x)})^*)\| \leqslant 1$, Proposition 3.2.7 of [29] implies that

$$\int_{X''}^{\oplus} p_n(\mathfrak{b}(\boldsymbol{A}^{(x)}), \mathfrak{b}(\boldsymbol{A}^{(x)})^*)\, d\mu(x) \xrightarrow{*s} \int_{X''}^{\oplus} j_k(x) I\, d\mu(x).$$

This combined with (5.4.4) gives $p_n(\mathfrak{b}(\boldsymbol{A}), \mathfrak{b}(\boldsymbol{A})^*) \xrightarrow{*s} M_k$ and consequently $M_k \in \mathcal{W}''(\boldsymbol{A})$. In this way we have shown that $\{X_1, X_2, \ldots\} \subset \mathfrak{N}$ where $\mathfrak{N}$ consists of all $B \in \mathfrak{M}''$ such that multiplication $M(B)$ by the characteristic function of B belongs to $\mathcal{W}''(\boldsymbol{A})$. Since $\mathfrak{N}$ is a σ-algebra, we finally obtain $\mathfrak{N} = \mathfrak{M}''$.

Since $\mathcal{W}''(\boldsymbol{A}) = \mathcal{W}(\mathfrak{b}(\boldsymbol{A}))$ and each entry of $\mathfrak{b}(\boldsymbol{A})$ is a decomposable operator, $\mathcal{W}''(\boldsymbol{A})$ consists of decomposable operators. If $B \in \mathfrak{M}$, $M(B \cap X'')$ is a diagonalizable operator and hence $M(B \cap X'') \in \mathcal{Z}(\mathcal{W}''(\boldsymbol{A}))$. So, $\int_B^{\oplus} \mathsf{A}^{(x)}\, d\mu(x)$ $(= \int_{B\cap X''}^{\oplus} \mathsf{A}^{(x)}\, d\mu(x))$ and $\int_{X\setminus B}^{\oplus} \mathsf{A}^{(x)}\, d\mu(x)$ correspond (by Proposition 2.3.1) to mutually orthogonal central projections in $\mathcal{W}''(\boldsymbol{A})$, from which we conclude that

$$\int_B^{\oplus} \mathsf{A}^{(x)}\, d\mu(x) \perp_u \int_{X\setminus B}^{\oplus} \mathsf{A}^{(x)}\, d\mu(x).$$

Now (5.4.3) follows from (di2). ∎

Remark 5.4.5. Since every Borel injection of a standard Borel space into a Souslin–Borel one has measurable image and is a Borel isomorphism between its domain and range (cf. Theorem A.6 and Corollary A.7 in [35]), condition (ii) of Proposition 5.4.4 may be weakened by replacing the assumption that $\Phi(X')$ *is a measurable domain and* $\Phi|_{X'}$ *is a Borel isomorphism* by $\Phi|_{X'}$ *is Borel and one-to-one and* $\Phi(X')$ *is contained in a measurable domain.*

For simplicity, let us call a σ-finite measure ν on a measurable set $\mathcal{B} \subset \mathfrak{F}_N$ a *regularity* measure ($\nu \in \operatorname{rgm}(\mathcal{B})$) if ν is standard and the identity field of $\mathcal{B}$ into $\mathfrak{F}_N$ is regular. Equivalently, $\nu \in \operatorname{rgm}(\mathcal{B})$ iff ν is concentrated on a measurable domain (since measurable domains are Souslin–Borel and all σ-finite measures on such sets are standard). To shorten statements, we shall write $(\mu, \Phi) \in \operatorname{RGS}(X, \mathfrak{M})$ when μ is a standard measure on $(X, \mathfrak{M})$ and $\Phi\colon X \to \mathfrak{F}_N$ is a regular field.

Suppose $(\mu, \Phi) \in \operatorname{RGS}(X, \mathfrak{M})$. Let X' be as in Proposition 5.4.4(ii). Define a measure $\nu = \Phi^*(\mu)\colon \mathfrak{B}(\mathfrak{F}_N) \to [0, \infty]$ by $\nu(\mathcal{B}) = \mu(\Phi^{-1}(\mathcal{B}) \cap X')$. Notice that $\nu \in \operatorname{rgm}(\mathfrak{F}_N)$ and $\int_X^{\boxplus} \Phi(x)\, d\mu(x) = \int_{\mathfrak{F}_N}^{\boxplus} \mathsf{F}\, d\nu(\mathsf{F})$, thanks to (DI4). This observation shows that it suffices to consider regularity measures instead of abstract regular fields.

The following result is a link between regular fields and central decompositions of von Neumann algebras.

Proposition 5.4.6. *Let* $(X, \mathfrak{M}, \mu)$ *be a standard measure space,* $\Phi\colon X \ni x \mapsto \boldsymbol{A}^{(x)} \in \bigcup_{n=1}^{n=\infty} \mathfrak{F}_N(\mathcal{H}_n)$ *an integrable field, and let*

$$\boldsymbol{A} = \int_X^{\oplus} \boldsymbol{A}^{(x)}\, d\mu(x).$$

Then the following conditions are equivalent:

(i) $\{\mathsf{A}^{(x)}\}_{x\in X}$ *is regular,*
(ii) $\{\mathsf{X} \in \mathcal{CDD}_N : \mathsf{X} \leqslant^s \mathsf{A}\} = \{\int_B^{\oplus} \mathsf{A}^{(x)}\, d\mu(x) : B \in \mathfrak{M}\}$,
(iii) $\int_X^{\oplus} \mathcal{W}''(\boldsymbol{A}^{(x)})\, d\mu(x)$ *is the central decomposition of* $\mathcal{W}''(\boldsymbol{A})$.

Proof. First of all, note that the field $\{\mathcal{W}''(\boldsymbol{A}^{(x)})\}_{x\in X}$ is measurable according to [29, Definition 3.2.9], since $\mathcal{W}''(\boldsymbol{A}^{(x)}) = \mathcal{W}(\mathfrak{b}(\boldsymbol{A}^{(x)}))$. Further, under the assumptions of the proposition, (iii) is equivalent to

(iii′) the von Neumann algebra $\mathcal{A}$ of all diagonalizable operators is contained in $\mathcal{W}''(\boldsymbol{A})$.

It is clear that (iii′) follows from (iii). Conversely, when (iii′) holds, $\mathcal{W}'(\boldsymbol{A})$ consists of (some) decomposable operators (thanks to [35, Corollary IV.8.16] or [19, Theorem 14.1.10]). We see that so does $\mathcal{W}''(\boldsymbol{A})$ (since $\mathfrak{b}(\boldsymbol{A})$ is an N-tuple of decomposable operators) and hence $\mathcal{A} \subset \mathcal{W}'(\boldsymbol{A})$. This yields $\mathcal{A} \subset \mathcal{Z}(\mathcal{W}''(\boldsymbol{A}))$. Now using the terminology of Kadison and Ringrose [19], we conclude that $\mathcal{W}''(\boldsymbol{A})$ is decomposable (Theorem 14.1.16 and Proposition 14.1.18 in [19]), i.e. $\mathcal{W}''(\boldsymbol{A}) = \int_X^{\oplus} \mathcal{M}_x\, d\mu(x)$ for some measurable field $\{\mathcal{M}_x\}_{x\in X}$ of von Neumann algebras. By the uniqueness of the decomposition $\mathfrak{b}(\boldsymbol{A}) = \int_X^{\oplus} \mathfrak{b}(\boldsymbol{A}^{(x)})\, d\mu(x)$ (cf. (di1), page 73), we obtain $\mathcal{W}(\mathfrak{b}(\boldsymbol{A}^{(x)})) \subset \mathcal{M}_x$ for μ-almost all $x \in X$ and thus $\int_X^{\oplus} \mathcal{W}''(\boldsymbol{A}^{(x)})\, d\mu(x) \subset \mathcal{W}''(\boldsymbol{A})$. Since the converse inclusion is immediate, we get $\mathcal{W}''(\boldsymbol{A}) = \int_X^{\oplus} \mathcal{W}''(\boldsymbol{A}^{(x)})\, d\mu(x)$. This proves (iii) because $\mathcal{W}''(\boldsymbol{A}^{(x)})$ is a factor for all $x \in X$ and consequently (by [35, Corollary IV.8.20]) $\mathcal{Z}(\mathcal{W}''(\boldsymbol{A})) = \int_X^{\oplus} \mathcal{Z}(\mathcal{W}''(\boldsymbol{A}^{(x)}))\, d\mu(x) = \mathcal{A}$.

We leave it as a simple exercise that the assertion of the proposition now easily follows. ∎

An important consequence of Proposition 5.4.6 is

Corollary 5.4.7. *Let* $(\mu, \Phi) \in \mathrm{RGS}(X, \mathfrak{M})$, $(\nu, \Psi) \in \mathrm{RGS}(Y, \mathfrak{N})$ *and let* $\widehat{\mu} = \Phi^*(\mu)$ *and* $\widehat{\nu} = \Psi^*(\nu)$. *For*

$$\mathsf{X} = \int_X^{\boxplus} \Phi(x)\, d\mu(x) \quad \text{and} \quad \mathsf{Y} = \int_Y^{\boxplus} \Psi(y)\, d\nu(y)$$

we have:

(a) $\mathsf{X} = \mathsf{Y} \Leftrightarrow \widehat{\mu} \ll \widehat{\nu} \ll \widehat{\mu}$,
(b) $\mathsf{X} \leqslant^s \mathsf{Y} \Leftrightarrow \widehat{\mu} \ll \widehat{\nu}$.

Proof. We know that $\mathsf{X} = \int_{\mathfrak{F}_N}^{\boxplus} \mathsf{F}\, d\widehat{\mu}(\mathsf{F})$ and $\mathsf{Y} = \int_{\mathfrak{F}_N}^{\boxplus} \mathsf{F}\, d\widehat{\nu}(\mathsf{F})$. Observe that (b) follows from (a) and Proposition 5.4.6, and the implication '$\Leftarrow$' in (a) is a consequence of (DI4). To prove the converse, assume $\boldsymbol{X} = \int_X^{\boxplus} \boldsymbol{A}^{(x)}\, d\mu(x)$ with $\mathsf{A}^{(x)} = \Phi(x)$ for μ-almost all $x \in X$, $\boldsymbol{Y} = \int_Y^{\boxplus} \boldsymbol{B}^{(y)}\, d\nu(y)$ with $\mathsf{B}^{(y)} = \Psi(y)$ for ν-almost all $y \in Y$, and U is a unitary operator such that $U \cdot \boldsymbol{X} \cdot U^{-1} = \boldsymbol{Y}$. It then follows from Proposition 5.4.6 that U sends the algebra of all diagonalizable operators on $\overline{\mathcal{D}}(\boldsymbol{X})$ onto the algebra of all diagonalizable operators on $\overline{\mathcal{D}}(\boldsymbol{Y})$. Thus, according to [35, Theorem IV.8.23], there is a Borel isomorphism $\kappa : Y \setminus Y_0 \to X \setminus X_0$ where $X_0 \in \mathcal{N}(\mu)$ and $Y_0 \in \mathcal{N}(\nu)$ such that

$$\kappa^*(\nu) \ll \mu \ll \kappa^*(\nu) \tag{5.4.7}$$

and U may be written in the form $U = \int_X^{\oplus} U_x \sqrt{\frac{d\kappa^*(\nu)}{d\mu}(x)}\, d\mu(x)$ where $\{U_x\}_{x\in X}$ is a certain measurable field of unitary operators (for details we refer to Takesaki's book [35]). Since $U \cdot \mathfrak{b}(\boldsymbol{X}) = \mathfrak{b}(\boldsymbol{Y}) \cdot U$, we conclude from (di1) (page 73) that

$$\int_X^{\oplus} U_x \cdot \mathfrak{b}(\boldsymbol{A}^{(x)}) \sqrt{\frac{d\kappa^*(\nu)}{d\mu}(x)}\, d\mu(x) = \int_X^{\oplus} \mathfrak{b}(\boldsymbol{B}^{(\kappa(x))}) \cdot U_x \sqrt{\frac{d\kappa^*(\nu)}{d\mu}(x)}\, d\mu(x).$$

Now thanks to the uniqueness of the decomposition of a bounded decomposable operator and the positivity of the function $\sqrt{d\kappa^*(\nu)/\,d\mu}$, the last equation implies that $U_x \cdot \mathfrak{b}(\boldsymbol{A}^{(x)}) = \mathfrak{b}(\boldsymbol{B}^{(\kappa(x))}) \cdot U_x$ for μ-almost all $x \in X$. Consequently, $\mathsf{B}^{(\kappa(x))} = \mathsf{A}^{(x)}$ for μ-almost all $x \in X$. We leave it as an exercise that this combined with (5.4.7) gives $\widehat{\mu} \ll \widehat{\nu} \ll \widehat{\mu}$, which finishes the proof. ∎

A similar result was obtained by Ernest (cf. [9, Theorem 3.8]). However, he worked with quasi-equivalence classes instead of unitary equivalence classes.

To avoid repetitions, let us say a function $f\colon X \to I_{\aleph_0}$ *fits* to $(\mu, \Phi) \in \mathrm{RGS}(X, \mathfrak{M})$ iff f is almost measurable and there are disjoint measurable sets X_1 and X_2 such that $\mu(X \setminus (X_1 \cup X_2)) = 0$, $f(X_1) \subset \mathrm{Card}$ and $\Phi(X_2) \subset \mathfrak{s}_N$. Note that if this happens, the function $f \odot \Phi$ given by $(f \odot \Phi)(x) = f(x) \odot \Phi(x)$ is well defined on $X_1 \cup X_2$.

LEMMA 5.4.8. *Let $(\mu, \Phi) \in \mathrm{RGS}(X, \mathfrak{M})$ and $f\colon X \to I_{\aleph_0} \setminus \{0\}$ be a function which fits to (μ, Φ). Then $(\mu, f \odot \Phi) \in \mathrm{RGS}(X, \mathfrak{M})$ as well.*

Proof. It follows from Theorem 5.3.8 that $f \odot \Phi$ is integrable. Further, we infer from (DI3) (page 73) that $\aleph_0 \odot \int_D^{\oplus} f(x) \odot \Phi(x)\, d\mu(x) = \int_D^{\oplus} (\aleph_0 \cdot f(x)) \odot \Phi(x)\, d\mu(x) = \aleph_0 \odot \int_D^{\oplus} \Phi(x)\, d\mu(x)$ and thus $\int_D^{\oplus} f(x) \odot \Phi(x)\, d\mu(x) \perp_u \int_{X\setminus D}^{\oplus} f(x) \odot \Phi(x)\, d\mu(x)$ since

$$\int_D^{\oplus} \Phi(x)\, d\mu(x) \perp_u \int_{X\setminus D}^{\oplus} \Phi(x)\, d\mu(x). \ \blacksquare$$

Whenever a function $f\colon X \to I_{\aleph_0}$ fits to $(\mu, \Phi) \in \mathrm{RGS}(X, \mathfrak{M})$, we define $\int_X^{\boxplus} f(x) \odot \Phi(x)\, d\mu(x)$ as follows. Put $s(f) = \{x \in X : f(x) > 0\}$ and take $X_0 \in \mathcal{N}(\mu)$ such that f is measurable on $X \setminus X_0$. If $\mu(s(f) \setminus X_0) > 0$, $\int_X^{\boxplus} f(x) \odot \Phi(x)\, d\mu(x)$ denotes $\int_{s(f)\setminus X_0}^{\boxplus} f(x) \odot \Phi(x)\, d\mu(x)$ (see Lemma 5.4.8). Otherwise let $\int_X^{\boxplus} f(x) \odot \Phi(x)\, d\mu(x) = \mathsf{O}$. The usage of '$\int^{\boxplus}$' here is justified by Lemma 5.4.8.

Below we formulate a variation of [9, Proposition 3.2]. We shall use it in our theorem on prime decomposition.

LEMMA 5.4.9. *Let $\mathsf{A} \in \mathcal{SEP}_N$ be the direct sum of a minimal N-tuple and a semiminimal one.*

(A) *There exists $\mu_{\mathsf{A}} \in \mathrm{rgm}(\mathfrak{p}_N)$ such that $\mathsf{A} = \int_{\mathfrak{p}_N}^{\boxplus} \mathsf{P}\, d\mu_{\mathsf{A}}(\mathsf{P})$. For $\mu \in \mathrm{rgm}(\mathfrak{p}_N)$, $\mathsf{A} = \int_{\mathfrak{p}_N}^{\boxplus} \mathsf{P}\, d\mu(\mathsf{P}) \Leftrightarrow \mu \ll \mu_{\mathsf{A}} \ll \mu$.*
(B) *For $\mathsf{B} \in \mathcal{SEP}_N$ the following conditions are equivalent:*

(i) $\mathsf{B} \ll \mathsf{A}$,

(ii) *there is an almost measurable function* $f\colon \mathfrak{p}_N \to I_{\aleph_0}$ *such that* $f(\mathfrak{a}_N) \subset \mathrm{Card}$, $f(\mathfrak{f}_N) \subset \{0, \aleph_0\}$ *and*

$$\mathsf{B} = \int_{\mathfrak{p}_N}^{\boxplus} f(\mathsf{P}) \odot \mathsf{P}\, d\mu_{\mathsf{A}}(\mathsf{P}). \tag{5.4.8}$$

(C) *Let* $(\mu, \Phi) \in \mathrm{RGS}(X, \mathfrak{M})$.

(a) *If* $\Phi(X) \subset \mathfrak{a}_N$, $\int_X^{\boxplus} \Phi(x)\, d\mu(x) \in \mathcal{MF}_N$.

(b) *If* $\Phi(X) \subset \mathfrak{f}_N$, $\int_X^{\boxplus} \Phi(x)\, d\mu(x) \in \mathcal{HIM}_N$.

(c) *If* $\Phi(X) \subset \mathfrak{s}_N$ *and* $f\colon X \to \mathbb{R}_+$ *is almost measurable,* $\int_X^{\boxplus} f(x) \odot \Phi(x)\, d\mu(x) \in \mathcal{SM}_N$.

Proof. Let $\mathsf{F} \in \mathcal{SEP}_N$ and let $\boldsymbol{F}$ be a representative of F. It follows from the reduction theory of von Neumann algebras (see e.g. [35, Theorem IV.8.21]) that there is a standard Borel space $(X, \mathfrak{M})$ with a probability Borel measure λ and a measurable field $\{\mathcal{M}_x\}_{x \in X}$ of factors (each of which acts on some $\mathcal{H}_n$) such that the von Neumann algebras $\mathcal{M} := \int_X^{\oplus} \mathcal{M}_x\, d\lambda(x)$ and $\mathcal{W}''(\boldsymbol{F})$ are spatially isomorphic. Write $\mathfrak{b}(\boldsymbol{F}) = (T_1, \ldots, T_N)$. Now, T_j corresponds (under the spatial isomorphism) to $T_j' \in \mathcal{M}$. Since then $\mathfrak{b}(\boldsymbol{F}) \equiv (T_1', \ldots, T_N')$, we see that there is $\boldsymbol{F}' \in \mathrm{CDD}_N$ such that $\mathfrak{b}(\boldsymbol{F}') = (T_1', \ldots, T_N')$ and consequently $\boldsymbol{F}' \equiv \boldsymbol{F}$. Thus replacing $\boldsymbol{F}$ by $\boldsymbol{F}'$, we may assume that $\mathcal{W}''(\boldsymbol{F}) = \mathcal{M}$. Write $T_j = \int_X^{\oplus} T_j^{(x)}\, d\lambda(x)$ where $T_j^{(x)} \in \mathcal{M}_x$ for λ-almost all $x \in X$. Since $\|T_j\| \leqslant 1$, we also have $\|T_j^{(x)}\| \leqslant 1$ λ-almost everywhere. Further, the function $x \mapsto \mathcal{N}(I - (T_j^{(x)})^* T_j^{(x)})$ is measurable (in the target space we consider the Effros Borel structure separately on each $\mathcal{H}_n$) and hence the set $X_0 = \{x \in X\colon\ \mathcal{N}(I - (T_j^{(x)})^* T_j^{(x)}) \neq \{0\}\}$ is measurable. Suppose $\lambda(X_0) > 0$. Then there exists a measurable vector field $x \mapsto \xi_x$ such that $\xi_x \in \mathcal{N}(I - (T_j^{(x)})^* T_j^{(x)})$ and $\|\xi_x\| \leqslant 1$ for λ-almost all $x \in X$, and $\int_X \|\xi_x\|^2\, d\lambda(x) > 0$ (see Corollary after Theorem 2 in [6]; or [35, Corollary IV.8.3]). We infer that $\xi = \int_X^{\oplus} \xi_x\, d\lambda(x)$ is well defined and nonzero, and $T_j^* T_j \xi = \xi$, which contradicts the fact that T_j is a value of the $\mathfrak{b}$-transform. This shows that $\lambda(X_0) = 0$ and hence for λ-almost all $x \in X$ there is an operator $F_j^{(x)} \in \mathrm{CDD}$ such that $\mathfrak{b}(F_j^{(x)}) = T_j^{(x)}$. Put $\boldsymbol{F}^{(x)} = (F_1^{(x)}, \ldots, F_N^{(x)})$ and observe that the function $x \mapsto \boldsymbol{F}^{(x)}$ is measurable (since the $\mathfrak{b}$-transform is an isomorphism) and $\boldsymbol{F} = \int_X^{\oplus} \boldsymbol{F}^{(x)}\, d\lambda(x)$. Since the field $x \mapsto \mathcal{W}''(\boldsymbol{F}^{(x)})$ is measurable and $\mathcal{W}''(\boldsymbol{F}^{(x)}) \subset \mathcal{M}_x$, $\int_X^{\oplus} \mathcal{W}''(\boldsymbol{F}^{(x)})\, d\lambda(x) \subset \mathcal{M} = \mathcal{W}''(\boldsymbol{F})$. At the same time, $T_1, \ldots, T_N \in \int_X^{\oplus} \mathcal{W}''(\boldsymbol{F}^{(x)})\, d\lambda(x)$ and therefore $\mathcal{W}''(\boldsymbol{F}) \subset \int_X^{\oplus} \mathcal{W}''(\boldsymbol{F}^{(x)})\, d\lambda(x)$ as well. We conclude that $\mathcal{W}''(\boldsymbol{F}^{(x)}) = \mathcal{M}_x$ for λ-almost all $x \in X$ and consequently $\int_X^{\oplus} \mathcal{W}''(\boldsymbol{F}^{(x)})\, d\lambda(x)$ is the central decomposition of $\mathcal{W}''(\boldsymbol{F})$. In particular, $\mathsf{F}^{(x)} \in \mathfrak{F}_N$ for λ-almost all $x \in X$. Now Proposition 5.4.6 implies that $\mathsf{F} = \int_X^{\boxplus} \Phi(x)\, d\lambda(x)$ where $\Phi\colon X \ni x \mapsto \mathsf{F}^{(x)} \in \mathfrak{F}_N$. Let $\mu_{\mathsf{F}} = \Phi^*(\lambda) \in \mathrm{rgm}(\mathfrak{F}_N)$. We know that then

$$\mathsf{F} = \int_{\mathfrak{F}_N}^{\boxplus} \mathsf{X}\, d\mu_{\mathsf{F}}(\mathsf{X}). \tag{5.4.9}$$

Further, since central decompositions of von Neumann algebras preserve the types ([18, Theorem 14.1.21] or [35, Corollary V.6.7]), we infer that F is type I, I^n, II, II^1, II^∞ or III iff μ_{F}-almost all $\mathsf{X} \in \mathfrak{F}_N$ are. In particular, if F is the direct sum of a minimal N-

tuple and a semiminimal one, $\mathcal{W}''(\boldsymbol{F})$ decomposes into type I_1, II_1 and III parts (and no other) and consequently μ_{F}-almost all $\mathsf{X} \in \mathfrak{F}_N$ are type I^1 (atoms) or II^1 (semiprimes), or III (fractals)—cf. Propositions 5.1.3 and 5.1.6. This proves the first claim of (A). The remainder of (A) follows from Corollary 5.4.7.

