Contents

Abstract

We consider the motion of incompressible mhd in a domain bounded a free surface. In the external domain there exists an electromagnetic field generated by some currents which keeps the mhd flow in the bounded domain. On the free surface transmission conditions for the electromagnetic fields are imposed. For sufficiently small initial velocity and vanishing external force the global existence is proved. The L_2-approach is used. This helps us to treat the transmission conditions.

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1. Introduction

In this paper we show global in time existence of solutions to a free boundary problem for incompressible magnetohydrodynamics. In a domain $\overset{1}{\Omega}_t \subset \mathbb{R}^3$ bounded by a free surface S_t we consider the motion of incompressible magnetohydrodymamics. A domain $\overset{2}{\Omega}_t$, which is bounded by a fixed boundary B, is external to $\overset{1}{\Omega}_t$. In $\overset{2}{\Omega}_t$ we have an electromagnetic field generated by some currents on B. Hence the electromagnetic field in $\overset{2}{\Omega}_t$ acts on the motion of magnetohydrodynamics in $\overset{1}{\Omega}_t$ through the free boundary S_t. Moreover, the motion in $\overset{1}{\Omega}_t$ is generated by the initial velocity and the initial magnetic field and also by the initial shape of the free surface S_t. Our aim is to prove global existence assuming that the initial data are sufficiently small. Our considerations base on [4], where local existence of solutions of this problem is proved.

In $\overset{1}{\Omega}_t$ the motion is described by

$$
\begin{aligned}
&v_t + v\cdot\nabla v - \operatorname{div}\mathbb{T}(v,p) - \operatorname{div}(\mu\mathbb{T}(\overset{1}{H})) = f \quad && \text{in } \tilde{\Omega}_1^T,\\
&\operatorname{div} v = 0 && \text{in } \tilde{\Omega}_1^T,
\end{aligned} \tag{1.1}
$$

$$
\begin{aligned}
&\mu(\overset{1}{H}_t + v\cdot\nabla\overset{1}{H} - \overset{1}{H}\cdot\nabla v) + \frac{1}{\sigma_1}\operatorname{rot}\operatorname{rot}\overset{1}{H} = 0 \quad && \text{in } \tilde{\Omega}_1^T,\\
&\operatorname{div}\overset{1}{H} = 0 && \text{in } \tilde{\Omega}_1^T,
\end{aligned} \tag{1.2}
$$

where $\tilde{\Omega}_1^T = \bigcup_{0\le t\le T}\overset{1}{\Omega}_t \times \{t\}$, $v = (v_1(x,t), v_2(x,t), v_3(x,t)) \in \mathbb{R}^3$ is the velocity of the fluid, $p = p(x,t) \in \mathbb{R}$ is the pressure, $x = (x_1, x_2, x_3)$ the global cartesian coordinates. $\overset{i}{H} = \overset{i}{H}(x,t) = (\overset{i}{H}_1(x,t), \overset{i}{H}_2(x,t), \overset{i}{H}_3(x,t)) \in \mathbb{R}^3$, $i = 1,2$, is the magnetic field, $f = f(x,t) = (f_1(x,t), f_2(x,t), f_3(x,t)) \in \mathbb{R}^3$ is the external force field, μ is the constant magnetic permeability and σ_i is the constant electric conductivity in $\overset{i}{\Omega}_t$, $i = 1,2$. We denote by $\mathbb{T}(v,p)$ the stress tensor of the form

$$\mathbb{T}(v,p) = \nu\mathbb{D}(v) - p\mathbb{I},$$

where ν is the constant viscosity coefficient, $\mathbb{I}$ is the unit matrix and

$$\mathbb{D}(v) = \{v_{ix_j} + v_{jx_i}\}_{i,j=1,2,3} \tag{1.3}$$

is the dilatation tensor.

Moreover, we denote by $\mathbb{T}(\overset{1}{H})$ the stress tensor of the magnetic field described by

$$\mathbb{T}(\overset{1}{H}) = \left\{\overset{1}{H}_i\overset{1}{H}_j - \frac{\overset{1}{H}{}^2}{2}\delta_{ij}\right\}_{i,j=1,2,3}. \tag{1.4}$$

In $\overset{2}{\Omega}_t$ there is no fluid motion but there is a fluid under a constant pressure p_0, where the electromagnetic field is described by the problem

$$
\begin{aligned}
&\mu\overset{2}{H}{}'_t + \frac{1}{\sigma_2}\operatorname{rot}\operatorname{rot}\overset{2}{H}{}' = 0 && \text{in } \tilde{\Omega}_2^T,\\
&\operatorname{div}\overset{2}{H}{}' = 0 && \text{in } \tilde{\Omega}_2^T,\\
&\overset{2}{H}{}' = H_* && \text{on } B,
\end{aligned}
\tag{1.5}
$$

where $\tilde{\Omega}_2^T = \bigcup_{0\le t\le T}\overset{2}{\Omega}_t\times\{t\}$. As the initial conditions we assume $\overset{i}{\Omega}_t|_{t=0} = \overset{i}{\Omega}$, $S_t|_{t=0} = S$, $i = 1,2$, $v|_{t=0} = v_0$, $\overset{1}{H}|_{t=0} = \overset{1}{H}_0$ in $\overset{1}{\Omega}$, $\overset{2}{H}|_{t=0} = \overset{2}{H}_0$ in $\overset{2}{\Omega}$.

It is difficult to examine problem (1.5) because the nonhomogeneous Dirichlet boundary condition on B implies that integration by parts will not give any estimate. Therefore, we construct an extension $\tilde{H}_*$ of H_* satisfying

$$
\tilde{H}_*|_B = H_*,
\tag{1.6}
$$

which is divergence free and vanishes in a neighborhood of S_t. The divergence free extension is possible if

$$
0 = \int_{\overset{2}{\Omega}_t}\operatorname{div}\tilde{H}_*\,dx = \int_B H_*\cdot\bar{n}\,dB,
\tag{1.7}
$$

where $\bar{n}$ is the unit normal external vector to B. Using the extension $\tilde{H}_*$ we can introduce the new function

$$
\overset{2}{H} = \overset{2}{H}{}' - \tilde{H}_*
\tag{1.8}
$$

which is a solution to the problem

$$
\begin{aligned}
&\mu\overset{2}{H}_t + \frac{1}{\sigma_2}\operatorname{rot}\operatorname{rot}\overset{2}{H} = -\mu\tilde{H}_{*t} + \frac{1}{\sigma_2}\operatorname{rot}\operatorname{rot}\tilde{H}_* \equiv G && \text{in } \tilde{\Omega}_2^T,\\
&\operatorname{div}\overset{2}{H} = 0 && \text{in } \tilde{\Omega}_2^T,\\
&\overset{2}{H} = 0 && \text{on } B,\\
&\overset{2}{H}|_{t=0} = \overset{2}{H}{}'(0) - \tilde{H}_*|_{t=0} \equiv \overset{2}{H}(0) && \text{in } \overset{2}{\Omega}.
\end{aligned}
\tag{1.9}
$$

To transform problem (1.9) to lagrangian coordinates we extend v on $\overset{2}{\Omega}_t$. The extension is denoted by v' and it is divergence free (see [11]). In this case we can express problem (1.9) in the form

$$
\begin{aligned}
&\mu(\overset{2}{H}_t + v'\cdot\nabla\overset{2}{H}) - \frac{1}{\sigma_2}\operatorname{rot}\operatorname{rot}\overset{2}{H} = G + \mu v'\cdot\nabla\overset{2}{H} && \text{in } \tilde{\Omega}_2^T,\\
&\operatorname{div}\overset{2}{H} = 0 && \text{in } \tilde{\Omega}_2^T,\\
&\overset{2}{H} = 0 && \text{on } B,\\
&\overset{2}{H}|_{t=0} = \overset{2}{H}(0) && \text{in } \overset{2}{\Omega}.
\end{aligned}
\tag{1.10}
$$

Examining problem (1.10) we can introduce in $\overset{2}{\Omega}_t$ the lagrangian coordinates ξ as the initial data to the problem

$$\frac{dx}{dt} = v'(x,t), \quad x|_{t=0} = \xi \in \overset{2}{\Omega}, \quad x \in \overset{2}{\Omega}_t. \tag{1.11}$$

Similarly, lagrangian coordinates in $\overset{1}{\Omega}_t$ are defined by the problem

$$\frac{dx}{dt} = v(x,t), \quad x|_{t=0} = \xi \in \overset{1}{\Omega}, \quad x \in \overset{1}{\Omega}_t. \tag{1.12}$$

To simplify the presentation of the results we introduce the notation

$$(v, v') = (\overset{1}{v}, \overset{2}{v}). \tag{1.13}$$

Finally, we introduce the following boundary and transmission conditions on S_t:

$$\begin{aligned} &\bar n \cdot \mathbb{T}(v,p) + \mu \bar n \cdot \mathbb{T}(\overset{1}{H}) = -p_0 \bar n \quad \text{on } S_t,\\ &v \cdot \bar n = -\frac{\varphi_t}{|\nabla\varphi|} \end{aligned} \tag{1.14}$$

and

$$[H] = 0, \quad \left(\frac{1}{\sigma_1}\operatorname{rot}\overset{1}{H} - \mu v \times \overset{1}{H}\right)_\tau = \frac{1}{\sigma_2}(\operatorname{rot}\overset{2}{H})_\tau \quad \text{on } S_t, \tag{1.15}$$

where the subscript τ means the tangent coordinates to S_t, $\varphi(x,t) = 0$ describes at least locally S_t and $[A] = \overset{1}{A} - \overset{2}{A}$, where $\overset{i}{A}$ is the quantity defined in $\overset{i}{\Omega}_t$, $i = 1,2$. The transmission conditions (1.15) couple the magnetic fields $\overset{1}{H}$ and $\overset{2}{H}$.

To prove global existence we need the following local result (see [4]).

Theorem 1. *Assume that* $v_0 \in H^2(\overset{1}{\Omega})$, $\overset{i}{H}_0 \in H^2(\overset{i}{\Omega})$, $i = 1,2$, $\partial_t^k v|_{t=0} \in L_2(\overset{1}{\Omega})$, $\partial_t^k \bar{\overset{i}{H}}|_{t=0} \in L_2(\overset{i}{\Omega})$, $i = 1,2$, $k \le 2$, $\partial_t^i \bar f \in L_2(\overset{1}{\Omega}{}^T)$, $i \le 2$, $\bar G, \bar G_t \in L_2(\overset{2}{\Omega}{}^T)$ *and* $S \in H^{5/2}$, *where* $\overset{i}{\Omega}{}^T = \overset{i}{\Omega} \times [0,T]$, $i = 1,2$. *Assume the transmission conditions* (1.15) *hold on* S_t. *Then for* T *sufficiently small there exists a solution to problem* (1.1), (1.2), (1.10), (1.15) *such that*

$$\begin{aligned} &\partial_t^i \bar v \in L_\infty(0,T; H^{2-i}(\overset{1}{\Omega})) \cap L_2(0,T; H^{3-i}(\overset{1}{\Omega})), \qquad i \le 2,\\ &\partial_t^k \bar{\overset{i}{H}} \in L_\infty(0,T; H^{2-k}(\overset{i}{\Omega})) \cap L_2(0,T; H^{3-k}(\overset{i}{\Omega})) \equiv X(\overset{i}{\Omega} \times [0,T]),\\ &\qquad\qquad k \le 2, \ i = 1,2, \end{aligned} \tag{1.16}$$

and

$$\begin{aligned} &\|\bar v\|_{X(\overset{1}{\Omega}{}^T)} + \|\bar p'\|^2_{\Gamma_0^2(\overset{1}{\Omega}{}^T)} + \sum_{i=1}^2 \|\bar{\overset{i}{H}}\|_{X(\overset{i}{\Omega}{}^T)}\\ &\le c\Big[\sum_{k=0}^2 \Big(\|\partial_t^k v|_{t=0}\|_{H^{2-k}(\overset{1}{\Omega})} + \sum_{i=1}^2 \|\partial_t^k \overset{i}{H}|_{t=0}\|_{H^{2-k}(\overset{i}{\Omega})}\Big) + \|\bar f\|_{L_2(0,T;H^1(\overset{1}{\Omega}))}\\ &\quad + \|\bar f_t\|_{L_2(\overset{1}{\Omega}{}^T)} + \|\bar f_{tt}\|_{L_2(\overset{1}{\Omega}{}^T)} + \|\bar G\|_{L_2(0,T;\overset{1}{H}(\overset{2}{\Omega}))} + \|\bar G_t\|_{L_2(\overset{2}{\Omega}{}^T)} + \|\bar G_{tt}\|_{L_2(\overset{2}{\Omega}{}^T)}\Big] \equiv \bar D, \end{aligned} \tag{1.17}$$

where $\bar G$ *is given by* (1.9), $\bar p' = \bar p - p_0$ *and* $\bar u = \bar u(\xi,t) = u(x(\xi,t),t)$ (*see* (1.11), (1.12)).

Main Theorem. *Let the assumptions of Theorem* 1 *hold. Assume that* $f = 0$, $\int_\Omega v_0\,dx = \int_\Omega v_0 \cdot \varphi_i\,dx = 0$, $i = 1,2,3$, *where* φ_i, $i = 1,2,3$ *are defined in* (6.4). *Assume that* D *from* (1.17) *is sufficiently small. Then there exists a global solution to the problem* (1.1), (1.2), (1.10), (1.15) *in* $X(\overset{i}{\Omega} \times \mathbb{R}_+)$, $i = 1,2$, *where* $X(\overset{i}{\Omega} \times \mathbb{R}_+)$ *is defined in* (1.16).

Remark 1.1. For $\bar{D}$ small Theorem 1 holds with suitably large T. Moreover, if $\bar{D}$ goes to zero then the existence time T converges to infinity.

2. Notation and auxiliary results

First we introduce the notation employed in this paper. We do not distinguish between norms of scalar and vector-valued functions. Let ω be a vector, $\omega = (\omega_1, \dots, \omega_n)$. Then

$$|\omega| = \left(\sum_{i=1}^n |\omega_i|^2\right)^{1/2}.$$

Let $L_p(\Omega) = \{u : \int_\Omega |u|^p\,dx < \infty\}$, $p \in [1,\infty]$. The space of functions with the norm

$$\|u\|_{V_2^0(\Omega^T)} = \|u\|_{L_\infty(0,T;L_2(\Omega))} + \|u\|_{L_2(0,T;H^1(\Omega))}$$

is denoted by $V_2^0(\Omega^T)$. We shall use the notation

$$\|u\|_{\Gamma_k^l(\Omega)} = \sum_{i \le l-k} \|\partial_t^i u\|_{H^{l-i}(\Omega)}, \quad l,k \in \mathbb{N},$$

where $H^l(\Omega) = \left\{u : \sum_{|\alpha| \le l} \|D_x^\alpha u\|_{L_2(\Omega)} < \infty\right\}$, and

$$\|u\|_{\Gamma_{k,r}^l(\Omega^T)} = \|u\|_{L_r(0,T;\Gamma_k^l(\Omega))}.$$

Let

$$\|u\|_{L_p^k(\Omega)} = \sum_{|\alpha|=k} \|D_x^\alpha u\|_{L_p(\Omega)}, \quad \|u\|_{L_2^1(\Omega^t)} = \|u_x\|_{L_2(\Omega^t)} + \|u_t\|_{L_2(\Omega^t)},$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multiindex, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, $\alpha_i \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$, $i = 1,2,3$, $D_x^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$.

By c we denote a generic constant which changes its value from formula to formula. Similarly we denote by φ a generic function which is always positive and increasing.

To examine free boundary problems in hydrodynamics we use lagrangian coordinates which are the initial data to the following Cauchy problem:

$$\frac{dx}{dt} = v(x,t), \quad x|_{t=0} = \xi \in \overset{1}{\Omega}. \tag{2.1}$$

Therefore,

$$x_v(\xi,t) = \xi + \int_0^t \bar{v}(\xi,s)\,ds, \tag{2.2}$$

where $\bar{v}(\xi,t) = v(x_v(\xi,t),t)$.

To define lagrangian coordinates in $\overset{2}{\Omega}_t$ we need

LEMMA 2.1 ([11]). *Let $X(\overset{1}{\Omega}_t)$ be some Sobolev space. Let $v \in X(\overset{1}{\Omega}_t)$ be divergence free. Then there exists an extension v' of v on $\overset{1}{\Omega}_t \cup \overset{2}{\Omega}_t$ such that v' is divergence free, $v'|_{\overset{1}{\Omega}_t} = v$ and there exists a constant c such that*

$$\|v'\|_{X(\overset{1}{\Omega}_t \cup \overset{2}{\Omega}_t)} \le c\|v\|_{X(\overset{1}{\Omega}_t)}. \tag{2.3}$$

In view of the definition of lagrangian coordinates we have

$$\overset{1}{\Omega}_t = \{x \in \mathbb{R}^3 : x = x_v(\xi,t),\, \xi \in \overset{1}{\Omega}\}, \qquad S_t = \{x \in \mathbb{R}^3 : x = x_v(\xi,t),\, \xi \in S\},$$
$$\overset{1}{\Omega}_t \cup \overset{1}{\Omega}_t = \{x \in \mathbb{R}^3 : x = x_{v'}(\xi,t),\, \xi \in \overset{1}{\Omega} \cup \overset{2}{\Omega}\}.$$

To formulate our problem in lagrangian coordinates we need the notation

$$\begin{aligned} &\nabla_{\bar v} = \frac{\partial \xi_k}{\partial x}\frac{\partial}{\partial \xi_k}, \qquad \mathbb{D}_{\bar v}\bar u = \nabla_{\bar v}\bar u + (\nabla_{\bar v}\bar u)^T,\\ &\mathbb{T}_{\bar v}(\bar u,\bar p) = \mathbb{D}_{\bar v}(\bar u) - \bar p\mathbb{I}, \quad \operatorname{div}_{\bar v}\bar v = \partial_{x_i}\xi_k\partial_{\xi_k}\bar v_i = \nabla_{\bar v}\cdot\bar v, \end{aligned} \tag{2.4}$$

where summation over repeated indices is assumed, $\xi = \xi(x,t)$ is the inverse transformation to $x = x_{\bar v}(\xi,t)$. From [13, 12] we have

LEMMA 2.2. *Let $\Omega \subset \mathbb{R}^3$ be a given bounded domain. Let $v \in L_2(\Omega)$ be such that*

$$E_\Omega(v) = \int_\Omega (v_{jx_i} + v_{ix_j})^2\, dx. \tag{2.5}$$

Then there exists a constant c such that

$$\|v\|^2_{H^1(\Omega)} \le c(E_\Omega(v) + \|v\|^2_{L_2(\Omega)}). \tag{2.6}$$

LEMMA 2.3. *Let (2.2) describe the relation between the eulerian x and lagrangian ξ coordinates. Then*

$$|x_\xi - I| \le \Big|\int_0^t \bar v_\xi(\xi,s)\, ds\Big|, \tag{2.7}$$

$$|\xi_x| \le \exp\int_0^t |\bar v_\xi(\xi,s)|\, ds. \tag{2.8}$$

Proof. Expressing (2.2) in the form

$$x = \xi + \int_0^t \bar v(\xi,s)\, ds, \tag{2.9}$$

we see that (2.7) is obvious. To show (2.8) we obtain from (2.9) the relation

$$\xi_x = I + \int_0^t \bar v_\xi(\xi,s)\xi_x\, ds,$$

so

$$\frac{d}{dt}\xi_x = \bar v_\xi \xi_x. \tag{2.10}$$

From (2.10) it follows that

$$\frac{1}{2}\frac{d}{dt}\ln|\xi_x|^2 \le |\bar v_\xi|, \tag{2.11}$$

so (2.8) follows. ∎

3. Differential inequality for velocity

To prove the differential inequality we need a sufficiently regular local solution.

LEMMA 3.1. *Assume that* $X_1 = \|v\|^2_{\Gamma^2_0(\overset{1}{\Omega}_t)} + \|\overset{1}{H}\|^2_{\Gamma^2_0(\overset{1}{\Omega}_t)} < \infty$, $f \in \Gamma^1_0(\overset{1}{\Omega}_t)$ $v(0) \in L_2(\overset{1}{\Omega})$, $\big|\int_0^t \int_{\overset{1}{\Omega}_{t'}} f\,dx\,dt'\big| < \infty$. *Then*

$$\frac{d}{dt}\|v\|^2_{\Gamma^2_0(\overset{1}{\Omega}_t)} + \|v\|^2_{\Gamma^3_1(\overset{1}{\Omega}_t)} + \|p'\|^2_{\Gamma^2_1(\overset{1}{\Omega}_t)} \le c\Big[X_1^2(1 + X_1 + X_1^2) + X_1(1+X_1)\|v_{tt}\|^2_{H^1(\overset{1}{\Omega}_t)} + X_1^2\|v_t\|^2_{H^2(\overset{1}{\Omega}_t)} + X_1^2\|f\|^2_{L_2(\overset{1}{\Omega}_t)} + \|f\|^2_{\Gamma^2_0(\overset{1}{\Omega}_t)} + \Big|\int_0^t\int_{\overset{1}{\Omega}_{t'}} f(x,t')\,dx\,dt'\Big|^2 + \Big|\int_{\overset{1}{\Omega}} v(0)\,dx\Big|^2\Big], \tag{3.1}$$

where $p' = p - p_0$.

Proof. Multiplying $(1.1)_1$ by v, integrating over $\overset{1}{\Omega}_t$ and using boundary conditions (1.14) yields

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{L_2(\overset{1}{\Omega}_t)} + E_{\overset{1}{\Omega}_t}(v) = \int_{\overset{1}{\Omega}_t} f\cdot v\,dx. \tag{3.2}$$

In view of the Korn inequality (6.18) we have

$$\frac{d}{dt}\|v\|^2_{L_2(\overset{1}{\Omega}_t)} + \|v\|^2_{H^1(\overset{1}{\Omega}_t)} \le c\|f\|^2_{L_2(\overset{1}{\Omega}_t)} + c\Big|\int_0^t\int_{\overset{1}{\Omega}_{t'}} f(x,t')\,dx\,dt'\Big|^2 + c\Big|\int_{\overset{1}{\Omega}} v(0)\,dx\Big|^2. \tag{3.3}$$

Differentiating $(1.1)_1$ with respect to t, multiplying the result by v_t and integrating over Ω_t gives

$$\frac{1}{2}\frac{d}{dt}\|v_t\|^2_{L_2(\overset{1}{\Omega}_t)} - \int_{\overset{1}{\Omega}_t} \operatorname{div}[\mathbb{T}(v,p) + \mu\mathbb{T}(\overset{1}{H})]_t\cdot v_t\,dx = -\int_{\overset{1}{\Omega}_t} v_t\cdot\nabla v\cdot v_t\,dx + \int_{\overset{1}{\Omega}_t} f_t\cdot v_t\,dx. \tag{3.4}$$

Integrating by parts, the second term on the l.h.s. equals

$$-\int_{S_t} \bar n\cdot[\mathbb{T}(v,p') + \mathbb{T}(\overset{1}{H})]_t\cdot v_t dS_t + E_{\overset{1}{\Omega}_t}(v_t) + \mu\int_{\overset{1}{\Omega}_t}\mathbb{T}(\overset{1}{H})_t\cdot\nabla v_t\,dx \equiv I_1 + E_{\overset{1}{\Omega}_t}(v_t) + I_2,$$

where in view of the boundary conditions

$$I_1 = \int_{S_t} \bar n_t(\mathbb{T}(v,p') + \mu\mathbb{T}(\overset{1}{H}))\cdot v_t\,dS_t,$$

so

$$\begin{aligned} |I_1| &\le c\int_{S_t} |v|(|\nabla v| + |p'| + |\overset{1}{H}|^2)|v_t|\,dS_t \\ &\le \varepsilon\|v_t\|^2_{H^1(\overset{1}{\Omega}_t)} + c(1/\varepsilon)\|v\|^2_{H^1(\overset{1}{\Omega}_t)}(\|v\|^2_{H^2(\overset{1}{\Omega}_t)} + \|v_t\|^2_{H^1(\overset{1}{\Omega}_t)} + \|\overset{1}{H}\|^4_{H^1(\overset{1}{\Omega}_t)}) \\ &\quad + \varepsilon\|p'\|^2_{H^1(\overset{1}{\Omega}_t)} \end{aligned}$$

and

$$|I_2|\le\varepsilon\|v_t\|^2_{H^1(\overset{1}{\Omega}_t)}+c(1/\varepsilon)\|\overset{1}{H}\|^2_{H^1(\overset{1}{\Omega}_t)}\|\overset{1}{H}_t\|^2_{H^1(\overset{1}{\Omega}_t)}.$$

Finally, the r.h.s. of (3.4) is bounded by

$$\varepsilon\|v_t\|^2_{H^1(\overset{1}{\Omega}_t)}+c(1/\varepsilon)(\|v_t\|^2_{L_2(\overset{1}{\Omega}_t)}\|\nabla v\|^2_{H^1(\overset{1}{\Omega}_t)}+\|f_t\|^2_{L_2(\overset{1}{\Omega}_t)}).$$

