1. Introduction

We are concerned with the global existence and uniqueness of solutions to a three-dimensional (3-D) system of equations describing thermomechanical evolution of shape memory alloys (SMA).

This system, derived on thermodynamical grounds in [39], is a generalization of the one-dimensional Falk model [18] of displacive phase transitions in SMA. It corresponds to the conservations laws of linear momentum and energy with constitutive relations for stress tensor, internal energy and energy flux accounting for interaction effects on phase interfaces and viscosity. These relations comply with the entropy principle, assuring that the model is thermodynamically consistent.

The governing equations to be considered are given by

$$\mathbf{u}_{tt} - \nu \mathbf{Q}\mathbf{u}_t + \frac{\kappa}{4}\mathbf{Q}\mathbf{Q}\mathbf{u} = \boldsymbol{\nabla} \cdot F_{/\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}, \theta) + \mathbf{b}, \tag{1.1}$$

$$c(\boldsymbol{\epsilon}, \theta)\theta_t - k\Delta\theta = \theta F_{/\theta\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}, \theta) : \boldsymbol{\epsilon}_t + \nu(\mathbf{A}\boldsymbol{\epsilon}_t) : \boldsymbol{\epsilon}_t + g \tag{1.2}$$

in $Q_T = (0,T) \times \Omega$, where

$$c(\boldsymbol{\epsilon}, \theta) = c_v - \theta F_{/\theta\theta}(\boldsymbol{\epsilon}, \theta), \tag{1.3}$$

with appropriate initial and boundary conditions. Here $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, is a bounded domain with a smooth boundary $\partial\Omega$, $\mathbf{u}$ denotes the displacement vector, $\boldsymbol{\epsilon} = (\epsilon_{ij})$ with $\epsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i/j} + u_{j/i})$ is the linearized strain tensor, $\boldsymbol{\epsilon}_t = \boldsymbol{\epsilon}(\mathbf{u}_t)$ is the strain rate tensor, and $\theta > 0$ is the absolute temperature. We use the notation $f_{/i} = \partial f/\partial x_i$, $f_t = \partial f/\partial t$.

The elastic energy density $F(\boldsymbol{\epsilon}, \theta)$ has a multiple-well form as a function of the strain tensor $\boldsymbol{\epsilon}$ (order parameter), with the shape changing qualitatively with the temperature θ. These changes correspond to the fact that the austenitic phase is the global minimizer above a critical temperature, both the austenitic phase and the martensitic variants have equal energy density at the critical temperature, and the martensitic variants are global minimizers below the critical temperature.

As a representative model of $F(\boldsymbol{\epsilon}, \theta)$ we use the Falk–Konopka elastic energy (see [20]) in the form of sixth order polynomial in ϵ_{ij}, expressed in terms of crystallographical invariants, which generalizes the well-known 1-D expression [18], [19]

$$F(\epsilon, \theta) = \alpha_1(\theta - \theta_c)\epsilon^2 - \alpha_2\epsilon^4 + \alpha_3\epsilon^6, \tag{1.4}$$

where $\alpha_i > 0$ are constant parameters and $\theta_c > 0$ is a critical temperature.

In our notation $\mathbf{A}$ is the fourth order tensor representing linear isotropic Hooke's law and $\mathbf{Q}$ stands for the second order differential operator of linearized elasticity admitting the representation

$$\mathbf{Q}\mathbf{u} = \mu \Delta \mathbf{u} + (\lambda + \mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}), \tag{1.5}$$

where λ, μ are the Lamé constants with ranges specified in Section 4.

The coefficient $c(\boldsymbol{\epsilon}, \theta)$ represents the specific heat, and the constant positive coefficients c_v, k correspond to the caloric specific heat and the heat conductivity, respectively. The terms $\mathbf{b}$ and g represent external body forces and heat sources.

The term $\nu \mathbf{Q}\mathbf{u}_t$ in the elasticity equation corresponds to the mechanical viscosity governed by Hooke's law

$$\boldsymbol{\sigma}^v = \nu \mathbf{A} \boldsymbol{\epsilon}_t, \tag{1.6}$$

where $\boldsymbol{\sigma}^v$ is the viscous stress and the constant $\nu > 0$ is the viscosity coefficient. The term $\nu(\mathbf{A}\boldsymbol{\epsilon}_t) : \boldsymbol{\epsilon}_t$ in the temperature equation represents heat production due to viscous dissipation.

The fourth order term $\mathbf{Q}\mathbf{Q}\mathbf{u}$ with constant coefficient $\kappa > 0$ corresponds to interaction effects on phase interfaces which are expressed by the particular strain gradient contribution $|\mathbf{Q}\mathbf{u}|^2$ to the free energy density

$$f(\boldsymbol{\epsilon}(\mathbf{u}), \boldsymbol{\nabla}\boldsymbol{\epsilon}(\mathbf{u}), \theta) = -c_v \theta \log \theta + F(\boldsymbol{\epsilon}(\mathbf{u}), \theta) + \frac{\kappa}{8}|\mathbf{Q}\mathbf{u}|^2. \tag{1.7}$$

As is well known (see e.g. [21]), the parameter $\kappa > 0$ acts as an additional length scale in the problem related to the regularization of the system by means of the strain gradient term.

We prove that the system is well posed for $(\mathbf{u}, \theta)$ in the space $\mathbf{W}_p^{4,2}(Q_T) \times W_p^{2,1}(Q_T)$ with $p \geq n + 2$. In that case the strain tensor $\boldsymbol{\epsilon}$, its gradient $\boldsymbol{\nabla}\boldsymbol{\epsilon}$ and the temperature θ are continuous functions in the space-time cylinder Q_T.

The approach used in this paper relies on the parabolic decomposition of the elastic part (1.1) and the subsequent application of the Leray–Schauder fixed point theorem to the decomposed system coupled with the temperature equation. The method has been devised in [54] for the treatment of one- and special two-dimensional SMA models.

The decomposition of (1.1) into two parabolic systems is possible due to the particular structure of (1.1) involving the differential operators $\mathbf{Q}$ and $\mathbf{Q}\mathbf{Q}$ which correspond to the viscous and interfacial terms respectively. The condition $0 < \sqrt{\kappa} \leq \nu$ connecting the viscosity and gradient energy coefficients assures that the decomposed system is real and parabolic.

The proof of existence depends in an essential way on the regularity theory of linear elasticity systems due to Nečas [35], and on the general theory of parabolic systems due to Solonnikov [49], [48], [16]. Moreover, an important part of the existence proof is the demonstration that the temperature is nonnegative. The proof of this fact is based on a slight modification of the classical stability result of [32] for parabolic equations. All these results rely on the at least C^2-regularity requirement concerning the boundary $\partial\Omega$. Since the regularity result of Nečas [35] refers to the case of displacement boundary conditions

$$\mathbf{u} = 0 \quad \text{on } S_T = (0, T) \times \partial\Omega, \tag{1.8}$$

we restrict our analysis to this case. It is possible to consider slightly more general boundary conditions, namely both displacement and traction ones, but imposed on disjoint parts of the boundary (see [7]).

Since the problem is fourth order in $\mathbf{u}$, an additional boundary condition is needed. We assume that

$$\mathbf{Q}\mathbf{u} = 0 \quad \text{on } S_T, \tag{1.9}$$

which results in a compatibility condition for parabolic decomposition.

For the temperature the homogeneous Neumann boundary condition corresponding to thermal isolation

$$\nabla\theta \cdot \mathbf{n} = 0 \quad \text{on } S_T, \tag{1.10}$$

is assumed, where $\mathbf{n}$ denotes the unit outward normal to $\partial\Omega$. This condition is imposed here for technical reasons connected with the arguments used in energy estimates. It can be modified to include heat exchange boundary conditions.

With (1.1), (1.2) we associate the initial conditions

$$\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{u}_t(0,\mathbf{x}) = \mathbf{u}_1(\mathbf{x}), \tag{1.11}$$

$$\theta(0,\mathbf{x}) = \theta_0(\mathbf{x}) \quad \text{in } \Omega. \tag{1.12}$$

We shall refer to system (1.1), (1.2) with boundary conditions (1.8), (1.9), (1.10) and initial conditions (1.11), (1.12) as *problem* (P).

The proof of uniqueness is based on energy estimates for the difference of solutions and application of Gronwall's inequality. This requires an additional regularity of solutions, namely the continuity of $\nabla\mathbf{u}_t$ and $\nabla\theta$ in the space-time cylinder. This holds for solutions $(\mathbf{u},\theta)$ in the space $\mathbf{W}_p^{4,2}(Q_T) \times \mathbf{W}_p^{2,1}(Q_T)$ with $p > n+2$.

We now comment on the related known results. In three dimensions there exist different continuum models describing thermomechanical evolution of SMA. The well-known Frémond model [23] is based on the strain tensor, the volumetric proportions of austenite and martensite, and the absolute temperature as state variables. The interfacial structure is there accounted for by the gradient of the strain tensor trace. The well-posedness of such a model has been studied by Colli *et al.* [8]–[11], Hoffmann *et al.* [29]; see also [6]. A different model has been derived by Fried and Gurtin [24] in an isothermal case within a thermodynamical theory of configurational forces. It is based on the strain tensor, a multicomponent order parameter and its gradient.

Concerning three-dimensional free energy models based on the strain tensor, the most known are due to Falk–Konopka [20], Ericksen–James [17], Barsch–Krumhansl [5]; see also Klouček and Luskin [31].

For a recent survey of continuum models of microstructure evolution in SMA we refer to Roubiček [43].

We mention also that in the isothermal case a phase transition model in the form of a multidimensional viscoelasticity system

$$\mathbf{u}_{tt} - \nu\Delta\mathbf{u}_t = \nabla \cdot \boldsymbol{\sigma}(\nabla\mathbf{u}), \tag{1.13}$$

where $\boldsymbol{\sigma}(\mathbf{C}) = F_{/\mathbf{C}}(\mathbf{C})$ and F is a multiple-well potential, has been studied by many authors, e.g. Ball *et al.* [4], Rybka [44]–[46], Swart and Holmes [51], Friesecke and Dolzmann [26]. We note that a hidden parabolic structure of (1.13) has been used in the analysis of this problem in [41], [44]–[46], [51]. The transformation of variables due to

Rybka [44], which is a generalization of the one-dimensional transformation of Pego [41], allows one to reformulate the problem (1.13) as a semilinear degenerate parabolic system.

A viscoelasticity system with capillarity

$$\mathbf{u}_{tt} - \nu \Delta \mathbf{u}_t + \kappa \Delta^2 \mathbf{u} = \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}(\boldsymbol{\nabla} \mathbf{u}) \tag{1.14}$$

has been justified as a relevant phase transition model by several authors (see e.g. [1], [53]), and studied e.g. in [47] in a quasi-steady approximation.

In the special 2-D case the system (1.14) coupled with the energy equation (1.2) has been considered in [54].

For related evolution problems in 3-D thermoelasticity we refer to Racke [42] and Nečas *et al.* [37], [36].

Our contribution consists in proving global-in-time existence and uniqueness results for system (1.1) containing both the viscosity and interfacial terms, expressed by means of the operator $\mathbf{Q}$, coupled with the temperature equation (1.2).

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2. Notation and preliminaries

Let $I = (0,T)$, $Q_t = (0,t) \times \Omega$, $\Omega_t = \{t\} \times \Omega$, $S_t = (0,t) \times \partial\Omega$, and let $\mathbf{n}$ stand for the unit outward normal to $\partial\Omega$. We use the Sobolev space notation of [32] and bold letters for vector- and tensor-valued mappings, similarly to [7]. The summation convention over repeated indices is used, and for vectors $\mathbf{a} = (a_i)$, $\mathbf{b} = (b_i)$ and tensors $\mathbf{B} = (B_{ij})$, $\widetilde{\mathbf{B}} = (\widetilde{B}_{ij})$, $\mathbf{C} = (C_{ijk})$, $\widetilde{\mathbf{C}} = (\widetilde{C}_{ijk})$, $\mathbf{A} = (A_{ijkl})$ we write:

$$\begin{gathered}
\mathbf{a} \cdot \mathbf{b} = a_i b_i, \qquad \mathbf{B} : \widetilde{\mathbf{B}} = B_{ij} \widetilde{B}_{ij}, \qquad \mathbf{C} : \widetilde{\mathbf{C}} = C_{ijk} \widetilde{C}_{ijk}, \\
\mathbf{aB} = (a_i B_{ij}), \qquad \mathbf{Ba} = (B_{ij} a_j), \qquad \mathbf{aC} = (a_i C_{ijk}), \\
\mathbf{Ca} = (C_{ijk} a_k), \qquad \mathbf{aA} = (a_i A_{ijkl}), \qquad \mathbf{Aa} = (A_{ijkl} a_l), \\
\mathbf{BC} = (B_{ij} C_{ijk}), \qquad \mathbf{CB} = (C_{ijk} B_{jk}), \qquad \mathbf{BA} = (B_{ij} A_{ijkl}), \qquad \mathbf{AB} = (A_{ijkl} B_{kl}), \\
|\mathbf{b}| = (b_i b_i)^{1/2}, \qquad |\mathbf{B}| = (B_{ij} B_{ij})^{1/2}, \qquad |\mathbf{C}| = (C_{ijk} C_{ijk})^{1/2}.
\end{gathered}$$

Moreover, for the gradient of a tensor we use the convention of the last position of the differentiation index, e.g., for a tensor $\mathbf{B} = (B_{ij})$,

$$\boldsymbol{\nabla} \mathbf{B} = (B_{ij/k}).$$

The linear map

$$\boldsymbol{\epsilon}(\mathbf{u}) \mapsto \mathbf{A}\boldsymbol{\epsilon}(\mathbf{u}) = \lambda \operatorname{trace} \boldsymbol{\epsilon}(\mathbf{u}) \mathbf{I} + 2\mu \boldsymbol{\epsilon}(\mathbf{u}), \tag{2.1}$$

where λ, μ are the Lamé constants, $\mathbf{I} = (\delta_{ij})$ is the unit matrix, represents Hooke's law for a homogeneous isotropic material. Here $\mathbf{A} = (A_{ijkl})$ with

$$A_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

is the fourth order elasticity tensor satisfying the following symmetry conditions:

(2.2) $$A_{ijkl} = A_{jikl}, \quad A_{ijkl} = A_{ijlk}, \quad A_{ijkl} = A_{klij}.$$

We note here the following properties: symmetry

(2.3) $$\boldsymbol{\epsilon}(\boldsymbol{\varphi}) : (\mathbf{A}\boldsymbol{\epsilon}(\mathbf{u})) = \epsilon_{ij}(\boldsymbol{\varphi}) A_{ijkl}\epsilon_{kl}(\mathbf{u}) = \epsilon_{kl}(\mathbf{u})A_{klij}\epsilon_{ij}(\boldsymbol{\varphi}) = \boldsymbol{\epsilon}(\mathbf{u}) : (\mathbf{A}\boldsymbol{\epsilon}(\boldsymbol{\varphi})),$$

as well as coercivity and boundedness,

(2.4) $$a_\star|\boldsymbol{\epsilon}|^2 \le (\mathbf{A}\boldsymbol{\epsilon}) : \boldsymbol{\epsilon} \le a^\star|\boldsymbol{\epsilon}|^2,$$

where

$$a_\star = \min[n\lambda + 2\mu, 2\mu], \qquad a^\star = \max[n\lambda + 2\mu, 2\mu].$$

By $\mathbf{Q}$ we shall denote the operator of linearized elasticity [7] which is defined by

(2.5) $$\mathbf{u} \mapsto \mathbf{Q}\mathbf{u} = \boldsymbol{\nabla} \cdot (\mathbf{A}\boldsymbol{\epsilon}(\mathbf{u})),$$

where $\boldsymbol{\nabla}\cdot$ denotes the divergence operator. We use the convention of contraction over the last index, i.e.,

$$\boldsymbol{\nabla} \cdot (\mathbf{A}\boldsymbol{\epsilon}(\mathbf{u})) = \partial_j(\, A_{ijkl}\epsilon_{kl}(\mathbf{u})\,) = A_{ijkl}\epsilon_{kl/j}(\mathbf{u}).$$

Hence, by (2.2),

(2.6) $$\mathbf{Q}\mathbf{u} = \epsilon_{kl/j}(\mathbf{u})A_{klji} = \boldsymbol{\nabla}\boldsymbol{\epsilon}(\mathbf{u})\mathbf{A}.$$

We note that by (2.1), $\mathbf{Q}$ admits a representation (1.5).

For further use we collect some identities involving the operator $\mathbf{Q}$. The formula of integration by parts

(2.7) $$\begin{aligned}\int_\Omega (\mathbf{Q}\mathbf{u}) \cdot (\mathbf{Q}\mathbf{v})\, dx = {} & \int_\Omega \mathbf{v} \cdot (\mathbf{Q}\mathbf{Q}\mathbf{u})\, dx \\ & - \int_{\partial\Omega} \mathbf{v} \cdot ((\mathbf{A}\boldsymbol{\epsilon}(\mathbf{Q}\mathbf{u}))\mathbf{n})\, dS + \int_{\partial\Omega} (\mathbf{Q}\mathbf{u}) \cdot ((\mathbf{A}\boldsymbol{\epsilon}(\mathbf{v}))\mathbf{n})\, dS\end{aligned}$$

is a consequence of the symmetry property (2.3) and a two-fold application of Green's formula (for $\boldsymbol{\varphi}$, $\boldsymbol{\sigma}$ sufficiently regular)

(2.8) $$\int_\Omega \boldsymbol{\varphi} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma})\, dx = -\int_\Omega \boldsymbol{\nabla}\boldsymbol{\varphi} : \boldsymbol{\sigma}\, dx + \int_{\partial\Omega} \boldsymbol{\varphi} \cdot (\boldsymbol{\sigma}\mathbf{n})\, dS.$$

The first variation $\delta|\mathbf{Q}\mathbf{u}|^2/\delta\mathbf{u}$ of the operator $\mathbf{u} \mapsto |\mathbf{Q}\mathbf{u}|^2$ is given by

(2.9) $$\begin{aligned}\int_\Omega \boldsymbol{\zeta} \cdot \frac{\delta}{\delta\mathbf{u}}|\mathbf{Q}\mathbf{u}|^2\, dx &= \frac{d}{d\alpha}\int_\Omega |\mathbf{Q}(\mathbf{u} + \alpha\boldsymbol{\zeta})|^2\, dx\Big|_{\alpha=0} \\ &= 2\int_\Omega \boldsymbol{\zeta} \cdot (\mathbf{Q}\mathbf{Q}\mathbf{u})\, dx \qquad \text{for } \boldsymbol{\zeta} \in \mathbf{C}_0^\infty(\Omega).\end{aligned}$$

It is obtained from Green's formula (2.8) and the symmetry property (2.3), since

$$\frac{d}{d\alpha}\int\limits_{\Omega}|\boldsymbol{\nabla}\cdot(\mathbf{A}\boldsymbol{\epsilon}(\mathbf{u}+\alpha\boldsymbol{\zeta}))|^2\,dx\Big|_{\alpha=0} = 2\int\limits_{\Omega}(\boldsymbol{\nabla}\cdot\mathbf{A}\boldsymbol{\epsilon}(\mathbf{u}))\cdot(\boldsymbol{\nabla}\cdot\mathbf{A}\boldsymbol{\epsilon}(\boldsymbol{\zeta}))\,dx$$
$$= -2\int\limits_{\Omega}\boldsymbol{\epsilon}(\mathbf{Q}\mathbf{u}):(\mathbf{A}\boldsymbol{\epsilon}(\boldsymbol{\zeta}))\,dx = -2\int\limits_{\Omega}\boldsymbol{\epsilon}(\boldsymbol{\zeta}):(\mathbf{A}\boldsymbol{\epsilon}(\mathbf{Q}\mathbf{u}))\,dx$$
$$= 2\int\limits_{\Omega}\boldsymbol{\zeta}\cdot(\boldsymbol{\nabla}\cdot(\mathbf{A}\boldsymbol{\epsilon}(\mathbf{Q}\mathbf{u})))\,dx = 2\int\limits_{\Omega}\boldsymbol{\zeta}\cdot(\mathbf{Q}\mathbf{Q}\mathbf{u})\,dx.$$

The above calculation also shows that the first variation of the operator

$$\boldsymbol{\epsilon}(\mathbf{u}) \mapsto |\boldsymbol{\nabla}\cdot(\mathbf{A}\boldsymbol{\epsilon}(\mathbf{Q}\mathbf{u}))|^2$$

is given by

$$\int\limits_{\Omega}\boldsymbol{\epsilon}(\boldsymbol{\zeta}):\frac{\delta}{\delta\boldsymbol{\epsilon}(\mathbf{u})}|\boldsymbol{\nabla}\cdot(\mathbf{A}\boldsymbol{\epsilon}(\mathbf{Q}\mathbf{u}))|^2\,dx = -2\int\limits_{\Omega}\boldsymbol{\epsilon}(\boldsymbol{\zeta}):(\mathbf{A}\boldsymbol{\epsilon}(\mathbf{Q}\mathbf{u}))\,dx \qquad \text{for } \boldsymbol{\zeta}\in\mathbf{C}_0^\infty(\Omega). \tag{2.10}$$

For further purposes we note that for the strain-gradient energy of the form (1.7) we have

$$f_{/\mathbf{D}\boldsymbol{\epsilon}} = \frac{\kappa}{4}\mathbf{A}(\boldsymbol{\nabla}\boldsymbol{\epsilon}(\mathbf{u})\mathbf{A}) = \frac{\kappa}{4}\mathbf{A}\mathbf{Q}\mathbf{u}, \tag{2.11}$$

since by (2.6),

$$f_{/\epsilon_{pq/r}} = \frac{\kappa}{4}\,\epsilon_{kl_{/j}}A_{klji}A_{pqri}.$$

Moreover, in view of (2.11),

$$f_{/\mathbf{D}\boldsymbol{\epsilon}}\mathbf{n} = \frac{\kappa}{4}\,(\mathbf{A}\mathbf{Q}\mathbf{u})\mathbf{n}. \tag{2.12}$$

Hence, in particular, the boundary condition (1.9) implies that

$$f_{/\mathbf{D}\boldsymbol{\epsilon}}\mathbf{n} = 0 \quad \text{on } S_T. \tag{2.13}$$

3. Thermodynamical framework

Problem (P) expresses conservation laws for linear momentum and energy (assuming constant mass density) with appropriate constitutive equations. The corresponding free energy density is assumed to be a function of the strain tensor, its first gradient and the absolute temperature,

$$f = \widehat{f}(\boldsymbol{\epsilon}, \mathbf{D}\boldsymbol{\epsilon}, \theta),$$

in the particular Ginzburg–Landau form (1.7) with the terms representing thermal energy, elastic energy and interfacial energy.