We turn to (B). First of all, note that (5.4.8) makes sense thanks to Lemma 5.4.8. Suppose that B is given by (5.4.8). We may assume that f is measurable. Then $s(f) = \{\mathsf{P} \in \mathfrak{p}_N : f(\mathsf{P}) > 0\} \in \mathfrak{B}_N$. It follows from (DI3) (page 73) that

$$\aleph_0 \odot \mathsf{B} = \int_{\mathfrak{p}_N}^{\oplus} (\aleph_0 \cdot f(\mathsf{P})) \odot \mathsf{P}\, d\mu_{\mathsf{A}}(\mathsf{P}) = \aleph_0 \odot \int_{s(f)}^{\oplus} \mathsf{P}\, d\mu_{\mathsf{A}}(\mathsf{P}) \leqslant \aleph_0 \odot \mathsf{A}$$

and thus $\mathsf{B} \ll \mathsf{A}$.

Now assume that $\mathsf{B} \ll \mathsf{A}$. Let $\mu_{\mathsf{B}} \in \operatorname{rgm}(\mathfrak{F}_N)$ be as in (5.4.9) with $\mathsf{F} = \mathsf{B}$. Since $\mathsf{B} \ll \mathsf{A}$ and $\mathsf{A}, \mathsf{B} \in \mathcal{SEP}_N$, $\aleph_0 \odot \mathsf{B} \leqslant^s \aleph_0 \odot \mathsf{A}$ (cf. Corollary 3.6.5). So, (PR6) (page 13) and Proposition 5.4.6 yield a measurable set $\mathcal{B} \subset \mathfrak{p}_N$ such that $\aleph_0 \odot \mathsf{B} = \aleph_0 \odot \int_{\mathcal{B}}^{\boxplus} \mathsf{P}\, d\mu_{\mathsf{A}}(\mathsf{P})$. Now we infer from (DI3) and Lemma 5.4.8 that

$$\int_{\mathfrak{F}_N}^{\boxplus} \aleph_0 \odot \mathsf{F}\, d\mu_{\mathsf{B}}(\mathsf{F}) = \int_{\mathcal{B}}^{\boxplus} \aleph_0 \odot \mathsf{P}\, d\mu_{\mathsf{A}}(\mathsf{P}). \tag{5.4.10}$$

An application of Proposition 5.4.4 shows that there are measurable domains $\mathcal{F}_0 \subset \mathcal{B}$ and $\mathcal{G}_0 \subset \mathfrak{F}_N$ such that $\mu_{\mathsf{A}}(\mathcal{B} \setminus \mathcal{F}_0) = 0$, $\mu_{\mathsf{B}}(\mathfrak{F}_N \setminus \mathcal{G}_0) = 0$, $\mathcal{F}_0^* = \{\aleph_0 \odot \mathsf{P} : \mathsf{P} \in \mathcal{F}_0\} \in \mathfrak{B}_N$, $\mathcal{G}_0^* = \{\aleph_0 \odot \mathsf{F} : \mathsf{F} \in \mathcal{G}_0\} \in \mathfrak{B}_N$, the sets $\mathcal{F}_0$, $\mathcal{G}_0$, $\mathcal{F}_0^*$ and $\mathcal{G}_0^*$ are standard Borel spaces and the functions $\Phi : \mathcal{F}_0 \ni \mathsf{P} \mapsto \aleph_0 \odot \mathsf{P} \in \mathcal{F}_0^*$ and $\Psi : \mathcal{G}_0 \ni \mathsf{F} \mapsto \aleph_0 \odot \mathsf{F} \in \mathcal{G}_0^*$ are Borel isomorphisms. Put $\mathcal{F} = \Phi^{-1}(\mathcal{F}_0^* \cap \mathcal{G}_0^*) \in \mathfrak{B}_N$ and $\mathcal{G} = \Psi^{-1}(\mathcal{F}_0^* \cap \mathcal{G}_0^*) \in \mathfrak{B}_N$. Let $\Theta = \Psi^{-1} \circ \Phi|_{\mathcal{F}}$. Observe that Θ is a Borel isomorphism of $\mathcal{F}$ onto $\mathcal{G}$. One may deduce from Corollary 5.4.7 and (5.4.10) that $\mu_{\mathsf{A}}(\mathcal{B} \setminus \mathcal{F}) = 0$ and $\mu_{\mathsf{B}}(\mathfrak{F}_N \setminus \mathcal{G}) = 0$, and $\lambda \ll \mu_{\mathsf{A}}|_{\mathcal{F}} \ll \lambda$ where $\lambda(\sigma) = \mu_{\mathsf{B}}(\Theta(\sigma \cap \mathcal{F}))$ for measurable $\sigma \subset \mathfrak{p}_N$. Consequently (by (DI4)),

$$\mathsf{B} = \int_{\mathcal{F}}^{\boxplus} \Theta(\mathsf{P})\, d\mu_{\mathsf{A}}(\mathsf{P}). \tag{5.4.11}$$

Since $\Theta(\mathsf{P}) \ll \mathsf{P}$ for any $\mathsf{P} \in \mathcal{F}$, we may define $f : \mathfrak{p}_N \to I_{\aleph_0}$ by $f(\mathsf{P}) = \Theta(\mathsf{P}) : \mathsf{P}$ for $\mathsf{P} \in \mathcal{F}$ and $f(\mathsf{P}) = 0$ for $\mathsf{P} \in \mathfrak{p}_N \setminus \mathcal{F}$. Thanks to (5.4.11), it suffices to show that $f|_{\mathcal{F}}$ is measurable. Since $\mathcal{F}$ and $\mathcal{G}$ are standard Borel spaces, the graph $\Gamma = \{(\mathsf{P}, \Theta(\mathsf{P})) : \mathsf{P} \in \mathcal{F}\}$ of Θ is a Borel subset of $\mathcal{F} \times \mathcal{G}$ and $u : \mathcal{F} \ni \mathsf{P} \mapsto (\mathsf{P}, \Theta(\mathsf{P})) \in \Gamma$ is a Borel isomorphism. Finally, since Div is Borel (see Chapter 5.3, page 70), so is the function $v : \Gamma \ni (\mathsf{A}, \mathsf{B}) \mapsto \mathsf{B} : \mathsf{A} \in I_{\aleph_0}$ (here it is important that $\mathcal{F}$ and $\mathcal{G}$ are standard Borel spaces). The observation that $f|_{\mathcal{F}} = v \circ u$ finishes the proof.

Finally, (C) follows from Proposition 5.4.6 and the fact that central decompositions of von Neumann algebras preserve the types. ■

The formula (5.4.9) corresponds to Ernest's central decomposition of a bounded operator [9, Chapter III]. It is not however of interest to us. Also a variation of (5.4.8) appears in [9, Lemma 4.4].

We shall need one more result.

Lemma 5.4.10. *For $\mu, \nu \in \operatorname{rgm}(\mathfrak{F}_N)$ the following conditions are equivalent:*

(i) $\int_{\mathfrak{F}_N}^{\boxplus} \mathsf{F}\, d\mu(\mathsf{F}) \perp_u \int_{\mathfrak{F}_N}^{\boxplus} \mathsf{F}\, d\nu(\mathsf{F})$,

(ii) *there are measurable sets* $\mathcal{A}, \mathcal{B} \subset \mathfrak{F}_N$ *such that* $\mu(\mathfrak{F}_N \setminus \mathcal{A}) = 0$, $\nu(\mathfrak{F}_N \setminus \mathcal{B}) = 0$ *and* $\mathcal{A} \perp_u \mathcal{B}$,

(iii) *there are measurable sets* $\mathcal{A}, \mathcal{B} \subset \mathfrak{F}_N$ *such that* $\mu(\mathfrak{F}_N \setminus \mathcal{A}) = 0$, $\nu(\mathfrak{F}_N \setminus \mathcal{B}) = 0$ *and* $\mathcal{A} \perp_s \mathcal{B}$,

(iv) $\mu \perp \nu$ *and* $\mu + \nu \in \operatorname{rgm}(\mathfrak{F}_N)$.

Proof. (i)⇒(iv): Put $\mathsf{A} = \int_{\mathfrak{F}_N}^{\boxplus} \mathsf{X}\, d\mu(\mathsf{X})$, $\mathsf{B} = \int_{\mathfrak{F}_N}^{\boxplus} \mathsf{X}\, d\nu(\mathsf{X})$ and $\mathsf{F} = \mathsf{A} \boxplus \mathsf{B}$, and let $\lambda = \mu_{\mathsf{F}}$ where μ_{F} is as in (5.4.9). Since $\mathsf{A}, \mathsf{B} \leqslant^s \mathsf{F}$, we infer from Corollary 5.4.7 that $\mu, \nu \ll \lambda$. So, $\mu + \nu \ll \lambda$ and therefore $\mu + \nu \in \operatorname{rgm}(\mathfrak{F}_N)$. Further, there are measurable sets $\mathcal{A}, \mathcal{B} \subset \mathfrak{F}_N$ such that $\mu \ll \lambda|_{\mathcal{A}} \ll \mu$ and $\nu \ll \lambda|_{\mathcal{B}} \ll \nu$ and consequently, again by Corollary 5.4.7, $\mathsf{A} = \int_{\mathcal{A}}^{\boxplus} \mathsf{X}\, d\lambda(\mathsf{X})$ and $\mathsf{B} = \int_{\mathcal{B}}^{\boxplus} \mathsf{X}\, d\lambda(\mathsf{X})$. Since then $\int_{\mathcal{A}\cap\mathcal{B}}^{\boxplus} \mathsf{X}\, d\lambda(\mathsf{X}) \leqslant^s \mathsf{A}, \mathsf{B}$, one has $\lambda(\mathcal{A} \cap \mathcal{B}) = 0$ and hence $\mu \perp \nu$.

(iv)⇒(iii): Put $\lambda = \mu + \nu$ and let $\mathcal{A}_0$ and $\mathcal{B}_0$ be disjoint measurable subsets of $\mathfrak{F}_N$ on which (respectively) μ and ν are concentrated. Since $\lambda \in \operatorname{rgm}(\mathfrak{F}_N)$, $\int_{\mathcal{A}_0}^{\oplus} \mathsf{F}\, d\lambda(\mathsf{F}) \perp_u \int_{\mathcal{B}_0}^{\oplus} \mathsf{F}\, d\lambda(\mathsf{F})$, which yields (cf. Proposition 5.2.3, and the proof of Proposition 5.4.4, or [29, Proposition 3.2.7]) that there exist a sequence $(p_n)_{n=1}^{\infty} \subset \mathcal{P}_1(N)$ and a set $\mathcal{Z} \in \mathcal{N}(\lambda)$ such that $p_n(\mathfrak{b}(\mathsf{F}), \mathfrak{b}(\mathsf{F})^*) \xrightarrow{*s} j(\mathsf{F})I$ for each $\mathsf{F} \in (\mathcal{A}_0 \cup \mathcal{B}_0) \setminus \mathcal{Z}$ where j is the characteristic function of $\mathcal{A}_0$. Consequently, μ and ν are concentrated on, respectively, $\mathcal{A} = \mathcal{A}_0 \setminus \mathcal{Z}$ and $\mathcal{B} = \mathcal{B}_0 \setminus \mathcal{Z}$, and $\mathcal{A} \perp_s \mathcal{B}$.

Since (ii) obviously follows from (iii), it remains to show that (ii) implies (i). Suppose (i) is false. This means that there are nontrivial N-tuples $\mathsf{A} \leqslant^s \int_{\mathfrak{F}_N}^{\boxplus} \mathsf{F}\, d\mu(\mathsf{F})$ and $\mathsf{B} \leqslant^s \int_{\mathfrak{F}_N}^{\boxplus} \mathsf{F}\, d\nu(\mathsf{F})$ such that $\aleph_0 \odot \mathsf{A} = \aleph_0 \odot \mathsf{B}$. By Corollary 5.4.7, there are measurable sets $\mathcal{A}_1, \mathcal{B}_1 \subset \mathfrak{F}_N$ such that $\mathsf{A} = \int_{\mathcal{A}_1}^{\boxplus} \mathsf{F}\, d\mu(\mathsf{F})$ and $\mathsf{B} = \int_{\mathcal{B}_1}^{\boxplus} \mathsf{F}\, d\nu(\mathsf{F})$. All these remarks combined with (DI3) (page 73) and Lemma 5.4.8 give

$$\int_{\mathcal{A}_1\cap\mathcal{A}}^{\boxplus} \aleph_0 \odot \mathsf{F}\, d\mu(\mathsf{F}) = \int_{\mathcal{B}_1\cap\mathcal{B}}^{\boxplus} \aleph_0 \odot \mathsf{F}\, d\nu(\mathsf{F}) \tag{5.4.12}$$

where $\mathcal{A}$ and $\mathcal{B}$ are as in (ii). Thanks to Proposition 5.4.4, we may assume that $\mathcal{F} = \{\aleph_0 \odot \mathsf{F}\colon\ \mathsf{F} \in \mathcal{A}_1 \cap \mathcal{A}\}$ and $\mathcal{G} = \{\aleph_0 \odot \mathsf{F}\colon \mathsf{F} \in \mathcal{B}_1 \cap \mathcal{B}\}$ are measurable. We conclude from the unitary disjointness of $\mathcal{A}$ and $\mathcal{B}$ that

$$\Phi^*(\mu)(\mathcal{G}) = 0 \quad \text{and} \quad \Phi^*(\nu)(\mathcal{F}) = 0 \tag{5.4.13}$$

where $\Phi\colon \mathfrak{F}_N \ni \mathsf{F} \mapsto \aleph_0 \odot \mathsf{F} \in \mathfrak{F}_N$. But (5.4.12) implies, by Corollary 5.4.7, that $\Phi^*(\mu) \ll \Phi^*(\nu) \ll \Phi^*(\mu)$. Consequently, it follows from (5.4.13) that $\mu(\mathcal{A}_1 \cap \mathcal{A}) = 0$ and $\nu(\mathcal{B} \cap \mathcal{B}_1) = 0$, contrary to the fact that A and B were nonzero. ∎

Taking into account the above result, for arbitrary two measures $\mu, \nu \in \operatorname{rgm}(\mathfrak{F}_N)$ we shall write $\mu \perp_s \nu$ iff any of the equivalent conditions (i)–(iv) of Lemma 5.4.10 is fulfilled.

5.5. 'Continuous' direct sums

Property (DI4) (page 74) suggests replacing standard measures μ by their null σ-ideals $\mathcal{N}(\mu)$. In this chapter we follow this idea. In that way we shall extend the notion of the (standard 'discrete') direct sum to a more general context.

DEFINITION 5.5.1. A *measurable space with nullity* is a triple $(\mathscr{X},\mathfrak{M},\mathcal{N})$ where $(\mathscr{X},\mathfrak{M})$ is a measurable space and $\mathcal{N}$ is a σ-ideal in $\mathfrak{M}$; that is, $\emptyset \in \mathcal{N} \subset \mathfrak{M}$, $\bigcup_{n=1}^{\infty} A_n \in \mathcal{N}$ whenever $\{A_n\}_{n=1}^{\infty} \subset \mathcal{N}$, and $\{B \in \mathfrak{M}\colon B \subset A\} \subset \mathcal{N}$ for every $A \in \mathcal{N}$.

Whenever $(\mathscr{X},\mathfrak{M},\mathcal{N})$ is a measurable space with nullity, $\overline{\mathcal{N}}$ denotes the family of all (possibly nonmeasurable) sets which are contained in members of $\mathcal{N}$. Members of $\overline{\mathcal{N}}$ are called *null* sets, other subsets of $\mathscr{X}$ are called *nonnull.* For $Y \in \mathfrak{M}$, $(Y,\mathfrak{M}|_Y,\mathcal{N}|_Y)$ is the induced measurable space with nullity, i.e. $\mathfrak{M}|_Y = \{B \in \mathfrak{M}\colon B \subset Y\}$ and $\mathcal{N}|_Y = \mathfrak{M}|_Y \cap \mathcal{N}$. The space $(\mathscr{X},\mathfrak{M},\mathcal{N})$ is *trivial* iff $\mathscr{X} \in \mathcal{N}$.

A function $\Phi\colon \mathscr{X}_1 \to \mathscr{X}_2$ is a *null-isomorphism* between measurable spaces with nullities $(\mathscr{X}_1,\mathfrak{M}_1,\mathcal{N}_1)$ and $(\mathscr{X}_2,\mathfrak{M}_2,\mathcal{N}_2)$ if Φ is a Borel isomorphism such that $\mathcal{N}_2 = \{\Phi(Z)\colon\ Z \in \mathcal{N}_1\}$. If $\Psi\colon X_1 \to \mathscr{X}_2$ (with $X_1 \subset \mathscr{X}_1$) is a function such that there are sets $Z_1 \in \mathcal{N}_1$ and $Z_2 \in \mathcal{N}_2$ for which $\mathscr{X}_1 \setminus Z_1 \subset X_1$ and $\Psi|_{\mathscr{X}_1\setminus Z_1}$ is a null-isomorphism of $(\mathscr{X}_1 \setminus Z_1, \mathfrak{M}_1|_{\mathscr{X}_1\setminus Z_1}, \mathcal{N}_1|_{\mathscr{X}_1\setminus Z_1})$ onto $(\mathscr{X}_2 \setminus Z_2, \mathfrak{M}_2|_{\mathscr{X}_2\setminus Z_2}, \mathcal{N}_2|_{\mathscr{X}_2\setminus Z_2})$, then Ψ is said to be an *almost null-isomorphism* and the spaces $(\mathscr{X}_1,\mathfrak{M}_1,\mathcal{N}_1)$ and $(\mathscr{X}_2,\mathfrak{M}_2,\mathcal{N}_2)$ are *almost isomorphic.* Similarly, a function $u\colon X \to Y$ (where $X \subset \mathscr{X}$, $(\mathscr{X},\mathfrak{M},\mathcal{N})$ is a measurable space with nullity and $(Y,\mathfrak{N})$ is a measurable space) is said to be *almost measurable* iff there is a set $X' \in \mathfrak{M}$ contained in X such that $X \setminus X' \in \overline{\mathcal{N}}$ and $u|_{X'}$ is measurable.

Of main interest to us are measurable spaces whose nullities come from certain measures. For this purpose we introduce

DEFINITION 5.5.2. Let $(\mathscr{X},\mathfrak{M},\mathcal{N})$ be a measurable space with nullity. A measurable set $A \subset \mathscr{X}$ is *standard* iff $(A,\mathfrak{M}|_A,\mathcal{N}|_A)$ is almost isomorphic to $(Y,\mathfrak{N},\mathcal{N}(\nu))$ for some standard measure space $(Y,\mathfrak{N},\nu)$. Standard sets are nonnull.

A family $\mathcal{B}$ is said to be a *base* of $(\mathscr{X},\mathfrak{M},\mathcal{N})$ iff the following two conditions are fulfilled:

- $\mathcal{B}$ consists of pairwise disjoint measurable sets and $\mathscr{X} \setminus \bigcup \mathcal{B} \in \mathcal{N}$,
- for any $A \subset \bigcup \mathcal{B}$ we have: $A \in \mathfrak{M}$ (respectively $A \in \mathcal{N}$) iff $A \cap B \in \mathfrak{M}$ ($A \cap B \in \mathcal{N}$) for any $B \in \mathcal{B}$.

A base is *standard* iff it consists of standard sets. $(\mathscr{X},\mathfrak{M},\mathcal{N})$ is called *multi-standard* iff it admits a standard base.