Employing the estimates in (3.4) yields

$$\begin{aligned}\frac12\frac{d}{dt}\|v_t\|^2_{L_2(\overset{1}{\Omega}_t)}+E_{\overset{1}{\Omega}_t}(v_t)&\le\varepsilon(\|v_t\|^2_{H^1(\overset{1}{\Omega}_t)}+\|p'\|^2_{H^1(\overset{1}{\Omega}_t)})\\&\quad+c(1/\varepsilon)(\|v\|^2_{H^2(\overset{1}{\Omega}_t)}\|v\|^2_{\Gamma_1^2(\overset{1}{\Omega}_t)}+\|v\|^2_{H^1(\overset{1}{\Omega}_t)}\|\overset{1}{H}\|^4_{H^1(\overset{1}{\Omega}_t)}\\&\quad+\|\overset{1}{H}\|^2_{H^1(\overset{1}{\Omega}_t)}\|\overset{1}{H}_t\|^2_{H^1(\overset{1}{\Omega}_t)}+\|f_t\|^2_{L_2(\overset{1}{\Omega}_t)}).\end{aligned}\tag{3.5}$$

In view of the Korn inequality (6.30) and ε sufficiently small we derive

$$\begin{aligned}&\frac{d}{dt}\|v_t\|^2_{L_2(\overset{1}{\Omega}_t)}+\|v_t\|^2_{H^1(\overset{1}{\Omega}_t)}\\&\le\varepsilon\|p'\|^2_{H^1(\overset{1}{\Omega}_t)}+c(1/\varepsilon)\big(\|v\|^2_{H^2(\overset{1}{\Omega}_t)}\|v\|^2_{\Gamma_1^2(\overset{1}{\Omega}_t)}+\|v\|^2_{H^1(\overset{1}{\Omega}_t)}\|\overset{1}{H}\|^4_{H^1(\overset{1}{\Omega}_t)}\\&\quad+\|\overset{1}{H}\|^2_{H^1(\overset{1}{\Omega}_t)}\|\overset{1}{H}_t\|^2_{H^1(\overset{1}{\Omega}_t)}+\|f_t\|^2_{L_2(\overset{1}{\Omega}_t)}+\|f\|^2_{L_2(\overset{1}{\Omega}_t)}\big).\end{aligned}\tag{3.6}$$

Differentiating $(1.1)_1$ twice with respect to t, multiplying the result by v_{tt} and integrating over Ω_t yields

$$\begin{aligned}&\frac12\frac{d}{dt}\int_{\overset{1}{\Omega}_t}v_{tt}^2\,dx+\int_{\overset{1}{\Omega}_t}(2v_t\cdot\nabla v_t\cdot v_{tt}+v_{tt}\cdot\nabla v\cdot v_{tt})\,dx\\&\quad-\int_{\overset{1}{\Omega}_t}\operatorname{div}(\mathbb{T}(v,p)+\mu\mathbb{T}(\overset{1}{H}))_{tt}\cdot v_{tt}\,dx=\int_{\overset{1}{\Omega}_t}f_{tt}\cdot v_{tt}\,dx.\end{aligned}\tag{3.7}$$

We bound the second term on the l.h.s. by

$$\varepsilon\|v_{tt}\|^2_{H^1(\overset{1}{\Omega}_t)}+c(1/\varepsilon)(\|v_{tt}\|^2_{L_2(\overset{1}{\Omega}_t)}\|v\|^2_{H^2(\overset{1}{\Omega}_t)}+\|v_t\|^4_{H^1(\overset{1}{\Omega}_t)}).$$

Integration by parts in the last term on the l.h.s. of (3.7) implies

$$\begin{aligned}-\int_{\overset{1}{\Omega}_t}\operatorname{div}(\mathbb{T}(v,p')+\mathbb{T}(\overset{1}{H}))_{tt}\cdot v_{tt}\,dx&=-\int_{S_t}\bar n\cdot(\mathbb{T}(v,p')+\mathbb{T}(\overset{1}{H}))_{tt}\cdot v_{tt}\,dS_t\\&\quad+\int_{\overset{1}{\Omega}_t}\mathbb{T}(v,p)_{tt}\cdot\nabla v_{tt}\,dx+\mu\int_{\overset{1}{\Omega}_t}\mathbb{T}(\overset{1}{H})_{tt}\cdot\nabla v_{tt}\,dx\equiv I_1+I_2+I_3.\end{aligned}$$

Now we estimate the integrals I_i, $i=1,2,3$. To employ the boundary conditions we have

$$\begin{aligned}I_1=&-\int_{S_t}(\bar n\cdot(\mathbb{T}(v,p')+\mathbb{T}(\overset{1}{H})))_{tt}\cdot v_{tt}\,dS_t+\int_{S_t}\bar n_{tt}\cdot(\mathbb{T}(v,p')+\mu\mathbb{T}(\overset{1}{H}))\cdot v_{tt}\,dS_t\\&+2\int_{S_t}\bar n_t\cdot(\mathbb{T}(v,p')+\mu\mathbb{T}(\overset{1}{H}))_t\cdot v_{tt}\,dS_t\equiv J_1+J_2+J_3.\end{aligned}$$

In view of the boundary conditions (1.14), J_1 vanishes. Next,

$$
\begin{aligned}
|J_2| &\le c\int\limits_{S_t}(|v_t|+|v|^2)(|\nabla v|+|p'|+|\overset{1}{H}|^2)|v_{tt}|\,dS_t\\
&\le \varepsilon\|v_{tt}\|^2_{H^1(\overset{1}{\Omega}_t)}+c(1/\varepsilon)(\|v_t\|^2_{H^1(\overset{1}{\Omega}_t)}+\|v\|^4_{H^1(\overset{1}{\Omega}_t)})(\|v\|^2_{H^2(\overset{1}{\Omega}_t)}+\|\overset{1}{H}\|^2_{H^1(\overset{1}{\Omega}_t)}\|\overset{1}{H}\|^2_{H^2(\overset{1}{\Omega}_t)})\\
&\quad+\varepsilon\|p'\|^2_{H^1(\overset{1}{\Omega}_t)}+c(1/\varepsilon)(\|v_t\|^2_{H^1(\overset{1}{\Omega}_t)}+\|v\|^4_{H^1(\overset{1}{\Omega}_t)})\|v_{tt}\|^2_{H^1(\overset{1}{\Omega}_t)}.
\end{aligned}
$$

Finally

$$
\begin{aligned}
|J_3| &\le c\int\limits_{S_t}|v_t|(|\nabla v_t|+|p_t|+|\overset{1}{H}|\,|\overset{1}{H}_t|)|v_{tt}|\,dS_t\\
&\le \varepsilon\|v_{tt}\|^2_{H^1(\overset{1}{\Omega}_t)}+c(1/\varepsilon)(\|v_t\|^2_{H^1(\overset{1}{\Omega}_t)}\|v_t\|^2_{H^2(\overset{1}{\Omega}_t)}+\|v_t\|^2_{H^1(\overset{1}{\Omega}_t)}\|\overset{1}{H}\|^2_{H^1(\overset{1}{\Omega}_t)}\|\overset{1}{H}_t\|^2_{H^1(\overset{1}{\Omega}_t)})\\
&\quad+\varepsilon\|p_t\|^2_{H^1(\overset{1}{\Omega}_t)}+c(1/\varepsilon)\|v_t\|^2_{H^1(\overset{1}{\Omega}_t)}\|v_{tt}\|^2_{H^1(\overset{1}{\Omega}_t)}.
\end{aligned}
$$

The term I_2 equals $I_2=E_{\Omega_t}(v_{tt})$ and

$$
\begin{aligned}
|I_3| &\le c\int\limits_{\overset{1}{\Omega}_t}(|\overset{1}{H}_{tt}|\,|\overset{1}{H}|+|\overset{1}{H}_t|^2)|\nabla v_{tt}|\,dx\\
&\le \varepsilon\|\nabla v_{tt}\|^2_{L_2(\overset{1}{\Omega}_t)}+c(1/\varepsilon)(\|\overset{1}{H}\|^2_{H^2(\overset{1}{\Omega}_t)}\|\overset{1}{H}_{tt}\|^2_{L_2(\overset{1}{\Omega}_t)}+\|\overset{1}{H}_t\|^4_{H^1(\overset{1}{\Omega}_t)}).
\end{aligned}
$$

Using the above considerations and estimates in (3.7) yields

$$
\begin{aligned}
&\frac12\frac{d}{dt}\|v_{tt}\|^2_{L_2(\overset{1}{\Omega}_t)}+E_{\overset{1}{\Omega}_t}(v_{tt})\le\varepsilon(\|v_{tt}\|^2_{H^1(\overset{1}{\Omega}_t)}+\|p'\|^2_{H^1(\overset{1}{\Omega}_t)}\\
&\quad+\|p_t\|^2_{H^1(\overset{1}{\Omega}_t)})+c(1/\varepsilon)\big[\|v_{tt}\|^2_{L_2(\overset{1}{\Omega}_t)}\|v\|^2_{H^2(\overset{1}{\Omega}_t)}+\|v_t\|^4_{H^1(\overset{1}{\Omega}_t)}\\
&\quad+(\|v_t\|^2_{H^1(\overset{1}{\Omega}_t)}+\|v\|^4_{H^1(\overset{1}{\Omega}_t)})(\|v\|^2_{H^2(\overset{1}{\Omega}_t)}+\|\overset{1}{H}\|^2_{H^1(\overset{1}{\Omega}_t)}\|\overset{1}{H}\|^2_{H^2(\overset{1}{\Omega}_t)})\\
&\quad+(\|v_t\|^2_{H^1(\overset{1}{\Omega}_t)}+\|v\|^4_{H^1(\overset{1}{\Omega}_t)})\|v_{tt}\|^2_{H^1(\overset{1}{\Omega}_t)}+\|v_t\|^2_{H^1(\overset{1}{\Omega}_t)}\|v_t\|^2_{H^2(\overset{1}{\Omega}_t)}\\
&\quad+\|v_t\|^2_{H^1(\overset{1}{\Omega}_t)}\|\overset{1}{H}\|^2_{H^1(\overset{1}{\Omega}_t)}\|\overset{1}{H}_t\|^2_{H^1(\overset{1}{\Omega}_t)}+\|v_t\|^2_{H^1(\overset{1}{\Omega}_t)}\|v_{tt}\|^2_{H^1(\overset{1}{\Omega}_t)}\\
&\quad+\|\overset{1}{H}\|^2_{H^2(\overset{1}{\Omega}_t)}\|\overset{1}{H}_{tt}\|^2_{L_2(\overset{1}{\Omega}_t)}+\|\overset{1}{H}_t\|^4_{H^1(\overset{1}{\Omega}_t)}+\|f_{tt}\|^2_{L_2(\overset{1}{\Omega}_t)}\big].
\end{aligned}
\tag{3.8}
$$

Simplifying, (3.8) takes the form

$$
\begin{aligned}
&\frac12\frac{d}{dt}\|v_{tt}\|^2_{L_2(\overset{1}{\Omega}_t)}+E_{\overset{1}{\Omega}_t}(v_{tt})\le\varepsilon(\|v_{tt}\|^2_{H^1(\overset{1}{\Omega}_t)}+\|p'\|^2_{H^1(\overset{1}{\Omega}_t)}\\
&\quad+\|p_t\|^2_{H^1(\overset{1}{\Omega}_t)})+c(1/\varepsilon)[\|v\|^4_{\Gamma_0^2(\overset{1}{\Omega}_t)}(1+\|v\|^2_{\Gamma_0^2(\overset{1}{\Omega}_t)})\\
&\quad+\|v\|^2_{\Gamma_0^2(\overset{1}{\Omega}_t)}(1+\|v\|^2_{\Gamma_0^2(\overset{1}{\Omega}_t)})\|\overset{1}{H}\|^4_{\Gamma_0^2(\overset{1}{\Omega}_t)}+\|\overset{1}{H}\|^4_{\Gamma_0^2(\overset{1}{\Omega}_t)}\\
&\quad+\|v\|^2_{\Gamma_0^2(\overset{1}{\Omega}_t)}(1+\|v\|^2_{\Gamma_0^2(\overset{1}{\Omega}_t)})\|v_{tt}\|^2_{H^1(\overset{1}{\Omega}_t)}+\|v\|^2_{\Gamma_0^2(\overset{1}{\Omega}_t)}\|v_t\|^2_{H^2(\overset{1}{\Omega}_t)}+\|f_{tt}\|^2_{L_2(\overset{1}{\Omega}_t)}].
\end{aligned}
\tag{3.9}
$$

We recall the Korn inequality (6.44):

$$\|v_{tt}\|^2_{H^1(\overset{1}{\Omega}_t)} \le cE_{\overset{1}{\Omega}_t}(v_{tt}) + c(\|v\|^2_{H^1(\overset{1}{\Omega}_t)}\|v_t\|^2_{H^1(\overset{1}{\Omega}_t)} + \|\overset{1}{H}\|^4_{\Gamma_0^2(\overset{1}{\Omega}_t)} + \|f_t\|^2_{L_2(\overset{1}{\Omega}_t)}). \tag{3.10}$$

Using (3.10) in (3.9) we obtain, for sufficiently small ε,

$$\begin{aligned}\frac{d}{dt}\|v_{tt}\|^2_{L_2(\overset{1}{\Omega}_t)} + \|v_{tt}\|^2_{H^1(\overset{1}{\Omega}_t)} &\le \varepsilon(\|p'\|^2_{H^1(\overset{1}{\Omega}_t)} + \|p_t\|^2_{H^1(\overset{1}{\Omega}_t)})\\ &\quad + c(1/\varepsilon)\big[\|v\|^4_{\Gamma_0^2(\overset{1}{\Omega}_t)}(1+\|v\|^2_{\Gamma_0^2(\overset{1}{\Omega}_t)})\\ &\quad + \|v\|^2_{\Gamma_0^2(\overset{1}{\Omega}_t)}(1+\|v\|^2_{\Gamma_0^2(\overset{1}{\Omega}_t)})\|\overset{1}{H}\|^4_{\Gamma_0^2(\overset{1}{\Omega}_t)} + \|\overset{1}{H}\|^4_{\Gamma_0^2(\overset{1}{\Omega}_t)}\\ &\quad + \|v\|^2_{\Gamma_0^2(\overset{1}{\Omega}_t)}(1+\|v\|^2_{\Gamma_0^2(\overset{1}{\Omega}_t)})\|v_{tt}\|^2_{H^1(\overset{1}{\Omega}_t)} + \|v\|_{\Gamma_0^2(\overset{1}{\Omega}_t)}\|v_t\|^2_{H^2(\overset{1}{\Omega}_t)}\\ &\quad + \|f_t\|^2_{L_2(\overset{1}{\Omega}_t)} + \|f_{tt}\|^2_{L_2(\overset{1}{\Omega}_t)}\big].\end{aligned} \tag{3.11}$$

To simplify the above expressions we introduce the notation

$$X_1 = \|v\|^2_{\Gamma_0^2(\overset{1}{\Omega}_t)} + \|\overset{1}{H}\|^2_{\Gamma_0^2(\overset{1}{\Omega}_t)}. \tag{3.12}$$

Then (3.6) takes the form

$$\begin{aligned}&\frac{d}{dt}\|v_t\|^2_{L_2(\overset{1}{\Omega}_t)} + \|v_t\|^2_{H^1(\overset{1}{\Omega}_t)}\\ &\qquad \le \varepsilon\|p'\|^2_{H^1(\overset{1}{\Omega}_t)} + c(1/\varepsilon)(X_1^2 + X_1^3 + \|f_t\|^2_{L_2(\overset{1}{\Omega}_t)} + \|f\|^2_{L_2(\overset{1}{\Omega}_t)})\end{aligned} \tag{3.13}$$

and (3.11) can be expressed as

$$\begin{aligned}\frac{d}{dt}\|v_{tt}\|^2_{L_2(\overset{1}{\Omega}_t)} + \|v_{tt}\|^2_{H^1(\overset{1}{\Omega}_t)} &\le \varepsilon(\|p'\|^2_{H^1(\overset{1}{\Omega}_t)} + \|p_t\|^2_{H^1(\overset{1}{\Omega}_t)})\\ &\quad + c(1/\varepsilon)\big[X_1^2(1+X_1+X_1^2) + X_1(1+X_1)\|v_{tt}\|^2_{H^1(\overset{1}{\Omega}_t)}\\ &\quad + X_1^2\|v_t\|^2_{H^2(\overset{1}{\Omega}_t)} + \|f_t\|^2_{L_2(\overset{1}{\Omega}_t)} + \|f_{tt}\|^2_{L_2(\overset{1}{\Omega}_t)}\big].\end{aligned} \tag{3.14}$$

Consider the following stationary Stokes problem:

$$\begin{aligned}&-\operatorname{div}\mathbb{T}(v,p') = -v_t - v\cdot\nabla v + \mu\operatorname{div}\mathbb{T}(\overset{1}{H}) + f && \text{in } \overset{1}{\Omega}_t,\\ &\operatorname{div} v = 0 && \text{in } \overset{1}{\Omega}_t,\\ &\bar n\cdot\mathbb{T}(v,p') = -\mu\bar n\cdot\mathbb{T}(\overset{1}{H}) && \text{on } S_t,\end{aligned} \tag{3.15}$$

where t is fixed. For solutions to problem (3.15) we have

$$\begin{aligned}&\|v\|_{H^2(\overset{1}{\Omega}_t)} + \|p'\|_{H^1(\overset{1}{\Omega}_t)}\\ &\quad \le c(\|v_t\|_{L_2(\overset{1}{\Omega}_t)} + \|v\|_{H^1(\overset{1}{\Omega}_t)}\|v\|_{H^2(\overset{1}{\Omega}_t)} + \|\overset{1}{H}\|_{H^1(\overset{1}{\Omega}_t)}\|\overset{1}{H}\|_{H^2(\overset{1}{\Omega}_t)} + \|f\|_{L_2(\overset{1}{\Omega}_t)})\\ &\quad \le c\|v_t\|_{L_2(\overset{1}{\Omega}_t)} + c(X_1 + \|f\|_{L_2(\overset{1}{\Omega}_t)}),\end{aligned} \tag{3.16}$$

where we have used that

$$\|\overset{1}{H}\overset{1}{H}\|_{H^{1/2}(S_t)} \le c\|\overset{1}{H}\overset{1}{H}\|_{H^1(\overset{1}{\Omega}_t)} \le c\|\overset{1}{H}\|_{H^1(\overset{1}{\Omega}_t)}\|\overset{1}{H}\|_{H^2(\overset{1}{\Omega}_t)}.$$

Using (3.16) in (3.13) and assuming that ε is sufficiently small we get

$$\frac{d}{dt}\|v_t\|^2_{L_2(\overset{1}{\Omega}_t)}+\|v_t\|^2_{H^1(\overset{1}{\Omega}_t)}\le c(X_1^2+X_1^3+\|f\|^2_{L_2(\overset{1}{\Omega}_t)}+\|f_t\|^2_{L_2(\overset{1}{\Omega}_t)}). \tag{3.17}$$

Adding (3.16) and (3.17) yields

$$\begin{aligned}\frac{d}{dt}\|v_t\|^2_{L_2(\overset{1}{\Omega}_t)}&+\|v\|^2_{H^2(\overset{1}{\Omega}_t)}+\|v_t\|^2_{H^1(\overset{1}{\Omega}_t)}+\|p'\|^2_{H^1(\overset{1}{\Omega}_t)}\\&\le c(X_1^2+X_1^3+\|f\|^2_{L_2(\overset{1}{\Omega}_t)}+\|f_t\|^2_{L_2(\overset{1}{\Omega}_t)}).\end{aligned} \tag{3.18}$$

Using the formula

$$\begin{aligned}\frac{d}{dt}\int\limits_{\overset{1}{\Omega}_t}v_x^2\,dx&=\int\limits_{\overset{1}{\Omega}_t}\partial_tv_x^2\,dx+\int\limits_{\overset{1}{\Omega}_t}v\cdot\nabla v_x^2\,dx\\&\le\varepsilon(\|v_{xt}\|^2_{L_2(\overset{1}{\Omega}_t)}+\|v_{xx}\|^2_{L_2(\overset{1}{\Omega}_t)})+c(1/\varepsilon)(\|v_x\|^2_{L_2(\overset{1}{\Omega}_t)}+\|v\|^2_{H^1(\overset{1}{\Omega}_t)}\|v\|^2_{H^2(\overset{1}{\Omega}_t)}),\end{aligned} \tag{3.19}$$

in (3.18) implies

$$\begin{aligned}\frac{d}{dt}(\|v_x\|^2_{L_2(\overset{1}{\Omega}_t)}+\|v_t\|^2_{L_2(\overset{1}{\Omega}_t)})&+\|v\|^2_{H^2(\overset{1}{\Omega}_t)}+\|v_t\|^2_{H^1(\overset{1}{\Omega}_t)}+\|p'\|^2_{H^1(\overset{1}{\Omega}_t)}\\&\le c\|v_x\|^2_{L_2(\overset{1}{\Omega}_t)}+c(X_1^2+X_2^3+\|f\|^2_{L_2(\overset{1}{\Omega}_t)}+\|f_t\|^2_{L_2(\overset{1}{\Omega}_t)}).\end{aligned} \tag{3.20}$$

Adding (3.3) and (3.20) gives

$$\begin{aligned}\frac{d}{dt}(\|v\|^2_{L_2(\overset{1}{\Omega}_t)}&+\|v_x\|^2_{L_2(\overset{1}{\Omega}_t)}+\|v_t\|^2_{L_2(\overset{1}{\Omega}_t)})+\|v\|^2_{H^2(\overset{1}{\Omega}_t)}+\|v_t\|^2_{H^1(\overset{1}{\Omega}_t)}+\|p'\|^2_{H^1(\overset{1}{\Omega}_t)}\\&\le c\Big(X_1^2+X_1^3+\|f\|^2_{L_2(\overset{1}{\Omega}_t)}+\|f_t\|^2_{L_2(\overset{1}{\Omega}_t)}+\Big|\int\limits_0^t\int\limits_{\overset{1}{\Omega}_{t'}}f\,dx\,dt'\Big|^2+\Big|\int\limits_{\overset{1}{\Omega}}v(0)\,dx\Big|^2\Big).\end{aligned} \tag{3.21}$$

Using (3.16) in (3.14) yields

$$\begin{aligned}\frac{d}{dt}\|v_{tt}\|^2_{L_2(\overset{1}{\Omega}_t)}+\|v_{tt}\|^2_{H^1(\overset{1}{\Omega}_t)}&\le\varepsilon(\|p'\|^2_{H^1(\overset{1}{\Omega}_t)}+\|p_t\|^2_{H^1(\overset{1}{\Omega}_t)})\\&+c(1/\varepsilon)\big[X_1^2(1+X_1+X_1^2)+X_1(1+X_1)\|v_{tt}\|^2_{H^1(\overset{1}{\Omega}_t)}\\&+X_1^2\|v_t\|^2_{H^2(\overset{1}{\Omega}_t)}+X_1^2\|f\|^2_{L_2(\overset{1}{\Omega}_t)}+\|f_t\|^2_{L_2(\overset{1}{\Omega}_t)}+\|f_{tt}\|^2_{L_2(\overset{1}{\Omega}_t)}\big].\end{aligned} \tag{3.22}$$

Differentiating (3.15) with respect to time gives

$$\begin{aligned}&-\operatorname{div}\mathbb{T}(v_t,p_t)=-v_{tt}-v_t\cdot\nabla v-v\cdot\nabla v_t+\mu\operatorname{div}\mathbb{T}(\overset{1}{H})_t+f_t,\\&\operatorname{div}v_t=0,\\&\bar n\cdot\mathbb{T}(v_t,p_t)=-\bar n_t\cdot\mathbb{T}(v,p')-\bar n_t\mathbb{T}(\overset{1}{H})-\mu\bar n\cdot\mathbb{T}(\overset{1}{H})_t.\end{aligned} \tag{3.23}$$

For solutions to problem (3.23) the following estimate is valid:

$$\begin{aligned}\|v_t\|_{H^2(\overset{1}{\Omega}_t)}+\|p_t\|_{H^1(\overset{1}{\Omega}_t)}&\le c[\|v_{tt}\|_{L_2(\overset{1}{\Omega}_t)}+\|v_t\|_{H^1(\overset{1}{\Omega}_t)}\|v\|_{H^2(\overset{1}{\Omega}_t)}\\&+\|\overset{1}{H}_t\|_{H^1(\overset{1}{\Omega}_t)}\|\overset{1}{H}\|_{H^2(\overset{1}{\Omega}_t)}+\|f_t\|_{L_2(\overset{1}{\Omega}_t)}+\|v\cdot\nabla v\|_{H^{1/2}(S_t)}\\&+\|vp'\|_{H^{1/2}(S_t)}+\|v\overset{1}{H}\overset{1}{H}\|_{H^{1/2}(S_t)}+\|\overset{1}{H}\overset{1}{H}_t\|_{H^{1/2}(S_t)}].\end{aligned} \tag{3.24}$$

Continuing,

$$
\begin{aligned}
(3.25)\qquad & \|v_t\|_{H^2(\overset{1}{\Omega}_t)} + \|p_t\|_{H^1(\overset{1}{\Omega}_t)} \\
&\le c[\|v_{tt}\|_{L_2(\overset{1}{\Omega}_t)} + \|v_t\|_{H^1(\overset{1}{\Omega}_t)}\|v\|_{H^2(\overset{1}{\Omega}_t)} + \|\overset{1}{H}_t\|_{H^1(\overset{1}{\Omega}_t)}\|\overset{1}{H}\|_{H^2(\overset{1}{\Omega}_t)} \\
&\quad + \|f_t\|_{L_2(\overset{1}{\Omega}_t)} + \|v\|^2_{H^2(\overset{1}{\Omega}_t)} + \|v\|_{H^2(\overset{1}{\Omega}_t)}\|p'\|_{H^1(\overset{1}{\Omega}_t)} + \|v\|_{H^2(\overset{1}{\Omega}_t)}\|\overset{1}{H}\|^2_{H^2(\overset{1}{\Omega}_t)} \\
&\quad + \|\overset{1}{H}\|_{H^2(\overset{1}{\Omega}_t)}\|\overset{1}{H}_t\|_{H^1(\overset{1}{\Omega}_t)}] \\
&\le c[\|v_{tt}\|_{L_2(\overset{1}{\Omega}_t)} + X_1 + X_1^{3/2} + \|v\|_{H^2(\overset{1}{\Omega}_t)}\|p'\|_{H^1(\overset{1}{\Omega}_t)} + \|f_t\|_{L_2(\overset{1}{\Omega}_t)}].
\end{aligned}
$$