As a prototype example of the elastic energy for 3-D SMA we consider the Falk–Konopka model [20]:

$$F(\boldsymbol{\epsilon},\theta) = \sum_{i=1}^{3}F_i^2(\theta)J_i^2(\boldsymbol{\epsilon}) + \sum_{i=1}^{5}F_i^4(\theta)J_i^4(\boldsymbol{\epsilon}) + \sum_{i=1}^{2}F_i^6(\theta)J_i^6(\boldsymbol{\epsilon}) \tag{3.1}$$

with

$$F_i^2(\theta) = \alpha_i^2(\theta-\theta_c), \qquad F_i^4(\theta) = \alpha_i^4(\theta-\theta_c), \qquad F_i^6(\theta) = \alpha_i^6.$$

Here α_i^k, θ_c are constants and $J_i^k(\boldsymbol{\epsilon}),\ i = 1, \dots, i^k$, denote kth order invariants defined as the following combinations of the strain components ϵ_{ij}:

$$
\begin{aligned}
&J_1^2 = \epsilon_{\bar{1}}^2, \quad J_2^2 = 3\epsilon_{\bar{2}}^2 + \epsilon_{\bar{3}}^2, \quad J_3^2 = \epsilon_{\bar{4}}^2 + \epsilon_{\bar{5}}^2 + \epsilon_{\bar{6}}^2,\\
&J_1^4 = (J_2^2)^2, \quad J_2^4 = \epsilon_{\bar{4}}^4 + \epsilon_{\bar{5}}^4 + \epsilon_{\bar{6}}^4, \quad J_3^4 = (J_3^2)^2, \quad J_4^4 = J_2^2 J_3^2,\\
&J_5^4 = \epsilon_{\bar{4}}^2(\epsilon_{\bar{2}} - \epsilon_{\bar{3}})^2 + \epsilon_{\bar{5}}^2(\epsilon_{\bar{2}} + \epsilon_{\bar{3}})^2 + 4\epsilon_{\bar{6}}^2\epsilon_{\bar{2}}^2, \quad J_1^6 = (J_2^2)^3, \quad J_2^6 = \epsilon_{\bar{2}}^2(\epsilon_{\bar{2}}^2 - \epsilon_{\bar{3}}^2)^2,
\end{aligned}
$$

with

$$
\begin{aligned}
&\epsilon_{\bar{1}} = \operatorname{trace}\boldsymbol{\epsilon}/3, \quad \epsilon_{\bar{2}} = (2\epsilon_{33} - \epsilon_{11} - \epsilon_{22})/6,\\
&\epsilon_{\bar{3}} = (\epsilon_{11} - \epsilon_{22})/2, \quad \epsilon_{\bar{4}} = \epsilon_{23}, \quad \epsilon_{\bar{5}} = \epsilon_{13}, \quad \epsilon_{\bar{6}} = \epsilon_{12}.
\end{aligned}
$$

The free energy (3.1) is invariant with respect to the cubic symmetry of the high temperature austenitic phase, that is, it satisfies the isotropy condition

$$F(\boldsymbol{\epsilon}, \theta) = F(\mathbf{G}\boldsymbol{\epsilon}\mathbf{G}^T, \theta) \tag{3.2}$$

for each of the 48 matrices $\mathbf{G}$ representing corresponding symmetry operations in $\mathbb{R}^3$.

Equation (1.1) corresponds to the linear momentum balance (for mass density $\varrho = 1$)

$$\mathbf{u}_{tt} - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = \mathbf{b}, \tag{3.3}$$

where the symmetric stress tensor $\boldsymbol{\sigma}$ is given through the constitutive equation

$$\boldsymbol{\sigma} = \frac{\delta f}{\delta \boldsymbol{\epsilon}} + \boldsymbol{\sigma}^v. \tag{3.4}$$

Here $\boldsymbol{\sigma}^v$ is the viscous stress tensor given by linear Hooke's law (1.6), and the expression $\delta f/\delta\boldsymbol{\epsilon}$ is the first variation of f with respect to $\boldsymbol{\epsilon}$,

$$\frac{\delta f}{\delta \boldsymbol{\epsilon}} = f_{/\boldsymbol{\epsilon}} - \boldsymbol{\nabla} \cdot f_{/\mathbf{D}\boldsymbol{\epsilon}}. \tag{3.5}$$

In case of free energy (1.7), by (2.11),

$$\frac{\delta f}{\delta \boldsymbol{\epsilon}} = F_{/\boldsymbol{\epsilon}} - \frac{\kappa}{4}\,\mathbf{A}\boldsymbol{\epsilon}(\mathbf{Q}\mathbf{u}), \tag{3.6}$$

which follows from (2.10). In view of (1.6), (3.6) the constitutive equation for the stress tensor takes on the form

$$\boldsymbol{\sigma} = F_{/\boldsymbol{\epsilon}} - \frac{\kappa}{4}\,\mathbf{A}\boldsymbol{\epsilon}(\mathbf{Q}\mathbf{u}) + \nu\mathbf{A}\boldsymbol{\epsilon}_t, \tag{3.7}$$

with three contributions corresponding to the elastic stress, interfacial stress (hyperstress) and viscous stress.

Relation (3.7) augments the conventional constitutive law for an elastic material in such a way that the stress tensor depends not only on the strain tensor $\boldsymbol{\epsilon}$, but also on the strain rate tensor $\boldsymbol{\epsilon}_t$ and second spatial gradients $\mathbf{D}^2\boldsymbol{\epsilon}$. The characteristic feature of (3.7) is the nonlinear dependence of the stress tensor on $\boldsymbol{\epsilon}$, while $\boldsymbol{\epsilon}_t$ and $\mathbf{D}^2\boldsymbol{\epsilon}$ enter linearly via Hooke's law. This constitutive equation generalizes to three dimensions the well-known one-dimensional laws (see e.g. [1], [53]).

We also point out that the particular form (3.7) is necessary for parabolic decomposition of the elasticity system.

From now on, we denote for simplicity by $\boldsymbol{\sigma}^h$ the third order tensor

$$\boldsymbol{\sigma}^h = f_{/\mathbf{D}\boldsymbol{\epsilon}} \quad \text{where} \quad \sigma^h_{ijk} = f_{/\epsilon_{ij/k}}.$$

Equation (1.2) corresponds to the energy balance

$$e_t + \nabla \cdot \mathbf{q} - \boldsymbol{\sigma} : \boldsymbol{\epsilon}_t = g. \tag{3.8}$$

Here $e = \widehat{e}(\boldsymbol{\epsilon}, \mathbf{D}\boldsymbol{\epsilon}, \theta)$ is the internal energy density which obeys the Gibbs relations

$$e = f + \theta s, \qquad s = -f_{/\theta}, \tag{3.9}$$

where s is the entropy. The energy flux $\mathbf{q}$ consists of the stationary and nonstationary parts

$$\mathbf{q} = \mathbf{q}_0 - \boldsymbol{\epsilon}_t \boldsymbol{\sigma}^h. \tag{3.10}$$

The stationary part

$$\mathbf{q}_0 = -k\nabla\theta \tag{3.11}$$

is the heat flux governed by Fourier's law with the heat conductivity coefficient $k > 0$. The unconventional nonstationary part

$$\boldsymbol{\epsilon}_t \boldsymbol{\sigma}^h = (\epsilon_{tij}\sigma^h_{ijk}) \tag{3.12}$$

is associated with evolving diffuse interfaces (see [2]). A similar flux, called interstitial work flux corresponding to working of phase interfaces, appears in Dunn–Serrin's [14] thermodynamical theory of higher grade thermoelastic materials.

In view of the Gibbs relations and the identity

$$\nabla \cdot (\boldsymbol{\epsilon}_t \boldsymbol{\sigma}^h) = \partial_k(\epsilon_{tij}\sigma^h_{ijk}) = \epsilon_{tij}\partial_k\sigma^h_{ijk} + \partial_k\epsilon_{tij}\sigma^h_{ijk} = \boldsymbol{\epsilon}_t : (\nabla \cdot \boldsymbol{\sigma}^h) + \boldsymbol{\sigma}^h : \nabla\boldsymbol{\epsilon}_t,$$

we have

$$\begin{aligned} e_t - \nabla \cdot (\boldsymbol{\epsilon}_t f_{/\mathbf{D}\boldsymbol{\epsilon}}) - \boldsymbol{\sigma} : \boldsymbol{\epsilon}_t &\\ = (\theta s_t + f_{\boldsymbol{\epsilon}} : \boldsymbol{\epsilon}_t + f_{/\mathbf{D}\boldsymbol{\epsilon}} : \nabla\boldsymbol{\epsilon}_t) &- ((\nabla \cdot f_{/\mathbf{D}\boldsymbol{\epsilon}}) : \boldsymbol{\epsilon}_t + f_{/\mathbf{D}\boldsymbol{\epsilon}} : \nabla\boldsymbol{\epsilon}_t)\\ - (f_{/\boldsymbol{\epsilon}} - \nabla \cdot f_{/\mathbf{D}\boldsymbol{\epsilon}} + \boldsymbol{\sigma}^v) : \boldsymbol{\epsilon}_t &\\ = \theta s_t - \sigma^v : \boldsymbol{\epsilon}_t. & \end{aligned} \tag{3.13}$$

Therefore the energy equation (3.8) takes on the form

$$\theta s_t + \nabla \cdot \mathbf{q}_0 = \boldsymbol{\sigma}^v : \boldsymbol{\epsilon}_t + g. \tag{3.14}$$

After introducing the specific heat coefficient

$$c(\boldsymbol{\epsilon}, \mathbf{D}\boldsymbol{\epsilon}, \theta) = -\theta f_{/\theta\theta}(\boldsymbol{\epsilon}, \mathbf{D}\boldsymbol{\epsilon}, \theta), \tag{3.15}$$

equation (3.14) becomes

$$\begin{aligned} c(\boldsymbol{\epsilon}, \mathbf{D}\boldsymbol{\epsilon}, \theta)\theta_t + \nabla \cdot \mathbf{q}_0 = \theta f_{/\theta\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}, \mathbf{D}\boldsymbol{\epsilon}, \theta) : \boldsymbol{\epsilon}_t &\\ + \theta f_{/\theta\mathbf{D}\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}, \mathbf{D}\boldsymbol{\epsilon}, \theta) : \nabla\boldsymbol{\epsilon}_t + \boldsymbol{\sigma}^v : \boldsymbol{\epsilon}_t + g.& \end{aligned} \tag{3.16}$$

For the free energy given by (1.7) this yields equation (1.2).

The constitutive equations for $\boldsymbol{\sigma}$, e and $\mathbf{q}$ presented above are a special case of more general relations. It has been shown in [39] that the stress relation admits the form

$$\boldsymbol{\sigma} = \frac{\delta f}{\delta\boldsymbol{\epsilon}} + \theta(\mathbf{h} - \boldsymbol{\sigma}^h)\nabla\left(\frac{1}{\theta}\right) + \boldsymbol{\sigma}^v, \tag{3.17}$$

where $\mathbf{h}$ is an arbitrary third order tensor. The constitutive relation for the energy flux is then

$$\mathbf{q} = \mathbf{q}_0 - \boldsymbol{\epsilon}_t \mathbf{h}. \tag{3.18}$$

For thermodynamical consistency the heat flux $\mathbf{q}_0$ and the viscous stress tensor $\boldsymbol{\sigma}^v$ have to satisfy the dissipation inequality

$$\boldsymbol{\epsilon}_t : \left(\frac{\boldsymbol{\sigma}^v}{\theta}\right) + \boldsymbol{\nabla}\left(\frac{1}{\theta}\right) \cdot \mathbf{q}_0 \geq 0 \quad \text{for all fields } (\mathbf{u}, \theta). \tag{3.19}$$

Clearly, condition (3.19) is satisfied by Fourier's and Hooke's laws.

For the model with general constitutive equations (3.17), (3.18) the following entropy inequality holds [39] for all $(\mathbf{u}, \theta)$:

$$\begin{aligned} s_t + \boldsymbol{\nabla} \cdot \boldsymbol{\psi} - \boldsymbol{\lambda}_u \cdot (\mathbf{u}_{tt} - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} - \mathbf{b}) - \lambda_\theta \left(\left(e + \frac{|\mathbf{u}_t|^2}{2} \right)_t + \boldsymbol{\nabla} \cdot (-\mathbf{u}_t \boldsymbol{\sigma} + \mathbf{q}) - \mathbf{b} \cdot \mathbf{u}_t \right) \\ = \boldsymbol{\epsilon}_t : \left(\frac{\boldsymbol{\sigma}^v}{\theta}\right) + \boldsymbol{\nabla}\left(\frac{1}{\theta}\right) \cdot \mathbf{q}_0 \geq 0, \end{aligned} \tag{3.20}$$

where the entropy flux $\boldsymbol{\psi}$ is given by

$$\boldsymbol{\psi} = \frac{1}{\theta}(\mathbf{q}_0 + \boldsymbol{\epsilon}_t(\boldsymbol{\sigma}^h - \mathbf{h})), \tag{3.21}$$

and the multipliers $\boldsymbol{\lambda}_u$ and λ_θ, conjugated respectively with linear momentum and total energy balances, are

$$\boldsymbol{\lambda}_u = -\lambda_\theta \mathbf{u}_t, \qquad \lambda_\theta = 1/\theta. \tag{3.22}$$

We note that problem (P) corresponds to the particular case $\mathbf{h} = f_{/\mathbf{D}\boldsymbol{\epsilon}}$. If the gradient term in free energy does not depend on θ, which is the case for (1.7), then $f_{/\theta \mathbf{D}\boldsymbol{\epsilon}} = 0$, and such a choice of $\mathbf{h}$ gives $\boldsymbol{\sigma}$ independent of $\boldsymbol{\nabla}\theta$.

By using (3.20) we see that for solutions of problem (P) the standard Clausius–Duhem inequality

$$s_t + \boldsymbol{\nabla} \cdot \left(\frac{\mathbf{q}_0}{\theta}\right) = \boldsymbol{\epsilon}_t : \left(\frac{\boldsymbol{\sigma}^v}{\theta}\right) + \boldsymbol{\nabla}\left(\frac{1}{\theta}\right) \cdot \mathbf{q}_0 + \frac{g}{\theta} \geq \frac{g}{\theta} \tag{3.23}$$

is satisfied for all $(\mathbf{u}, \theta)$.

Finally we add a comment on Hooke's constitutive law (1.6). As is known (see Fosdick and Serrin [22]), the linear stress response function is incompatible with the principle of frame-indifference, therefore an exact linear constitutive theory for elastic solids is impossible. On the other hand, Hooke's law is commonly used in the theory of linearized elasticity based on the small strain assumption, as an approximation to a more complicated realistic description (see [7]). We point out that Hooke's law satisfies the invariance condition of an isotropic function (see e.g. [28], p. 235). In particular, for the viscosity given by (1.6) the following condition holds:

$$\mathbf{R}\boldsymbol{\sigma}^v(\boldsymbol{\epsilon}_t)\mathbf{R}^T = \boldsymbol{\sigma}^v(\mathbf{R}\boldsymbol{\epsilon}_t\mathbf{R}^T)$$

for any proper orthogonal tensor $\mathbf{R}$ of the second order.

4. Assumptions and main results

Below we list our assumptions grouped into several categories.

(D) Domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, with boundary $\partial\Omega$ of class C^3.

The additional C^3-regularity is needed in Lemma 7.3, which concerns the inheritance of the right-hand side differentiability properties by the solutions of parabolic systems.

(LP) Linear part: the coefficients of the operator $\mathbf{Q}$ satisfy

$$\mu > 0, \qquad n\lambda + 2\mu > 0.$$

This is needed for ensuring the coercivity of the algebraic operator $\mathbf{A}$ in (2.4). These conditions also imply the strong ellipticity of the operator $\mathbf{Q}$ (see Section 7.2) and the parabolicity of the evolution system with the operator $\mathbf{Q}$ (see Section 7.1). Both properties require $\lambda + 2\mu > 0$.

The next assumptions concern the elastic energy:

(FE-1) Structure: $F(\boldsymbol{\epsilon}, \theta)$ is of class C^3 on $S^2 \times [0, \infty)$, where S^2 denotes the set of symmetric tensors of second order in $\mathbb{R}^n$. We assume the splitting into entropic and energetic parts

$$F(\boldsymbol{\epsilon}, \theta) = F_1(\boldsymbol{\epsilon}, \theta) + F_2(\boldsymbol{\epsilon}),$$

where $F_1(\boldsymbol{\epsilon}, \theta)$ is a concave function with respect to θ,

$$F_{1/\theta\theta}(\boldsymbol{\epsilon}, \theta) \le 0 \quad \text{ for } (\boldsymbol{\epsilon}, \theta) \in S^2 \times [0, \infty),$$

such that $F_1(\boldsymbol{\epsilon}, \theta)$ is linear in θ over a certain interval $[0, \theta_1)$, $\theta_1 = \text{const}$, and has the polynomial growth θ^r for $\theta \ge \theta_1$.

(FE-2) Growth conditions: There exists a positive constant Λ such that for $\theta \ge \theta_1$ and large values of ϵ_{ij} the following conditions are satisfied:

$$\begin{aligned}
|F_{1/\boldsymbol{\epsilon\epsilon}}(\boldsymbol{\epsilon}, \theta)| &\le \Lambda \theta^r |\boldsymbol{\epsilon}|^{q-1}, & |F_{2/\boldsymbol{\epsilon\epsilon}}(\boldsymbol{\epsilon})| &\le \Lambda |\boldsymbol{\epsilon}|^{\overline{q}-1}, \\
|F_{1/\boldsymbol{\epsilon}\theta}(\boldsymbol{\epsilon}, \theta)| &\le \Lambda \theta^{r-1} |\boldsymbol{\epsilon}|^{q}, & |F_{1/\theta\theta}(\boldsymbol{\epsilon}, \theta)| &\le \Lambda \theta^{r-2} |\boldsymbol{\epsilon}|^{q+1}, \\
|F_{1/\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}, \theta)| &\le \Lambda \theta^r |\boldsymbol{\epsilon}|^{q}, & |F_{2/\boldsymbol{\epsilon}}(\boldsymbol{\epsilon})| &\le \Lambda |\boldsymbol{\epsilon}|^{\overline{q}},
\end{aligned}$$

with

$$0 < r < \frac{1}{2}, \qquad 1 < \overline{q} \le \frac{q_n p_n}{4n}, \qquad 0 < q \le (\overline{q} + 1)\left(\frac{1}{2} - r\right),$$

where $p_n = n + 2$, and q_n is the Sobolev exponent for which the imbedding of $W_2^1(\Omega)$ into $L_{q_n}(\Omega)$ is continuous, that is, $q_n = 2n/(n-2)$ for $n \ge 3$ and q_n is any finite number for $n = 2$. We note that

$$0 < q \le \frac{q_n p_n}{2n}\left(\frac{1}{2} - r\right).$$

The above conditions imply the following growth of $F(\boldsymbol{\epsilon}, \theta)$:

$$|F_1(\boldsymbol{\epsilon}, \theta)| \le \Lambda + \Lambda \theta^r |\boldsymbol{\epsilon}|^{q+1}, \qquad |F_2(\boldsymbol{\epsilon})| \le \Lambda + \Lambda |\boldsymbol{\epsilon}|^{\overline{q}+1}.$$

We add some comments on the above conditions. The restrictions concern the θ-growth exponent of F_1, the ϵ-growth exponent of F_2 and the condition relating the ϵ-growth of F_1 with its θ-growth and with the ϵ-growth of F_2.