Let $\mathscr{F} = \{(\mathscr{X}_s,\mathfrak{M}_s,\mathcal{N}_s)\}_{s\in S}$ be a family of measurable spaces with nullities. The direct sum of $\mathscr{F}$, denoted by $\bigoplus_{s\in S}(\mathscr{X}_s,\mathfrak{M}_s,\mathcal{N}_s)$, is a measurable space with nullity $(\mathscr{X},\mathfrak{M},\mathcal{N})$ defined as follows: $\mathscr{X} = \bigcup_{s\in S}(\mathscr{X}_s \times \{s\})$; $\pi\colon \mathscr{X} \to \bigcup_{s\in S}\mathscr{X}_s$ is given by $\pi(x,s) = x$; $A \in \mathfrak{M}$ (respectively $A \in \mathcal{N}$) iff $\pi(A \cap (\mathscr{X}_s \times \{s\})) \in \mathfrak{M}_s$ ($\pi(A \cap (\mathscr{X}_s \times \{s\})) \in \mathcal{N}_s$) for every $s \in S$. Note that $\{\mathscr{X}_s \times \{s\}\}_{s\in S}$ is a base of $\bigoplus_{s\in S}(\mathscr{X}_s,\mathfrak{M}_s,\mathcal{N}_s)$. We call π the *canonical projection.*

Let $(\mathscr{X},\mathfrak{M},\mathcal{N})$ be a multi-standard measurable space with nullity. Let $\mathscr{X}^d$ be the set of all points $x \in \mathscr{X}$ such that $\{x\} \notin \overline{\mathcal{N}}$. One may show that $\mathscr{X}^d \in \mathfrak{M}$ (since $\mathscr{X}$ is multi-standard), $\mathfrak{M}|_{\mathscr{X}^d}$ is the power set of $\mathscr{X}^d$ and $\mathcal{N}|_{\mathscr{X}^d} = \{\emptyset\}$. Points of $\mathscr{X}^d$ are called *atoms*, while $\mathscr{X}^d$ and its complement $\mathscr{X}^c$ are called, respectively, the *discrete* and *continuous* parts of $\mathscr{X}$. Further, if $(Y,\mathfrak{N},\mu)$ is a nonatomic standard measure space, then there is $Z \in \mathcal{N}(\mu)$ such that $(Y \setminus Z, \mathfrak{N}|_{Y\setminus Z}, \mathcal{N}(\mu)|_{Y\setminus Z})$ is isomorphic to $([0,1],\mathfrak{B}([0,1]),\mathcal{L}_0)$ where $\mathcal{L}_0$ is the σ-ideal of all Borel subsets of $[0,1]$ whose Lebesgue measure is equal to 0 (by

Theorem 14.3.9 on page 270 in [27]). Using this fact, one may check that there is a base of $(\mathscr{X},\mathfrak{M},\mathcal{N})$ each of whose members either consists of a single point belonging to $\mathscr{X}^d$ or is isomorphic to $([0,1],\mathfrak{B}([0,1]),\mathcal{L}_0)$. Since every base of the last space is countable (finite or not; see the proof of Lemma 5.5.4 below), one deduces that either $\mathscr{X}^c$ is null or is a standard set, or every standard base of $(\mathscr{X},\mathfrak{M},\mathcal{N})$ contains the same, uncountable, number of sets almost isomorphic to $([0,1],\mathfrak{B}([0,1]),\mathcal{L}_0)$. We define two characteristic cardinal numbers related to $\mathscr{X}$ as follows: $\iota^d(\mathscr{X}) = \operatorname{card}(\mathscr{X}^d)$ and $\iota^c(\mathscr{X})$ is either 0 (if $\mathscr{X}^c$ is null) or $\aleph_0$ (if $\mathscr{X}^c$ is standard), or is equal to the uncountable number of members of a standard base which are almost isomorphic to $([0,1],\mathfrak{B}([0,1]),\mathcal{L}_0)$. We see that two multi-standard measurable spaces with nullities $\mathscr{X}$ and $\mathscr{Y}$ are almost isomorphic iff $\iota^d(\mathscr{X}) = \iota^d(\mathscr{Y})$ and $\iota^c(\mathscr{X}) = \iota^c(\mathscr{Y})$. What is more, for any $\alpha \in \mathrm{Card}$ and $\beta \in \mathrm{Card}_\infty \cup \{0\}$ there is a multi-standard measurable space with nullity $\mathscr{Z}$ for which $\iota^d(\mathscr{Z}) = \alpha$ and $\iota^c(\mathscr{Z}) = \beta$. (Indeed, take a set D of cardinality α and a set S disjoint from D whose cardinality is either β if $\beta \neq \aleph_0$ or 1 if $\beta = \aleph_0$. For each $s \in S$ let $(I_s,\mathfrak{M}_s,\mathcal{N}_s)$ be a copy of $([0,1],\mathfrak{B}([0,1]),\mathcal{L}_0)$ and for $d \in D$ let $(I_d,\mathfrak{M}_d,\mathcal{N}_d)$ be a standard one-point measurable space with nullity. Now it suffices to define $\mathscr{Z}$ as $\bigoplus_{x\in D\cup S}(I_x,\mathfrak{M}_x,\mathcal{N}_x)$.)

From now on, $(\mathscr{X},\mathfrak{M},\mathcal{N})$ and $(\mathscr{X}',\mathfrak{M}',\mathcal{N}')$ denote multi-standard measurable spaces with nullities. Let $\Phi\colon \mathscr{X} \ni x \mapsto \mathsf{B}^{(x)} \in \mathcal{SEP}_N$ be any function. If there exist $Z \in \mathcal{N}$ and an integrable field $\mathscr{X} \setminus Z \ni x \mapsto \boldsymbol{A}^{(x)} \in \bigcup_{n=1}^{n=\infty} \mathrm{CDD}_N(\mathcal{H}_n)$ such that $\mathsf{A}^{(x)} = \Phi(x)$ for all $x \in \mathscr{X} \setminus Z$, we call Φ a *summable field* and define $\bigoplus_{x\in\mathscr{X}}^{\mathcal{N}} \mathsf{B}^{(x)}$ as follows. If $\mathscr{X}$ is trivial, we put $\bigoplus_{x\in\mathscr{X}}^{\mathcal{N}} \mathsf{B}^{(x)} = \mathsf{O}$. Otherwise let $\mathcal{B}$ be a standard base of $(\mathscr{X},\mathfrak{M},\mathcal{N})$. For every $B \in \mathcal{B}$ there is a standard measure μ_B on $(B,\mathfrak{M}|_B)$ such that $\mathcal{N}(\mu_B) = \mathcal{N}|_B$. We put

$$\bigoplus_{x\in\mathscr{X}}^{\mathcal{N}} \mathsf{B}^{(x)} = \bigoplus_{B\in\mathcal{B}} \int_B^{\oplus} \mathsf{B}^{(x)}\, d\mu_B(x). \tag{5.5.1}$$

The next result shows that $\bigoplus_{x\in\mathscr{X}}^{\mathcal{N}} \mathsf{B}^{(x)}$ is well defined.

PROPOSITION 5.5.3. *Formula* (5.5.1) *well defines* $\bigoplus_{x\in\mathscr{X}}^{\mathcal{N}} \mathsf{B}^{(x)}$. *That is, the right-hand side of* (5.5.1) *is independent of the choice of a standard base* $\mathcal{B}$ *and standard measures* μ_B; *and* $\{\mathsf{B}^{(x)}\}_{x\in B}$ *is an integrable* (*with respect to* μ_B) *field for each* $B \in \mathcal{B}$.

Proof. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be standard bases for $(\mathscr{X},\mathfrak{M},\mathcal{N})$ and $\{\mu_B^{(j)}\colon B \in \mathcal{B}_j\}$ $(j = 1,2)$ corresponding families of standard measures. For each $D \in \mathcal{B}_j$ let $D' \in \mathfrak{M}|_D$ be such that $D \setminus D' \in \mathcal{N}$ and $(D',\mathfrak{M}|_{D'})$ is a standard Borel space. Then the set

$$I(D',\mathcal{B}_{3-j}) = \{E \in \mathcal{B}_{3-j}\colon D' \cap E \neq \emptyset\} \quad \text{is countable} \tag{5.5.2}$$

(see the last fragment of the proof of Lemma 5.5.4 below). Additionally put

$$\mathcal{I} = \{(D_1,D_2) \in \mathcal{B}_1 \times \mathcal{B}_2\colon D_1' \cap D_2' \notin \mathcal{N}\}.$$

Thanks to (DI3) (page 73) and (5.5.2) we obtain

$$\begin{aligned}\bigoplus_{A\in\mathcal{B}_1} \int_A^{\oplus} \mathsf{B}^{(x)}\, d\mu_A^{(1)}(x) &= \bigoplus_{A\in\mathcal{B}_1} \Big(\bigoplus\Big\{\int_{A'\cap B'}^{\oplus} \mathsf{B}^{(x)}\, d\mu_A^{(1)}(x)\colon (A,B) \in \mathcal{I}\Big\}\Big)\\ &= \bigoplus\Big\{\int_{A'\cap B'}^{\oplus} \mathsf{B}^{(x)}\, d\mu_A^{(1)}(x)\colon (A,B) \in \mathcal{I}\Big\}\end{aligned}$$

and similarly

$$\bigoplus_{B\in\mathcal{B}_2}\int_B^{\oplus} \mathsf{B}^{(x)}\,d\mu_B^{(2)}(x) = \bigoplus\Big\{\int_{A'\cap B'}^{\oplus} \mathsf{B}^{(x)}\,d\mu_B^{(2)}(x)\colon (A,B)\in\mathcal{I}\Big\}.$$

Now the fact that $\mathcal{N}(\mu_A^{(1)}|_{A'\cap B'}) = \mathcal{N}(\mu_B^{(2)}|_{A'\cap B'})$ combined with (DI4) (page 74) yields

$$\bigoplus_{A\in\mathcal{B}_1}\int_A^{\oplus} \mathsf{B}^{(x)}\,d\mu_A^{(1)}(x) = \bigoplus_{B\in\mathcal{B}_2}\int_B^{\oplus} \mathsf{B}^{(x)}\,d\mu_B^{(2)}(x).$$

The remainder is left to the reader. ∎

It is easily seen that the restriction of a summable field to a measurable set is summable as well. Thanks to Proposition 5.5.3, we may rewrite (5.5.1) in a new form: whenever $\mathcal{B}$ is a standard base of $(\mathscr{X},\mathfrak{M},\mathcal{N})$ and $\{\mathsf{A}^{(x)}\}_{x\in\mathscr{X}}$ is summable,

$$\bigoplus_{x\in\mathscr{X}}^{\mathcal{N}} \mathsf{A}^{(x)} = \bigoplus_{B\in\mathcal{B}}\Big(\bigoplus_{x\in B}^{\mathcal{N}} \mathsf{A}^{(x)}\Big). \tag{5.5.3}$$

Using this, one may prove that (5.5.3) is satisfied for an arbitrary (not necessarily standard) base $\mathcal{B}$.

Our next goal is to extend the notion of summability to a more general context. In what follows, we equip $\mathbb{R}_+\cup\mathrm{Card}$ with the Borel structure induced by the order topology (precisely, each of the sets I_α with $\alpha\in\mathrm{Card}_\infty$ is equipped with this Borel structure).

Lemma 5.5.4. *For a function $f\colon \mathscr{X}\to\mathbb{R}_+\cup\mathrm{Card}$ the following conditions are equivalent:*

(i) *f is almost measurable,*
(ii) *there is $Z\in\mathcal{N}$ with the following properties:*
  (a) *$A=f^{-1}(\mathbb{R}_+)\setminus Z\in\mathfrak{M}$ and $f|_A\colon A\to\mathbb{R}_+$ is measurable,*
  (b) *for every $\alpha\in\mathrm{Card}_\infty$, $f^{-1}(\{\alpha\})\setminus Z\in\mathfrak{M}$,*
  (c) *for each standard set $B\in\mathfrak{M}$ there exists $Z_B\in\mathcal{N}$ such that $f(B\setminus Z_B)\cap\mathrm{Card}_\infty$ is countable (finite or not).*

Proof. Suppose all conditions of (ii) are fulfilled. In what follows we preserve the notation of (ii). Let $\mathcal{B}$ be a standard base of $\mathscr{X}$. Put $\mathscr{Z}=Z\cup\bigcup_{B\in\mathcal{B}}(B\cap Z_B)$. Then $\mathscr{Z}\in\mathcal{N}$ and (a)–(c) imply that $f|_{\mathscr{X}\setminus\mathscr{Z}}$ is measurable.

Now assume that $Z\in\mathcal{N}$ is such that $f|_{\mathscr{X}\setminus Z}$ is measurable. It is clear that (a) and (b) are satisfied. To show (c), it suffices to prove the following claim: if $(Y,\mathfrak{N})$ is a standard Borel space and $u\colon Y\to I_\gamma$ is measurable, then $D=u(Y)\cap\mathrm{Card}_\infty$ is countable. Since D is well ordered, D is countable iff so is the subset D_0 of D consisting of all elements of D which have an immediate predecessor (relative to D) in D. Note that if $\alpha\in D_0$, then $\{\alpha\}$ is open in D with respect to the topology inherited from I_γ. Consequently, every subset of D_0 is open in D and hence $Y_0=u^{-1}(D_0)$ is Borel and $u|_{Y_0}$ is a Borel function of Y_0 (which is a standard Borel space) onto the discrete space D_0. It therefore follows from the theory of Souslin sets that D_0 is countable. (Indeed, if D_0 were uncountable, there would exist a continuous mapping of D_0 onto a non-Souslin subset of $[0,1]$. It would then follow that a non-Souslin subset of $[0,1]$ could be the image of a standard Borel space under a Borel function, which is impossible.) ∎

Lemma 5.5.4 has two important consequences: if $f, g\colon \mathscr{X} \to \mathbb{R}_+ \cup \mathrm{Card}$ are almost measurable and $\alpha \in \mathrm{Card}$, the functions $f+g$ and $\alpha \cdot f$ are almost measurable as well. We shall use these facts several times.

In the next two paragraphs, $\Phi\colon \mathscr{D} \to \mathcal{SEP}_N$ is a summable field and $f\colon \mathscr{D} \to \mathbb{R}_+ \cup \mathrm{Card}$ is an almost measurable function where $\mathscr{D} \in \mathfrak{M}$ (notice that $\mathscr{D}$ is multi-standard).

We say that f *fits* to Φ iff there are two disjoint measurable sets D_1 and D_2 such that $\mathscr{D} \setminus (D_1 \cup D_2) \in \mathcal{N}$, $f(D_1) \subset \mathrm{Card}$ and $\Phi(D_2) \subset \mathcal{SM}_N$. (If f fits to Φ, $f(x) \odot \Phi(x)$ makes sense for almost all $x \in \mathscr{D}$.)

There is $Z \in \mathcal{N}$ such that the sets $A = f^{-1}(I_{\aleph_0} \setminus \{0\}) \setminus Z$ and $A_\alpha = f^{-1}(\{\alpha\}) \setminus Z$ with uncountable α's are measurable and the function $f|_A\colon A \to I_{\aleph_0}$ is Borel. We call the pair (f, Φ) *summable* if f fits to Φ and the field $A \ni x \mapsto f(x) \odot \Phi(x) \in \mathcal{SEP}_N$ is summable. If this is the case, we define $\bigoplus_{x\in\mathscr{D}}^{\mathcal{N}} f(x) \odot \Phi(x)$ by

$$\bigoplus_{x\in\mathscr{D}}^{\mathcal{N}} f(x) \odot \Phi(x) = \Big(\bigoplus_{x\in A}^{\mathcal{N}} f(x) \odot \Phi(x)\Big) \oplus \bigoplus_{\alpha>\aleph_0} \Big(\alpha \odot \bigoplus_{x\in A_\alpha}^{\mathcal{N}} \Phi(x)\Big).$$

It is clear that the summability of (f, Φ) and the formula for

$$\bigoplus_{x\in\mathscr{D}}^{\mathcal{N}} f(x) \odot \Phi(x)$$

are independent of the choice of Z. Notice that the summability of Φ is equivalent to the summability of (δ, Φ) where $\delta\colon \mathscr{D} \to \mathbb{R}_+ \cup \mathrm{Card}$ is constantly equal to 1.

The following properties follow from (DI0)–(DI4) (page 73) and (5.5.3). Everywhere below, $(f, \{\mathsf{A}^{(x)}\}_{x\in\mathscr{X}})$ is a summable pair.

(CS0) For each $\mathscr{D} \in \mathfrak{M}$ the pair $(f, \{\mathsf{A}^{(x)}\}_{x\in\mathscr{D}})$ is summable as well and $\bigoplus_{x\in\mathscr{D}}^{\mathcal{N}} f(x) \odot \mathsf{A}^{(x)} = \mathsf{O}$ iff $s_{\mathscr{D}}(f) := \{x \in \mathscr{D}\colon f(x) \neq 0\} \in \overline{\mathcal{N}}$; $\bigoplus_{x\in\mathscr{D}}^{\mathcal{N}} f(x) \odot \mathsf{A}^{(x)} \in \mathcal{SEP}_N$ iff there is $Z \in \mathcal{N}$ such that $s_{\mathscr{D}}(f) \setminus Z$ is standard and $f(\mathscr{D} \setminus Z) \subset I_{\aleph_0}$.

(CS1) The pair $(f, \{\mathfrak{b}(\mathsf{A}^{(x)})\}_{x\in\mathscr{X}})$ is summable and

$$\mathfrak{b}\Big(\bigoplus_{x\in\mathscr{X}}^{\mathcal{N}} f(x) \odot \mathsf{A}^{(x)}\Big) = \bigoplus_{x\in\mathscr{X}}^{\mathcal{N}} f(x) \odot \mathfrak{b}(\mathsf{A}^{(x)}).$$

(CS2) Whenever $\mathcal{B}$ is a base of $(\mathscr{X}, \mathfrak{M}, \mathcal{N})$,

$$\bigoplus_{x\in\mathscr{X}}^{\mathcal{N}} f(x) \odot \mathsf{A}^{(x)} = \bigoplus_{B\in\mathcal{B}} \Big(\bigoplus_{x\in B}^{\mathcal{N}} f(x) \odot \mathsf{A}^{(x)}\Big).$$

(CS3) (A) If $(f, \{\mathsf{B}^{(x)}\}_{x\in\mathscr{X}})$ is summable, so is $(f, \{\mathsf{A}^{(x)} \oplus \mathsf{B}^{(x)}\}_{x\in\mathscr{X}})$ and

$$\bigoplus_{x\in\mathscr{X}}^{\mathcal{N}} f(x) \odot (\mathsf{A}^{(x)} \oplus \mathsf{B}^{(x)}) = \Big(\bigoplus_{x\in\mathscr{X}}^{\mathcal{N}} f(x) \odot \mathsf{A}^{(x)}\Big) \oplus \Big(\bigoplus_{x\in\mathscr{X}}^{\mathcal{N}} f(x) \odot \mathsf{B}^{(x)}\Big).$$

(B) For every $\alpha \in \mathrm{Card}$, the pair $(\alpha \cdot f, \{\mathsf{A}^{(x)}\}_{x\in\mathscr{X}})$ is summable and

$$\bigoplus_{x\in\mathscr{X}}^{\mathcal{N}} (\alpha \cdot f(x)) \odot \mathsf{A}^{(x)} = \alpha \odot (\bigoplus_{x\in\mathscr{X}}^{\mathcal{N}} f(x) \odot \mathsf{A}^{(x)}).$$

(C) If in addition $(g, \{\mathsf{A}^{(x)}\}_{x\in\mathscr{X}})$ is summable, so is the pair $(f+g, \{\mathsf{A}^{(x)}\}_{x\in\mathscr{X}})$

and

$$\bigoplus_{x\in\mathscr{X}}^{\mathcal{N}} (f(x)+g(x)) \odot \mathsf{A}^{(x)} = \Big(\bigoplus_{x\in\mathscr{X}}^{\mathcal{N}} f(x) \odot \mathsf{A}^{(x)}\Big) \oplus \Big(\bigoplus_{x\in\mathscr{X}}^{\mathcal{N}} g(x) \odot \mathsf{A}^{(x)}\Big).$$

(CS4) If $\psi\colon \mathscr{X}' \to \mathscr{X}$ is an almost null-isomorphism, the pair $(f\circ\psi, \{\mathsf{A}^{(\psi(x'))}\}_{x'\in\mathscr{X}'})$ is summable and

$$\bigoplus_{x'\in\mathscr{X}'}^{\mathcal{N}'} f(\psi(x')) \odot \mathsf{A}^{(\psi(x'))} = \bigoplus_{x\in\mathscr{X}}^{\mathcal{N}} f(x) \odot \mathsf{A}^{(x)}.$$

Since properties (CS3)(B) and (CS3)(C) are of importance to us and are not so easy, let us prove them. It is quite simple that both the pairs appearing in the two assertions are summable. Thanks to (CS2), we may assume that $\mathscr{X}$ is standard. It then follows from Lemma 5.5.4 that we may also assume that both $f(\mathscr{X})\cap\mathrm{Card}_\infty$ and $g(\mathscr{X})\cap\mathrm{Card}_\infty$ are countable and f and g are Borel.

We start with (CS3)(B). Observe that (DI3) yields the assertion for $\alpha \leqslant \aleph_0$. So, $\aleph_0 \odot (\bigoplus_{x\in\mathscr{X}}^{\mathcal{N}} f(x)\odot\mathsf{A}^{(x)}) = \bigoplus_{x\in\mathscr{X}}^{\mathcal{N}} (\aleph_0\cdot f(x))\odot\mathsf{A}^{(x)}$. This implies that we may further assume that $f(\mathscr{X})\subset\mathrm{Card}_\infty$ (replacing f by $\aleph_0\cdot f$ and reducing $\mathscr{X}$ to $s(f)=s_{\mathscr{X}}(f)$). But then the assertion easily follows from (CS2) and the countability of $f(\mathscr{X})$.

We now turn to (CS3)(C). Put $A_f(\aleph_0) = f^{-1}(I_{\aleph_0})$ and $A_f(\alpha) = f^{-1}(\{\alpha\})$ for uncountable α. In the same way define $A_g(\beta)$ (corresponding to g) for $\beta\in\mathrm{Card}_\infty$. Notice that the sets $I_f = \{\alpha\in\mathrm{Card}_\infty\colon A_f(\alpha)\neq\emptyset\}$ and $I_g = \{\alpha\in\mathrm{Card}_\infty\colon A_g(\alpha)\neq\emptyset\}$ are countable and hence the family $\{A_f(\alpha)\cap A_g(\beta)\colon (\alpha,\beta)\in I_f\times I_g\}$ is a base of $(\mathscr{X},\mathfrak{M},\mathcal{N})$. Therefore—using again (CS2)—we may assume that I_f and I_g consist of single cardinals. The case $I_f = I_g = \{\aleph_0\}$ follows from (DI3), while the one when $\aleph_0\notin I_f\cup I_g$ is obvious. Finally, if e.g. $I_f=\{\aleph_0\}$ and $I_g=\{\alpha\}$ for some $\alpha>\aleph_0$, then (by (CS3)(B)) $\bigoplus_{x\in\mathscr{X}}^{\mathcal{N}}(f(x)+g(x))\odot\mathsf{A}^{(x)} = \bigoplus_{x\in\mathscr{X}}^{\mathcal{N}} g(x)\odot\mathsf{A}^{(x)} = \alpha\odot\bigoplus_{x\in\mathscr{X}}^{\mathcal{N}}\mathsf{A}^{(x)} \geqslant \bigoplus_{x\in\mathscr{X}}^{\mathcal{N}}\aleph_0\odot\mathsf{A}^{(x)}$ and (again by (CS2) and (CS3)(B))

$$\begin{aligned}\bigoplus_{x\in\mathscr{X}}^{\mathcal{N}} \aleph_0\odot\mathsf{A}^{(x)} &= \Big(\bigoplus_{x\in\mathscr{X}}^{\mathcal{N}} (\aleph_0\cdot f(x))\odot\mathsf{A}^{(x)}\Big) \oplus \Big(\bigoplus_{x\notin s(f)}^{\mathcal{N}} \aleph_0\odot\mathsf{A}^{(x)}\Big)\\ &\geq \aleph_0\odot\Big(\bigoplus_{x\in\mathscr{X}}^{\mathcal{N}} f(x)\odot\mathsf{A}^{(x)}\Big) \geqslant \bigoplus_{x\in\mathscr{X}}^{\mathcal{N}} f(x)\odot\mathsf{A}^{(x)},\end{aligned}$$

which finishes the proof.