Using (3.16) in (3.25) yields

$$
\begin{aligned}
(3.26)\qquad & \|v_t\|_{H^2(\overset{1}{\Omega}_t)} + \|p_t\|_{H^1(\overset{1}{\Omega}_t)} \\
&\le c[\|v_{tt}\|_{L_2(\overset{1}{\Omega}_t)} + X_1 + X_1^{3/2} + \|v\|_{H^2(\overset{1}{\Omega}_t)}\|f\|_{L_2(\overset{1}{\Omega}_t)} + \|f_t\|_{L_2(\overset{1}{\Omega}_t)}].
\end{aligned}
$$

Adding (3.16), (3.22) and (3.26) yields

$$
\begin{aligned}
(3.27)\qquad & \frac{d}{dt}\|v_{tt}\|^2_{L_2(\overset{1}{\Omega}_t)} + \|v_{tt}\|^2_{H^1(\overset{1}{\Omega}_t)} + \|v_t\|^2_{H^2(\overset{1}{\Omega}_t)} + \|v\|^2_{H^2(\overset{1}{\Omega}_t)} + \|p_t\|^2_{H^1(\overset{1}{\Omega}_t)} \\
&\le c[\|v_t\|^2_{L_2(\overset{1}{\Omega}_t)} + X_1^2(1 + X_1 + X_1^2) + X_1(1 + X_1)\|v_{tt}\|^2_{H^1(\overset{1}{\Omega}_t)} \\
&\quad + X_1^2\|v_t\|^2_{H^2(\overset{1}{\Omega}_t)} + X_1\|f\|^2_{L_2(\overset{1}{\Omega}_t)} + \|f\|^2_{L_2(\overset{1}{\Omega}_t)} + \|f_t\|^2_{L_2(\overset{1}{\Omega}_t)}].
\end{aligned}
$$

Hence, (3.21) and (3.27) yield

$$
\begin{aligned}
(3.28)\qquad & \frac{d}{dt}(\|v\|^2_{L_2(\overset{1}{\Omega}_t)} + \|v_x\|^2_{L_2(\overset{1}{\Omega}_t)} + \|v_t\|^2_{L_2(\overset{1}{\Omega}_t)} + \|v_{tt}\|^2_{L_2(\overset{1}{\Omega}_t)}) \\
&\quad + \|v\|^2_{H^2(\overset{1}{\Omega}_t)} + \|v_t\|^2_{H^2(\overset{1}{\Omega}_t)} + \|v_{tt}\|^2_{H^1(\overset{1}{\Omega}_t)} + \|p'\|^2_{H^1(\overset{1}{\Omega}_t)} + \|p_t\|^2_{H^1(\overset{1}{\Omega}_t)} \\
&\le c\Big[X_1^2(1 + X_1 + X_1^2) + X_1(1 + X_1)\|v_{tt}\|^2_{H^1(\overset{1}{\Omega}_t)} \\
&\quad + X_1^2\|v_t\|^2_{H^2(\overset{1}{\Omega}_t)} + X_1^2\|f\|^2_{L_2(\overset{1}{\Omega}_t)} + \|f\|^2_{L_2(\overset{1}{\Omega}_t)} + \|f_t\|^2_{L_2(\overset{1}{\Omega}_t)} \\
&\quad + \Big|\int_0^t\int_{\overset{1}{\Omega}_{t'}} f\,dx\,dt'\Big|^2 + \Big|\int_{\overset{1}{\Omega}} v(0)\,dx\Big|\Big].
\end{aligned}
$$

Next, we need

$$
\begin{aligned}
(3.29)\qquad & \frac{d}{dt}\int_{\overset{1}{\Omega}_t} v_{xt}^2\,dx = \int_{\overset{1}{\Omega}_t}[(v_{xt}^2)_t + v\cdot\nabla(v_{xt}^2)]\,dx = 2\int_{\overset{1}{\Omega}_t}(v_{xt}v_{xtt} + vv_{xxt}v_{xt})\,dx \\
&\le \varepsilon(\|v_{xtt}\|^2_{L_2(\overset{1}{\Omega}_t)} + \|v_{xxt}\|^2_{L_2(\overset{1}{\Omega}_t)}) + c(1/\varepsilon)(\|v_{xt}\|^2_{L_2(\overset{1}{\Omega}_t)} + \|v\|^2_{H^2(\overset{1}{\Omega}_t)}\|v_{xt}\|^2_{L_2(\overset{1}{\Omega}_t)}).
\end{aligned}
$$

Adding (3.17) and (3.29) yields

$$
\begin{aligned}
(3.30)\qquad & \frac{d}{dt}\|v_t\|^2_{H^1(\overset{1}{\Omega}_t)} + \|v_t\|^2_{H^1(\overset{1}{\Omega}_t)} \le \varepsilon(\|v_{xtt}\|^2_{L_2(\overset{1}{\Omega}_t)} + \|v_{xxt}\|^2_{L_2(\overset{1}{\Omega}_t)}) \\
&\quad + c(1/\varepsilon)(X_1^2 + X_1^3 + \|f\|^2_{L_2(\overset{1}{\Omega}_t)} + \|f_t\|^2_{L_2(\overset{1}{\Omega}_t)}).
\end{aligned}
$$

Next, we examine

$$
\begin{aligned}
(3.31)\qquad \frac{d}{dt}\int_{\overset{1}{\Omega}_t} v_{xx}^2\,dx &= \int_{\overset{1}{\Omega}_t}[(v_{xx}^2)_t + v\cdot\nabla(v_{xx}^2)]\,dx = 2\int_{\overset{1}{\Omega}_t}[v_{xx}v_{xxt} + vv_{xxx}v_{xx}]\,dx\\
&\le \varepsilon(\|v_{xxt}\|^2_{L_2(\overset{1}{\Omega}_t)} + \|v_{xxx}\|^2_{L_2(\overset{1}{\Omega}_t)}) + c(1/\varepsilon)(\|v_{xx}\|^2_{L_2(\overset{1}{\Omega}_t)} + \|v\|^2_{H^2(\overset{1}{\Omega}_t)}\|v_{xx}\|^2_{L_2(\overset{1}{\Omega}_t)}).
\end{aligned}
$$

Adding (3.18) and (3.31) gives

$$
\begin{aligned}
(3.32)\qquad &\frac{d}{dt}(\|v_t\|^2_{L_2(\overset{1}{\Omega}_t)} + \|v_{xx}\|^2_{L_2(\overset{1}{\Omega}_t)}) + \|v\|^2_{H^2(\overset{1}{\Omega}_t)} + \|v_t\|^2_{H^1(\overset{1}{\Omega}_t)} + \|p'\|^2_{H^1(\overset{1}{\Omega}_t)}\\
&\quad\le \varepsilon(\|v_{xxt}\|^2_{L_2(\overset{1}{\Omega}_t)} + \|v_{xxx}\|^2_{L_2(\overset{1}{\Omega}_t)}) + c(1/\varepsilon)(X_1^2 + X_1^3 + \|f\|^2_{L_2(\overset{1}{\Omega}_t)} + \|f_t\|^2_{L_2(\overset{1}{\Omega}_t)}).
\end{aligned}
$$

For solutions to (3.15) we have

$$
\begin{aligned}
(3.33)\qquad &\|v\|^2_{H^3(\overset{1}{\Omega}_t)} + \|p'\|^2_{H^2(\overset{1}{\Omega}_t)}\\
&\quad\le c[\|v_t\|^2_{H^1(\overset{1}{\Omega}_t)} + \|v\|^4_{H^2(\overset{1}{\Omega}_t)} + \|\overset{1}{H}\|^4_{H^2(\overset{1}{\Omega}_t)} + \|f\|^2_{H^1(\overset{1}{\Omega}_t)} + \|\bar{n}\cdot\mathbb{T}(\overset{1}{H})\|^2_{H^{3/2}(\overset{1}{\Omega}_t)}]\\
&\quad\le c[\|v_t\|^2_{H^1(\overset{1}{\Omega}_t)} + X_1^2 + \|f\|^2_{H^1(\overset{1}{\Omega}_t)}].
\end{aligned}
$$

Adding (3.17) and (3.33) yields

$$
\begin{aligned}
(3.34)\qquad &\frac{d}{dt}\|v_t\|^2_{L_2(\overset{1}{\Omega}_t)} + \|v_t\|^2_{H^1(\overset{1}{\Omega}_t)} + \|v\|^2_{H^3(\overset{1}{\Omega}_t)} + \|p'\|^2_{H^2(\overset{1}{\Omega}_t)}\\
&\qquad\qquad\le c[X_1^2 + X_1^3 + \|f\|^2_{H^1(\overset{1}{\Omega}_t)} + \|f_t\|^2_{L_2(\overset{1}{\Omega}_t)}].
\end{aligned}
$$

From (3.28), (3.30), (3.32) and (3.34), for sufficiently small ε we derive

$$
\begin{aligned}
(3.35)\qquad &\frac{d}{dt}\|v\|^2_{\Gamma_0^2(\overset{1}{\Omega}_t)} + \|v\|^2_{\Gamma_1^3(\overset{1}{\Omega}_t)} + \|p'\|^2_{\Gamma_1^2(\overset{1}{\Omega}_t)}\\
&\le c\Big[X_1^2(1 + X_1 + X_1^2) + X_1(1+X_1)\|v_{tt}\|^2_{H^1(\overset{1}{\Omega}_t)} + X_1^2\|v_t\|^2_{H^2(\overset{1}{\Omega}_t)} + X_1^2\|f\|^2_{L_2(\overset{1}{\Omega}_t)}\\
&\quad + \|f\|^2_{H^1(\overset{1}{\Omega}_t)} + \|f_t\|^2_{L_2(\overset{1}{\Omega}_t)} + \|f_{tt}\|^2_{L_2(\overset{1}{\Omega}_t)} + \Big|\int_0^t\int_{\overset{1}{\Omega}_{t'}} f\,dx\,dt'\Big|^2 + \Big|\int_{\overset{1}{\Omega}} v(0)\,dx\Big|^2\Big].
\end{aligned}
$$

This implies (3.1) and concludes the proof. ∎

4. Differential inequality for the magnetic field

In this section we derive a differential inequality for magnetic fields which is similar to the inequality (3.1) from Section 3.

THEOREM 4.1. *Assume that the magnetic fields* $\overset{1}{H}$, $\overset{2}{H}$ *satisfy equations* (1.2), (1.9) *and the transmission conditions* (1.15). *Assume that the assumptions of Lemmas* 4.2–4.8 *are satisfied. Then*

$$\frac{d}{dt}\Big[\sum_{i=1}^2\|\overset{i}{H}\|^2_{\Gamma_0^2(\overset{1}{\Omega}_t)}+\sum_{j=1}^3X_{2j}\Big]+\sum_{i=1}^2\|\overset{i}{H}\|^2_{\Gamma_1^3(\overset{i}{\Omega}_t)} \tag{4.1}$$
$$\le c\|v\|^2_{\Gamma_1^2(\overset{1}{\Omega}_t)}(1+\|v\|^2_{\Gamma_1^2(\overset{1}{\Omega}_t)})\sum_{i=1}^2\|\overset{i}{H}\|^2_{\Gamma_1^2(\overset{i}{\Omega}_t)}$$
$$+c\|v\|^2_{\Gamma_0^2(\overset{1}{\Omega}_t)}\sum_{i=1}^2\|\overset{i}{H}\|^2_{\Gamma_1^3(\overset{i}{\Omega}_t)}+c(\|G\|^2_{\Gamma_0^1(\overset{2}{\Omega}_t)}+\|G_{tt}\|^2_{L_{6/5}(\overset{2}{\Omega}_t)}),$$

where

$$X_{21}+X_{22}+X_{23}=\sum_{i=1}^2\Big[\|\overset{i}{H}_x\|^2_{L_2(\overset{i}{\Omega}_t)}+\|\overset{i}{H}_{xx}\|^2_{L_2(\overset{i}{\Omega}_t)}+\|\overset{i}{H}_{xt}\|^2_{L_2(\overset{i}{\Omega}_t)} \tag{4.2}$$
$$+\sum_{k\in\mathcal{M}}(\|\overset{i,k}{\tilde H}_x\|^2_{L_2(\overset{i}{\Omega}_k)}+\|\overset{i,k}{\tilde H}_{xx}\|^2_{L_2(\overset{i}{\Omega}_k)}+\|\overset{i,k}{\tilde H}_{xt}\|^2_{L_2(\overset{i}{\Omega}_k)})$$
$$+\sum_{k\in\mathcal{N}}(\|\overset{i,k}{\tilde H}_\tau\|^2_{L_2(\overset{i}{\hat\Omega}_k)}+\|\overset{i,k}{\tilde H}_{\tau\tau}\|^2_{L_2(\overset{i}{\hat\Omega}_k)}+\|\overset{i,k}{\tilde H}_{t\tau}\|^2_{L_2(\overset{i}{\hat\Omega}_k)})\Big].$$

To describe X_{2j}, $j=1,2,3$, we need a partition of unity $\overset{i}{\zeta}_k\in C^\infty$, $0\le\overset{i}{\zeta}_k(x)\le1$, $\operatorname{supp}\overset{i}{\zeta}_k=\overset{i}{\Omega}_k\subset\overset{i}{\Omega}_t$, $i=1,2$, such that $\overset{i}{\Omega}_t=\bigcup_{k\in\mathcal{M}\cup\mathcal{N}}\overset{i}{\Omega}_k$, $i=1,2$. By $\overset{i}{\Omega}_k$, $k\in\mathcal{M}$, we denote an interior subdomain of $\overset{i}{\Omega}_t$ and by $\overset{i}{\hat\Omega}_k$, $k\in\mathcal{N}$, a boundary subdomain. In some neighborhoods of S_t and B we introduce a system of curvilinear coordinates (τ_1,τ_2,n), where τ_1, τ_2 are tangent coordinates and n is the normal.

Moreover, τ replaces either τ_1 or τ_2. Finally $\overset{i,k}{\tilde H}=\overset{i}{H}\overset{i}{\zeta}_k$.

Lemma 4.2. *Assume that* $\|v\|_{H^1(\overset{1}{\Omega}_t)}+\sum_{i=1}^2\|\overset{i}{H}\|_{H^1(\overset{i}{\Omega}_t)}<\infty$, $G\in L_{6/5}(\overset{2}{\Omega}_t)$ *and assume the transmission conditions hold on* S_t. *Then*

$$\frac{d}{dt}\Big(\sum_{i=1}^2\|\overset{i}{H}\|^2_{L_2(\overset{i}{\Omega}_t)}\Big)+\sum_{i=1}^2\|\overset{i}{H}\|^2_{H^1(\overset{i}{\Omega}_t)}\le c\|v\|^2_{H^1(\overset{1}{\Omega}_t)}\sum_{i=1}^2\|\overset{i}{H}\|^2_{H^1(\overset{i}{\Omega}_t)}+c\|G\|^2_{L_{6/5}(\overset{2}{\Omega}_t)}, \tag{4.3}$$

where we have used that v' *is an extension of* v *into* $\overset{2}{\Omega}_t$ *such that*

$$\|v'\|_{L_3(\overset{2}{\Omega}_t)}\le c\|v'\|_{H^1(\overset{2}{\Omega}_t)}\le c\|v\|_{H^1(\overset{1}{\Omega}_t)}. \tag{4.4}$$

Proof. We multiply (1.2) by $\overset{1}{H}$ and integrate the result over $\overset{1}{\Omega}_t$. Next we multiply $(1.9)_1$ by $\overset{2}{H}$ and integrate the result over $\overset{2}{\Omega}_t$. Adding, we get

$$\mu\int_{\overset{1}{\Omega}_t}[(\overset{1}{H}{}^2)_t+v\cdot\nabla\overset{1}{H}{}^2]\,dx+\mu\int_{\overset{2}{\Omega}_t}[(\overset{2}{H}{}^2)_t+v'\cdot\nabla\overset{2}{H}{}^2]\,dx \tag{4.5}$$
$$+\frac{1}{\sigma_1}\int_{\overset{1}{\Omega}_t}\operatorname{rot}\operatorname{rot}\overset{1}{H}\cdot\overset{1}{H}\,dx+\frac{1}{\sigma_2}\int_{\overset{2}{\Omega}_t}\operatorname{rot}\operatorname{rot}\overset{2}{H}\cdot\overset{2}{H}\,dx$$
$$=\int_{\overset{2}{\Omega}_t}G\cdot\overset{2}{H}\,dx+\mu\int_{\overset{1}{\Omega}_t}\overset{1}{H}\cdot\nabla v\cdot\overset{1}{H}\,dx+\mu\int_{\overset{2}{\Omega}_t}v'\cdot\nabla\overset{2}{H}\cdot\overset{2}{H}\,dx.$$

In view of (4.75) and the transmission conditions we get

$$\begin{aligned}(4.6)\qquad &\mu\frac{d}{dt}\int\limits_{\overset{1}{\Omega}_t}\overset{1}{H}{}^2\,dx+\mu\frac{d}{dt}\int\limits_{\overset{2}{\Omega}_t}\overset{2}{H}{}^2\,dx+\frac{1}{\sigma_1}\int\limits_{\overset{1}{\Omega}_t}|\operatorname{rot}\overset{1}{H}|^2\,dx+\frac{1}{\sigma_2}\int\limits_{\overset{2}{\Omega}_t}|\operatorname{rot}\overset{2}{H}|^2\,dx\\ &\le\varepsilon_1\|\overset{1}{H}\|^2_{L_6(\overset{1}{\Omega}_t)}+c(1/\varepsilon_1)\|\overset{1}{H}\|^2_{L_3(\overset{1}{\Omega}_t)}\|\nabla v\|^2_{L_2(\overset{1}{\Omega}_t)}\\ &\quad+\varepsilon_2\|\overset{2}{H}\|^2_{L_6(\overset{2}{\Omega}_t)}+c(1/\varepsilon_2)\Big(\|G\|^2_{L_{6/5}(\overset{2}{\Omega}_t)}\\ &\quad+\|v'\|^2_{L_3(\overset{2}{\Omega}_t)}\|\nabla\overset{2}{H}\|^2_{L_2(\overset{2}{\Omega}_t)}+\Big|\int\limits_{S_t}v\times\overset{1}{H}\cdot\bar n\times\overset{1}{H}\,dS_t\Big|\Big).\end{aligned}$$

The last term is bounded by

$$(4.7)\qquad \varepsilon_3\|\overset{1}{H}\|^2_{H^1(\overset{1}{\Omega}_t)}+c(1/\varepsilon_3)\|v\|^2_{H^1(\overset{1}{\Omega}_t)}\|\overset{1}{H}\|^2_{H^1(\overset{1}{\Omega}_t)}.$$

Employing (4.7) in (4.6) and assuming that ε_1, ε_2, ε_3 are sufficiently small implies

$$\begin{aligned}(4.8)\qquad &\frac{d}{dt}\int\limits_{\overset{1}{\Omega}_t}\overset{1}{H}{}^2\,dx+\frac{d}{dt}\int\limits_{\overset{2}{\Omega}_t}\overset{2}{H}{}^2\,dx+\|\overset{1}{H}\|^2_{H^1(\overset{1}{\Omega}_t)}+\|\overset{2}{H}\|^2_{H^1(\overset{2}{\Omega}_t)}\\ &\le c(\|\overset{1}{H}\|^2_{H^1(\overset{1}{\Omega}_t)}\|v\|^2_{H^1(\overset{1}{\Omega}_t)}+\|v'\|^2_{H^1(\overset{2}{\Omega}_t)}\|\overset{2}{H}\|^2_{H^1(\overset{2}{\Omega}_t)}+\|G\|^2_{L_{6/5}(\overset{2}{\Omega}_t)}).\end{aligned}$$

Hence (4.8) yields (4.3). ∎

LEMMA 4.3. *Assume that* $\|v\|_{\Gamma_1^2(\overset{1}{\Omega}_t)}+\sum_{i=1}^2\|\overset{i}{H}\|_{\Gamma_1^2(\overset{i}{\Omega}_t)}<\infty$, $G_t\in L_{6/5}(\overset{2}{\Omega}_t)$ *and assume the transmission conditions hold on* S_t. *Then*

$$(4.9)\qquad \frac{d}{dt}\sum_{i=1}^2\|\overset{i}{H}_t\|^2_{L_2(\overset{i}{\Omega}_t)}+\sum_{i=1}^2\|\overset{i}{H}_t\|^2_{H^1(\overset{i}{\Omega}_t)}\le c\Big[\sum_{i=1}^2\|\overset{i}{v}\|^2_{\Gamma_1^2(\overset{i}{\Omega}_t)}\|\overset{i}{H}\|^2_{\Gamma_1^2(\overset{i}{\Omega}_t)}+\|G_t\|^2_{L_{6/5}(\overset{2}{\Omega}_t)}\Big],$$

where $v'=v^2$.

Proof. Differentiating $(1.2)_1$ with respect to time, multiplying the result by $\overset{1}{H}_t$ and integrating over $\overset{1}{\Omega}_t$ we get

$$\begin{aligned}(4.10)\qquad &\mu\int\limits_{\overset{1}{\Omega}_t}[\partial_t(\overset{1}{H}{}_t^2)+v\cdot\nabla(\overset{1}{H}{}_t^2)]\,dx+\frac{1}{\sigma_1}\int\limits_{\overset{1}{\Omega}_t}\operatorname{rot}\operatorname{rot}\overset{1}{H}_t\cdot\overset{1}{H}_t\,dx\\ &=-\mu\int\limits_{\overset{1}{\Omega}_t}v_t\cdot\nabla\overset{1}{H}\cdot\overset{1}{H}_t\,dx+\mu\int\limits_{\overset{1}{\Omega}_t}(\overset{1}{H}_t\cdot\nabla v+\overset{1}{H}\cdot\nabla v_t)\cdot\overset{1}{H}_t\,dx.\end{aligned}$$

Using formula (4.75) implies

$$\begin{aligned}(4.11)\qquad &\mu\frac{d}{dt}\int\limits_{\overset{1}{\Omega}_t}\overset{1}{H}{}_t^2\,dx+\frac{1}{\sigma_1}\int\limits_{\overset{1}{\Omega}_t}\operatorname{rot}\operatorname{rot}\overset{1}{H}_t\cdot\overset{1}{H}_t\,dx\le\varepsilon_1\|\overset{1}{H}_t\|^2_{L_6(\overset{1}{\Omega}_t)}\\ &+c(1/\varepsilon_1)[\|v_t\|^2_{L_3(\overset{1}{\Omega}_t)}\|\nabla\overset{1}{H}\|^2_{L_2(\overset{1}{\Omega}_t)}+\|\overset{1}{H}_t\|^2_{L_3(\overset{1}{\Omega}_t)}\|\nabla v\|^2_{L_2(\overset{1}{\Omega}_t)}+\|\overset{1}{H}\|^2_{L_3(\overset{1}{\Omega}_t)}\|\nabla v_t\|^2_{L_2(\overset{1}{\Omega}_t)}].\end{aligned}$$

Differentiating $(1.9)_1$ with respect to time, multiplying the result by $\overset{2}{H}_t$ and integrating over $\overset{2}{\Omega}_t$ gives

$$\mu\int_{\overset{2}{\Omega}_t}[\partial_t\overset{2}{H}{}_t^2+v'\cdot\nabla(\overset{2}{H}{}_t^2)]\,dx+\frac{1}{\sigma_2}\int_{\overset{2}{\Omega}_t}\operatorname{rot}\operatorname{rot}\overset{2}{H}_t\cdot\overset{2}{H}_t\,dx = \mu\int_{\overset{2}{\Omega}_t}v'\cdot\nabla\overset{2}{H}_t\cdot\overset{2}{H}_t\,dx+\int_{\overset{2}{\Omega}_t}G_t\cdot\overset{2}{H}_t\,dx. \tag{4.12}$$

In view of formula (4.75) we obtain

$$\mu\frac{d}{dt}\int_{\overset{2}{\Omega}_t}\overset{2}{H}_t\,dx+\frac{1}{\sigma_2}\int_{\overset{2}{\Omega}_t}\operatorname{rot}\operatorname{rot}\overset{2}{H}_t\cdot\overset{2}{H}_t\,dx\le\varepsilon_2\|\nabla\overset{2}{H}_t\|^2_{L_2(\overset{2}{\Omega}_t)} + c(1/\varepsilon_2)\|v'\|^2_{L_4(\overset{2}{\Omega}_t)}\|\overset{2}{H}_t\|^2_{L_4(\overset{2}{\Omega}_t)}+\varepsilon_3\|\overset{2}{H}_t\|^2_{L_6(\overset{2}{\Omega}_t)}+c(1/\varepsilon_3)\|G_t\|^2_{L_{6/5}(\overset{2}{\Omega}_t)}. \tag{4.13}$$

Adding (4.11) and (4.13) and employing the transmission conditions, we obtain for sufficiently small ε_1–ε_3 the inequality

$$\begin{aligned}\mu\frac{d}{dt}\int_{\overset{1}{\Omega}_t}\overset{1}{H}{}_t^2\,dx+\mu\frac{d}{dt}\int_{\overset{2}{\Omega}_t}\overset{2}{H}{}_t^2\,dx+\|\overset{1}{H}_t\|^2_{H^1(\overset{1}{\Omega}_t)}+\|\overset{2}{H}_t\|^2_{H^1(\overset{2}{\Omega}_t)}&\\ \le\Big|\int_{S_t}(v\times\overset{1}{H})_t\cdot\bar n\times\overset{1}{H}_t\,dS_t\Big|&\\ +c[\|v_t\|^2_{L_3(\overset{1}{\Omega}_t)}\|\nabla\overset{1}{H}\|^2_{L_2(\overset{1}{\Omega}_t)}+\|\overset{1}{H}_t\|^2_{L_3(\overset{1}{\Omega}_t)}\|\nabla v\|^2_{L_2(\overset{1}{\Omega}_t)}&\\ +\|\overset{1}{H}\|^2_{L_3(\overset{1}{\Omega}_t)}\|\nabla v_t\|^2_{L_2(\overset{1}{\Omega}_t)}+\|v'\|^2_{L_4(\overset{2}{\Omega}_t)}\|\overset{2}{H}_t\|^2_{L_4(\overset{2}{\Omega}_t)}+\|G_t\|^2_{L_{6/5}(\overset{2}{\Omega}_t)}].&\end{aligned} \tag{4.14}$$

The first term on the r.h.s. of (4.14) is estimated by

$$\varepsilon\|\overset{1}{H}_t\|^2_{L_3(S_t)}+c(1/\varepsilon)(\|v_t\|^2_{L_3(S_t)}\|\overset{1}{H}\|^2_{L_3(S_t)}+\|v\|^2_{L_3(S_t)}\|\overset{1}{H}_t\|^2_{L_3(S_t)}). \tag{4.15}$$

Then, for a sufficiently small ε, we derive from (4.14) the inequality

$$\begin{aligned}&\frac{d}{dt}\|\overset{1}{H}_t\|^2_{L_2(\overset{1}{\Omega}_t)}+\frac{d}{dt}\|\overset{2}{H}_t\|^2_{L_2(\overset{2}{\Omega}_t)}+\|\overset{1}{H}_t\|^2_{H^1(\overset{1}{\Omega}_t)}+\|\overset{2}{H}_t\|^2_{H^1(\overset{2}{\Omega}_t)}\\ &\le c[\|v'\|^2_{H^1(\overset{2}{\Omega}_t)}\|\overset{2}{H}_t\|^2_{H^1(\overset{2}{\Omega}_t)}+\|v_t\|^2_{H^1(\overset{1}{\Omega}_t)}\|\overset{1}{H}\|^2_{H^1(\overset{1}{\Omega}_t)}+\|v\|^2_{H^1(\overset{1}{\Omega}_t)}\|\overset{1}{H}_t\|^2_{H^1(\overset{1}{\Omega}_t)}+\|G_t\|^2_{L_{6/5}(\overset{2}{\Omega}_t)}]\\ &\le c(\|v\|^2_{\Gamma_1^2(\overset{1}{\Omega}_t)}\|\overset{1}{H}\|^2_{\Gamma_1^2(\overset{1}{\Omega}_t)}+\|v'\|^2_{H^1(\overset{2}{\Omega}_t)}\|\overset{2}{H}\|^2_{\Gamma_1^2(\overset{2}{\Omega}_t)}+\|G_t\|^2_{L_{6/5}(\overset{2}{\Omega}_t)}).\end{aligned} \tag{4.16}$$

Now (4.16) yields (4.9). ■

Next we obtain an estimate for second time derivatives.