The most restrictive is the condition $r < 1/2$, and $\overline{q} \leq 5/2$ in 3-D. In 2-D, since q_n is any finite number, arbitrary polynomial growth is admissible.

In particular, in 3-D the above conditions are satisfied for

$$\overline{q} = \frac{5}{2}, \qquad q = 1, \qquad r = \frac{3}{14}.$$

Moreover, we assume the structural lower bound for the energetic part $F_2(\boldsymbol{\epsilon})$ of the free energy.

(FE-3) There exist positive constants c, Λ such that

$$c|\boldsymbol{\epsilon}|^{\overline{q}+1} - \Lambda \leq F_2(\boldsymbol{\epsilon}).$$

This is satisfied by the model example (3.1) with the growth restriction (FE-2).

The next assumption concerns the structural simplification of the energy equation by neglecting the nonlinear elastic contribution $-\theta F_{1/\theta\theta}(\boldsymbol{\epsilon}, \theta)$ in the specific heat coefficient. This allows us to apply the classical parabolic theory in the existence proof.

We point out that because of the technique applied we were unable either to allow $F_1(\boldsymbol{\epsilon}, \theta)$ linear in θ or, assuming the θ-growth condition, to incorporate the arising nonlinearity in the specific heat coefficient.

(SH) The elastic energy contribution $-\theta F_{1/\theta\theta}(\boldsymbol{\epsilon}, \theta)$ to the specific heat coefficient due to the nonlinearity of F_1 in θ is neglected, that is, we set

$$c(\boldsymbol{\epsilon}, \theta) = c_v = \text{const} > 0.$$

We are looking for the solution in the anisotropic Sobolev space

$$V(p) = \{(\mathbf{u}, \theta) \in \mathbf{W}_p^{4,2}(Q_T) \times W_p^{2,1}(Q_T)\},$$

with p relating to L_p-integrability. The assumptions on the initial data and the source terms correspond to this space.

(BV-p) The initial conditions satisfy for $1 < p < \infty$ the inclusions

$$\mathbf{u}_0 \in \mathbf{W}_p^{4-2/p}(\Omega), \qquad \mathbf{u}_1 \in \mathbf{W}_p^{2-2/p}(\Omega),$$
$$0 \leq \theta_0 \in W_p^{2-2/p}(\Omega),$$

and the compatibility relations. The source terms satisfy

$$\mathbf{b} \in \mathbf{L}_p(Q_T), \qquad g \in L_p(Q_T), \qquad g \geq 0 \quad \text{a.e. in } Q_T.$$

The first main result concerns the existence of solutions to problem (P).

Theorem 4.1 (Existence). *Under assumptions* (D), (LP), (FE-1)–(FE-3), (SH), (BV-p) *and the condition*

$$0 < \sqrt{\kappa} \leq \nu,$$

there exists for $p_n \leq p < \infty$ *a solution* $(\mathbf{u},\theta) \in V(p)$ *to problem* (P) *for any* $T > 0$. *Moreover*, $\theta \geq 0$ *in* Q_T, *and the following a priori estimates hold:*

$$\|\mathbf{u}\|_{\mathbf{W}_p^{4,2}(Q_T)} \leq \Lambda, \quad \|\theta\|_{W_p^{2,1}(Q_T)} \leq \Lambda \tag{4.1}$$

with a constant Λ *depending on the data of the problem*, Ω *and time* T.

We note some properties of the solution which follow directly from the classical imbeddings (see Section 7.5).

Corollary 4.1. *For a solution to problem* (P) *the following holds:* $\mathbf{u}$, $\boldsymbol{\nabla}\mathbf{u}$, $\boldsymbol{\nabla}^2\mathbf{u}$, $\mathbf{u}_t$, θ *are Hölder continuous in* Q_T, $\boldsymbol{\nabla}^3\mathbf{u}, \boldsymbol{\nabla}\mathbf{u}_t, \boldsymbol{\nabla}\theta \in \mathbf{L}_p(Q_T)$, $p_n \leq p < \infty$, *and*

$$|\mathbf{u}|, |\boldsymbol{\nabla}\mathbf{u}|, |\boldsymbol{\nabla}^2\mathbf{u}|, |\mathbf{u}_t| \leq \Lambda, \quad 0 \leq \theta \leq \Lambda \quad \text{in } Q_T, \tag{4.2}$$

$$\|\boldsymbol{\nabla}^3\mathbf{u}\|_{\mathbf{L}_p(Q_T)}, \|\boldsymbol{\nabla}\mathbf{u}_t\|_{\mathbf{L}_p(Q_T)}, \|\boldsymbol{\nabla}\theta\|_{\mathbf{L}_p(Q_T)} \leq \Lambda. \tag{4.3}$$

In order to prove the uniqueness of the solution, the continuity property of $\boldsymbol{\nabla}\mathbf{u}_t$ in Q_T is needed. This holds provided $p > p_n$.

Theorem 4.2 (Uniqueness). *Let the assumptions of Theorem* 4.1 *be satisfied for* $p_n < p < \infty$. *Then the solution to problem* (P) *is unique for any* $T > 0$.

We now collect a priori bounds which follow from the imbeddings.

Corollary 4.2. *The solution to problem* (P) *has in case* $p_n < p < \infty$ *the following properties:* $\boldsymbol{\nabla}^3\mathbf{u}$, $\boldsymbol{\nabla}\mathbf{u}_t$, $\boldsymbol{\nabla}\theta$ *are Hölder continuous in* Q_T *and satisfy the bounds*

$$|\boldsymbol{\nabla}^3\mathbf{u}|, |\boldsymbol{\nabla}\mathbf{u}_t|, |\boldsymbol{\nabla}\theta| \leq \Lambda \quad \text{in } Q_T. \tag{4.4}$$

5. Existence proof

As mentioned in the introduction, in the proof we shall use a parabolic decomposition of the elasticity system (1.1) and the Leray–Schauder fixed point theorem, following the strategy outlined in [54] for some special cases.

From now on Λ denotes a generic constant, different in various instances. In general, Λ can depend on the data of the problem, domain Ω and time T. The proof consists of several steps which are described below.

Step 1: Parabolic decomposition of (1.1). Choosing numbers α, β so that

$$\alpha + \beta = \nu, \quad \alpha\beta = \kappa/4, \tag{5.1}$$

the system (1.1) with initial conditions (1.11) and boundary conditions (1.8), (1.9) decomposes into the following systems of BVP's for a vector field $\mathbf{w}$:

$$\begin{aligned} &\mathbf{w}_t - \beta\mathbf{Q}\mathbf{w} = \boldsymbol{\nabla} \cdot F_{/\boldsymbol{\epsilon}}(\boldsymbol{\epsilon},\theta) + \mathbf{b} && \text{in } Q_T,\\ &\mathbf{w}(0,\mathbf{x}) = \mathbf{u}_1(\mathbf{x}) - \alpha\mathbf{Q}\mathbf{u}_0(\mathbf{x}) && \text{in } \Omega,\\ &\mathbf{w} = 0 && \text{on } S_T, \end{aligned} \tag{5.2}$$

and the displacement $\mathbf{u}$:

$$
(5.3)\qquad
\begin{aligned}
&\mathbf{u}_t - \alpha\mathbf{Q}\mathbf{u} = \mathbf{w} && \text{in } Q_T,\\
&\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) && \text{in } \Omega,\\
&\mathbf{u} = 0 && \text{on } S_T.
\end{aligned}
$$

The condition $0 < \sqrt{\kappa} \le \nu$ assures that $\alpha, \beta > 0$.

System (5.2), (5.3) for $\mathbf{w}, \mathbf{u}$ is coupled to the BVP for θ:

$$
(5.4)\qquad
\begin{aligned}
&c_v\theta_t - k\Delta\theta = \theta F_{/\theta\boldsymbol{\epsilon}}(\boldsymbol{\epsilon},\theta) : \boldsymbol{\epsilon}_t + \nu(\mathbf{A}\boldsymbol{\epsilon}_t) : \boldsymbol{\epsilon}_t + g && \text{in } Q_T,\\
&\theta(0,\mathbf{x}) = \theta_0(\mathbf{x}) && \text{in } \Omega,\\
&\boldsymbol{\nabla}\theta \cdot \mathbf{n} = 0 && \text{on } S_T.
\end{aligned}
$$

We note that a solution $(\mathbf{u},\theta) \in V(p)$ to system (5.2)–(5.4) satisfies, thanks to (5.1) and continuity of $\mathbf{u}$, $\boldsymbol{\nabla}\mathbf{u}$, $\boldsymbol{\nabla}^2\mathbf{u}$, $\mathbf{u}_t$ in Q_T, the following BVP for $\mathbf{u}$:

$$
(5.5)\qquad
\begin{aligned}
&\mathbf{u}_{tt} - \nu\mathbf{Q}\mathbf{u}_t + \frac{\kappa}{4}\mathbf{Q}\mathbf{Q}\mathbf{u} = \boldsymbol{\nabla}\cdot F_{/\boldsymbol{\epsilon}}(\boldsymbol{\epsilon},\theta) + \mathbf{b} && \text{in } Q_T,\\
&\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{u}_t(0,\mathbf{x}) = \mathbf{u}_1(\mathbf{x}) && \text{in } \Omega,\\
&\mathbf{u} = 0, \quad \mathbf{Q}\mathbf{u} = 0 && \text{on } S_T,
\end{aligned}
$$

and BVP (5.4) for θ. Therefore it is a solution to problem (P) in $V(p)$.

Step 2. To this system we apply the Leray–Schauder fixed point theorem, recalled here in one of its equivalent formulations for the reader's convenience.

THEOREM 5.1 (see [12]). *Let $\mathcal{B}$ be a Banach space. Assume that $T : [0,1] \times \mathcal{B} \to \mathcal{B}$ is a map with the following properties:*

(i) *For any fixed $\tau \in [0,1]$ the map $T(\tau,\cdot) : \mathcal{B} \to \mathcal{B}$ is completely continuous.*

(ii) *For every bounded subset $\mathcal{C}$ of $\mathcal{B}$, the family of maps $T(\cdot,\chi) : [0,1] \to \mathcal{B}$, $\chi \in \mathcal{C}$, is uniformly equicontinuous.*

(iii) *There is a bounded subset $\mathcal{C}$ of $\mathcal{B}$ such that any fixed point in $\mathcal{B}$ of $T(\tau,\cdot)$, $0 \le \tau \le 1$, is contained in $\mathcal{C}$.*

(iv) *$T(0,\cdot)$ has precisely one fixed point in $\mathcal{B}$.*

Then $T(1,\cdot)$ has at least one fixed point in $\mathcal{B}$.

We now define the map T_τ from $V(p)$ into $V(p)$,

$$
T_\tau : (\overline{\mathbf{u}}, \bar{\theta}) \mapsto (\mathbf{u},\theta), \qquad \tau \in [0,1],
$$

by means of the following three problems: BVP for $\mathbf{w}$:

$$
(5.6)\qquad
\begin{aligned}
&\mathbf{w}_t - \beta\mathbf{Q}\mathbf{w} = \tau[\boldsymbol{\nabla}\cdot F_{/\boldsymbol{\epsilon}}(\overline{\boldsymbol{\epsilon}},\bar{\theta}) + \mathbf{b}] && \text{in } Q_T,\\
&\mathbf{w}(0,\mathbf{x}) = \tau[\mathbf{u}_1(\mathbf{x}) - \alpha\mathbf{Q}\mathbf{u}_0(\mathbf{x})] && \text{in } \Omega,\\
&\mathbf{w} = 0 && \text{on } S_T,
\end{aligned}
$$

BVP for $\mathbf{u}$:

$$
(5.7)\qquad
\begin{aligned}
&\mathbf{u}_t - \alpha\mathbf{Q}\mathbf{u} = \mathbf{w} && \text{in } Q_T,\\
&\mathbf{u}(0,\mathbf{x}) = \tau\mathbf{u}_0(\mathbf{x}) && \text{in } \Omega,\\
&\mathbf{u} = 0 && \text{on } S_T,
\end{aligned}
$$

and BVP for θ:

$$
(5.8)\qquad
\begin{aligned}
&c_v\theta_t - k\Delta\theta = \tau[\bar\theta F_{/\theta\boldsymbol{\epsilon}}(\boldsymbol{\epsilon},\bar\theta) : \boldsymbol{\epsilon}_t + \nu(\mathbf{A}\boldsymbol{\epsilon}_t) : \boldsymbol{\epsilon}_t + g] &&\text{in } Q_T,\\
&\theta(0,\mathbf{x}) = \tau\theta_0(\mathbf{x}) &&\text{in } \Omega,\\
&\boldsymbol{\nabla}\theta\cdot\mathbf{n} = 0 &&\text{on } S_T,
\end{aligned}
$$

where $\bar{\boldsymbol{\epsilon}} = \boldsymbol{\epsilon}(\bar{\mathbf{u}})$.

Clearly, a fixed point of T_1 in $V(p)$ is equivalent to a solution $(\mathbf{u},\theta)$ in $V(p)$ of system (5.2)–(5.4), and thus is a solution to problem (P) in $V(p)$.

In further steps of the proof we shall verify the assumptions of Theorem 5.1.

Step 3. First we show that T_τ is well defined in $V(p)$, i.e. the image $T_\tau(V(p))$ belongs to $V(p)$. To establish the existence of solutions to system (5.6)–(5.8) in $V(p)$ we make use of the fundamental fact (see Lemma 7.1) that the system

$$
(5.9)\qquad
\begin{aligned}
&\mathbf{u}_t - \mathbf{Q}\mathbf{u} = \mathbf{f} &&\text{in } Q_T,\\
&\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) &&\text{in } \Omega,\\
&\mathbf{u} = 0 &&\text{on } S_T
\end{aligned}
$$

is parabolic in the general (Solonnikov) sense. This allows us to apply the Solonnikov theory of parabolic systems (see Remark 7.1). By this theory (see Corollary 7.1) the solution of system (5.9) satisfies for $1 < p < \infty$ the inequality

$$
(5.10)\qquad \|\mathbf{u}\|_{\mathbf{W}^{2,1}_p(Q_T)} \le \Lambda\{\|\mathbf{f}\|_{\mathbf{L}_p(Q_T)} + \|\mathbf{u}_0\|_{\mathbf{W}^{2-2/p}_p(\Omega)}\}.
$$

By imbedding it follows that $\bar{\boldsymbol{\epsilon}}$, $\boldsymbol{\nabla}\bar{\boldsymbol{\epsilon}}$, $\bar\theta$ are continuous in Q_T, $\nabla\bar\theta$ is in $\mathbf{L}_p(Q_T)$ for $p_n \le p < \infty$, and their corresponding norms are bounded by the $V(p)$-norm of $(\bar{\mathbf{u}},\bar\theta)$. This fact together with the equality

$$
(5.11)\qquad \boldsymbol{\nabla}\cdot F_{/\boldsymbol{\epsilon}}(\bar{\boldsymbol{\epsilon}},\bar\theta) = F_{/\epsilon_{ij}\epsilon_{kl}}(\bar{\boldsymbol{\epsilon}},\bar\theta)\bar\epsilon_{kl/j} + F_{/\epsilon_{ij}\theta}(\bar{\boldsymbol{\epsilon}},\bar\theta)\bar\theta_{/j}
$$

implies that the right-hand side of (5.6) can be majorized in $\mathbf{L}_p(Q_T)$-norm by $(\bar{\mathbf{u}},\bar\theta)$ in $V(p)$-norm. In consequence, the application of estimate (5.10) to BVP (5.6) implies that $\mathbf{w}\in\mathbf{W}^{2,1}_p(Q_T)$ and the corresponding bound holds. Applying subsequently this estimate to (5.7) we obtain the boundedness of $\mathbf{u}$ in $\mathbf{W}^{4,2}_p(Q_T)$.

As concerns the thermal part, we note that since by imbedding, $\boldsymbol{\epsilon}_t \in \mathbf{L}_p(Q_T)$ for $p_n \le p < \infty$, the right-hand side of (5.8) in $L_p(Q_T)$-norm can be estimated by $(\bar{\mathbf{u}},\bar\theta)$ in $V(p)$-norm. Then the classical parabolic theory (see [32, Thm. 9.1], [48]) implies that $\theta\in W^{2,1}_p(Q_T)$ and the corresponding bound holds. Therefore $T_\tau(\bar{\mathbf{u}},\bar\theta)\in V(p)$.

Step 4. We verify equicontinuity of T_τ with respect to τ. Let $(\bar{\mathbf{u}},\bar\theta)$ be in a bounded set in $V(p)$, and $(\mathbf{w}^i,\mathbf{u}^i,\theta^i)$, $i=1,2$, be two solutions of (5.6)–(5.8) corresponding to $\tau^i\in[0,1]$. By virtue of (5.10) we have

$$
(5.12)\qquad \|\mathbf{w}^1-\mathbf{w}^2\|_{\mathbf{W}^{2,1}_p(Q_T)},\quad \|\mathbf{u}^1-\mathbf{u}^2\|_{\mathbf{W}^{4,2}_p(Q_T)} \le \Lambda|\tau^1-\tau^2|.
$$

The difference $\eta = \theta^1-\theta^2$ satisfies the BVP

$$
(5.13)\qquad \begin{aligned}
&c_v\eta_t - k\Delta\eta = (\tau^1-\tau^2)P^1 + \tau^2(P^1-P^2) && \text{in } Q_T,\\
&\eta(0,\mathbf{x}) = (\tau^1-\tau^2)\theta_0(\mathbf{x}) && \text{in } \Omega,\\
&\nabla\eta\cdot\mathbf{n} = 0 && \text{on } S_T,
\end{aligned}
$$

where

$$P^i = \bar\theta F_{/\theta\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}^i,\bar\theta) : \boldsymbol{\epsilon}_t^i + \nu(\mathbf{A}\boldsymbol{\epsilon}_t^i) : \boldsymbol{\epsilon}_t^i + g.$$

By using estimate (5.12) we may bound from above the difference

$$
\begin{aligned}
P^1 - P^2 = \bar\theta(F_{/\theta\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}^1,\bar\theta) - F_{/\theta\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}^2,\bar\theta)) : \boldsymbol{\epsilon}_t^1 + \bar\theta F_{/\theta\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}^2,\bar\theta) : (\boldsymbol{\epsilon}_t^1 - \boldsymbol{\epsilon}_t^2)\\
+ \nu(\mathbf{A}(\boldsymbol{\epsilon}_t^1 - \boldsymbol{\epsilon}_t^2)) : \boldsymbol{\epsilon}_t^1 + \nu(\mathbf{A}\boldsymbol{\epsilon}_t^2) : (\boldsymbol{\epsilon}_t^1 - \boldsymbol{\epsilon}_t^2)
\end{aligned}
$$

in $L_p(Q_T)$-norm by $\Lambda|\tau^1-\tau^2|$. In consequence, by the classical parabolic theory,

$$(5.14)\qquad \|\theta^1-\theta^2\|_{\mathbf{W}_p^{2,1}(Q_T)} \le \Lambda|\tau^1-\tau^2|.$$

Thus assumption (ii) of the Leray–Schauder theorem is satisfied.

Step 5. We now show the uniqueness of the fixed point of T_τ for $\tau = 0$. By the regularity of the problem for $\tau = 0$ system (5.6)–(5.8) has the unique solution $(\mathbf{w},\mathbf{u},\theta) = (0,0,0)$. Therefore $V(p) \ni (\mathbf{u},\theta) = (0,0)$ is the unique fixed point of $T_0(\cdot)$.

Step 6. The essential part of the proof is the verification of assumption (iii) in the Leray–Schauder theorem, that is, finding an a priori bound for a fixed point of T_τ. Without loss of generality we may set $\tau = 1$. Let then $(\mathbf{u},\theta) \in V(p)$, $p_n \le p < \infty$, be a fixed point of T_1.