We now repeat the idea of the previous chapter. Let $(f,\{\mathsf{A}^{(x)}\}_{x\in\mathscr{X}})$ be a summable pair. If

$$\bigoplus_{x\in\mathscr{D}'}^{\mathcal{N}} f(x)\odot\mathsf{A}^{(x)} \perp_u \bigoplus_{x\in\mathscr{D}''}^{\mathcal{N}} f(x)\odot\mathsf{A}^{(x)} \tag{5.5.4}$$

for any two disjoint sets $\mathscr{D}',\mathscr{D}''\in\mathfrak{M}$, we call the pair $(f,\{\mathsf{A}^{(x)}\}_{x\in\mathscr{X}})$ *regular* and we write $\boxplus_{x\in\mathscr{X}}^{\mathcal{N}} f(x)\odot\mathsf{A}^{(x)}$ in place of $\bigoplus_{x\in\mathscr{X}}^{\mathcal{N}} f(x)\odot\mathsf{A}^{(x)}$.

Similarly, a summable field $\{\mathsf{A}^{(x)}\}_{x\in\mathscr{X}}$ is regular iff (5.5.4) is satisfied with f constantly equal to 1. As usual, using $\boxplus_{x\in\mathscr{X}}^{\mathcal{N}} f(x)\odot\mathsf{A}^{(x)}$ presupposes that $(f,\{\mathsf{A}^{(x)}\}_{x\in\mathscr{X}})$ is regular. Note that, by definition, regular pairs and fields are summable.

The next result collects fundamental facts on the notion defined above.

THEOREM 5.5.5. *Let* $\Phi\colon \mathscr{X} \ni x \mapsto \mathsf{A}^{(x)} \in \mathfrak{F}_N$ *be any function.*

(I) *The following conditions are equivalent:*

 (i) *the field* $\{\mathsf{A}^{(x)}\}_{x\in\mathscr{X}}$ *is regular,*

 (ii) *for every standard set* $A \in \mathfrak{M}$ *there is* $Z \in \mathcal{N}$ *such that* $\Phi(A \setminus Z)$ *is a measurable domain and* $\Phi|_{A\setminus Z}$ *is a Borel isomorphism of* $A \setminus Z$ *onto* $\Phi(A \setminus Z)$.

(II) *If* Φ *satisfies condition* (I)(ii) *and* $f\colon \mathscr{X} \to \mathbb{R}_+ \cup \operatorname{Card}$ *is an almost measurable function which fits to* Φ, *then* $(f, \{\mathsf{A}^{(x)}\}_{x\in\mathscr{X}})$ *is regular. Moreover,*

$$\Big\{\mathsf{Y} \in \mathcal{CDD}_N : \mathsf{Y} \leqslant^s \boxplus_{x\in\mathscr{X}}^{\mathcal{N}} f(x) \odot \mathsf{A}^{(x)}\Big\} = \Big\{\boxplus_{x\in\mathscr{D}}^{\mathcal{N}} f(x) \odot \mathsf{A}^{(x)} : \mathscr{D} \in \mathfrak{M}\Big\}. \tag{5.5.5}$$

Proof. The implication '(i)⇒(ii)' in (I) follows immediately from Proposition 5.4.4. To prove the converse, first note that Φ is summable because of (ii), the existence of a standard base of $\mathscr{X}$ and Proposition 5.4.4. Further, take two disjoint nonnull measurable sets $\mathscr{D}_1$ and $\mathscr{D}_2$. Let $\mathcal{B}_j$ be a standard base of $\mathscr{D}_j$. Since $B_1 \cup B_2$ is standard for $B_j \in \mathscr{D}_j$, we infer from (ii) and Proposition 5.4.4 that $\bigoplus_{x\in B_1}^{\mathcal{N}} \mathsf{A}^{(x)} \perp_u \bigoplus_{x\in B_2}^{\mathcal{N}} \mathsf{A}^{(x)}$. Consequently, $\bigoplus_{B\in\mathcal{B}_1}(\bigoplus_{x\in B}^{\mathcal{N}} \mathsf{A}^{(x)}) \perp_u \bigoplus_{B\in\mathcal{B}_2}(\bigoplus_{x\in B}^{\mathcal{N}} \mathsf{A}^{(x)})$ and hence (i) follows from (CS2).

Now assume Φ and f are as in (II). We may assume that f is Borel. Define $f_0\colon \mathscr{X} \to I_{\aleph_0} \setminus \{0\}$ by $f_0(x) = f(x)$ if $f(x) \in I_{\aleph_0} \setminus \{0\}$ and $f_0(x) = 1$ otherwise. The function f_0 is Borel and fits to Φ. Let $B \in \mathfrak{M}$ be a standard set. Then there is a standard measure μ on $(B, \mathfrak{M}|_B)$ such that $\mathcal{N}(\mu) = \mathcal{N}|_B$. We infer from the assumptions that $(\mu, \Phi|_B) \in \operatorname{RGS}(B, \mathfrak{M}|_B)$. Hence, Lemma 5.4.8 implies that

$$(\mu, (f_0 \odot \Phi)|_B) \in \operatorname{RGS}(B, \mathfrak{M}|_B). \tag{5.5.6}$$

Consequently, if B_1 and B_2 are two disjoint standard (measurable) subsets of $\mathscr{X}$, then

$$\bigoplus_{x\in B_1}^{\mathcal{N}} f_0(x) \odot \mathsf{A}^{(x)} \perp_u \bigoplus_{x\in B_2}^{\mathcal{N}} f_0(x) \odot \mathsf{A}^{(x)}. \tag{5.5.7}$$

We also conclude from (5.5.6) that (f_0, Φ) is summable on every standard subset of $\mathscr{X}$. Since $\mathscr{X}$ is multi-standard, (f_0, Φ) is therefore summable. It now follows from the definitions of f_0 and of summability that (f, Φ) is summable as well.

Further, if B is a standard subset of $\mathscr{X}$, it follows from Lemma 5.5.4 and the definitions of f_0 and of $\bigoplus_{x\in B}^{\mathcal{N}} f(x) \odot \Phi(x)$ that $\bigoplus_{x\in B}^{\mathcal{N}} f(x) \odot \Phi(x) \ll \bigoplus_{x\in B}^{\mathcal{N}} f_0(x) \odot \mathsf{A}^{(x)}$. This, combined with (5.5.7), yields

$$\bigoplus_{x\in B_1}^{\mathcal{N}} f(x) \odot \mathsf{A}^{(x)} \perp_u \bigoplus_{x\in B_2}^{\mathcal{N}} f(x) \odot \mathsf{A}^{(x)} \tag{5.5.8}$$

for any two disjoint standard sets $B_1, B_2 \subset \mathscr{X}$. Now if $\mathscr{D}'$ and $\mathscr{D}''$ are two arbitrary disjoint nonnull Borel subsets of $\mathscr{X}$, the fact that they are multi-standard together with (CS2) and (5.5.8) gives (5.5.4). It therefore suffices to check (5.5.5). We have already shown the inclusion '⊃' in (5.5.5) (cf. (CS2)). Finally, fix $\mathsf{Y} \in \mathcal{CDD}_N$ such that

$$\mathsf{Y} \leqslant^s \boxplus_{x\in\mathscr{X}}^{\mathcal{N}} f(x) \odot \mathsf{A}^{(x)}. \tag{5.5.9}$$

Let $\mathcal{B}_0$ be a standard base of $\mathscr{X}$. Thanks to Lemma 5.5.4, for every $B \in \mathcal{B}_0$ there are pairwise disjoint measurable subsets $W_0^B, W_1^B, \ldots$ of B such that $B \setminus \bigcup_{n=0}^{\infty} W_n^B \in \mathcal{N}$, $f(W_0^B) \subset \mathbb{R}_+ \setminus \{0\}$ and $f|_{W_n^B}$ is constantly equal to some $\alpha \in \mathrm{Card}_\infty \cup \{0\}$. Notice that then $\mathcal{B} = \{W_n^B \colon B \in \mathcal{B}_0,\ n \geqslant 0\} \setminus \mathcal{N}$ is a standard base of $\mathscr{X}$ as well. Denote by $\mathcal{B}_f$ the set of all $B \in \mathcal{B}$ for which $f(B) \subset \mathbb{R}_+ \setminus \{0\}$ and let $\mathcal{B}' = \mathcal{B} \setminus \mathcal{B}_f$. For each $B \in \mathcal{B}'$ there is a (unique) $\alpha_B \in \mathrm{Card}_\infty \cup \{0\}$ such that $f(B) = \{\alpha_B\}$. Now (CS2), (CS3) and a part of (II) already proved give

$$\boxplus_{x \in \mathscr{X}}^{\mathcal{N}} f(x) \odot \mathsf{A}^{(x)} = \Big[\boxplus_{B \in \mathcal{B}_f} \Big(\boxplus_{x \in B}^{\mathcal{N}} f(x) \odot \mathsf{A}^{(x)}\Big)\Big] \boxplus \Big[\boxplus_{B \in \mathcal{B}'} \alpha_B \odot \Big(\boxplus_{x \in B}^{\mathcal{N}} \mathsf{A}^{(x)}\Big)\Big]. \tag{5.5.10}$$

It may be deduced from (5.5.9) and (5.5.10) (using e.g. Proposition 3.1.4 and Theorem 3.1.1) that Y is of the form

$$\mathsf{Y} = \Big(\boxplus_{B \in \mathcal{B}_f} \mathsf{Y}_B\Big) \boxplus \Big(\boxplus_{B \in \mathcal{B}'} \widetilde{\mathsf{Y}}_B\Big)$$

where $\mathsf{Y}_B \leqslant^s \boxplus_{x \in B}^{\mathcal{N}} f(x) \odot \mathsf{A}^{(x)}$ for $B \in \mathcal{B}_f$ and $\widetilde{\mathsf{Y}}_B \leqslant^s \alpha_B \odot \boxplus_{x \in B}^{\mathcal{N}} \mathsf{A}^{(x)}$ for $B \in \mathcal{B}'$. Further, by (PR6) (page 13), for each $B \in \mathcal{B}'$ there is $\mathsf{Y}_B \leqslant^s \boxplus_{x \in B}^{\mathcal{N}} \mathsf{A}^{(x)}$ such that $\widetilde{\mathsf{Y}}_B = \alpha_B \odot \mathsf{Y}_B$. Since $\mathcal{B}$ consists of standard sets, we infer from Proposition 5.4.6 that for every $B \in \mathcal{B}$ there exists a measurable set $\mathscr{D}_B \subset B$ for which $\mathsf{Y}_B = \boxplus_{x \in \mathscr{D}_B}^{\mathcal{N}} f(x) \odot \mathsf{A}^{(x)}$ provided $B \in \mathcal{B}_f$ and $\mathsf{Y}_B = \boxplus_{x \in \mathscr{D}_B}^{\mathcal{N}} \mathsf{A}^{(x)}$ if $B \in \mathcal{B}'$. Put $\mathscr{D} = \bigcup_{B \in \mathcal{B}} \mathscr{D}_B$ and note that $\mathscr{D}$ is Borel since $\mathcal{B}$ is a base. Finally, the family $\{\mathscr{D}_B \colon B \in \mathcal{B}\}$ is a base of $\mathscr{D}$ and hence we deduce from (CS2) and (CS3) that

$$\boxplus_{x \in \mathscr{D}}^{\mathcal{N}} f(x) \odot \mathsf{A}^{(x)} = \boxplus_{B \in \mathcal{B}} \Big(\boxplus_{x \in \mathscr{D}_B}^{\mathcal{N}} f(x) \odot \mathsf{A}^{(x)}\Big) = \Big(\boxplus_{B \in \mathcal{B}_f} \mathsf{Y}_B\Big) \boxplus \Big(\boxplus_{B \in \mathcal{B}'} \alpha_B \odot \mathsf{Y}_B\Big) = \mathsf{Y},$$

and we are done. ∎

Similarly to the previous chapter, for a field $\Phi \colon \mathscr{X} \to \mathfrak{F}_N$ we shall write $\Phi \in \mathrm{RGS}_{\mathrm{loc}}$ or $\Phi \in \mathrm{RGS}_{\mathrm{loc}}(\mathscr{X})$ if Φ satisfies condition (ii) of Theorem 5.5.5.

5.6. Prime decomposition

Semiprimes are those members of $\mathfrak{p}_N$ which make the issue of prime decomposition of N-tuples more complicated and ambiguous. To shape this in a way similar to that in the ring of natural numbers, we have to allow multiplicity functions to take real values (beside infinite cardinals) instead of (only) integer ones. Such an approach is therefore similar to Ernest's multiplicity theory (Chapter 4 of [9]) and will enable us to propose the prime decomposition of an arbitrary N-tuple in an (essentially) unique form (see Theorem 5.6.14). We consider this a more attractive manner of 'factorial decomposing' of N-tuples than Ernest's central decomposition [9].

In this chapter $(\mathscr{X}, \Phi)$ is a fixed pair such that $(\mathscr{X}, \mathfrak{M}, \mathcal{N})$ is a multi-standard measurable space with nullity and $\Phi \in \mathrm{RGS}_{\mathrm{loc}}(\mathscr{X})$ is such that $\Phi(\mathscr{X}) \subset \mathfrak{p}_N$. After removing from $\mathscr{X}$ a null measurable set, we may assume Φ is measurable. Let

$$\mathscr{X}_I = \Phi^{-1}(\mathfrak{a}_N), \qquad \mathscr{X}_{II} = \Phi^{-1}(\mathfrak{s}_N), \qquad \mathscr{X}_{III} = \Phi^{-1}(\mathfrak{f}_N).$$

Notice that $\mathscr{X}_I$, $\mathscr{X}_{II}$ and $\mathscr{X}_{III}$ are measurable, pairwise disjoint and $\mathscr{X}_I \cup \mathscr{X}_{II} \cup \mathscr{X}_{III} = \mathscr{X}$.

DEFINITION 5.6.1. A function $f\colon \mathscr{D} \to \mathbb{R}_+ \cup \mathrm{Card}$, where $\mathscr{D} \in \mathfrak{M}$, is *admissible* for Φ iff f is almost measurable, $f(\mathscr{X}_I \cap \mathscr{D}) \subset \mathrm{Card}$ and $f(\mathscr{X}_{III} \cap \mathscr{D}) \subset \{0\} \cup \mathrm{Card}_\infty$. The class of all admissible functions on $\mathscr{X}$ is denoted by $\mathscr{A}(\mathscr{X}, \Phi)$ or briefly by $\mathscr{A}(\mathscr{X})$.

For each $f \in \mathscr{A}(\mathscr{X})$, $s(f)$ is the *support* of f, i.e. $s(f) = \{x \in \mathscr{X} : f(x) \neq 0\}$ ($s(f)$ is measurable provided so is f).

Note that each admissible function fits to Φ. Thus, by Theorem 5.5.5, for every $f \in \mathscr{A}(\mathscr{X})$ we may write $\boxplus_{x\in\mathscr{X}}^{\mathcal{N}} f(x) \odot \Phi(x)$. As is practised in measure theory, the term *almost everywhere*, abbreviated *a.e.*, will mean that the relevant property (relation, etc.) holds on $\mathscr{X} \setminus Z$ for some $Z \in \overline{\mathcal{N}}$.

As a consequence of Lemma 5.5.4 we obtain

COROLLARY 5.6.2. *For $f, g \in \mathscr{A}(\mathscr{X})$,*

(a) $f + g$, $f \cdot g$, $f \vee g$, $f \wedge g \in \mathscr{A}(\mathscr{X})$ *where* $f \vee g = \max(f, g)$ *and* $f \wedge g = \min(f, g)$,
(b) $\alpha \cdot f \in \mathscr{A}(\mathscr{X})$ *for each* $\alpha \in \mathrm{Card}$,
(c) *if* $f(\mathscr{X}_I \cup \mathscr{X}_{III}) \subset \{0\}$, $t \cdot f \in \mathscr{A}(\mathscr{X})$ *for every* $t \in \mathbb{R}_+$,
(d) *if* $f \leqslant g$ *a.e., there is* $u \in \mathscr{A}(\mathscr{X})$ *such that* $g = f + u$ *a.e.*

We leave the proof of Corollary 5.6.2 as an exercise. A part of it may be strengthened:

LEMMA 5.6.3. *Whenever $f_1, f_2, \ldots$ are admissible functions, so are $\bigwedge_{n\geqslant 1} f_n\colon \mathscr{X} \ni x \mapsto \inf_{n\geqslant 1} f_n(x) \in \mathbb{R}_+ \cup \mathrm{Card}$ and $\bigvee_{n\geqslant 1} f_n\colon \mathscr{X} \ni x \mapsto \sup_{n\geqslant 1} f_n(x) \in \mathbb{R}_+ \cup \mathrm{Card}$. In particular, $\sum_{n=1}^\infty f_n \in \mathscr{A}(\mathscr{X})$ (where $(\sum_{n=1}^\infty f_n)(x) = \sum_{n=1}^\infty f_n(x)$).*

Proof. We leave it as an exercise that it is enough to show, thanks to Lemma 5.5.4, that the closure of any countable subset K of Card_∞ (in $I_\gamma \supset K$ for any $\gamma \in \mathrm{Card}_\infty$) is countable as well (recall that countable compact Hausdorff spaces are metrizable, by [9, Theorem 3.1.9]). But this is quite simple: for every element x (except the last) of $L = (\mathrm{cl}\, K) \setminus K$ there exists $c_x \in K$ which lies between x and its immediate successor (relative to L) in L. Since $L \ni x \mapsto c_x \in K$ is one-to-one, the assertion follows. ∎

PROPOSITION 5.6.4. *For $f, g \in \mathscr{A}(\mathscr{X})$,*

$$\boxplus_{x\in\mathscr{X}}^{\mathcal{N}} f(x) \odot \Phi(x) = \boxplus_{x\in\mathscr{X}}^{\mathcal{N}} g(x) \odot \Phi(x) \tag{5.6.1}$$

iff $f = g$ a.e.

Proof. The 'if' part is clear. Suppose (5.6.1) holds. It follows from (CS3) (page 85) that $\boxplus_{x\in\mathscr{B}}^{\mathcal{N}} u(x) \odot \Phi(x) \ll \boxplus_{x\in\mathscr{B}}^{\mathcal{N}} \Phi(x)$ for each $\mathscr{B} \in \mathfrak{M}$ and $u \in \{f, g\}$. Since $\boxplus_{x\in\mathscr{B}}^{\mathcal{N}} \Phi(x) \perp_u \boxplus_{x\notin\mathscr{B}}^{\mathcal{N}} \Phi(x)$, (5.6.1) and (CS2) imply therefore that

$$\boxplus_{x\in\mathscr{B}}^{\mathcal{N}} f(x) \odot \Phi(x) = \boxplus_{x\in\mathscr{B}}^{\mathcal{N}} g(x) \odot \Phi(x) \tag{5.6.2}$$

for any $\mathscr{B} \in \mathfrak{M}$. Let $\mathscr{D} \in \mathfrak{M}$ be standard. It suffices to check that $f = g$ almost everywhere on $\mathscr{D}$. Thanks to Lemma 5.5.4 we may assume that $f|_{\mathscr{D}}$ and $g|_{\mathscr{D}}$ are Borel and

$$(f(\mathscr{D}) \cup g(\mathscr{D})) \cap \mathrm{Card}_\infty \text{ is countable.} \tag{5.6.3}$$

By (5.6.3), the sets $\mathscr{D}_+ = \{x \in \mathscr{D}\colon f(x) < g(x)\}$ and $\mathscr{D}_- = \{x \in \mathscr{D}\colon f(x) > g(x)\}$ are Borel. Suppose, to the contrary, that e.g. $\mathscr{D}_+ \notin \mathcal{N}$. We consider two cases.

Assume there are a nonnull measurable set $\mathscr{B} \subset \mathscr{D}_+$ and two cardinals α and β such that $f(\mathscr{B}) = \{\alpha\}$ and $g(\mathscr{B}) = \{\beta\}$. Let $\mathsf{B} = \boxplus_{x\in\mathscr{B}}^{\mathcal{N}} \Phi(x)$. We infer from (CS0) that $\mathsf{B} \neq \mathsf{O}$. Moreover, since $\Phi(\mathscr{X}) \subset \mathfrak{p}_N$ and $\Phi \in \mathrm{RGS}_{\mathrm{loc}}$, and $\mathscr{B}$ is standard, Lemma 5.4.9 implies that B is the direct sum of a minimal N-tuple and a semiminimal one. Consequently, $\alpha \odot \mathsf{B} < \beta \odot \mathsf{B}$ (use e.g. Theorem 3.6.1 and (AO4), page 32, if applicable). But this contradicts (5.6.2) because $\boxplus_{x\in\mathscr{B}}^{\mathcal{N}} f(x) \odot \Phi(x) = \alpha \odot \mathsf{B}$ and $\boxplus_{x\in\mathscr{B}}^{\mathcal{N}} g(x) \odot \Phi(x) = \beta \odot \mathsf{B}$.