LEMMA 4.4. *Assume that*

$$\|v\|_{\Gamma_1^2(\overset{1}{\Omega}_t)}+\sum_{i=1}^2(\|\overset{i}{H}\|_{H^2(\overset{i}{\Omega}_t)}+\|\overset{i}{H}_t\|^2_{H^2(\overset{i}{\Omega}_t)}+\|\overset{i}{H}_{tt}\|_{H^1(\overset{i}{\Omega}_t)})<\infty,$$

$G_{tt} \in L_{6/5}(\overset{2}{\Omega}_t)$ and assume the transmission conditions hold on S_t. Then

$$
\frac{d}{dt}\Big(\sum_{i=1}^{2}\|\overset{i}{H}_{tt}\|^2_{L_2(\overset{i}{\Omega}_t)}\Big)+\sum_{i=1}^{2}\|\overset{i}{H}_{tt}\|^2_{H^1(\overset{2}{\Omega}_t)}
\le c\|v\|^2_{\Gamma_1^2(\overset{1}{\Omega}_t)}\sum_{i=1}^{2}(\|\overset{i}{H}\|^2_{H^2(\overset{i}{\Omega}_t)}+\|\overset{i}{H}_t\|^2_{H^2(\overset{2}{\Omega}_t)}+\|\overset{i}{H}_{tt}\|^2_{H^1(\overset{2}{\Omega}_t)})+c\|G_{tt}\|^2_{L_{6/5}(\overset{2}{\Omega}_t)}, \tag{4.17}
$$

where in the above inequalities we have used the extension

$$
\|v'\|_{\Gamma_1^2(\overset{2}{\Omega}_t)}\le c\|v\|_{\Gamma_1^2(\overset{1}{\Omega}_t)}. \tag{4.18}
$$

Proof. Differentiating $(1.2)_1$ twice with respect to t, multiplying by $\overset{1}{H}_{tt}$ and integrating over $\overset{1}{\Omega}_t$ we obtain

$$
\mu\int_{\overset{1}{\Omega}_t}[\partial_t(\overset{1}{H}{}^2_{tt})+v\cdot\nabla(\overset{1}{H}{}^2_{tt})]\,dx+\frac{1}{\sigma_1}\int_{\overset{1}{\Omega}_t}\operatorname{rot}\operatorname{rot}\overset{1}{H}_{tt}\cdot\overset{1}{H}_{tt}\,dx
=-\mu\int_{\overset{1}{\Omega}_t}(v_{tt}\cdot\nabla\overset{1}{H}+2v_t\cdot\nabla\overset{1}{H}_t-\overset{1}{H}_{tt}\cdot\nabla v-2\overset{1}{H}_t\cdot\nabla v_t)\cdot\overset{1}{H}_{tt}\,dx. \tag{4.19}
$$

In view of formula (4.75) we have

$$
\begin{aligned}
&\mu\frac{d}{dt}\int_{\overset{1}{\Omega}_t}\overset{1}{H}{}^2_{tt}\,dx+\frac{1}{\sigma_1}\int_{\overset{1}{\Omega}_t}\operatorname{rot}\operatorname{rot}\overset{1}{H}_{tt}\cdot\overset{1}{H}_{tt}\,dx\\
&\le\varepsilon\|\overset{1}{H}_{tt}\|^2_{L_6(\overset{1}{\Omega}_t)}+c(1/\varepsilon)[\|v_{tt}\|^2_{L_2(\overset{1}{\Omega}_t)}\|\nabla\overset{1}{H}\|^2_{L_3(\overset{1}{\Omega}_t)}+\|v_t\|^2_{L_3(\overset{1}{\Omega}_t)}\|\nabla\overset{1}{H}_t\|^2_{L_2(\overset{1}{\Omega}_t)}\\
&\quad+\|\overset{1}{H}_{tt}\|^2_{L_2(\overset{1}{\Omega}_t)}\|\nabla v\|^2_{L_3(\overset{1}{\Omega}_t)}+\|\overset{1}{H}_t\|^2_{L_3(\overset{1}{\Omega}_t)}\|\nabla v_t\|^2_{L_2(\overset{1}{\Omega}_t)}]\\
&\le\varepsilon\|\overset{1}{H}_{tt}\|^2_{L_6(\overset{1}{\Omega}_t)}+c(1/\varepsilon)\|v\|^2_{\Gamma_0^2(\overset{1}{\Omega}_t)}\|\overset{1}{H}\|^2_{\Gamma_0^2(\overset{1}{\Omega}_t)}.
\end{aligned} \tag{4.20}
$$

Differentiating $(1.9)_1$ twice with respect to t, multiplying the result by $\overset{2}{H}_{tt}$ and integrating over $\overset{2}{\Omega}_t$ yields

$$
\mu\int_{\overset{2}{\Omega}_t}[\partial_t(\overset{2}{H}{}^2_{t})+v'\cdot\nabla(\overset{2}{H}{}^2_{tt})]\,dx+\frac{1}{\sigma_2}\int_{\overset{2}{\Omega}_t}\operatorname{rot}\operatorname{rot}\overset{2}{H}_{tt}\cdot\overset{2}{H}_{tt}\,dx
=\int_{\overset{2}{\Omega}_t}(\mu v'\cdot\nabla\overset{2}{H}_{tt}\cdot\overset{2}{H}_{tt}+G_{tt}\cdot\overset{2}{H}_{tt})\,dx. \tag{4.21}
$$

In view of (4.75) we get

$$
\mu\frac{d}{dt}\int_{\overset{2}{\Omega}_t}\overset{2}{H}{}^2_{tt}\,dx+\frac{1}{\sigma_2}\int_{\overset{2}{\Omega}_t}\operatorname{rot}\operatorname{rot}\overset{2}{H}_{tt}\cdot\overset{2}{H}_{tt}\,dx
\le\varepsilon\|\overset{2}{H}_{xtt}\|^2_{L_2(\overset{2}{\Omega}_t)}+c(1/\varepsilon)(\|v'\|^2_{H^2(\overset{2}{\Omega}_t)}\|\overset{2}{H}_{tt}\|^2_{L_2(\overset{2}{\Omega}_t)}+\|G_{tt}\|^2_{L_{6/5}(\overset{2}{\Omega}_t)}). \tag{4.22}
$$

Adding (4.20) and (4.22), and integrating by parts, we derive the inequality

$$
\begin{aligned}
(4.23)\qquad &\frac{d}{dt}\int\limits_{\overset{1}{\Omega}_t}\overset{1}{H}{}^2_{tt}\,dx+\frac{d}{dt}\int\limits_{\overset{2}{\Omega}_t}\overset{2}{H}{}^2_{tt}\,dx+\|\overset{1}{H}_{tt}\|^2_{H^1(\overset{1}{\Omega}_t)}+\|\overset{2}{H}_{tt}\|^2_{H^1(\overset{2}{\Omega}_t)}\\
&\quad-\sum_{i=1}^{2}\frac{1}{\sigma_i}\int\limits_{S_t}\operatorname{rot}\overset{i}{H}_{tt}\cdot\overset{i}{\bar n}\times\overset{i}{H}_{tt}\,dS_t\\
&\le c\big[\|v\|^2_{\Gamma_0^2(\overset{1}{\Omega}_t)}\|\overset{1}{H}\|^2_{\Gamma_0^2(\overset{1}{\Omega}_t)}+\|v'\|^2_{H^2(\overset{2}{\Omega}_t)}\|\overset{2}{H}_{tt}\|^2_{L_2(\overset{2}{\Omega}_t)}+\|G_{tt}\|^2_{L_{6/5}(\overset{2}{\Omega}_t)}\big],
\end{aligned}
$$

where we have used that ε is sufficiently small.

Now we examine the last two terms on the l.h.s. of (4.23). In view of the transmission conditions they are equal to

$$
\begin{aligned}
I&\equiv\sum_{i=1}^{2}\frac{1}{\sigma_i}\int\limits_{S_t}\operatorname{rot}\overset{i}{H}_{tt}\times\overset{i}{\bar n}\cdot\overset{i}{H}_{tt}\,dS_t\\
&=\int\limits_{S_t}\bigg[\frac{1}{\sigma_1}\operatorname{rot}\overset{1}{H}_{tt}\times\overset{1}{\bar n}\cdot\overset{1}{H}_{tt}-\frac{1}{\sigma_2}\operatorname{rot}\overset{2}{H}_{tt}\times\overset{2}{\bar n}\cdot\overset{2}{H}_{tt}\bigg]\,dS_t\\
&=\int\limits_{S_t}\bigg[\frac{1}{\sigma_1}(\operatorname{rot}\overset{1}{H}\times\overset{1}{\bar n})_{tt}-\frac{1}{\sigma_2}(\operatorname{rot}\overset{2}{H}\times\overset{2}{\bar n})_{tt}\bigg]\cdot\overset{1}{H}_{tt}\,dS_t\\
&\quad+\int\limits_{S_t}\bigg[\frac{1}{\sigma_1}\operatorname{rot}\overset{1}{H}\times\overset{1}{\bar n}_{tt}+\frac{2}{\sigma_1}\operatorname{rot}\overset{1}{H}_t\times\overset{1}{\bar n}_t-\frac{1}{\sigma_2}\operatorname{rot}\overset{2}{H}\times\overset{2}{\bar n}_{tt}-\frac{2}{\sigma_2}\operatorname{rot}\overset{2}{H}_t\times\overset{2}{\bar n}_t\bigg]\cdot\overset{1}{H}_{tt}\,dS_t\\
&\equiv I_1+I_2.
\end{aligned}
$$

Employing the transmission conditions we have

$$
I_1=\int\limits_{St}\mu(v\times\overset{1}{H})_{tt}\cdot\overset{1}{H}_{tt}\,dS_t=\mu\int\limits_{S_t}(v_{tt}\times\overset{1}{H}+2v_t\times\overset{1}{H}_t)\cdot\overset{1}{H}_{tt}\,dS_t.
$$

Hence

$$
|I_1|\le\varepsilon\|\overset{1}{H}_{tt}\|^2_{H^1(\overset{1}{\Omega}_t)}+c(1/\varepsilon)(\|v_{tt}\|^2_{H^1(\overset{1}{\Omega}_t)}\|\overset{1}{H}\|^2_{H^1(\overset{1}{\Omega}_t)}+\|v_t\|^2_{H^1(\overset{1}{\Omega}_t)}\|\overset{1}{H}_t\|^2_{H^1(\overset{1}{\Omega}_t)})
$$

and

$$
\begin{aligned}
|I_2|\le{}&\varepsilon\|\overset{1}{H}_{tt}\|^2_{H^1(\overset{1}{\Omega}_t)}+c(1/\varepsilon)[\|v_t\|^2_{H^1(\overset{1}{\Omega}_t)}(\|\overset{1}{H}\|^2_{H^2(\overset{1}{\Omega}_t)}+\|\overset{2}{H}\|^2_{H^2(\overset{2}{\Omega}_t)})\\
&+\|v\|^2_{H^1(\overset{1}{\Omega}_t)}(\|\overset{1}{H}_t\|^2_{H^2(\overset{1}{\Omega}_t)}+\|\overset{2}{H}_t\|^2_{H^2(\overset{2}{\Omega}_t)})].
\end{aligned}
$$

Choosing ε sufficiently small inequality (4.23) takes the form

$$
\begin{aligned}
(4.24)\qquad &\frac{d}{dt}\|\overset{1}{H}_{tt}\|^2_{L_2(\overset{1}{\Omega}_t)}+\frac{d}{dt}\|\overset{2}{H}_{tt}\|^2_{L_2(\overset{2}{\Omega}_t)}+\|\overset{1}{H}_{tt}\|^2_{H^1(\overset{1}{\Omega}_t)}+\|\overset{2}{H}_{tt}\|^2_{H^1(\overset{2}{\Omega}_t)}\\
&\le c(\|v\|^2_{\Gamma_0^2(\overset{1}{\Omega}_t)}\|\overset{1}{H}\|^2_{\Gamma_0^2(\overset{1}{\Omega}_t)}+\|v_{tt}\|^2_{H^1(\overset{1}{\Omega}_t)}\|\overset{1}{H}\|^2_{H^1(\overset{1}{\Omega}_t)}\\
&\quad+\|v\|^2_{H^1(\overset{1}{\Omega}_t)}\|\overset{1}{H}_t\|^2_{H^2(\overset{1}{\Omega}_t)}+\|v'\|^2_{H^2(\overset{2}{\Omega}_t)}\|\overset{2}{H}_{tt}\|^2_{H^1(\overset{2}{\Omega}_t)}+\|G_{tt}\|^2_{L_{6/5}(\overset{2}{\Omega}_t)}).
\end{aligned}
$$

From (4.24) we deduce (4.17). ∎

Now we obtain estimates for the space derivatives. First we make

Remark 4.5. Let us introduce a partition of unity $\overset{i}{\zeta}_k(x)$, $i=1,2$, $k\in\mathcal{M}\cup\mathcal{N}$, where $\operatorname{supp}\overset{i}{\zeta}_k$, $k\in\mathcal{M}$, is an interior subdomain of $\overset{i}{\Omega}_t$ and $\operatorname{supp}\overset{i}{\zeta}_k$, $k\in\mathcal{N}$, is a boundary subdomain. Then $\overset{i}{\zeta}_k$, $k\in\mathcal{N}$, are equal to 1 in some neighborhood of S_t. Moreover $\overset{1}{\zeta}_k(x)=\overset{2}{\zeta}_k(x)$, $x\in S_t$, $k\in\mathcal{N}$. Let $\xi_k\in S_t$, $k\in\mathcal{N}$, be in $\operatorname{supp}\overset{i}{\zeta}_k\cap S_t$, $i=1,2$.

In the subdomain $\operatorname{supp}\overset{i}{\zeta}_k$, $k\in\mathcal{N}$, we introduce curvilinear coordinates $\tau_1(x)$, $\tau_2(x)$, $n(x)$ such that for $x\in S_t$, $\tau_1(x)$, $\tau_2(x)$ are tangent coordinates to S_t and $n(x)$ the normal coordinate. With these coordinates there is connected a system of orthonormal vectors such that $\bar\tau_1(x)$, $\bar\tau_2(x)$, $x\in S_t$, are vectors tangent to S_t and $\bar n(x)$, $x\in S_t$ is a normal vector to S_t. Moreover, we assume that $\nabla\tau_\alpha=\bar\tau_\alpha$, $\nabla n=\bar n$, $|\bar\tau_\alpha|=1$, $|\bar n|=1$, $\alpha=1,2$, and $\bar\tau_1$, $\bar\tau_2$, $\bar n$ are orthogonal.

Finally, we introduce the notation $\overset{i,k}{\tilde H}=\overset{i}{H}\overset{i}{\zeta}_k$, $i=1,2$, $k\in\mathcal{M}\cup\mathcal{N}$.

Using the above notation we express problems $(1.2)_1$ and $(1.9)_1$ in the forms

$$
\begin{aligned}
&\mu\big(\overset{1,k}{\tilde H}_t+v\cdot\nabla\overset{1,k}{\tilde H}-\overset{1,k}{\tilde H}\cdot\nabla v\big)+\frac{1}{\sigma_1}\operatorname{rot}\operatorname{rot}\overset{1,k}{\tilde H}\\
&\quad=\mu\overset{1}{H}\overset{1}{\zeta}_{kt}+\mu\overset{1}{H}v\cdot\nabla\overset{1}{\zeta}_k+\frac{1}{\sigma_1}\operatorname{rot}(\nabla\overset{1}{\zeta}_k\times\overset{1}{H})+\frac{1}{\sigma_1}\nabla\overset{1}{\zeta}_k\times\operatorname{rot}\overset{1}{H},\\
&\overset{1,k}{\tilde H}|_{t=0}=\overset{1,k}{\tilde H}(0)
\end{aligned}
\tag{4.25}
$$

and

$$
\begin{aligned}
&\mu\big(\overset{2,k}{\tilde H}_t+v'\cdot\nabla\overset{2,k}{\tilde H}\big)+\frac{1}{\sigma_2}\operatorname{rot}\operatorname{rot}\overset{2,k}{\tilde H}=\mu\overset{2}{H}\overset{2}{\zeta}_{kt}+\mu v'\cdot\nabla\overset{2}{\zeta}_k\overset{2}{H}\\
&\quad+\frac{1}{\sigma_2}\operatorname{rot}(\nabla\overset{2}{\zeta}_k\times\overset{2}{H})+\frac{1}{\sigma_2}\nabla\overset{2}{\zeta}_k\times\operatorname{rot}\overset{2}{H}+\mu v'\cdot\nabla\overset{2}{H}\overset{2}{\zeta}_k+\overset{2,k}{\tilde G},\\
&\overset{2,k}{\tilde H}|_B=0,\quad\overset{2,k}{\tilde H}|_{t=0}=\overset{2,k}{\tilde H}(0),
\end{aligned}
\tag{4.26}
$$

where $k\in\mathcal{M}\cup\mathcal{N}$.

Lemma 4.6. *Assume that*

$$\sum_{i=1}^2\big(\|\overset{i}{H}_{xt}\|_{L_2(\overset{i}{\Omega}_t)}+\|\overset{i}{H}\|_{\Gamma_0^1(\overset{i}{\Omega}_t)}+\|\overset{i}{H}_t\|_{L_2(\overset{i}{\Omega}_t)}\big)+\|v\|_{H^2(\overset{1}{\Omega}_t)}<\infty,$$

$G\in L_2(\overset{2}{\Omega}_t)$ *and assume the transmission conditions hold on* S_t. *Then*

$$
\begin{aligned}
\frac{d}{dt}X_{21}+\sum_{i=1}^2\|\overset{i}{H}\|^2_{H^2(\overset{i}{\Omega}_t)}&\le c\sum_{i=1}^2\|\overset{i}{H}\|^2_{\Gamma_0^1(\overset{i}{\Omega}_t)}+\varepsilon\sum_{i=1}^2\|\overset{i}{H}_{xt}\|^2_{L_2(\overset{i}{\Omega}_t)}\\
&\quad+c\|v\|^2_{H^2(\overset{1}{\Omega}_t)}\sum_{i=1}^2\|\overset{i}{H}\|^2_{H^1(\overset{i}{\Omega}_t)}+c\|G\|^2_{L_2(\overset{2}{\Omega}_t)},
\end{aligned}
\tag{4.27}
$$

where

$$X_{21}=\sum_{i=1}^{2}\Big(\|\overset{i}{H}_x\|^2_{L_2(\overset{i}{\Omega}_t)}+\sum_{k\in\mathcal{M}}\|\overset{i,k}{\tilde{H}}_x\|^2_{L_2(\overset{i}{\Omega}_k)}+\sum_{k\in\mathcal{N}}\|\overset{i,k}{\tilde{H}}_\tau\|^2_{L_2(\overset{i}{\hat{\Omega}}_k)}\Big).$$

Proof. To prove the lemma we shall restrict ourselves to neighborhoods located near S_t because calculations in neighborhoods at a positive distance from S_t are similar but much simpler. Hence we consider problems (4.25) and (4.26) for $k\in\mathcal{N}$. Differentiating (4.25) with respect to $\partial_\tau=\bar{\tau}\cdot\nabla$, where $\bar{\tau}|_{S_t}$ is a vector tangent to S_t, next multiplying by $\overset{1,k}{\tilde{H}}_\tau$ and integrating the result over $\overset{1}{\hat{\Omega}}_k=\overset{i}{\Omega}_t\cap\operatorname{supp}\overset{i}{\zeta}_k$, $i=1,2$, yields

$$\begin{aligned}
(4.28)\qquad &\mu\int\limits_{\overset{1}{\hat{\Omega}}_k}(\overset{1,k}{\tilde{H}}_{\tau t}+v\cdot\nabla\overset{1,k}{\tilde{H}}_\tau)\cdot\overset{1,k}{\tilde{H}}_\tau\,dx+\frac{1}{\sigma_1}\int\limits_{\overset{1}{\hat{\Omega}}_k}\operatorname{rot}\operatorname{rot}\overset{1,k}{\tilde{H}}_\tau\cdot\overset{1,k}{\tilde{H}}_\tau\,dx\\
&=\mu\int\limits_{\overset{1}{\hat{\Omega}}_k}(\bar{\tau}_t\cdot\nabla\overset{1,k}{\tilde{H}}-\bar{\tau}\cdot\nabla(v)\cdot\nabla\overset{1,k}{\tilde{H}}+v\cdot\nabla(\bar{\tau}\cdot\nabla)\overset{1,k}{\tilde{H}})\cdot\overset{1,k}{\tilde{H}}_\tau\,dx\\
&\quad+\mu\int\limits_{\overset{1}{\hat{\Omega}}_k}\partial_\tau(\overset{1}{H}\overset{1}{\zeta}_{kt}+\overset{1}{H}v\cdot\nabla\overset{1}{\zeta}_k+\overset{1}{H}\cdot\nabla v\overset{1}{\zeta}_k)\cdot\overset{1,k}{\tilde{H}}_\tau\,dx\\
&\quad+\frac{1}{\sigma_1}\int\limits_{\overset{1}{\hat{\Omega}}_k}\partial_\tau(\operatorname{rot}(\nabla\overset{1}{\zeta}_k\times\overset{1}{H})+\nabla\overset{1}{\zeta}_k\times\operatorname{rot}\overset{1}{H})\cdot\overset{1,k}{\tilde{H}}_\tau\,dx\\
&\quad-\frac{1}{\sigma_1}\int\limits_{\overset{1}{\hat{\Omega}}_t}[\partial_\tau(\operatorname{rot}\operatorname{rot}\overset{1,k}{\tilde{H}})-\operatorname{rot}\operatorname{rot}\partial_\tau\overset{1,k}{\tilde{H}}]\cdot\overset{1,k}{\tilde{H}}_\tau\,dx.
\end{aligned}$$

