# Contents

# Abstract

Let $\Phi$ be a concave function on $(0,\infty)$ of strictly critical lower type index $p_\Phi \in (0,1]$ and $\omega \in A_\infty^{\rm loc}(\mathbb{R}^n)$ (the class of local weights introduced by V. S. Rychkov). We introduce the weighted local Orlicz–Hardy space $h_\omega^\Phi(\mathbb{R}^n)$ via the local grand maximal function. Let $\rho(t) \equiv t^{-1}/\Phi^{-1}(t^{-1})$ for all $t \in (0,\infty)$. We also introduce the BMO-type space $\mathrm{bmo}_{\rho,\,\omega}(\mathbb{R}^n)$ and establish the duality between $h_\omega^\Phi(\mathbb{R}^n)$ and $\mathrm{bmo}_{\rho,\,\omega}(\mathbb{R}^n)$. Characterizations of $h_\omega^\Phi(\mathbb{R}^n)$, including the atomic characterization, the local vertical and the local nontangential maximal function characterizations, are presented. Using the atomic characterization, we prove the existence of finite atomic decompositions achieving the norm in some dense subspaces of $h_\omega^\Phi(\mathbb{R}^n)$, from which we further deduce that for a given admissible triplet $(\rho,\,q,\,s)_\omega$ and a $\beta$-quasi-Banach space $\mathcal{B}_\beta$ with $\beta \in (0,1]$, if $T$ is a $\mathcal{B}_\beta$-sublinear operator, and maps all $(\rho,\,q,\,s)_\omega$-atoms and $(\rho,\,q)_\omega$-single-atoms with $q<\infty$ (or all continuous $(\rho,\,q,\,s)_\omega$-atoms with $q=\infty$) into uniformly bounded elements of $\mathcal{B}_\beta$, then $T$ uniquely extends to a bounded $\mathcal{B}_\beta$-sublinear operator from $h_\omega^\Phi(\mathbb{R}^n)$ to $\mathcal{B}_\beta$. As applications, we show that the local Riesz transforms are bounded on $h_\omega^\Phi(\mathbb{R}^n)$, the local fractional integrals are bounded from $h_{\omega^p}^p(\mathbb{R}^n)$ to $L_{\omega^q}^q(\mathbb{R}^n)$ when $q>1$ and from $h_{\omega^p}^p(\mathbb{R}^n)$ to $h_{\omega^q}^q(\mathbb{R}^n)$ when $q\le 1$, and some pseudo-differential operators are also bounded on both $h_\omega^\Phi(\mathbb{R}^n)$. All results for any general $\Phi$ even when $\omega\equiv 1$ are new.



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# 1. Introduction

It is well known that the theory of the classical local Hardy spaces, originally introduced by Goldberg [18], plays an important role in partial differential equations and harmonic analysis; see, for example, [18, 6, 43, 51, 52, 53] and their references. In particular, pseudo-differential operators are bounded on local Hardy spaces $h^p(\mathbb{R}^n)$ with $p \in (0,1]$, but they are not bounded on Hardy spaces $H^p(\mathbb{R}^n)$ with $p \in (0,1]$; see [18] (also [52, 53]). In [6], Bui studied the weighted version $h^p_\omega(\mathbb{R}^n)$ of the local Hardy space $h^p(\mathbb{R}^n)$ with $\omega \in A_\infty(\mathbb{R}^n)$, where and in what follows, $A_q(\mathbb{R}^n)$ for $q \in [1,\infty]$ denotes the *class of Muckenhoupt's weights*; see, for example, [17] for their definitions and properties.

Rychkov [43] introduced and studied a class of local weights, denoted by $A^{\mathrm{loc}}_\infty(\mathbb{R}^n)$ (see also Definition 2.1 below), and the weighted Besov–Lipschitz spaces and Triebel–Lizorkin spaces with weights belonging to $A^{\mathrm{loc}}_\infty(\mathbb{R}^n)$, which contains $A_\infty(\mathbb{R}^n)$ weights and exponential weights introduced by Schott [44] as special cases. In particular, Rychkov [43] generalized some of the theory of weighted local Hardy spaces developed by Bui [6] to $A^{\mathrm{loc}}_\infty(\mathbb{R}^n)$ weights. In fact, Rychkov established the local vertical and the local nontangential maximal function characterizations of weighted local Hardy spaces with $A^{\mathrm{loc}}_\infty(\mathbb{R}^n)$ weights. Very recently, Tang [49] established the weighted atomic decomposition characterization of the weighted local Hardy space $h^p_\omega(\mathbb{R}^n)$ with $\omega \in A^{\mathrm{loc}}_\infty(\mathbb{R}^n)$ via the local grand maximal function. Motivated by [5], Tang also established some criterions for boundedness of $\mathcal{B}_\beta$-sublinear operators on $h^p_\omega(\mathbb{R}^n)$ (see Section 6 for the notion of $\mathcal{B}_\beta$-sublinear operators, which was first introduced in [56]). As applications, Tang [49, 50] proved that some strongly singular integrals, pseudo-differential operators and their commutators are bounded on $h^p_\omega(\mathbb{R}^n)$. It is worth pointing out that in recent years, many papers are focused on criterions for boundedness of (sub)linear operators on various Hardy spaces with different underlying spaces (see, for example, [4, 35, 57, 5, 20, 56, 42, 49]), and on entropy and approximation numbers of embeddings of function spaces with Muckenhoupt weight (see, for example, [21, 22, 23, 24]).

It is also well known that the classical BMO space (the *space of functions with bounded mean oscillation*) originally introduced by John and Nirenberg [29] and the classical Morrey space originally introduced by Morrey [37] play an important role in the study of partial differential equations and harmonic analysis; see, for example, [15, 11, 14, 38]. In particular, Fefferman and Stein [15] proved that $\mathrm{BMO}(\mathbb{R}^n)$ is the dual space of the Hardy space $H^1(\mathbb{R}^n)$. Moveover, Goldberg [18] introduced the space $\mathrm{bmo}(\mathbb{R}^n)$ and proved that $\mathrm{bmo}(\mathbb{R}^n)$ is the dual space of the local Hardy space $h^1(\mathbb{R}^n)$.

On the other hand, as the generalization of $L^p(\mathbb{R}^n)$, the Orlicz space was introduced by Birnbaum–Orlicz in [2] and Orlicz in [39]; since then, the theory of the Orlicz spaces themselves has been well developed and these spaces have been widely used in probability, statistics, potential theory, partial differential equations, as well as harmonic analysis and some other fields of analysis; see, for example, [40, 41, 7, 34, 25]. Moreover, Orlicz–Hardy spaces are also suitable substitutes of the Orlicz spaces in dealing with many problems of analysis; see, for example, [26, 55, 47, 27]. Recall that Orlicz–Hardy spaces and their dual spaces were studied by Janson [26] on $\mathbb{R}^n$ and Viviani [55] on spaces of homogeneous type in the sense of Coifman and Weiss [10]. Recently, Orlicz–Hardy spaces associated with some differential operators and their dual spaces were introduced and studied in [28, 27].

Let $\omega \in A_\infty^{\mathrm{loc}}(\mathbb{R}^n)$, $\Phi$ be a concave function on $(0,\infty)$ of strictly critical lower type index $p_\Phi \in (0,1]$ (see (2.6) below for the definition) and

$$\rho(t) \equiv t^{-1}/\Phi^{-1}(t^{-1})$$

for all $t \in (0,\infty)$, where $\Phi^{-1}$ is the inverse function of $\Phi$. A typical example of such Orlicz functions is $\Phi(t) \equiv t^p$ for all $t \in (0,\infty)$ and $p \in (0,1]$. Motivated by [43, 49, 18, 28, 27, 5], in this paper, we introduce the weighted local Orlicz–Hardy space $h_\omega^\Phi(\mathbb{R}^n)$ via the local grand maximal function. We also introduce the BMO-type space $\mathrm{bmo}_{\rho,\,\omega}(\mathbb{R}^n)$ and establish the duality between $h_\omega^\Phi(\mathbb{R}^n)$ and $\mathrm{bmo}_{\rho,\,\omega}(\mathbb{R}^n)$. Characterizations of $h_\omega^\Phi(\mathbb{R}^n)$, including the atomic characterization, the local vertical and the local nontangential maximal function characterizations, are presented. Using the atomic characterization, we prove the existence of finite atomic decompositions achieving the norm in some dense subspaces of $h_\omega^\Phi(\mathbb{R}^n)$, from which we further deduce that for a given admissible triplet $(\rho,\, q,\, s)_\omega$ and a $\beta$-quasi-Banach space $\mathcal{B}_\beta$ with $\beta \in (0,1]$, if $T$ is a $\mathcal{B}_\beta$-sublinear operator, and maps all $(\rho,\, q,\, s)_\omega$-atoms and $(\rho,\, q)_\omega$-single-atoms with $q < \infty$ (or all continuous $(\rho,\, q,\, s)_\omega$-atoms with $q = \infty$) into uniformly bounded elements of $\mathcal{B}_\beta$, then $T$ uniquely extends to a bounded $\mathcal{B}_\beta$-sublinear operator from $h_\omega^\Phi(\mathbb{R}^n)$ to $\mathcal{B}_\beta$. As applications, we show that the local Riesz transforms are bounded on $h_\omega^\Phi(\mathbb{R}^n)$, the local fractional integrals are bounded from $h_{\omega^p}^p(\mathbb{R}^n)$ to $L_{\omega^q}^q(\mathbb{R}^n)$ when $q > 1$ and from $h_{\omega^p}^p(\mathbb{R}^n)$ to $h_{\omega^q}^q(\mathbb{R}^n)$ when $q \le 1$, and some pseudo-differential operators are also bounded on both $h_\omega^\Phi(\mathbb{R}^n)$ and $\mathrm{bmo}_{\rho,\,\omega}(\mathbb{R}^n)$. We point out that the Orlicz–Hardy spaces $h_\omega^\Phi(\mathbb{R}^n)$ include the classical local Hardy spaces of Goldberg [18], the weighted local Hardy spaces of Bui [6] and the weighted local Hardy spaces of Tang [49] as special cases. Moreover, the method of obtaining atomic decompositions used in this paper (see the proof of Theorem 5.6 below) is different from the classical methods in [18, 6]. Indeed, following Bownik [3] (see also [5, 49]), we give a direct proof for the weighted atomic characterization of $h_\omega^\Phi(\mathbb{R}^n)$, without invoking the atomic characterization of $H_\omega^\Phi(\mathbb{R}^n)$. All results of this paper for any general $\Phi$ even when $\omega \equiv 1$ are new.

Precisely, this paper is organized as follows. In Section 2, we first recall some definitions and notation concerning local weights introduced in [43, 49], then describe some basic assumptions and present some properties of Orlicz functions considered in this paper.

In Section 3, we first introduce the weighted local Orlicz–Hardy space $h_{\omega,\,N}^\Phi(\mathbb{R}^n)$ via the local grand maximal function, and then the weighted atomic local Orlicz–Hardy space $h_\omega^{\rho,\,q,\,s}(\mathbb{R}^n)$ for any admissible triplet $(\rho,\, q,\, s)_\omega$ (see Definition 3.4 below). We point out that when $\Phi(t) \equiv t^p$ for all $t \in (0,\infty)$ and $p \in (0,1]$, the weighted local Orlicz–Hardy space $h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)$ is just the weighted local Hardy space $h^{p}_{\omega,\,N}(\mathbb{R}^n)$ introduced by Tang in [49]. Next, we establish the local vertical and the local nontangential maximal function characterizations of $h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)$ via a local Calderón reproducing formula and some useful estimates established by Rychkov [43], which generalizes [43, Theorem 2.24] by taking $\Phi(t) \equiv t^p$ for all $t \in (0,\infty)$ and $p \in (0,1]$; see Theorems 3.12 and 3.14 and Remark 3.13 below. Finally, we present some properties of these weighted local Orlicz–Hardy spaces $h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)$ and weighted atomic local Orlicz–Hardy spaces $h^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)$.

Throughout the paper, as usual, $\mathcal{D}(\mathbb{R}^n)$ denotes the *set of all $C^\infty(\mathbb{R}^n)$ functions on $\mathbb{R}^n$ with compact support*, endowed with the inductive limit topology, and $\mathcal{D}'(\mathbb{R}^n)$ its *topological dual space*, endowed with the weak$^*$ topology. Let $\lfloor r\rfloor$ for any $r \in \mathbb{R}$ denote the *maximal integer not more than $r$*. In Section 4, for any given $f \in \mathcal{D}'(\mathbb{R}^n)$, integer

$$s \geq \lfloor n(q_\omega/p_\Phi - 1)\rfloor$$

and $\lambda > \inf_{x\in\mathbb{R}^n} \mathcal{G}_{N,\,\widetilde{R}}(f)(x)$, where $q_\omega$, $p_\Phi$ and $\mathcal{G}_{N,\,\widetilde{R}}(f)$ are respectively as in (2.4), (2.6) and (3.2) below, and $\widetilde{R} = 2^{3(10+n)}$, following [46, 3, 5, 49], via a Whitney decomposition of $\Omega_\lambda$ in (4.1), we obtain the Calderón–Zygmund decomposition $f \equiv g + \sum_i b_i$ in $\mathcal{D}'(\mathbb{R}^n)$ of degree $s$ and height $\lambda$ associated with the local grand maximal function $\mathcal{G}_{N,\,\widetilde{R}}(f)$. The main task of Section 4 is to establish some subtle estimates for $g$ and $\{b_i\}_i$. Precisely, Lemmas 4.2 through 4.5 are estimates on $\{b_i\}_i$, the bad part of $f$, while Lemmas 4.6 and 4.7 on $g$, the good part of $f$. As an application of these estimates, we obtain the density of $L^q_\omega(\mathbb{R}^n) \cap h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)$ in $h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)$, where $q \in (q_\omega,\infty)$ (see Corollary 4.8 below). With a different proof from [49, Lemma 4.9], via an application of the boundedness of the local vector-valued Hardy–Littlewood maximal operator obtained by Rychkov [43] (see also Lemma 3.10 below), our Lemma 4.7 below improves [49, Lemma 4.9] even when $\Phi(t) \equiv t^p$ for all $t \in (0,\infty)$ and $p \in (0,1]$, which is necessary for Corollary 4.8.

In Section 5, we prove that for any given admissible triplet $(\rho,\,q,\,s)_\omega$,

$$h^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n) = h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)$$

with equivalent norms when positive integer $N \geq N_{\Phi,\,\omega}$ (see (3.25) below for the definition of $N_{\Phi,\,\omega}$), by using the Calderón–Zygmund decomposition and some subtle estimates obtained in Section 4, which completely covers [49, Theorem 5.1] by taking $\Phi(t) \equiv t^p$ for all $p \in (0,1]$ and $t \in (0,\infty)$; see Theorem 5.6 and Remark 5.7 below. It is worth pointing out that we show Theorem 5.6 by a way different from the methods in [18, 6], but close to those in [3, 5, 49]. For simplicity, in the rest of this introduction, we denote by $h^{\Phi}_{\omega}(\mathbb{R}^n)$ the *weighted local Orlicz–Hardy space $h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)$ with $N \geq N_{\Phi,\,\omega}$*.

Assume that $(\rho,\,q,\,s)_\omega$ is an admissible triplet. Let $h^{\rho,\,q,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)$ be the *space of all finite linear combinations of $(\rho,\,q,\,s)_\omega$-atoms and $(\rho,\,q)_\omega$-single-atoms* (see Definition 6.1 below), and $h^{\rho,\,\infty,\,s}_{\omega,\,\mathrm{fin},\,c}(\mathbb{R}^n)$ the *space of all $f \in h^{\rho,\,\infty,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)$ with compact support*. In Section 6, we prove that $\|\cdot\|_{h^{\rho,\,q,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)}$ and $\|\cdot\|_{h^{\Phi}_{\omega}(\mathbb{R}^n)}$ are equivalent quasi-norms on $h^{\rho,\,q,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)$ when $q < \infty$, and $\|\cdot\|_{h^{\rho,\,\infty,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)}$ and $\|\cdot\|_{h^{\Phi}_{\omega}(\mathbb{R}^n)}$ are equivalent quasi-norms on $h^{\rho,\,\infty,\,s}_{\omega,\,\mathrm{fin},\,c}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ when $q = \infty$ (see Theorem 6.2 below). As an application, we prove that for a given admissible triplet $(\rho,\,q,\,s)_\omega$ and a $\beta$-quasi-Banach space $\mathcal{B}_\beta$ with $\beta \in (0,1]$, if $\Phi$ has an upper type $\widetilde{p}$ satisfying $0 < \widetilde{p} \leq \beta$, and $T$ is a $\mathcal{B}_\beta$-sublinear operator mapping all $(\rho,\,q,\,s)_\omega$-atoms and $(\rho,\,q)_\omega$-single-atoms with $q\in(q_\omega,\infty)$ (or all continuous $(\rho,\,q,\,s)_\omega$-atoms with $q=\infty$) into uniformly bounded elements of $\mathcal{B}_\beta$, then $T$ uniquely extends to a bounded $\mathcal{B}_\beta$-sublinear operator from $h^\Phi_\omega(\mathbb{R}^n)$ to $\mathcal{B}_\beta$ which coincides with $T$ on these $(\rho,\,q,\,s)_\omega$-atoms and $(\rho,\,q)_\omega$-single-atoms; see Theorem 6.4 below. We remark that this extends both the results of Meda–Sjögren–Vallarino [35] and Yang–Zhou [57] to the setting of weighted local Orlicz–Hardy spaces. We also point out that Theorems 6.2(i) and 6.4(i) below completely cover [49, Theorems 6.1 and 6.2], respectively, by taking $\Phi(t)\equiv t^p$ for all $t\in(0,\infty)$ and $p\in(0,1]$; and Theorems 6.2(ii) and 6.4(ii) are new even when $\Phi(t)\equiv t^p$ for all $t\in(0,\infty)$ and $p\in(0,1]$; see Remark 6.5 below.

Let $(\rho,\,q,\,s)_\omega$ be an admissible triplet, $q'$ the dual exponent of $q$ and $q_\omega$ the critical index of $\omega$. In Section 7, we introduce the BMO-type space $\mathrm{bmo}^{q'}_{\rho,\,\omega}(\mathbb{R}^n)$ and prove that

$$[h^{\rho,\,q,\,s}_\omega(\mathbb{R}^n)]^*=\mathrm{bmo}^{q'}_{\rho,\,\omega}(\mathbb{R}^n),$$

where $[h^{\rho,\,q,\,s}_\omega(\mathbb{R}^n)]^*$ denotes the dual space of $h^{\rho,\,q,\,s}_\omega(\mathbb{R}^n)$; see Theorem 7.5 below. Denote $\mathrm{bmo}^1_{\rho,\,\omega}(\mathbb{R}^n)$ simply by $\mathrm{bmo}_{\rho,\,\omega}(\mathbb{R}^n)$. As applications of Theorems 5.6 and 7.5, we see that for $q\in[1,\frac{q_\omega}{q_\omega-1})$, $\mathrm{bmo}^q_{\rho,\,\omega}(\mathbb{R}^n)=\mathrm{bmo}_{\rho,\,\omega}(\mathbb{R}^n)$ with equivalent norms and

$$[h^\Phi_\omega(\mathbb{R}^n)]^*=\mathrm{bmo}_{\rho,\,\omega}(\mathbb{R}^n);$$

see Corollaries 7.6 and 7.7 below.