Finally, if there is no set $\mathscr{B}$ with all above-mentioned properties, it may be deduced from (5.6.3) that there exists a nonnull measurable set $\mathscr{B} \subset \mathscr{D}_+ \cap \mathscr{X}_{II}$ such that $f(\mathscr{B}) \subset \mathbb{R}_+$. Let $\mathsf{B} = \boxplus_{x\in\mathscr{B}}^{\mathcal{N}} f(x) \odot \Phi(x)$. As before, an application of Lemma 5.4.9 shows that

$$\mathsf{B} \in \mathcal{SM}_N. \tag{5.6.4}$$

On the other hand, there is a measurable function $u\colon \mathscr{B} \to (\mathbb{R}_+ \cup \mathrm{Card}) \setminus \{0\}$ such that $g(x) = f(x) + u(x)$ for all $x \in \mathscr{B}$. Then (CS3) combined with (5.6.2) gives $\mathsf{B} = \boxplus_{x\in\mathscr{B}}^{\mathcal{N}} g(x) \odot \Phi(x) = \mathsf{B} \oplus (\boxplus_{x\in\mathscr{B}}^{\mathcal{N}} u(x) \odot \Phi(x))$, which means, thanks to (5.6.4), that $\boxplus_{x\in\mathscr{B}}^{\mathcal{N}} u(x) \odot \Phi(x) = \mathsf{O}$ (cf. (AO4)), contrary to (CS0). ∎

THEOREM 5.6.5. *Let* $\mathsf{T} = \boxplus_{x\in\mathscr{X}}^{\mathcal{N}} \Phi(x)$. *Then*

$$\Big\{\boxplus_{x\in\mathscr{X}}^{\mathcal{N}} f(x) \odot \Phi(x)\colon f \in \mathscr{A}(\mathscr{X}, \Phi)\Big\} = \{\mathsf{X} \in \mathcal{CDD}_N : \mathsf{X} \ll \mathsf{T}\}.$$

Proof. It easily follows from (CS3) (page 85) that $\boxplus_{x\in\mathscr{X}}^{\mathcal{N}} f(x) \odot \Phi(x) \ll \mathsf{T}$ for every $f \in \mathscr{A}(\mathscr{X})$. We fix $\mathsf{X} \in \mathcal{CDD}_N$ such that $\mathsf{X} \ll \mathsf{T}$. Let $\{B_s\}_{s\in S}$ be a standard base of $\mathscr{X}$. We may assume that $\bigcup_{s\in S} B_s = \mathscr{X}$. For each $s \in S$ put $\mathsf{T}_s = \boxplus_{x\in B_s}^{\mathcal{N}} \Phi(x)$. We infer from (CS0) that $\mathsf{T}_s \in \mathcal{SEP}_N$ and from (CS2) that $\mathsf{T} = \boxplus_{s\in S} \mathsf{T}_s$. Let $\mathsf{X}_s = \mathsf{E}(\mathsf{X}|\mathsf{T}_s)$. Observe that $\mathsf{X} = \boxplus_{s\in S} \mathsf{X}_s$ and $\mathsf{X}_s \ll \mathsf{T}_s$. Suppose for each $s \in S$ there is an admissible function $f_s\colon B_s \to \mathbb{R}_+ \cup \mathrm{Card}$ such that $\mathsf{X}_s = \boxplus_{x\in B_s}^{\mathcal{N}} f_s(x) \odot \Phi(x)$. Then the union $f\colon \mathscr{X} \to \mathbb{R}_+ \cup \mathrm{Card}$ of f_s's is admissible as well and it follows from (CS2) that

$$\boxplus_{x\in\mathscr{X}}^{\mathcal{N}} f(x) \odot \Phi(x) = \boxplus_{s\in S}\Big(\boxplus_{x\in B_s}^{\mathcal{N}} f_s(x) \odot \Phi(x)\Big) = \boxplus_{s\in S} \mathsf{X}_s = \mathsf{X}.$$

The above argument reduces the problem to the case when $\mathscr{X}$ is standard. Then there is a standard measure μ on $\mathfrak{M}$ such that $\mathcal{N}(\mu) = \mathcal{N}$. Consequently,

$$\boxplus_{x\in\mathscr{X}}^{\mathcal{N}} f(x) \odot \Phi(x) = \int_{\mathscr{X}}^{\boxplus} f(x) \odot \Phi(x)\, d\mu(x) \tag{5.6.5}$$

for every Borel function $f\colon \mathscr{X} \to I_{\aleph_0}$ which fits to Φ. Recall that for each $\mathsf{A} \in \mathcal{CDD}_N$, $s(\mathsf{A})$ is given by (4.4.5) (page 47) and $s(\mathsf{A}) = \bigwedge\{\mathsf{E} \leqslant^s \mathsf{J}\colon \mathsf{A} \ll \mathsf{E}\}$. Since $\mathsf{T} \in \mathcal{SEP}_N$ (because $\mathscr{X}$ is standard), $s(\mathsf{T}) \in \mathcal{SEP}_N$ as well. So, if $\mathsf{X} \ll \mathsf{T}$, then $s(\mathsf{X}) \leqslant^s s(\mathsf{T})$ and consequently the set $J = \{(i,\alpha) \in \Upsilon\colon \mathsf{E}_\alpha^i(\mathsf{X}) \neq \mathsf{O}\}$ is countable.

We infer from Lemma 5.4.9 that:

- T is the direct sum of a minimal N-tuple and a semiminimal one,
- there is $\lambda \in \operatorname{rgm}(\mathfrak{p}_N)$ such that $\mathsf{T} = \int_{\mathfrak{p}_N}^{\boxplus} \mathsf{P}\, d\lambda(\mathsf{P})$,
- for each $(i,\alpha) \in J$ there is a Borel function $u_\alpha^i \colon \mathfrak{p}_N \to I_{\aleph_0}$ such that $u_\alpha^i(\mathfrak{a}_N) \subset \mathrm{Card}$, $u_\alpha^i(\mathfrak{f}_N) \subset \{0, \aleph_0\}$ and

$$\begin{aligned} \mathsf{E}_\alpha^i(\mathsf{X}) &= \int_{\mathfrak{p}_N}^{\boxplus} u_\alpha^i(\mathsf{P}) \odot \mathsf{P}\, d\lambda(\mathsf{P}) \quad \text{if } (i,\alpha) \neq (\mathit{II}, 1), \\ \mathsf{E}_{sm}(\mathsf{X}) &= \int_{\mathfrak{p}_N}^{\boxplus} u_\alpha^i(\mathsf{P}) \odot \mathsf{P}\, d\lambda(\mathsf{P}) \quad \text{if } (i,\alpha) = (\mathit{II}, 1). \end{aligned} \tag{5.6.6}$$

Further, it follows from Corollary 5.4.7 that

$$\Phi^*(\mu) \ll \lambda \ll \Phi^*(\mu) \tag{5.6.7}$$

(cf. (5.6.5)). Since $\mathscr{X}$ is standard and $\Phi \in \mathrm{RGS}_{\mathrm{loc}}(\mathscr{X})$, we may assume that Φ is a Borel isomorphism of $\mathscr{X}$ onto a measurable domain. Put $g_\alpha^i = u_\alpha^i \circ \Phi$ for $(i,\alpha) \in J$ and note that $g_\alpha^i \in \mathscr{A}(\mathscr{X})$. Now (5.6.5), (5.6.6) and (5.6.7) combined with (DI4) (page 74) for every $(i,\alpha) \in J$ yield

$$\mathsf{E}_\alpha^i(\mathsf{X}) = \bigboxplus_{x \in \mathscr{X}}^{\mathcal{N}} g_\alpha^i(x) \odot \Phi(x) \quad \text{if } (i,\alpha) \neq (\mathit{II}, 1), \tag{5.6.8}$$

$$\mathsf{E}_{sm}(\mathsf{X}) = \bigboxplus_{x \in \mathscr{X}}^{\mathcal{N}} g_\alpha^i(x) \odot \Phi(x) \quad \text{if } (i,\alpha) = (\mathit{II}, 1). \tag{5.6.9}$$

Let (i,α) and (i',α') be distinct elements of J. Suppose $s(g_\alpha^i) \cap s(g_{\alpha'}^{i'}) \notin \mathcal{N}$ ($s(g_\alpha^i)$'s are measurable since g_α^i's are). Then there is a nonnull measurable set $\mathscr{B}$ which is contained in $s(g_\alpha^i) \cap s(g_{\alpha'}^{i'})$. Consequently, thanks to (CS3) and (5.6.8)–(5.6.9), $\aleph_0 \odot \boxplus_{x\in\mathscr{B}}^{\mathcal{N}} \Phi(x) \leqslant \aleph_0 \odot \mathsf{E}_\alpha^i(\mathsf{X})$ as well as $\aleph_0 \odot \boxplus_{x\in\mathscr{B}}^{\mathcal{N}} \Phi(x) \leqslant \aleph_0 \odot \mathsf{E}_{\alpha'}^{i'}(\mathsf{X})$, which is impossible since $\mathsf{E}_\alpha^i(\mathsf{X}) \perp_u \mathsf{E}_{\alpha'}^{i'}(\mathsf{X})$ and $\boxplus_{x\in\mathscr{B}}^{\mathcal{N}} \Phi(x) \neq \mathsf{O}$. This proves that $s(g_\alpha^i) \cap s(g_{\alpha'}^{i'}) \in \mathcal{N}$ for any distinct (i,α) and (i',α') in J. It then follows from the countability of J that there is $\mathscr{Z} \in \mathcal{N}$ such that the sets $\mathscr{S}_\alpha^i = s(g_\alpha^i) \setminus \mathscr{Z}$ ($(i,\alpha) \in J$) are pairwise disjoint. Now we define $f\colon \mathscr{X} \to \mathbb{R}_+ \cup \mathrm{Card}$ by the rules: $f(x) = \alpha \cdot g_\alpha^i(x)$ for $x \in \mathscr{S}_\alpha^i$ with $(i,\alpha) \in J \setminus \{(\mathit{II},1)\}$; $f(x) = g_1^{\mathit{II}}(x)$ for $x \in \mathscr{S}_1^{\mathit{II}}$ provided $(\mathit{II},1) \in J$; and $f(x) = 0$ for $x \notin \bigcup_{(i,\alpha)\in J} \mathscr{S}_\alpha^i$. It follows from the construction that $f \in \mathscr{A}(\mathscr{X})$. Finally, Theorem 3.6.1, (5.6.8)–(5.6.9), (CS2) and (CS3) (page 85) give $\mathsf{X} = \boxplus_{x\in\mathscr{X}}^{\mathcal{N}} f(x) \odot \Phi(x)$. ∎

Theorem 5.6.5 asserts that $\mathcal{I}(\Phi) = \{\boxplus_{x\in\mathscr{X}}^{\mathcal{N}} f(x) \odot \Phi(x)\colon f \in \mathscr{A}(\mathscr{X})\}$ is an ideal. We call a quadruple $(\mathscr{Y}, \mathfrak{N}, \mathcal{Z}, \Psi)$ or a pair $(\mathscr{Y}, \Psi)$ a *covering* for an ideal $\mathcal{A} \subset \mathcal{CDD}_N$ iff $(\mathscr{Y}, \mathfrak{N}, \mathcal{Z})$ is a multi-standard measurable space with nullity, $\Psi \in \mathrm{RGS}_{\mathrm{loc}}(\mathscr{Y})$, $\Psi(\mathscr{Y}) \subset \mathfrak{p}_N$ and $\mathcal{I}(\Psi) = \mathcal{A}$ (with this terminology we are inspired by condition (ii) of Theorem 5.5.5). Whenever the ideal $\mathcal{A}$ is irrelevant, we shall speak briefly of a *covering*. A *full covering* is a covering for $\mathcal{CDD}_N$.

As usual, whenever $\mathscr{D} \in \mathfrak{M}$, $j_{\mathscr{D}}$ stands for the characteristic function of $\mathscr{D}$.

Corollary 5.6.6. *Let $f,g,h_1,h_2,\ldots \in \mathscr{A}(\mathscr{X})$, $\mathsf{X} = \boxplus_{x\in\mathscr{X}}^{\mathcal{N}} f(x)\odot\Phi(x)$ and $\mathsf{Y} = \boxplus_{x\in\mathscr{X}}^{\mathcal{N}} g(x)\odot\Phi(x)$.*

(A) $\mathsf{X} \leqslant \mathsf{Y}$ *iff* $f \leqslant g$ *a.e.*
(B) $\mathsf{X} \perp_u \mathsf{Y}$ *iff* $f\cdot g = 0$ *a.e.*
(C) $\mathsf{X} \ll \mathsf{Y}$ *iff* $s(f)\setminus s(g) \in \overline{\mathcal{N}}$.
(D) $\mathsf{X} \leqslant^s \mathsf{Y}$ *iff* $f = g\cdot j_{\mathscr{D}}$ *a.e. for some* $\mathscr{D}\in\mathfrak{M}$.
(E) $\boxplus_{x\in\mathscr{X}}^{\mathcal{N}}[\sum_{n=1}^{\infty} h_n(x)]\odot\Phi(x) = \bigoplus_{n=1}^{\infty}[\boxplus_{x\in\mathscr{X}}^{\mathcal{N}} h_n(x)\odot\Phi(x)]$.

Proof. Observe that (D) is an immediate consequence of (5.5.5) (page 87) and Proposition 5.6.4; (B) follows from (A) and Theorem 5.6.5; (E) is a consequence of (A), (CS3) (page 85) and (AO6) (page 32); while (C) follows from (CS3) and (B). It therefore suffices to prove (A). The implication '$\Leftarrow$' is a consequence of (CS3) and Corollary 5.6.2(d). Finally, the converse follows from Proposition 5.6.4 and Theorem 5.6.5. Indeed, if $\mathsf{X}\leqslant\mathsf{Y}$, there is $\mathsf{A}\in\mathcal{CDD}_N$ such that $\mathsf{Y} = \mathsf{X}\oplus\mathsf{A}$. Then $\mathsf{A}\in\mathcal{I}(\Phi)$ and consequently there is $h\in\mathscr{A}(\mathscr{X})$ for which $\mathsf{A} = \boxplus_{x\in\mathscr{X}}^{\mathcal{N}} h(x)\odot\Phi(x)$. We now deduce from (CS3) that $\boxplus_{x\in\mathscr{X}}^{\mathcal{N}} g(x)\odot\Phi(x) = \boxplus_{x\in\mathscr{X}}^{\mathcal{N}}(f+h)(x)\odot\Phi(x)$ and hence, by Proposition 5.6.4, $g = f+h$ a.e. ∎

For the next result, we put $\mathscr{X}_{I_n} = \Phi^{-1}(\mathfrak{a}_N(n))$, $\mathscr{X}_{II_1} = \Phi^{-1}(\mathfrak{s}_N(1))$ and $\mathscr{X}_{II_\infty} = \Phi^{-1}(\mathfrak{s}_N(\infty))$. Observe that these sets are pairwise disjoint, $\mathscr{X}_I = \bigcup_{n=1}^{n=\infty}\mathscr{X}_{I_n}$ and $\mathscr{X}_{II} = \mathscr{X}_{II_1}\cup\mathscr{X}_{II_\infty}$. For simplicity, we assume that Φ is measurable, which implies that all these sets are measurable as well.

Corollary 5.6.7. *Let $f\in\mathscr{A}(\mathscr{X})$ and $\mathsf{A} = \boxplus_{x\in\mathscr{X}}^{\mathcal{N}} f(x)\odot\Phi(x)$.*

(a) $\mathsf{A}\in\mathcal{MF}_N$ (*respectively* $\mathsf{A}\in\mathcal{HIM}_N$; $\mathsf{A}\in\mathcal{SM}_N$) *iff* $f = j_{\mathscr{D}}$ *a.e. for some measurable* $\mathscr{D}\subset\mathscr{X}_I$ (*respectively* $f = \aleph_0\cdot j_{\mathscr{D}}$ *a.e. for some measurable* $\mathscr{D}\subset\mathscr{X}_{III}$; *there is* $\mathscr{Z}\in\mathcal{N}$ *such that* $f((\mathscr{X}_I\cup\mathscr{X}_{III})\setminus\mathscr{Z})\subset\{0\}$ *and* $f(\mathscr{X}_{II}\setminus\mathscr{Z})\subset\mathbb{R}_+$). *In particular,* $\boxplus_{x\in\mathscr{X}}^{\mathcal{N}}\Phi(x)$ *is the direct sum of a minimal N-tuple and a semiminimal one.*
(b) $\mathsf{A}\in\mathcal{SEP}_N$ (*respectively* $\mathsf{A}\in\mathfrak{a}_N$; $\mathsf{A}\in\mathfrak{f}_N$; $\mathsf{A}\in\mathfrak{s}_N$; $\mathsf{A}\in\mathfrak{F}_N$) *iff there is* $\mathscr{Z}\in\mathcal{N}$ *such that* $s(f)\setminus\mathscr{Z}$ *is standard and* $f(\mathscr{X}\setminus\mathscr{Z})\subset I_{\aleph_0}$ (*respectively* $f = j_{\{x\}}$ *a.e. for some* $x\in\mathscr{X}_I\cap\mathscr{X}^d$; $f = \aleph_0\cdot j_{\{x\}}$ *a.e. for some* $x\in\mathscr{X}_{III}\cap\mathscr{X}^d$; $f = t\cdot j_{\{x\}}$ *a.e. for some* $x\in\mathscr{X}_{II}\cap\mathscr{X}^d$ *and* $t\in\mathbb{R}_+\setminus\{0\}$; $f = s\cdot j_{\{x\}}$ *a.e. for some* $x\in\mathscr{X}^d$ *and* $s\in I_{\aleph_0}\setminus\{0\}$).
(c) A *is type* I; I^n; II; II^1; II^∞; III *iff, respectively,* $s(f)\setminus\mathscr{X}_I$; $s(f)\setminus\mathscr{X}_{I_n}$; $s(f)\setminus\mathscr{X}_{II}$; $s(f)\setminus\mathscr{X}_{II_1}$; $s(f)\setminus\mathscr{X}_{II_\infty}$; $s(f)\setminus\mathscr{X}_{III}$ *is a member of* $\overline{\mathcal{N}}$.
(d) $\mathsf{A}^d = \boxplus_{x\in\mathscr{X}^d} f(x)\odot\Phi(x)$ *and* $\mathsf{A}^c = \boxplus_{x\in\mathscr{X}^c}^{\mathcal{N}} f(x)\odot\Phi(x)$.
(e) *Let* $\mathscr{Z}\in\mathcal{N}$ *be such that* $f|_{\mathscr{X}\setminus\mathscr{Z}}$ *is Borel and* $\mathscr{X}\setminus\mathscr{Z}$ *is the union of a base* $\mathcal{B}$ *consisting of sets each of which is isomorphic either to* $([0,1],\mathfrak{B}([0,1]),\mathcal{L}_0)$ *or to a one-point nontrivial measurable space with nullity* (*there exists such* $\mathscr{Z}$). *Put* $\mathscr{E}_{sm} = f^{-1}(\mathbb{R}_+\setminus\{0\})\cap\mathscr{X}_{II}\setminus\mathscr{Z}$ *and* $\mathscr{E}_\alpha^i = f^{-1}(\{\alpha\})\cap\mathscr{X}_i\setminus\mathscr{Z}$ *for* $(i,\alpha)\in\Upsilon_*$. *Then* $\mathcal{E} = \{\mathscr{E}_\alpha^i\colon (i,\alpha)\in\Upsilon_*\}\cup\{\mathscr{E}_{sm}\}$ *is a base of* $\mathscr{X}$; *and* $\mathsf{E}_{sm}(\mathsf{A}) = \boxplus_{x\in\mathscr{E}_{sm}}^{\mathcal{N}} f(x)\odot\Phi(x)$, $\mathsf{E}_\alpha^i(\mathsf{A}) = \boxplus_{x\in\mathscr{E}_\alpha^i}^{\mathcal{N}}\Phi(x)$ *for* $(i,\alpha)\in\Upsilon$ *with* $i\neq II$ *and* $\alpha\neq 0$, *and* $\mathsf{E}_\alpha^{II}(\mathsf{A}) = \aleph_0\odot\boxplus_{x\in\mathscr{E}_\alpha^{II}}^{\mathcal{N}}\Phi(x)$ *for* $\alpha\in\mathrm{Card}_\infty$.

Proof. Points (a)–(d) are left as exercises. They are almost immediate consequences of Propositions 4.5.4, 5.4.6 and the fact that central decompositions of von Neumann algebras preserve the types. Note also that $\boxplus_{x\in\mathscr{X}^d}^{\mathcal{N}} f(x)\odot\Phi(x) = \boxplus_{x\in\mathscr{X}^d} f(x)\odot\Phi(x)$ since $\mathcal{N}|_{\mathscr{X}^d} = \{\emptyset\}$.

To prove (e), it suffices to show that $\mathcal{E}$ is a base of $\mathscr{X}$, since then the remainder will follow from (CS2), (CS3) (page 85), (a) and the uniqueness in Theorem 3.6.1. It is clear that $\mathcal{E}$ consists of pairwise disjoint, measurable sets (because f is measurable on $\mathscr{X}\setminus\mathscr{Z}$) and $\mathscr{X}\setminus\bigcup\mathcal{E} = \mathscr{Z}$. Now assume $A\subset\mathscr{X}\setminus\mathscr{Z}$ is such that $A\cap\mathscr{E}\in\mathfrak{M}$ (respectively $A\cap\mathscr{E}\in\mathcal{N}$) for any $\mathscr{E}\in\mathcal{E}$. Let $\mathcal{B}$ be as in (e). It follows from the proof of Lemma 5.5.4 that $f(B)\cap\mathrm{Card}_\infty$ is countable for each $B\in\mathcal{B}$. Consequently, also the set $\mathcal{E}(B) = \{E\in\mathcal{E}\colon E\cap B\neq\emptyset\}$ is countable and thus $A\cap B = \bigcup_{E\in\mathcal{E}(B)}[(A\cap E)\cap B]$ is a member of $\mathfrak{M}$ (respectively $\mathcal{N}$) for any $B\in\mathcal{B}$. Since $\mathcal{B}$ is a base, we obtain $A\in\mathfrak{M}$ ($A\in\mathcal{N}$) and we are done. ∎

Remark 5.6.8. For every measurable set $\mathscr{D}\subset\mathscr{X}$, let $\mathsf{j}_{\mathscr{D}}$ denote an admissible function which is 0 off $\mathscr{D}$, 1 on $\mathscr{D}\setminus\mathscr{X}_{III}$ and $\aleph_0$ on $\mathscr{D}\cap\mathscr{X}_{III}$.

Using Corollary 5.6.6(E) as well as (CS2) and (CS3)(B) (page 85), one may show that whenever $(\mathscr{X},\mathfrak{M},\mathcal{N},\Phi)$ is a covering, the regular (continuous) direct sums of the form $\boxplus_{x\in\mathscr{X}}^{\mathcal{N}} f(x)\odot\Phi(x)$ with $f\in\mathscr{A}(\mathscr{X})$ may be defined by axioms (AX0)–(AX3) stated below. Namely, it is now quite easy to prove that if $\Psi\colon\mathscr{A}(\mathscr{X})\to\mathcal{CDD}_N$ is an assignment such that

(AX0) for every $\mathscr{D}\in\mathfrak{M}$, $\Psi(\mathsf{j}_{\mathscr{D}}) = \boxplus_{x\in\mathscr{D}}^{\mathcal{N}}\Phi(x)$,
(AX1) whenever $\mathcal{B}$ is a base of $\mathscr{X}$, $\Psi(f) = \bigoplus_{B\in\mathcal{B}}\Psi(\mathsf{j}_B\cdot f)$ for every $f\in\mathscr{A}(\mathscr{X})$,
(AX2) $\Psi(\alpha\cdot f) = \alpha\odot\Psi(f)$ for any $\alpha\in\mathrm{Card}$ and $f\in\mathscr{A}(\mathscr{X})$,
(AX3) $\Psi(\sum_{n=1}^\infty f_n) = \bigoplus_{n=1}^\infty\Psi(f_n)$ for all $f_1, f_2,\ldots\in\mathscr{A}(\mathscr{X})$,

then $\Psi(f) = \boxplus_{x\in\mathscr{X}}^{\mathcal{N}} f(x)\odot\Phi(x)$ for any $f\in\mathscr{A}(\mathscr{X})$ (to show this, use Corollary 5.6.7(e) and the fact that a real-valued measurable function may be written as a series of rational-valued simple functions). However, at this moment we do not know whether Φ is uniquely determined (up to a.e. equality) by 'its' continuous direct sums appearing in (AX0). This (and even more) will be proved later, in Theorem 5.6.17.

The next result follows from Corollary 5.6.7 and its proof is left to the reader.