Similarly we obtain

$$\begin{aligned}
(4.29)\qquad &\mu\int\limits_{\overset{2}{\hat{\Omega}}_k}(\overset{2,k}{\tilde{H}}_\tau+v'\cdot\nabla\overset{2,k}{\tilde{H}}_\tau)\cdot\overset{2,k}{\tilde{H}}_\tau\,dx+\frac{1}{\sigma_2}\int\limits_{\overset{2}{\hat{\Omega}}_k}\operatorname{rot}\operatorname{rot}\overset{2,k}{\tilde{H}}_\tau\cdot\overset{2,k}{\tilde{H}}_\tau\,dx\\
&=\mu\int\limits_{\overset{2}{\hat{\Omega}}_k}(\bar{\tau}_t\cdot\nabla\overset{2,k}{\tilde{H}}-\bar{\tau}\cdot\nabla(v')\cdot\nabla\overset{2,k}{\tilde{H}}+v'\cdot\nabla(\bar{\tau}\cdot\nabla)\overset{2,k}{\tilde{H}}_\tau)\cdot\overset{2,k}{\tilde{H}}_\tau\,dx\\
&\quad+\mu\int\limits_{\overset{2}{\hat{\Omega}}_k}\partial_\tau(\overset{2}{H}\overset{2}{\zeta}_{kt}+v'\cdot\nabla\overset{2}{\zeta}_k\overset{2}{H}+v'\cdot\nabla\overset{2}{H}\overset{2}{\zeta}_k)\cdot\overset{2,k}{\tilde{H}}_\tau\,dx\\
&\quad+\frac{1}{\sigma_2}\int\limits_{\overset{2}{\hat{\Omega}}_k}\partial_\tau(\operatorname{rot}(\nabla\overset{2}{\zeta}_k\times\overset{2}{H})+\nabla\overset{2}{\zeta}_k\times\operatorname{rot}\overset{2}{H})\cdot\overset{2,k}{\tilde{H}}\,dx+\int\limits_{\overset{2}{\hat{\Omega}}_k}\overset{k}{\tilde{G}}_\tau\cdot\overset{2,k}{\tilde{H}}_\tau\,dx\\
&\quad+\frac{1}{\sigma_2}\int\limits_{\overset{2}{\hat{\Omega}}_k}[\operatorname{rot}\operatorname{rot}\overset{2,k}{\tilde{H}}_\tau-\partial_\tau(\operatorname{rot}\operatorname{rot}\overset{2,k}{\tilde{H}})]\cdot\overset{2,k}{\tilde{H}}_\tau\,dx.
\end{aligned}$$

We estimate the terms from the r.h.s. of (4.28). The first term is estimated by

$$\varepsilon_1\|\overset{1,k}{\tilde{H}}_\tau\|^2_{H^1(\overset{1}{\hat{\Omega}}_k)}+c(1/\varepsilon_1)\|v\|^2_{H^2(\overset{1}{\hat{\Omega}}_k)}\|\overset{1,k}{\tilde{H}}\|^2_{H^1(\overset{1}{\hat{\Omega}}_k)},$$

and the second by

$$\varepsilon_2\|\overset{1,k}{\tilde H}_\tau\|^2_{H^1(\overset{1}{\hat\Omega}_k)}+c(1/\varepsilon_2)\|v_\tau\|^2_{H^1(\overset{1}{\hat\Omega}_k)}\|\overset{1}{H}_\tau\|^2_{H^1(\overset{1}{\hat\Omega}_k)}.$$

Integrating by parts in the third term we can estimate it by

$$\varepsilon_3\|\overset{1,k}{\tilde H}_{\tau\tau}\|^2_{L_2(\overset{1}{\hat\Omega}_k)}+c(1/\varepsilon_3)\|\overset{1}{H}\|^2_{H^1(\overset{1}{\hat\Omega}_k)}.$$

Finally, the last term is bounded by

$$\varepsilon_4\|\overset{1,k}{\tilde H}_\tau\|^2_{H^1(\overset{1}{\hat\Omega}_k)}+c(1/\varepsilon_4)\|v\|^2_{H^2(\overset{1}{\hat\Omega}_k)}\|\overset{1,k}{\tilde H}\|^2_{H^2(\overset{1}{\hat\Omega}_k)}.$$

All terms in (4.29) can be estimated in a similar way.

Adding (4.28), (4.29) and using the above estimates we obtain

$$\begin{aligned}(4.30)\quad &\sum_{i=1}^2\bigg(\mu\frac{d}{dt}\|\overset{i,k}{\tilde H}_\tau\|^2_{L_2(\overset{i}{\hat\Omega}_k)}+\frac{1}{\sigma_i}\int\limits_{\overset{i}{\hat\Omega}_k}\operatorname{rot}\operatorname{rot}\overset{i,k}{\tilde H}_\tau\cdot\overset{i,k}{\tilde H}_\tau\,dx\bigg)\\
&\le\sum_{i=1}^2\varepsilon\|\overset{i,k}{\tilde H}_\tau\|^2_{H^1(\overset{i}{\hat\Omega}_k)}+c(1/\varepsilon)\|\overset{i}{v}\|^2_{H^2(\overset{i}{\hat\Omega}_k)}(\|\overset{i,k}{\tilde H}\|^2_{H^1(\overset{i}{\hat\Omega}_k)}+\|\overset{i}{H}_\tau\|^2_{H^1(\overset{i}{\hat\Omega}_k)})\\
&\quad+c(1/\varepsilon)\Big(\|\overset{k}{\tilde G}_\tau\|^2_{L_2(\overset{2}{\hat\Omega}_k)}+\sum_{i=1}^2\|\overset{i}{H}\|^2_{H^1(\overset{i}{\hat\Omega}_k)}\Big),\end{aligned}$$

where $v=\overset{1}{v}$, $v'=\overset{2}{v}$.

Next we examine the last two terms on the l.h.s. of (4.30). Integrating by parts they are equal to

$$\frac{1}{\sigma_1}\int\limits_{\overset{1}{\hat\Omega}_k}|\operatorname{rot}\overset{1,k}{\tilde H}_\tau|^2\,dx+\frac{1}{\sigma_2}\int\limits_{\overset{2}{\hat\Omega}_k}|\operatorname{rot}\overset{2,k}{\tilde H}_\tau|^2\,dx-\sum_{i=1}^2\frac{1}{\sigma_i}\int\limits_{S_k}\operatorname{rot}\overset{i,k}{\tilde H}_\tau\cdot\bar n\times\overset{i,k}{H}_\tau\,dS_t\equiv I,$$

where $\overset{i}{\bar n}$ is the exterior normal to S_t and $S_k=\operatorname{supp}\zeta_k^1\cap S_t$.

Employing the transmission conditions we obtain

$$\begin{aligned}(4.31)\quad &\bigg|\sum_{i=1}^2\frac{1}{\sigma_i}\int\limits_{S_k}\operatorname{rot}\overset{i,k}{\tilde H}_\tau\cdot\overset{i}{\bar n}\times\overset{i,k}{\tilde H}_\tau\,dS_k\bigg|\le\int\limits_{S_k}|(v\times\overset{1,k}{\tilde H})_\tau\cdot\overset{1}{\bar n}\times\overset{1,k}{\tilde H}_\tau|\,dS_k\\
&\quad+\varepsilon_1\|\overset{1,k}{\tilde H}_\tau\|^2_{H^1(\overset{1}{\hat\Omega}_k)}+c(1/\varepsilon_1)\int\limits_0^t\|v\|^2_{H^2(\overset{1}{\hat\Omega}_k)}\,dt'\,\|\overset{1,k}{\tilde H}\|^2_{H^2(\overset{1}{\hat\Omega}_k)}\\
&\le\varepsilon\|\overset{1,k}{\tilde H}_\tau\|^2_{H^1(\overset{1}{\hat\Omega}_k)}+c(1/\varepsilon)\Big(\|\overset{1,k}{\tilde H}_\tau\|^2_{H^1(\overset{1}{\hat\Omega}_k)}\|v\|^2_{H^1(\overset{1}{\hat\Omega}_k)}\\
&\quad+\|\overset{1,k}{\tilde H}\|^2_{H^1(\overset{1}{\hat\Omega}_k)}\|v_\tau\|^2_{H^1(\overset{1}{\hat\Omega}_k)}+\int\limits_0^t\|v\|^2_{H^2(\overset{1}{\hat\Omega}_k)}\,dt'\,\|\overset{1,k}{\tilde H}\|^2_{H^2(\overset{1}{\hat\Omega}_k)}\Big).\end{aligned}$$

From the form of the last two terms on the l.h.s. of (4.31) we see that we need to find an estimate for $\|\overset{i,k}{\tilde H}_n\|_{H^1(\overset{i}{\hat\Omega}_k)}$, $i=1,2$. For this purpose we express the Laplace operator in the curvilinear coordinates τ_1, τ_2, n such that $\frac{\partial\tau_\alpha}{\partial x_i}\cdot\frac{\partial\tau_\beta}{\partial x_i}=\delta_{\alpha\beta}$, $\alpha,\beta=1,2$, $\frac{\partial\tau_\alpha}{\partial x_i}\cdot\frac{\partial n}{\partial x_i}=0$, $\frac{\partial n}{\partial x_i}\cdot\frac{\partial n}{\partial x_i}=1$, where summation over i is assumed. Then we have

$$\Delta u = u_{\tau_\alpha\tau_\alpha} + u_{nn} + \Delta\tau_\alpha u_{\tau_\alpha} + \Delta n u_n,$$

where summation over $\alpha = 1,2$ is assumed.

Using the above formula in (4.25) and (4.26) yields

$$\begin{aligned}(4.32)\qquad \overset{1,k}{\tilde H}_{nn} = & -\overset{1,k}{\tilde H}_{\tau_\alpha\tau_\alpha} - \Delta\tau_\alpha\overset{1,k}{\tilde H}_{\tau_\alpha} - \Delta n\overset{1,k}{\tilde H}_n + \mu\sigma_1(\overset{1,k}{\tilde H}_t + v\cdot\nabla\overset{1,k}{\tilde H}) \\ & - \sigma_1\overset{1}{H}\overset{1}{\zeta}_{kt} - \sigma_1 v\cdot\nabla\overset{1}{\zeta}_k\overset{1}{H} - \mu\overset{1,k}{\tilde H}\cdot\nabla v + 2\nabla\overset{1}{H}\nabla\overset{1}{\zeta}_k + \overset{1}{H}\Delta\overset{1}{\zeta}_k\end{aligned}$$

and

$$\begin{aligned}(4.33)\qquad \overset{2,k}{\tilde H}_{nn} = & -\overset{2,k}{\tilde H}_{\tau_\alpha\tau_\alpha} - \Delta\tau_\alpha\overset{2,k}{\tilde H}_{\tau_\alpha} - \Delta n\overset{2,k}{\tilde H}_n - \sigma_2\overset{2}{H}\overset{2}{\zeta}_{kt} \\ & + \mu\sigma_2\overset{2,k}{\tilde H}_t + 2\nabla\overset{2}{\zeta}_k\nabla\overset{2}{H} + \Delta\overset{2}{\zeta}_k\overset{2}{H} - \overset{k}{\tilde G}.\end{aligned}$$

Taking the L_2-norm of (4.32) we have

$$\begin{aligned}(4.34)\qquad & \|\overset{1,k}{\tilde H}_{nn}\|^2_{L_2(\overset{1}{\hat\Omega}_k)} \\ & \le c(\|\overset{1,k}{\tilde H}_{\tau\tau}\|^2_{L_2(\overset{1}{\hat\Omega}_k)} + \|\overset{1,k}{\tilde H}_t\|^2_{L_2(\overset{1}{\hat\Omega}_k)} + \|\overset{1}{H}\|^2_{H^1(\overset{1}{\hat\Omega}_k)} + \|v\|^2_{H^2(\overset{1}{\hat\Omega}_k)}\|\overset{1}{H}\|^2_{H^1(\overset{1}{\hat\Omega}_k)}).\end{aligned}$$

Similarly, the L_2-norm of (4.33) yields

$$\begin{aligned}(4.35)\qquad \|\overset{2,k}{\tilde H}_{nn}\|^2_{L_2(\overset{2}{\hat\Omega}_k)} \le & \, c(\|\overset{2,k}{\tilde H}_{\tau\tau}\|^2_{L_2(\overset{2}{\hat\Omega}_k)} + \|\overset{2,k}{\tilde H}_t\|^2_{L_2(\overset{2}{\hat\Omega}_k)} \\ & + \|\overset{2}{H}\|^2_{H^1(\overset{2}{\hat\Omega}_k)} + \|\overset{k}{\tilde G}\|^2_{L_2(\overset{2}{\hat\Omega}_k)} + \|v'\|^2_{H^1(\overset{2}{\hat\Omega}_k)}\|\overset{2}{H}\|^2_{H^1(\overset{2}{\hat\Omega}_k)}).\end{aligned}$$

Adding (4.30), (4.31), (4.34) and (4.35) gives

$$\begin{aligned}(4.36)\qquad & \frac{d}{dt}\|\overset{1,k}{\tilde H}_\tau\|^2_{L_2(\overset{1}{\hat\Omega}_k)} + \frac{d}{dt}\|\overset{2,k}{\tilde H}_\tau\|^2_{L_2(\overset{2}{\hat\Omega}_k)} + \|\overset{1,k}{\tilde H}\|^2_{H^2(\overset{1}{\hat\Omega}_k)} + \|\overset{2,k}{\tilde H}\|^2_{H^2(\overset{2}{\hat\Omega}_k)} \\ & \le c\sum_{i=1}^2[\|\overset{i}{v}\|^2_{H^2(\overset{i}{\hat\Omega}_k)}(\|\overset{i,k}{\tilde H}\|^2_{H^2(\overset{i}{\hat\Omega}_k)} + \|\overset{i}{H}_\tau\|^2_{H^1(\overset{i}{\hat\Omega}_k)}) \\ & \quad + \|\overset{i,k}{\tilde H}_t\|^2_{L_2(\overset{i}{\hat\Omega}_k)}\|\overset{i,k}{\tilde H}_{\tau\tau}\|^2_{L_2(\overset{i}{\hat\Omega}_k)}] + c\Big(\|\overset{k}{\tilde G}\|^2_{H^1(\overset{2}{\hat\Omega}_k)} + \sum_{i=1}^2\|\overset{i}{H}\|^2_{H^1(\overset{i}{\hat\Omega}_k)}\Big).\end{aligned}$$

In interior subdomains similar estimates can be calculated separately in $\overset{1}{\Omega}_t$ and $\overset{2}{\Omega}_t$. Therefore the transmission conditions are not needed. Moreover, under the time derivative norms, all first derivatives appear.

However, in neighborhoods near S_t, the normal derivative to S_t does not appear under the time derivative (see (4.36)). To add such derivatives we consider the expression

$$\begin{aligned}(4.37)\qquad J & = \frac{d}{dt}\sum_{i=1}^2\|\overset{i}{H}_x\|^2_{L_2(\overset{i}{\Omega}_t)} = \sum_{i=1}^2\frac{d}{dt}\int_{\overset{i}{\Omega}_t}\overset{i}{H}{}^2_x\,dx \\ & = \sum_{i=1}^2\frac{d}{dt}\int_{\overset{i}{\Omega}}(\overset{i}{H}{}^2_x)(x(\xi,t),t)\,d\xi = \sum_{i=1}^2\int_{\overset{i}{\Omega}}[(\overset{i}{H}{}^2_x)_x\overset{i}{v} + (\overset{i}{H}{}^2_x)_t]\,d\xi\end{aligned}$$

$$= \sum_{i=1}^{2} \int_{\overset{i}{\Omega}_t} [(\overset{i}{H}{}_x^2)_x \overset{i}{v} + (\overset{i}{H}{}_x^2)_t]\, dx = 2\sum_{i=1}^{2} \int_{\overset{i}{\Omega}_t} (\overset{i}{H}_x \overset{i}{H}_{xx} \overset{i}{v} + \overset{i}{H}_x \overset{i}{H}_{xt})\, dx$$
$$\le \varepsilon \sum_{i=1}^{2} (\|\overset{i}{H}_{xx}\|^2_{L_2(\overset{i}{\Omega}_t)} + \|\overset{i}{H}_{xt}\|^2_{L_2(\overset{i}{\Omega}_t)})$$
$$+ c(1/\varepsilon) \sum_{i=1}^{2} (\|\overset{i}{v}\|^2_{H^1(\overset{i}{\Omega}_t)} \|\overset{i}{H}_x\|^2_{H^1(\overset{i}{\Omega}_t)} + \|\overset{i}{H}_x\|^2_{L_2(\overset{i}{\Omega}_t)}),$$

where $\overset{1}{v} = v$, $\overset{2}{v} = v'$.

Summing the above estimates over all subdomains of the partition of unity (4.37) and assuming that ε is sufficiently small we obtain

$$\frac{d}{dt} X_{21}(t) + \sum_{i=1}^{2} \|\overset{i}{H}\|^2_{H^2(\overset{i}{\Omega}_t)} \tag{4.38}$$
$$\le \varepsilon \sum_{i=1}^{2} \|\overset{i}{H}_{xt}\|^2_{L_2(\overset{i}{\Omega}_t)} + c[\|\overset{1}{v}\|^2_{H^2(\overset{1}{\Omega}_t)} \|\overset{1}{H}\|^2_{H^2(\overset{1}{\Omega}_t)} + \|G\|^2_{L_2(\overset{2}{\Omega}_t)}]$$
$$+ c(1/\varepsilon) \sum_{i=1}^{2} \|\overset{i}{v}\|^2_{H^2(\overset{i}{\Omega}_t)} \|\overset{i}{H}\|^2_{H^2(\overset{i}{\Omega}_t)} + c(1/\varepsilon) \sum_{i=1}^{2} (\|\overset{i}{H}\|^2_{H^1(\overset{i}{\Omega}_t)} + \|\overset{i}{H}_t\|^2_{L_2(\overset{i}{\Omega}_t)}),$$

where

$$X_{21}(t) = \sum_{i=1}^{2} \Big(\|\overset{i}{H}_x\|^2_{L_2(\overset{i}{\Omega}_k)} + \sum_{k\in\mathcal{M}} \|\overset{i,k}{\tilde{H}}_x\|^2_{L_2(\overset{i}{\Omega}_k)} + \sum_{k\in\mathcal{N}} \|\overset{i,k}{\tilde{H}}_\tau\|^2_{L_2(\overset{i}{\hat{\Omega}}_k)} \Big). \tag{4.39}$$

Then inequality (4.38) yields (4.27). ■

Lemma 4.7. *Assume that*

$$\sum_{i=1}^{2} (\|\overset{i}{H}_{xxt}\|_{L_2(\overset{i}{\Omega}_t)} + \|\overset{i}{H}\|_{\Gamma_1^2(\overset{i}{\Omega}_t)} + \|\overset{i}{H}\|_{H^2(\overset{i}{\Omega}_t)}) + \|v\|_{H^2(\overset{1}{\Omega}_t)} < \infty,$$

$G \in H^1(\overset{2}{\Omega}_t)$ *and assume the transmission conditions hold on* S_t. *Then*

$$\frac{d}{dt} X_{22} + \sum_{i=1}^{2} \|\overset{i}{H}\|^2_{H^3(\overset{i}{\Omega}_t)} \le \varepsilon \sum_{i=1}^{2} \|\overset{i}{H}_{xxt}\|^2_{L_2(\overset{i}{\Omega}_t)} \tag{4.40}$$
$$+ c(1/\varepsilon) \sum_{i=1}^{2} \|\overset{i}{H}\|^2_{\Gamma_1^2(\overset{i}{\Omega}_t)} + c(1/\varepsilon) \|v\|^2_{H^2(\overset{1}{\Omega}_t)} \sum_{i=1}^{2} \|\overset{i}{H}\|^2_{H^2(\overset{i}{\Omega}_t)} + c\|G\|^2_{H^1(\overset{2}{\Omega}_t)},$$

where

$$X_{22} = \sum_{i=1}^{2} \Big(\|\overset{i}{H}_{xx}\|^2_{L_2(\overset{i}{\Omega}_t)} + \sum_{k\in\mathcal{M}} \|\overset{ik}{\tilde{H}}_{xx}\|^2_{L_2(\overset{i}{\Omega}_k)} + \sum_{k\in\mathcal{N}} \|\overset{i,k}{\tilde{H}}_{\tau\tau}\|^2_{L_2(\overset{i}{\hat{\Omega}}_k)} \Big).$$

Proof. Similarly to the proof of Lemma 4.6 we shall restrict ourselves to neighborhoods of S_t. First we differentiate (4.25) twice with respect to τ. To obtain an equation appropriate for derivation of an energy estimate we need

$$
\begin{aligned}
&\bar\tau\cdot\nabla(\bar\tau\cdot\nabla\overset{l,k}{H}_t)=\overset{l,k}{H}_{\tau\tau t}-\bar\tau_t\cdot\nabla(\bar\tau)\cdot\nabla\overset{l,k}{H}-\bar\tau\cdot\nabla(\bar\tau_t\cdot\nabla\overset{l,k}{H})+\bar\tau\cdot\nabla(\bar\tau_t)\cdot\nabla\overset{l,k}{H},\\
&\bar\tau\cdot\nabla(\bar\tau\cdot\nabla(v\cdot\overset{l,k}{H}))=\bar\tau\cdot\nabla(\bar\tau\cdot\nabla(v)\cdot\nabla\overset{l,k}{H})+\bar\tau\cdot\nabla(v_j\bar\tau\cdot\nabla\nabla_j\overset{l,k}{H})\\
&\quad=\partial_\tau(v_{j\tau}\cdot\nabla_j\overset{l,k}{H})+\bar\tau\cdot\nabla(v_j)\bar\tau\cdot\nabla\nabla_j\overset{l,k}{H}+v_j\bar\tau\cdot\nabla(\bar\tau\cdot\nabla\nabla_j\overset{l,k}{H})\\
&\quad=v\cdot\nabla\overset{l,k}{H}_{\tau\tau}+\partial_\tau(v_{j\tau}\cdot\nabla_j\overset{l,k}{H})+\bar\tau\cdot\nabla(v_j)\bar\tau\nabla\nabla_j\overset{l,k}{H}-v_j\nabla_j(\tau_l\tau_i)\nabla_l\nabla_i\overset{l,k}{H},\\
&\bar\tau\cdot\nabla(\bar\tau\cdot\nabla)\Delta\overset{l,k}{H}=\nabla_i(\bar\tau\cdot\nabla(\bar\tau\cdot\nabla)\nabla_i\overset{l,k}{H})-\nabla_i(\bar\tau\cdot\nabla(\bar\tau\cdot\nabla))\nabla_i\overset{l,k}{H}\\
&\quad=\Delta(\bar\tau\cdot\nabla(\bar\tau\cdot\nabla)\overset{l,k}{H})-\nabla_i(\nabla_i(\bar\tau\cdot\nabla(\bar\tau\cdot\nabla))\overset{l,k}{H})-\nabla_i(\bar\tau\cdot\nabla(\bar\tau\cdot\nabla))\nabla_i\overset{l,k}{H}\\
&\quad=\Delta\overset{l,k}{H}_{\tau\tau}+\overset{l,k}{A},\\
&\bar\tau\cdot\nabla(\bar\tau\cdot\nabla)\nabla\operatorname{div}\overset{l,k}{H}=\nabla_i(\bar\tau\cdot\nabla(\bar\tau\cdot\nabla)\nabla\overset{l,k}{H}_i)-\nabla_i(\bar\tau\cdot\nabla(\bar\tau\cdot\nabla))\nabla\overset{l,k}{H}_i\\
&\quad=\nabla\operatorname{div}\overset{l,k}{H}_{\tau\tau}-\nabla(\nabla_i\bar\tau\cdot\nabla(\bar\tau\cdot\nabla))\overset{l,k}{H}_i-\nabla_i(\bar\tau\cdot\nabla(\bar\tau\cdot\nabla))\nabla\overset{l,k}{H}_i\\
&\quad\equiv\nabla\operatorname{div}\overset{l,k}{H}_{\tau\tau}+\overset{l,k}{B},\quad l=1,2.
\end{aligned}
\tag{4.41}
$$

In view of (4.41) we obtain from (4.25) the equations

$$
\begin{aligned}
&\mu(\overset{1,k}{\tilde H}_{\tau\tau t}+v\cdot\nabla\overset{1,k}{\tilde H}_{\tau\tau})+\frac{1}{\sigma_1}\operatorname{rot}\operatorname{rot}\overset{1,k}{\tilde H}_{\tau\tau}\\
&=\mu(\bar\tau_t\cdot\nabla(\bar\tau)\cdot\nabla\overset{1,k}{\tilde H}+\bar\tau\cdot\nabla(\bar\tau_t\cdot\nabla\overset{1,k}{\tilde H})\\
&\quad-\bar\tau\cdot\nabla(\bar\tau_t)\cdot\overset{1,k}{\tilde H}-\partial_\tau(v_{j\tau}\cdot\nabla_j\overset{1,k}{\tilde H})-\bar\tau\cdot\nabla(v_j)\bar\tau\cdot\nabla\nabla_j\overset{1,k}{\tilde H}\\
&\quad+v_j\nabla_j(\tau_l\tau_j)\nabla_l\nabla_i\overset{1,k}{\tilde H})-\frac{1}{\sigma_1}(\overset{1,k}{A}+\overset{1,k}{B})+\mu\partial_\tau^2(\overset{1}{H}\overset{1}{\zeta}_{kt}+v\cdot\nabla\overset{1}{\zeta}_k\overset{1}{H}+\overset{1,k}{\tilde H}\cdot\nabla v)\\
&\quad+\frac{1}{\sigma_1}\operatorname{rot}(\nabla\overset{1}{\zeta}_k\times\overset{1}{H})+\frac{1}{\sigma_1}\nabla\overset{1}{\zeta}_k\times\operatorname{rot}\overset{1}{H}.
\end{aligned}
\tag{4.42}
$$