First we shall prove that the temperature is nonnegative, i.e., $\theta \ge 0$ in Q_T.

LEMMA 5.1. *If $(\mathbf{u},\theta)$ is a fixed point of T_1 in $V(p)$, $p_n \le p < \infty$, then $\theta \ge 0$ in Q_T.*

Proof. We consider the parabolic problem for η:

$$
(5.15)\qquad \begin{aligned}
&c_v\eta_t - k\Delta\eta - a\eta = f && \text{in } Q_T,\\
&\eta(0,\mathbf{x}) = \theta_0(\mathbf{x}) && \text{in } \Omega,\\
&\boldsymbol{\nabla}\eta\cdot\mathbf{n} = 0 && \text{on } S_T,
\end{aligned}
$$

where

$$a = F_{/\theta\boldsymbol{\epsilon}}(\boldsymbol{\epsilon},\theta) : \boldsymbol{\epsilon}_t, \qquad f = \nu(\mathbf{A}\boldsymbol{\epsilon}_t) : \boldsymbol{\epsilon}_t + g \ge 0.$$

First we check that (5.15) satisfies the assumptions of the classical stability theory [32] (see Thm. 7.4). Take $q = 7/4$, $q_1 = 2$, $r_1 = 1$ while $r = 7/3$ for $n = 2$ and $r = 7$ for $n = 3$. Then

$$
\begin{aligned}
\|a\|_{q,r,Q_T}^r &= \int_0^T \Big(\int_\Omega |a|^{7/4}\,dx\Big)^{r/q} dt \le \Lambda\int_0^T (|\Omega|^{9/16}\|\boldsymbol{\epsilon}_t\|_{\mathbf{L}_4(\Omega)}^{7/4})^{r/q}\,dt \le \Lambda,\\
\|f\|_{q_1,r_1,Q_T}^{r_1} &\le \Lambda(\|g\|_{q_1,r_1,Q_T} + \|(\mathbf{A}\boldsymbol{\epsilon}_t) : \boldsymbol{\epsilon}_t\|_{q_1,r_1,Q_T})\\
&\le \Lambda(T^{1/2}\|g\|_{L_2(Q_T)} + T^{1/2}\|\boldsymbol{\epsilon}_t\|_{\mathbf{L}_4(Q_T)}^2) \le \Lambda,
\end{aligned}
$$

where we have used the fact that by imbedding, θ, $\boldsymbol{\epsilon}$ are continuous in Q_T and $\boldsymbol{\epsilon}_t \in \mathbf{L}_p(\Omega)$, $p_n \le p < \infty$, for any $t \in I$.

Thus problem (5.15) has according to [32] (see Thm. 7.2) a unique solution in the space $V_2^{1,1/2}(Q_T)$. Now we take smooth functions a^m, f^m, θ_0^m converging to a, f, θ_0 in appropriate norms. The classical solution η^m to (5.15) with coefficients replaced by their regular counterparts is nonnegative. This follows from the maximum principle (see Corollary 7.2) and the fact that $\theta_0^m \geq 0$, $f^m \geq 0$. In addition, by the stability result (see Thm. 7.4), the following convergence in $V_2^{1,0}(Q_T)$ holds:

$$\lim_{m\to\infty} |\eta^m - \eta|_{Q_T} = 0.$$

Let ϕ be any nonnegative smooth function on Ω. Then

$$0 \leq \int_{\Omega_t} \phi\eta^m \, dx = \int_{\Omega_t} \phi(\eta^m - \eta)\, dx + \int_{\Omega_t} \phi\eta \, dx.$$

But

$$\int_{\Omega_t} \phi|\eta^m - \eta| \, dx \leq \Lambda |\eta^m - \eta|_{Q_T} \to 0.$$

Hence

$$\int_{\Omega_t} \phi\eta \, dx \geq 0$$

for any $t \in I$ and smooth $\phi \geq 0$. Therefore $\eta \geq 0$ a.e. in Q_T. It is now enough to observe that η coincides with θ, since (5.15) is equivalent to (5.4). ∎

The proof that $(\mathbf{u}, \theta)$ is a priori bounded in $V(p)$ requires a sequence of estimates which will be iteratively improved. The first are, as usual, the energy estimates.

LEMMA 5.2. *A fixed point of T_1 satisfies for any $t \in I$ the bound*

$$\int_{\Omega_t} \Big(c_v\theta + \frac{\gamma}{2}|\mathbf{u}_t|^2 + \frac{\gamma\kappa}{8}|\mathbf{Q}\mathbf{u}|^2 + \gamma c|\boldsymbol{\epsilon}|^{\bar{q}+1}\Big) dx + \frac{1}{2} a_\star \nu(\gamma - 1) \int_{Q_t} |\boldsymbol{\epsilon}_t|^2 dx\, dt' \leq \Lambda \tag{5.16}$$

with some positive constants c, $\gamma > 1$, and Λ depending on the initial data, the sources $\mathbf{b}, g$ and time horizon T.

Proof. Integrating the temperature equation in (5.4) over Q_t and using the boundary conditions gives

$$c_v \int_{Q_t} \frac{d}{dt}\theta \, dx\, dt' = \int_{Q_t} \theta F_{1/\theta\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}, \theta) : \boldsymbol{\epsilon}_t \, dx\, dt' + \nu \int_{Q_t} (\mathbf{A}\boldsymbol{\epsilon}_t) : \boldsymbol{\epsilon}_t \, dx\, dt' + \int_{Q_t} g \, dx\, dt'. \tag{5.17}$$

Multiplying the elasticity equation (5.5) by $\gamma\mathbf{u}_t$, $\gamma = \text{const} > 1$, and integrating over Q_t gives

$$\begin{aligned} \frac{\gamma}{2}\int_{Q_t} \frac{d}{dt}|\mathbf{u}_t|^2 \, dx\, dt' - \gamma\nu \int_{Q_t} (\mathbf{Q}\mathbf{u}_t)\cdot \mathbf{u}_t \, dx\, dt' + \frac{\gamma\kappa}{4}\int_{Q_t} (\mathbf{Q}\mathbf{Q}\mathbf{u})\cdot \mathbf{u}_t \, dx\, dt' \\ - \gamma \int_{Q_t} (\boldsymbol{\nabla}\cdot F_{/\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}, \theta))\cdot \mathbf{u}_t \, dx\, dt' = \gamma \int_{Q_t} \mathbf{b}\cdot \mathbf{u}_t \, dx\, dt'. \end{aligned} \tag{5.18}$$

The second integral on the left-hand side of (5.18), after integration by parts and using the boundary condition for $\mathbf{u}$, is

$$-\gamma\nu \int\limits_{Q_t} (\mathbf{Q}\mathbf{u}_t)\cdot \mathbf{u}_t\,dx\,dt' = \gamma\nu \int\limits_{Q_t} (\mathbf{A}\boldsymbol{\epsilon}_t) : \boldsymbol{\epsilon}_t\,dx\,dt'. \tag{5.19}$$

The third integral, after applying the integration by parts (2.7) and the boundary condition for $\mathbf{Qu}$, becomes

$$\frac{\gamma\kappa}{4} \int\limits_{Q_t} (\mathbf{QQu})\cdot \mathbf{u}_t\,dx\,dt' = \frac{\gamma\kappa}{4} \int\limits_{Q_t} (\mathbf{Qu})\cdot(\mathbf{Q}\mathbf{u}_t)\,dx\,dt' = \frac{\gamma\kappa}{8} \int\limits_{Q_t} \frac{d}{dt}|\mathbf{Qu}|^2\,dx\,dt'. \tag{5.20}$$

Finally, if we integrate by parts and use the boundary condition for $\mathbf{u}$, the fourth integral gives

$$\begin{aligned} -\gamma \int\limits_{Q_t} (\boldsymbol{\nabla}\cdot F_{/\boldsymbol{\epsilon}}(\boldsymbol{\epsilon},\theta))\cdot \mathbf{u}_t\,dx\,dt' &= \gamma \int\limits_{Q_t} F_{/\boldsymbol{\epsilon}}(\boldsymbol{\epsilon},\theta) : \boldsymbol{\epsilon}_t\,dx\,dt' \\ &= \gamma \int\limits_{Q_t} \frac{d}{dt}F_2(\boldsymbol{\epsilon})\,dx\,dt' + \gamma \int\limits_{Q_t} F_{1/\boldsymbol{\epsilon}}(\boldsymbol{\epsilon},\theta) : \boldsymbol{\epsilon}_t\,dx\,dt'. \end{aligned} \tag{5.21}$$

Using (5.19)–(5.21) in (5.18) and combining with (5.17) gives the identity

$$\begin{aligned} &\int\limits_{Q_t} \frac{d}{dt}\Big(c_v\theta + \frac{\gamma}{2}|\mathbf{u}_t|^2 + \frac{\gamma\kappa}{8}|\mathbf{Qu}|^2 + \gamma F_2(\boldsymbol{\epsilon})\Big)\,dx\,dt' + \nu(\gamma-1)\int\limits_{Q_t} (\mathbf{A}\boldsymbol{\epsilon}_t) : \boldsymbol{\epsilon}_t\,dx\,dt' \\ &\qquad = \int\limits_{Q_t} (\theta F_{1/\theta\boldsymbol{\epsilon}}(\boldsymbol{\epsilon},\theta) - \gamma F_{1/\boldsymbol{\epsilon}}(\boldsymbol{\epsilon},\theta)) : \boldsymbol{\epsilon}_t\,dx\,dt' + \int\limits_{Q_t} (g + \gamma\mathbf{b}\cdot\mathbf{u}_t)\,dx\,dt'. \end{aligned} \tag{5.22}$$

Hence, using assumption (FE-3) and the bound (2.4) we obtain

$$\begin{aligned} &\int\limits_{\Omega_t} \Big(c_v\theta + \frac{\gamma}{2}|\mathbf{u}_t|^2 + \frac{\gamma\kappa}{8}|\mathbf{Qu}|^2 + \gamma c|\boldsymbol{\epsilon}|^{\bar q+1}\Big)\,dx + a_\star\nu(\gamma-1)\int\limits_{Q_t} |\boldsymbol{\epsilon}_t|^2\,dx\,dt' \\ &\le \Lambda + \int\limits_{\Omega} \Big(c_v\theta_0 + \frac{\gamma}{2}|\mathbf{u}_1|^2 + \frac{\gamma\kappa}{8}|\mathbf{Qu}_0|^2 + \gamma F_2(\boldsymbol{\epsilon}_0)\Big)\,dx \\ &\quad + \int\limits_{Q_t} \Big(g + \frac{\gamma}{2}|\mathbf{b}|^2\Big)\,dx\,dt' + \int\limits_{Q_t} \frac{\gamma}{2}|\mathbf{u}_t|^2\,dx\,dt' + \int\limits_{Q_t} (\theta F_{1/\theta\boldsymbol{\epsilon}}(\boldsymbol{\epsilon},\theta) - \gamma F_{1/\boldsymbol{\epsilon}}(\boldsymbol{\epsilon},\theta)) : \boldsymbol{\epsilon}_t\,dx\,dt', \end{aligned} \tag{5.23}$$

where $\boldsymbol{\epsilon}_0 = \boldsymbol{\epsilon}(\mathbf{u}_0)$.

By Young's inequality the last integral on the right-hand side of (5.23) is estimated by

$$\begin{aligned} &\int\limits_{Q_t} (\theta F_{1/\theta\boldsymbol{\epsilon}}(\boldsymbol{\epsilon},\theta) - \gamma F_{1/\boldsymbol{\epsilon}}(\boldsymbol{\epsilon},\theta)) : \boldsymbol{\epsilon}_t\,dx\,dt' \\ &\qquad \le \frac{\delta}{2}\int\limits_{Q_t} |\boldsymbol{\epsilon}_t|^2\,dx\,dt' + \frac{1}{2\delta}\int\limits_{Q_t} |\theta F_{1/\theta\boldsymbol{\epsilon}}(\boldsymbol{\epsilon},\theta) - \gamma F_{1/\boldsymbol{\epsilon}}(\boldsymbol{\epsilon},\theta)|^2\,dx\,dt', \end{aligned} \tag{5.24}$$

where, with the appropriate choice of δ, the δ-integral is absorbed by the left-hand side of (5.23).

Applying assumptions (FE-1), (FE-2) to the second integral on the right-hand side of (5.24) gives

$$\int\limits_{Q_t} |\theta F_{1/\theta\boldsymbol{\epsilon}}(\boldsymbol{\epsilon},\theta) - \gamma F_{1/\boldsymbol{\epsilon}}(\boldsymbol{\epsilon},\theta)|^2\,dx\,dt' \le \Lambda + \Lambda \int\limits_{Q_t} \theta^{2r}|\boldsymbol{\epsilon}|^{2q}\,dx\,dt' \le \Lambda + \frac{\Lambda}{p_1}\int\limits_{Q_t}\theta^{2rp_1}\,dx\,dt' + \frac{\Lambda}{p_2}\int\limits_{Q_t}|\boldsymbol{\epsilon}|^{2qp_2}\,dx\,dt' \le \Lambda + \frac{\Lambda}{p_1}\int\limits_{Q_t}\theta\,dx\,dt' + \frac{\Lambda}{p_2}\int\limits_{Q_t}|\boldsymbol{\epsilon}|^{2q/(1-2r)}\,dx\,dt', \tag{5.25}$$

where we have used Young's inequality with $p_1 = 1/(2r)$, $p_2 = 1/(1-2r)$. By the condition on q the last integral in (5.25) is estimated by

$$\int\limits_{Q_t} |\boldsymbol{\epsilon}|^{2q/(1-2r)}\,dx\,dt' \le \int\limits_{Q_t} |\boldsymbol{\epsilon}|^{\bar q+1}\,dx\,dt'. \tag{5.26}$$

Consequently, combining (5.23) and (5.24)–(5.26) gives

$$\int\limits_{\Omega_t}\Big(c_v\theta + \frac{\gamma}{2}|\mathbf{u}_t|^2 + \frac{\gamma\kappa}{8}|\mathbf{Q}\mathbf{u}|^2 + \gamma c|\boldsymbol{\epsilon}|^{\bar q+1}\Big)\,dx + \frac{1}{2}a_\star\nu(\gamma-1)\int\limits_{Q_t}|\boldsymbol{\epsilon}_t|^2\,dx\,dt' \le \Lambda + \Lambda\int\limits_{Q_t}(\theta + |\mathbf{u}_t|^2 + |\boldsymbol{\epsilon}|^{\bar q+1})\,dx\,dt'. \tag{5.27}$$

Since, by Lemma 5.1, $\theta \ge 0$, applying Gronwall's inequality in (5.27) yields the assertion. ■

The energy estimates allow us to obtain more refined bounds for the fixed point. By the strong ellipticity property of the operator $\mathbf{Q}$ and the result of Nečas (see Lemma 7.4) it follows from (5.16) that

$$\|\mathbf{u}\|_{L_\infty(I;\mathbf{W}_2^2(\Omega))} \le \Lambda. \tag{5.28}$$

Consequently,

$$\|\boldsymbol{\epsilon}\|_{L_\infty(I;\mathbf{W}_2^1(\Omega))} \le \Lambda, \tag{5.29}$$

and by imbedding,

$$\|\boldsymbol{\epsilon}\|_{L_\infty(I;\mathbf{L}_{q_n}(\Omega))} \le \Lambda, \tag{5.30}$$

where q_n is the Sobolev exponent. Moreover, (5.16) gives

$$\mathbf{u} \in \mathbf{W}^{2,1}_{2,\infty}(Q_T) \quad \text{and} \quad \|\mathbf{u}\|_{\mathbf{W}^{2,1}_{2,\infty}(Q_T)} \le \Lambda. \tag{5.31}$$

Hence,

$$\boldsymbol{\epsilon} \in \mathbf{W}^{1,1/2}_{2,\infty}(Q_T) \quad \text{and} \quad \|\boldsymbol{\epsilon}\|_{\mathbf{W}^{1,1/2}_{2,\infty}(Q_T)} \le \Lambda, \tag{5.32}$$

so, by imbedding,

$$\boldsymbol{\epsilon} \in \mathbf{L}_p(Q_T) \quad \text{and} \quad \|\boldsymbol{\epsilon}\|_{\mathbf{L}_p(Q_T)} \le \Lambda \quad \text{for } p = \frac{2p_n}{n-2} = \frac{q_n p_n}{n}. \tag{5.33}$$

Our strategy now is to improve the estimates for $\boldsymbol{\epsilon}$. For this purpose we use the regularity property of parabolic systems (see Lemma 7.2). Applied to solutions of BVP (5.2), it gives the representation

$$\mathbf{w} - \mathbf{w}(0) = \mathbf{w}^0 + \sum_{i=1}^n \frac{\partial \mathbf{w}^i}{\partial x_i}, \tag{5.34}$$

where $\mathbf{w}(0) = \mathbf{u}_1 - \alpha\mathbf{Q}\mathbf{u}_0$, and $\mathbf{w}^i$, $i = 0, 1, \ldots, n$, are the solutions of the problems

$$(5.35)\qquad \begin{aligned} &\mathbf{w}^i_t - \beta\mathbf{Q}\mathbf{w}^i = \mathbf{h}^i && \text{in } Q_T,\\ &\mathbf{w}^i(0,\mathbf{x}) = 0 && \text{in } \Omega,\ 0 \le i \le n,\\ &\mathbf{w}^i = 0 && \text{on } S_T \text{ for } 0 \le i \le n-1,\\ &\frac{\partial \mathbf{w}^n}{\partial \mathbf{n}} = 0 && \text{on } S_T \text{ for } i = n, \end{aligned}$$

with $\mathbf{h}^0 = \mathbf{b} + \beta\mathbf{Q}\mathbf{w}(0)$, $\mathbf{h}^i = (F_{/\epsilon_{ki}}(\boldsymbol{\epsilon}, \theta))_{k=1,\ldots,n}$, and the estimate

$$(5.36)\qquad \|\mathbf{w} - \mathbf{w}(0)\|_{\mathbf{W}^{1,1/2}_p(Q_t)} \le \Lambda(\|\mathbf{b}\|_{\mathbf{L}_p(Q_t)} + \|\beta\mathbf{Q}\mathbf{w}(0)\|_{\mathbf{L}_p(\Omega)} + \|F_{/\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}, \theta)\|_{\mathbf{L}_p(Q_t)})$$

with the constant Λ depending on p, T, Ω.

We start utilizing (5.36) for $p = 4$. For this purpose we first estimate $\|F_{/\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}, \theta)\|_{\mathbf{L}_4(Q_t)}$ in terms of the norm $\|\theta\|_{L_2(Q_t)}$ which will be bounded later on in Lemma 5.4.

LEMMA 5.3. *The following inequality holds:*

$$(5.37)\qquad \|F_{/\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}, \theta)\|_{\mathbf{L}_4(Q_t)} \le \Lambda + \Lambda\|\theta\|^{1/2}_{L_2(Q_t)}.$$

Proof. We have

$$(5.38)\qquad \|F_{/\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}, \theta)\|^4_{\mathbf{L}_4(Q_t)} \le \Lambda(\|F_{1/\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}, \theta)\|^4_{\mathbf{L}_4(Q_t)} + \|F_{2/\boldsymbol{\epsilon}}(\boldsymbol{\epsilon})\|^4_{\mathbf{L}_4(Q_t)}).$$

Applying the growth condition (FE-2) and estimate (5.33) gives

$$(5.39)\qquad \begin{aligned} \|F_{1/\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}, \theta)\|^4_{\mathbf{L}_4(Q_t)} &\le \Lambda + \Lambda \int_{Q_t} \theta^{4r}|\boldsymbol{\epsilon}|^{4q}\,dx\,dt'\\ &\le \Lambda + \frac{\Lambda}{p_1}\int_{Q_t}\theta^{4rp_1}\,dx\,dt' + \frac{\Lambda}{p_2}\int_{Q_t}|\boldsymbol{\epsilon}|^{4qp_2}\,dx\,dt'\\ &\le \Lambda + \frac{\Lambda}{p_1}\int_{Q_t}\theta^{2}\,dx\,dt' + \frac{\Lambda}{p_2}\int_{Q_t}|\boldsymbol{\epsilon}|^{4q/(1-2r)}\,dx\,dt' \end{aligned}$$

for $p_1 = 1/(2r)$, $p_2 = 1/(1-2r) < \infty$. Since, by the assumption on q,

$$(5.40)\qquad \frac{4q}{1-2r} \le \frac{q_n p_n}{n},$$

the last integral in (5.39) is, due to (5.33), bounded by a constant Λ. Similarly, recalling the assumption on $\overline{q}$, we get

$$(5.41)\qquad \|F_{2/\boldsymbol{\epsilon}}(\boldsymbol{\epsilon})\|^4_{\mathbf{L}_4(Q_t)} \le \Lambda + \Lambda\int_{Q_t}|\boldsymbol{\epsilon}|^{4\overline{q}}\,dx\,dt' \le \Lambda + \Lambda\int_{Q_t}|\boldsymbol{\epsilon}|^{q_np_n/n}\,dx\,dt' \le \Lambda.$$

Combining (5.38), (5.39) and (5.41) gives estimate (5.37). ∎

By (5.37) and assumption (BV-p) it follows from (5.36) that

$$(5.42)\qquad \|\mathbf{w}\|_{\mathbf{W}^{1,1/2}_4(Q_t)} \le \Lambda + \Lambda\|\theta\|^{1/2}_{L_2(Q_t)}.$$

From this, using the regularity property of parabolic systems in Lemma 7.3, we get

$$(5.43)\qquad \|\nabla\mathbf{u}\|_{\mathbf{W}^{2,1}_4(Q_t)} \le \Lambda + \Lambda\|\theta\|^{1/2}_{L_2(Q_t)},$$

so

$$\|\boldsymbol{\epsilon}\|_{\mathbf{W}_4^{2,1}(Q_t)} \le \Lambda + \Lambda \|\theta\|_{L_2(Q_t)}^{1/2}. \tag{5.44}$$

With this estimate we are ready to prove the temperature bounds.