Corollary 5.6.9. *Let $(\mathscr{Y},\Psi)$ be a covering for an ideal $\mathcal{A}\subset\mathcal{CDD}_N$ and let $\mathsf{B} = \mathsf{J}(\mathcal{A})$. Then $\iota^d(\mathscr{Y}) = \mathrm{card}(\{\mathsf{X}\in\mathfrak{F}_N\colon\mathsf{X}\leqslant^s\mathsf{B}\})$ and $\iota^c(\mathscr{Y}) = \dim(\mathsf{B}^c)$.*

Our next aim is to establish (in a sense) the uniqueness (Theorem 5.6.10 and Corollary 5.6.11 below) and the existence (Proposition 5.6.13) of coverings for arbitrary ideals in $\mathcal{CDD}_N$.

Theorem 5.6.10. *Let $(\mathscr{X}_1,\mathfrak{M}_1,\mathcal{N}_1,\Phi_1)$ and $(\mathscr{X}_2,\mathfrak{M}_2,\mathcal{N}_2,\Phi_2)$ be two coverings such that*

$$\boxplus_{x\in\mathscr{X}_1}^{\mathcal{N}_1}\Phi_1(x) = \boxplus_{x\in\mathscr{X}_2}^{\mathcal{N}_2}\Phi_2(x). \tag{5.6.10}$$

Then there are sets $\mathscr{Z}_j\in\mathcal{N}_j$ $(j=1,2)$ and a null-isomorphism $\tau\colon\mathscr{X}_1\setminus\mathscr{Z}_1\to\mathscr{X}_2\setminus\mathscr{Z}_2$ such that $\Phi_1(x) = \Phi_2(\tau(x))$ for each $x\in\mathscr{X}_1\setminus\mathscr{Z}_1$.

Proof. Let $\mathcal{B}_j$ be a standard base of $\mathscr{X}_j$. For $B \in \mathcal{B}_j$ put $\mathsf{T}_B^{(j)} = \boxplus_{x\in B}^{\mathcal{N}_j} \Phi_j(x)$. It follows from (CS2) and (5.6.10) that

$$\boxplus_{B\in\mathcal{B}_1} \mathsf{T}_B^{(1)} = \boxplus_{B\in\mathcal{B}_2} \mathsf{T}_B^{(2)}. \tag{5.6.11}$$

Let $I = \{(B_1,B_2) \in \mathcal{B}_1 \times \mathcal{B}_2\colon \mathsf{T}_{B_1,B_2} := \mathsf{T}_{B_1}^{(1)} \wedge \mathsf{T}_{B_2}^{(2)} \neq \mathsf{O}\}$. We conclude from (5.6.11) that

$$\mathsf{T}_B^{(1)} = \boxplus\{\mathsf{T}_{B,B'}\colon (B,B') \in I\} \quad (B \in \mathcal{B}_1), \tag{5.6.12}$$

$$\mathsf{T}_B^{(2)} = \boxplus\{\mathsf{T}_{B',B}\colon (B,B') \in I\} \quad (B \in \mathcal{B}_2). \tag{5.6.13}$$

It follows from Corollary 5.6.6 and (5.6.12)–(5.6.13) that for any $B' \in \mathcal{B}_1$ and $B'' \in \mathcal{B}_2$ the sets $I_2(B') = \{B_2 \in \mathcal{B}_2\colon (B',B_2) \in I\}$ and $I_1(B'') = \{B_1 \in \mathcal{B}_1\colon (B_1,B'') \in I\}$ are countable (since $\mathsf{T}_B^{(j)} \in \mathcal{SEP}_N$) and thus there are families of pairwise disjoint sets $\{D^1_{B',B}\}_{B\in I_2(B')} \subset \mathfrak{M}_1$ and $\{D^2_{B,B''}\}_{B\in I_1(B'')} \subset \mathfrak{M}_2$ such that $B' = \bigcup_{B\in I_2(B')} D^1_{B',B}$, $B'' = \bigcup_{B\in I_1(B'')} D^2_{B,B''}$ and

$$\mathsf{T}_{B_1,B_2} = \boxplus_{x\in D^1_{B_1,B_2}}^{\mathcal{N}_1} \Phi_1(x) = \boxplus_{x\in D^2_{B_1,B_2}}^{\mathcal{N}_2} \Phi_2(x) \tag{5.6.14}$$

for any $(B_1,B_2) \in I$ (cf. Corollary 5.6.6 or Theorem 5.5.5). We also infer from the countability of $I_1(B_2)$ and $I_2(B_1)$ that

$$\{D^j_{B_1,B_2}\colon (B_1,B_2) \in I\} \text{ is a base of } \mathscr{X}_j. \tag{5.6.15}$$

Fix $(B_1,B_2) \in I$. Since $D^j_{B_1,B_2}$ is standard and $\Phi_j \in \mathrm{RGS}_{\mathrm{loc}}$, there is a Borel set $G_j \subset D^j_{B_1,B_2}$ such that $D^j_{B_1,B_2} \setminus G_j \in \mathcal{N}_j$, $\Phi_j(G_j)$ is a measurable domain and $\Phi_j|_{G_j}$ is a Borel isomorphism of G_j onto $\Phi_j(G_j)$. Let μ_j be a standard measure on $\mathfrak{M}_j|_{G_j}$ for which $\mathcal{N}_j|_{G_j} = \mathcal{N}(\mu_j)$. Relation (5.6.14) shows that $\int_{G_1}^{\boxplus} \Phi_1(x)\,d\mu_1(x) = \int_{G_2}^{\boxplus} \Phi_2(x)\,d\mu_2(x)$. Hence Corollary 5.4.7 implies that $\widehat{\mu}_1 \ll \widehat{\mu}_2 \ll \widehat{\mu}_1$ where $\widehat{\mu}_j(\mathcal{F}) = \mu_j(\Phi_j^{-1}(\mathcal{F}) \cap G_j)$ for $\mathcal{F} \in \mathfrak{B}(\mathfrak{p}_N)$. Consequently, $Z^j_{B_1,B_2} = D^j_{B_1,B_2} \setminus [\Phi_j^{-1}(\Phi_1(G_1) \cap \Phi_2(G_2)) \cap G_j] \in \mathcal{N}_j$ and $\tau_{B_1,B_2}\colon D^1_{B_1,B_2} \setminus Z^1_{B_1,B_2} \ni x \mapsto (\Phi_2|_{G_2})^{-1}(\Phi_1(x)) \in D^2_{B_1,B_2} \setminus Z^2_{B_1,B_2}$ is a well defined null-isomorphism such that

$$\begin{cases} \tau_{B_1,B_2}\colon D^1_{B_1,B_2} \setminus Z^1_{B_1,B_2} \to D^2_{B_1,B_2} \setminus Z^2_{B_1,B_2}, \\ \Phi_2 \circ \tau_{B_1,B_2} = \Phi_1|_{D^1_{B_1,B_2} \setminus Z^1_{B_1,B_2}}. \end{cases} \tag{5.6.16}$$

Now it suffices to put $\mathscr{Z}_j = (\mathscr{X}_j \setminus \bigcup \mathcal{B}_j) \cup \bigcup_{(B_1,B_2)\in I} Z^j_{B_1,B_2}$ and to define $\tau\colon \mathscr{X}_1 \setminus \mathscr{Z}_1 \to \mathscr{X}_2 \setminus \mathscr{Z}_2$ as the union of $\{\tau_{B_1,B_2}\}_{(B_1,B_2)\in I}$. It follows from (5.6.15) and (5.6.16) that $\mathscr{Z}_j \in \mathcal{N}_j$ and τ is a null-isomorphism we searched for. ■

Corollary 5.6.11. *Let $\mathcal{A}$ be an ideal and $(\mathscr{X}^1,\mathfrak{M}^1,\mathcal{N}^1,\Phi^1)$ and $(\mathscr{X}^2,\mathfrak{M}^2,\mathcal{N}^2,\Phi^2)$ be two coverings for $\mathcal{A}$. Then there are sets $\mathscr{Z}^j \in \mathcal{N}^j$ $(j = 1,2)$, a Borel function $u\colon \mathscr{X}^1 \to \mathbb{R}_+ \setminus \{0\}$ with $u(\mathscr{X}^1_I \cup \mathscr{X}^1_{III}) \subset \{1\}$, and a null-isomorphism $\tau\colon \mathscr{X}^1 \setminus \mathscr{Z}^1 \to \mathscr{X}^2 \setminus \mathscr{Z}^2$ such that $\Phi^2(\tau(x)) = u(x) \odot \Phi^1(x)$ for every $x \in \mathscr{X}^1 \setminus \mathscr{Z}^1$.*

Proof. Let $\mathsf{T}_j = \boxplus_{x\in\mathscr{X}^j}^{\mathcal{N}^j} \Phi^j(x)$. It follows from the assumptions and Theorem 5.6.5 that

$$\mathsf{T}_1 \ll \mathsf{T}_2 \tag{5.6.17}$$

and there is $f \in \mathscr{A}(\mathscr{X}^1)$ such that

$$\mathsf{T}_2 = \boxplus_{x\in\mathscr{X}^1}^{\mathcal{N}^1} f(x) \odot \Phi^1(x). \tag{5.6.18}$$

Now Corollary 5.6.7 implies that T_2 is the direct sum of a minimal N-tuple and a semiminimal one, and consequently there is $Z \in \mathcal{N}^1$ such that $A := s(f) \setminus Z \in \mathfrak{M}^1$, $f|_A$ is Borel, $f(A\cap\mathscr{X}^1_I) \subset \{1\}$, $f(A\cap\mathscr{X}^1_{III}) \subset \{\aleph_0\}$ and $f(A\cap\mathscr{X}^1_{II}) \subset \mathbb{R}_+ \setminus \{0\}$. Further, Corollary 5.6.6 combined with (5.6.17) shows that $\mathscr{X}^1 \setminus s(f) \in \overline{\mathcal{N}^1}$ and hence $\mathscr{X}^1 \setminus A \in \mathcal{N}^1$. Define $u\colon \mathscr{X}^1 \to \mathbb{R}_+ \setminus \{0\}$ by $u(x) = f(x)$ for $x \in A \setminus \mathscr{X}_{III}$ and $u(x) = 1$ otherwise. Observe that u is Borel and fits to Φ^1, and $u(x) \odot \Phi^1(x) = f(x) \odot \Phi^1(x)$ for $x \in A$. So, (5.6.18) gives

$$\boxplus_{x\in\mathscr{X}^2}^{\mathcal{N}^2} \Phi^2(x) = \boxplus_{x\in\mathscr{X}^1}^{\mathcal{N}^1} u(x) \odot \Phi^1(x). \tag{5.6.19}$$

Finally, since u is real-valued, $(u \odot \Phi^1)(\mathscr{X}^1) \subset \mathfrak{p}_N$ and we deduce from Theorem 5.5.5 that $(\mathscr{X}^1, u \odot \Phi^1)$ is a covering. So, the assertion follows from Theorem 5.6.10, thanks to (5.6.19). ∎

To establish the existence of coverings, we need the following

LEMMA 5.6.12. *Let $\mathscr{E} \subset \operatorname{rgm}(\mathfrak{p}_N)$ be a family such that*

$$\mu \perp_s \nu \quad \textit{if} \quad \mu \neq \nu \textit{ and } \mu, \nu \in \mathscr{E}. \tag{5.6.20}$$

Let $(\mathscr{X}, \mathfrak{M}, \mathcal{N}) = \bigoplus_{\mu\in\mathscr{E}}(\mathfrak{p}_N, \mathfrak{B}(\mathfrak{p}_N), \mathcal{N}(\mu))$ and $\Phi\colon \mathscr{X} \to \mathfrak{p}_N$ be the canonical projection. Then $(\mathscr{X}, \Phi)$ is a covering and

$$\boxplus_{x\in\mathscr{X}}^{\mathcal{N}} \Phi(x) = \boxplus_{\mu\in\mathscr{E}} \int_{\mathfrak{p}_N}^{\boxplus} \mathsf{P}\, d\mu(\mathsf{P}). \tag{5.6.21}$$

Proof. First of all, the usage of '$\boxplus_{\mu\in\mathscr{E}}$' is allowed by Lemma 5.4.10, thanks to (5.6.20). Further, since regularity measures are concentrated on measurable domains which are Souslin–Borel sets, $(\mathscr{X}, \mathfrak{M}, \mathcal{N})$ is a multi-standard measurable space with nullity and $\{\mathfrak{p}_N \times \{\mu\}\}_{\mu\in\mathscr{E}}$ is a standard base of $\mathscr{X}$. Thus, it suffices to check that $\Phi \in \mathrm{RGS}_{\mathrm{loc}}(\mathscr{X})$ (then (5.6.21) will automatically be satisfied). It is clear that Φ is Borel.

Let $A \in \mathfrak{M}$ be standard. We will show that condition (ii) of Theorem 5.5.5 is fulfilled. Since A is standard, the set $\mathscr{E}' = \{\mu \in \mathscr{E}\colon \Phi(A) \notin \mathcal{N}(\mu)\}$ is countable. Observe that $Z_0 = A \cap [\bigcup_{\mu\notin\mathscr{E}'}(\mathfrak{p}_N \times \{\mu\})] \in \mathcal{N}$. Since $A \setminus Z_0 \subset \mathfrak{p}_N \times \mathscr{E}' \in \mathfrak{M}$, we may assume that

$$A = \mathfrak{p}_N \times \mathscr{E}'. \tag{5.6.22}$$

For $\mu \in \mathscr{E}'$ let $\mathsf{T}_\mu = \int_{\mathfrak{p}_N}^{\boxplus} \mathsf{P}\, d\mu(\mathsf{P})$. Put $\mathsf{T} = \boxplus_{\mu\in\mathscr{E}'} \mathsf{T}_\mu$. It follows from Lemma 5.4.9(C) that T_μ ($\mu \in \mathscr{E}'$) is the direct sum of a minimal N-tuple and a semiminimal one, and thus so is T. Moreover, since $\mathscr{E}'$ is countable, $\mathsf{T} \in \mathcal{SEP}_N$ ($\mathsf{T} \neq \mathsf{O}$ because standard sets are nonnull). Now Lemma 5.4.9(A) asserts that there is a measure $\lambda \in \operatorname{rgm}(\mathfrak{p}_N)$ such that $\mathsf{T} = \int_{\mathfrak{p}_N}^{\boxplus} \mathsf{P}\, d\lambda(\mathsf{P})$. Since $\mathsf{T}_\mu \leqslant^s \mathsf{T}$, we conclude from Corollary 5.4.7 that

$$\mu \ll \lambda \quad (\mu \in \mathscr{E}'). \tag{5.6.23}$$

Further, it follows from (5.6.20) and the countability of $\mathscr{E}'$ that there is a collection $\{S_\mu\}_{\mu \in \mathscr{E}'}$ of pairwise disjoint measurable subsets of $\mathfrak{p}_N$ such that $\mu(\mathfrak{p}_N \setminus S_\mu) = 0$ for every $\mu \in \mathscr{E}'$. Finally, let $\mathcal{F} \subset \mathfrak{p}_N$ be a measurable domain such that $\lambda(\mathfrak{p}_N \setminus \mathcal{F}) = 0$. Put

$$D = \bigcup_{\mu \in \mathscr{E}'} [(S_\mu \cap \mathcal{F}) \times \{\mu\}].$$

Observe that $D \subset A$ (by (5.6.22)), $A \setminus D \in \mathcal{N}$ ($\mathfrak{p}_N \setminus (S_\mu \cap \mathcal{F}) \in \mathcal{N}(\mu)$ by (5.6.23)), $\Phi|_D$ is one-to-one (since the S_μ's are pairwise disjoint) and $\Phi(D) \subset \mathcal{F}$. So, Remark 5.4.5 finishes the proof. ∎

Proposition 5.6.13. *Let $\mathsf{T} \in \mathcal{CDD}_N$ be the direct sum of a minimal N-tuple and a semiminimal one. There is a covering $(\mathscr{X}, \mathfrak{M}, \mathcal{N}, \Phi)$ such that*

$$\mathsf{T} = \boxplus_{x \in \mathscr{X}}^{\mathcal{N}} \Phi(x).$$

Proof. By Zorn's lemma, there is a maximal family $\mathscr{E} \subset \operatorname{rgm}(\mathfrak{p}_N)$ such that (5.6.20) is satisfied and $\mathsf{T}_\mu := \int_{\mathfrak{p}_N}^{\boxplus} \mathsf{P}\, d\mu(\mathsf{P}) \leqslant^s \mathsf{T}$ for each $\mu \in \mathscr{E}$ (since $\mathsf{T}_\mu \neq \mathsf{O}$; cf. Lemma 3.4.1). It follows from Lemma 5.6.12 and its proof that $\boxplus_{\mu \in \mathscr{E}} \mathsf{T}_\mu \leqslant^s \mathsf{T}$ and that it is enough to show that $\mathsf{X} := \mathsf{T} \boxminus (\boxplus_{\mu \in \mathscr{E}} \mathsf{T}_\mu)$ is equal to O. Suppose, to the contrary, that $\mathsf{X} \neq \mathsf{O}$. Since $\mathsf{T} \leqslant \mathsf{J}$, we infer from Proposition 3.4.10 that there is $\mathsf{Y} \in \mathcal{SEP}_N$ such that $\mathsf{Y} \leqslant^s \mathsf{X}$. Then Y is the direct sum of a minimal N-tuple and a semiminimal one (because $\mathsf{X} \leqslant^s \mathsf{T}$). Now Lemma 5.4.9 yields $\nu \in \operatorname{rgm}(\mathfrak{p}_N)$ such that $\int_{\mathfrak{p}_N}^{\boxplus} \mathsf{P}\, d\nu(\mathsf{P}) = \mathsf{Y}$ ($\leqslant^s \mathsf{T}$). Finally, since $\mathsf{Y} \perp_u \mathsf{T}_\mu$ for every $\mu \in \mathscr{E}$, Lemma 5.4.10 asserts that $\nu \perp_s \mu$ for any $\mu \in \mathscr{E}$, contrary to the fact that $\mathscr{E}$ is maximal. ∎

The next theorem is an immediate consequence of all previously established properties. This result may be formulated for arbitrary coverings. Of main interest to us, however, are the full coverings. To make the theorem most transparent, we repeat some of the properties proved earlier.

Theorem 5.6.14 (Prime Decomposition).

(I) *There exists a full covering. What is more, for every $\mathsf{T} \in \mathcal{SM}_N$ with $\aleph_0 \odot \mathsf{T} = \mathsf{J}_{II}$ there is a full covering $(\mathscr{X}, \mathfrak{M}, \mathcal{N}, \Phi)$ such that $\boxplus_{x \in \mathscr{X}}^{\mathcal{N}} \Phi(x) = \mathsf{J}_I \boxplus \mathsf{T} \boxplus \mathsf{J}_{III}$.*

(II) *Let $(\mathscr{X}^1, \mathfrak{M}^1, \mathcal{N}^1, \Phi^1)$ and $(\mathscr{X}^2, \mathfrak{M}^2, \mathcal{N}^2, \Phi^2)$ be full coverings. There are a Borel function $u\colon \mathscr{X}^1 \to \mathbb{R}_+ \setminus \{0\}$ such that $u(\mathscr{X}^1_I \cup \mathscr{X}^1_{III}) = \{1\}$ and an almost null-isomorphism $\tau\colon \mathscr{X}^1 \to \mathscr{X}^2$ such that $\Phi^2 \circ \tau = u \odot \Phi^1$ a.e. In particular, for every $f \in \mathscr{A}(\mathscr{X}^2)$, $(f \circ \tau)u \in \mathscr{A}(\mathscr{X}^1)$ and*

$$\boxplus_{x \in \mathscr{X}^2}^{\mathcal{N}^2} f(x) \odot \Phi^2(x) = \boxplus_{x \in \mathscr{X}^1}^{\mathcal{N}^1} [(f \circ \tau)u](x) \odot \Phi^1(x).$$

(III) *Let $(\mathscr{X}, \mathfrak{M}, \mathcal{N}, \{\mathsf{P}_x\}_{x \in \mathscr{X}})$ be a full covering.*

(A) *For each* $\mathsf{A} \in \mathcal{CDD}_N$ *there is* $f \in \mathscr{A}(\mathscr{X})$ *such that* $\mathsf{A} = \boxplus_{x\in\mathscr{X}}^{\mathcal{N}} f(x) \odot \mathsf{P}_x$.
(B) *For any* $f_1, f_2, f_3, \ldots \in \mathscr{A}(\mathscr{X})$,

$$\boxplus_{x\in\mathscr{X}}^{\mathcal{N}} \Big[\sum_{n=1}^{\infty} f_n(x)\Big] \odot \mathsf{P}_x = \bigoplus_{n=1}^{\infty} \Big[\boxplus_{x\in\mathscr{X}}^{\mathcal{N}} f_n(x) \odot \mathsf{P}_x \Big].$$

(C) *Let* $f, g \in \mathscr{A}(\mathscr{X})$. *Put* $\mathsf{X} = \boxplus_{x\in\mathscr{X}}^{\mathcal{N}} f(x) \odot \mathsf{P}_x$ *and* $\mathsf{Y} = \boxplus_{x\in\mathscr{X}}^{\mathcal{N}} g(x) \odot \mathsf{P}_x$. *Then:*

(a) $\mathsf{X} = \mathsf{Y} \Leftrightarrow f = g$ *a.e.*,
(b) $\mathsf{X} \leqslant \mathsf{Y} \Leftrightarrow f \leqslant g$ *a.e.*,
(c) $\mathsf{X} \leqslant^s \mathsf{Y} \Leftrightarrow f = g \cdot j_{\mathscr{D}}$ *a.e. for some* $\mathscr{D} \in \mathfrak{M}$,
(d) $\mathsf{X} \ll \mathsf{Y} \Leftrightarrow s(f) \setminus s(g) \in \overline{\mathcal{N}}$,
(e) $\mathsf{X} \perp_u \mathsf{Y} \Leftrightarrow f \cdot g = 0$ *a.e.* $\Leftrightarrow s(f) \cap s(g) \in \overline{\mathcal{N}}$,
(f) $\alpha \odot \mathsf{X} = \boxplus_{x\in\mathscr{X}}^{\mathcal{N}} (\alpha \cdot f)(x) \odot \mathsf{P}_x$ *for any* $\alpha \in \mathrm{Card}$,
(g) $\mathsf{X} \in \mathcal{SM}_N \Leftrightarrow s(f) \setminus \mathscr{X}_{II} \in \overline{\mathcal{N}}$ *and* $f^{-1}(\mathrm{Card}_\infty) \in \overline{\mathcal{N}}$; *if* $\mathsf{X} \in \mathcal{SM}_N$, *then* $t \odot \mathsf{X} = \boxplus_{x\in\mathscr{X}}^{\mathcal{N}} [t \cdot f(x)] \odot \mathsf{P}_x$ *for each* $t \in \mathbb{R}_+$,
(h) $\mathsf{X} \in \mathcal{SEP}_N$ *iff there is* $\mathscr{Z} \in \mathcal{N}$ *such that* $s(f) \setminus \mathscr{Z}$ *is standard and* $f(\mathscr{X} \setminus \mathscr{Z}) \subset I_{\aleph_0}$.