Multiplying (4.42) by $\overset{1,k}{\tilde H}_{\tau\tau}$ and integrating over $\overset{1}{\hat\Omega}_k$ we get

$$
\begin{aligned}
&\mu\frac{d}{dt}\int\limits_{\overset{1}{\hat\Omega}_k}|\overset{1,k}{\tilde H}_{\tau\tau}|^2\,dx+\frac{1}{\sigma_1}\int\limits_{\overset{1}{\hat\Omega}_k}\operatorname{rot}\operatorname{rot}\overset{1,k}{\tilde H}_{\tau\tau}\cdot\overset{1,k}{\tilde H}_{\tau\tau}\,dx\\
&\qquad=\int\limits_{\overset{1}{\hat\Omega}_k}[\bar\tau_t\cdot\nabla(\bar\tau)\cdot\nabla\overset{1,k}{\tilde H}-\bar\tau\cdot\nabla(\bar\tau_t)\cdot\nabla\overset{1,k}{\tilde H}\\
&\qquad\quad-\bar\tau\cdot\nabla(v_j)\bar\tau\cdot\nabla\nabla_j\overset{1,k}{\tilde H}+v_j\nabla_j(\tau_l\tau_i)\nabla_l\nabla_i\overset{1,k}{\tilde H}]\cdot\overset{1,k}{\tilde H}_{\tau\tau}\,dx\\
&\qquad\quad+\int\limits_{\overset{1}{\hat\Omega}_k}\partial_\tau(\bar\tau_t\cdot\nabla\overset{1,k}{\tilde H}-v_{j\tau}\nabla_j\overset{1,k}{\tilde H})\cdot\overset{1,k}{\tilde H}_{\tau\tau}\,dx-\int\limits_{\overset{1}{\hat\Omega}_k}\frac{1}{\sigma_1}(\overset{1,k}{A}+\overset{1,k}{B})\cdot\overset{1,k}{\tilde H}_{\tau\tau}\,dx\\
&\qquad\quad+\int\limits_{\overset{1}{\hat\Omega}_k}\partial_\tau^2(\overset{1}{H}\overset{1}{\zeta}_{kt}+v\cdot\nabla\overset{1}{\zeta}_k\overset{1}{H}+\overset{1,k}{\tilde H}\cdot\nabla v\\
&\qquad\quad+\frac{1}{\sigma_1}(\operatorname{rot}(\nabla\overset{1}{\zeta}_k\times\overset{1}{H})+\nabla\overset{1}{\zeta}_k\times\operatorname{rot}\overset{1}{H}))\cdot\overset{1,k}{\tilde H}_{\tau\tau}\,dx.
\end{aligned}
\tag{4.43}
$$

Now we estimate the terms from the r.h.s. of (4.43). The first term is estimated by

$$\varepsilon_1\|\overset{1,k}{\tilde H}_{\tau\tau}\|^2_{H^1(\overset{1}{\hat\Omega}_k)}+c(1/\varepsilon)\|v\|^2_{H^2(\overset{1}{\hat\Omega}_k)}\|\overset{1,k}{\tilde H}\|^2_{H^2(\overset{1}{\hat\Omega}_k)}.$$

Integrating by parts in the second term we can bound it by

$$\varepsilon_2\|\overset{1,k}{\tilde H}_{\tau\tau\tau}\|^2_{L_2(\overset{1}{\hat\Omega}_k)}+c(1/\varepsilon_2)\|v\|^2_{H^2(\overset{1}{\hat\Omega}_k)}\|\overset{1,k}{\tilde H}\|^2_{H^2(\overset{1}{\hat\Omega}_k)}.$$

We estimate the third term by

$$\varepsilon_3\|\overset{1,k}{\tilde H}\|^2_{L_2(\overset{1}{\hat\Omega}_k)}+c(1/\varepsilon_3)\|\overset{1,k}{\tilde H}\|^2_{H^2(\overset{1}{\hat\Omega}_k)}.$$

Integrating by parts in the last term we can bound it by

$$\varepsilon_4\|\overset{1,k}{\tilde H}_{\tau\tau\tau}\|^2_{L_2(\overset{1}{\hat\Omega}_k)}+c(1/\varepsilon_4)\big(\|v\|^2_{H^2(\overset{1}{\hat\Omega}_k)}(\|\overset{1}{H}\|^2_{H^1(\overset{1}{\hat\Omega}_k)}+\|\overset{1,k}{\tilde H}\|^2_{H^1(\overset{1}{\hat\Omega}_k)})+\|\overset{1}{H}\|^2_{H^2(\overset{1}{\hat\Omega}_k)}\big).$$

Using (4.41), from (4.26) we obtain

$$
\begin{aligned}
(4.44)\qquad \mu(\overset{2,k}{\tilde H}_{\tau\tau t}+v'\cdot\nabla\overset{2,k}{\tilde H}_{\tau\tau})+\frac{1}{\sigma_2}\operatorname{rot}\operatorname{rot}\overset{2,k}{\tilde H}_{\tau\tau}&=\mu[\bar\tau_t\cdot\nabla(\bar\tau)\cdot\nabla\overset{2,k}{\tilde H}\\
&+\bar\tau\cdot\nabla(\bar\tau_t\cdot\nabla\overset{2,k}{\tilde H})-\bar\tau\cdot\nabla(\bar\tau_t)\cdot\nabla\overset{2,k}{\tilde H}-\partial_\tau(v'_{j\tau}\cdot\nabla_j\overset{2,k}{\tilde H})\\
&-\bar\tau\cdot\nabla(v'_j)\bar\tau\cdot\nabla\nabla_j\overset{2,k}{\tilde H}+v'_j\nabla_j(\tau_l\tau_i)\nabla_l\nabla_i\overset{2,k}{\tilde H}]-\frac{1}{\sigma_2}(\overset{2,k}{A}+\overset{2,k}{B})\\
&+\mu\partial_\tau^2(\overset{k}{\tilde G}+\overset{2}{H}\overset{2}{\zeta}_{kt}+v'\cdot\nabla\overset{2}{\zeta}_k\overset{2}{H}+v'\cdot\nabla\overset{2}{H}\overset{2}{\zeta}_k)\\
&+\frac{1}{\sigma_2}\operatorname{rot}(\nabla\overset{2}{\zeta}_k\times\overset{2}{H})+\frac{1}{\sigma_2}\nabla\overset{2}{\zeta}_k\times\operatorname{rot}\overset{2}{H}.
\end{aligned}
$$

Multiplying (4.44) by $\overset{2,k}{\tilde H}_{\tau\tau}$ and integrating over $\overset{2}{\hat\Omega}_k$ we obtain

$$
\begin{aligned}
(4.45)\qquad &\mu\frac{d}{dt}\int_{\overset{2}{\hat\Omega}_k}|\overset{2,k}{\tilde H}_{\tau\tau}|^2\,dx+\frac{1}{\sigma_2}\int_{\overset{2}{\hat\Omega}_k}\operatorname{rot}\operatorname{rot}\overset{2,k}{\tilde H}_{\tau\tau}\cdot\overset{2,k}{\tilde H}_{\tau\tau}\,dx\\
&=\int_{\overset{2}{\hat\Omega}_k}[\bar\tau_t\cdot\nabla(\bar\tau)\cdot\nabla\overset{2,k}{\tilde H}-\bar\tau\cdot\nabla(\bar\tau_t)\cdot\nabla\overset{2,k}{\tilde H}-\bar\tau(v'_j)\bar\tau\cdot\nabla\nabla_j\overset{2,k}{\tilde H}\\
&\quad+v'_j\cdot\nabla_j(\tau_l\tau_i)\nabla_l\nabla_i\overset{2,k}{\tilde H}]\cdot\overset{2,k}{\tilde H}_{\tau\tau}\,dx\\
&\quad+\int_{\overset{2}{\hat\Omega}_k}\partial_\tau(\bar\tau_t\cdot\nabla\overset{2,k}{\tilde H}-v'_{j\tau}\nabla_j\overset{2,k}{\tilde H})\cdot\overset{2,k}{\tilde H}_{\tau\tau}\,dx\\
&\quad-\int_{\overset{2}{\hat\Omega}_k}\frac{1}{\sigma_1}(\overset{2,k}{A}+\overset{2,k}{B})\cdot\overset{2,k}{\tilde H}_{\tau\tau}\,dx+\int_{\overset{2}{\hat\Omega}_k}\partial_\tau^2\big(\overset{k}{\tilde G}+\overset{2}{H}\overset{2}{\zeta}_{kt}+v'\cdot\nabla\overset{2}{\zeta}_k\overset{2}{H}\\
&\quad+v'\cdot\nabla\overset{2}{H}\overset{2}{\zeta}_k+v'\cdot\nabla\overset{2}{H}\overset{2}{\zeta}_k+\frac{1}{\sigma_2}(\operatorname{rot}(\nabla\overset{2}{\zeta}_k\times\overset{2}{H})+\nabla\overset{2}{\zeta}_k\times\operatorname{rot}\overset{2}{H})\big)\cdot\overset{2,k}{\tilde H}_{\tau\tau}\,dx.
\end{aligned}
$$

All terms of the r.h.s. in (4.45) can be estimated in a similar way to (4.43).

Summing the above inequalities we obtain

$$
\sum_{i=1}^{2}\bigg(\mu\frac{d}{dt}\|\overset{i,k}{\tilde H}_{\tau\tau}\|^2_{L_2(\overset{i}{\hat\Omega}_k)}+\frac{1}{\sigma_i}\int\limits_{\overset{i}{\hat\Omega}_k}\operatorname{rot}\operatorname{rot}\overset{i,k}{\tilde H}_{\tau\tau}\cdot\overset{i,k}{\tilde H}_{\tau\tau}\,dx\bigg)
\tag{4.46}
$$

$$
\begin{aligned}
&\le\sum_{i=1}^{2}[\varepsilon\|\overset{i,k}{\tilde H}\|^2_{H^3(\overset{i}{\hat\Omega}_k)}+c(1/\varepsilon)(\|\overset{i}{v}\|^2_{H^2(\overset{i}{\hat\Omega}_k)}(\|\overset{i,k}{\tilde H}\|^2_{H^2(\overset{i}{\hat\Omega}_k)}+\|\overset{i}{H}\|^2_{H^2(\overset{i}{\hat\Omega}_k)})\\
&\quad+\|\overset{i}{H}\|^2_{H^2(\overset{i}{\hat\Omega}_k)})]+\|\overset{k}{\tilde G}_{\tau\tau}\|^2_{L_2(\overset{2}{\hat\Omega}_k)}.
\end{aligned}
$$

Next we examine the last two terms on the l.h.s. of (4.46). Integrating by parts they are equal to

$$
\sum_{i=1}^{2}\frac{1}{\sigma_i}\bigg(\int\limits_{\overset{i}{\hat\Omega}_k}|\operatorname{rot}\overset{i,k}{H}_{\tau\tau}|^2\,dx-\int\limits_{S_k}\operatorname{rot}\overset{i,k}{\tilde H}_{\tau\tau}\cdot\overset{i}{\bar n}\times\overset{i,k}{\tilde H}\,dS_t\bigg).
$$

Employing the transmission conditions we get

$$
\bigg|\sum_{i=1}^{2}\frac{1}{\sigma_i}\int\limits_{S_k}(\operatorname{rot}\overset{i,k}{\tilde H}\cdot\bar\tau_\mu\bar\tau_\mu+\operatorname{rot}\overset{i,k}{\tilde H}\cdot\overset{i}{\bar n}\overset{i}{\bar n})_{\tau\tau}\cdot\overset{i}{\bar n}\times\overset{i,k}{\tilde H}_{\tau\tau}\,dS_t\bigg|
\tag{4.47}
$$

$$
\begin{aligned}
&=\bigg|\sum_{i=1}^{2}\frac{1}{\sigma_i}\int\limits_{S_k}(\operatorname{rot}\overset{i,k}{\tilde H}\cdot\bar\tau_\mu)_{\tau\tau}\bar\tau_\mu\cdot\overset{i}{\bar n}\times\overset{i,k}{\tilde H}_{\tau\tau}\,dS_t\bigg|\le\bigg|\int\limits_{S_k}(v\times\overset{1,k}{\tilde H})_{\tau\tau}\bar\tau_\mu\cdot\overset{1}{\bar n}\times\overset{1,k}{\tilde H}_{\tau\tau}\,dS_t\bigg|\\
&\quad+\bigg|\sum_{i=1}^{2}\frac{1}{\sigma_i}\int\limits_{S_k}[(\operatorname{rot}\overset{i,k}{\tilde H}\cdot\bar\tau_\mu)_\tau\bar\tau_{\mu\tau}+(\operatorname{rot}\overset{i,k}{\tilde H}\cdot\overset{i}{\bar n})_\tau\overset{i}{\bar n}_\tau+\operatorname{rot}\overset{i,k}{\tilde H}\cdot\overset{i}{\bar n}\overset{i}{\bar n}_{\tau\tau}]\cdot\overset{i}{\bar n}\times\overset{i,k}{\tilde H}_{\tau\tau}\,dS_t\bigg|\\
&\le\sum_{i=1}^{2}[\varepsilon\|\overset{i,k}{\tilde H}_{\tau\tau}\|^2_{H^1(\overset{i}{\hat\Omega}_k)}+\varepsilon\|\overset{i,k}{\tilde H}_\tau\|^2_{H^2(\overset{i}{\hat\Omega}_k)}]\\
&\quad+c(1/\varepsilon)\|\overset{i,k}{\tilde H}\|^2_{L_2(\overset{i}{\hat\Omega}_k)}+c(1/\varepsilon)(\|v_{\tau\tau}\|^2_{H^1(\overset{1}{\hat\Omega}_k)}\|\overset{1,k}{\tilde H}\|_{H^1(\overset{1}{\hat\Omega}_k)}\\
&\quad+\|v_\tau\|^2_{H^1(\overset{1}{\hat\Omega}_k)}\|\overset{1,k}{\tilde H}_\tau\|^2_{H^1(\overset{1}{\hat\Omega}_k)}+\|v\|^2_{H^1(\overset{1}{\hat\Omega}_k)}\|\overset{1,k}{\tilde H}_{\tau\tau}\|^2_{H^1(\overset{1}{\hat\Omega}_k)}).
\end{aligned}
$$

From (4.32) and (4.33) we have

$$
\begin{aligned}
\sum_{i=1}^{2}\|\overset{i,k}{\tilde H}_{nn}\|^2_{H^1(\overset{i}{\hat\Omega}_k)}&\le c\sum_{i=1}^{2}(\|\overset{i,k}{\tilde H}_{\tau\tau}\|^2_{H^1(\overset{i}{\hat\Omega}_k)}+\|\overset{i,k}{\tilde H}_t\|^2_{H^1(\overset{i}{\hat\Omega}_k)}\\
&\quad+\|\overset{i}{H}\|^2_{H^2(\overset{i}{\hat\Omega}_k)}+\|\overset{i}{v}\|^2_{H^2(\overset{i}{\hat\Omega}_k)}\|\overset{i}{H}\|^2_{H^2(\overset{i}{\hat\Omega}_k)})+c\|\overset{k}{\tilde G}\|^2_{H^1(\overset{2}{\hat\Omega}_k)}.
\end{aligned}
\tag{4.48}
$$

To have all second derivatives under the time derivative we have to repeat the considerations from (4.37).

Therefore, we examine the expression

$$(4.49)\quad J' = \frac{d}{dt}\sum_{i=1}^{2}\|\overset{i}{H}_{xx}\|^2_{L_2(\overset{i}{\Omega}_t)} = \sum_{i=1}^{2}\frac{d}{dt}\int\limits_{\overset{i}{\Omega}_t}\overset{i}{H}{}^2_{xx}\,dx = \sum_{i=1}^{2}\frac{d}{dt}\int\limits_{\overset{i}{\Omega}}(\overset{i}{H}{}^2_{xx})(x(\xi,t),t)\,d\xi$$
$$= \sum_{i=1}^{2}\int\limits_{\overset{i}{\Omega}}[(\overset{i}{H}{}^2_{xx})_x(x(\xi,t),t)\overset{i}{v} + (\overset{i}{H}{}^2_{xx})_t]\,d\xi$$
$$= \sum_{i=1}^{2}\int\limits_{\overset{i}{\Omega}_t}(\overset{i}{H}_{xx}\overset{i}{H}_{xxx}\overset{i}{v} + \overset{i}{H}_{xx}\overset{i}{H}_{xxt})\,dx \le \varepsilon\sum_{i=1}^{2}(\|\overset{i}{H}_{xxx}\|^2_{L_2(\overset{i}{\Omega}_t)} + \|\overset{i}{H}_{xxt}\|^2_{L_2(\overset{i}{\Omega}_t)})$$
$$+ c(1/\varepsilon)\sum_{i=1}^{2}(\|\overset{i}{v}\|^2_{L_\infty(\overset{i}{\Omega}_t)}\|\overset{i}{H}_{xx}\|^2_{L_2(\overset{i}{\Omega}_t)} + \|\overset{i}{H}_{xx}\|^2_{L_2(\overset{i}{\Omega}_t)}).$$

Repeating the considerations for interior subdomains we obtain

$$(4.50)\quad \sum_{i=1}^{2}\Big(\frac{d}{dt}\|\overset{i,k}{\tilde{H}}_{xx}\|^2_{L_2(\overset{i}{\Omega}_k)} + \|\overset{i,k}{\tilde{H}}\|^2_{H^3(\overset{i}{\Omega}_k)}\Big)$$
$$\le c\sum_{i=1}^{2}[\|\overset{i,k}{\tilde{H}}_t\|^2_{H^1(\overset{i}{\Omega}_k)} + \|\overset{i}{v}_\tau\|^2_{H^2(\overset{i}{\Omega}_k)}(\|\overset{i}{H}\|^2_{H^2(\overset{i}{\Omega}_k)} + \|\overset{i,k}{\tilde{H}}\|^2_{H^2(\overset{i}{\Omega}_k)})$$
$$+ \|\overset{i}{H}\|^2_{H^2(\overset{i}{\Omega}_k)} + \|\overset{i,k}{\tilde{H}}\|_{H^2(\overset{1}{\Omega}_k)} + \|\overset{i}{v}\|^2_{H^2(\overset{i}{\Omega}_k)}(\|\overset{i}{H}_\tau\|^2_{H^2(\overset{i}{\Omega}_k)} + \|\overset{i}{H}_\tau\|^2_{H^2(\overset{i}{\Omega}_k)})] + c\|\overset{k}{\tilde{G}}\|^2_{H^2(\overset{2}{\Omega}_k)},$$

where $k \in \mathcal{M}$.

Summing (4.46)–(4.50) over all subdomains of the partition of unity and using (4.49) we obtain

$$(4.51)\quad \frac{d}{dt}X_{22}(t) + \sum_{i=1}^{2}\|\overset{i}{H}\|^2_{H^3(\overset{i}{\Omega}_t)}$$
$$\le \varepsilon\sum_{i=1}^{2}\|\overset{i}{H}_{xxt}\|^2_{L_2(\overset{i}{\Omega}_t)} + c(1/\varepsilon)\sum_{i=1}^{2}(\|\overset{i}{v}\|^2_{H^2(\overset{i}{\Omega}_t)}\|\overset{i}{H}\|^2_{H^3(\overset{i}{\Omega}_t)}$$
$$+ \|\overset{i}{v}\|^2_{H^3(\overset{i}{\Omega}_t)}\|\overset{i}{H}\|^2_{H^2(\overset{i}{\Omega}_t)} + \|\overset{i}{H}\|^2_{\Gamma_1^2(\overset{i}{\Omega}_t)}) + c\|G\|^2_{H^2(\overset{2}{\Omega}_t)},$$

where

$$(4.52)\quad X_{22}(t) = \sum_{i=1}^{2}\Big(\sum_{k\in\mathcal{M}}\|\overset{i,k}{\tilde{H}}_{xx}\|^2_{L_2(\overset{i}{\Omega}_k)} + \sum_{k\in\mathcal{N}}\|\overset{i,k}{\tilde{H}}_{\tau\tau}\|^2_{L_2(\overset{i}{\Omega}_k)} + \|\overset{i}{H}_{xx}\|^2_{L_2(\overset{i}{\Omega}_t)}\Big).$$

Then inequality (4.51) implies (4.40). ■

Adding (4.3), (4.9) and (4.27) we derive the inequality

$$(4.53)\quad \frac{d}{dt}\Big[\sum_{i=1}^{2}(\|\overset{i}{H}\|^2_{L_2(\overset{i}{\Omega}_t)} + \|\overset{i}{H}_t\|^2_{L_2(\overset{i}{\Omega}_t)}) + X_{21}\Big] + \sum_{i=1}^{2}\|\overset{i}{H}\|^2_{\Gamma_1^2(\overset{i}{\Omega}_t)}$$
$$\le c\|v\|^2_{\Gamma_1^2(\overset{1}{\Omega}_t)}\sum_{i=1}^{2}\|\overset{i}{H}\|^2_{\Gamma_1^2(\overset{i}{\Omega}_t)} + c(\|G\|^2_{L_2(\overset{2}{\Omega}_t)} + \|G_t\|^2_{L_{6/5}(\overset{2}{\Omega}_t)}).$$

Adding (4.40) and (4.53) gives

$$
\begin{aligned}
(4.54)\qquad &\frac{d}{dt}\Big[\sum_{i=1}^{2}\sigma_i(\|\overset{i}{H}\|^2_{L_2(\overset{i}{\Omega}_t)}+\|\overset{i}{H}_t\|^2_{L_2(\overset{i}{\Omega}_t)})+X_{21}+X_{22}\Big]\\
&\quad+\sum_{i=1}^{2}(\|\overset{i}{H}\|^2_{\Gamma_1^2(\overset{i}{\Omega}_t)}+\|\overset{i}{H}\|^2_{H^3(\overset{i}{\Omega}_t)})\le\varepsilon\sum_{i=1}^{2}\|\overset{i}{H}_{xxt}\|^2_{L_2(\overset{i}{\Omega}_t)}\\
&\quad+c(1/\varepsilon)\|v\|^2_{\Gamma_1^2(\overset{1}{\Omega}_t)}\sum_{i=1}^{2}\|\overset{i}{H}\|^2_{\Gamma_1^2(\overset{i}{\Omega}_t)}+c(\|G\|^2_{L_2(\overset{2}{\Omega}_t)}+\|G_t\|^2_{L_{6/5}(\overset{2}{\Omega}_t)}).
\end{aligned}
$$

LEMMA 4.8. *Assume that*

$$
\sum_{i=1}^{2}(\|\overset{i}{H}_{xtt}\|_{L_2(\overset{i}{\Omega}_t)}+\|\overset{i}{H}\|_{\Gamma_1^2(\overset{i}{\Omega}_t)}+\|\overset{i}{H}\|_{H^3(\overset{i}{\Omega}_t)})+\|v\|_{\Gamma_1^2(\overset{1}{\Omega}_t)}+\|v\|_{H^2(\overset{1}{\Omega}_t)}<\infty,
$$

$G_t\in L_2(\overset{2}{\Omega}_t)$ *and assume the transmission conditions hold on* S_t. *Then*

$$
\begin{aligned}
(4.55)\qquad &\frac{d}{dt}X_{23}+\sum_{i=1}^{2}\|\overset{i}{H}_{xxt}\|^2_{L_2(\overset{i}{\Omega}_t)}\le\varepsilon\sum_{i=1}^{2}\|\overset{i}{H}_{xtt}\|^2_{L_2(\overset{i}{\Omega}_t)}+c\sum_{i=1}^{2}\|\overset{i}{H}\|^2_{\Gamma_1^2(\overset{i}{\Omega}_t)}\\
&\quad+c(1/\varepsilon)\|v\|^2_{\Gamma_1^2(\overset{1}{\Omega}_t)}(1+\|v\|^2_{\Gamma_1^2(\overset{1}{\Omega}_t)})\sum_{i=1}^{2}\|\overset{i}{H}\|^2_{\Gamma_1^2(\overset{i}{\Omega}_t)}\\
&\quad+c\|v\|^2_{H^2(\overset{1}{\Omega}_t)}\sum_{i=1}^{2}\|\overset{i}{H}\|^2_{H^3(\overset{i}{\Omega}_t)}+c\|G_t\|^2_{L_2(\overset{2}{\Omega}_t)},
\end{aligned}
$$

where

$$
X_{23}=\sum_{i=1}^{2}\Big(\|\overset{i}{H}_{xt}\|^2_{L_2(\overset{i}{\Omega}_k)}+\sum_{k\in\mathcal{M}}\|\overset{i,k}{\tilde{H}}_{xt}\|^2_{L_2(\overset{i}{\Omega}_t)}+\sum_{k\in\mathcal{N}}\|\overset{i,k}{\tilde{H}}_{\tau t}\|^2_{L_2(\overset{i}{\hat{\Omega}}_k)}\Big).
$$