Lemma 5.4. *If* (5.44) *holds then there exists a constant Λ depending on the data such that the solution of BVP* (5.4) *satisfies for any $t \in I$ the following estimate:*

$$\int_{\Omega_t} \theta^2 \, dx + \int_{Q_t} |\nabla\theta|^2 \, dx\, dt' \le \Lambda. \tag{5.45}$$

Proof. We multiply the temperature equation (5.4) by θ and integrate over Q_t using the boundary conditions to get

$$\begin{aligned} \frac{c_v}{2} \int_{Q_t} \frac{d}{dt} \theta^2 \, dx\, dt' + k \int_{Q_t} |\nabla\theta|^2 \, dx\, dt' &= \int_{Q_t} \theta^2 F_{1/\theta\boldsymbol{\epsilon}}(\boldsymbol{\epsilon},\theta) : \boldsymbol{\epsilon}_t \, dx\, dt' \\ &\quad + \nu \int_{Q_t} \theta(\mathbf{A}\boldsymbol{\epsilon}_t) : \boldsymbol{\epsilon}_t \, dx\, dt' + \int_{Q_t} \theta g \, dx\, dt'. \end{aligned} \tag{5.46}$$

By Young's inequality the first integral on the right-hand side of (5.46) is estimated by

$$\int_{Q_t} \theta^2 F_{1/\theta\boldsymbol{\epsilon}}(\boldsymbol{\epsilon},\theta) : \boldsymbol{\epsilon}_t \, dx\, dt' \le \frac{1}{4} \int_{Q_t} |\boldsymbol{\epsilon}_t|^4 \, dx\, dt' + \frac{3}{4} \int_{Q_t} |\theta^2 F_{1/\theta\boldsymbol{\epsilon}}(\boldsymbol{\epsilon},\theta)|^{4/3} \, dx\, dt', \tag{5.47}$$

where, by (5.44), the $\boldsymbol{\epsilon}_t$-integral is bounded by

$$\Lambda + \Lambda \int_{Q_t} \theta^2 \, dx\, dt', \tag{5.48}$$

and for the second integral, using the growth condition, we have

$$\begin{aligned} \int_{Q_t} |\theta^2 F_{1/\theta\boldsymbol{\epsilon}}(\boldsymbol{\epsilon},\theta)|^{4/3} \, dx\, dt' &\le \Lambda + \int_{Q_t} \theta^{4(r+1)/3} |\boldsymbol{\epsilon}|^{4q/3} \, dx\, dt' \\ &\le \Lambda + \frac{\Lambda}{p_1} \int_{Q_t} \theta^{4(r+1)p_1/3} \, dx\, dt' + \frac{\Lambda}{p_2} \int_{Q_t} |\boldsymbol{\epsilon}|^{4qp_2/3} \, dx\, dt' \\ &\le \Lambda + \frac{\Lambda}{p_1} \int_{Q_t} \theta^2 \, dx\, dt' + \frac{\Lambda}{p_2} \int_{Q_t} |\boldsymbol{\epsilon}|^{4q/(1-2r)} \, dx\, dt' \end{aligned} \tag{5.49}$$

for $p_1 = 3/(2r+2)$, $p_2 = 3/(1-2r)$. In view of (5.40) and estimate (5.33), the right-hand side of (5.49) is bounded by the expression (5.48).

Now, by (5.44), the second integral on the right-hand side of (5.46) is estimated as follows:

$$\int_{Q_t} \theta(\mathbf{A}\boldsymbol{\epsilon}_t) : \boldsymbol{\epsilon}_t \, dx\, dt' \le \Lambda \int_{Q_t} |\boldsymbol{\epsilon}_t|^4 \, dx\, dt' + \Lambda \int_{Q_t} \theta^2 \, dx\, dt' \le \Lambda + \Lambda \int_{Q_t} \theta^2 \, dx\, dt'. \tag{5.50}$$

Clearly, the last term in (5.46) is also majorized by (5.48). Returning to (5.46) and incorporating the above gives

$$\frac{c_v}{2} \int_{\Omega_t} \theta^2 \, dx + k \int_{Q_t} |\nabla\theta|^2 \, dx\, dt' \le \frac{c_v}{2} \int_{\Omega} \theta_0^2 \, dx + \Lambda + \Lambda \int_{Q_t} \theta^2 \, dx\, dt'. \tag{5.51}$$

Now the application of Gronwall's inequality yields the assertion. ■

Utilizing temperature estimates (5.45) in (5.44) gives the bound

$$\|\boldsymbol{\epsilon}\|_{\mathbf{W}_4^{2,1}(Q_t)} \le \Lambda. \tag{5.52}$$

Hence, by imbedding, $\boldsymbol{\epsilon}$ is Hölder continuous in Q_T, and

$$|\boldsymbol{\epsilon}| \le \Lambda \quad \text{in } Q_T. \tag{5.53}$$

Moreover, by the imbedding of the space $L_\infty(I; L_2(\Omega)) \cap L_2(I; W_2^1(\Omega))$ in $L_p(Q_t)$ for $p > 2$ (see [13]), from (5.45) we have

$$\|\theta\|_{L_{2p_n/n}(Q_t)} \le \Lambda. \tag{5.54}$$

Thanks to (5.53) and (5.54), in the 3-D case we can further improve the estimates. Now we have

$$\|F_{/\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}, \theta)\|_{\mathbf{L}_{p_n}(Q_t)} \le \Lambda, \tag{5.55}$$

which results from the estimates

$$\begin{aligned} \int\limits_{Q_t} |F_{1/\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}, \theta)|^{p_n}\, dx\, dt' &\le \Lambda + \frac{\Lambda}{p_1} \int\limits_{Q_t} \theta^{r p_n p_1}\, dx\, dt' + \frac{\Lambda}{p_2} \int\limits_{Q_t} |\boldsymbol{\epsilon}|^{q p_n p_2}\, dx\, dt' \\ &\le \Lambda + \frac{\Lambda}{p_1} \int\limits_{Q_t} \theta^{2p_n/n}\, dx\, dt' + \frac{\Lambda}{p_2} \int\limits_{Q_t} |\boldsymbol{\epsilon}|^{2q p_n/(2-nr)}\, dx\, dt' \le \Lambda \end{aligned} \tag{5.56}$$

for $p_1 = 2/(nr)$, $p_2 = 2/(2 - nr)$, and

$$\int\limits_{Q_t} |F_{2/\boldsymbol{\epsilon}}(\boldsymbol{\epsilon})|^{p_n}\, dx\, dt' \le \Lambda + \Lambda \int\limits_{Q_t} |\boldsymbol{\epsilon}|^{\bar{q} p_n}\, dx\, dt' \le \Lambda. \tag{5.57}$$

So, returning to (5.36) gives

$$\|\mathbf{w}\|_{\mathbf{W}_{p_n}^{1,1/2}(Q_t)} \le \Lambda, \tag{5.58}$$

and subsequently,

$$\|\boldsymbol{\epsilon}\|_{\mathbf{W}_{p_n}^{2,1}(Q_T)} \le \Lambda. \tag{5.59}$$

By imbedding, we conclude from (5.59) that

$$\boldsymbol{\nabla}\boldsymbol{\epsilon} \in \mathbf{L}_p(Q_t) \quad \text{and} \quad \|\boldsymbol{\nabla}\boldsymbol{\epsilon}\|_{\mathbf{L}_p(Q_t)} \le \Lambda \quad \text{for } p_n \le p < \infty. \tag{5.60}$$

Our further procedure consists in applying to BVP (5.4) the classical parabolic theory ([32, Thm. IV.9.1]). We have

Lemma 5.5. *The following bound holds for the right-hand side of the temperature equation* (5.4)*:*

$$\|\theta F_{1/\theta\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}, \theta) : \boldsymbol{\epsilon}_t + \nu(\mathbf{A}\boldsymbol{\epsilon}_t) : \boldsymbol{\epsilon}_t + g\|_{L_{p_n/2}(Q_t)} \le \Lambda. \tag{5.61}$$

Proof. We have

$$\int\limits_{Q_t} |\theta F_{1/\theta\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}, \theta) : \boldsymbol{\epsilon}_t|^{p_n/2}\, dx\, dt' \le \frac{1}{2} \int\limits_{Q_t} |\boldsymbol{\epsilon}_t|^{p_n}\, dx\, dt' + \frac{1}{2} \int\limits_{Q_t} |\theta F_{1/\theta\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}, \theta)|^{p_n}\, dx\, dt',$$

where the first term on the right-hand side is, by (5.59), bounded by Λ. For the second one we have

$$\begin{aligned}\int_{Q_t} |\theta F_{1/\theta\boldsymbol{\epsilon}}(\boldsymbol{\epsilon},\theta)|^{p_n}\,dx\,dt' &\le \Lambda + \Lambda \int_{Q_t} \theta^{rp_n}|\boldsymbol{\epsilon}|^{qp_n}\,dx\,dt' \\ &\le \Lambda + \frac{\Lambda}{p_1}\int_{Q_t} \theta^{rp_np_1}\,dx\,dt' + \frac{\Lambda}{p_2}\int_{Q_t} |\boldsymbol{\epsilon}|^{qp_np_2}\,dx\,dt' \\ &\le \Lambda + \frac{\Lambda}{p_1}\int_{Q_t} \theta^{2p_n/n}\,dx\,dt' + \frac{\Lambda}{p_2}\int_{Q_t} |\boldsymbol{\epsilon}|^{2qp_n/(2-nr)}\,dx\,dt' \le \Lambda\end{aligned}$$

for $p_1 = 2/(nr)$, $p_2 = 2/(2-nr)$, where we have used the bounds (5.54) and (5.53).

Similarly, utilizing (5.59), we have

$$\int_{Q_t} |(\mathbf{A}\boldsymbol{\epsilon}_t) : \boldsymbol{\epsilon}_t|^{p_n/2}\,dx\,dt' \le \frac{1}{2}\int_{Q_t} |\mathbf{A}\boldsymbol{\epsilon}_t|^{p_n}\,dx\,dt' + \frac{1}{2}\int_{Q_t} |\boldsymbol{\epsilon}_t|^{p_n}\,dx\,dt' \le \Lambda.$$

This shows the assertion. ∎

Hence, the parabolic theory implies that

$$\theta \in W^{2,1}_{p_n/2}(Q_t) \quad \text{and} \quad \|\theta\|_{W^{2,1}_{p_n/2}(Q_t)} \le \Lambda, \tag{5.62}$$

so, by imbedding,

$$\nabla\theta \in \mathbf{L}_{p_n}(Q_t), \qquad \theta \in L_p(Q_t),$$

and

$$\|\nabla\theta\|_{\mathbf{L}_{p_n}(Q_t)} \le \Lambda, \qquad \|\theta\|_{L_p(Q_t)} \le \Lambda \qquad \text{for } p_n/2 \le p < \infty. \tag{5.63}$$

Now we are ready to improve iteratively a priori bounds. For this purpose we return to the decomposed system and estimate the right-hand side of the $\mathbf{w}$-equation (5.2).

Lemma 5.6. *The following bound holds for the right-hand side of* (5.2)*:*

$$\|\boldsymbol{\nabla}\cdot F_{/\boldsymbol{\epsilon}}(\boldsymbol{\epsilon},\theta)\|_{\mathbf{L}_{p_n}(Q_t)} \le \Lambda. \tag{5.64}$$

Proof. Applying the bound

$$\|\boldsymbol{\nabla}\cdot F_{/\boldsymbol{\epsilon}}(\boldsymbol{\epsilon},\theta)\|_{\mathbf{L}_{p_n}(Q_t)} \le \|\boldsymbol{\nabla}\cdot F_{1/\boldsymbol{\epsilon}}(\boldsymbol{\epsilon},\theta)\|_{\mathbf{L}_{p_n}(Q_t)} + \|\boldsymbol{\nabla}\cdot F_{2/\boldsymbol{\epsilon}}(\boldsymbol{\epsilon})\|_{\mathbf{L}_{p_n}(Q_t)}, \tag{5.65}$$

and using equality (5.11), we get

$$\begin{aligned}\int_{Q_t} |\boldsymbol{\nabla}\cdot F_{1/\boldsymbol{\epsilon}}(\boldsymbol{\epsilon},\theta)|^{p_n}\,dx\,dt' &\le \Lambda\int_{Q_t} |F_{1/\epsilon_{ij}\epsilon_{kl}}(\boldsymbol{\epsilon},\theta)\partial_j\epsilon_{kl}|^{p_n}\,dx\,dt' \\ &\quad + \Lambda\int_{Q_t} |F_{1/\epsilon_{ij}\theta}(\boldsymbol{\epsilon},\theta)\partial_j\theta|^{p_n}\,dx\,dt' \equiv I_1 + I_2.\end{aligned} \tag{5.66}$$

In view of the growth conditions, the term I_1 is estimated by

$$\begin{aligned}I_1 &\le \Lambda\int_{Q_t} (1+\theta^{rp_n}|\boldsymbol{\epsilon}|^{(q-1)p_n})|\boldsymbol{\nabla}\boldsymbol{\epsilon}|^{p_n}\,dx\,dt' \\ &\le \Lambda + \Lambda\int_{Q_t} \theta^{2rp_n}|\boldsymbol{\epsilon}|^{2(q-1)p_n}\,dx\,dt' + \int_{Q_t} |\boldsymbol{\nabla}\boldsymbol{\epsilon}|^{2p_n}\,dx\,dt' \le \Lambda,\end{aligned} \tag{5.67}$$

where in the last inequality we have used the bounds (5.53), (5.63) and (5.60).

Finally, recalling (FE-1), (FE-2), we get

$$I_2 \le \Lambda \int\limits_{Q_t \cap \{\theta < \theta_1\}} (1 + |\boldsymbol{\epsilon}|^{qp_n}) |\boldsymbol{\nabla}\theta|^{p_n} \, dx \, dt' + \Lambda \int\limits_{Q_t \cap \{\theta \ge \theta_1\}} \theta^{(r-1)p_n} |\boldsymbol{\epsilon}|^{qp_n} |\boldsymbol{\nabla}\theta|^{p_n} \, dx \, dt' \le \Lambda, \tag{5.68}$$

where we have used the continuity of θ and estimates (5.53), (5.63). Combining the above estimates yields the assertion. ■

Estimate (5.64) allows us to apply the Solonnikov theory of parabolic systems to BVP (5.2) and to conclude that

$$\mathbf{w} \in \mathbf{W}^{2,1}_{p_n}(Q_t) \quad \text{and} \quad \|\mathbf{w}\|_{\mathbf{W}^{2,1}_{p_n}(Q_t)} \le \Lambda. \tag{5.69}$$

Now, an application of this theory to BVP (5.3) provides

$$\mathbf{u} \in \mathbf{W}^{4,2}_{p_n}(Q_t) \quad \text{and} \quad \|\mathbf{u}\|_{\mathbf{W}^{4,2}_{p_n}(Q_t)} \le \Lambda. \tag{5.70}$$

Hence, by imbedding,

$$\boldsymbol{\epsilon}_t \in \mathbf{L}_p(Q_t) \quad \text{and} \quad \|\boldsymbol{\epsilon}_t\|_{\mathbf{L}_p(Q_t)} \le \Lambda \quad \text{for } p_n \le p < \infty. \tag{5.71}$$

With this estimate we return to the temperature equation and estimate its right-hand side in $L_p(Q_t)$-norm. We obtain, for any $p \ge p_n$,

$$\begin{aligned} \int\limits_{Q_t} |\theta F_{1/\theta \boldsymbol{\epsilon}}(\boldsymbol{\epsilon}, \theta) : \boldsymbol{\epsilon}_t|^p \, dx \, dt' &\le \frac{1}{2} \int\limits_{Q_t} |\theta F_{1/\theta \boldsymbol{\epsilon}}(\boldsymbol{\epsilon}, \theta)|^{2p} \, dx \, dt' + \frac{1}{2} \int\limits_{Q_t} |\boldsymbol{\epsilon}_t|^{2p} \, dx \, dt' \\ &\le \Lambda + \Lambda \int\limits_{Q_t} \theta^{2rp} |\boldsymbol{\epsilon}|^{2qp} \, dx \, dt' + \Lambda \int\limits_{Q_t} |\boldsymbol{\epsilon}_t|^{2p} \, dx \, dt' \le \Lambda, \end{aligned} \tag{5.72}$$

where we have used the pointwise estimates (5.53) on $\boldsymbol{\epsilon}$ and the L_p-estimates (5.63), (5.71) on θ and $\boldsymbol{\epsilon}_t$. Hence, by assumption (BV-p), the classical parabolic theory assures that

$$\theta \in W^{2,1}_p(Q_t) \quad \text{and} \quad \|\theta\|_{W^{2,1}_p(Q_t)} \le \Lambda \quad \text{for } p_n \le p < \infty. \tag{5.73}$$

Again, by imbedding it follows that θ is continuous in Q_t and

$$\theta \le \Lambda \quad \text{in } Q_t, \tag{5.74}$$

as well as

$$\boldsymbol{\nabla}\theta \in L_p(Q_t) \quad \text{and} \quad \|\boldsymbol{\nabla}\theta\|_{L_p(Q_t)} \le \Lambda \quad \text{for } p_n \le p < \infty. \tag{5.75}$$

In the last step, using the same arguments as in Lemma 5.6 and taking advantage of (5.74), (5.75) we estimate

$$\|\boldsymbol{\nabla} \cdot F_{/\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}, \theta)\|_{\mathbf{L}_p(Q_t)} \le \Lambda \quad \text{for } p_n \le p < \infty. \tag{5.76}$$

Therefore, returning to system (5.2)–(5.4) together with the Solonnikov theory gives the final estimates

$$\|\mathbf{w}\|_{\mathbf{W}^{2,1}_p(Q_t)} \le \Lambda \quad \text{and} \quad \|\mathbf{u}\|_{\mathbf{W}^{4,2}_p(Q_t)} \le \Lambda \quad \text{for } p_n \le p < \infty. \tag{5.77}$$

This completes the derivation of an a priori bound for a fixed point of the transformation T_1, meaning that assumption (iii) of the Leray–Schauder Theorem 5.1 is satisfied.

In the last part of the proof we demonstrate assumption (i) by showing that for fixed $\tau \in [0,1]$, T_τ maps bounded subsets into precompact subsets in $V(p)$.