In Section 8, as applications of Theorem 6.2, we obtain the boundedness of some operators from $h^\Phi_\omega(\mathbb{R}^n)$ to some $\beta$-quasi-Banach space $\mathcal{B}_\beta$ with $\beta\in(0,1]$. First, we prove that the local Riesz transforms are bounded on $h^\Phi_\omega(\mathbb{R}^n)$ if $p_\Phi=p^+_\Phi$ and (2.5) holds for $p^+_\Phi$ with $t\in[1,\infty)$ (see Section 2 below for the definitions of $p^+_\Phi$), which completely covers [49, Lemma 8.3] by taking $\Phi(t)\equiv t$ for all $t\in(0,\infty)$; see Theorem 8.2 and Remark 8.4 below. Then we introduce the local fractional integral operator $I^{\mathrm{loc}}_\alpha$ and show that $I^{\mathrm{loc}}_\alpha$ is bounded from $h^p_{\omega^p}(\mathbb{R}^n)$ to $L^q_{\omega^q}(\mathbb{R}^n)$ when $\alpha\in(0,n)$, $p\in\big[\frac{n}{n+\alpha},1\big]$, $\frac1q=\frac1p-\frac{\alpha}{n}$, and $\omega^{\frac{nr}{nr-n-r\alpha}}\in A^{\mathrm{loc}}_1(\mathbb{R}^n)$ for some $r\in(\frac{n}{n-\alpha},\infty)$ and $\int_{\mathbb{R}^n}[\omega(x)]^p\,dx=\infty$ (see Theorem 8.10 below); furthermore, when $\alpha\in(0,1)$, $p\in(0,\frac{n}{n+\alpha}]$, $\frac1q=\frac1p-\frac{\alpha}{n}$, and $\omega$ satisfies the same conditions, we prove that $I^{\mathrm{loc}}_\alpha$ is bounded from $h^p_{\omega^p}(\mathbb{R}^n)$ to $h^q_{\omega^q}(\mathbb{R}^n)$ (see Theorem 8.11 below). To the best of our knowledge, Theorems 8.10 and 8.11 are new even when $\omega\equiv1$. Finally, let $\omega\in A_\infty(\phi)$, a class introduced by Tang [50] (see also Definition 8.13 below), and $T$ be an $S^0_{1,\,0}(\mathbb{R}^n)$ pseudo-differential operator. We prove that $T$ is bounded on $h^\Phi_\omega(\mathbb{R}^n)$ if $p_\Phi=p^+_\Phi$ and (2.5) holds for $p^+_\Phi$ with $t\in[1,\infty)$; see Theorem 8.18 below. We point out that $A_\infty(\phi)\subset A^{\mathrm{loc}}_\infty(\mathbb{R}^n)$ but $A_\infty(\phi)\supset A_\infty(\mathbb{R}^n)$. We also remark that Theorem 8.18 below extends [18, Theorem 4] to the setting of weighted local Orlicz–Hardy spaces, and completely covers [49, Theorem 7.3] by taking $\Phi(t)\equiv t^p$ for all $t\in(0,\infty)$ and $p\in(0,1]$ and also [32, Theorem 2] by taking $\Phi(t)\equiv t$ for all $t\in(0,\infty)$ and $\omega\in A_1(\mathbb{R}^n)$; see Remark 8.19 below. As an application of Theorems 7.5 and 8.18, we also find that $T$ is bounded on $\mathrm{bmo}_{\rho,\,\omega}(\mathbb{R}^n)$; see Corollary 8.20 below.

The main motivation of this paper is to pave the way for the study of weighted Orlicz–Hardy spaces associated with divergence operators on strongly Lipschitz domains of $\mathbb{R}^n$. The corresponding Hardy spaces associated with divergence operators on strongly Lipschitz domains of $\mathbb{R}^n$ were originally studied by Auscher and Russ [1], where the atomic characterization of the classical Hardy spaces plays a key role. Earlier works on Hardy spaces on domains can be found, for example, in [31, 36, 9, 8, 54]. It was shown in those papers that the theory of Hardy spaces on domains plays an important role in partial differential equations and harmonic analysis.

Finally we make some conventions on notation. Throughout the paper, we denote by $C$ a positive constant which is independent of the main parameters, but it may vary from line to line. We also use $C(\gamma, \beta, \ldots)$ to denote a positive constant depending on the indicated parameters $\gamma$, $\beta$, .... The symbol $A \lesssim B$ means that $A \le CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. The symbol $\lfloor s \rfloor$ for $s \in \mathbb{R}$ denotes the maximal integer not more than $s$. For any given normed spaces $\mathcal{A}$ and $\mathcal{B}$ with the corresponding norms $\|\cdot\|_{\mathcal{A}}$ and $\|\cdot\|_{\mathcal{B}}$, the symbol $\mathcal{A} \subset \mathcal{B}$ means that if $f \in \mathcal{A}$, then $f \in \mathcal{B}$ and $\|f\|_{\mathcal{B}} \lesssim \|f\|_{\mathcal{A}}$. For any subset $G$ of $\mathbb{R}^n$, we denote by $G^{\complement}$ the set $\mathbb{R}^n \setminus G$; for a measurable set $E$, denote by $\chi_E$ the characteristic function of $E$. We also set $\mathbb{N} \equiv \{1, 2, \ldots\}$ and $\mathbb{Z}_+ \equiv \mathbb{N} \cup \{0\}$. For any $\theta \equiv (\theta_1, \ldots, \theta_n) \in \mathbb{Z}_+^n$, let $|\theta| \equiv \theta_1 + \cdots + \theta_n$ and $\partial_x^\theta \equiv \partial^{|\theta|}/\partial x_1^{\theta_1} \cdots \partial x_n^{\theta_n}$. Given a function $g$ on $\mathbb{R}^n$, if $\int_{\mathbb{R}^n} g(x)\,dx \neq 0$, let $L_g \equiv -1$; otherwise, let $L_g \in \mathbb{Z}_+$ be the maximal integer such that $g$ has vanishing moments up to order $L_g$, namely, $\int_{\mathbb{R}^n} g(x) x^\alpha\,dx = 0$ for all multi-indices $\alpha$ with $|\alpha| \le L_g$.

## 2. Preliminaries

In this section, we first recall some notions and notation concerning local weights introduced in [43, 49], then describe some basic assumptions and present some properties of Orlicz functions considered in this paper.

**2.1. $A_p^{\mathrm{loc}}(\mathbb{R}^n)$ weights.** In this subsection, we recall some notions and properties of local weights. Let $Q$ be a cube in $\mathbb{R}^n$; we denote its Lebesgue measure by $|Q|$. Throughout the paper, *all cubes are assumed to be closed and their sides parallel to the coordinate axes.*

DEFINITION 2.1. Let $p \in (1, \infty)$. The *weight class* $A_p^{\mathrm{loc}}(\mathbb{R}^n)$ is defined to be the set of all nonnegative locally integrable functions $\omega$ on $\mathbb{R}^n$ such that

$$A_p^{\mathrm{loc}}(\omega) \equiv \sup_{|Q|\le 1} \frac{1}{|Q|^p} \int_Q \omega(x)\,dx \left( \int_Q [\omega(y)]^{-p'/p}\,dy \right)^{p/p'} < \infty, \tag{2.1}$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with $|Q| \le 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

When $p = 1$, the *weight class* $A_1^{\mathrm{loc}}(\mathbb{R}^n)$ is defined to be the set of all nonnegative locally integrable functions $\omega$ on $\mathbb{R}^n$ such that

$$A_1^{\mathrm{loc}}(\omega) \equiv \sup_{|Q|\le 1} \frac{1}{|Q|} \int_Q \omega(x)\,dx \left( \operatorname*{ess\,sup}_{y \in Q} [\omega(y)]^{-1} \right) < \infty, \tag{2.2}$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with $|Q| \le 1$.

When $p = \infty$, the *weight class* $A_\infty^{\mathrm{loc}}(\mathbb{R}^n)$ is defined to be the set of all nonnegative locally integrable functions $\omega$ on $\mathbb{R}^n$ such that for any $\alpha \in (0, 1)$,

$$A_\infty^{\mathrm{loc}}(\omega;\, \alpha) \equiv \sup_{|Q|\le 1} \left[ \sup_{F \subset Q,\, |F| \ge \alpha|Q|} \frac{\omega(Q)}{\omega(F)} \right] < \infty, \tag{2.3}$$

where $F$ runs through all measurable sets in $\mathbb{R}^n$ with the indicated properties, the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with $|Q| \le 1$ and $\omega(Q) \equiv \int_Q \omega(x)\,dx$.

REMARK 2.2. (i) We point out that the weight class $A_p^{\rm loc}(\mathbb{R}^n)$ for $p \in (1,\infty]$ was introduced by Rychkov [43] and $A_1^{\rm loc}(\mathbb{R}^n)$ by Tang [49]. By Hölder's inequality, we see that $A_{p_1}^{\rm loc}(\mathbb{R}^n) \subset A_{p_2}^{\rm loc}(\mathbb{R}^n) \subset A_\infty^{\rm loc}(\mathbb{R}^n)$, if $1 \le p_1 < p_2 < \infty$. Conversely, it was proved in [43, Lemma 1.3] that if $\omega \in A_\infty^{\rm loc}(\mathbb{R}^n)$, then $\omega \in A_p^{\rm loc}(\mathbb{R}^n)$ for some $p \in (1,\infty)$. Thus, we have $A_\infty^{\rm loc}(\mathbb{R}^n) = \bigcup_{1\le p<\infty} A_p^{\rm loc}(\mathbb{R}^n)$.

(ii) For any constant $\widetilde{C} \in (0,\infty)$, the condition $|Q| \le 1$ can be replaced by $|Q| \le \widetilde{C}$ in (2.1), (2.2) and (2.3); see [43, Remark 1.5]. In this case, $A_p^{\rm loc}(\omega)$ with $p \in [1,\infty)$ and $A_\infty^{\rm loc}(\omega,\, \alpha)$ depend on $\widetilde{C}$.

In what follows, $Q(x,t)$ denotes the *closed cube centered at $x$ and of sidelength $t$*. Similarly, given $Q = Q(x,t)$ and $\lambda \in (0,\infty)$, we write $\lambda Q$ for the $\lambda$-*dilated cube*, which is the cube with the same center $x$ and with sidelength $\lambda t$. Given a Lebesgue measurable set $E$ and a weight $\omega \in A_\infty^{\rm loc}(\mathbb{R}^n)$, let $\omega(E) \equiv \int_E \omega(x)\,dx$. For any $\omega \in A_\infty^{\rm loc}(\mathbb{R}^n)$, the space $L^p_\omega(\mathbb{R}^n)$ with $p \in (0,\infty)$ denotes the set of all measurable functions $f$ such that

$$\|f\|_{L^p_\omega(\mathbb{R}^n)} \equiv \left\{\int_{\mathbb{R}^n} |f(x)|^p \omega(x)\,dx\right\}^{1/p} < \infty,$$

and $L^\infty_\omega(\mathbb{R}^n) \equiv L^\infty(\mathbb{R}^n)$. The symbol $L^{1,\infty}_\omega(\mathbb{R}^n)$ denotes the set of all measurable functions $f$ such that

$$\|f\|_{L^{1,\infty}_\omega(\mathbb{R}^n)} \equiv \sup_{\lambda>0}\{\lambda\omega(\{x \in \mathbb{R}^n : |f(x)| > \lambda\})\} < \infty.$$

For a positive constant $\widetilde{C}$, any locally integrable function $f$ and $x \in \mathbb{R}^n$, the *local Hardy–Littlewood maximal function* $M^{\rm loc}_{\widetilde{C}}(f)$ is defined by

$$M^{\rm loc}_{\widetilde{C}}(f)(x) \equiv \sup_{Q\ni x,\, |Q|\le \widetilde{C}} \frac{1}{|Q|}\int_Q |f(y)|\,dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ such that $Q \ni x$ and $|Q| \le \widetilde{C}$. If $\widetilde{C} = 1$, we denote $M^{\rm loc}_{\widetilde{C}}(f)$ simply by $M^{\rm loc}(f)$.

Next, we recall some properties of weights in $A_\infty^{\rm loc}(\mathbb{R}^n)$ and $A_p(\mathbb{R}^n)$; here and in what follows, $A_p(\mathbb{R}^n)$ for $p \in [1,\infty)$ denotes the classical *Muckenhoupt weights*; see [17, 46] for their definitions.

LEMMA 2.3.

(i) *Let $p \in [1,\infty)$, $\omega \in A_p^{\rm loc}(\mathbb{R}^n)$, and $Q$ be a unit cube, namely, $l(Q) = 1$. Then there exist an $\overline{\omega} \in A_p(\mathbb{R}^n)$ such that $\overline{\omega} = \omega$ on $Q$, and a positive constant $C$ independent of $Q$ such that $A_p(\overline{\omega}) \le C A_p^{\rm loc}(\omega)$, where $A_p(\overline{\omega})$ denotes the weight constant of $\overline{\omega}$, which is as in* (2.1) *and* (2.2) *after removing the restriction $l(Q) \le 1$.*

(ii) *If $\omega \in A_p^{\rm loc}(\mathbb{R}^n)$ with $p \in (1,\infty)$, then there exist $\eta_1, \eta_2 \in (0,\infty)$ such that $\omega \in A^{\rm loc}_{p-\eta_1}(\mathbb{R}^n)$ with $p - \eta_1 \in (1,\infty)$, and $\omega^{1+\eta_2} \in A_p^{\rm loc}(\mathbb{R}^n)$.*

(iii) *If $1 \le p_1 < p_2 < \infty$, then $A^{\rm loc}_{p_1}(\mathbb{R}^n) \subset A^{\rm loc}_{p_2}(\mathbb{R}^n)$.*

(iv) *For* $p \in (1, \infty)$, $\omega \in A_p^{\rm loc}(\mathbb{R}^n)$ *if and only if* $\omega^{-1/(p-1)} \in A_{p'}^{\rm loc}(\mathbb{R}^n)$, *where*

$$1/p + 1/p' = 1.$$

(v) *For* $\omega \in A_\infty^{\rm loc}(\mathbb{R}^n)$ *and* $Q = Q(x_0, l(Q))$, *there exists a positive constant* $C$ *such that* $\omega(2Q) \le C\omega(Q)$ *when* $l(Q) < 1$, *and* $\omega(Q(x_0, l(Q)+1)) \le C\omega(Q)$ *when* $l(Q) \ge 1$.

(vi) *If* $p \in (1, \infty)$ *and* $\omega \in A_p^{\rm loc}(\mathbb{R}^n)$, *then the local Hardy–Littlewood maximal operator* $M^{\rm loc}$ *is bounded on* $L^p_\omega(\mathbb{R}^n)$.

(vii) *If* $\omega \in A_1^{\rm loc}(\mathbb{R}^n)$, *then* $M^{\rm loc}$ *is bounded from* $L^1_\omega(\mathbb{R}^n)$ *to* $L^{1,\infty}_\omega(\mathbb{R}^n)$.

(viii) *If* $\omega \in A_p(\mathbb{R}^n)$ *with* $p \in [1, \infty)$, *then there exists a positive constant* $C$ *such that for all cubes* $Q_1, Q_2 \subset \mathbb{R}^n$ *with* $Q_1 \subset Q_2$,

$$\frac{\omega(Q_2)}{\omega(Q_1)} \le C\left(\frac{|Q_2|}{|Q_1|}\right)^p.$$

Lemma 2.3(i) is just [43, Lemma 1.1]. The statements (ii) through (vii) of Lemma 2.3 are just Lemma 2.1 and Corollary 2.1 in [49], which are deduced from Lemma 2.3(i) and the properties of $A_p(\mathbb{R}^n)$; see the proofs of [49, Lemma 2.1, Corollary 2.1]. Lemma 2.3(viii) is included, for example, in [16, 17, 46].

Remark 2.4. Let $\widetilde{C}$ be a positive constant. It was pointed out in [43, Remark 1.5] and [49] that (i) through (vii) of Lemma 2.3 are also true if $l(Q) = 1$, $l(Q) \ge 1$, $l(Q) < 1$, $Q(x_0, l(Q)+1)$ and $M^{\rm loc}$ are respectively replaced by $l(Q) = \widetilde{C}$, $l(Q) \ge \widetilde{C}$, $l(Q) < \widetilde{C}$, $Q(x_0, l(Q)+\widetilde{C})$ and $M^{\rm loc}_{\widetilde{C}}$. In this case, the constants appearing in (i), (vi) and (vii) of Lemma 2.3 depend on $\widetilde{C}$.

For any given $\omega \in A_\infty^{\rm loc}(\mathbb{R}^n)$, define the *critical index* of $\omega$ by

$$q_\omega \equiv \inf\{p \in [1, \infty) : \omega \in A_p^{\rm loc}(\mathbb{R}^n)\}. \tag{2.4}$$

Remark 2.5. Obviously, $q_\omega \in [1, \infty)$. If $q_\omega \in (1, \infty)$, by Lemma 2.2(ii), it is easy to know that $\omega \notin A^{\rm loc}_{q_\omega}(\mathbb{R}^n)$. Moreover, there exists an $\omega \notin A_1^{\rm loc}(\mathbb{R}^n)$ such that $q_\omega = 1$. Indeed, for $t \in \mathbb{R} \setminus \{0\}$, let $\omega(t) \equiv [\ln(1/|t|)]^{-1}$. Johnson and Neugebauer [30, p. 254, Remark] showed that $\omega \in (\bigcap_{p>1} A_p(\mathbb{R}^n)) \setminus A_1(\mathbb{R}^n)$. By the fact that $A_p(\mathbb{R}^n) \subset A_p^{\rm loc}(\mathbb{R}^n)$ for all $p \in [1, \infty)$, which is obvious by the definitions, we see that $\omega \in \bigcap_{p>1} A_p^{\rm loc}(\mathbb{R}^n)$. We claim that $\omega \notin A_1^{\rm loc}(\mathbb{R}^n)$. In fact, taking $x \in (0, 1/2)$, we have

$$M^{\rm loc}(\omega)(x) \ge \frac{1}{2}\int_{x-1}^{x+1} \omega(t)\,dt \ge \int_0^{1/2}\left[\ln\left(\frac{1}{t}\right)\right]^{-1} dt \equiv \infty.$$

Moreover, it is easy to see that $\omega(x) \to 0$ as $x \to 0$. Thus, by (2.2), we know that $\omega \notin A_1^{\rm loc}(\mathbb{R}^n)$.

For $\mathcal{D}(\mathbb{R}^n)$, $\mathcal{D}'(\mathbb{R}^n)$ and $L^q_\omega(\mathbb{R}^n)$, we have the following conclusions.