We leave the proofs of (g) and of a part of (II) as exercises.

Theorem 5.6.14 says that after fixing $\mathsf{T} \in \mathcal{SM}_N$ such that $\aleph_0 \odot \mathsf{T} = \mathsf{J}_{II}$, there is a unique (up to almost null-isomorphism) full covering $(\mathscr{X}, \mathfrak{M}, \mathcal{N}, \Phi)$ such that

$$\boxplus_{x\in\mathscr{X}_{II}}^{\mathcal{N}} \Phi(x) = \mathsf{T}.$$

Then for every $\mathsf{A} \in \mathcal{CDD}_N$ there is a unique (up to almost everywhere equality) function $\mathfrak{m} \in \mathscr{A}(\mathscr{X})$ such that

$$\mathsf{A} = \boxplus_{x\in\mathscr{X}}^{\mathcal{N}} \mathfrak{m}(x) \odot \Phi(x). \tag{5.6.24}$$

The function $\mathfrak{m}$ is called the *multiplicity function of* A (*relative to* T) (compare with Chapter 4 of [9]) and the formula (5.6.24) is called the *prime decomposition of* A (*relative to* T). One may check that $\mathsf{A} \in \mathcal{CDD}_N$ has a multiplicity function (respectively a prime decomposition) of a unique (i.e. independent of the choice of T) form iff $\mathsf{E}_{sm}(\mathsf{A}) = 0$ (respectively $\mathsf{A} \perp_u \mathsf{J}_{II}$).

Since $\mathfrak{a}_N(n)$ for finite n consists of bounded N-tuples, Theorem 5.6.14 implies that every N-tuple X whose type I^∞, II and III parts vanish admits a decomposition in the form $\mathsf{X} = \bigoplus_{n=1}^{\infty} \mathsf{X}^{(n)}$ where each $\mathsf{X}^{(n)}$ is bounded. So, in the notation of Examples 4.5.2, every such X belongs to $\mathcal{I}[\mathrm{cl}\,\Omega(\mathrm{bd})]$.

Remark 5.6.15. Theorem 5.6.14 implies that all measurable spaces with nullities being ingredients of full coverings are almost isomorphic. One may therefore ask about their (common) characteristic numbers ι^d and ι^c. Using the results of the next chapter and Corollary 5.6.9 one may show that both numbers are equal to $2^{\aleph_0}$. Even more: whenever $(\mathscr{X}, \Phi)$ is a full covering, for $\mathscr{Y} \in \{\mathscr{X}, \mathscr{X}_I, \mathscr{X}_{I_1}, \mathscr{X}_{I_2}, \ldots, \mathscr{X}_{I_\infty}, \mathscr{X}_{II}, \mathscr{X}_{II_1}, \mathscr{X}_{II_\infty}, \mathscr{X}_{III}\}$ one has $\iota^d(\mathscr{Y}) = \iota^c(\mathscr{Y}) = 2^{\aleph_0}$.

REMARK 5.6.16. There is a striking resemblance between Theorems 4.4.2 and 5.6.14, and between the forms of $\Lambda(\Omega)$ (where Ω is an underlying model space) and of $\mathscr{A}(\mathscr{X},\Psi)$ (where $(\mathscr{X},\Psi)$ is a full covering). It is not a coincidence. When $(\mathscr{X},\mathfrak{M},\mathcal{N},\Psi)$ is a full covering, $\mathcal{A} = L^\infty(\mathscr{X},\mathfrak{M},\mathcal{N})$ is a $\mathcal{W}^*$-algebra (since $\mathscr{X}$ is multi-standard—see the first paragraph of §1.18 in [29]). Now if Ω is the Gelfand spectrum of $\mathcal{A}$, there is a one-to-one correspondence between clopen subsets of Ω and members of $\mathfrak{M}$ which naturally correspond to N-tuples X such that $\mathsf{X} \leqslant^s \widetilde{\mathsf{T}} := \mathsf{J}_I \boxplus \mathsf{T} \boxplus \mathsf{J}_{III}$ where $\mathsf{T} := \boxplus^{\mathcal{N}}_{x\in\mathscr{X}_{II}} \Psi(x)$. Since $\mathcal{Z}(\mathcal{W}''(\widetilde{\boldsymbol{T}}))$ is isomorphic to $\mathcal{Z}(\mathcal{W}''(\boldsymbol{J}))$ (because $\widetilde{\mathsf{T}} \ll \mathsf{J} \ll \widetilde{\mathsf{T}}$; cf. (PR6), page 13), Ω is therefore homeomorphic to the Gelfand spectrum of $\mathcal{Z}(\mathcal{W}''(\boldsymbol{J}))$, that is, Ω is an underlying model space. Further, using results of Chapters 4.4 and 5.6, one may show that there is a 'natural' correspondence, $f \mapsto \widehat{f}$, between $\Lambda(\Omega)$ and $\mathscr{A}(\mathscr{X})$ (induced by the isomorphism between $\mathcal{C}(\Omega)$ and $\mathcal{A}$) where in $\mathscr{A}(\mathscr{X})$ we identify functions which are equal almost everywhere. One may check then that the assignment

$$\Lambda(\Omega) \ni f \mapsto \boxplus^{\mathcal{N}}_{x\in\mathscr{X}} \widehat{f}(x) \odot \Psi(x)$$

is inverse to Φ_{T} introduced in Theorem 4.4.2. Thus $\mathscr{A}(\mathscr{X})$ may be considered as a 'concrete realization' of $\Lambda(\Omega)$. With such an approach, the multiplicity function $\mathfrak{m} \in \mathscr{A}(\mathscr{X})$ (relative to T) of $\mathsf{X} \in \mathcal{SM}_N$ corresponds to $d\mathsf{X}/d\mathsf{T}$.

THEOREM 5.6.17. *Let $(\mathscr{X},\mathfrak{M},\mathcal{N})$ be a multi-standard measurable space with nullity.*

(I) *Let $\Phi\colon \mathscr{X} \to \mathfrak{p}_N$ be such that $(\mathscr{X},\Phi)$ is a covering and let $\mu\colon \mathfrak{M} \to \mathcal{CDD}_N$ be given by*

$$\mu(\mathscr{A}) = \boxplus^{\mathcal{N}}_{x\in\mathscr{A}} \Phi(x) \qquad (\mathscr{A} \in \mathfrak{M}). \tag{5.6.25}$$

Then:

(M1) *$\mu(\mathscr{X})$ is the direct sum of a minimal N-tuple and a semiminimal one,*
(M2) *for every $\mathscr{A} \in \mathfrak{M}$, $\mu(\mathscr{A}) = \mathsf{O} \Leftrightarrow \mathscr{A} \in \mathcal{N}$,*
(M3) *whenever $\mathscr{A}$ and $\mathscr{B}$ are two measurable disjoint sets, $\mu(\mathscr{A} \cup \mathscr{B}) = \mu(\mathscr{A}) \boxplus \mu(\mathscr{B})$,*
(M4) *for every $\mathsf{A} \in \mathcal{CDD}_N$ such that $\mathsf{A} \leqslant^s \mu(\mathscr{X})$ there exists $\mathscr{A} \in \mathfrak{M}$ for which $\mu(\mathscr{A}) = \mathsf{A}$.*

(II) *For every function $\mu\colon \mathfrak{M} \to \mathcal{CDD}_N$ satisfying conditions* (M0)–(M3) *there exists a unique* (*up to almost everywhere equality*) *function $\Phi\colon \mathscr{X} \to \mathfrak{p}_N$ such that $(\mathscr{X},\Phi)$ is a covering and* (5.6.25) *is satisfied.*

Proof. Point (I) is left to the reader.

Let μ be as in (II). Put $\mathsf{T} = \mu(\mathscr{X})$. Observe that:

(M4) for any $\mathscr{A},\mathscr{B} \in \mathfrak{M}$, $\mu(\mathscr{A}) \leqslant^s \mu(\mathscr{B})$ iff $\mathscr{A} \setminus \mathscr{B} \in \mathcal{N}$,
(M5) $\{\mu(\mathscr{A})\colon \mathscr{A} \in \mathfrak{M}\} = \{\mathsf{A} \in \mathcal{CDD}_N\colon \mathsf{A} \leqslant^s \mathsf{T}\}$.

Indeed, (M5) easily follows from (M2) and (M3), because $\mathsf{T} = \mu(\mathscr{A}) \boxplus \mu(\mathscr{X} \setminus \mathscr{A})$ for every measurable $\mathscr{A}$. To prove (M4), first of all note that

$$\mu(\mathscr{A}) = \mu(\mathscr{B}) \Leftrightarrow (\mathscr{A} \setminus \mathscr{B}) \cup (\mathscr{B} \setminus \mathscr{A}) \in \mathcal{N}, \tag{5.6.26}$$

since, by (M2), $\mu(\mathscr{A}) = \mu(\mathscr{A} \setminus \mathscr{B}) \boxplus \mu(\mathscr{A} \cap \mathscr{B})$, $\mu(\mathscr{B}) = \mu(\mathscr{B} \setminus \mathscr{A}) \boxplus \mu(\mathscr{A} \cap \mathscr{B})$ and (again thanks to (M2)) $\mu(\mathscr{A} \setminus \mathscr{B}) \perp_u \mu(\mathscr{B} \setminus \mathscr{A})$. These remarks combined with (M1) give (5.6.26). Now if $\mathscr{A} \setminus \mathscr{B} \in \mathcal{N}$, we infer from (5.6.26) that $\mu(\mathscr{A}) = \mu(\mathscr{A} \cap \mathscr{B})$ and hence, by (M2), $\mu(\mathscr{B}) = \mu(\mathscr{B} \setminus \mathscr{A}) \boxplus \mu(\mathscr{A})$, which yields $\mu(\mathscr{A}) \leqslant^s \mu(\mathscr{B})$. Conversely, if the last inequality is satisfied, we conclude from (M5) that there is $\mathscr{C} \in \mathfrak{M}$ such that $\mu(\mathscr{C}) = \mu(\mathscr{B}) \boxminus \mu(\mathscr{A})$. Since then (again by (M2)) $\mu(\mathscr{C} \cup \mathscr{A}) = \mu(\mathscr{C}) \boxplus \mu(\mathscr{A} \setminus \mathscr{C}) = \mu(\mathscr{A}) \boxplus \mu(\mathscr{C} \setminus \mathscr{A})$, $\mu(\mathscr{C}) \perp_u \mu(\mathscr{A})$ and $\mu(\mathscr{A} \setminus \mathscr{C}) \perp_u \mu(\mathscr{C} \setminus \mathscr{A})$, we get $\mu(\mathscr{A}) = \mu(\mathscr{A} \setminus \mathscr{C})$ and consequently $\mu(\mathscr{A} \cup \mathscr{C}) = \mu(\mathscr{C}) \boxplus \mu(\mathscr{A}) = \mu(\mathscr{B})$. So, (5.6.26) yields the assertion of (M4).

Further, it follows from (M0) and Proposition 5.6.13 that there exists a covering $(\mathscr{X}', \mathfrak{M}', \mathcal{N}', \Psi)$ such that $\mathsf{T} = \boxplus^{\mathcal{N}'}_{x \in \mathscr{X}'} \Psi(x)$. Put $\mu' \colon \mathfrak{M}' \ni \mathscr{A} \mapsto \boxplus^{\mathcal{N}'}_{x \in \mathscr{A}} \Psi(x) \in \mathcal{CDD}_N$. It may be inferred from Proposition 5.6.4, Theorem 5.6.5 and Corollary 5.6.6 that conditions (M4) and (M5) as well as (5.6.26) are satisfied when μ, $\mathfrak{M}$ and $\mathcal{N}$ are replaced by (respectively) μ', $\mathfrak{M}'$ and $\mathcal{N}'$. Let $\mathcal{M}$ and $\mathcal{M}'$ denote the quotient (abstract) Boolean σ-algebras $\mathfrak{M}/\mathcal{N}$ and $\mathfrak{M}'/\mathcal{N}'$ (respectively). We shall denote the equivalence class in $\mathcal{M}$ (in $\mathcal{M}'$) of $\mathscr{A} \in \mathfrak{M}$ (of $\mathscr{A} \in \mathfrak{M}'$) by $[\mathscr{A}]_{\mathcal{N}}$ (by $[\mathscr{A}]_{\mathcal{N}'}$). (M4), (M5) and (5.6.26) for both μ and μ' imply that the rule

$$\tau([\mathscr{A}]_{\mathcal{N}}) = [\mathscr{B}]_{\mathcal{N}'} \iff \mu(\mathscr{A}) = \mu'(\mathscr{B})$$

well defines an order isomorphism $\tau \colon \mathcal{M} \to \mathcal{M}'$. Hence whenever $\tau([\mathscr{A}]_{\mathcal{N}}) = [\mathscr{B}]_{\mathcal{N}'}$, then

$$\mathscr{A} \text{ is standard } \iff \mathscr{B} \text{ is standard.}$$

Since τ is an order isomorphism, it is an isomorphism of Boolean σ-algebras as well. Now an application of [27, Corollary 14.4.12] separately for every member of a standard base of $\mathscr{X}$ shows that there are sets $\mathscr{Z} \in \mathcal{N}$ and $\mathscr{Z}' \in \mathcal{N}'$, and a null-isomorphism $\varphi \colon \mathscr{X} \setminus \mathscr{Z} \to \mathscr{X}' \setminus \mathscr{Z}'$ such that $\tau([\mathscr{A}]_{\mathcal{N}}) = [\varphi(\mathscr{A} \setminus \mathscr{Z})]_{\mathcal{N}'}$ for every $\mathscr{A} \in \mathfrak{M}$. In particular, $\mu(\mathscr{A}) = \mu'(\varphi(\mathscr{A} \setminus \mathscr{Z}))$ or, equivalently,

$$\mu(\mathscr{A}) = \boxplus^{\mathcal{N}'}_{y \in \varphi(\mathscr{A} \setminus \mathscr{Z})} \Psi(y) = \boxplus^{\mathcal{N}}_{x \in \mathscr{A} \setminus \mathscr{Z}} (\Psi \circ \varphi)(x)$$

for any $\mathscr{A} \in \mathfrak{M}$. So, to obtain (5.6.25) it suffices to define $\Phi \colon \mathscr{X} \to \mathfrak{p}_N$ as an arbitrary extension of $\Psi \circ \varphi$.

Now assume that $\Phi' \colon \mathscr{X} \to \mathfrak{p}_N$ is another function such that $(\mathscr{X}, \Phi')$ is a covering and $\mu(\mathscr{A}) = \boxplus^{\mathcal{N}}_{x \in \mathscr{A}} \Phi'(x)$ for every $\mathscr{A} \in \mathfrak{M}$. Then $\boxplus^{\mathcal{N}}_{x \in \mathscr{X}} \Phi(x) = \boxplus^{\mathcal{N}}_{x \in \mathscr{X}} \Phi'(x)$ and consequently—by Theorem 5.6.10—there is an almost null-isomorphism $\kappa \colon \mathscr{X} \to \mathscr{X}$ such that $\Phi' = \Phi \circ \kappa$ almost everywhere. It suffices to check that $\kappa(x) = x$ for almost all $x \in \mathscr{X}$. Take $\mathscr{Z} \in \mathcal{N}$ such that $\kappa|_{\mathscr{X} \setminus \mathscr{Z}}$ is a null isomorphism of $\mathscr{X} \setminus \mathscr{Z}$ onto its (measurable) range. For simplicity, for every $\mathscr{A} \in \mathfrak{M}$ put $\mathscr{A}_* = \mathscr{A} \setminus \mathscr{Z}$. Notice that then

$$\boxplus^{\mathcal{N}}_{x \in \mathscr{A}_*} \Phi(x) = \boxplus^{\mathcal{N}}_{x \in \kappa(\mathscr{A}_*)} \Phi(x).$$

This implies (cf. Proposition 5.6.4) that $(\mathscr{A}_* \setminus \kappa(\mathscr{A}_*)) \cup (\kappa(\mathscr{A}_*) \setminus \mathscr{A}_*) \in \mathcal{N}$. Equivalently, $[\mathscr{A}_*]_{\mathcal{N}} = [\kappa(\mathscr{A}_*)]_{\mathcal{N}}$ for every $\mathscr{A} \in \mathfrak{M}$. Since $\mathscr{X}$ is multi-standard, it follows from the uniqueness in [27, Theorem 14.4.10] that $\kappa(x) = x$ almost everywhere and we are done. ∎

6. CLASSIFICATION OF IDEALS

6.1. Types of isomorphisms

This is the only part where we will compare ideals of tuples of different lengths (that is, ideals in $\mathcal{CDD}_N$ as well as in $\mathcal{CDD}_{N'}$ with $N' \neq N$).

We begin with

EXAMPLE 6.1.1. It is known that every properly infinite or type I von Neumann algebra acting in a separable Hilbert space is singly generated ([38], [28], [11]). There are also examples of singly generated type II_1 factors ([11]). Also tensor products of two singly generated von Neumann algebras acting in separable Hilbert spaces are singly generated ([28, Corollary 2.1]). Further, according to [29, Theorem 2.6.6], the $\mathcal{W}^*$-tensor product of a type I_n, II_1, II_∞ or III $\mathcal{W}^*$-algebra and $L^\infty([0,1])$ is of the same type. Also, for a factor $\mathcal{M}$,

$$\mathcal{Z}(\mathcal{M} \mathbin{\bar{\otimes}} L^\infty([0,1])) \cong L^\infty([0,1]), \tag{6.1.1}$$

by [29, Proposition 2.6.7] or [35, Corollary IV.5.11]. Finally, if T is a bounded operator and $\boldsymbol{T} = (T,\dots,T) \in \mathrm{CDD}_N$, then $\mathcal{W}(T) = \mathcal{W}(\boldsymbol{T})$. All this shows that the ideals $\mathcal{I}^c_{I_n}$, $\mathcal{I}^c_{II_1}$, $\mathcal{I}^c_{II_\infty}$ and $\mathcal{I}^c_{III}$ are nonntrivial. (Indeed, take a singly generated factor $\mathcal{M}$ acting in a separable Hilbert space of a fixed type i and let T be a generator of $\mathcal{M}\bar{\otimes}L^\infty([0,1])$. Then $\boldsymbol{T} = (T,\dots,T) \in \mathcal{I}^c_i$, by (6.1.1).)

COROLLARY 6.1.2. *Let Ω denote the underlying model space for $\mathcal{CDD}_N$. Each of the spaces Ω, Ω_I, Ω_{I_n} $(n = 1,2,\dots,\infty)$, Ω_{II}, Ω_{II_1}, Ω_{II_∞} and Ω_{III} is homeomorphic to the topological disjoint union of $\beta D(2^{\aleph_0})$ and $\beta[D(2^{\aleph_0}) \times \mathfrak{X}]$ where $D(2^{\aleph_0})$ is the discrete space of size $2^{\aleph_0}$ and $\mathfrak{X}$ is the Gelfand spectrum of $L^\infty([0,1])$.*

Proof. By Theorem 5.1.12, it suffices to show that $\kappa_c(E) = 2^{\aleph_0}$ where E denotes any of the spaces in question. Equivalently (cf. Proposition 4.4.5), this is to say that $\dim(\mathsf{J}(\mathcal{A}))$ $= 2^{\aleph_0}$ where $\mathcal{A}$ is one of $\mathcal{I}^c_{I_n}$, $\mathcal{I}^c_{II_1}$, $\mathcal{I}^c_{II_\infty}$, $\mathcal{I}^c_{III}$. To simplify the argument, let $(i,k,\mathfrak{z})$ be one of $(I,n,\mathfrak{a}_N(n))$ (where $n \in \{1,2,\dots,\infty\}$), $(II,1,\mathfrak{s}_N(1))$, $(II,\infty,\mathfrak{s}_N(\infty))$, $(III,\infty,\mathfrak{f}_N)$ and let $\mathcal{A} = \mathcal{I}^c_{i_k}$. By Example 6.1.1 we know $\mathcal{A}$ is nontrivial. Hence (e.g. by Proposition 3.4.10) there is $\mathsf{A} \in \mathcal{A} \cap \mathcal{SEP}_N$ which is either minimal or semiminimal. Now according to Lemma 5.4.9, there is $\mu \in \operatorname{rgm}(\mathfrak{p}_N)$ such that

$$\mathsf{A} = \int_{\mathfrak{p}_N}^{\boxplus} \mathsf{P}\, d\mu(\mathsf{P}).$$

There is a measurable domain $\mathcal{F}$ on which μ is concentrated. Since μ is standard, we may assume that $\mathcal{F}$ is a standard Borel space, and that $\mathcal{F} \subset \mathfrak{Z}$, by Corollary 5.6.7(c). We infer from $\mathsf{A}^c = \mathsf{A}$ that μ is nonatomic and consequently that $\mathcal{F}$ is uncountable. So, $\mathcal{F}$ is Borel isomorphic to $[0,1]$, which implies that there is a family $\{\lambda_t\}_{t\in\mathbb{R}}$ of probability nonatomic Borel measures on $\mathcal{F}$ which are mutually singular. Since every measure on $\mathcal{F}$ is a regularity measure, Lemma 5.4.10 shows that $\mathsf{X}_s := \int_{\mathcal{F}}^{\boxplus} \mathsf{P}\, d\lambda_s(\mathsf{P}) \perp_u \int_{\mathcal{F}}^{\boxplus} \mathsf{P}\, d\lambda_t(\mathsf{P}) = \mathsf{X}_t$ for any distinct real numbers s and t. Finally, again thanks to Corollary 5.6.7, $\mathsf{X}_s \in \mathcal{A}$ (because $\mathcal{F} \subset \mathfrak{Z}$ and λ_s is nonatomic) and X_s is minimal or semiminimal for every $s \in \mathbb{R}$. Consequently, $\mathsf{X} := \boxplus_{s\in\mathbb{R}} \mathsf{X}_s$ is a minimal or semiminimal member of $\mathcal{A}$ as well. This gives $\mathsf{X} \leqslant \mathsf{J}(\mathcal{A})$ and therefore $\dim(\mathsf{J}(\mathcal{A})) \geqslant \dim(\mathsf{X}) = 2^{\aleph_0}$ (since $\mathsf{X}_s \in \mathcal{SEP}_N$ for each $s \in \mathbb{R}$). ■

An important consequence of Corollary 6.1.2 is that the underlying model space for $\mathcal{CDD}_N$ and its 'characteristic' subsets are independent of N. This will be crucial to our investigations. Hence, we may briefly speak of an *underlying model space*.

Everywhere below, $\mathcal{A}$ and $\mathcal{B}$ denote arbitrary ideals in $\mathcal{CDD}_N$ and $\mathcal{CDD}_{N'}$ (respectively).