Step 7: Complete continuity of T_τ. Let $(\overline{\mathbf{u}}^n, \bar{\theta}^n)$ be a bounded sequence in $V(p)$ such that

$$(\overline{\mathbf{u}}^n, \bar{\theta}^n) \rightharpoonup (\overline{\mathbf{u}}, \bar{\theta}) \quad \text{weakly in } V(p) \text{ as } n \to \infty.$$

We shall show that for the images of T_τ,

$$(\mathbf{u}^n, \theta^n) = T_\tau(\overline{\mathbf{u}}^n, \bar{\theta}^n), \tag{5.78}$$

we have

$$\mathbf{u}^n \to \mathbf{u} \quad \text{strongly in } \mathbf{W}_p^{4,2}(Q_T), \tag{5.79}$$

$$\theta^n \to \theta \quad \text{strongly in } W_p^{2,1}(Q_T) \tag{5.80}$$

as $n \to \infty$, where

$$(\mathbf{u}, \theta) = T_\tau(\overline{\mathbf{u}}, \bar{\theta}). \tag{5.81}$$

Due to the Aubin compactness theorem (see Thm. 7.6),

$$\overline{\mathbf{u}}^n \to \overline{\mathbf{u}} \quad \text{in } L_p(I, \mathbf{W}_p^3(\Omega)), \qquad \bar{\theta}^n \to \bar{\theta} \quad \text{in } L_p(I, W_p^1(\Omega)), \tag{5.82}$$

with both convergences being strong. By (5.11) it follows, in view of (5.82) and the regularity assumptions on $F(\boldsymbol{\epsilon}, \theta)$, that

$$\boldsymbol{\nabla} \cdot F_{/\boldsymbol{\epsilon}}(\overline{\boldsymbol{\epsilon}}^n, \bar{\theta}^n) \to \boldsymbol{\nabla} \cdot F_{/\boldsymbol{\epsilon}}(\overline{\boldsymbol{\epsilon}}, \bar{\theta}) \quad \text{strongly in } \mathbf{L}_p(Q_T), \tag{5.83}$$

where

$$\overline{\boldsymbol{\epsilon}}^n = \boldsymbol{\epsilon}(\overline{\mathbf{u}}^n), \qquad \overline{\boldsymbol{\epsilon}} = \boldsymbol{\epsilon}(\overline{\mathbf{u}}).$$

According to Solonnikov theory, the convergence (5.83) of the right-hand side of the equation for $\mathbf{w}$ in the definition of T_τ implies the convergence of the corresponding solutions $\mathbf{w}^n$:

$$\mathbf{w}^n \to \mathbf{w} \quad \text{strongly in } \mathbf{W}_p^{2,1}(Q_T).$$

Consequently, for solutions $\mathbf{u}^n$ of (5.7) the convergence (5.79) holds.

Let us now consider the convergence of θ^n. By the same arguments as in (5.72) we have, for any $p_n \le p < \infty$,

$$\|\bar{\theta}^n F_{/\theta\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}^n, \bar{\theta}^n) : \boldsymbol{\epsilon}_t^n + \nu(\mathbf{A}\boldsymbol{\epsilon}_t^n) : \boldsymbol{\epsilon}_t^n\|_{L_p(Q_T)} \le \Lambda.$$

Hence, by the classical parabolic theory,

$$\theta^n \in W_p^{2,1}(Q_T) \quad \text{and} \quad \|\theta^n\|_{W_p^{2,1}(Q_T)} \le \Lambda. \tag{5.84}$$

For brevity, set

$$P^n(\boldsymbol{\epsilon}^n, \bar{\theta}^n, \tau) = \tau[\bar{\theta}^n F_{/\theta\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}^n, \bar{\theta}^n) : \boldsymbol{\epsilon}_t^n + \nu(\mathbf{A}\boldsymbol{\epsilon}_t^n) : \boldsymbol{\epsilon}_t^n + g],$$
$$P(\boldsymbol{\epsilon}, \bar{\theta}, \tau) = \tau[\bar{\theta} F_{/\theta\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}, \bar{\theta}) : \boldsymbol{\epsilon}_t + \nu(\mathbf{A}\boldsymbol{\epsilon}_t) : \boldsymbol{\epsilon}_t + g],$$

where

$$\boldsymbol{\epsilon}^n = \boldsymbol{\epsilon}(\mathbf{u}^n), \qquad \boldsymbol{\epsilon} = \boldsymbol{\epsilon}(\mathbf{u}).$$

Then the difference $\eta^n = \theta^n - \theta$ satisfies the BVP

$$\begin{aligned} & c_v \eta^n_t - k\Delta\eta^n = P^n(\boldsymbol{\epsilon}^n, \bar\theta^n, \tau) - P(\boldsymbol{\epsilon}, \bar\theta, \tau) && \text{in } Q_T,\\ & \eta^n(0, \mathbf{x}) = 0 && \text{in } \Omega,\\ & \boldsymbol{\nabla}\eta^n \cdot \mathbf{n} = 0 && \text{on } S_T. \end{aligned} \tag{5.85}$$

In order to prove that $\eta^n \to 0$ strongly in $W^{2,1}_p(Q_T)$, by the classical theory, it is enough to show that the right-hand side of (5.85) converges to 0 in $L_p(Q_T)$-norm. We have

$$\begin{aligned} \int_{Q_t} & |P^n(\boldsymbol{\epsilon}^n, \bar\theta^n, \tau) - P(\boldsymbol{\epsilon}, \bar\theta, \tau)|^p \, dx\, dt' \\ & \le \int_{Q_t} |\bar\theta^n - \bar\theta|^p |F_{/\theta\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}^n, \bar\theta^n) : \boldsymbol{\epsilon}^n_t|^p \, dx\, dt' \\ & \quad + \int_{Q_t} \bar\theta^p |(F_{/\theta\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}^n, \bar\theta^n) - F_{/\theta\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}, \bar\theta)) : \boldsymbol{\epsilon}^n_t|^p \, dx\, dt' \\ & \quad + \int_{Q_t} \bar\theta^p |F_{/\theta\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}, \bar\theta) : (\boldsymbol{\epsilon}^n_t - \boldsymbol{\epsilon}_t)|^p \, dx\, dt' \\ & \quad + \int_{Q_t} (|(\mathbf{A}\boldsymbol{\epsilon}^n_t - \mathbf{A}\boldsymbol{\epsilon}_t) : \boldsymbol{\epsilon}^n_t|^p + |(\mathbf{A}\boldsymbol{\epsilon}_t) : (\boldsymbol{\epsilon}^n_t - \boldsymbol{\epsilon}_t)|^p) \, dx\, dt' \to 0 \end{aligned}$$

as $n \to \infty$, due to (5.79), (5.82), the continuity of $F_{/\theta\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}, \theta)$ and (5.71). This shows (5.80), and thereby the complete continuity of T_τ. The proof of existence is thus finished. ∎

6. Uniqueness proof

The proof consists in the direct comparison of two solutions by means of energy estimates and the application of Gronwall's inequality. Let $(\mathbf{u}^1, \theta^1), (\mathbf{u}^2, \theta^2) \in V(p)$ be two solutions corresponding to the same data. To simplify notation we set, for $i = 1, 2$,

$$\mathbf{v} = \mathbf{u}^2 - \mathbf{u}^1, \quad \eta = \theta^2 - \theta^1, \quad \boldsymbol{\epsilon}^i = \boldsymbol{\epsilon}(\mathbf{u}^i), \quad \boldsymbol{\epsilon}^i_t = \boldsymbol{\epsilon}(\mathbf{u}^i_t), \quad F^i_{/\boldsymbol{\epsilon}} = F_{/\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}^i, \theta^i), \quad F^i_{/\theta\boldsymbol{\epsilon}} = F_{/\theta\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}^i, \theta^i).$$

The difference $(\mathbf{v}, \eta) \in V(p)$ satisfies the BVP

$$\mathbf{v}_{tt} - \nu\mathbf{Q}\mathbf{v}_t + \frac{\kappa}{4}\mathbf{Q}\mathbf{Q}\mathbf{v} = \boldsymbol{\nabla} \cdot (F^2_{/\boldsymbol{\epsilon}} - F^1_{/\boldsymbol{\epsilon}}), \tag{6.1}$$

$$\begin{aligned} c_v\eta_t - k\Delta\eta &= (\theta^2 F^2_{/\theta\boldsymbol{\epsilon}} : \boldsymbol{\epsilon}^2_t - \theta^1 F^1_{/\theta\boldsymbol{\epsilon}} : \boldsymbol{\epsilon}^1_t) + (\nu(\mathbf{A}\boldsymbol{\epsilon}^2_t) : \boldsymbol{\epsilon}^2_t - \nu(\mathbf{A}\boldsymbol{\epsilon}^1_t) : \boldsymbol{\epsilon}^1_t) \\ &\equiv R_1 + R_2 \quad \text{in } Q_T, \end{aligned} \tag{6.2}$$

$$\mathbf{v}(0, x) = 0, \quad \mathbf{v}_t(0, x) = 0, \quad \eta(0, x) = 0 \quad \text{in } \Omega, \tag{6.3}$$

$$\mathbf{v} = \mathbf{Q}\mathbf{v} = 0, \quad \boldsymbol{\nabla}\eta \cdot \mathbf{n} = 0 \quad \text{on } S_T. \tag{6.4}$$

In the first step we obtain energy estimates for the mechanical part in terms of the L_2-norm of η. To this end we multiply (6.1) by $\mathbf{v}_t$ and integrate over Q_t to get

$$\frac{1}{2}\int\limits_{Q_t}\frac{d}{dt}|\mathbf{v}_t|^2\,dx\,dt' - \nu\int\limits_{Q_t}(\mathbf{Q}\mathbf{v}_t)\cdot\mathbf{v}_t\,dx\,dt' + \frac{\kappa}{4}\int\limits_{Q_t}(\mathbf{Q}\mathbf{Q}\mathbf{v})\cdot\mathbf{v}_t\,dx\,dt' - \int\limits_{Q_t}(\boldsymbol{\nabla}\cdot(F^2_{/\boldsymbol{\epsilon}} - F^1_{/\boldsymbol{\epsilon}}))\cdot\mathbf{v}_t\,dx\,dt' = 0. \tag{6.5}$$

Integration by parts (as in the proof of Lemma 5.2) and the use of initial conditions (6.3) yield

$$\int\limits_{\Omega_t}\left(\frac{1}{2}|\mathbf{v}_t|^2 + \frac{\kappa}{8}|\mathbf{Q}\mathbf{v}|^2\right)dx + \nu\int\limits_{Q_t}(\mathbf{A}\boldsymbol{\epsilon}(\mathbf{v}_t)):\boldsymbol{\epsilon}(\mathbf{v}_t)\,dx\,dt' = -\int\limits_{Q_t}(F^2_{/\boldsymbol{\epsilon}} - F^1_{/\boldsymbol{\epsilon}}):\boldsymbol{\epsilon}(\mathbf{v}_t)\,dx\,dt'. \tag{6.6}$$

Moreover, thanks to (6.3), we have

$$\frac{1}{2}\int\limits_{\Omega_t}|\boldsymbol{\epsilon}(\mathbf{v})|^2\,dx = \frac{1}{2}\int\limits_{Q_t}\frac{d}{dt}|\boldsymbol{\epsilon}(\mathbf{v})|^2\,dx\,dt' = \int\limits_{Q_t}\boldsymbol{\epsilon}(\mathbf{v}):\boldsymbol{\epsilon}(\mathbf{v}_t)\,dx\,dt'. \tag{6.7}$$

Combining (6.6), (6.7), and using the estimate

$$|F^2_{/\boldsymbol{\epsilon}} - F^1_{/\boldsymbol{\epsilon}}| \le \Lambda(|\boldsymbol{\epsilon}(\mathbf{v})| + |\eta|), \tag{6.8}$$

which follows from the regularity assumption on $F(\boldsymbol{\epsilon},\theta)$ and the uniform bounds on $\boldsymbol{\epsilon}^i,\theta^i$ in Q_T, by Young's inequality we arrive at

$$\int\limits_{\Omega_t}\left(\frac{1}{2}|\mathbf{v}_t|^2 + |\boldsymbol{\epsilon}(\mathbf{v})|^2 + \frac{\kappa}{8}|\mathbf{Q}\mathbf{v}|^2\right)dx + a_\star\nu\int\limits_{Q_t}|\boldsymbol{\epsilon}(\mathbf{v}_t)|^2\,dx\,dt' \le (\delta_1+\delta_2)\int\limits_{Q_t}|\boldsymbol{\epsilon}(\mathbf{v}_t)|^2\,dx\,dt' + \Lambda(\delta_1^{-1}+\delta_2^{-1})\int\limits_{Q_t}(|\boldsymbol{\epsilon}(\mathbf{v})|^2 + |\eta|^2)\,dx\,dt'. \tag{6.9}$$

With an appropriate choice of δ_i the $\boldsymbol{\epsilon}(\mathbf{v}_t)$-term is absorbed by the left-hand side. Next, the application of Gronwall's inequality implies that

$$\|\mathbf{v}_t\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))} + \|\boldsymbol{\epsilon}(\mathbf{v})\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))} + \|\mathbf{Q}\mathbf{v}\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))} + \|\boldsymbol{\epsilon}(\mathbf{v}_t)\|_{\mathbf{L}_2(Q_T)} \le \Lambda\|\eta\|_{L_2(Q_T)}. \tag{6.10}$$

Hence, from the ellipticity property of the operator $\mathbf{Q}$, it follows that

$$\|\mathbf{v}\|_{L_\infty(0,T;\mathbf{W}_2^2(\Omega))} \le \Lambda\|\eta\|_{L_2(Q_T)}. \tag{6.11}$$

The energy estimates for the thermal part follow by multiplying equation (6.2) by η and integrating over Q_t:

$$\frac{c_v}{2}\int\limits_{Q_t}\frac{d}{dt}\eta^2\,dx\,dt' + k\int\limits_{Q_t}|\boldsymbol{\nabla}\eta|^2\,dx\,dt' = \int\limits_{Q_t}(R_1+R_2)\eta\,dx\,dt'. \tag{6.12}$$

Because of the uniform bounds on $\boldsymbol{\epsilon}^i$, θ^i, $\boldsymbol{\epsilon}^i_t$ and C^3-regularity of $F_1(\boldsymbol{\epsilon},\theta)$ we have

$$|R_1|,|R_2| \le \Lambda(|\eta| + |\boldsymbol{\epsilon}(\mathbf{v})| + |\boldsymbol{\epsilon}(\mathbf{v}_t)|). \tag{6.13}$$

Hence, thanks to (6.10), the right-hand side of (6.12) is estimated by

$$\int\limits_{Q_t}(R_1+R_2)\eta\,dx\,dt' \le \Lambda\int\limits_{Q_t}\eta^2\,dx\,dt'. \tag{6.14}$$

Since $\eta(0,\mathbf{x}) = 0$ in Ω, the application of Gronwall's inequality to (6.12) implies that $\eta = 0$ in Q_t. Simultaneously, by inequality (6.11), $\mathbf{v} = 0$ in Q_T. This completes the proof of uniqueness. ■

7. Auxiliary results

The results obtained in the previous sections are based on several properties of solutions to parabolic systems. Most of them are contained in [32], [49], [48], [27] and only the ones which are frequently used are recalled here. However, many of these facts were formulated for other boundary conditions or in slightly different situations. Their adaptation often required modifications of proofs. In order to make the presentation complete, we show explicitly these changes and the modified parts of the proofs.

7.1. Parabolicity of systems with elasticity operator. We consider the system of equations

$$\mathbf{u}_t - \mathbf{Q}\mathbf{u} = \mathbf{f} \quad \text{in } Q_T \tag{7.1}$$

for the two- and three-dimensional cases. Its explicit forms are as follows: for $n = 2$,

$$\begin{aligned} u_{1/t} - [\gamma u_{1/11} + \mu u_{1/22} + (\lambda+\mu)u_{2/12}] &= f_1,\\ u_{2/t} - [\mu u_{2/11} + \gamma u_{2/22} + (\lambda+\mu)u_{1/12}] &= f_2, \end{aligned} \tag{7.2}$$

for $n = 3$,

$$\begin{aligned} u_{1/t} - [\gamma u_{1/11} + \mu u_{1/22} + \mu u_{1/33} + (\lambda+\mu)(u_{2/12} + u_{3/13})] &= f_1,\\ u_{2/t} - [\mu u_{2/11} + \gamma u_{2/22} + \mu u_{2/33} + (\lambda+\mu)(u_{1/12} + u_{3/23})] &= f_2,\\ u_{3/t} - [\mu u_{3/11} + \mu u_{3/22} + \gamma u_{3/33} + (\lambda+\mu)(u_{1/13} + u_{2/23})] &= f_3. \end{aligned} \tag{7.3}$$

Here $\gamma = \lambda + 2\mu > 0$ and $\mu > 0$.

We show the following fact:

LEMMA 7.1. *Let $n = 2$ and the domain be of the form*

$$\Omega = \{\mathbf{x} = (x_1, x_2) \mid x_2 > 0\}, \quad Q_T = (0,T)\times\Omega, \quad S_T = (0,T)\times\partial\Omega.$$

System (7.1) *with boundary conditions*

$$\text{(BU):} \quad \mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \quad \textit{in } \Omega, \quad \mathbf{u}(t,\mathbf{x}) = \mathbf{g} \quad \textit{on } S_T,$$

or

$$\text{(BN):} \quad \mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \quad \textit{in } \Omega, \quad \nabla\mathbf{u}\cdot\mathbf{n} = 0 \quad \textit{on } S_T,$$

is parabolic in the general (*Solonnikov*) *sense.*

Proof. We use the results of [49] and [16]. We must check that the differential operator in (7.2) is parabolic in the sense of Solonnikov, and the initial and boundary conditions satisfy the complementarity requirements.

Parabolicity. Rewrite the system in the form

$$\begin{aligned} &\mathcal{L}_0(p,\boldsymbol{\xi})\mathbf{u} = \mathbf{f} && \text{in } Q_T,\\ &C(p,\boldsymbol{\xi})\mathbf{u} = \mathbf{u}_0 && \text{in } \Omega,\\ &B_u(p,\boldsymbol{\xi})\mathbf{u} = \mathbf{g} \quad \text{or} \quad B_n(p,\boldsymbol{\xi})\mathbf{u} = 0 && \text{on } S_T. \end{aligned} \tag{7.4}$$

Here p denotes $\partial/\partial t$, and $\boldsymbol{\xi} = (\xi_1, \xi_2)$ corresponds to $D_x = (\partial/\partial x_1, \partial/\partial x_2)$. It is obvious that

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_u = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_n = \begin{bmatrix} \xi_2 & 0 \\ 0 & \xi_2 \end{bmatrix},$$

$$\mathcal{L}_0 = \begin{bmatrix} p - \gamma\xi_1^2 - \mu\xi_2^2 & -(\lambda+\mu)\xi_1\xi_2 \\ -(\lambda+\mu)\xi_1\xi_2 & p - \mu\xi_1^2 - \gamma\xi_2^2 \end{bmatrix}.$$

The condition of parabolicity states that all the roots of the equation

$$L_0(p, i\boldsymbol{\xi}) = \det[\mathcal{L}_0(p, i\boldsymbol{\xi})] = 0 \tag{7.5}$$

must satisfy uniformly in Q_T the inequality

$$\Re p_i < -\delta|\boldsymbol{\xi}|^2, \quad i = 1, 2, \tag{7.6}$$

for some constant $\delta > 0$. Equation (7.5) takes on the form

$$p^2 + p(\gamma+\mu)|\boldsymbol{\xi}|^2 + \gamma\mu|\boldsymbol{\xi}|^4 = 0,$$

which has roots

$$p_1 = -\mu|\boldsymbol{\xi}|^2, \quad p_2 = -\gamma|\boldsymbol{\xi}|^2,$$

and therefore condition (7.6) is satisfied for any $\delta \in (0, \min[\mu, \gamma])$.

Initial condition. Let $\widehat{\mathcal{L}}_0$ be the matrix associated with $\mathcal{L}_0$, that is,

$$\widehat{\mathcal{L}}_0(p, i\boldsymbol{\xi}) = L_0(p, i\boldsymbol{\xi}) \cdot \mathcal{L}_0^{-1}(p, i\boldsymbol{\xi}).$$

The rows of the matrix

$$\mathcal{P}(p) = C(p, 0) \cdot \widehat{\mathcal{L}}_0(p, 0)$$

should be linearly independent modulo the polynomial $L_0(p, 0) = p^2$. But

$$\mathcal{P}(p) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}$$

and thus the complementarity condition is obviously satisfied.

Boundary conditions. We assume that the pair $(\boldsymbol{\eta}, p)$ satisfies the inequalities

$$\Re p \geq -\delta|\boldsymbol{\eta}|^2, \quad |p| + |\boldsymbol{\eta}|^2 > 0, \quad \delta \in (0, \min[\mu, \gamma]). \tag{7.7}$$

Here we denote by $\boldsymbol{\eta} = (\xi_1, 0)$ a vector tangent to $\partial\Omega$. Let in addition $\boldsymbol{\nu}$ be the unit inward normal vector to $\partial\Omega$, $\boldsymbol{\nu} = (0, 1)$. We consider the equation in terms of τ:

$$L_0(p, i(\boldsymbol{\eta} + \tau\boldsymbol{\nu})) = 0$$

which has 2 roots with positive imaginary parts, τ_1^+, τ_2^+. We put

$$L_0^+(\tau) = (\tau - \tau_1^+)(\tau - \tau_2^+) = \tau^2 - a_1\tau + a_0,$$

where $a_1 = \tau_1^+ + \tau_2^+$, $a_0 = \tau_1^+\tau_2^+$. The condition which must be satisfied is that the columns of the matrix

$$\mathcal{R}(p, \xi_1, \tau) = B(p, i(\boldsymbol{\eta} + \tau\boldsymbol{\nu})) \cdot \widehat{\mathcal{L}}_0(p, i(\boldsymbol{\eta} + \tau\boldsymbol{\nu}))$$

considered as τ-polynomials should be linearly independent modulo the polynomial $L_0^+(\tau)$ for any $(\boldsymbol{\eta}, p)$ satisfying (7.7). Here B denotes B_u or B_n.