Lemma 2.6. *Let* $\omega \in A_\infty^{\rm loc}(\mathbb{R}^n)$, $q_\omega$ *be as in* (2.4) *and* $p \in (q_\omega, \infty]$.

(i) *If* $1/p + 1/p' = 1$, *then* $\mathcal{D}(\mathbb{R}^n) \subset L^{p'}_{\omega^{-1/(p-1)}}(\mathbb{R}^n)$.

(ii) $L^p_\omega(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$ *and the inclusion is continuous.*

(iii) *Let* $\phi \in \mathcal{D}(\mathbb{R}^n)$ *and* $\int_{\mathbb{R}^n} \phi(x)\,dx = 1$. *If* $q \in (q_\omega, \infty)$, *then for any* $f \in L^q_\omega(\mathbb{R}^n)$, $f * \phi_t \to f$ *in* $L^q_\omega(\mathbb{R}^n)$ *as* $t \to 0$*; here and in what follows,* $\phi_t(x) \equiv (1/t^n)\phi(x/t)$ *for all* $t \in (0, \infty)$ *and* $x \in \mathbb{R}^n$.

We remark that (i) and (ii) of Lemma 2.6, and Lemma 2.6(iii), are, respectively, Lemma 2.2 and Proposition 2.1 in [49].

**2.2. Orlicz functions.** Let $\Phi$ be a positive function on $\mathbb{R}_+ \equiv (0,\infty)$. The function $\Phi$ is said to be of *upper type* $p$ (resp. *lower type* $p$) for some $p \in [0,\infty)$ if there exists a positive constant $C$ such that for all $t \in [1,\infty)$ (resp. $t \in (0,1]$) and $s \in (0,\infty)$,

$$\Phi(st) \le Ct^p\Phi(s). \tag{2.5}$$

Obviously, if $\Phi$ is of lower type $p$ for some $p \in (0,\infty)$, then $\lim_{t\to 0_+}\Phi(t) = 0$. So for the sake of convenience, if necessary we may assume that $\Phi(0) = 0$. If $\Phi$ is of both upper type $p_1$ and lower type $p_0$, then $\Phi$ is said to be of *type* $(p_0,\, p_1)$. Let

$$p_\Phi^+ \equiv \inf\{p \in (0,\infty) : \text{there exists } C \in (0,\infty) \\ \text{such that (2.5) holds for all } t \in [1,\infty) \text{ and } s \in (0,\infty)\},$$

and

$$p_\Phi^- \equiv \sup\{p \in (0,\infty) : \text{there exists } C \in (0,\infty) \\ \text{such that (2.5) holds for all } t \in (0,1] \text{ and } s \in (0,\infty)\}.$$

The function $\Phi$ is said to be of *strictly lower type* $p$ if for all $t \in (0,1)$ and $s \in (0,\infty)$, $\Phi(st) \le t^p\Phi(s)$, and we define

$$p_\Phi \equiv \sup\{p \in (0,\infty) : \Phi(st) \le t^p\Phi(s) \text{ holds for all } t \in (0,1) \text{ and } s \in (0,\infty)\}. \tag{2.6}$$

It is easy to see that $p_\Phi \le p_\Phi^- \le p_\Phi^+$ for all $\Phi$. In what follows, $p_\Phi$, $p_\Phi^-$ and $p_\Phi^+$ are respectively called the *strictly critical lower type index*, the *critical lower type index* and the *critical upper type index* of $\Phi$. We point out that if $p_\Phi$ is defined as in (2.6), then $\Phi$ is also of strictly critical lower type $p_\Phi$; see [27] for the proof.

Throughout the paper, we always assume that $\Phi$ satisfies the following assumption.

ASSUMPTION (A). Let $\Phi$ be a positive function defined on $\mathbb{R}_+$, which is of strictly lower type with strictly critical lower type index $p_\Phi \in (0,1]$. Also assume that $\Phi$ is continuous, strictly increasing, subadditive and concave.

Notice that if $\Phi$ satisfies Assumption (A), then $\Phi(0) = 0$ and $\Phi$ is obviously of upper type 1. For any concave and positive function $\widetilde{\Phi}$ of strictly lower type $p$, if we set $\Phi(t) \equiv \int_0^t (\widetilde{\Phi}(s)/s)\,ds$ for $t \in [0,\infty)$, then by [55, Proposition 3.1], $\Phi$ is equivalent to $\widetilde{\Phi}$, namely, there exists a positive constant $C$ such that $C^{-1}\widetilde{\Phi}(t) \le \Phi(t) \le C\widetilde{\Phi}(t)$ for all $t \in [0,\infty)$; moreover, $\Phi$ is strictly increasing, concave, subadditive and continuous function of strictly lower type $p$. Notice that all our results are invariant under taking equivalent functions satisfying Assumption (A). From this, we deduce that all results in this paper with $\Phi$ as in Assumption (A) also hold for all concave and positive functions $\widetilde{\Phi}$ of the same strictly critical lower type $p_\Phi$ as $\Phi$.

Let $\Phi$ satisfy Assumption (A) and $\omega \in A_\infty^{\rm loc}(\mathbb{R}^n)$. A measurable function $f$ on $\mathbb{R}^n$ is said to belong to the space $L_\omega^\Phi(\mathbb{R}^n)$ if $\int_{\mathbb{R}^n}\Phi(|f(x)|)\omega(x)\,dx < \infty$. Moreover, for any

$f \in L^{\Phi}_{\omega}(\mathbb{R}^n)$, define

$$\|f\|_{L^{\Phi}_{\omega}(\mathbb{R}^n)} \equiv \inf\left\{\lambda \in (0,\infty) : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right)\omega(x)\,dx \le 1\right\}.$$

Since $\Phi$ is strictly increasing, we define the function $\rho$ on $\mathbb{R}_+$ by setting, for all $t \in (0,\infty)$,

$$\rho(t) \equiv \frac{t^{-1}}{\Phi^{-1}(t^{-1})}, \tag{2.7}$$

where $\Phi^{-1}$ is the inverse function of $\Phi$. Then the types of $\Phi$ and $\rho$ have the following relation. Let $0 < p_0 \le p_1 \le 1$ and $\Phi$ be an increasing function. Then $\Phi$ is of type $(p_0, p_1)$ if and only if $\rho$ is of type $(p_1^{-1} - 1, p_0^{-1} - 1)$; see [55] for the proof. Moreover, it is easy to see that for all $t \in (0,\infty)$,

$$t\Phi\left(\frac{1}{t\rho(t)}\right) = 1, \tag{2.8}$$

which is used in what follows.

## 3. Weighted local Orlicz–Hardy spaces and their maximal function characterizations

In this section, we introduce the weighted local Orlicz–Hardy space $h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)$ via the local grand maximal function and establish its local vertical and nontangential maximal function characterizations via a local Calderón reproducing formula and some useful estimates obtained by Rychkov [43]. We also introduce the weighted atomic local Orlicz–Hardy space $h^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)$ and give some basic properties of these spaces.

First, we introduce some local maximal functions. For $N \in \mathbb{Z}_+$ and $R \in (0,\infty)$, let

$$\mathcal{D}_{N,\,R}(\mathbb{R}^n) \equiv \Big\{\psi \in \mathcal{D}(\mathbb{R}^n) : \operatorname{supp}(\psi) \subset B(0,R), \\ \|\psi\|_{\mathcal{D}_N(\mathbb{R}^n)} \equiv \sup_{x\in\mathbb{R}^n} \sup_{\alpha\in\mathbb{Z}^n_+,\,|\alpha|\le N} |\partial^{\alpha}\psi(x)| \le 1\Big\}.$$

DEFINITION 3.1. Let $N \in \mathbb{Z}_+$ and $R \in (0,\infty)$. For any $f \in \mathcal{D}'(\mathbb{R}^n)$, the *local nontangential grand maximal function* $\widetilde{\mathcal{G}}_{N,\,R}(f)$ of $f$ is defined by setting, for all $x \in \mathbb{R}^n$,

$$\widetilde{\mathcal{G}}_{N,\,R}(f)(x) \equiv \sup\{|\psi_t * f(z)| : |x - z| < t < 1,\ \psi \in \mathcal{D}_{N,\,R}(\mathbb{R}^n)\}, \tag{3.1}$$

and the *local vertical grand maximal function* $\mathcal{G}_{N,\,R}(f)$ of $f$ is defined by setting, for all $x \in \mathbb{R}^n$,

$$\mathcal{G}_{N,\,R}(f)(x) \equiv \sup\{|\psi_t * f(x)| : t \in (0,1),\ \psi \in \mathcal{D}_{N,\,R}(\mathbb{R}^n)\}. \tag{3.2}$$

For convenience's sake, when $R = 1$, we denote $\mathcal{D}_{N,\,R}(\mathbb{R}^n)$, $\widetilde{\mathcal{G}}_{N,\,R}(f)$ and $\mathcal{G}_{N,\,R}(f)$ simply by $\mathcal{D}^0_N(\mathbb{R}^n)$, $\widetilde{\mathcal{G}}^0_N(f)$ and $\mathcal{G}^0_N(f)$, respectively; when $R = 2^{3(10+n)}$, we denote $\mathcal{D}_{N,\,R}(\mathbb{R}^n)$, $\widetilde{\mathcal{G}}_{N,\,R}(f)$ and $\mathcal{G}_{N,\,R}(f)$ simply by $\mathcal{D}_N(\mathbb{R}^n)$, $\widetilde{\mathcal{G}}_N(f)$ and $\mathcal{G}_N(f)$, respectively. For any $N \in \mathbb{Z}_+$ and $x \in \mathbb{R}^n$, obviously,

$$\mathcal{G}^0_N(f)(x) \le \mathcal{G}_N(f)(x) \le \widetilde{\mathcal{G}}_N(f)(x).$$

For the local grand maximal function $\mathcal{G}^0_N(f)$, we have the following proposition, which is just [49, Proposition 2.2].

PROPOSITION 3.2. *Let $N \geq 2$.*

(i) *Then there exists a positive constant $C$ such that for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap \mathcal{D}'(\mathbb{R}^n)$ and almost every $x \in \mathbb{R}^n$,*

$$|f(x)| \leq \mathcal{G}^0_N(f)(x) \leq M^{\mathrm{loc}}(f)(x).$$

(ii) *If $\omega \in A^{\mathrm{loc}}_p(\mathbb{R}^n)$ with $p \in (1, \infty)$, then $f \in L^p_\omega(\mathbb{R}^n)$ if and only if $f \in \mathcal{D}'(\mathbb{R}^n)$ and $\mathcal{G}^0_N(f) \in L^p_\omega(\mathbb{R}^n)$; moreover,*

$$\|f\|_{L^p_\omega(\mathbb{R}^n)} \sim \|\mathcal{G}^0_N(f)\|_{L^p_\omega(\mathbb{R}^n)}.$$

(iii) *If $\omega \in A^{\mathrm{loc}}_1(\mathbb{R}^n)$, then $\mathcal{G}^0_N$ is bounded from $L^1_\omega(\mathbb{R}^n)$ to $L^{1,\infty}_\omega(\mathbb{R}^n)$.*

Now we introduce the weighted local Orlicz–Hardy space via the local grand maximal function as follows.

DEFINITION 3.3. Let $\Phi$ satisfy Assumption (A), $\omega \in A^{\mathrm{loc}}_\infty(\mathbb{R}^n)$, $q_\omega$ and $p_\Phi$ be respectively as in (2.4) and (2.6), and $\widetilde{N}_{\Phi,\,\omega} \equiv \lfloor n(q_\omega/p_\Phi - 1)\rfloor + 2$. For each $N \in \mathbb{N}$ with $N \geq \widetilde{N}_{\Phi,\,\omega}$, the *weighted local Orlicz–Hardy space* is defined by

$$h^\Phi_{\omega,\,N}(\mathbb{R}^n) \equiv \{f \in \mathcal{D}'(\mathbb{R}^n) : \mathcal{G}_N(f) \in L^\Phi_\omega(\mathbb{R}^n)\}.$$

Moreover, let $\|f\|_{h^\Phi_{\omega,\,N}(\mathbb{R}^n)} \equiv \|\mathcal{G}_N(f)\|_{L^\Phi_\omega(\mathbb{R}^n)}$.

We remark that when $\Phi(t) \equiv t^p$ for all $t \in (0, \infty)$ and $p \in (0, 1]$, $h^\Phi_{\omega,\,N}(\mathbb{R}^n)$ above is the weighted local Hardy space $h^p_{\omega,\,N}(\mathbb{R}^n)$ introduced by Tang [49]. Obviously, for any integers $N_1$ and $N_2$ with $N_1 \geq N_2 \geq \widetilde{N}_{\Phi,\,\omega}$,

$$h^\Phi_{\omega,\,\widetilde{N}_{\Phi,\,\omega}}(\mathbb{R}^n) \subset h^\Phi_{\omega,\,N_2}(\mathbb{R}^n) \subset h^\Phi_{\omega,\,N_1}(\mathbb{R}^n),$$

and the inclusions are continuous. We also point out that Theorem 3.14 below further implies that

$$\|\mathcal{G}^0_N(f)\|_{L^\Phi_\omega(\mathbb{R}^n)} \sim \|\widetilde{\mathcal{G}}^0_N(f)\|_{L^\Phi_\omega(\mathbb{R}^n)} \sim \|\mathcal{G}_N(f)\|_{L^\Phi_\omega(\mathbb{R}^n)} \sim \|\widetilde{\mathcal{G}}_N(f)\|_{L^\Phi_\omega(\mathbb{R}^n)}$$

for all $N \in \mathbb{N}$ with $N \geq N_{\Phi,\,\omega}$ (see (3.25) for the definition of $N_{\Phi,\,\omega}$).

Next, we introduce the weighted local atoms, via which we introduce the weighted atomic local Orlicz–Hardy space.

DEFINITION 3.4. Let $\Phi$ satisfy Assumption (A), $\omega \in A^{\mathrm{loc}}_\infty(\mathbb{R}^n)$ and $q_\omega$, $\rho$ be respectively as in (2.4) and (2.7). A triplet $(\rho,\, q,\, s)_\omega$ is called *admissible* if $q \in (q_\omega, \infty]$, $s \in \mathbb{Z}_+$ and $s \geq \lfloor n(q_\omega/p_\Phi - 1)\rfloor$. A function $a$ on $\mathbb{R}^n$ is called a $(\rho,\, q,\, s)_\omega$*-atom* if there exists a cube $Q \subset \mathbb{R}^n$ such that

(i) $\mathrm{supp}(a) \subset Q$;
(ii) $\|a\|_{L^q_\omega(\mathbb{R}^n)} \leq [\omega(Q)]^{\frac{1}{q}-1}[\rho(\omega(Q))]^{-1}$;
(iii) $\int_{\mathbb{R}^n} a(x)x^\alpha\,dx = 0$ for all $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| \leq s$, when $l(Q) < 1$.

Moreover, a function $a$ on $\mathbb{R}^n$ is called a $(\rho,\, q)_\omega$*-single-atom* with $q \in (q_\omega, \infty]$ if

$$\|a\|_{L^q_\omega(\mathbb{R}^n)} \leq [\omega(\mathbb{R}^n)]^{1/q-1}[\rho(\omega(\mathbb{R}^n))]^{-1}.$$

We point out that when $\Phi(t) \equiv t^p$ for all $t \in (0, \infty)$ and $p \in (0, 1]$, $(\rho,\, q,\, s)_\omega$-atoms and $(\rho,\, q)_\omega$-single-atoms are respectively $(p,\, q,\, s)_\omega$-atoms and $(p,\, q)_\omega$-single-atoms, introduced by Tang [49].

DEFINITION 3.5. Let $\Phi$ satisfy Assumption (A), $\omega \in A_\infty^{\mathrm{loc}}(\mathbb{R}^n)$, $q_\omega$ and $\rho$ be respectively as in (2.4) and (2.7), and $(\rho,\, q,\, s)_\omega$ be admissible. The *weighted atomic local Orlicz–Hardy space* $h_\omega^{\rho,\, q,\, s}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{D}'(\mathbb{R}^n)$ satisfying

$$f = \sum_{i=0}^{\infty} \lambda_i a_i$$

in $\mathcal{D}'(\mathbb{R}^n)$, where $\{a_i\}_{i\in\mathbb{N}}$ are $(\rho,\, q,\, s)_\omega$-atoms with $\mathrm{supp}(a_i) \subset Q_i$, $a_0$ is a $(\rho,\, q)_\omega$-single-atom, $\{\lambda_i\}_{i\in\mathbb{Z}_+} \subset \mathbb{C}$, and

$$\sum_{i=1}^{\infty} \omega(Q_i)\Phi\left(\frac{|\lambda_i|}{\omega(Q_i)\rho(\omega(Q_i))}\right) + \omega(\mathbb{R}^n)\Phi\left(\frac{|\lambda_0|}{\omega(\mathbb{R}^n)\rho(\omega(\mathbb{R}^n))}\right) < \infty.$$

Moreover, letting

$$\Lambda(\{\lambda_i a_i\}_i) \equiv \inf\left\{\lambda \in (0,\infty) : \sum_{i=1}^{\infty} \omega(Q_i)\Phi\left(\frac{|\lambda_i|}{\lambda\omega(Q_i)\rho(\omega(Q_i))}\right)\right.$$
$$\left. + \,\omega(\mathbb{R}^n)\Phi\left(\frac{|\lambda_0|}{\lambda\omega(\mathbb{R}^n)\rho(\omega(\mathbb{R}^n))}\right) \le 1\right\},$$

the quasi-norm of $f \in h_\omega^{\rho,\, q,\, s}(\mathbb{R}^n)$ is defined by

$$\|f\|_{h_\omega^{\rho,\, q,\, s}(\mathbb{R}^n)} \equiv \inf\{\Lambda(\{\lambda_i a_i\}_{i\in\mathbb{Z}_+})\},$$

where the infimum is taken over all the decompositions of $f$ as above.

REMARK 3.6. (i) Notice that the definition of $\Lambda(\{\lambda_i a_i\}_{i\in\mathbb{Z}_+})$ above is different from that in [55]. In fact, if $p \in (0, 1]$ and $\Phi(t) \equiv t^p$ for all $t \in (0, \infty)$, then $\Lambda(\{\lambda_i a_i\}_{i\in\mathbb{Z}_+})$ coincides with $(\sum_{i\in\mathbb{Z}_+} |\lambda_i|^p)^{1/p}$.