Proof. Differentiating (4.25) with respect to τ and t, multiplying by $\overset{1,k}{\tilde{H}}_{\tau t}$, next integrating the result over $\overset{1}{\hat{\Omega}}_k$ yields

$$
\begin{aligned}
(4.56)\qquad &\mu\int_{\overset{1}{\hat{\Omega}}_k}(\overset{1,k}{\tilde{H}}_{\tau tt}+v\cdot\nabla\overset{1,k}{\tilde{H}}_{\tau t})\cdot\overset{1,k}{\tilde{H}}_{\tau t}\,dx+\frac{1}{\sigma_1}\int_{\overset{1}{\hat{\Omega}}_k}\operatorname{rot}\operatorname{rot}\overset{1,k}{\tilde{H}}_{\tau t}\cdot\overset{1,k}{\tilde{H}}_{\tau t}\,dx\\
&=\mu\int_{\overset{1}{\hat{\Omega}}_k}v_t\cdot\nabla\overset{1,k}{\tilde{H}}_{\tau}\cdot\overset{1,k}{\tilde{H}}_{\tau t}\,dx+\int_{\overset{1}{\hat{\Omega}}_k}\partial_t\big(\bar{\tau}_t\cdot\nabla\overset{1,k}{\tilde{H}}-\bar{\tau}\cdot\nabla(v)\cdot\nabla\overset{1,k}{\tilde{H}}\\
&\quad+v\cdot\nabla(\bar{\tau}\cdot\nabla)\overset{1,k}{\tilde{H}}\big)\cdot\overset{1,k}{\tilde{H}}_{\tau t}\,dx+\mu\int_{\overset{1}{\hat{\Omega}}_k}\partial_t\partial_\tau(\overset{1}{H}\overset{1}{\zeta}_{kt}+\overset{1}{H}\cdot\nabla\overset{1}{v}\zeta_k+\overset{1}{H}v\cdot\nabla\overset{1}{\zeta}_k)\cdot\overset{1,k}{\tilde{H}}_{\tau t}\,dx\\
&\quad+\frac{1}{\sigma_1}\int_{\overset{1}{\hat{\Omega}}_k}\partial_t\partial_\tau(\operatorname{rot}(\nabla\overset{1}{\zeta}_k\times\overset{1}{H})+\nabla\overset{1}{\zeta}_k\times\operatorname{rot}\overset{1}{H})\cdot\overset{1,k}{\tilde{H}}_{\tau t}\,dx.
\end{aligned}
$$

Now we estimate the terms from the r.h.s. of (4.56). The first term is estimated by

$$\varepsilon_1\|\overset{1,k}{\tilde H}_{\tau t}\|^2_{H^1(\overset{1}{\hat\Omega}_k)}+c(1/\varepsilon_1)\|v\|^2_{H^2(\overset{1}{\hat\Omega}_k)}\|\overset{1,k}{\tilde H}_{\tau}\|^2_{H^2(\overset{1}{\hat\Omega}_k)}.$$

We estimate the second term by

$$\varepsilon_2\|\overset{1,k}{\tilde H}_{\tau t}\|^2_{H^1(\overset{1}{\hat\Omega}_k)}+c(1/\varepsilon_2)(\|v\|^2_{\Gamma_1^2(\overset{1}{\hat\Omega}_k)}\|\overset{1,k}{\tilde H}\|^2_{\Gamma_1^2(\overset{1}{\hat\Omega}_k)}+\|\overset{1,k}{\tilde H}\|^2_{H^2(\overset{1}{\hat\Omega}_k)}),$$

the third by

$$\varepsilon_3\|\overset{1,k}{\tilde H}_{\tau t}\|^2_{H^1(\overset{1}{\hat\Omega}_k)}+c(1/\varepsilon_3)(\|v\|^2_{\Gamma_1^3(\overset{1}{\hat\Omega}_k)}\|\overset{1}{H}\|^2_{\Gamma_1^2(\overset{1}{\hat\Omega}_k)}+\|v\|^2_{\Gamma_1^2(\overset{1}{\hat\Omega}_k)}\|\overset{1}{H}\|^2_{\Gamma_1^3(\overset{1}{\hat\Omega}_k)}),$$

and the last one by

$$\varepsilon_4\|\overset{1,k}{\tilde H}_{\tau t}\|^2_{H^1(\overset{1}{\hat\Omega}_k)}+c(1/\varepsilon_4)\|\overset{1}{H}\|^2_{\Gamma_1^1(\overset{1}{\hat\Omega}_k)}.$$

Employing the above estimates in (4.56) yields

$$\begin{aligned}&\mu\frac{d}{dt}\|\overset{1,k}{\tilde H}_{\tau t}\|^2_{L_2(\overset{1}{\hat\Omega}_k)}+\frac{1}{\sigma_1}\int_{\overset{1}{\hat\Omega}_k}\operatorname{rot}\operatorname{rot}\overset{1,k}{\tilde H}_{\tau t}\cdot\overset{1,k}{\tilde H}_{\tau t}\,dx\\
&\quad\le\varepsilon\|\overset{1,k}{\tilde H}_{\tau t}\|^2_{H^1(\overset{1}{\hat\Omega}_k)}+c(1/\varepsilon)[\|v\|^2_{\Gamma_1^3(\overset{1}{\hat\Omega}_k)}\|\overset{1}{H}\|^2_{\Gamma_1^2(\overset{1}{\hat\Omega}_t)}+\|v\|^2_{\Gamma_1^2(\overset{1}{\hat\Omega}_k)}(\|\overset{1}{H}\|^2_{\Gamma_1^3(\overset{1}{\hat\Omega}_k)}+\|\overset{1,k}{\tilde H}\|^2_{\Gamma_1^2(\overset{1}{\hat\Omega}_k)})\\
&\qquad+\|\overset{1}{H}\|^2_{\Gamma_1^2(\overset{1}{\hat\Omega}_k)}+\|\overset{1,k}{\tilde H}\|^2_{H^2(\overset{1}{\hat\Omega}_k)}].\end{aligned}$$

After similar considerations in $\overset{2}{\Omega}_t$ we derive

$$\begin{aligned}&\mu\frac{d}{dt}\|\overset{2,k}{\tilde H}_{\tau t}\|^2_{L_2(\overset{2}{\hat\Omega}_k)}+\frac{1}{\sigma_2}\int_{\overset{2}{\hat\Omega}_k}\operatorname{rot}\operatorname{rot}\overset{2,k}{\tilde H}_{\tau t}\cdot\overset{2,k}{\tilde H}_{\tau t}\,dx\\
&\quad\le\varepsilon\|\overset{2,k}{\tilde H}_{\tau t}\|^2_{H^1(\overset{2}{\hat\Omega}_k)}+c(1/\varepsilon)[\|v'\|^2_{\Gamma_1^3(\overset{2}{\hat\Omega}_k)}\|\overset{2}{H}\|^2_{\Gamma_1^2(\overset{2}{\hat\Omega}_k)}+\|v\|^2_{\Gamma_1^2(\overset{2}{\hat\Omega}_k)}(\|\overset{2}{H}\|^2_{\Gamma_1^3(\overset{2}{\hat\Omega}_k)}+\|\overset{2,k}{\tilde H}\|^2_{\Gamma_1^2(\overset{2}{\hat\Omega}_k)})\\
&\qquad+\|\overset{2}{H}\|^2_{\Gamma_1^2(\overset{2}{\hat\Omega}_k)}+\|\overset{2,k}{\tilde H}\|^2_{H^2(\overset{2}{\hat\Omega}_k)}+\|\overset{k}{\tilde G}\|^2_{\Gamma_0^1(\overset{2}{\hat\Omega}_k)}].\end{aligned}\tag{4.57}$$

Next adding (4.56), (4.57) we examine the term

$$\begin{aligned}&\sum_{i=1}^{2}\frac{1}{\sigma_i}\int_{\overset{2}{\hat\Omega}_k}\operatorname{rot}\operatorname{rot}\overset{i,k}{\tilde H}_{\tau t}\cdot\overset{i,k}{\tilde H}_{\tau t}\,dx\\
&\qquad=\sum_{i=1}^{2}\frac{1}{\sigma_i}\int_{\overset{i}{\hat\Omega}_k}|\operatorname{rot}\overset{i,k}{\tilde H}_{\tau t}|^2\,dx+\sum_{i=1}^{2}\frac{1}{\sigma_i}\int_{S_k}\operatorname{rot}\overset{i,k}{\tilde H}_{\tau t}\cdot\bar n\times\overset{i,k}{\tilde H}_{\tau t}\,dS_k.\end{aligned}\tag{4.58}$$

We estimate the last term on the r.h.s. of (4.58). Using the transmission conditions, we obtain

$$
\begin{aligned}
(4.59)\quad &\Big|\sum_{i=1}^2\frac{1}{\sigma_i}\int_{S_k}(\operatorname{rot}\overset{i,k}{\tilde H}\cdot\bar\tau_\mu\bar\tau_\mu+\operatorname{rot}\overset{i,k}{\tilde H}\cdot\overset{i}{\bar n}\overset{i}{\bar n})_{\tau t}\cdot\overset{i}{\bar n}\times\overset{i,k}{\tilde H}_{\tau t}\,dS_k\Big|\\
&=\Big|\sum_{i=1}^2\frac{1}{\sigma_i}\int_{S_k}(\operatorname{rot}\overset{i,k}{\tilde H}\cdot\bar\tau_\mu)_{\tau t}\bar\tau_\mu\cdot\overset{i}{\bar n}\times\overset{i,k}{\tilde H}_{\tau t}dS_k\Big|\le\Big|\int_{S_k}(v\times\overset{1,k}{\tilde H})_{\tau t}\bar\tau_\mu\cdot\overset{1}{\bar n}\times\overset{1,k}{\tilde H}_{\tau t}\,dS_k\Big|\\
&\quad+\Big|\sum_{i=1}^2\frac{1}{\sigma_i}\int_{S_k}[(\operatorname{rot}\overset{i,k}{\tilde H}\cdot\bar\tau_\mu)_t\bar\tau_{\mu\tau}+(\operatorname{rot}\overset{i,k}{\tilde H}\cdot\bar\tau_\mu)_\tau\bar\tau_{\mu t}+(\operatorname{rot}\overset{i,k}{\tilde H}\cdot\overset{i}{\bar n})_\tau\overset{i}{\bar n}_t\\
&\quad+(\operatorname{rot}\overset{i,k}{\tilde H}\cdot\overset{i}{\bar n})_t\cdot\overset{i}{\bar n}_\tau]\cdot\overset{i}{\bar n}\times\overset{i,k}{\tilde H}_{\tau t}\,dS_k\Big|\\
&\le\sum_{i=1}^2[\varepsilon\|\overset{i,k}{\tilde H}_{\tau t}\|^2_{H^1(\overset{i}{\hat\Omega}_k)}+c(1/\varepsilon)(\|\overset{i}{v}_t\|^2_{H^2(\overset{i}{\hat\Omega}_k)}\|\overset{i,k}{\tilde H}\|^2_{H^2(\overset{i}{\hat\Omega}_k)}+\|\overset{i}{v}\|^2_{H^2(\overset{i}{\hat\Omega}_k)}\|\overset{i,k}{\tilde H}_t\|^2_{H^2(\overset{i}{\hat\Omega}_k)}\\
&\quad+\|\overset{i,k}{\tilde H}_t\|^2_{L_2(\overset{i}{\hat\Omega}_k)})].
\end{aligned}
$$

In virtue of (4.56)–(4.59) we obtain the inequality

$$
\begin{aligned}
(4.60)\quad &\sum_{i=1}^2\bigg(\frac{d}{dt}\|\overset{i,k}{\tilde H}_{\tau t}\|^2_{L_2(\overset{i}{\hat\Omega}_k)}+\|\overset{i,k}{\tilde H}_{\tau t}\|^2_{H^1(\overset{i}{\hat\Omega}_k)}\bigg)\\
&\le c\sum_{i=1}^2[(\|\overset{i}{v}\|^2_{\Gamma_1^3(\overset{i}{\hat\Omega}_k)}(\|\overset{i}{H}\|^2_{\Gamma_1^2(\overset{i}{\hat\Omega}_k)}+\|\overset{i,k}{\tilde H}\|^2_{\Gamma_1^2(\overset{i}{\hat\Omega}_k)})+\|\overset{i}{v}\|^2_{\Gamma_1^2(\overset{i}{\hat\Omega}_k)}(\|\overset{i}{H}\|^2_{\Gamma_1^3(\overset{i}{\hat\Omega}_k)}+\|\overset{i,k}{\tilde H}\|^2_{\Gamma_1^2(\overset{i}{\hat\Omega}_k)})\\
&\quad+\|\overset{i}{H}\|^2_{\Gamma_1^2(\overset{i}{\hat\Omega}_k)}+\|\overset{i,k}{\tilde H}\|^2_{H^2(\overset{i}{\hat\Omega}_k)})]+\|\overset{k}{\tilde G}\|^2_{\Gamma_0^1(\overset{2}{\hat\Omega}_k)}=A_k^2(t).
\end{aligned}
$$

From (4.32) and (4.33) we have

$$
\begin{aligned}
(4.61)\quad \sum_{i=1}^2\|\overset{i,k}{\tilde H}_{nnt}\|^2_{L_2(\overset{i}{\hat\Omega}_k)}&\le c\sum_{i=1}^2\Big(\|\overset{i,k}{\tilde H}_{\tau\tau t}\|^2_{L_2(\overset{i}{\hat\Omega}_k)}\\
&\quad+\|\overset{1}{v}_t\overset{1}{H}+\overset{1}{v}\overset{1}{H}_t+\overset{1}{H}_t\overset{1}{v}_x+\overset{1}{H}_t\overset{1}{v}_{xt}\|^2_{L_2(\overset{1}{\hat\Omega}_k)}\\
&\quad+\|\overset{i}{v}_{xx}\overset{i,k}{\tilde H}_x+\overset{i,k}{\tilde H}_{xt}+\overset{i,k}{\tilde H}_{tt}+\overset{i}{v}_t\overset{i,k}{\tilde H}_x+\overset{i}{v}_x\overset{i,k}{\tilde H}_t+\overset{i}{v}_{xt}\overset{i,k}{\tilde H}\|^2_{L_2(\overset{i}{\hat\Omega}_k)}\\
&\quad+\sum_{i=1}^2\|\overset{i}{H}_t+\overset{i}{H}_{xt}\|^2_{L_2(\overset{i}{\hat\Omega}_k)}+\|\overset{k}{\tilde G}_t\|^2_{L_2(\overset{2}{\hat\Omega}_k)}\Big)\\
&\le c\sum_{i=1}^2\Big(\|\overset{i,k}{\tilde H}_{\tau\tau t}\|^2_{L_2(\overset{i}{\hat\Omega}_k)}+\|\overset{i,k}{\tilde H}\|^2_{\Gamma_0^2(\overset{i}{\hat\Omega}_k)}+\|\overset{i}{v}\|^2_{H^2(\overset{i}{\hat\Omega}_k)}\|\overset{i,k}{\tilde H}\|^2_{H^3(\overset{i}{\hat\Omega}_k)}\\
&\quad+\|\overset{i}{v}\|^2_{\Gamma_1^2(\overset{i}{\hat\Omega}_k)}\|\overset{i,k}{\tilde H}\|^2_{\Gamma_1^2(\overset{i}{\hat\Omega}_k)}+\sum_{i=1}^2\|\overset{i}{H}_t\|^2_{H^1(\overset{i}{\hat\Omega}_k)}+\|\overset{1}{v}\|^2_{\Gamma_1^2(\overset{1}{\hat\Omega}_k)}\|\overset{1}{H}\|^2_{\Gamma_1^2(\overset{1}{\hat\Omega}_k)}\Big)\\
&\quad+c\|\overset{k}{\tilde G}_t\|^2_{H^1(\overset{2}{\hat\Omega}_k)}.
\end{aligned}
$$

From (4.60), (4.61) we get

$$\frac{d}{dt}\sum_{i=1}^{2}\|\overset{i,k}{\tilde H}_{\tau t}\|^2_{L_2(\overset{i}{\hat\Omega}_k)}+\sum_{i=1}^{2}\|\overset{i,k}{\tilde H}_{t}\|^2_{H^2(\overset{i}{\hat\Omega}_k)}\le A_k^2(t). \tag{4.62}$$

To have all space derivatives under the time derivative we consider the expression

$$\begin{aligned}J&=\frac{d}{dt}\sum_{i=1}^{2}\frac{\sigma_i}{2}\|\overset{i}{H}_{xt}\|^2_{L_2(\overset{i}{\Omega}_t)}=\sum_{i=1}^{2}\sigma_i\frac{d}{dt}\int_{\overset{i}{\Omega}_t}\overset{i}{H}{}^2_{xt}\,dx=\sum_{i=1}^{2}\frac{\sigma_i}{2}\frac{d}{dt}\int_{\overset{i}{\Omega}}\overset{i}{H}{}^2_{xt}(x(\xi,t),t)\,d\xi\\
&=2\sum_{i=1}^{2}\frac{\sigma_i}{2}\int_{\overset{i}{\Omega}}(\overset{i}{H}_{xxt}\overset{i}{H}_{xt}\overset{i}{v}+\overset{i}{H}_{xtt}\overset{i}{H}_{xt})(x(\xi,t),t)\,d\xi\\
&=2\sum_{i=1}^{2}\frac{\sigma_i}{2}\int_{\overset{i}{\Omega}_t}(\overset{i}{H}_{xxt}\overset{i}{H}_{xt}\overset{i}{v}+\overset{i}{H}_{xtt}\overset{i}{H}_{xt})\,dx\\
&\le\varepsilon\sum_{i=1}^{2}(\|\overset{i}{H}_{xxt}\|^2_{L_2(\overset{i}{\Omega}_t)}+\|\overset{i}{H}_{xtt}\|^2_{L_2(\overset{i}{\Omega}_t)})\\
&\quad+c(1/\varepsilon)\sum_{i=1}^{2}(\|\overset{i}{v}\|^2_{H^2(\overset{i}{\Omega}_t)}\|\overset{i}{H}_{xt}\|^2_{L_2(\overset{i}{\Omega}_t)}+\|\overset{i}{H}_{xt}\|^2_{L_2(\overset{i}{\Omega}_t)}).\end{aligned}\tag{4.63}$$

For interior subdomains, so for $k\in\mathcal M$, we obtain instead of (4.62) the inequality

$$\frac{d}{dt}\sum_{i=1}^{2}\frac{\sigma_i}{2}\|\overset{i,k}{\tilde H}_{xt}\|^2_{L_2(\overset{i}{\Omega}_k)}+\sum_{i=1}^{2}\|\overset{i,k}{\tilde H}_{t}\|^2_{H^2(\overset{i}{\Omega}_k)}\le cA_k^2(t). \tag{4.64}$$

To show (4.64) we do not need the transmission conditions because $\overset{i}{\Omega}_k$, $k\in\mathcal M$, are separated from S_t.

Summing (4.62), (4.64) over all subdomains of the partition of unity, using (4.63) and assuming that ε is sufficiently small we obtain

$$\begin{aligned}&\frac{d}{dt}X_{23}(t)+\sum_{i=1}^{2}\|\overset{i}{H}_{xxt}\|^2_{L_2(\overset{i}{\Omega}_t)}\\
&\quad\le\sum_{i=1}^{2}(\varepsilon\|\overset{i}{H}_t\|^2_{H^2(\overset{i}{\Omega}_t)}+\|\overset{i}{v}\|^2_{\Gamma_1^3(\overset{i}{\Omega}_t)}\|\overset{i}{H}\|^2_{\Gamma_1^2(\overset{i}{\Omega}_t)}+\|\overset{i}{v}\|^2_{\Gamma_1^2(\overset{i}{\Omega}_t)}\|\overset{i}{H}\|^2_{\Gamma_1^3(\overset{i}{\Omega}_t)}\\
&\qquad+\|\overset{i}{H}\|^2_{\Gamma_1^2(\overset{i}{\Omega}_t)}+\|\overset{i}{H}\|^2_{H^2(\overset{i}{\Omega}_t)})+\|G\|^2_{\Gamma_0^1(\overset{2}{\Omega}_t)},\end{aligned}\tag{4.65}$$

where

$$X_{23}(t)=\sum_{i=1}^{2}\Big(\|\overset{i}{H}_{xt}\|^2_{L_2(\overset{i}{\Omega}_t)}+\sum_{k\in\mathcal M}\|\overset{i}{H}_{xt}\|^2_{L_2(\overset{i}{\Omega}_k)}+\sum_{k\in\mathcal N}\|\overset{i}{H}_{\tau t}\|^2_{L_2(\overset{i}{\hat\Omega}_k)}\Big). \tag{4.66}$$

Then inequality (4.65) implies (4.55). ∎

Proof of Theorem 4.1. Adding (4.54) and (4.55) implies

$$(4.67)\quad \frac{d}{dt}\Big(\sum_{i=1}^{2}\|\overset{i}{H}\|^2_{\Gamma_1^2(\overset{i}{\Omega}_t)}+\sum_{j=1}^{3}X_{2j}\Big)+\sum_{i=1}^{2}\|\overset{i}{H}\|^2_{\Gamma_2^3(\overset{i}{\Omega}_t)}$$
$$\le\varepsilon\sum_{i=1}^{2}\|\overset{i}{H}_{xtt}\|^2_{L_2(\overset{i}{\Omega}_t)}+c(1/\varepsilon)\|v\|^2_{\Gamma_1^2(\overset{1}{\Omega}_t)}(1+\|v\|^2_{\Gamma_1^2(\overset{1}{\Omega}_t)})\sum_{i=1}^{2}\|\overset{i}{H}\|^2_{\Gamma_1^2(\overset{i}{\Omega}_t)}$$
$$+c(1/\varepsilon)\|v\|^2_{H^2(\overset{1}{\Omega}_t)}\sum_{i=1}^{2}\|\overset{i}{H}\|^2_{H^3(\overset{i}{\Omega}_t)}+c\|G\|^2_{\Gamma_0^1(\overset{2}{\Omega}_t)}.$$

Finally, adding (4.9) and (4.17), assuming that ε is sufficiently small we obtain

$$(4.68)\quad \frac{d}{dt}\Big(\sum_{i=1}^{2}\|\overset{i}{H}\|^2_{\Gamma_0^2(\overset{i}{\Omega}_t)}+\sum_{j=1}^{3}X_{2j}\Big)+\sum_{i=1}^{2}\|\overset{i}{H}\|^2_{\Gamma_1^3(\overset{i}{\Omega}_t)}$$
$$\le c\|v\|^2_{\Gamma_1^2(\overset{1}{\Omega}_t)}(1+\|v\|^2_{\Gamma_1^2(\overset{1}{\Omega}_t)})\sum_{i=1}^{2}\|\overset{i}{H}\|^2_{\Gamma_1^2(\overset{i}{\Omega}_t)}$$
$$+c\|v\|^2_{\Gamma_0^2(\overset{1}{\Omega}_t)}\sum_{i=1}^{2}\|\overset{i}{H}\|^2_{\Gamma_1^3(\overset{i}{\Omega}_t)}+c(\|G\|^2_{\Gamma_0^1(\overset{2}{\Omega}_t)}+\|G_{tt}\|^2_{L_{6/5}(\overset{2}{\Omega}_t)}).$$

The above inequality implies (4.1). ■

REMARK 4.9. Let ξ be the lagrangian coordinate and x the eulerian one. They are connected by the problem

$$(4.69)\quad \frac{dx}{dt}=v(x,t),\quad x|_{t=0}=\xi.$$

Solving (4.69) we get

$$(4.70)\quad x=\xi+\int_0^t v(x,t')\,dt'=\xi+\int_0^t\bar v(\xi,t')\,dt'\equiv x_1(\xi,t),\quad \xi\in\overset{1}{\Omega}.$$

Hence, we have

$$(4.71)\quad \overset{1}{\Omega}_t=\{x=x_1(\xi,t):\xi\in\overset{1}{\Omega}\},\quad S_t=\{x=x_1(\xi,t)=x_2(\xi,t):\xi\in S\},$$

because $v=v'$ on S_t,

$$(4.72)\quad \overset{2}{\Omega}_t=\{x=x_2(\xi,t):\xi\in\overset{2}{\Omega}\}.$$

In the case (4.72) instead of (4.70) we have

$$(4.73)\quad x=\xi+\int_0^t\bar v'(\xi,t')\,dt'\equiv x_2(\xi,t),\quad \xi\in\overset{2}{\Omega}.$$

By $\{x_\xi\}$ we denote the Jacobi matrix of transformations (4.70) and (4.73). Let $J=\det\{x_\xi\}$ be the Jacobian of the transformations. Then

$$(4.74)\quad \frac{dJ}{dt}=J\operatorname{div}v.$$

For the divergence free vector v we have $J(t)=1$.