Let us analyze $B = B_u$ first. Then

$$\mathcal{R}(p,\xi_1,\tau) = \begin{bmatrix} p+\mu\xi_1^2+\gamma\tau^2 & -(\lambda+\mu)\xi_1\tau \\ -(\lambda+\mu)\xi_1\tau & p+\gamma\xi_1^2+\mu\tau^2 \end{bmatrix}.$$

Next we construct a matrix of remainders resulting from division by $L_0^+(\tau)$:

$$\mathcal{R}_m = \begin{bmatrix} a_1\gamma\tau-\gamma a_0+(p+\mu\xi_1^2) & -(\lambda+\mu)\xi_1\tau \\ -(\lambda+\mu)\xi_1\tau & a_1\mu\tau-\mu a_0+(p+\gamma\xi_1^2) \end{bmatrix}.$$

Finally, we form the matrix of 4 columns containing the coefficients of τ and of the constant terms:

$$\mathcal{R}'_m = \begin{bmatrix} a_1\gamma & -(\lambda+\mu)\xi_1 & -a_0\gamma+(p+\mu\xi_1^2) & 0 \\ -(\lambda+\mu)\xi_1 & a_1\mu & 0 & -a_0\mu+(p+\gamma\xi_1^2) \end{bmatrix}.$$

This matrix should have rank 2. Considering the columns {1,3} and {2,4} we see that, for $\xi_1 \neq 0$ and $\lambda+\mu \neq 0$, if the corresponding minors vanished simultaneously, then the following equalities would hold:

$$-a_0\gamma+(p+\mu\xi_1^2)=0,$$
$$-a_0\mu+(p+\gamma\xi_1^2)=0.$$

Hence,

$$p=-(\gamma+\mu)\xi_1^2,$$

and condition (7.7) is violated. For $\xi_1=0$ or $\lambda+\mu=0$ it is enough to consider the minor {1,2}, since a_1 cannot be 0 as a sum of terms with positive imaginary part. Therefore at least one of these minors must be nonzero.

The matrix $B_n(i(\boldsymbol{\eta}+\tau\boldsymbol{\nu}))$ is proportional to B_u, $B_n(i(\boldsymbol{\eta}+\tau\boldsymbol{\nu}))=i\tau B_u$. The matrix $\mathcal{R}_m$ takes on the form

$$\mathcal{R}_m = \begin{bmatrix} [\gamma(a_1-a_0)+(p+\mu\xi_1^2)]\tau+\gamma a_1a_0 & -a_1(\lambda+\mu)\xi_1\tau+a_0(\lambda+\mu)\xi_1 \\ -a_1(\lambda+\mu)\xi_1\tau+a_0(\lambda+\mu)\xi_1 & [\mu(a_1-a_0)+(p+\gamma\xi_1^2)]\tau+\mu a_1a_0 \end{bmatrix}.$$

As in the former case, the matrix of coefficients of τ and constant terms is

$$\mathcal{R}'_m = \begin{bmatrix} \gamma(a_1-a_0)+p+\mu\xi_1^2 & -a_1(\lambda+\mu)\xi_1 & \gamma a_1a_0 & a_0(\lambda+\mu)\xi_1 \\ -a_1(\lambda+\mu)\xi_1 & \mu(a_1-a_0)+p+\gamma\xi_1^2 & a_0(\lambda+\mu)\xi_1 & \mu a_1a_0 \end{bmatrix}.$$

For $\xi_1=0$ or $\lambda+\mu=0$ we consider the minor $\{3,4\}$ and obviously $\gamma\mu(a_1a_0)^2\neq 0$. If $\xi_1\neq 0$ and $\lambda+\mu\neq 0$ the minors $\{2,3\}$ and $\{1,4\}$ lead to simultaneous equations

$$\gamma p+\gamma^2\xi_1^2+\gamma\mu(a_1-a_0)+(\gamma-\mu)^2\xi_1^2=0,$$
$$\mu p+\mu^2\xi_1^2+\gamma\mu(a_1-a_0)+(\gamma-\mu)^2\xi_1^2=0,$$

and consequently to

$$(\gamma-\mu)p+(\gamma^2-\mu^2)\xi_1^2=0,$$

or $p=-(\gamma+\mu)\xi_1^2$, which contradicts (7.7). ■

The assertion of this lemma is generalized, using a partition of unity and standard arguments, to the case of domains with C^2-boundaries. This also allows us to obtain two important corollaries. The first one concerns the regularity of solutions.

COROLLARY 7.1. *The solution of system* (7.2), *for a domain with* C^2*-boundary, satisfies the following inequalities: for the case BU:*

$$\|\mathbf{u}\|_{\mathbf{W}_p^{2,1}(Q_T)} \le \Lambda\{\|\mathbf{f}\|_{\mathbf{L}_p(Q_T)} + \|\mathbf{u}_0\|_{\mathbf{W}_p^{2-2/p}(\Omega)} + \|\mathbf{g}\|_{\mathbf{W}_p^{2-1/p,1-1/2p}(S_T)}\},$$

and for the case BN:

$$\|\mathbf{u}\|_{\mathbf{W}_p^{2,1}(Q_T)} \le \Lambda\{\|\mathbf{f}\|_{\mathbf{L}_p(Q_T)} + \|\mathbf{u}_0\|_{\mathbf{W}_p^{2-2/p}(\Omega)}\},$$

where $p > 1$ *and* Λ *is a constant depending on* Ω *and* T.

The proof follows from Lemma 7.1 and the results of [48, Thm. 5.4, with $t_j = 2$, $s_k = 0$, $\varrho_\alpha = -2$, $l = 1$, $\sigma_q = -2$, $j, k, q, \alpha = 1, 2$] (see also [27, Thm. 4.3 and Corollary 4.5]).

We remark that Corollary 7.1 follows from the properties of the general Green functions for appropriate systems of equations. Let us mention their particular exact form for the case $n = 2$ according to [55], which clearly shows the role of the coefficients μ and γ. If we set ($r^2 = x_1^2 + x_2^2$)

$$\begin{aligned}
a(t,\mathbf{x}) &= \frac{1}{2\gamma t r^2}\left[\gamma x_2^2 \exp\left(-\frac{r^2}{4\mu t}\right) + \mu x_1^2 \exp\left(-\frac{r^2}{4\gamma t}\right)\right],\\
b(t,\mathbf{x}) &= \frac{1}{2\gamma t r^2}\left[\gamma x_1^2 \exp\left(-\frac{r^2}{4\mu t}\right) + \mu x_2^2 \exp\left(-\frac{r^2}{4\gamma t}\right)\right],\\
c(t,\mathbf{x}) &= \mu\,\frac{x_2^2 - x_1^2}{r^4}\left[\exp\left(-\frac{r^2}{4\mu t}\right) - \exp\left(-\frac{r^2}{4\gamma t}\right)\right],\\
d(t,\mathbf{x}) &= \frac{x_1 x_2}{2\gamma t r^2}\left[\mu \exp\left(-\frac{r^2}{4\gamma t}\right) - \gamma \exp\left(-\frac{r^2}{4\mu t}\right)\right],\\
e(t,\mathbf{x}) &= 2\mu\,\frac{x_1 x_2}{r^4}\left[\exp\left(-\frac{r^2}{4\gamma t}\right) - \exp\left(-\frac{r^2}{4\mu t}\right)\right],
\end{aligned}$$

then the matrix potential function may be written as

$$G(x,t) = \frac{1}{2\pi\mu}\begin{bmatrix} a+c & d+e \\ d+e & b-c \end{bmatrix}. \tag{7.8}$$

Finally, let us comment on the three-dimensional case.

REMARK 7.1. *System* (7.1) *for* $n = 3$ *with initial and boundary conditions as in Lemma* 7.1 *is parabolic in the general* (*Solonnikov*) *sense.*

Proof. The polynomial (7.5) deciding about parabolicity may be written as

$$p^3 + p^2(\gamma + 2\mu)|\boldsymbol{\xi}|^2 + p\mu(2\gamma + \mu)|\boldsymbol{\xi}|^4 + \mu^2\gamma|\boldsymbol{\xi}|^6 = 0.$$

It is easy to see that it has roots

$$p_1 = p_2 = -\mu|\boldsymbol{\xi}|^2, \qquad p_3 = -\gamma|\boldsymbol{\xi}|^2,$$

which satisfy inequality (7.6). We may check the complementarity requirements for the initial and boundary conditions as in Lemma 7.1. ∎

In consequence, the three-dimensional equivalent of Corollary 7.1 holds.

7.2. Additional regularity of solutions. In this section we shall describe situations when the solutions to parabolic and elliptic systems are more regular, provided the right-hand sides have some additional properties.

The first result is a generalization of the Friedman–Nečas result (see Lemma 2.1 in [25]) to the case of parabolic systems.

LEMMA 7.2. *Let* $\mathbf{u}$ *be a solution for* $n = 2$ *or* $n = 3$ *of the system*

$$\begin{aligned} &\mathbf{u}_t - \mathbf{Q}\mathbf{u} = \mathbf{f} + \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} && \text{in } Q_T,\\ &\mathbf{u}(0,\mathbf{x}) = 0 && \text{in } \Omega,\\ &\mathbf{u} = 0 && \text{on } S_T, \end{aligned} \tag{7.9}$$

where $\mathbf{f} = (f_i)$ *and* $\boldsymbol{\sigma} = (\sigma_{ij})$. *If* $\mathbf{f} \in \mathbf{L}_p(Q_T)$, $\boldsymbol{\sigma} \in \mathbf{L}_p(Q_T)$ *for* $1 < p < \infty$, *then, for every* $0 < t < T$,

$$\|\mathbf{u}\|_{\mathbf{W}_p^{1,1/2}(Q_t)} \le \Lambda_p\{\|\mathbf{f}\|_{\mathbf{L}_p(Q_t)} + \|\boldsymbol{\sigma}\|_{\mathbf{L}_p(Q_t)}\}, \tag{7.10}$$

where the constant Λ_p *depends on* Ω, γ, μ, T *and* p. *In addition we have the representation*

$$\mathbf{u} = \mathbf{u}^0 + \sum_{i=1}^n \frac{\partial \mathbf{u}^i}{\partial x_i}, \tag{7.11}$$

where $\mathbf{u}^i$, $i = 0, 1, \ldots, n$, *satisfy the BVP's*

$$\begin{aligned} &\mathbf{u}^i_t - \mathbf{Q}\mathbf{u}^i = \mathbf{h}^i && \text{in } Q_T,\\ &\mathbf{u}^i(0,\mathbf{x}) = 0 && \text{in } \Omega,\ 0 \le i \le n,\\ &\mathbf{u}^i = 0 && \text{on } S_T \text{ for } 0 \le i \le n-1,\\ &\frac{\partial \mathbf{u}^n}{\partial \mathbf{n}} = 0 && \text{on } S_T \text{ for } i = n, \end{aligned} \tag{7.12}$$

with $\mathbf{h}^0 = \mathbf{f}$, *and* $\mathbf{h}^i = (\sigma_{ki})_{k=1,\ldots,n}$ *for* $i = 1, \ldots, n$.

Proof. We follow the arguments of [25] where the single equation is considered.

First assume that $\Omega = \{x_n > 0\}$. Let $\mathbf{u}^i$, $i = 0, 1, \ldots, n$, be the solutions of the following auxiliary BVP's:

$$\begin{aligned} &\mathbf{u}^i_t - \mathbf{Q}\mathbf{u}^i = \mathbf{h}^i && \text{in } Q_T,\\ &\mathbf{u}^i(0,\mathbf{x}) = 0 && \text{in } \Omega,\ 0 \le i \le n,\\ &\mathbf{u}^i = 0 && \text{on } (0,T)\times\{x_n = 0\} \text{ for } 0 \le i \le n-1,\\ &\frac{\partial \mathbf{u}^n}{\partial x_n} = 0 && \text{on } (0,T)\times\{x_n = 0\} \text{ for } i = n. \end{aligned} \tag{7.13}$$

Then the solution of (7.9) is given by (7.11). Thanks to the regularity result in Corollary 7.1 we have

$$\|\mathbf{u}^i\|_{\mathbf{W}_p^{2,1}(Q_T)} \le \Lambda\|\mathbf{h}^i\|_{\mathbf{L}_p(Q_T)}, \qquad 0 \le i \le n,$$

so, by imbedding,

$$\left\|\frac{\partial \mathbf{u}^i}{\partial x_i}\right\|_{\mathbf{W}_p^{1,1/2}(Q_T)} \le \Lambda\|\mathbf{h}^i\|_{\mathbf{L}_p(Q_T)}, \qquad 0 \le i \le n.$$

Hence, if we recall (7.11), estimate (7.10) follows. By standard arguments using a partition of unity, (7.10) and (7.13) are extended to general domains Ω. ∎

The next lemma concerns the situation when the functions represented in the form (7.11) become in turn the right-hand sides of equations or systems.

We consider two BVP's:

$$\begin{aligned} &u_t - \Delta u = f && \text{in } Q_T,\\ &u(0,\mathbf{x}) = u_0 && \text{in } \Omega,\\ &u = 0 && \text{on } S_T, \end{aligned} \tag{7.14}$$

and

$$\begin{aligned} &\mathbf{u}_t - \mathbf{Q}\mathbf{u} = \mathbf{f} && \text{in } Q_T,\\ &\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0 && \text{in } \Omega,\\ &\mathbf{u} = 0 && \text{on } S_T. \end{aligned} \tag{7.15}$$

In addition we assume that $u_0 \in W_p^{4-2/p}(\Omega)$, $\mathbf{u}_0 \in \mathbf{W}_p^{4-2/p}(\Omega)$, $1 < p < \infty$, and the data f, $\mathbf{f}$, u_0, $\mathbf{u}_0$ satisfy the compatibility requirements corresponding to the solvability of (7.14) and (7.15) in $W_p^{4,2}(Q_T)$ or $\mathbf{W}_p^{4,2}(Q_T)$. Then we have the following.

LEMMA 7.3. *Assume that* $f = D_{x_i}F$, $\mathbf{f} = D_{x_i}\mathbf{F}$ *for some* $i \in \{1,\dots,n\}$, *and*

$$\|F\|_{W_p^{2,1}(Q_T)} \le K_1, \qquad \|\mathbf{F}\|_{\mathbf{W}_p^{2,1}(Q_T)} \le K_2.$$

Then, for any $j \in \{1,\dots,n\}$,

$$\|D_{x_i}u\|_{W_p^{2,1}(Q_T)} \le \Lambda(K_1 + \|u_0\|_{W_p^{4-2/p}(\Omega)}), \qquad \|D_{x_i}\mathbf{u}\|_{\mathbf{W}_p^{2,1}(Q_T)} \le \Lambda(K_2 + \|\mathbf{u}_0\|_{\mathbf{W}_p^{4-2/p}(\Omega)}).$$

Proof. Let again $\Omega = \{\mathbf{x} \in \mathbb{R}^n \mid x_n > 0\}$, $Q_T = (0,T) \times \Omega$ and $S_T = (0,T) \times \partial\Omega$. For simplicity we shall prove the statement in case $n = 2$, since the method for $n = 3$ is exactly the same. The claim is obvious for $w = D_{x_1}u$ because it satisfies the BVP

$$\begin{aligned} &w_t - \Delta w = D_{x_1} f && \text{in } Q_T,\\ &w(0,\mathbf{x}) = D_{x_1} u_0 && \text{in } \Omega,\\ &w = 0 && \text{on } S_T, \end{aligned} \tag{7.16}$$

and the regularity result of Corollary 7.1 applies, since $D_{x_1}f = D_{x_1x_i}F \in L_p(Q_T)$. The same concerns $\mathbf{w} = D_{x_1}\mathbf{u}$. It remains to consider $v = D_{x_2}u$ and $\mathbf{v} = D_{x_2}\mathbf{u}$. We claim that v satisfies another BVP,

$$\begin{aligned} &v_t - \Delta v = D_{x_2} f && \text{in } Q_T,\\ &v(0,\mathbf{x}) = D_{x_2} u_0 && \text{in } \Omega,\\ &\frac{\partial v}{\partial \mathbf{n}} = f && \text{on } S_T. \end{aligned} \tag{7.17}$$

Informally this follows by observing that $u_t = 0$, $u_{/11} = 0$ on S_T, hence from the equation itself we may get

$$-u_{/22} = -v_{/2} = \frac{\partial v}{\partial \mathbf{n}} = f.$$

This can be made rigorous by considering smooth $\widetilde{F}$ approximating F in $W_p^{2,1}(Q_T)$, analyzing the classical solution and a density argument.

But f on S_T is the trace of $D_{x_i}F$ with $F \in W_p^{2,1}(Q_T)$, hence it satisfies the regularity requirements for application of the classical regularity theory.

Now consider $\mathbf{v} = D_{x_2}\mathbf{u}$. By the same reasoning it satisfies

$$\begin{aligned} &\mathbf{v}_t - \mathbf{Q}\mathbf{v} = D_{x_2}\mathbf{f} && \text{in } Q_T,\\ &\mathbf{v}(0,\mathbf{x}) = D_{x_2}\mathbf{u}_0 && \text{in } \Omega,\\ &\mathbf{B}(\boldsymbol{\xi})\mathbf{v} = -\mathbf{f} && \text{on } S_T, \end{aligned} \tag{7.18}$$

where $\boldsymbol{\xi} = (\xi_1,\xi_2)$ and $\xi_1 = D_{x_1}$, $\xi_2 = D_{x_2}$. The boundary operator has the form

$$\mathbf{B}(\boldsymbol{\xi}) = \begin{bmatrix} \mu\xi_2 & (\lambda+\mu)\xi_1 \\ (\lambda+\mu)\xi_1 & \gamma\xi_2 \end{bmatrix}.$$

As in Lemma 7.1 it may be checked that system (7.18) is parabolic, so Corollary 7.1 applies. Finally, by bounding the $W_p^{2,1}$- or $\mathbf{W}_p^{2,1}(Q_T)$-norms of solutions to (7.16), (7.17), (7.18) in terms of data we get the required estimates. This result extends to general domains in the same way as Lemma 7.2 does. ■

The last lemma concerning the properties of strongly elliptic operators is due to Nečas (see [35, p. 260], also [7, p. 296]).

We say that the differential operator $\mathbf{L}$ acting on $\mathbf{u} = (u_1,u_2,u_3)$ defined by

$$\mathbf{L}\mathbf{u} = \frac{\partial}{\partial x_i} A_{rsij} \frac{\partial u_s}{\partial x_j}$$

is *strongly elliptic* if there exists a constant $\delta > 0$ such that

$$A_{rsij}\xi_i\xi_j\eta_r\eta_s \ge \delta|\boldsymbol{\xi}|^2|\boldsymbol{\eta}|^2 \quad \text{or} \quad A_{rsij}\xi_i\xi_j\eta_r\eta_s \le -\delta\,|\boldsymbol{\xi}|^2|\boldsymbol{\eta}|^2$$

for any $\boldsymbol{\xi},\boldsymbol{\eta} \in \mathbb{R}^n \setminus \{0\}$. Now in order to make the presentation selfcontained we shall show the known result (see e.g. [28, Sect. 29, Ex. 3]) that for $\mu > 0$ and $\gamma = \lambda + 2\mu > 0$ the operator $\mathbf{Q}$ has this property. Indeed, from (7.3) it can be easily deduced that

$$\begin{aligned} A_{1111} = A_{2222} = A_{3333} &= \gamma,\\ A_{1122} = A_{1133} = A_{2211} = A_{2233} = A_{3311} = A_{3322} &= \mu,\\ A_{1212} = A_{1313} = A_{2121} = A_{2323} = A_{3131} = A_{3232} &= \lambda+\mu. \end{aligned}$$

Define a symmetric matrix $B(\boldsymbol{\xi}) = [b_{rs}]$ by

$$b_{rs} = A_{rsij}\xi_i\xi_j.$$

With this notation

$$A_{rsij}\xi_i\xi_j\eta_r\eta_s = \boldsymbol{\eta}^T B(\boldsymbol{\xi})\boldsymbol{\eta}.$$

Assume for the moment that the eigenvalues of $B(\boldsymbol{\xi})$ are all negative, say $-\lambda_1(\boldsymbol{\xi})$, $-\lambda_2(\boldsymbol{\xi})$, $-\lambda_3(\boldsymbol{\xi})$. Then

$$\boldsymbol{\eta}^T B(\boldsymbol{\xi})\boldsymbol{\eta} \le -\min[\lambda_1,\lambda_2,\lambda_3]\,|\boldsymbol{\eta}|^2. \tag{7.19}$$

But, as is easily seen, $B(\boldsymbol{\xi}) = -\mathcal{L}_0(0,\boldsymbol{\xi})$ (see (7.4)), and the eigenvalue problem for it is the same as (7.5), namely

$$\det[\mathcal{L}_0(p,i\boldsymbol{\xi})] = 0,$$

with roots $p_1 = p_2 = -\mu|\boldsymbol{\xi}|^2$, $p_3 = -\gamma|\boldsymbol{\xi}|^2$. Hence

$$\min[\lambda_1, \lambda_2, \lambda_3] = \min[\mu, \gamma]|\boldsymbol{\xi}|^2.$$

Substituting this equality into (7.19) gives the required result,

$$\boldsymbol{\eta}^T B(\boldsymbol{\xi})\boldsymbol{\eta} \le -\min[\mu, \gamma]|\boldsymbol{\xi}|^2|\boldsymbol{\eta}|^2.$$

Therefore the operator $\mathbf{Q}$ is strongly elliptic and the result of Nečas applies.