(ii) Let $\{\lambda_i^k\}_{i,\, k}$ and $\{a_i^k\}_{i,\, k}$ satisfy $\Lambda(\{\lambda_i^k a_i^k\}_{i\in\mathbb{Z}_+}) < \infty$, where $k = 1,\, 2$. Since $\Phi$ is subadditive and of strictly lower type $p_\Phi$, we have

$$[\Lambda(\{\lambda_i^1 a_i^1,\ \lambda_i^2 a_i^2\}_{i\in\mathbb{Z}_+})]^{p_\Phi} \le \sum_{k=1}^{2} [\Lambda(\{\lambda_i^k a_i^k\}_{i\in\mathbb{Z}_+})]^{p_\Phi}.$$

(iii) Since $\Phi$ is concave, it is of upper type 1. Thus, for any $f \in h_\omega^{\rho,\, q,\, s}(\mathbb{R}^n)$, there exist $\{a_i\}_{i\in\mathbb{Z}_+}$ and $\{\lambda_i\}_{i\in\mathbb{Z}_+}$ as in Definition 3.5 such that

$$\sum_{i\in\mathbb{Z}_+} |\lambda_i| \lesssim \Lambda(\{\lambda_i a_i\}_{i\in\mathbb{Z}_+}) \lesssim \|f\|_{h_\omega^{\rho,\, q,\, s}(\mathbb{R}^n)}.$$

Next, we introduce some local vertical, tangential and nontangential maximal functions, and then establish the characterizations of the weighted local Orlicz–Hardy space $h_{\omega,\, N}^{\Phi}(\mathbb{R}^n)$ via these local maximal functions.

DEFINITION 3.7. Let

$$\psi_0 \in \mathcal{D}(\mathbb{R}^n) \text{ with } \int_{\mathbb{R}^n} \psi_0(x)\, dx \ne 0. \tag{3.3}$$

For $j \in \mathbb{Z}_+$, $A, B \in [0, \infty)$ and $y \in \mathbb{R}^n$, let $m_{j,\,A,\,B}(y) \equiv (1+2^j|y|)^A 2^{B|y|}$. The *local vertical maximal function* $\psi_0^+(f)$ of $f$ associated to $\psi_0$ is defined by setting, for all $x \in \mathbb{R}^n$,

$$\psi_0^+(f)(x) \equiv \sup_{j \in \mathbb{Z}_+} |(\psi_0)_j * f(x)|, \tag{3.4}$$

the *local tangential Peetre-type maximal function* $\psi_{0,\,A,\,B}^{**}(f)$ of $f$ associated to $\psi_0$ is defined by setting, for all $x \in \mathbb{R}^n$,

$$\psi_{0,\,A,\,B}^{**}(f)(x) \equiv \sup_{j \in \mathbb{Z}_+,\, y \in \mathbb{R}^n} \frac{|(\psi_0)_j * f(x-y)|}{m_{j,\,A,\,B}(y)} \tag{3.5}$$

and the *local nontangential maximal function* $(\psi_0)_\triangledown^*(f)$ of $f$ associated to $\psi_0$ is defined by setting, for all $x \in \mathbb{R}^n$,

$$(\psi_0)_\triangledown^*(f)(x) \equiv \sup_{|x-y|<t<1} |(\psi_0)_t * f(y)|; \tag{3.6}$$

here and in what follows, for all $x \in \mathbb{R}^n$, $(\psi_0)_j(x) \equiv 2^{jn}\psi_0(2^j x)$ for all $j \in \mathbb{Z}_+$ and $(\psi_0)_t(x) \equiv (1/t^n)\psi_0(x/t)$ for all $t \in (0, \infty)$.

Obviously, for any $x \in \mathbb{R}^n$, we have

$$\psi_0^+(f)(x) \le (\psi_0)_\triangledown^*(f)(x) \lesssim \psi_{0,\,A,\,B}^{**}(f)(x).$$

We remark that the local tangential Peetre-type maximal function $\psi_{0,\,A,\,B}^{**}(f)$ was introduced by Rychkov [43].

In order to establish the local vertical and the local nontangential maximal function characterizations of $h_{\omega,\,N}^{\Phi}(\mathbb{R}^n)$, we first establish some relations in the norm of $L_\omega^\Phi(\mathbb{R}^n)$ of the local maximal functions $\psi_{0,\,A,\,B}^{**}(f)$, $\psi_0^+(f)$ and $\widetilde{\mathcal{G}}_{N,\,R}(f)$, which further imply the desired characterizations. We begin with some technical lemmas.

Lemma 3.8. *Let $\psi_0$ be as in (3.3) and $\psi(x) \equiv \psi_0(x) - (1/2^n)\psi_0(x/2)$ for all $x \in \mathbb{R}^n$. Then for any given integer $L \in \mathbb{Z}_+$, there exist $\eta_0, \eta \in \mathcal{D}(\mathbb{R}^n)$ such that $L_\eta \ge L$ and*

$$f = \eta_0 * \psi_0 * f + \sum_{j=1}^{\infty} \eta_j * \psi_j * f$$

*in $\mathcal{D}'(\mathbb{R}^n)$ for all $f \in \mathcal{D}'(\mathbb{R}^n)$.*

Lemma 3.8 is just [43, Theorem 1.6].

Remark 3.9. Let $\psi_0$, $\psi$, $\eta_0$ and $\eta$ be as in Lemma 3.8. From the proof of [43, Theorem 1.6], it is easy to deduce that for any $j \in \mathbb{Z}_+$ and $f \in \mathcal{D}'(\mathbb{R}^n)$,

$$f = (\eta_0)_j * (\psi_0)_j * f + \sum_{k=j+1}^{\infty} \eta_k * \psi_k * f$$

in $\mathcal{D}'(\mathbb{R}^n)$ (see also [43, (2.11)]). We omit the details.

For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $B \in [0, \infty)$ and $x \in \mathbb{R}^n$, let

$$K_B f(x) \equiv \int_{\mathbb{R}^n} |f(y)| 2^{-B|x-y|}\, dy; \tag{3.7}$$

here and in what follows, $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ denotes the *set of all locally integrable functions on* $\mathbb{R}^n$.

Lemma 3.10. *Let $p \in (1, \infty)$, $q \in (1, \infty]$, and $\omega \in A_p^{\mathrm{loc}}(\mathbb{R}^n)$. Then there exists a positive constant $C$ such that for any sequence $\{f^j\}_j$ of measurable functions,*

$$\|\{M^{\mathrm{loc}}(f^j)\}_j\|_{L^p_\omega(l_q)} \le C\|\{f^j\}_j\|_{L^p_\omega(l_q)}; \tag{3.8}$$

*here and in what follows,*

$$\|\{f^j\}_j\|_{L^p_\omega(l_q)} \equiv \Big\|\Big\{\sum_j |f^j|^q\Big\}^{1/q}\Big\|_{L^p_\omega(\mathbb{R}^n)}.$$

*Also, there exist positive constants $C$ and $B_0 \equiv B_0(\omega, n)$ such that for all $B \ge B_0/p$,*

$$\|\{K_B(f^j)\}_j\|_{L^p_\omega(l_q)} \le C\|\{f^j\}_j\|_{L^p_\omega(l_q)}. \tag{3.9}$$

Lemma 3.10 is just [43, Lemma 2.11]. Moreover, from the proof of [43, Lemma 2.11], it is easy to deduce that (3.8) also holds for $M^{\mathrm{loc}}_{\widetilde{C}}$ with any given positive constant $\widetilde{C}$. In this case, the positive constant $C$ in Lemma 3.10 depends on $\widetilde{C}$.

Lemma 3.11. *Let $\psi_0$ be as in (3.3) and $r \in (0, \infty)$. Then there exists a positive constant $A_0$ depending only on the support of $\psi_0$ such that for any $A \in (\max\{A_0, n/r\}, \infty)$ and $B \in [0, \infty)$, there exists a positive constant $C$, depending only on $n$, $r$, $\psi_0$, $A$ and $B$, such that for all $f \in \mathcal{D}'(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}_+$,*

$$\begin{aligned}[(\psi_0)^*_{j,A,B}(f)(x)]^r \le C\sum_{k=j}^{\infty} 2^{(j-k)(Ar-n)}\{&M^{\mathrm{loc}}(|(\psi_0)_k * f|^r)(x)\\ &+ K_{Br}(|(\psi_0)_k * f|^r)(x)\},\end{aligned}$$

*where*

$$(\psi_0)^*_{j,A,B}(f)(x) \equiv \sup_{y\in\mathbb{R}^n} \frac{|(\psi_0)_j * f(x-y)|}{m_{j,A,B}(y)}$$

*for all $x \in \mathbb{R}^n$.*

*Proof.* Lemma 3.11 is a modified version of [43, Lemma 2.10], and was essentially obtained by Rychkov in the proof of [43, Theorem 2.24]. Let $\psi$ be as in Lemma 3.8. Indeed, Rychkov [43] showed Lemma 3.11 under the assumption that $f \in \mathcal{S}'_e$, namely, there exist a positive constant $A_f$ and a nonnegative integer $N_f$ such that for all $\gamma \in \mathcal{D}(\mathbb{R}^n)$,

$$|\langle f, \gamma\rangle| \le A_f \sup\{|\partial^\alpha\gamma(x)|e^{N_f|x|} :\ x \in \mathbb{R}^n,\ \alpha \in \mathbb{Z}^n_+ \text{ and } |\alpha| \le N_f\},$$

which guarantees that for all $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}_+$,

$$M_{A,B}(x, j) \equiv \sup_{k\ge j,\, y\in\mathbb{R}^n} 2^{(j-k)A}\frac{|\psi_k * f(x-y)|}{m_{j,A,B}(y)} < \infty.$$

By [19, Proposition 2.3.4(a)], for any $f \in \mathcal{D}'(\mathbb{R}^n)$, we have $M_{A,B}(x, j) < \infty$ for all $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}_+$, provided $A > A_0$, where $A_0$ is a positive constant depending only on the support of $\psi_0$. This finishes the proof. ∎

Theorem 3.12. *Let $\Phi$ satisfy Assumption* (A), *$\omega \in A^{\mathrm{loc}}_\infty(\mathbb{R}^n)$, $R \in (0, \infty)$, $\psi_0$, $q_\omega$ and $p_\Phi$ be respectively as in (3.3), (2.4) and (2.6), and $\psi_0^+(f)$, $\psi^{**}_{0,A,B}(f)$, and $\widetilde{\mathcal{G}}_{N,R}(f)$ be respectively as in (3.4), (3.5) and (3.1). Let*

$$A_1 \equiv \max\{A_0,\, nq_\omega/p_\Phi\},$$

*$B_1 \equiv B_0/p_\Phi$ and $N_0 \equiv \lfloor 2A_1 \rfloor + 1$, where $A_0$ and $B_0$ are respectively as in Lemmas* 3.3 *and* 3.2*. Then for any $A \in (A_1, \infty)$, $B \in (B_1, \infty)$ and integer $N \ge N_0$, there exists a positive constant $C$, depending only on $A$, $B$, $N$, $R$, $\psi_0$, $\Phi$, $\omega$ and $n$, such that for all $f \in \mathcal{D}'(\mathbb{R}^n)$,*

$$\|\psi_{0,\,A,\,B}^{**}(f)\|_{L^\Phi_\omega(\mathbb{R}^n)} \le C\|\psi_0^+(f)\|_{L^\Phi_\omega(\mathbb{R}^n)}, \tag{3.10}$$

*and*

$$\|\widetilde{\mathcal{G}}_{N,\,R}(f)\|_{L^\Phi_\omega(\mathbb{R}^n)} \le C\|\psi_0^+(f)\|_{L^\Phi_\omega(\mathbb{R}^n)}. \tag{3.11}$$

*Proof.* Let $f \in \mathcal{D}'(\mathbb{R}^n)$. First, we prove (3.10). Let $A \in (A_1, \infty)$ and $B \in (B_1, \infty)$. By $A_1 \equiv \max\{A_0, nq_\omega/p_\Phi\}$ and $B_1 \equiv B_0/p_\Phi$, we know that there exists $r_0 \in (0, p_\Phi/q_\omega)$ such that $A > n/r_0$ and $Br_0 > B_0/q_\omega$, where $A_0$ and $B_0$ are respectively as in Lemmas 3.3 and 3.10. Thus, by Lemma 3.11, for all $x \in \mathbb{R}^n$, we have

$$\begin{aligned}[][(\psi_0)^*_{j,\,A,\,B}(f)(x)]^{r_0} \lesssim \sum_{k=j}^{\infty} 2^{(j-k)(Ar_0-n)}\{&M^{\mathrm{loc}}(|(\psi_0)_k * f|^{r_0})(x)\\ &+ K_{Br_0}(|(\psi_0)_k * f|^{r_0})(x)\}.\end{aligned} \tag{3.12}$$

Let $\psi_0^+(f)$ and $\psi_{0,\,A,\,B}^{**}(f)$ be respectively as in (3.4) and (3.5). We notice that for any $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}_+$,

$$|(\psi_0)_k * f(x)| \le \psi_0^+(f)(x),$$

which together with (3.12) implies that for all $x \in \mathbb{R}^n$,

$$[\psi_{0,\,A,\,B}^{**}(f)(x)]^{r_0} \lesssim M^{\mathrm{loc}}([\psi_0^+(f)]^{r_0})(x) + K_{Br_0}([\psi_0^+(f)]^{r_0})(x). \tag{3.13}$$

By (3.12) and the subadditivity of $\Phi$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} &\Phi(\psi_{0,\,A,\,B}^{**}(f)(x))\omega(x)\,dx\\ &\lesssim \int_{\mathbb{R}^n} \Phi(\{M^{\mathrm{loc}}([\psi_0^+(f)]^{r_0})(x)\}^{1/r_0})\omega(x)\,dx\\ &\quad+ \int_{\mathbb{R}^n} \Phi(\{K_{Br_0}([\psi_0^+(f)]^{r_0})(x)\}^{1/r_0})\omega(x)\,dx \equiv \mathrm{I}_1 + \mathrm{I}_2.\end{aligned} \tag{3.14}$$

First, we estimate $\mathrm{I}_1$. As $r_0 < p_\Phi/q_\omega$, we know that there exists $q \in (q_\omega, \infty)$ such that $r_0 q < p_\Phi$ and $\omega \in A_q^{\mathrm{loc}}(\mathbb{R}^n)$. For any $\alpha \in (0, \infty)$ and $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, let

$$g = g\chi_{\{x\in\mathbb{R}^n:\ |g(x)|\le\alpha\}} + g\chi_{\{x\in\mathbb{R}^n:\ |g(x)|>\alpha\}} \equiv g_1 + g_2.$$

It is easy to see that

$$\{x \in \mathbb{R}^n : M^{\mathrm{loc}}(g)(x) > 2\alpha\} \subset \{x \in \mathbb{R}^n : M^{\mathrm{loc}}(g_2)(x) > \alpha\},$$

which together with Lemma 2.3(vi) implies that

$$\begin{aligned}\omega(&\{x \in \mathbb{R}^n : M^{\mathrm{loc}}(g)(x) > 2\alpha\})\\ &\le \omega(\{x \in \mathbb{R}^n : M^{\mathrm{loc}}(g_2)(x) > \alpha\}) \le \frac{1}{\alpha^q}\int_{\mathbb{R}^n} [M^{\mathrm{loc}}(g_2)(x)]^q\omega(x)\,dx\\ &\lesssim \frac{1}{\alpha^q}\int_{\mathbb{R}^n} |g_2(x)|^q\omega(x)\,dx \sim \frac{1}{\alpha^q}\int_{\{x\in\mathbb{R}^n:\,|g(x)|>\alpha\}} |g(x)|^q\omega(x)\,dx.\end{aligned} \tag{3.15}$$

Thus, for any $\alpha \in (0,\infty)$, by (3.15), we have

$$\begin{aligned}
\omega(\{x \in \mathbb{R}^n :\, & [M^{\mathrm{loc}}([\psi_0^+(f)]^{r_0})(x)]^{1/r_0} > \alpha\}) \\
&\lesssim \frac{1}{\alpha^{r_0 q}} \int_{\{x\in\mathbb{R}^n:\, [\psi_0^+(f)(x)]^{r_0} > \alpha^{r_0}/2\}} [\psi_0^+(f)(x)]^{r_0 q} \omega(x)\, dx \\
&\sim \sigma_{\psi_0^+(f)}\left(\frac{\alpha}{2^{1/r_0}}\right) + \frac{1}{\alpha^{r_0 q}} \int_{\alpha/2^{1/r_0}}^\infty r_0 q s^{r_0 q-1} \sigma_{\psi_0^+(f)}(s)\, ds; \qquad (3.16)
\end{aligned}$$

here and in what follows,

$$\sigma_{\psi_0^+(f)}(s) \equiv \omega(\{x \in \mathbb{R}^n :\, \psi_0^+(f)(x) > s\}).$$

From the fact that $\Phi$ is concave and of lower type $p_\Phi$, we infer that $\Phi(t) \sim \int_0^t (\Phi(s)/s)\, ds$ for all $t \in (0,\infty)$. By this, (3.16) and the lower type $p_\Phi$ property of $\Phi$, the fact $r_0 q < p_\Phi$ and Fubini's theorem, we have

$$\begin{aligned}
\mathrm{I}_1 &\sim \int_{\mathbb{R}^n} \left\{ \int_0^{\{M^{\mathrm{loc}}([\psi_0^+(f)]^{r_0})(x)\}^{1/r_0}} \frac{\Phi(t)}{t}\, dt \right\} \omega(x)\, dx \\
&\sim \int_0^\infty \frac{\Phi(t)}{t} \sigma_{\{M^{\mathrm{loc}}([\psi_0^+(f)]^{r_0})\}^{1/r_0}}(t)\, dt \\
&\lesssim \int_0^\infty \frac{\Phi(t)}{t} \left\{ \sigma_{\psi_0^+(f)}\left(\frac{t}{2^{1/r_0}}\right) + \frac{1}{t^{r_0 q}} \int_{t/2^{1/r_0}}^\infty r_0 q s^{r_0 q-1} \sigma_{\psi_0^+(f)}(s)\, ds \right\} dt \\
&\lesssim \mathrm{J}_f + \int_0^\infty r_0 q s^{r_0 q-1} \sigma_{\psi_0^+(f)}(s) \left\{ \int_0^{2^{1/r_0} s} \frac{\Phi(t)}{t} \frac{1}{t^{r_0 q}}\, dt \right\} ds \\
&\sim \mathrm{J}_f + \int_0^\infty r_0 q s^{r_0 q-1} \sigma_{\psi_0^+(f)}(s) \Phi(2^{1/r_0} s) \left\{ \int_0^{2^{1/r_0} s} \left(\frac{t}{2^{1/r_0} s}\right)^{p_\Phi} \frac{1}{t^{r_0 q+1}}\, dt \right\} ds \\
&\sim \mathrm{J}_f \sim \int_{\mathbb{R}^n} \Phi(\psi_0^+(f)(x)) \omega(x)\, dx, \qquad (3.17)
\end{aligned}$$

where $\mathrm{J}_f \equiv \int_0^\infty (\Phi(t)/t) \sigma_{\psi_0^+(f)}(t)\, dt$.