Definition 6.1.3. A function $\Phi\colon \mathcal{A} \to \mathcal{B}$ is an *isomorphism* iff Φ is a bijection and $\Phi(\bigoplus_{s\in S} \mathsf{A}_s) = \bigoplus_{s\in S} \Phi(\mathsf{A}_s)$ for every collection $\{\mathsf{A}_s\}_{s\in S} \subset \mathcal{A}$ (where, of course, S is a set). An isomorphism $\Phi\colon \mathcal{A} \to \mathcal{B}$ is

- an *s-isomorphism* iff $\dim \Phi(\mathsf{A}) = \dim \mathsf{A}$ for every $\mathsf{A} \in \mathcal{A}$,
- a *t-isomorphism* iff for each $\mathsf{A} \in \mathcal{A}$ the following condition is fulfilled: $\Phi(\mathsf{A})$ is of type i^k iff so is A, where i^k is one of I^n $(n = 1, 2, \dots, \infty)$, II^1, II^∞, III^∞.

Two ideals are *isomorphic*, *s-isomorphic* or *t-isomorphic* if there exists a suitable isomorphism between them.

Let 'i' be the empty, 's' or 't' prefix. We write $\mathcal{A} \cong^i \mathcal{B}$ iff $\mathcal{A}$ and $\mathcal{B}$ are i-isomorphic. Additionally, we write $\mathcal{A} \preccurlyeq^i \mathcal{B}$ if $\mathcal{A} \cong^i \mathcal{B}'$ for some ideal $\mathcal{B}' \subset \mathcal{B}$.

As is easily seen, every t-isomorphism is an s-isomorphism. Therefore:

$$\mathcal{A} \cong^t \mathcal{B} \Rightarrow \mathcal{A} \cong^s \mathcal{B} \Rightarrow \mathcal{A} \cong \mathcal{B},$$
$$\mathcal{A} \preccurlyeq^t \mathcal{B} \Rightarrow \mathcal{A} \preccurlyeq^s \mathcal{B} \Rightarrow \mathcal{A} \preccurlyeq \mathcal{B}.$$

It is also clear that '$\preccurlyeq^i$' is transitive, while '$\cong^i$' is an equivalence relation.

The main tool of this part is the following

Theorem 6.1.4. *If $\Phi\colon \mathcal{A} \to \mathcal{B}$ is a bijection such that*

$$\Phi(\mathsf{X} \oplus \mathsf{Y}) = \Phi(\mathsf{X}) \oplus \Phi(\mathsf{Y}) \tag{6.1.2}$$

for any $\mathsf{X}, \mathsf{Y} \in \mathcal{A}$, then Φ is an isomorphism and Φ preserves all notions, features and operations appearing in (ST1)–(ST17) (pp. 36–37).

The above result is a generalization of Proposition 4.2.1 and its proof goes similarly (see Chapter 4.2). In particular, for every isomorphism $\Phi\colon \mathcal{A} \to \mathcal{B}$ and each $\mathsf{A} \in \mathcal{A}$ one has: $\dim \Phi(\mathsf{A})$ is uncountable iff so is $\dim(\mathsf{A})$, and if this is the case, they are equal. So, Φ is an s-isomorphism if Φ preserves 'dim' for members of $\mathcal{SEP}$ (the prefix 's' is from 'separable'). One may also check that Φ preserves atoms, fractals, semiprimes (using their definitions and the observation on page 59 after Definition 5.1.4), factor tuples (by Proposition 5.1.2) and types I, II and III. Consequently, $\Phi(\mathcal{A}^d) = \mathcal{B}^d$ and $\Phi(\mathcal{A}^c) = \mathcal{B}^c$.

6.2. Classification of ideals up to isomorphism

We shall now define characteristics of ideals which will turn out to be sufficient to determine whether $\mathcal{A} \cong^i \mathcal{B}$ or $\mathcal{A} \preccurlyeq^i \mathcal{B}$.

DEFINITION 6.2.1. For any $D \in \{I, I_1, I_2, \ldots, I_\infty, II, II_1, II_\infty, III\}$ let

$$\chi_D^d(\mathcal{A}) = \operatorname{card}(\{\mathsf{X}\colon \mathsf{X} \in \mathfrak{F}_N \cap \mathcal{I}_D,\ \mathsf{X} \leqslant^s \mathsf{J}(\mathcal{A})\}),$$

$\chi_D^c(\mathcal{A}) = \dim(\mathsf{J}(\mathcal{A}^c \cap \mathcal{I}_D))$ and $\chi_D(\mathcal{A}) = (\chi_D^d(\mathcal{A}), \chi_D^c(\mathcal{A}))$. Finally, let

$$\chi(\mathcal{A}) = (\chi_I(\mathcal{A}); \chi_{II}(\mathcal{A}); \chi_{III}(\mathcal{A})),$$
$$\chi_s(\mathcal{A}) = (\chi_{I_1}^d(\mathcal{A}), \chi_{I_2}^d(\mathcal{A}), \ldots, \chi_{I_\infty}^d(\mathcal{A})),$$
$$\chi_t(\mathcal{A}) = (\chi_{I_1}(\mathcal{A}); \chi_{I_2}(\mathcal{A}); \ldots, \chi_{I_\infty}(\mathcal{A}); \chi_{II_1}(\mathcal{A}); \chi_{II_\infty}(\mathcal{A})).$$

When comparing sequences (finite or infinite) of the same length whose entries are cardinals, '$\leqslant$' will denote the coordinatewise order.

Let Ω be an underlying model space and let $\Psi_N = \Phi_{\mathsf{T}}\colon \mathcal{CDD}_N \to \Lambda(\Omega)$ be as in Theorem 4.4.2. For $E = \operatorname{supp}_\Omega \mathcal{A}$ we have (under the notation of Definition 6.2.1) $\chi_D^d(\mathcal{A}) = \kappa_d(E \cap \Omega_D)$ and $\chi_D^c(\mathcal{A}) = \kappa_c(E \cap \Omega_D)$ (cf. Proposition 4.4.5). So, according to Theorem 5.1.12 (page 62; below '$\cong$' means 'homeomorphic'),

$$\Omega_D \cap \operatorname{supp}_\Omega \mathcal{A} \cong \Omega_D \cap \operatorname{supp}_\Omega \mathcal{B} \iff \chi_D(\mathcal{A}) = \chi_D(\mathcal{B}). \tag{6.2.1}$$

As an application of Theorem 6.1.4, Theorem 4.4.2, Corollary 6.1.2 and (6.2.1) we obtain

THEOREM 6.2.2. *Let N and N' be positive integers, and $\mathcal{A} \subset \mathcal{CDD}_N$ and $\mathcal{B} \subset \mathcal{CDD}_{N'}$ be ideals.*

- (I) $\mathcal{CDD}_N \cong^t \mathcal{CDD}_{N'}$. *What is more, each entry of $\chi(\mathcal{CDD}_N)$, of $\chi_s(\mathcal{CDD}_N)$ as well as of $\chi_t(\mathcal{CDD}_N)$ is equal to $2^{\aleph_0}$.*
- (II) $\mathcal{A} \cong \mathcal{B} \Leftrightarrow \chi(\mathcal{A}) = \chi(\mathcal{B})$; $\mathcal{A} \preccurlyeq \mathcal{B} \Leftrightarrow \chi(\mathcal{A}) \leqslant \chi(\mathcal{B})$.
- (III) $\mathcal{A} \cong^s \mathcal{B} \Leftrightarrow \chi(\mathcal{A}) = \chi(\mathcal{B})$ *and* $\chi_s(\mathcal{A}) = \chi_s(\mathcal{B})$; $\mathcal{A} \preccurlyeq^s \mathcal{B} \Leftrightarrow \chi(\mathcal{A}) \leqslant \chi(\mathcal{B})$ *and* $\chi_s(\mathcal{A}) \leqslant \chi_s(\mathcal{B})$.
- (IV) $\mathcal{A} \cong^t \mathcal{B} \Leftrightarrow \chi(\mathcal{A}) = \chi(\mathcal{B})$ *and* $\chi_t(\mathcal{A}) = \chi_t(\mathcal{B})$; $\mathcal{A} \preccurlyeq^t \mathcal{B} \Leftrightarrow \chi(\mathcal{A}) \leqslant \chi(\mathcal{B})$ *and* $\chi_t(\mathcal{A}) \leqslant \chi_t(\mathcal{B})$.
- (V) *Up to isomorphism (resp. t-isomorphism), there are only γ (resp. $2^{\aleph_0}$) different ideals where $\gamma = \operatorname{card}(\{\alpha \in \mathrm{Card}\colon \alpha \leqslant 2^{\aleph_0}\})$.*

Proof. The second claim of (I) follows from Corollary 6.1.2 and Proposition 5.1.8. Since (V) and the remainder of (I) follow from (IV), it is sufficient to prove (II)–(IV). Since their proofs are based on the same idea, we only handle (IV).

Since representatives of members of $\mathcal{I}^c$ act in infinite-dimensional Hilbert spaces, Theorem 6.1.4 shows that if $\Phi\colon \mathcal{A} \to \mathcal{A}' \subset \mathcal{B}$ is a t-isomorphism, then necessarily $\chi(\mathcal{A}) = \chi(\mathcal{A}') \leqslant \chi(\mathcal{B})$ and $\chi_t(\mathcal{A}) = \chi_t(\mathcal{A}') \leqslant \chi_t(\mathcal{B})$. Conversely, if $\chi(\mathcal{A}) \leqslant \chi(\mathcal{B})$ and $\chi_t(\mathcal{A}) \leqslant \chi_t(\mathcal{B})$ (respectively $\chi(\mathcal{A}) = \chi(\mathcal{B})$ and $\chi_t(\mathcal{A}) = \chi_t(\mathcal{B})$), then there is an ideal $\mathcal{A}' \subset \mathcal{B}$ ($\mathcal{A}' = \mathcal{B}$) for which $\chi(\mathcal{A}') = \chi(\mathcal{A})$ and $\chi_t(\mathcal{A}') = \chi_t(\mathcal{A})$ (this may be deduced e.g. from (6.2.1); $\mathcal{A}'$ may be defined as $\mathcal{I}[F]$ for suitable clopen set $F \subset \operatorname{supp}_\Omega \mathcal{B}$). Now Theorem 5.1.12 combined with (6.2.1) implies that there are homeomorphisms $h_D\colon \Omega_D \cap \operatorname{supp}_\Omega \mathcal{A} \to \Omega_D \cap \operatorname{supp}_\Omega \mathcal{A}'$ where D runs over $I_1, I_2, \ldots, I_\infty, II_1, II_\infty, III$. Define a homeomorphism $H\colon \operatorname{supp}_\Omega \mathcal{A} \to \operatorname{supp}_\Omega \mathcal{B}$ as the unique continuous extension of the union of all h_D's. Finally let $\Phi\colon \mathcal{A} \to \mathcal{B}$ be defined as follows. For $\mathsf{A} \in \mathcal{A}$ put $f = \Psi_N(\mathsf{A}) \in \Lambda(\Omega)$. Since $\operatorname{supp} f \subset \operatorname{supp}_\Omega \mathcal{A}$, the rules $g = f \circ H^{-1}$ on $\operatorname{supp}_\Omega \mathcal{B}$ and $g = 0$ elsewhere well define $g \in \Lambda(\Omega)$ such that $\operatorname{supp} g \subset \operatorname{supp}_\Omega \mathcal{B}$. We put $\Phi(\mathsf{A}) = \Psi_{N'}^{-1}(g)$. It is easily seen that Φ is a well defined bijection. What is more, Φ satisfies condition (6.1.2), by Theorem 4.4.2(D4$'$). Consequently, Theorem 6.1.4 shows that Φ is an isomorphism. It follows from the construction that Φ is in fact a t-isomorphism. ∎

Corollary 6.2.3. *If $\mathcal{A} \preccurlyeq^i \mathcal{B}$ and $\mathcal{B} \preccurlyeq^i \mathcal{A}$, then $\mathcal{A} \cong^i \mathcal{B}$.*

Corollary 6.2.4. *$\mathcal{CDD}_N \cong^t \mathcal{I}(1)$ where $\mathcal{I}(1) \subset \mathcal{CDD}$ is the ideal of all contraction operators.*

Proof. Thanks to Theorem 6.2.2 we may assume that $N = 1$. Observe that the $\mathfrak{b}$-transform is a t-isomorphism of $\mathcal{CDD}$ onto a subideal of $\mathcal{I}(1)$. So, the assertion follows from Corollary 6.2.3. ∎

Corollary 6.2.5. *Let $\mathcal{U}$ be the ideal of all single unitary operators.*

(1) *The ideal $\mathcal{I}_{I_1}$ of all normal N-tuples is t-isomorphic to $\mathcal{U}$.*
(2) *The ideal $\mathcal{I}_I$ of all N-tuples of type I is s-isomorphic to $\mathcal{U}$.*

Proof. Observe that all entries of the suitable characteristics of the ideals in question coincide (and each is either 0 or $2^{\aleph_0}$) and apply Theorem 6.2.2. ∎

The above corollaries say that whatever can be said about single (unitary) contraction operators in the language of 'discrete' direct sums, this will have its natural counterpart for arbitrary (type I) N-tuples.

Remark 6.2.6. Since $\mathcal{CDD}_N \cong^t \mathcal{CDD}_{N'}$ for any N and N', we may also speak of *spatially i-isomorphic* ideals. Precisely, ideals $\mathcal{A} \subset \mathcal{CDD}_N$ and $\mathcal{A}' \subset \mathcal{CDD}_{N'}$ are spatially i-isomorphic (as usual, 'i' is the empty, 's' or 't' prefix) iff there is an i-isomorphism $\Phi\colon \mathcal{CDD}_N \to \mathcal{CDD}_{N'}$ which sends $\mathcal{A}$ onto $\mathcal{A}'$. However, this idea brings nothing new. Indeed, it is quite easy to check that $\mathcal{A}$ and $\mathcal{A}'$ are spatially i-isomorphic iff $\mathcal{A} \cong^i \mathcal{A}'$ and $\mathcal{A}^\perp \cong^i (\mathcal{A}')^\perp$. So, we only have to double the length of characteristics. However, one relevant information may be interesting: up to spatial isomorphism, there are only $\operatorname{card}(\{\alpha \in \mathrm{Card}\colon \alpha \leqslant 2^{\aleph_0}\})$ different ideals. So, under the continuum hypothesis, this number is countable.

6.3. Concluding remarks

6.3.1. Finite-dimensional tuples. The results of Chapters 5.4–5.6, especially Lemmas 5.4.9 and 5.6.12, prove that it is good to know how to recognize regularity measures, especially in finite-dimensional case, since Proposition 5.4.2 simply characterizes summable fields of N-tuples. The author is not aware of the existence of any result in this direction. We make

CONJECTURE. *Every σ-finite* (*Borel*) *measure on $\mathfrak{a}_N(n)$ for finite n is concentrated on a measurable domain.*

Below we confirm the conjecture for $n = 1$. (This is surely well known. However, we could not find anything about it in the literature.) Let us first make some comments on consequences of the conjecture. If it is true, then every pair $(\mathscr{X}, \{\mathsf{P}_x\}_{x\in\mathscr{X}})$ where $(\mathscr{X}, \mathcal{M}, \mathcal{N})$ is standard and $\mathscr{X} \ni x \mapsto \mathsf{P}_x \in \bigcup_{n=1}^{\infty} \mathfrak{a}_N(n)$ is a one-to-one Borel function is a regular system, i.e. P_x's form the prime decomposition of some $\mathsf{X} \in \mathcal{SEP}_N$. Indeed, the sets $\mathscr{X}_n = \{x \in \mathscr{X} \colon \dim(\mathsf{P}_x) = n\}$ $(n = 1, 2, \ldots)$ are measurable and there is a finite Borel measure μ_n on $\mathfrak{a}_N(n)$ such that $\mathscr{X}_n \ni x \mapsto \mathsf{P}_x \in \mathfrak{a}_N(n)$ is an almost null-isomorphism between $(\mathscr{X}_n, \mathfrak{M}|_{\mathscr{X}_n}, \mathcal{N}|_{\mathscr{X}_n})$ and $(\mathfrak{a}_N(n), \mathfrak{B}(\mathfrak{a}_N(n)), \mathcal{N}(\mu_n))$. Now it follows from the conjecture that $\mu_n \in \operatorname{rgm}(\mathfrak{a}_N(n))$ and consequently $\{\mathsf{P}_x\}_{x\in\mathscr{X}_n} \in \mathrm{RGS}_{\mathrm{loc}}$. Put $\mathsf{X}_n = \boxplus_{x\in\mathscr{X}_n}^{\mathcal{N}} \mathsf{P}_x (= \int_{\mathfrak{a}_N(n)}^{\boxplus} \mathsf{P}\, d\mu_n(\mathsf{P}))$. We conclude from Corollary 5.6.7 that $\mathsf{X}_n \in \mathcal{I}_{I_n}$. So, $\mathsf{X}_n \perp_u \mathsf{X}_m$ for $n \neq m$ and therefore $\mu_n \perp_s \mu_m$, thanks to Lemma 5.4.10. Now it suffices to apply Lemma 5.6.12 to deduce that $\{\mathsf{P}_x\}_{x\in\mathscr{X}} \in \mathrm{RGS}_{\mathrm{loc}}$ (and $\boxplus_{n=1}^{\infty} \mathsf{X}_n = \boxplus_{x\in\mathscr{X}}^{\mathcal{N}} \mathsf{P}_x$).

The work of Ernest shows that there are standard Borel measures on $\mathfrak{p}_N \cap \mathcal{SEP}_N(\infty)$ which are not concentrated on measurable domains (see Propositions 1.53 and 3.13 in [9]).

Let us now show that every σ-finite Borel measure μ on $\mathbb{C}^N$ is concentrated on a measurable domain. Since there is a finite Borel measure ν on $\mathbb{C}^N$ such that $\mu \ll \nu$, we may assume μ is finite. First assume μ is concentrated on a compact set. Put $\boldsymbol{T} = \int_{\mathbb{C}^N}^{\oplus} \xi\, d\mu(\xi)$. It follows from the Stone–Weierstrass theorem that $M_f \in \mathcal{W}(\boldsymbol{T})$ for every $f \in \mathcal{C}(K)$ where M_f is multiplication by f. This implies that $M_u \in \mathcal{W}(\boldsymbol{T})$ for every $u \in L^\infty(\mu)$ as well. Consequently, $M_u \in \mathcal{Z}(\mathcal{W}(\boldsymbol{T}))$ (since $\mathcal{W}(\boldsymbol{T})$ consists of decomposable operators) and hence $\int_A^{\oplus} \xi\, d\mu(\xi) \perp_u \int_{\mathbb{C}^N\setminus A}^{\oplus} \xi\, d\mu(\xi)$, which shows that $\mathsf{T} = \int_{\mathbb{C}^N}^{\boxplus} \xi\, d\mu(\xi)$ and thus $\mu \in \operatorname{rgm}(\mathbb{C}^N)$.

Now if μ is arbitrary, there is a sequence $(K_n)_{n=1}^{\infty}$ of compact pairwise disjoint subsets of $\mathbb{C}^N$ such that $\mu(\mathbb{C}^N \setminus \bigcup_{n=1}^{\infty} K_n) = 0$. The above argument proves that $\mu|_{K_n} \in \operatorname{rgm}(\mathbb{C}^N)$ for every n. Put $\mathsf{X}_n = \int_{K_n}^{\boxplus} \xi\, d\mu(\xi)$. Now we repeat an earlier argument: $\mathsf{X}_n \perp_u \mathsf{X}_m$ for $n \neq m$ (by Lemma 5.4.10) and thus $\mu \in \operatorname{rgm}(\mathbb{C}^N)$, thanks to Lemma 5.6.12.

6.3.2. Problem of axiomatization. Theorem 5.6.17 (cf. also Remark 5.6.8) establishes a one-to-one correspondence between coverings and functions $\mu \colon \mathfrak{M} \to \mathcal{CDD}_N$ satisfying conditions (M0)–(M3) (see Theorem 5.6.17). These conditions are purely 'discrete', i.e. they need no measure-theoretic nor topological background and are formulated in terms of the direct sum operation for pairs. So, it seems to be interesting (and may turn out to be relevant) which topological or measure-theoretic notions (operations, features, tools, etc.) are sufficient to reconstruct from μ the covering to which it corresponds.

6.3.3. 'Continuous' ideals. Just as we defined continuous direct sums, one may try to define 'continuous' ideals in $\mathcal{CDD}_N$. This may be done in a few ways. Here we propose only one of them. Let us call an ideal $\mathcal{A} \subset \mathcal{CDD}_N$ *continuous* if $\mathcal{A}$ satisfies the following condition. Whenever $(\mathscr{X}, \mathfrak{M}, \mathcal{N}, \Phi)$ is a full covering and $\mathsf{A} = \boxplus_{x\in\mathscr{X}}^{\mathcal{N}} \mathfrak{m}(x) \odot \Phi(x)$ for some $\mathfrak{m} \in \mathscr{A}(\mathscr{X})$, then $\mathsf{A} \in \mathcal{A}$ if and only if there is a set $\mathscr{Z} \in \mathcal{N}$ such that $\Phi(x) \in \mathcal{A}$ for every $x \in s(\mathfrak{m}) \setminus \mathscr{Z}$. Using Theorem 5.6.14 one may easily check that it suffices to verify the above condition for a fixed full covering and only for $\mathsf{A} \in \mathcal{SEP}_N$. For example, $\mathcal{I}_i$ is a continuous ideal for each $i \in \{I, I_1, I_2, \ldots, I_\infty, II, II_1, II_\infty, III\}$, while $\mathcal{I}_i^c$ and $\mathcal{I}_i^d$ are not. A *p-isomorphism* (the 'p' refers to 'prime decomposition') between continuous ideals is an isomorphism $\Psi\colon \mathcal{A} \to \mathcal{B}$ such that whenever $\mathsf{A} = \boxplus_{x\in\mathscr{X}}^{\mathcal{N}} \mathfrak{m}(x) \odot \mathsf{P}_x$ is a prime decomposition of $\mathsf{A} \in \mathcal{A}$, a prime decomposition of $\Psi(\mathsf{A})$ may be written in the form $\boxplus_{x\in\mathscr{X}}^{\mathcal{N}} \mathfrak{m}(x) \odot \Psi(\mathsf{P}_x)$, and the same for Ψ^{-1}. The following problem may be interesting.

QUESTION. *Are $\mathcal{CDD}_N$ and $\mathcal{CDD}_{N'}$ p-isomorphic?*

6.3.4. Length of tuples. Our last remark is about the length of tuples. Readers interested in sequences (that is, countable infinite families) of closed densely defined operators acting in common Hilbert spaces may verify that most of the results (with no changes in proofs) of this work remain true also in that case, i.e. for $N = \infty$. (However, when working with uncountable families, a counterpart of crucial Theorem 2.2.4 fails to be true, which causes the whole theory to break down in that case.) Since infinite sequences are rarely investigated, we restricted our study to finite collections.

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