Let Ω_t replace either $\overset{1}{\Omega}_t$ or $\overset{2}{\Omega}_t$. Then

$$
\begin{aligned}
(4.75)\qquad \frac{d}{dt}\int_{\Omega_t} f(x,t)\,dx &= \frac{d}{dt}\int_{\Omega} f(x(\xi,t),t)J\,d\xi \\
&= \int_{\Omega}\Big(\frac{\partial f}{\partial t} + v\cdot\nabla f + f\operatorname{div} v\Big)J\,d\xi = \int_{\Omega_t}(f_t(x,t) + v\cdot\nabla f + f\operatorname{div} v)\,dx.
\end{aligned}
$$

5. Global existence

From Lemma 3.1 and Theorem 4.1 we get

Lemma 5.1. *For a sufficiently smooth solution* $(v,p,\overset{1}{H},\overset{2}{H})$ *of* (1.1)–(1.10) *we have*

$$
\begin{aligned}
(5.1)\qquad \frac{d}{dt}\varphi + \phi \le c\Big(&\varphi(1+\varphi+\varphi^2)\phi + \|f\|^4_{L_2(\overset{1}{\Omega}_t)} + \|f\|^2_{\Gamma^2_0(\overset{1}{\Omega}_t)} \\
&+ \Big|\int_0^t\int_{\overset{1}{\Omega}_{t'}} f(x,t')\,dx\,dt'\Big|^2 + \Big|\int_{\overset{1}{\Omega}} v(0)\,dx\Big|^2 + \|G\|^2_{\Gamma^1_0(\overset{2}{\Omega}_t)} + \|G_{tt}\|^2_{L_{6/5}(\overset{2}{\Omega}_t)}\Big),
\end{aligned}
$$

where

$$
(5.2)\qquad
\begin{aligned}
\varphi(t) &= \|v\|^2_{\Gamma^2_0(\overset{1}{\Omega}_t)} + \sum_{i=1}^2 \|\overset{i}{H}\|^2_{\Gamma^2_0(\overset{i}{\Omega}_t)} + X, \qquad X = \sum_{i=1}^3 X_{2i},\\
\phi(t) &= \|v\|^2_{\Gamma^3_1(\overset{1}{\Omega}_t)} + \sum_{i=1}^2 \|\overset{i}{H}\|^2_{\Gamma^3_1(\overset{2}{\Omega}_t)} + \|p'\|^2_{\Gamma^2_1(\overset{1}{\Omega}_t)}.
\end{aligned}
$$

To prove global existence we introduce the spaces

$$
\mathcal{N}(t) = \{(v,p,\overset{1}{H},\overset{2}{H}) : \varphi(t) < \infty\}, \qquad \mathcal{M}(t) = \Big\{(v,p,\overset{1}{H},\overset{2}{H}) : \varphi(t) + \int_0^t \phi(\tau)\,d\tau < \infty\Big\}.
$$

From Theorem 1 we get

Lemma 5.2. *Assume that* $(v(0),p(0),\overset{1}{H}(0),\overset{2}{H}(0)) \in \mathcal{N}(0)$ *and* $\varphi(0) < \varepsilon_1$. *Then we have* $(v(t),p(t),\overset{1}{H}(t),\overset{2}{H}(t)) \in \mathcal{M}(t)$, $t \le T$, *where* T *is the time of local existence and*

$$
(5.3)\qquad \varphi(t) + \int_0^t \phi(\tau)\,d\tau \le D \equiv c(\varepsilon_1 + \beta(t)),
$$

where D *is given by* (1.17).

Lemma 5.3. *Assume that there exists a local solution to* (1.1), (1.2), (1.10), (1.15) *in* $\mathcal{M}(t)$, $t \le T$, *with initial data in* $\mathcal{N}(0)$ *sufficiently small and*

$$
(5.4)\qquad \alpha(t) = \|G\|^2_{\Gamma^1_0(\overset{2}{\Omega}_t)} + \|G_{tt}\|^2_{L_{6/5}(\overset{2}{\Omega}_t)} \le e^{-\mu t},
$$

$t \le T$ *and* $\mu > 1/2$. *Then*

$$
(5.5)\qquad \varphi(t) \le ce^{-t/2}\Big(\varphi(0) + \frac{1}{\mu - \frac{1}{2}}\Big).
$$

Proof. From (5.1) and (5.4) we get

$$\frac{d}{dt}\varphi + \phi \le c\varphi(1+\varphi+\varphi^2)\phi + ce^{-\mu t}. \tag{5.6}$$

From Lemma 5.2 we have $c\varphi(1+\varphi+\varphi^2)\phi \le \frac{1}{2}\phi$ if α and ε_1 are sufficiently small. Then from (5.4) we get

$$\frac{d}{dt}\varphi + \frac{1}{2}\phi \le ce^{-\mu t}. \tag{5.7}$$

We have $\varphi < \phi$. Then from (5.7),

$$\frac{d}{dt}\varphi + \frac{1}{2}\varphi \le ce^{-\mu t}. \tag{5.8}$$

Inequality (5.8) implies (5.5). ■

LEMMA 5.4. *Let the assumptions of Lemma* 5.3 *be satisfied and* $\varphi(0) \le \varepsilon_1$. *Then* $\varphi(kT) \le \varepsilon_1$, *where* T *is the time of local existence and* $k \in \mathbb{N}$.

Proof. If T and $\mu > 1/2$ are sufficiently large, then from (5.5) we obtain

$$\varphi(T) \le ce^{-T/2}\bigg(\varphi(0) + \frac{1}{\mu - 1/2}\bigg) \le \varphi(0).$$

Now we consider problem (1.1)–(1.10) for $t \in [kT, (k+1)T]$. Then (5.1) implies

$$\frac{d}{dt}\varphi + \phi \le c(\varphi(1+\varphi+\varphi^2)\phi + \|G\|^2_{\Gamma^1_0(\Omega_t)} + \|G_{tt}\|^2_{L_{6/5}(\Omega_t)}), \qquad t \in [kT, (k+1)T]. \tag{5.9}$$

Let $\varphi(kT) \le \varphi(0)$. Then from (5.9) similarly to (5.5) we get, for $t \in [kT, (k+1)T]$

$$\varphi(t) \le \frac{c}{\mu - 1/2}e^{kT(\frac{1}{2}-\mu)-\frac{1}{2}t} + c\varphi(kT)e^{\frac{1}{2}(kT-t)} \tag{5.10}$$

and then

$$\varphi((k+1)T) \le \frac{c}{\mu - 1/2}e^{-T(\frac{1}{2}+\mu k)} + c\varphi(kT)e^{-\frac{1}{2}T} \le e^{-\frac{1}{2}T}\Big(\frac{c}{\mu - 1/2}e^{-T\mu k} + c\varphi(0)\Big) \le \varphi(0)$$

if T, μ are sufficiently large.

From (5.10) we also obtain

$$\varphi(kT) \le e^{-\frac{1}{2}kT}\bigg(\frac{c}{\mu - 1/2}\cdot\frac{e^{T(\mu-1/2)}}{e^{T(\mu-1/2)} - 1} + \varphi(0)\bigg), \qquad k \in \mathbb{N}. \tag{5.11}$$

From (5.11) we have

$$\sum_{i=0}^{\infty}\varphi(iT) \le \frac{e^{-\frac{1}{2}T}}{1 - e^{-\frac{1}{2}T}}\Big(\frac{c}{\mu - 1/2}\cdot\frac{e^{(\mu-1/2)}}{e^{T(\mu-1/2)} - 1} + \varphi(0)\Big) + \varphi(0) \equiv \mathcal{A}. \tag{5.12}$$

From (5.9) we get

$$\varphi(t) + \int_{kT}^{t}\phi\,dt' \le \frac{c}{\mu}e^{-\mu kT} + c\varphi(kT)$$

and then

$$\int_{kT}^{(k+1)T}\phi\,dt \le \frac{c}{\mu}e^{-\mu kT} + c\varphi(kT). \tag{5.13}$$

Finally (5.13) and (5.12) imply

$$\sum_{i=0}^{k-1} \|v\|^2_{L_2(iT,(i+1)T;H^3(\Omega_t))} \le \sum_{i=0}^{k-1} \int\limits_{iT}^{(i+1)T} \phi(t)\,dt \le \sum_{i=0}^{\infty} \left(\frac{c}{\mu} e^{-\mu iT} + \varphi(iT)\right) \le \frac{c}{\mu(1-e^{-\mu T})} + \mathcal{A} \equiv I. \tag{5.14}$$ ∎

Proof of Main Theorem. The theorem is proved step by step using local existence in a fixed time interval. Under the assumption that

$$(v(0), p(0), H(0)) \in \mathcal{N}(0), \tag{5.15}$$

Theorem 1 and Lemma 5.1 yield local existence of solutions of (1.1), (1.2), (1.10), (1.15).

By (5.15) and Lemma 5.1 the local solution belongs to $\mathcal{M}(t)$, $t \le T$. For small ε_1 and $\beta(t)$ the existence time T is suitably large, so we can assume it is a fixed positive large number. To prove the Main Theorem we need the Korn inequalities (see Section 6) and imbedding theorems over the domain Ω_t. The constants in those theorems depend on Ω_t, the shape of S_t and $\int_0^t \|v\|^2_{H^3(\Omega_\tau)}\,d\tau$, so generally they are functions of t.

But in view of (5.3) with sufficiently small ε_1, β we obtain

$$\Big|\int\limits_0^t v\,d\tau\Big| \le c(\varepsilon_1 + \beta(t)), \quad t \in [0,T]. \tag{5.16}$$

Hence from the relation

$$x = \xi + \int\limits_0^t v(x(\xi,\tau),\tau)\,d\tau, \quad \xi \in S, \quad t \le T, \tag{5.17}$$

for sufficiently small ε_1, β and fixed T, the shape of S_t, $t \le T$, does not change too much, so the constants from the imbedding theorems can be chosen independent of time. Now we wish to extend the solution to the interval $[T, 2T]$. Using Lemma 5.4 and (5.9)–(5.13) we can prove the existence of a local solution in $\mathcal{M}(t)$, $T \le t \le 2T$. To prove

$$\varphi(2T) \le \varepsilon_1 \tag{5.18}$$

we need inequality (5.11), where the constants depend on the constants from the imbedding theorems and Korn inequalities for $t \in [T, 2T]$. Therefore we show that the shape of S_t and $\int_0^t \|v\|^2_{H^3(\Omega_\tau)}\,d\tau$, $t \le 2T$, do not change more than for $t \le T$. Then for $k = 2$, ε_1 sufficiently small and μ sufficiently large we see that $\int_0^t v(x(\xi,t),t)dt$ is small for any $t \in [0, 2T]$, so (5.14) imply that the constants of the imbedding theorems do not change more than in $[0,T]$, and then the differential inequality (5.1) can also be shown for this interval with the same constants. Hence in view of Lemma 5.1 the solution of (1.1), (1.2), (1.10), (1.15) belongs to $\mathcal{M}(t)$, $t \in [T, 2T]$. Continuing, we find that there exists a local solution in the interval $[0, kT]$.

Showing that $\int_0^t v(x,\tau)\,d\tau$ is sufficiently small for all t we have $\operatorname{dist}\{S_t, S_0\} \le |\int_0^t v\,d\tau| \le I$, where I defined in (5.14) is small. Hence the domains Ω_t and Ω are close to each other and all the imbedding theorems can be applied and results for elliptic problems (3.15), (3.23) are valid for all Ω_t, $t > 0$.

Repeating the above discussion for the intervals $[kT,(k+1)T]$, $k\ge 2$, we prove the existence for all $t\in\mathbb{R}_+$. ∎

6. Korn inequality

LEMMA 6.1. *Let Ω_t be a bounded domain. Let (v,p) be a solution of* (1.1) *and*

$$E_{\Omega_t}=\int\limits_{\Omega_t}(\partial_{x_i}v_j+\partial_{x_j}v_i)^2\,dx<\infty. \tag{6.1}$$

Then there exists a constant c such that

$$\|v\|^2_{H^1(\Omega_t)}\le c(E_{\Omega_t}(v)+\|v\|^2_{L_2(\Omega_t)}). \tag{6.2}$$

Proof. Set

$$u=\sum_{i=1}^3 b_i\varphi_i(x)+v, \tag{6.3}$$

where

$$\begin{aligned}&\varphi_i=(x-\bar x)\times e_i,\quad e_i=(\delta_{i1},\delta_{i2},\delta_{i3}),\quad i=1,2,3,\\ &\bar x=\frac{1}{|\Omega_t|}\Big(\int\limits_{\Omega_t}x_1\,dx,\int\limits_{\Omega_t}x_2\,dx,\int\limits_{\Omega_t}x_3\,dx\Big).\end{aligned} \tag{6.4}$$

Define $b=(b_1,b_2,b_3)$ by

$$b=\frac{1}{2|\Omega_t|}\int\limits_{\Omega_t}\operatorname{rot}v\,dx. \tag{6.5}$$

Since $\operatorname{rot}\varphi_i=-2e_i$, $i=1,2,3$, equations (6.3), (6.4) imply

$$\int\limits_{\Omega_t}\operatorname{rot}u\,dx=0. \tag{6.6}$$

From (6.4) we have $\int_{\Omega_t}\varphi_i\,dx=0$ for $i=1,2,3$ so

$$\int\limits_{\Omega_t}u\,dx=\int\limits_{\Omega_t}v\,dx\quad\text{and also}\quad E_{\Omega_t}(\varphi_i)=0,\ i=1,2,3, \tag{6.7}$$

so

$$E_{\Omega_t}(u)=E_{\Omega_t}(v). \tag{6.8}$$

By Theorem 1 of [10] we have

$$\partial_{x_j}w_i=\varepsilon_{ikt}\partial_{x_k}S_{jk},\quad i=1,2,3,\quad w=\operatorname{rot}u,\quad S_{ij}=\partial_{x_i}u_j+\partial_{x_j}u_i, \tag{6.9}$$

so (6.6) implies that

$$\|\operatorname{rot}u\|^2_{L_2(\Omega_t)}\le c\sum_{i,j=1}^3\|S_{ij}\|^2_{L_2(\Omega_t)}=cE_{\Omega_t}(u)=E_{\Omega_t}(v). \tag{6.10}$$

Employing the identity

$$\partial_{x_j}u_i=\tfrac12(\partial_{x_j}u_i+\partial_{x_i}u_j)+\tfrac12(\partial_{x_j}u_i-\partial_{x_i}u_j) \tag{6.11}$$

and (6.10) we have

$$\|\nabla u\|^2_{L_2(\Omega_t)} \le c(E_{\Omega_t}(u) + \|\operatorname{rot} u\|^2_{L_2(\Omega_t)}) \le cE_{\Omega_t}(v). \tag{6.12}$$

Using (6.3) we obtain

$$\|\nabla v\|^2_{L_2(\Omega_t)} \le c(E_{\Omega_t}(v) + |b|^2). \tag{6.13}$$

Using (6.3) we get

$$\sum_{i=1}^3 b_i \int\limits_{\Omega_t} \varphi_i \cdot \varphi_j \, dx = \int\limits_{\Omega_t} (u-v)\cdot \varphi_i \, dx. \tag{6.14}$$

Since $\det \Gamma \neq 0$, where $\Gamma = \{\Gamma_{ij}\}$, $\Gamma_{ij} = \int_{\Omega_t} \varphi_i \cdot \varphi_j \, dx$ we can calculate b from (6.14), so

$$|b|^2 \le c(\|u\|^2_{L_2(\Omega_t)} + \|v\|^2_{L_2(\Omega_t)}). \tag{6.15}$$

Now by the Poincaré inequality and (6.7), (6.8), we obtain

$$\begin{aligned} \|u\|^2_{L_2(\Omega_t)} &\le 2\bigg\| u - \frac{1}{|\Omega_t|}\int\limits_{\Omega_t} u\, dx \bigg\|^2_{L_2(\Omega_t)} + 2\bigg\| \frac{1}{|\Omega_t|}\int\limits_{\Omega_t} u\, dx \bigg\|^2_{L_2(\Omega_t)} \\ &\le c\bigg(\|\nabla u\|^2_{L_2(\Omega_t)} + \bigg\| \frac{1}{|\Omega_t|}\int\limits_{\Omega_t} v\, dx \bigg\|^2_{L_2(\Omega_t)} \bigg) \le c(E_{\Omega_t}(v) + \|v\|^2_{L_2(\Omega_t)}), \end{aligned} \tag{6.16}$$

where (6.12) is used. From (6.13), (6.15) and (6.16) we get (6.2). ∎

Lemma 6.2. *Let $\Omega_t \subset \mathbb{R}^3$ be bounded, $v \in H^1(\Omega_t)$ satisfy* (1.1) *and*

$$E_{\Omega_t}(v) = \int\limits_{\Omega_t} (v_{ix_j} + v_{jx_i})^2 \, dx < \infty. \tag{6.17}$$

Then there exists a constant c such that

$$\|v\|^2_{H^1(\Omega_t)} \le c\bigg[E_{\Omega_t}(v) + \bigg| \int_0^t \int\limits_{\Omega} f(x,t')\, dx\, dt' \bigg|^2 + \bigg| \int\limits_{\Omega} v(0)\, dx \bigg|^2 \bigg]. \tag{6.18}$$

Proof. From $(1.1)_1$ we have

$$\int\limits_{\Omega_t} v\, dx = \int_0^t \int\limits_{\Omega_{t'}} f\, dx\, dt' + \int\limits_{\Omega} v_0 \, dx, \tag{6.19}$$

$$\int\limits_{\Omega_t} v\cdot \varphi_i\, dx = \int_0^t \int\limits_{\Omega_{t'}} f\varphi_i\, dx\, dt' + \int\limits_{\Omega} v_0 \cdot \varphi_i \, dx, \quad i = 1,2,3. \tag{6.20}$$

From (6.14) we have

$$\sum_{k=1}^3 b_k \int\limits_{\Omega_t} \varphi_k \cdot \varphi_i \, dx = \int\limits_{\Omega_t} u\cdot \varphi_i\, dx + \int_0^t \int\limits_{\Omega_{t'}} f\varphi_i\, dx\, dt' + \int\limits_{\Omega} v_0 \cdot \varphi_i \, dx. \tag{6.21}$$

Hence

$$|b| \le c\bigg(\|u\|_{L_2(\Omega_t)} + \bigg| \int_0^t \int\limits_{\Omega_{t'}} f\, dx\, dt' \bigg| + \bigg| \int\limits_{\Omega} v(0)\, dx \bigg| \bigg). \tag{6.22}$$

Hence, by (6.13) we have

$$\|\nabla v\|^2_{L_2(\Omega_t)} \le c\Big(E_{\Omega_t}(v) + \|u\|^2_{L_2(\Omega_t)} + \Big|\int_0^t\int_\Omega f\,dx\,dt'\Big|^2 + \Big|\int_\Omega v(0)\,dx\Big|^2\Big). \tag{6.23}$$

From (6.3) and (6.22) we have

$$\|v\|^2_{L_2(\Omega_t)} \le c\Big(\|u\|^2_{L_2(\Omega_t)} + \Big|\int_0^t\int_{\Omega_{t'}} f\,dx\,dt'\Big|^2 + \Big|\int_\Omega v(0)\,dx\Big|^2\Big). \tag{6.24}$$

Employing (6.7) in (6.19) yields

$$\Big|\int_{\Omega_t} u\,dx\Big|^2 \le \Big|\int_0^t\int_{\Omega_{t'}} f\,dx\,dt'\Big|^2 + \Big|\int_\Omega v(0)\,dx\Big|^2. \tag{6.25}$$

Hence

$$\begin{aligned}\|u\|^2_{L_2(\Omega_t)} &\le c\Big\|u - \fint_{\Omega_t} u\,dx\Big\|^2_{L_2(\Omega_t)} + c\Big|\fint_{\Omega_t} u\,dx\Big|^2 \\ &\le c\Big(\|\nabla u\|^2_{L_2(\Omega_t)} + \Big|\int_0^t\int_{\Omega_{t'}} f\,dx\,dt'\Big|^2 + \Big|\int_\Omega v(0)\,dx\Big|^2\Big) \\ &\le c\Big(E_{\Omega_t}(v) + \Big|\int_0^t\int_{\Omega_{t'}} f\,dx\,dt'\Big|^2 + \Big|\int_\Omega v(0)\,dx\Big|^2\Big).\end{aligned} \tag{6.26}$$

From (6.23), (6.24) and (6.26) we obtain (6.18). ∎

From $(1.1)_1$ we have

$$\int_{\Omega_t} v_t\,dx = -\int_{\Omega_t} v\cdot\nabla v\,dx + \int_{\Omega_t} f\,dx, \tag{6.27}$$

$$\int_{\Omega_t} v_t\cdot\varphi_i\,dx = \int_{\Omega_t} v\cdot\nabla v\cdot\varphi_i\,dx + \int_{\Omega_t} f\cdot\varphi_i\,dx. \tag{6.28}$$

Lemma 6.3. *Let $\Omega_t\subset\mathbb{R}^3$ be a bounded domain. Let* (6.27) *and* (6.28) *hold. Let*

$$E_{\Omega_t}(v_t) = \int_{\Omega_t}(v_{jx_it} + v_{ix_jt})^2\,dx < \infty. \tag{6.29}$$

Then there exists a constant c such that

$$\|v_t\|^2_{H^1(\Omega_t)} \le c(E_{\Omega_t}(v_t) + \|v\|^4_{H^1(\Omega_t)} + \|f\|^2_{L_2(\Omega_t)}). \tag{6.30}$$

Proof. Set

$$u = \sum_{i=1}^3 b_i\varphi_i(x) + v_t, \tag{6.31}$$

where φ_i, $i=1,2,3$, are introduced in Lemma 6.1.

Define $b = (b_1, b_2, b_3)$ by

$$b = \frac{1}{2|\Omega_t|} \int_{\Omega_t} \operatorname{rot} v_t \, dx. \tag{6.32}$$

Since $\operatorname{rot} \varphi_i = -2e_i$, $i = 1, 2, 3$, (6.31) implies

$$\int_{\Omega_t} \operatorname{rot} u \, dx = 0. \tag{6.33}$$

From the form of φ_i we have $\int_{\Omega_t} \varphi_i \, dx = 0$. Hence

$$\int_{\Omega_t} u \, dx = \int_{\Omega_t} v_t \, dx \quad \text{and} \quad E_{\Omega_t}(\varphi_i) = 0, \ i = 1, 2, 3. \tag{6.34}$$

Then

$$E_{\Omega_t}(u) = E_{\Omega_t}(v_t). \tag{6.35}$$

As in Lemma 6.1, we have

$$\|\nabla u\|^2_{L_2(\Omega_t)} \le c E_{\Omega_t}(v_t). \tag{6.36}$$

From (6.31) we have

$$\sum_{i=1}^{3} b_i \int_{\Omega_t} \varphi_i \cdot \varphi_j \, dx = \int_{\Omega_t} u \cdot \varphi_j \, dx - \int_{\Omega_t} v_t \cdot \varphi_j \, dx. \tag{6.37}$$

Using (6.28) yields

$$|b|^2 \le c(\|u\|^2_{L_2(\Omega_t)} + \|v\|^4_{H^1(\Omega_t)} + \|f\|^2_{L_2(\Omega_t)}). \tag{6.38}$$

By the Poincaré inequality we have

$$\begin{aligned} \|u\|^2_{L_2(\Omega_t)} &\le 2\Big\| u - \fint_{\Omega_t} u \, dx \Big\|^2_{L_2(\Omega_t)} + 2\Big\| \fint_{\Omega_t} u \, dx \Big\|^2_{L_2(\Omega_t)} \\ &\le c\Big(\|\nabla u\|^2_{L_2(\Omega_t)} + \Big\| \fint_{\Omega_t} v_t \, dx \Big\|^2_{L_2(\Omega_t)} \Big) \\ &\le c(E_{\Omega_t}(v_t) + \|v\|^4_{H^1(\Omega_t)} + \|f\|^2_{L_2(\Omega_t)}). \end{aligned} \tag{6.39}$$

Using

$$\|v_t\|^2_{L_2(\Omega_t)} \le c(\|u\|^2_{L_2(\Omega_t)} + |b|^2), \tag{6.40}$$

we obtain from (6.36), (6.38) and (6.39) inequality (6.30). ∎

From $(1.1)_1$ we have

$$\begin{aligned} \int_{\Omega_t} v_{tt} \, dx = &- \int_{\Omega_t} (v_t \cdot \nabla v + v \cdot \nabla v_t) \, dx + \int_{\Omega_t} (H_t \cdot \nabla H + H \cdot \nabla H_t) \, dx \\ &- \int_{\Omega_t} (H_{it} \nabla H_i + H_i \nabla H_{it}) \, dx + \int_{\Omega_t} f_t \, dx \end{aligned} \tag{6.41}$$

and

$$\int\limits_{\Omega_t} v_{tt}\cdot\varphi_i\,dx = -\int\limits_{\Omega_t}(v_t\cdot\nabla v+v\cdot\nabla v_t)\cdot\varphi_i\,dx + \int\limits_{\Omega_t}(H_t\cdot\nabla H+H\cdot\nabla H_t)\varphi_i\,dx - \int\limits_{\Omega_t}(H_{jt}\cdot\nabla H_j+H_j\cdot\nabla H_{jt})\cdot\varphi_i\,dx + \int\limits_{\Omega_t} f_t\cdot\varphi_i\,dx. \tag{6.42}$$

Lemma 6.4. *Let $\Omega_t\subset\mathbb{R}^3$ be bounded. Let* (6.41) *and* (6.42) *hold. Let*

$$E_{\Omega_t}(v_{tt}) = \int\limits_{\Omega_t}(v_{ittx_j}+v_{jttx_i})^2\,dx<\infty. \tag{6.43}$$

Then

$$\|v_{tt}\|^2_{H^1(\Omega_t)} \le c(E_{\Omega_t}(v_{tt}) + \|v_t\|^2_{H^1(\Omega_t)}\|v\|^2_{H^1(\Omega_t)} + \|H_t\|^2_{H^1(\Omega_t)}\|H\|^2_{H^1(\Omega_t)} + \|f_t\|^2_{L_2(\Omega_t)}). \tag{6.44}$$

Proof. In this case we set

$$u=\sum_{i=1}^3 b_i\varphi_i(x)+v_{tt}. \tag{6.45}$$

Repeating the considerations from Lemma 6.3 we conclude the proof. ∎

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