Next, we estimate $\mathrm{I}_2$. For any $\alpha \in (0,\infty)$ and $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, let $g_1$ and $g_2$ be as above. For $H \in [B_0/q, \infty)$, let $\int_{\mathbb{R}^n} 2^{-H|x-y|}\, dy \equiv c_H$. It is easy to see that for all $x \in \mathbb{R}^n$, $K_H(g_1)(x) \le c_H \alpha$, which implies that

$$\{x \in \mathbb{R}^n :\, K_H(g)(x) > (c_H+1)\alpha\} \subset \{x \in \mathbb{R}^n :\, K_H(g_2)(x) > \alpha\},$$

where $K_H$ is as in (3.7). Thus, by Lemma 3.10, we have

$$\begin{aligned}
\omega(\{x \in \mathbb{R}^n :\, K_H g(x) > (c_H+1)\alpha\}) &\le \omega(\{x \in \mathbb{R}^n :\, K_H g_2(x) > \alpha\}) \\
&\lesssim \frac{1}{\alpha^q} \int_{\{x\in\mathbb{R}^n:\, |g(x)|>\alpha\}} |g(x)|^q \omega(x)\, dx.
\end{aligned}$$

Similarly to (3.16), from the above estimate, $Br_0 > B_0/q$ and Lemma 3.2, we also deduce that

$$\begin{aligned}
\omega(\{x \in \mathbb{R}^n :\, & [K_{Br_0}([\psi_0^+(f)]^{r_0})(x)]^{1/r_0} > \alpha\}) \\
&\lesssim \sigma_{\psi_0^+(f)}\left(\frac{\alpha}{(c_{Br_0}+1)^{1/r_0}}\right) + \frac{1}{\alpha^{r_0 q}} \int_{\alpha/(c_{Br_0}+1)^{1/r_0}}^\infty r_0 q s^{r_0 q-1} \sigma_{\psi_0^+(f)}(s)\, ds.
\end{aligned}$$

From this, similarly to the estimate of $\mathrm{I}_1$, we also have

$$\mathrm{I}_2 \lesssim \int_{\mathbb{R}^n} \Phi(\psi_0^+(f)(x))\omega(x)\,dx. \tag{3.18}$$

Thus, we deduce from (3.14), (3.17) and (3.18) that

$$\int_{\mathbb{R}^n} \Phi(\psi_{0,A,B}^{**}(f)(x))\omega(x)\,dx \lesssim \int_{\mathbb{R}^n} \Phi(\psi_0^+(f)(x))\omega(x)\,dx.$$

Replacing $f$ by $f/\lambda$ with $\lambda \in (0,\infty)$ in the above inequality, and noticing that

$$\Phi(\psi_{0,A,B}^{**}(f/\lambda)) = \Phi(\psi_{0,A,B}^{**}(f)/\lambda)$$

and $\Phi(\psi_0^+(f/\lambda)) = \Phi(\psi_0^+(f)/\lambda)$, we have

$$\int_{\mathbb{R}^n} \Phi(\psi_{0,A,B}^{**}(f)(x)/\lambda)\omega(x)\,dx \lesssim \int_{\mathbb{R}^n} \Phi(\psi_0^+(f)(x)/\lambda)\omega(x)\,dx, \tag{3.19}$$

which together with the arbitrariness of $\lambda \in (0,\infty)$ implies (3.10).

Now, we prove (3.11). By $N_0 \equiv \lfloor 2A_1 \rfloor + 1$, we know that there exists $A \in (A_1,\infty)$ such that $2A < N_0$. In the rest of this proof, we fix $A \in (A_1,\infty)$ satisfying $2A < N_0$ and $B \in (B_1,\infty)$. Pick an integer $N \ge N_0$ and $R \in (0,\infty)$. For any $\gamma \in \mathcal{D}_{N,R}(\mathbb{R}^n)$, $t \in (0,1)$ and $j \in \mathbb{Z}_+$, from Lemma 3.8 and Remark 3.9, it follows that

$$\gamma_t * f = \gamma_t * (\eta_0)_j * (\psi_0)_j * f + \sum_{k=j+1}^{\infty} \gamma_t * \eta_k * \psi_k * f, \tag{3.20}$$

where $\eta_0, \eta \in \mathcal{D}(\mathbb{R}^n)$ with $L_\eta \ge N$ and $\psi$ is as in Lemma 3.8.

For any given $t \in (0,1)$ and $x \in \mathbb{R}^n$, let $2^{-j_0-1} \le t < 2^{-j_0}$ for some $j_0 \in \mathbb{Z}_+$, and $z \in \mathbb{R}^n$ satisfy $|z-x| < t$. Then, by (3.20), we have

$$\begin{aligned}
|\gamma_t * f(z)| &\le |\gamma_t * (\eta_0)_{j_0} * (\psi_0)_{j_0} * f(z)| + \sum_{k=j_0+1}^{\infty} |\gamma_t * \eta_k * \psi_k * f(z)| \\
&\le \int_{\mathbb{R}^n} |\gamma_t * (\eta_0)_{j_0}(y)|\,|(\psi_0)_{j_0} * f(z-y)|\,dy \\
&\quad + \sum_{k=j_0+1}^{\infty} \int_{\mathbb{R}^n} |\gamma_t * \eta_k(y)|\,|\psi_k * f(z-y)|\,dy \equiv \mathrm{I}_3 + \mathrm{I}_4.
\end{aligned} \tag{3.21}$$

To estimate $\mathrm{I}_3$, from

$$\begin{aligned}
\psi_{0,A,B}^{**}(f)(x) &= \sup_{j\in\mathbb{Z}_+,\, y\in\mathbb{R}^n} \frac{|(\psi_0)_j * f(x-y)|}{m_{j,A,B}(y)} \\
&= \sup_{j\in\mathbb{Z}_+,\, y\in\mathbb{R}^n} \frac{|(\psi_0)_j * f(x-(y+x-z))|}{m_{j,A,B}(y+x-z)} = \sup_{j\in\mathbb{Z}_+,\, y\in\mathbb{R}^n} \frac{|(\psi_0)_j * f(z-y)|}{m_{j,A,B}(y+x-z)},
\end{aligned}$$

we infer that

$$|(\psi_0)_{j_0} * f(z-y)| \le \psi_{0,A,B}^{**}(f)(x) m_{j_0,A,B}(y+x-z),$$

which, together with the facts that

$$m_{j_0,A,B}(y+x-z) \le m_{j_0,A,B}(x-z) m_{j_0,A,B}(y)$$

and $m_{j_0,A,B}(x-z) \lesssim 2^A$, implies that

$$|(\psi_0)_{j_0} * f(z-y)| \lesssim 2^A \psi_{0,A,B}^{**}(f)(x) m_{j_0,A,B}(y).$$

Thus, we have

$$\mathrm{I}_3 \lesssim 2^A \left\{ \int_{\mathbb{R}^n} |\gamma_t * (\eta_0)_{j_0}(y)| m_{j_0,\,A,\,B}(y)\,dy \right\} \psi^{**}_{0,\,A,\,B}(f)(x).$$

To estimate $\mathrm{I}_4$, by the definition of $\psi$, it is easy to see that for any $k \in \mathbb{N}$,

$$|\psi_k * f(z-y)| \le |(\psi_0)_k * f(z-y)| + |(\psi_0)_{k-1} * f(z-y)|.$$

By the definition of $\psi^{**}_{0,\,A,\,B}(f)$ and the facts that

$$m_{k,\,A,\,B}(y+x-z) \le m_{k,\,A,\,B}(x-z) m_{k,\,A,\,B}(y)$$

for any $k \in \mathbb{N}$ and $m_{k,\,A,\,B}(x-z) \lesssim 2^{(k-j_0)A}$, we conclude that

$$\begin{aligned}
|(\psi_0)_k * f(z-y)| &\le \psi^{**}_{0,\,A,\,B}(f)(x) m_{k,\,A,\,B}(y+x-z) \\
&\le \psi^{**}_{0,\,A,\,B}(f)(x) m_{k,\,A,\,B}(x-z) m_{k,\,A,\,B}(y) \\
&\lesssim 2^{(k-j_0)A} m_{k,\,A,\,B}(y) \psi^{**}_{0,\,A,\,B}(f)(x).
\end{aligned}$$

Similarly, we also have

$$|(\psi_0)_{k-1} * f(z-y)| \lesssim 2^{(k-j_0)A} m_{k,\,A,\,B}(y) \psi^{**}_{0,\,A,\,B}(f)(x).$$

Thus,

$$\mathrm{I}_4 \lesssim \sum_{k=j_0+1}^{\infty} 2^{(k-j_0)A} \left\{ \int_{\mathbb{R}^n} |\gamma_t * \eta_k(y)| m_{k,\,A,\,B}(y)\,dy \right\} \psi^{**}_{0,\,A,\,B}(f)(x).$$

From (3.21) and the above estimates of $\mathrm{I}_3$ and $\mathrm{I}_4$, it follows that

$$\begin{aligned}
|\gamma_t * f(z)| \lesssim \Bigg\{ &\int_{\mathbb{R}^n} |\gamma_t * (\eta_0)_{j_0}(y)| m_{j_0,\,A,\,B}(y)\,dy \\
&+ \sum_{k=j_0+1}^{\infty} 2^{(k-j_0)A} \int_{\mathbb{R}^n} |\gamma_t * \eta_k(y)| m_{k,\,A,\,B}(y)\,dy \Bigg\} \psi^{**}_{0,\,A,\,B}(f)(x). \quad (3.22)
\end{aligned}$$

Assume that $\mathrm{supp}(\eta_0) \subset B(0, R_0)$. Then $\mathrm{supp}((\eta_0)_j) \subset B(0, 2^{-j}R_0)$ for all $j \in \mathbb{Z}_+$. Moreover, as $\mathrm{supp}(\gamma) \subset B(0, R)$ and $2^{-j_0-1} \le t < 2^{-j_0}$, we see that

$$\mathrm{supp}(\gamma_t) \subset B(0, 2^{-j_0}R).$$

From this, we further deduce that $\mathrm{supp}(\gamma_t * (\eta_0)_{j_0}) \subset B(0, 2^{-j_0}(R_0+R))$ and

$$|\gamma_t * (\eta_0)_{j_0}(y)| \lesssim \int_{\mathbb{R}^n} |\gamma_t(s)|\,|(\psi_0)_{j_0}(y-s)|\,ds \lesssim 2^{j_0 n} \int_{\mathbb{R}^n} |\gamma_t(s)|\,ds \sim 2^{j_0 n},$$

which implies that

$$\int_{\mathbb{R}^n} |\gamma_t * (\eta_0)_{j_0}(y)| m_{j_0,\,A,\,B}(y)\,dy \lesssim 2^{j_0 n} \int_{B(0,2^{-j_0}(R_0+R))} (1+2^{j_0}|y|)^A 2^{B|y|}\,dy \lesssim 1. \quad (3.23)$$

Moreover, since $\eta$ has vanishing moments up to order $N$, it was proved in [43, (2.13)] that

$$\|\gamma_t * \eta_k\|_{L^\infty(\mathbb{R}^n)} \lesssim 2^{(j_0-k)N} 2^{j_0 n}$$

for all $k \in \mathbb{N}$ with $k \ge j_0+1$, which, together with the facts that $N > 2A$ and

$$\mathrm{supp}(\gamma_t * \eta_k) \subset B(0, 2^{-j_0}R_0 + 2^{-k}R),$$

implies that

$$\begin{aligned}\sum_{k=j_0+1}^{\infty} 2^{(k-j_0)A} \int_{\mathbb{R}^n} |\gamma_t * \eta_k(y)| m_{k,\,A,\,B}(y)\,dy \\ &\lesssim \sum_{k=j_0+1}^{\infty} 2^{(k-j_0)A} 2^{(j_0-k)N} 2^{j_0 n} (2^{-j_0}R_0 + 2^{-k}R)^n \\ &\quad \times [1+2^k(2^{-j_0}R_0+2^{-k}R)]^A 2^{(2^{-j_0}R_0+2^{-k}R)B} \\ &\lesssim \sum_{k=j_0+1}^{\infty} 2^{(j_0-k)(N-2A)} \lesssim 1. \end{aligned} \tag{3.24}$$

Thus, from (3.22), (3.23) and (3.24), we deduce that $|\gamma_t * f(z)| \lesssim \psi^{**}_{0,\,A,\,B}(f)(x)$. Then, by the arbitrariness of $t \in (0,1)$ and $z \in B(x,t)$, we know that

$$\widetilde{\mathcal{G}}_{N,R}(f)(x) \lesssim \psi^{**}_{0,\,A,\,B}(f)(x),$$

which together with (3.19) implies that for any $\lambda \in (0,\infty)$,

$$\int_{\mathbb{R}^n} \Phi(\widetilde{\mathcal{G}}_{N,\,R}(f)(x)/\lambda)\omega(x)\,dx \lesssim \int_{\mathbb{R}^n} \Phi(\psi_0^+(f)(x)/\lambda)\omega(x)\,dx.$$

From this, we infer that (3.11) holds, which completes the proof of Theorem 3.12. ∎

REMARK 3.13. Let $p \in (0,1]$. We point out that Theorem 3.12 when $R \equiv 1$ and $\Phi(t) \equiv t^p$ for all $t \in (0,\infty)$ was obtained by Rychkov [43, Theorem 2.24].

As a corollary of Theorem 3.1, we immediately deduce that the local vertical and the local nontangential maximal function characterizations of $h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)$ with $N \ge N_{\Phi,\,\omega}$ as follows. Here and in what follows,

$$N_{\Phi,\,\omega} \equiv \max\{\widetilde{N}_{\Phi,\,\omega},\, N_0\}, \tag{3.25}$$

where $\widetilde{N}_{\Phi,\,\omega}$ and $N_0$ are respectively as in Definition 3.3 and Theorem 3.12.

THEOREM 3.14. *Let $\Phi$ satisfy Assumption* (A), $\omega \in A^{\mathrm{loc}}_{\infty}(\mathbb{R}^n)$, $\psi_0$ *and* $N_{\Phi,\,\omega}$ *be respectively as in* (3.3) *and* (3.25). *Then for any integer $N \ge N_{\Phi,\,\omega}$, the following are equivalent:*

(i) $f \in h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)$;
(ii) $f \in \mathcal{D}'(\mathbb{R}^n)$ *and* $\psi_0^+(f) \in L^{\Phi}_{\omega}(\mathbb{R}^n)$;
(iii) $f \in \mathcal{D}'(\mathbb{R}^n)$ *and* $(\psi_0)^*_{\triangledown}(f) \in L^{\Phi}_{\omega}(\mathbb{R}^n)$;
(iv) $f \in \mathcal{D}'(\mathbb{R}^n)$ *and* $\widetilde{\mathcal{G}}_N(f) \in L^{\Phi}_{\omega}(\mathbb{R}^n)$;
(v) $f \in \mathcal{D}'(\mathbb{R}^n)$ *and* $\widetilde{\mathcal{G}}^0_N(f) \in L^{\Phi}_{\omega}(\mathbb{R}^n)$;
(vi) $f \in \mathcal{D}'(\mathbb{R}^n)$ *and* $\mathcal{G}^0_N(f) \in L^{\Phi}_{\omega}(\mathbb{R}^n)$.

*Moreover, for all $f \in h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)$,*

$$\begin{aligned}\|f\|_{h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)} &\sim \|\psi_0^+(f)\|_{L^{\Phi}_{\omega}(\mathbb{R}^n)} \sim \|(\psi_0)^*_{\triangledown}(f)\|_{L^{\Phi}_{\omega}(\mathbb{R}^n)} \\ &\sim \|\widetilde{\mathcal{G}}_N(f)\|_{L^{\Phi}_{\omega}(\mathbb{R}^n)} \sim \|\widetilde{\mathcal{G}}^0_N(f)\|_{L^{\Phi}_{\omega}(\mathbb{R}^n)} \sim \|\mathcal{G}^0_N(f)\|_{L^{\Phi}_{\omega}(\mathbb{R}^n)}, \end{aligned} \tag{3.26}$$

*where the implicit constants are independent of $f$.*

*Proof.* (i)⇒(ii). Pick an integer $N \ge N_{\Phi,\,\omega}$ and $f \in h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)$. Let $\widetilde{\psi}_0$ satisfy (3.3) and $\widetilde{\psi}_0 \in \mathcal{D}_N(\mathbb{R}^n)$. Then from the definition of $\mathcal{G}_N(f)$, we infer that $\widetilde{\psi}_0^+(f) \le \mathcal{G}_N(f)$ and hence $\widetilde{\psi}_0^+(f) \in L^\Phi_\omega(\mathbb{R}^n)$. For any $\psi_0$ satisfying (3.3), assume that $\mathrm{supp}(\psi_0) \subset B(0,R)$. Then, by (3.11) and the above argument, we have

$$\|\widetilde{\mathcal{G}}_{N,\,R}(f)\|_{L^\Phi_\omega(\mathbb{R}^n)} \lesssim \|\widetilde{\psi}_0^+(f)\|_{L^\Phi_\omega(\mathbb{R}^n)} \lesssim \|f\|_{h^\Phi_{\omega,\,N}(\mathbb{R}^n)},$$

which together with $\psi_0^+(f) \lesssim \widetilde{\mathcal{G}}_{N,\,R}(f)$ implies that $\psi_0^+(f) \in L^\Phi_\omega(\mathbb{R}^n)$ and

$$\|\psi_0^+(f)\|_{L^\Phi_\omega(\mathbb{R}^n)} \lesssim \|f\|_{h^\Phi_{\omega,\,N}(\mathbb{R}^n)}.$$

(ii)⇒(iii). Let $f \in \mathcal{D}'(\mathbb{R}^n)$ satisfy $\psi_0^+(f) \in L^\Phi_\omega(\mathbb{R}^n)$, where $\psi_0$ is as in (3.3). Then from the fact that

$$\psi_0^+(f) \le (\psi_0)^*_\triangledown(f) \lesssim \psi^{**}_{0,\,A,\,B}(f)$$

and (3.10), we deduce that $(\psi_0)^*_\triangledown(f) \in L^\Phi_\omega(\mathbb{R}^n)$ and

$$\|(\psi_0)^*_\triangledown(f)\|_{L^\Phi_\omega(\mathbb{R}^n)} \lesssim \|\psi_0^+(f)\|_{L^\Phi_\omega(\mathbb{R}^n)}.$$

(iii)⇒(iv). Let $f \in \mathcal{D}'(\mathbb{R}^n)$ satisfy $(\psi_0)^*_\triangledown(f) \in L^\Phi_\omega(\mathbb{R}^n)$, where $\psi_0$ is as in (3.3). By (3.11),

$$\|\widetilde{\mathcal{G}}_N(f)\|_{L^\Phi_\omega(\mathbb{R}^n)} \lesssim \|\psi_0^+(f)\|_{L^\Phi_\omega(\mathbb{R}^n)},$$

which together with the fact that

$$\psi_0^+(f) \le (\psi_0)^*_\triangledown(f)$$

and the assumption that $(\psi_0)^*_\triangledown(f) \in L^\Phi_\omega(\mathbb{R}^n)$ implies $\widetilde{\mathcal{G}}_N(f) \in L^\Phi_\omega(\mathbb{R}^n)$ and

$$\|\widetilde{\mathcal{G}}_N(f)\|_{L^\Phi_\omega(\mathbb{R}^n)} \lesssim \|(\psi_0)^*_\triangledown(f)\|_{L^\Phi_\omega(\mathbb{R}^n)}.$$

(iv)⇒(v)⇒(vi). Since $\mathcal{G}^0_N(f) \le \widetilde{\mathcal{G}}^0_N(f) \le \widetilde{\mathcal{G}}_N(f)$ for any $f \in \mathcal{D}'(\mathbb{R}^n)$ and $\Phi$ is increasing, we see that all the conclusions hold. Moreover, it is obvious that

$$\|\mathcal{G}^0_N(f)\|_{L^\Phi_\omega(\mathbb{R}^n)} \le \|\widetilde{\mathcal{G}}^0_N(f)\|_{L^\Phi_\omega(\mathbb{R}^n)} \le \|\widetilde{\mathcal{G}}_N(f)\|_{L^\Phi_\omega(\mathbb{R}^n)}.$$

(vi)⇒(i). Let $f \in \mathcal{D}'(\mathbb{R}^n)$ satisfy $\mathcal{G}^0_N(f) \in L^\Phi_\omega(\mathbb{R}^n)$. Let $\psi_1$ satisfy (3.3) and $\psi_1 \in \mathcal{D}^0_N(\mathbb{R}^n)$. Then by (3.10), we have

$$\|\widetilde{\mathcal{G}}_N(f)\|_{L^\Phi_\omega(\mathbb{R}^n)} \lesssim \|\psi_1^+(f)\|_{L^\Phi_\omega(\mathbb{R}^n)},$$

which together with the facts that $\psi_1^+(f) \le \mathcal{G}^0_N(f)$ and $\mathcal{G}_N(f) \le \widetilde{\mathcal{G}}_N(f)$ implies that

$$\|\mathcal{G}_N(f)\|_{L^\Phi_\omega(\mathbb{R}^n)} \lesssim \|\mathcal{G}^0_N(f)\|_{L^\Phi_\omega(\mathbb{R}^n)}.$$

Thus, by the definition of $h^\Phi_{\omega,\,N}(\mathbb{R}^n)$, we know that $f \in h^\Phi_{\omega,\,N}(\mathbb{R}^n)$ and

$$\|f\|_{h^\Phi_{\omega,\,N}(\mathbb{R}^n)} \lesssim \|\mathcal{G}^0_N(f)\|_{L^\Phi_\omega(\mathbb{R}^n)},$$

which completes the proof of Theorem 3.14. ∎

As a corollary of Theorems 3.12 and 3.14, we have the following local tangential maximal function characterization of $h^\Phi_{\omega,\,N}(\mathbb{R}^n)$. We omit the details.