LEMMA 7.4. *Assume $\partial\Omega \in C^2$ and $\mu > 0$, $\lambda + 2\mu > 0$. Then for solutions of the problem*

$$\mathbf{Qu} = \mathbf{f} \quad \text{in } \Omega,$$
$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega,$$

where $\mathbf{f} \in \mathbf{L}_2(\Omega)$, we have the bound

$$\|\mathbf{u}\|_{\mathbf{W}_2^2(\Omega)} \le \Lambda\|\mathbf{f}\|_{\mathbf{L}_2(\Omega)}.$$

7.3. The maximum principle for regular data. This section contains some conclusions which may be drawn from the results of [32]. For a domain Ω with C^2-boundary we consider the equation

$$u_t - a_{ij}u_{/ij} + au = f \quad \text{in } Q_T, \tag{7.20}$$

with initial condition

$$u(0, \mathbf{x}) = \psi_0(\mathbf{x}) \quad \text{in } \Omega, \tag{7.21}$$

and boundary conditions

$$b_i u_{/i} + bu = \psi \quad \text{on } S_T. \tag{7.22}$$

The assumptions concerning the data are as follows:

A1: the coefficients a_{ij}, a of (7.20) are bounded,
A2: $a_{ij}\xi_i\xi_j \ge 0$ for any $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$,
A3: the functions f, b_i, b are bounded,
A4: $\mathbf{b}\cdot\mathbf{n} \ge \delta > 0$ uniformly on S_T, where $\mathbf{b} = (b_1, \ldots, b_n)$,
A5: $b > 0$ on S_T.

Then we have ([32, Theorem I.2.2])

THEOREM 7.1. *For any $t_1 \in [0, T]$ the classical solution to problem* (7.20)–(7.22) *satisfies the inequality*

$$u(t_1, \mathbf{x}) \ge \sup_{\lambda > a_0} \min\left\{0; \min_{S_{t_1}} \frac{\psi e^{\lambda(t_1 - t)}}{b}; e^{\lambda t_1} \min_{\Omega} \psi_0; \frac{1}{\lambda - a_0} \min_{Q_{t_1}} f e^{\lambda(t_1 - t)}\right\},$$

where

$$a_0 = \max_{Q_{t_1}}(-a(t, \mathbf{x})).$$

From this theorem we immediately obtain two corollaries.

COROLLARY 7.2. *If $\psi_0 \ge 0$, $\psi \ge 0$, $f \ge 0$, then $u(t_1, \mathbf{x}) \ge 0$.*

COROLLARY 7.3. *The assertion of Corollary* 7.2 *holds also for $b = 0$.*

Proof. We imitate here [32]. For a domain with C^2-boundary there exists a function $\phi(\mathbf{x})$, twice differentiable, satisfying for $k>0$,

$$\boldsymbol{\nabla}\phi\cdot\mathbf{n}=-k,\qquad \phi=1\quad\text{on }\partial\Omega,\qquad 1\le\phi\le 2\quad\text{in }\Omega.$$

We introduce the function $w=u\phi$, which satisfies the same type of equation as u, and

$$b_i u_{/i}=\frac{1}{\phi}\,b_i w_{/i}-\frac{1}{\phi^2}\,b_i w\phi_{/i}\qquad\text{on }S_T.$$

From the definition of ϕ, $\boldsymbol{\nabla}\phi=-k\mathbf{n}$ on S_T. Hence, due to A4, $-b_i\phi_{/i}=\widetilde{b}\ge k\delta>0$, and the boundary condition (7.22) takes on the form

$$b_i w_{/i}+\widetilde{b}w=\psi\qquad\text{on }S_T.$$

Hence $w\ge 0$ and $u\ge 0$. ∎

7.4. The stability of solutions. At the beginning we define, following [32], some function spaces on the given space-time cylinder Q_T.

- $L_{q,r}(Q_T)$ — the space of functions for which the following norm is finite:

$$\|u\|_{q,r,Q_T}=\Big(\int_0^T\Big(\int_\Omega|u|^q\,dx\Big)^{r/q}dt\Big)^{1/r}.$$

- $V_2(Q_T)$ — the space with the norm

$$|u|_{Q_T}=\operatorname*{ess\,sup}_{0\le t\le T}\|u(t,\mathbf{x})\|_{L_2(\Omega)}+\|\boldsymbol{\nabla}u\|_{\mathbf{L}_2(Q_T)}.$$

- $V_2^{1,0}(Q_T)$ — the subspace of $V_2(Q_T)$ containing functions continuous from $[0,T]$ into $L_2(\Omega)$ with the norm

$$|u|_{Q_T}=\max_{0\le t\le T}\|u(t,\mathbf{x})\|_{L_2(\Omega)}+\|\boldsymbol{\nabla}u\|_{\mathbf{L}_2(Q_T)}.$$

- $W_2^{1,0}(Q_T)$ — the Hilbert space with the scalar product

$$(u,v)_{W_2^{1,0}(Q_T)}=\int_{Q_T}(uv+\boldsymbol{\nabla}u\cdot\boldsymbol{\nabla}v)\,dx\,dt.$$

- $W_2^{1,1}(Q_T)$ — the Hilbert space with the scalar product

$$(u,v)_{W_2^{1,1}(Q_T)}=\int_{Q_T}(uv+\boldsymbol{\nabla}u\cdot\boldsymbol{\nabla}v+u_tv_t)\,dx\,dt.$$

- $V_2^{1,1/2}(Q_T)$ — the subspace of $V_2^{1,0}(Q_T)$ containing functions for which

$$\lim_{h\to 0}\int_0^{T-h}\int_\Omega h^{-1}[u(t+h,\mathbf{x})-u(t,\mathbf{x})]^2\,dx\,dt=0.$$

In addition we recall that $V_2^{1,0}(Q_T)$ is the closure of $W_2^{1,1}(Q_T)$ in the norm of $V_2(Q_T)$.

We shall consider the parabolic equation with Neumann-type boundary conditions, namely

$$\mathcal{L}u=-f,\qquad (a_{ij}u_{/i})n_j=0\quad\text{on }S_T,\qquad u(0,\mathbf{x})=\psi_0(\mathbf{x}),\tag{7.23}$$

where

$$\mathcal{L}u = u_t - (a_{ij}u_{/i})_{/j} + au,$$

and

$$\nu_0|\boldsymbol{\xi}|^2 \le a_{ij}\xi_i\xi_j \le \mu_0|\boldsymbol{\xi}|^2 \quad \text{for all } \boldsymbol{\xi} = (\xi_i)_{i=1,\ldots,n}$$

for constants $0 < \nu_0 < \mu_0$.

We shall also make some assumptions concerning the data.

S1: The following norm of the coefficient a is bounded:

$$\|a\|_{q,r,Q_T} \le \mu_1,$$

where, for $n \ge 2$,

$$q \in \left(\frac{n}{2}, \infty\right], \qquad r \in [1,\infty), \qquad \frac{1}{r} + \frac{n}{2q} = 1.$$

S2: The right-hand side satisfies

$$\|f\|_{q_1,r_1,Q_T} \le \mu_2,$$

where, for $n = 2$,

$$q_1 \in (1,2], \qquad r_1 \in [1,2),$$

and for $n \ge 3$,

$$q_1 \in \left[\frac{2n}{n+2}, 2\right], \qquad r_1 \in [1,2].$$

We shall call a function u a *generalized solution* of problem (7.23) in $V_2(Q_T)$ (or $V_2^{1,0}(Q_T)$, or $V_2^{1,1/2}(Q_T)$) if it satisfies the integral identity

$$-\int_{Q_T} u\eta_t\,dx\,dt + \int_0^T [\mathcal{L}_1(u,\eta) + \mathcal{L}_2(f,\eta)]\,dt = \int_\Omega \psi_0\eta(0,\mathbf{x})\,dx, \tag{7.24}$$

where

$$\mathcal{L}_1(u,\eta) = \int_\Omega [a_{ij}u_{/i}\eta_{/j} + au\eta]\,dx, \qquad \mathcal{L}_2(u,\eta) = \int_\Omega f\eta\,dx,$$

and η is any function from $W_2^{1,1}(Q_T)$ with zero final value, $\eta(T,\mathbf{x}) = 0$.

Using this definition, we recall the following existence result ([32, Theorem III.5.1]).

THEOREM 7.2. *If the domain Ω has a piecewise C^2-boundary, and assumptions* S1 *and* S2 *are satisfied, then there exists a unique generalized solution to problem* (7.23) *in the space $V_2^{1,1/2}(Q_T)$. Any solution in $V_2(Q_T)$ belongs to $V_2^{1,1/2}(Q_T)$.*

In addition, the following useful inequalities have been proved in [32].

LEMMA 7.5. *Let the numbers r, q satisfy*

$$r \in [2,\infty), \qquad q \in \left[2, \frac{2n}{n-2}\right] \quad \textit{for } n > 2,$$
$$r \in (2,\infty], \qquad q \in [2,\infty) \qquad \textit{for } n = 2,$$

and Ω be as in Theorem 7.2. *The following inequalities hold for functions in* $V_2(Q_T)$: *if* u *vanishes on the boundary*, *then*

$$\|u\|_{q,r,Q_T} \le \beta_1(n,q)|u|_{Q_T}, \tag{7.25}$$

and without this condition,

$$\|u\|_{q,r,Q_T} \le \beta_1(n,q,T,|\Omega|)|u|_{Q_T}. \tag{7.26}$$

On this basis, we may formulate a modification of Lemma 2.1, Chapter III, in [32].

LEMMA 7.6. *Let the domain Ω have piecewise C^2-boundary and let $u \in V_2(Q_T)$ satisfy for any $t_1, t_2 \in [0,T]$ the inequality*

$$\frac{1}{2}\int_\Omega u^2\,dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} [\mathcal{L}_1(u,u) + \mathcal{L}_2(f,u)]\,dt \le 0.$$

In addition, let the coefficient a and the right-hand side f satisfy assumptions S1, S2. *Then*

$$|u|_{Q_T} \le C(\|u(0,\mathbf{x})\|_{L_2(\Omega)} + \|f\|_{q_1,r_1,Q_T}), \tag{7.27}$$

with the constant C of the form $C = C(n,\nu_0,\mu_0,\mu_1,q,T,|\Omega|)$.

Proof. The proof is exactly the same as in [32] for Lemma 2.1, Chapter III, where it concerns functions vanishing on the boundary. The only difference is that we use inequality (7.26) instead of (7.25), and therefore the constant depends on $T, |\Omega|$. ∎

Now we may formulate a result which is a slight modification of Theorem III.2.1 of [32], where it concerns solutions of problems with Dirichlet type homogeneous boundary conditions.

THEOREM 7.3. *For domains with piecewise C^2-boundary and under assumptions* S1, S2, *any function $u \in V_2^{1,0}(Q_T)$ satisfying problem* (7.23) *fulfils the inequality*

$$|u|_{Q_T} \le C(\|\psi_0\|_{L_2(\Omega)} + \|f\|_{q_1,r_1,Q_T}),$$

with C as in Lemma 7.6.

Proof. Let $u \in V_2^{1,0}(Q_T)$ be a generalized solution of (7.23). By definition, it satisfies the integral identity

$$-\int_{Q_T} u\eta_t\,dx\,dt + \int_{Q_T} [(a_{ij}u_{/i})\eta_{/j} + (au+f)\eta]\,dx\,dt = \int_\Omega \psi_0\eta(0,\mathbf{x})\,dx,$$

for any $\eta \in W_2^{1,1}(Q_T)$, $\eta(T,\mathbf{x}) = 0$. We substitute

$$\eta := \widehat{\eta}_h = \frac{1}{h}\int_{t-h}^{t} \widehat{\eta}(\tau,\mathbf{x})\,d\tau,$$

where $\widehat{\eta} \in W_2^{1,1}(Q_{-h,T})$, $Q_{-h,T} = (-h,T)\times\Omega$, and $\widehat{\eta} = 0$ for $t \le 0$, $t \ge T-h$. Then we define

$$u_h = \frac{1}{h}\int_t^{t+h} u(\tau,\mathbf{x})\,d\tau.$$

As a result the integral identity transforms, as in [32, Remark 2.1, Chapter III], to the form

$$\int\limits_{Q_{T-h}} [\, u_{h/t}\widehat{\eta} + (a_{ij}u_{/i})_h\widehat{\eta}_{/j} + (au+f)_h\widehat{\eta}]\, dx\, dt = 0.$$

In fact, this identity holds for more general functions $\widehat{\eta}$ defined as

$$\widehat{\eta}(t,\mathbf{x}) = \begin{cases} \eta(t,\mathbf{x}) & \text{for } t \in [0,t_1], \\ 0 & \text{for } t > t_1, \end{cases}$$

where $t_1 \le T-h$, and $\eta(t,\mathbf{x})$ is any function in $V_2^{1,0}(Q_{t_1})$. The derivation is identical as in [32], but concerns functions which do not vanish on the boundary S_T. The crucial point is the density of $W_2^{1,1}(Q_{-h,T})$ in $V_2^{1,0}(Q_{-h,T})$ and an additional construction, allowing one to find a sequence $\widehat{\eta}_{m,k} \in W_2^{1,1}(Q_{-h,T})$, which, as $m,k \to \infty$, converges in the norm of $V_2(Q_{-h,T})$ to the function $\widehat{\eta}$ defined above. Hence

$$\int\limits_{Q_{t_1}} [\, u_{h/t}\eta + (a_{ij}u_{/i})_h\eta_{/j} + (au+f)_h\eta]\, dx\, dt = 0$$

for any $\eta \in V_2^{1,0}(Q_{t_1})$. Taking $\eta = u_h$ and letting $h \to 0$, we get

$$\frac{1}{2}\int\limits_{\Omega} u^2\, dx \,\Big|_0^{t_1} + \int\limits_0^{t_1}[\mathcal{L}_1(u,u) + \mathcal{L}_2(f,u)]\, dt = 0$$

for any $t_1 \le T$. Thus the assumptions of Lemma 7.6 are fulfilled, and the assertion follows. ■

These preparatory results allow us to formulate the main stability property. It is a modification of Theorem III.4.5 of [32], with the difference that the equation has the Neumann-type boundary conditions, and the main part of the differential operator is not perturbed.

THEOREM 7.4. *Consider the parabolic problem* (7.23) *and its perturbed counterpart,*

$$\mathcal{L}^m u^m = -f^m, \qquad (a_{ij}u^m_{/i})n_j = 0 \quad \text{on } S_T, \qquad u^m(0,\mathbf{x}) = \psi_0^m(\mathbf{x}),$$

where

$$\mathcal{L}^m u^m = u^m_t - (a_{ij}u^m_{/i})_{/j} + a^m u^m.$$

Assume that

$$a^m \to a \quad \text{in } L_{q,r}(Q_T), \qquad f^m \to f \quad \text{in } L_{q_1,r_1}(Q_T), \qquad \psi_0^m \to \psi_0 \quad \text{in } L_2(\Omega),$$

where q, r, q_1, r_1 are specified by assumptions S1, S2. *Then the sequence u^m converges strongly to u in $V_2^{1,0}(Q_T)$.*

Proof. According to Theorem 7.2 both the original and the perturbed problems have solutions satisfying the integral identities

$$\int\limits_{Q_T} [\, -u\eta_t + a_{ij}u_{/i}\eta_{/j} + (au+f)\eta\,]\, dx\, dt = \int\limits_{\Omega} \psi_0\eta(0,\mathbf{x})\, dx,$$

$$\int\limits_{Q_T} [\, -u^m\eta_t + a_{ij}u^m_{/i}\eta_{/j} + (a^m u^m + f^m)\eta\,]\, dx\, dt = \int\limits_{\Omega} \psi_0^m\eta(0,\mathbf{x})\, dx.$$

Therefore for the difference $v^m = u^m - u$ we may write

$$\int\limits_{Q_T} [-v^m\eta_t + a_{ij}v^m_{/i}\eta_{/j} + (a^m v^m + F^m)\eta]\,dx\,dt = \int\limits_{\Omega} \Psi^m\eta(0,\mathbf{x})\,dx,$$

where

$$F^m = (a^m - a)u + f^m - f, \qquad \Psi^m = \psi_0^m - \psi_0.$$

We see that v^m is a solution of the same type of parabolic problem, and the coefficients are close (by assumption) to their unperturbed counterparts in appropriate norms. Hence Theorem 7.3 is applicable, and

$$|v^m|_{Q_T} \le C(\|\Psi^m\|_{L_2(\Omega)} + \|F^m\|_{q_1,r_1,Q_T}).$$

It may be proved, similarly to [32], that $\|F^m\|_{q_1,r_1,Q_T} \to 0$, which completes the proof. ■

7.5. Imbeddings and compactness. In this section we recall, for completeness of presentation, some classical results. The first one concerns imbeddings [32, Lemma II.3.3].

THEOREM 7.5. *Let $u \in W_q^{2l,l}(Q_T)$, l an integer. Then*

$$\|D_t^r D_x^s u\|_{L_p(Q_T)} \le C_1\delta^\alpha \langle\!\langle u\rangle\!\rangle^{(2l)}_{q,Q_T} + C_2\delta^{(\alpha-2l)}\|u\|_{L_q(Q_T)},$$

provided

$$p \ge q, \qquad \alpha \equiv 2l - 2r - s - \left(\frac{1}{q} - \frac{1}{p}\right)(n+2) \ge 0.$$

Moreover, if

$$\beta \equiv 2l - 2r - s - \frac{n+2}{q} > 0$$

then for $0 \le \lambda < \beta$,

$$\langle D_t^r D_x^s u\rangle^{(\lambda)}_{Q_T} \le C_3\delta^{(\beta-\lambda)}\langle\!\langle u\rangle\!\rangle^{(2l)}_{q,Q_T} + C_4\delta^{(\beta-\lambda-2l)}\|u\|_{L_q(Q_T)}.$$

In case β is not an integer, the above inequality is satisfied also for $\lambda = \beta$.

Here $\delta \in (0, \min(\sqrt{T}, d)]$, d is the altitude of the cone in the statement of the cone condition satisfied by Ω,

$$\langle\!\langle u\rangle\!\rangle^{(j)}_{q,Q_T} = \sum_{2r+s=j} \|D_t^r D_x^s u\|_{L_q(Q_T)},$$

and

$$\langle u\rangle^{(\lambda)}_{Q_T} = \sup_{(\mathbf{x},t),(\mathbf{x}',t)\in\overline{Q}_T,\,|\mathbf{x}-\mathbf{x}'|\le\varrho_0} \frac{|u(\mathbf{x},t) - u(\mathbf{x}',t)|}{|\mathbf{x}-\mathbf{x}'|^\lambda} + \sup_{(\mathbf{x},t),(\mathbf{x},t')\in\overline{Q}_T,\,|t-t'|\le\varrho_0} \frac{|u(\mathbf{x},t) - u(\mathbf{x},t')|}{|t-t'|^{\lambda/2}}. \ ■$$

We also recall the compactness theorem (see [3], [52]) used in the proof of existence.

THEOREM 7.6. *Let X_0, X, X_1 be Banach spaces, X_0 and X_1 reflexive, for which the following imbeddings hold:*

$$X_0 \xrightarrow{\textit{compact}} X \xrightarrow{\textit{continuous}} X_1.$$

Assuming $p_0, p_1 > 1$, define the space

$$Y = \{u \mid u \in L_{p_0}(I, X_0),\ u_t \in L_{p_1}(I, X_1)\}$$

with an appropriate norm. Then the imbedding $Y \to L_{p_0}(I, X)$ is compact. ■

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