Corollary 3.15. *Let $\Phi$ satisfy Assumption* (A), *$\psi_0$ be as in* (3.3), *$\omega \in A^{\mathrm{loc}}_\infty(\mathbb{R}^n)$, $N_{\Phi,\,\omega}$ be as in* (3.25), *$A$ and $B$ be as in Theorem* 3.12. *Then for any integer $N \ge N_{\Phi,\,\omega}$,*

$$f \in h^\Phi_{\omega,\,N}(\mathbb{R}^n)$$

*if and only if $f \in \mathcal{D}'(\mathbb{R}^n)$ and $\psi^{**}_{0,\,A,\,B}(f) \in L^\Phi_\omega(\mathbb{R}^n)$; moreover,*

$$\|f\|_{h^\Phi_{\omega,\,N}(\mathbb{R}^n)} \sim \|\psi^{**}_{0,\,A,\,B}(f)\|_{L^\Phi_\omega(\mathbb{R}^n)}.$$

Next, we give some basic properties concerning $h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)$ and $h^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)$.

PROPOSITION 3.16. *Let $\Phi$ satisfy Assumption* (A), $\omega\in A^{\rm loc}_{\infty}(\mathbb{R}^n)$ *and* $N_{\Phi,\,\omega}$ *be as in* (3.25). *For any integer $N\ge N_{\Phi,\,\omega}$, the inclusion $h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)\hookrightarrow\mathcal{D}'(\mathbb{R}^n)$ is continuous.*

*Proof.* Let $f\in h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)$. For any given $\phi\in\mathcal{D}(\mathbb{R}^n)$, assume that $\mathrm{supp}(\phi)\subset B(0,R)$ with $R\in(0,\infty)$. Then we have

$$|\langle f,\phi\rangle|=|f*\widetilde{\phi}(0)|\le\|\widetilde{\phi}\|_{\mathcal{D}_{N,\,R}(\mathbb{R}^n)}\inf_{x\in B(0,1)}\widetilde{\mathcal{G}}_{N,\,R}(f)(x),\tag{3.27}$$

where $\widetilde{\mathcal{G}}_{N,\,R}(f)$ is as in (3.2) and $\widetilde{\phi}(x)\equiv\phi(-x)$ for all $x\in\mathbb{R}^n$. Now, to prove Proposition 3.16, we consider the following two cases for $\|f\|_{h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)}$.

*Case (i):* $\|f\|_{h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)}\ge1$. In this case, by the upper type 1 property of $\Phi$ and Theorems 3.12 and 3.14, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n}&\Phi(\widetilde{\mathcal{G}}_{N,\,R}(f)(x))\omega(x)\,dx\\&\lesssim\|f\|_{h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)}\int_{\mathbb{R}^n}\Phi\left(\frac{\widetilde{\mathcal{G}}_{N,\,R}(f)(x)}{\|f\|_{h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)}}\right)\omega(x)\,dx\lesssim\|f\|_{h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)}.\end{aligned}\tag{3.28}$$

Notice that the upper type 1 property of $\Phi$ implies that for $t\in(0,1]$,

$$\Phi(1)=\Phi\left(t\frac{1}{t}\right)\lesssim\frac{1}{t}\Phi(t)$$

and hence $\Phi(t)\gtrsim t$. Thus, when $\inf_{x\in B(0,1)}\widetilde{\mathcal{G}}_{N,\,R}(f)(x)\le1$, from (3.27) and (3.28), we deduce that

$$\begin{aligned}|\langle f,\phi\rangle|&\lesssim\|\phi\|_{\mathcal{D}_{N,\,R}(\mathbb{R}^n)}\Phi\Big(\inf_{x\in B(0,1)}\widetilde{\mathcal{G}}_{N,\,R}(f)(x)\Big)\\&\lesssim\|\phi\|_{\mathcal{D}_{N,\,R}(\mathbb{R}^n)}\frac{1}{\omega(B(0,1))}\int_{B(0,1)}\Phi(\widetilde{\mathcal{G}}_{N,\,R}(f)(y))\omega(y)\,dy\\&\lesssim\|\phi\|_{\mathcal{D}_{N,\,R}(\mathbb{R}^n)}\frac{1}{\omega(B(0,1))}\|f\|_{h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)}.\end{aligned}\tag{3.29}$$

Let $p_\Phi$ be as in (2.6). Since $\Phi$ is of lower type $p_\Phi$, for $t\in(1,\infty)$, we have

$$\Phi(1)=\Phi\left(t\frac{1}{t}\right)\lesssim\frac{1}{t^{p_\Phi}}\Phi(t)$$

and hence $t\lesssim[\Phi(t)]^{1/p_\Phi}$. Thus, when $\inf_{x\in B(0,1)}\widetilde{\mathcal{G}}_{N,\,R}(f)(x)>1$, by (3.26) and (3.27), we conclude that

$$\begin{aligned}|\langle f,\phi\rangle|&\lesssim\|\phi\|_{\mathcal{D}_{N,\,R}(\mathbb{R}^n)}\Big\{\Phi\Big(\inf_{x\in B(0,1)}\widetilde{\mathcal{G}}_{N,\,R}(f)(x)\Big)\Big\}^{1/p_\Phi}\\&\lesssim\|\phi\|_{\mathcal{D}_{N,\,R}(\mathbb{R}^n)}[\omega(B(0,1))]^{-1/p_\Phi}\\&\quad\times\left\{\int_{B(0,1)}\Phi(\widetilde{\mathcal{G}}_{N,\,R}(f)(y))\omega(y)\,dy\right\}^{1/p_\Phi}\\&\lesssim\|\phi\|_{\mathcal{D}_{N,\,R}(\mathbb{R}^n)}[\omega(B(0,1))]^{-1/p_\Phi}\|f\|^{1/p_\Phi}_{h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)}.\end{aligned}\tag{3.30}$$

*Case (ii):* $\|f\|_{h^{\Phi}_{\omega,N}(\mathbb{R}^n)} < 1$. In this case, by the lower type $p_\Phi$ property of $\Phi$ and Theorems 3.12 and 3.14, we see that

$$\int_{\mathbb{R}^n} \Phi(\widetilde{\mathcal{G}}_{N,R}(f)(x))\omega(x)\,dx \lesssim \|f\|^{p_\Phi}_{h^{\Phi}_{\omega,N}(\mathbb{R}^n)} \int_{\mathbb{R}^n} \Phi\bigg(\frac{\widetilde{\mathcal{G}}_{N,R}(f)(x)}{\|f\|_{h^{\Phi}_{\omega,N}(\mathbb{R}^n)}}\bigg)\omega(x)\,dx \lesssim \|f\|^{p_\Phi}_{h^{\Phi}_{\omega,N}(\mathbb{R}^n)}.$$

Thus, from this fact and (3.27), similarly to the proof of (3.29) and (3.30), we infer that if $\inf_{x\in B(0,1)} \widetilde{\mathcal{G}}_{N,R}(f)(x) \le 1$, then

$$|\langle f,\phi\rangle| \lesssim \|\phi\|_{\mathcal{D}_{N,R}(\mathbb{R}^n)} \frac{1}{\omega(B(0,1))} \|f\|^{p_\Phi}_{h^{\Phi}_{\omega,N}(\mathbb{R}^n)},$$

and if $\inf_{x\in B(0,1)} \widetilde{\mathcal{G}}_{N,R}(f)(x) > 1$, then

$$|\langle f,\phi\rangle| \lesssim \|\phi\|_{\mathcal{D}_{N,R}(\mathbb{R}^n)} [\omega(B(0,1))]^{-1/p_\Phi} \|f\|_{h^{\Phi}_{\omega,N}(\mathbb{R}^n)}.$$

Thus, $f \in \mathcal{D}'(\mathbb{R}^n)$ and the inclusion is continuous, which completes the proof of Proposition 3.16. ∎

PROPOSITION 3.17. *Let $\Phi$ satisfy Assumption* (A), $\omega \in A^{\mathrm{loc}}_\infty(\mathbb{R}^n)$ *and $N_{\Phi,\omega}$ be as in* (3.25). *For any integer $N \ge N_{\Phi,\omega}$, the space $h^{\Phi}_{\omega,N}(\mathbb{R}^n)$ is complete.*

*Proof.* For any $\psi \in \mathcal{D}_N(\mathbb{R}^n)$ and $\{f_i\}_{i\in\mathbb{N}} \subset \mathcal{D}'(\mathbb{R}^n)$ such that $\{\sum_{i=1}^j f_i\}_{j\in\mathbb{N}}$ converges in $\mathcal{D}'(\mathbb{R}^n)$ to a distribution $f$ as $j\to\infty$, the series $\{\sum_{i=1}^j f_i * \psi\}_{j\in\mathbb{N}}$ converges to $f*\psi$ also pointwise as $j\to\infty$. By Assumption (A), we know that $\Phi$ is strictly increasing and subadditive, which together with the continuity of $\Phi$ implies that for all $x\in\mathbb{R}^n$,

$$\Phi(\mathcal{G}_N(f)(x)) \le \Phi\Big(\sum_{i=1}^{\infty} \mathcal{G}_N(f_i)(x)\Big) \le \sum_{i=1}^{\infty} \Phi(\mathcal{G}_N(f_i)(x)).$$

If $\sum_{i=1}^\infty \|f_i\|^{p_\Phi}_{h^{\Phi}_{\omega,N}(\mathbb{R}^n)} < \infty$ and we let $\lambda_i = \|f_i\|^{p_\Phi}_{h^{\Phi}_{\omega,N}(\mathbb{R}^n)}$, then by the strictly lower type $p_\Phi$ property of $\Phi$ and the Levi lemma, we know that

$$\begin{aligned}
\int_{\mathbb{R}^n} \Phi\bigg(\frac{\mathcal{G}_N(f)(x)}{(\sum_{j=1}^\infty \lambda_j)^{1/p_\Phi}}\bigg)\omega(x)\,dx
&\le \sum_{i=1}^{\infty} \int_{\mathbb{R}^n} \Phi\bigg(\frac{\mathcal{G}_N(f_i)(x)}{(\sum_{j=1}^\infty \lambda_j)^{1/p_\Phi}}\bigg)\omega(x)\,dx \\
&\le \sum_{i=1}^{\infty} \frac{\lambda_i}{\sum_{j=1}^\infty \lambda_j} \int_{\mathbb{R}^n} \Phi\bigg(\frac{\mathcal{G}_N(f_i)(x)}{\lambda_i^{1/p_\Phi}}\bigg)\omega(x)\,dx \le \sum_{i=1}^{\infty} \frac{\lambda_i}{\sum_{j=1}^\infty \lambda_j} = 1,
\end{aligned}$$

which further implies that

$$\|f\|^{p_\Phi}_{h^{\Phi}_{\omega,N}(\mathbb{R}^n)} \le \sum_{i=1}^{\infty} \|f_i\|^{p_\Phi}_{h^{\Phi}_{\omega,N}(\mathbb{R}^n)}. \tag{3.31}$$

To prove that $h^{\Phi}_{\omega,N}(\mathbb{R}^n)$ is complete, it suffices to show that for every sequence $\{f_j\}_{j\in\mathbb{N}}$ with $\|f_j\|_{h^{\Phi}_{\omega,N}(\mathbb{R}^n)} < 2^{-j}$ for any $j\in\mathbb{N}$, the series $\{f_j\}_{j\in\mathbb{N}}$ converges in $h^{\Phi}_{\omega,N}(\mathbb{R}^n)$. Since $\{\sum_{i=1}^j f_i\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $h^{\Phi}_{\omega,N}(\mathbb{R}^n)$, by Proposition 3.16 and the completeness of $\mathcal{D}'(\mathbb{R}^n)$, $\{\sum_{i=1}^j f_i\}_{j\in\mathbb{N}}$ is also a Cauchy sequence in $\mathcal{D}'(\mathbb{R}^n)$ and thus converges to some $f \in \mathcal{D}'(\mathbb{R}^n)$. Therefore, by (3.31),

$$\Big\|f - \sum_{i=1}^{j} f_i\Big\|_{h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)}^{p_\Phi} = \Big\|\sum_{i=j+1}^{\infty} f_i\Big\|_{h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)}^{p_\Phi} \le \sum_{i=j+1}^{\infty} 2^{-ip_\Phi} \to 0$$

as $j \to \infty$, which completes the proof of Proposition 3.17. ■

THEOREM 3.18. *Let $\Phi$ satisfy Assumption* (A), *$\omega \in A^{\mathrm{loc}}_{\infty}(\mathbb{R}^n)$ and $N_{\Phi,\,\omega}$ be as in* (3.25). *If $(\rho,\,q,\,s)_\omega$ is admissible* (*see Definition* 3.4), *then for any integer $N \ge N_{\Phi,\,\omega}$,*

$$h^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n) \subset h^{\Phi}_{\omega,\,N_{\Phi,\,\omega}}(\mathbb{R}^n) \subset h^{\Phi}_{\omega,\,N}(\mathbb{R}^n),$$

*and moreover there exists a positive constant $C$ such that for all $f \in h^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)$,*

$$\|f\|_{h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)} \le \|f\|_{h^{\Phi}_{\omega,\,N_{\Phi,\,\omega}}(\mathbb{R}^n)} \le C\|f\|_{h^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)}.$$

*Proof.* Obviously, by Definition 3.3, we only need to prove that $h^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n) \subset h^{\Phi}_{\omega,\,N_{\Phi,\,\omega}}(\mathbb{R}^n)$, and for all $f \in h^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)$,

$$\|f\|_{h^{\Phi}_{\omega,\,N_{\Phi,\,\omega}}(\mathbb{R}^n)} \lesssim \|f\|_{h^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)}.$$

To this end, by Theorem 3.14 and Definition 3.5, it suffices to prove that for any $(\rho,\,q)_\omega$-single-atom $a$ and $\lambda \in \mathbb{C}$,

$$\int_{\mathbb{R}^n} \Phi(\mathcal{G}^0_{N_{\Phi,\,\omega}}(\lambda a)(x))\omega(x)\,dx \lesssim \omega(\mathbb{R}^n)\Phi\bigg(\frac{|\lambda|}{\omega(\mathbb{R}^n)\rho(\omega(\mathbb{R}^n))}\bigg), \tag{3.32}$$

and for any $(\rho,\,q,\,s)_\omega$-atom $a$ supported in the cube $Q$ and $\lambda \in \mathbb{C}$,

$$\int_{\mathbb{R}^n} \Phi(\mathcal{G}^0_{N_{\Phi,\,\omega}}(\lambda a)(x))\omega(x)\,dx \lesssim \omega(Q)\Phi\bigg(\frac{|\lambda|}{\omega(Q)\rho(\omega(Q))}\bigg). \tag{3.33}$$

Indeed, for any $f \in h^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)$,

$$f = \sum_{i=0}^{\infty} \lambda_i a_i$$

in $\mathcal{D}'(\mathbb{R}^n)$, where $\{\lambda_i\}_{i=0}^{\infty} \subset \mathbb{C}$, $a_0$ is a $(\rho,\,q)_\omega$-single-atom and for any $i \in \mathbb{N}$, $a_i$ is a $(\rho,\,q,\,s)_\omega$-atom supported in the cube $Q_i$. Then, for any $\lambda \in (0,\infty)$, from the facts that $\mathcal{G}^0_{N_{\Phi,\,\omega}}(f/\lambda) = \mathcal{G}^0_{N_{\Phi,\,\omega}}(f)/\lambda$ and $\Phi$ is strictly increasing, subadditive and continuous, and from (3.32) and (3.33), we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Phi\bigg(\frac{\mathcal{G}^0_{N_{\Phi,\,\omega}}(f)(x)}{\lambda}\bigg)\omega(x)\,dx \\
&\quad = \int_{\mathbb{R}^n} \Phi\Big(\mathcal{G}^0_{N_{\Phi,\,\omega}}\Big(\frac{f}{\lambda}\Big)(x)\Big)\omega(x)\,dx \le \sum_{i=0}^{\infty} \int_{\mathbb{R}^n} \Phi\bigg(\mathcal{G}^0_{N_{\Phi,\,\omega}}\Big(\frac{\lambda_i a_i}{\lambda}\Big)(x)\bigg)\omega(x)\,dx \\
&\quad \lesssim \omega(\mathbb{R}^n)\Phi\bigg(\frac{|\lambda_0|}{\lambda\omega(\mathbb{R}^n)\rho(\omega(\mathbb{R}^n))}\bigg) + \sum_{i=1}^{\infty} \omega(Q_i)\Phi\bigg(\frac{|\lambda_i|}{\lambda\omega(Q_i)\rho(\omega(Q_i))}\bigg),
\end{aligned}$$

which together with Theorem 3.14 implies that $\|f\|_{h^{\Phi}_{\omega,\,N_{\Phi,\,\omega}}(\mathbb{R}^n)} \lesssim \|f\|_{h^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)}$.

We now prove (3.32). Since $q \in (q_\omega, \infty]$, by the definition of $q_\omega$, we have $\omega \in A^{\mathrm{loc}}_{q}(\mathbb{R}^n)$. Let $a$ be a $(\rho,\,q)_\omega$-single-atom and $\lambda \in \mathbb{C}$. When $\omega(\mathbb{R}^n) = \infty$, by the definition of the single atom, we know that $a = 0$ for almost every $x \in \mathbb{R}^n$. In this case, it is easy to see that (3.32) holds. When $\omega(\mathbb{R}^n) < \infty$, since $\Phi$ is concave, from Jensen's inequality, Hölder's inequality and Proposition 3.2(ii), we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Phi(\mathcal{G}^0_{N_\Phi,\,\omega}(\lambda a)(x))\omega(x)\,dx \\
&\qquad \le \omega(\mathbb{R}^n)\Phi\bigg(\frac{1}{\omega(\mathbb{R}^n)}\int_{\mathbb{R}^n} \mathcal{G}^0_{N_\Phi,\,\omega}(\lambda a)(x)\omega(x)\,dx\bigg) \\
&\qquad \le \omega(\mathbb{R}^n)\Phi\bigg(\frac{1}{[\omega(\mathbb{R}^n)]^{1/q}}\bigg\{\int_{\mathbb{R}^n} [\mathcal{G}^0_{N_\Phi,\,\omega}(\lambda a)(x)]^q\omega(x)\,dx\bigg\}^{1/q}\bigg) \\
&\qquad \lesssim \omega(\mathbb{R}^n)\Phi\bigg(\frac{1}{[\omega(\mathbb{R}^n)]^{1/q}}|\lambda|\,\|a\|_{L^q_\omega(\mathbb{R}^n)}\bigg) \lesssim \omega(\mathbb{R}^n)\Phi\bigg(\frac{|\lambda|}{\omega(\mathbb{R}^n)\rho(\omega(\mathbb{R}^n))}\bigg).
\end{aligned}$$

That is, (3.32) holds.

Next, we prove (3.33). Let $a$ be a $(\rho,\,q,\,s)_\omega$-atom supported in the cube $Q \equiv Q(x_0, r)$, and $\lambda \in \mathbb{C}$. We consider the following two cases for $Q$.

*Case 1:* $|Q| < 1$. In this case, letting $\widetilde{Q} \equiv 2\sqrt{n}Q$, we have

$$\int_{\mathbb{R}^n} \Phi(\mathcal{G}^0_{N_\Phi,\,\omega}(\lambda a)(x))\omega(x)\,dx = \int_{\widetilde{Q}} \Phi(\mathcal{G}^0_{N_\Phi,\,\omega}(\lambda a)(x))\omega(x)\,dx + \int_{\widetilde{Q}^\complement} \cdots \equiv \mathrm{I}_1 + \mathrm{I}_2. \tag{3.34}$$

For $\mathrm{I}_1$, by Jensen's inequality, Hölder's inequality, Lemma 2.3(v) and Proposition 3.2(ii), we have

$$\begin{aligned}
\mathrm{I}_1 &\le \omega(\widetilde{Q})\Phi\bigg(\frac{1}{\omega(\widetilde{Q})}\int_{\widetilde{Q}} \mathcal{G}^0_{N_\Phi,\,\omega}(\lambda a)(x)\omega(x)\,dx\bigg) \\
&\le \omega(\widetilde{Q})\Phi\bigg(\frac{1}{[\omega(\widetilde{Q})]^{1/q}}\bigg\{\int_{\widetilde{Q}} [\mathcal{G}^0_{N_\Phi,\,\omega}(\lambda a)(x)]^q\omega(x)\,dx\bigg\}^{1/q}\bigg) \\
&\lesssim \omega(\widetilde{Q})\Phi\bigg(\frac{1}{[\omega(\widetilde{Q})]^{1/q}}|\lambda|\,\|a\|_{L^q_\omega(\mathbb{R}^n)}\bigg) \\
&\lesssim \omega(\widetilde{Q})\Phi\bigg(\frac{|\lambda|}{\omega(Q)\rho(\omega(Q))}\bigg) \lesssim \omega(Q)\Phi\bigg(\frac{|\lambda|}{\omega(Q)\rho(\omega(Q))}\bigg),
\end{aligned}\tag{3.35}$$

which is the desired estimate for $\mathrm{I}_1$.

To estimate $\mathrm{I}_2$, we claim that for all $x \in \widetilde{Q}^\complement$,

$$\begin{aligned}
\mathcal{G}^0_{N_\Phi,\,\omega}(\lambda a)(x) \lesssim |\lambda|\,|Q|^{(s_0+n+1)/n}&[\omega(Q)\rho(\omega(Q))]^{-1} \\
&\times |x - x_0|^{-(s_0+n+1)}\chi_{B(x_0,2\sqrt{n})}(x),
\end{aligned}\tag{3.36}$$

where $s_0 \equiv \lfloor n(q_\omega/p_\Phi - 1)\rfloor$. Indeed, for any $\psi \in \mathcal{D}^0_N(\mathbb{R}^n)$ and $t \in (0,1)$, let $P$ be the Taylor expansion of $\psi$ about $(x - x_0)/t$ with degree $s_0$. By Taylor's remainder theorem, for any $y \in \mathbb{R}^n$, we have

$$\begin{aligned}
&\bigg|\psi\Big(\frac{x-y}{t}\Big) - P\Big(\frac{x-y}{t}\Big)\bigg| \\
&\qquad\qquad \lesssim \sum_{\substack{\alpha\in\mathbb{Z}^n_+\\ |\alpha|=s_0+1}} \bigg|(\partial^\alpha\psi)\Big(\frac{\theta(x-y)+(1-\theta)(x-x_0)}{t}\Big)\bigg|\,\bigg|\frac{x_0-y}{t}\bigg|^{s_0+1},
\end{aligned}$$

where $\theta \in (0,1)$. By $t \in (0,1)$ and $x \in \widetilde{Q}^\complement$, we see that $\mathrm{supp}(a * \psi_t) \subset B(x_0, 2\sqrt{n})$ and that $a * \psi_t(x) \neq 0$ implies that $t > |x - x_0|/2$. Thus, from the above facts, Definition 3.4 and (2.1), it follows that for all $x \in \widetilde{Q}^{\complement}$,

$$\begin{aligned}
|a * \psi_t(x)| &\le \frac{1}{t^n}\left\{\int_Q |a(y)|\left|\psi\left(\frac{x-y}{t}\right) - P\left(\frac{x-y}{t}\right)\right| dy\right\}\chi_{B(x_0,2\sqrt{n})}(x)\\
&\lesssim |x-x_0|^{-(s_0+n+1)}\left\{\int_Q |a(y)|\,|x_0-y|^{s_0+1}\,dy\right\}\chi_{B(x_0,2\sqrt{n})}(x)\\
&\lesssim |Q|^{(s_0+1)/n}\|a\|_{L^q_\omega(\mathbb{R}^n)}\left(\int_Q [\omega(y)]^{-q'/q}\,dy\right)^{1/q'}|x-x_0|^{-(s_0+n+1)}\chi_{B(x_0,2\sqrt{n})}(x)\\
&\lesssim |Q|^{(s_0+n+1)/n}[\omega(Q)\rho(\omega(Q))]^{-1}|x-x_0|^{-(s_0+n+1)}\chi_{B(x_0,2\sqrt{n})}(x),
\end{aligned}$$

which together with the arbitrariness of $\psi \in \mathcal{D}^0_N(\mathbb{R}^n)$ implies (3.36). Thus, the claim holds.

Let $Q_k \equiv 2^k\sqrt{n}Q$ for all $k \in \mathbb{N}$ and $k_0 \in \mathbb{N}$ satisfy $2^{k_0}r \le 4 < 2^{k_0+1}r$. As

$$s_0 = \lfloor n(q_\omega/p_\Phi - 1)\rfloor,$$

we know that there exists $q_0 \in (q_\omega, \infty)$ such that $p_\Phi(s_0+n+1) > nq_0$. From Lemma 2.3, it follows that there exists an $\overline{\omega} \in A_{p_0}(\mathbb{R}^n)$ such that $\omega = \overline{\omega}$ on $Q(x_0, 8\sqrt{n})$. From this fact, (3.36), the lower type $p_\Phi$ property of $\Phi$ and Lemma 2.3(viii), we conclude that

$$\begin{aligned}
\mathrm{I}_2 &\le \int_{\sqrt{n}r\le|x-x_0|<2\sqrt{n}} \Phi(\mathcal{G}^0_{N_{\Phi,\,\omega}}(\lambda a)(x))\omega(x)\,dx\\
&\lesssim \int_{\sqrt{n}r\le|x-x_0|<2\sqrt{n}} \Phi\big(|\lambda|\,|Q|^{(s_0+n+1)/n}[\omega(Q)\rho(\omega(Q))]^{-1}|x-x_0|^{-(s_0+n+1)}\big)\overline{\omega}(x)\,dx\\
&\lesssim \sum_{k=1}^{k_0}\int_{Q_{k+1}\setminus Q_k} \Phi\big(|\lambda|2^{-k(s_0+n+1)}[\omega(Q)\rho(\omega(Q))]^{-1}\big)\overline{\omega}(x)\,dx\\
&\lesssim \sum_{k=1}^{k_0} 2^{-k(s_0+n+1)p_\Phi}\overline{\omega}(Q_{k+1})\Phi\left(\frac{|\lambda|}{\omega(Q)\rho(\omega(Q))}\right)\\
&\lesssim \sum_{k=1}^{k_0} 2^{-k[(s_0+n+1)p_\Phi - nq_0]}\omega(Q)\Phi\left(\frac{|\lambda|}{\omega(Q)\rho(\omega(Q))}\right)\\
&\lesssim \omega(Q)\Phi\left(\frac{|\lambda|}{\omega(Q)\rho(\omega(Q))}\right),
\end{aligned}$$

which together with (3.34) and (3.35) implies (3.33) in Case 1.

*Case 2:* $|Q| \ge 1$. In this case, let $Q^* \equiv Q(x_0, r+2)$. Thus, from

$$\mathrm{supp}(\mathcal{G}^0_{N_{\Phi,\,\omega}}(\lambda a)) \subset Q^*,$$

Jensen's inequality, Hölder's inequality, Lemma 2.3(v), and Proposition 3.2(ii), we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Phi(\mathcal{G}^0_{N_{\Phi,\,\omega}}(\lambda a)(x))\omega(x)\,dx\\
&= \int_{Q^*} \Phi(\mathcal{G}^0_{N_{\Phi,\,\omega}}(\lambda a)(x))\omega(x)\,dx \le \omega(Q^*)\Phi\left(\frac{1}{\omega(Q^*)}\int_{Q^*} \mathcal{G}^0_{N_{\Phi,\,\omega}}(\lambda a)(x)\omega(x)\,dx\right)
\end{aligned}$$

$$\leq \omega(Q^*)\Phi\bigg(\frac{1}{[\omega(Q^*)]^{1/q}}\bigg\{\int_{Q^*}[\mathcal{G}^0_{N_{\Phi,\,\omega}}(\lambda a)(x)]^q\omega(x)\,dx\bigg\}^{1/q}\bigg)$$
$$\lesssim \omega(Q^*)\Phi\bigg(\frac{|\lambda|}{[\omega(Q^*)]^{1/q}}\|a\|_{L^q_\omega(\mathbb{R}^n)}\bigg) \lesssim \omega(Q^*)\Phi\bigg(\frac{|\lambda|}{\omega(Q)\rho(\omega(Q))}\bigg)$$
$$\lesssim \omega(Q)\Phi\bigg(\frac{|\lambda|}{\omega(Q)\rho(\omega(Q))}\bigg),$$

which proves (3.33) in Case 2. This finishes the proof of Theorem 3.18. ■

## 4. Calderón–Zygmund decompositions

In this section, we establish some subtle estimates for the Calderón–Zygmund decomposition associated with local grand maximal functions on the weighted Euclidean space $\mathbb{R}^n$ given in [49]. Notice that the construction of the Calderón–Zygmund decomposition in [49] is similar to those in [46, 3, 5].

Let $\Phi$ be a positive function on $\mathbb{R}_+$ satisfying Assumption (A), $\omega \in A^{\mathrm{loc}}_\infty(\mathbb{R}^n)$ and $q_\omega$ be as in (2.4). For an integer $N \geq 2$, let $\mathcal{G}_N(f)$ and $\mathcal{G}^0_N(f)$ be as in (3.2).

Throughout this section, let $f \in \mathcal{D}'(\mathbb{R}^n)$ be such that for all $\lambda \in (0,\infty)$,

$$\omega(\{x \in \mathbb{R}^n : \mathcal{G}_N(f)(x) > \lambda\}) < \infty.$$

For a given $\lambda > \inf_{x\in\mathbb{R}^n} \mathcal{G}_N(f)(x)$, we set

$$\Omega_\lambda \equiv \{x \in \mathbb{R}^n : \mathcal{G}_N(f)(x) > \lambda\}. \tag{4.1}$$

It is obvious that $\Omega_\lambda$ is a proper open subset of $\mathbb{R}^n$. First, we recall the usual Whitney decomposition of $\Omega_\lambda$ given in [49] (see also [46, 3, 5]). We can find closed cubes $\{Q_i\}_i$ such that

$$\Omega_\lambda = \bigcup_i Q_i, \tag{4.2}$$

their interiors are away from $\Omega_\lambda^\complement$ and

$$\mathrm{diam}(Q_i) \leq 2^{-(6+n)}\,\mathrm{dist}(Q_i, \Omega_\lambda^\complement) \leq 4\,\mathrm{diam}(Q_i).$$

In what follows, fix $a \equiv 1 + 2^{-(11+n)}$ and denote $aQ_i$ by $Q_i^*$ for all $i$. Then we have $Q_i \subset Q_i^*$. Moveover, $\Omega_\lambda = \bigcup_i Q_i^*$, and $\{Q_i^*\}_i$ have the bounded interior property, namely, every point in $\Omega_\lambda$ is contained in at most a fixed number of $\{Q_i^*\}_i$.

Now we take a function $\xi \in \mathcal{D}(\mathbb{R}^n)$ such that $0 \leq \xi \leq 1$, $\mathrm{supp}(\xi) \subset aQ(0,1)$ and $\xi \equiv 1$ on $Q(0,1)$. For $x \in \mathbb{R}^n$, set $\xi_i(x) \equiv \xi((x - x_k)/l_i)$. Here and in what follows, $x_i$ is the center of the cube $Q_i$ and $l_i$ its sidelength. Obviously, by the construction of $\{Q_i^*\}_i$ and $\{\xi_i\}_i$, for any $x \in \mathbb{R}^n$, we have $1 \leq \sum_i \xi_i(x) \leq L$, where $L$ is a fixed positive integer independent of $x$. Let

$$\zeta_i \equiv \frac{\xi_i}{\sum_j \xi_j}. \tag{4.3}$$

Then $\{\zeta_i\}_i$ form a smooth partition of unity for $\Omega_\lambda$ subordinate to the locally finite cover $\{Q_i^*\}_i$ of $\Omega_\lambda$, namely, $\chi_{\Omega_\lambda} = \sum_i \zeta_i$ with each $\zeta_i \in \mathcal{D}(\mathbb{R}^n)$ supported in $Q_i^*$.

Let $s \in \mathbb{Z}_+$ be some fixed integer and $\mathcal{P}_s(\mathbb{R}^n)$ denote the *linear space of polynomials in $n$ variables of degrees no more than $s$*. For each $i \in \mathbb{N}$ and $P \in \mathcal{P}_s(\mathbb{R}^n)$, set

$$\|P\|_i \equiv \left[\frac{1}{\int_{\mathbb{R}^n} \zeta_i(y)\,dy} \int_{\mathbb{R}^n} |P(x)|^2 \zeta_i(x)\,dx\right]^{1/2}. \tag{4.4}$$

Then it is easy to see that $(\mathcal{P}_s(\mathbb{R}^n), \|\cdot\|_i)$ is a finite-dimensional Hilbert space. Let $f \in \mathcal{D}'(\mathbb{R}^n)$. Since $f$ induces a linear functional on $\mathcal{P}_s(\mathbb{R}^n)$ via

$$P \mapsto \frac{1}{\int_{\mathbb{R}^n} \zeta_i(y)\,dy} \langle f, P\zeta_i\rangle,$$

by the Riesz representation theorem, there exists a unique polynomial

$$P_i \in \mathcal{P}_s(\mathbb{R}^n) \tag{4.5}$$

for each $i$ such that $\langle f, P\zeta_i\rangle = \langle P_i, P\zeta_i\rangle$ for all $P \in \mathcal{P}_s(\mathbb{R}^n)$. For each $i$, define the distribution

$$b_i \equiv (f - P_i)\zeta_i \quad \text{when } l_i \in (0,1), \qquad b_i \equiv f\zeta_i \quad \text{when } l_i \in [1,\infty). \tag{4.6}$$

We show that for suitable choices of $s$ and $N$, the series $\sum_i b_i$ converges in $\mathcal{D}'(\mathbb{R}^n)$, and in this case, we let $g \equiv f - \sum_i b_i$ in $\mathcal{D}'(\mathbb{R}^n)$. We point out that the representation

$$f = g + \sum_i b_i, \tag{4.7}$$

where $g$ and $b_i$ are as above, is called a *Calderón–Zygmund decomposition* of $f$ of degree $s$ and height $\lambda$ associated with $\mathcal{G}_N(f)$.

The rest of this section consists of a series of lemmas. Lemma 4.1 gives a property of the smooth partition of unity $\{\zeta_i\}_i$, Lemmas 4.2 through 4.5 are devoted to some estimates for the bad parts $\{b_i\}_i$, and Lemmas 4.6 and 4.7 give some controls over the good part $g$. Finally, Corollary 4.8 shows the density of $L^q_\omega(\mathbb{R}^n) \cap h^\Phi_{\omega,N}(\mathbb{R}^n)$ in $h^\Phi_{\omega,N}(\mathbb{R}^n)$, where $q \in (q_\omega, \infty)$. Lemmas 4.1 through 4.3, and Lemmas 4.5 and 4.6, are respectively Lemmas 4.2 through 4.5, and Lemmas 4.7 and 4.8 in [49].

LEMMA 4.1. *There exists a positive constant $C_1$ such that for all $f \in \mathcal{D}'(\mathbb{R}^n)$, all*

$$\lambda > \inf_{x\in\mathbb{R}^n} \mathcal{G}_N(f)(x)$$

*and all $l_i \in (0,1)$, we have*

$$\sup_{y\in\mathbb{R}^n} |P_i(y)\zeta_i(y)| \le C_1\lambda.$$

LEMMA 4.2. *There exists a positive constant $C_2$ such that for all $i \in \mathbb{N}$ and $x \in Q_i^*$,*

$$\mathcal{G}^0_N(b_i)(x) \le C_2 \mathcal{G}_N(f)(x). \tag{4.8}$$

LEMMA 4.3. *Assume that integers $s$ and $N$ satisfy $0 \le s < N$ and $N \ge 2$. Then there exist positive constants $C$, $C_3$ and $C_4$ such that for all $i \in \mathbb{N}$ and $x \in (Q_i^*)^\complement$,*

$$\mathcal{G}^0_N(b_i)(x) \le C \frac{\lambda l_i^{n+s+1}}{(l_i + |x - x_i|)^{n+s+1}} \chi_{B(x_i, C_3)}(x), \tag{4.9}$$

*where $x_i$ is the center of the cube $Q_i$. Moreover, if $x \in (Q_i^*)^\complement$ and $l_i \in [C_4, \infty)$, then $\mathcal{G}^0_N(b_i)(x) = 0$.*

Lemma 4.4. *Let $\Phi$ satisfy Assumption* (A), $\omega \in A_\infty^{\mathrm{loc}}(\mathbb{R}^n)$, $q_\omega$ *and* $p_\Phi$ *be respectively as in* (2.4) *and* (2.6). *If integers* $s, N$ *satisfy* $s \ge \lfloor n(q_\omega/p_\Phi - 1)\rfloor$, $N > s$ *and* $N \ge N_{\Phi,\,\omega}$, *where* $N_{\Phi,\,\omega}$ *is as in* (3.25), *then there exists a positive constant* $C_5$ *such that for all* $f \in h^\Phi_{\omega,\,N}(\mathbb{R}^n)$, $\lambda > \inf_{x\in\mathbb{R}^n} \mathcal{G}_N(f)(x)$ *and* $i \in \mathbb{N}$,

$$\int_{\mathbb{R}^n} \Phi(\mathcal{G}^0_N(b_i)(x))\omega(x)\,dx \le C_5 \int_{Q_i^*} \Phi(\mathcal{G}_N(f)(x))\omega(x)\,dx. \tag{4.10}$$

*Moreover, the series* $\sum_i b_i$ *converges in* $h^\Phi_{\omega,\,N}(\mathbb{R}^n)$ *and*

$$\int_{\mathbb{R}^n} \Phi\Big(\mathcal{G}^0_N\Big(\sum_i b_i\Big)(x)\Big)\omega(x)\,dx \le C_5 \int_{\Omega_\lambda} \Phi(\mathcal{G}_N(f)(x))\omega(x)\,dx. \tag{4.11}$$

*Proof.* By Lemmas 4.2 and 4.3, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \Phi(\mathcal{G}^0_N(b_i)(x))\omega(x)\,dx &\lesssim \int_{Q_i^*} \Phi(\mathcal{G}_N(f)(x))\omega(x)\,dx \\ &\quad+ \int_{(2C_3Q_i^0)\setminus Q_i^*} \Phi(\mathcal{G}^0_N(b_i)(x))\omega(x)\,dx, \end{aligned}\tag{4.12}$$

where $Q_i^0 \equiv Q(x_i, 1)$. Notice that $s \ge \lfloor n(q_\omega/p_\Phi - 1)\rfloor$ implies $(s+n+1)p_\Phi > nq_\omega$. Thus, we take $q_0 \in (q_\omega, \infty)$ such that $(s+n+1)p_\Phi > nq_0$ and $\omega \in A_{q_0}^{\mathrm{loc}}(\mathbb{R}^n)$. By Lemma 2.3(i), we know that there exists an $\widetilde{\omega} \in A_{q_0}(\mathbb{R}^n)$ such that $\widetilde{\omega} = \omega$ on $2C_3Q_i^0$ and $A_{q_0}(\widetilde{\omega}) \lesssim A_{q_0}^{\mathrm{loc}}(\omega)$. Using Lemma 4.3, the lower $p_\Phi$ property of $\Phi$, Lemma 2.3(viii) and the fact that $\mathcal{G}_N(f) > \lambda$ for all $x \in Q_i^*$, we conclude that

$$\begin{aligned}&\int_{(2C_3Q_i^0)\setminus Q_i^*} \Phi(\mathcal{G}^0_N(b_i)(x))\omega(x)\,dx \\ &\quad\le \sum_{k=1}^{k_0} \int_{2^kQ_i^*\setminus 2^{k-1}Q_i^*} \Phi(\mathcal{G}^0_N(b_i)(x))\widetilde{\omega}(x)\,dx \lesssim \sum_{k=1}^{k_0} \Phi\bigg(\frac{\lambda}{2^{k(n+s+1)}}\bigg) \int_{2^kQ_i^*} \widetilde{\omega}(x)\,dx \\ &\quad\lesssim \sum_{k=1}^{k_0} \Phi(\lambda)\frac{1}{2^{k(n+s+1)p_\Phi}} \int_{2^kQ_i^*} \widetilde{\omega}(x)\,dx \lesssim \sum_{k=1}^{k_0} \Phi(\lambda)2^{-k[(n+s+1)p_\Phi - nq_0]}\widetilde{\omega}(Q_i^*) \\ &\quad\lesssim \int_{Q_i^*} \Phi(\mathcal{G}_N(f)(x))\widetilde{\omega}(x)\,dx \sim \int_{Q_i^*} \Phi(\mathcal{G}_N(f)(x))\omega(x)\,dx, \end{aligned}\tag{4.13}$$

where $k_0 \in \mathbb{N}$ satisfies $2^{k_0-2} \le C_3 < 2^{k_0-1}$. From (4.12) and (4.13), we deduce that (4.10) holds. Then, by (4.10), we see that

$$\begin{aligned}\sum_i \int_{\mathbb{R}^n} \Phi(\mathcal{G}^0_N(b_i)(x))\omega(x)\,dx &\lesssim \sum_i \int_{Q_i^*} \Phi(\mathcal{G}_N(f)(x))\omega(x)\,dx \\ &\lesssim \int_{\Omega_\lambda} \Phi(\mathcal{G}_N(f)(x))\omega(x)\,dx.\end{aligned}$$

Combining the above inequality with the completeness of $h^\Phi_{\omega,\,N}(\mathbb{R}^n)$, we infer that $\sum_i b_i$ converges in $h^\Phi_{\omega,\,N}(\mathbb{R}^n)$. So by Proposition 3.16, the series $\sum_i b_i$ converges in $\mathcal{D}'(\mathbb{R}^n)$ and hence

$$\mathcal{G}^0_N\Big(\sum_i b_i\Big)(x) \le \sum_i \mathcal{G}^0_N(b_i)(x)$$

for all $x \in \mathbb{R}^n$, which gives (4.11). This finishes the proof of Lemma 4.4. ∎

Lemma 4.5. *Let $\omega \in A_\infty^{\rm loc}(\mathbb{R}^n)$ and $q_\omega$ be as in* (2.4)*, $s \in \mathbb{Z}_+$ and integer $N \ge 2$. If $q \in (q_\omega, \infty]$ and $f \in L^q_\omega(\mathbb{R}^n)$, then the series $\sum_i b_i$ converges in $L^q_\omega(\mathbb{R}^n)$ and there exists a positive constant $C_6$, independent of $f$, such that*

$$\Big\|\sum_i |b_i|\Big\|_{L^q_\omega(\mathbb{R}^n)} \le C_6\|f\|_{L^q_\omega(\mathbb{R}^n)}.$$

Lemma 4.6. *Let integers $s$ and $N$ satisfy $0 \le s < N$ and $N \ge 2$, $f \in \mathcal{D}'(\mathbb{R}^n)$ and $\lambda > \inf_{x\in\mathbb{R}^n} \mathcal{G}_N(f)(x)$. If $\sum_i b_i$ converges in $\mathcal{D}'(\mathbb{R}^n)$, then there exists a positive constant $C_7$, independent of $f$ and $\lambda$, such that for all $x \in \mathbb{R}^n$,*

$$\mathcal{G}_N^0(g)(x) \le \mathcal{G}_N^0(f)(x)\chi_{\Omega_\lambda^\complement}(x) + C_7\lambda\sum_i \frac{l_i^{n+s+1}}{(l_i+|x-x_i|)^{n+s+1}}\chi_{B(x_i,C_3)}(x),$$

*where $x_i$ is the center of $Q_i$ and $C_3$ is as in Lemma* 4.3.

Lemma 4.7. *Let $\Phi$ satisfy Assumption* (A)*, $\omega \in A_\infty^{\rm loc}(\mathbb{R}^n)$, $q_\omega$ and $p_\Phi$ be respectively as in* (2.4) *and* (2.6)*, $N \ge N_{\Phi,\,\omega}$, where $N_{\Phi,\,\omega}$ is as in* (3.25)*, and $q \in (q_\omega, \infty)$.*

(i) *If integers $s$ and $N$ satisfy $N > s \ge \lfloor n(q_\omega/p_\Phi - 1)\rfloor$ and $f \in h^\Phi_{\omega,\,N}(\mathbb{R}^n)$, then $\mathcal{G}_N^0(g) \in L^q_\omega(\mathbb{R}^n)$ and there exists a positive constant $C_8$, independent of $f$ and $\lambda$, such that*

$$\int_{\mathbb{R}^n}[\mathcal{G}_N^0(g)(x)]^q\omega(x)\,dx \le C_8\begin{cases}\lambda^{q-1}\int_{\mathbb{R}^n}\Phi(\mathcal{G}_N(f)(x))\omega(x)\,dx, & \lambda\in(0,1),\\ \lambda^{q-p_\Phi}\int_{\mathbb{R}^n}\Phi(\mathcal{G}_N(f)(x))\omega(x)\,dx, & \lambda\in[1,\infty).\end{cases} \tag{4.14}$$

(ii) *If $f \in L^q_\omega(\mathbb{R}^n)$, then $g \in L^\infty_\omega(\mathbb{R}^n)$ and there exists a positive constant $C_9$, independent of $f$ and $\lambda$, such that $\|g\|_{L^\infty_\omega(\mathbb{R}^n)} \le C_9\lambda$.*

*Proof.* We first prove (i). Let $f \in h^\Phi_{\omega,\,N}(\mathbb{R}^n)$. By Lemma 4.4 and Proposition 3.16, $\sum_i b_i$ converges in both $h^\Phi_{\omega,\,N}(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^n)$. By $s \ge \lfloor n(q_\omega/p_\Phi - 1)\rfloor$, we know that there exists $q_0 \in (q_\omega,\infty)$ such that $(s+n+1)p_\Phi > nq_0$ and $\omega \in A_{q_0}^{\rm loc}(\mathbb{R}^n)$. Let

$$\mathrm{J} \equiv \int_{\Omega_\lambda^\complement}[\mathcal{G}_N(f)(x)]^q\omega(x)\,dx.$$

From Lemmas 4.6 and 3.10, we infer that

$$\begin{aligned}\int_{\mathbb{R}^n}[\mathcal{G}_N^0(g)(x)]^q\omega(x)\,dx &\lesssim \lambda^q\int_{\mathbb{R}^n}\left[\sum_i\frac{l_i^{n+s+1}}{(l_i+|x-x_i|)^{n+s+1}}\chi_{B(x_i,C_3)}(x)\right]^q\omega(x)\,dx + \mathrm{J}\\ &\lesssim \lambda^q\int_{\mathbb{R}^n}\Big(\sum_i[M_{2C_3}^{\rm loc}(\chi_{Q_i})(x)]^{(n+s+1)/n}\Big)^q\omega(x)\,dx + \mathrm{J}\\ &\lesssim \lambda^q\int_{\mathbb{R}^n}\Big(\sum_i[\chi_{Q_i}(x)]^{(n+s+1)/n}\Big)^q\omega(x)\,dx + \mathrm{J}\\ &\lesssim \lambda^q\int_{\Omega_\lambda}\omega(x)\,dx + \mathrm{J} \sim \lambda^q\omega(\Omega_\lambda) + \mathrm{J}.\end{aligned}$$

Now, we consider the following two cases for $\lambda$.

*Case 1:* $\lambda \ge 1$. In this case, since $\Phi$ has lower type $p_\Phi$, we have

$$\lambda^q \omega(\Omega_\lambda) \le \lambda^{q-p_\Phi}\omega(\Omega_\lambda)\Big[\inf_{x\in\Omega_\lambda} \mathcal{G}_N(f)(x)\Big]^{p_\Phi} \le \lambda^{q-p_\Phi}\omega(\Omega_\lambda)\Phi\Big(\inf_{x\in\Omega_\lambda} \mathcal{G}_N(f)(x)\Big)$$
$$\le \lambda^{q-p_\Phi}\int_{\Omega_\lambda} \Phi(\mathcal{G}_N(f)(x))\omega(x)\,dx.$$

Recall that

$$\Omega_1 \equiv \{x\in\mathbb{R}^n : \mathcal{G}_N(f)(x) > 1\}.$$

From the fact that $\Phi$ has lower type $p_\Phi$ and upper type 1, it follows that

$$\mathrm{J} = \int_{\Omega_\lambda^{\complement}\cap\Omega_1} [\mathcal{G}_N(f)(x)]^q\omega(x)\,dx + \int_{\Omega_\lambda^{\complement}\cap\Omega_1^{\complement}} \cdots$$
$$\lesssim \int_{\Omega_\lambda^{\complement}} [\mathcal{G}_N(f)(x)]^{q-p_\Phi}\Phi(\mathcal{G}_N(f)(x))\omega(x)\,dx + \int_{\Omega_\lambda^{\complement}} [\mathcal{G}_N(f)(x)]^{q-1}\Phi(\mathcal{G}_N(f)(x))\omega(x)\,dx$$
$$\lesssim (\lambda^{q-p_\Phi} + \lambda^{q-1})\int_{\Omega_\lambda^{\complement}} \Phi(\mathcal{G}_N(f)(x))\omega(x)\,dx \lesssim \lambda^{q-p_\Phi}\int_{\Omega_\lambda^{\complement}} \Phi(\mathcal{G}_N(f)(x))\omega(x)\,dx,$$

which together with the estimate of $\lambda^q\omega(\Omega_\lambda)$ implies (4.10) in Case 1.

*Case 2:* $\lambda \in (0,1)$. In this case, for any $x\in\Omega_\lambda$, if $\mathcal{G}_N(f)(x) \ge 1 > \lambda$, using the fact that $\Phi$ has lower type $p_\Phi$, we conclude that

$$\lambda^q \le \lambda^{q-p_\Phi}[\mathcal{G}_N(f)(x)]^{p_\Phi} \lesssim \lambda^{q-p_\Phi}\Phi(\mathcal{G}_N(f)(x)) \lesssim \lambda^{q-1}\Phi(\mathcal{G}_N(f)(x)).$$

If $\mathcal{G}_N(f)(x) < 1$ and $\mathcal{G}_N(f)(x) > \lambda$, by the fact that $\Phi$ has upper type 1, we see that

$$\lambda^q \le \lambda^{q-1}\mathcal{G}_N(f)(x) \lesssim \lambda^{q-1}\Phi(\mathcal{G}_N(f)(x)).$$

From these estimates, we deduce that

$$\lambda^q\omega(\Omega_\lambda) \lesssim \lambda^{q-1}\int_{\Omega_\lambda} \Phi(\mathcal{G}_N(f)(x))\omega(x)\,dx.$$

For J, since $\lambda\in(0,1)$, $\mathcal{G}_N(f)(x) \le \lambda$ for all $x\in\Omega_\lambda^{\complement}$ and $\Phi$ has upper type 1, we know that

$$\mathrm{J} \le \lambda^{q-1}\int_{\Omega_\lambda^{\complement}} \mathcal{G}_N(f)(x)\omega(x)\,dx \lesssim \lambda^{q-1}\int_{\Omega_\lambda^{\complement}} \Phi(\mathcal{G}_N(f)(x))\omega(x)\,dx,$$

which together with the estimate of $\lambda^q\omega(\Omega_\lambda)$ implies (4.14) in Case 2. Thus, (i) holds.

Now we prove (ii). If $f\in L^q_\omega(\mathbb{R}^n)$, then $g$ and $\{b_i\}_i$ are functions. By Lemma 4.5, we know that $\sum_i b_i$ converges in $L^q_\omega(\mathbb{R}^n)$ and hence in $\mathcal{D}'(\mathbb{R}^n)$ by Lemma 2.6(ii). Write

$$g = f - \sum_i b_i = f\Big(1 - \sum_i \zeta_i\Big) + \sum_{i\in F} P_i\zeta_i = f\chi_{\Omega_\lambda^{\complement}} + \sum_{i\in F} P_i\zeta_i,$$

where $F \equiv \{i\in\mathbb{N} : l_i \in (0,1)\}$. By Lemma 4.1, we have $|g(x)| \lesssim \lambda$ for all $x\in\Omega_\lambda$, which combined with Proposition 3.2(i) yields

$$|g(x)| = |f(x)| \le \mathcal{G}_N(f)(x) \le \lambda$$

for almost every $x\in\Omega_\lambda^{\complement}$, Thus, $\|g\|_{L^\infty_\omega(\mathbb{R}^n)} \lesssim \lambda$. This shows (ii) and hence finishes the proof of Lemma 4.7. ∎

COROLLARY 4.8. *Let $\Phi$ satisfy Assumption* (A), $\omega \in A_{\infty}^{\rm loc}(\mathbb{R}^n)$, $q_\omega$ *be as in* (2.4),

$$q \in (q_\omega, \infty)$$

*and* $N \geq N_{\Phi,\,\omega}$, *where* $N_{\Phi,\,\omega}$ *is as in* (3.25). *Then* $h^{\Phi}_{\omega,\,N}(\mathbb{R}^n) \cap L^q_\omega(\mathbb{R}^n)$ *is dense in* $h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)$.

*Proof.* Let $f \in h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)$. For any $\lambda > \inf_{x\in\mathbb{R}^n} \mathcal{G}_N(f)(x)$, let

$$f = g^\lambda + \sum_i b_i^\lambda$$

be the Calderón–Zygmund decomposition of $f$ of degree $s$ with $\lfloor n(q_\omega/p_\Phi - 1)\rfloor \leq s < N$ and height $\lambda$ associated to $\mathcal{G}_N(f)$. By Lemma 4.4,

$$\int_{\mathbb{R}^n} \Phi\Big(\mathcal{G}^0_N\Big(\sum_i b_i^\lambda\Big)(x)\Big)\omega(x)\,dx \lesssim \int_{\{x\in\mathbb{R}^n:\,\mathcal{G}_N(f)(x)>\lambda\}} \Phi(\mathcal{G}_N(f)(x))\omega(x)\,dx.$$

Hence, $g^\lambda \to f$ in $h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)$ as $\lambda \to \infty$. Moreover, by Lemma 4.7(i), we have $\mathcal{G}^0_N(g^\lambda) \in L^q_\omega(\mathbb{R}^n)$, which together with Proposition 3.2(ii) implies $g^\lambda \in L^q_\omega(\mathbb{R}^n)$. This finishes the proof of Corollary 4.8. ■

## 5. Weighted atomic decompositions of $h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)$

In this section, we establish the equivalence between $h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)$ and $h^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)$ by using the Calderón–Zygmund decomposition associated to the local grand maximal function stated in Section 4.

Let $\Phi$ satisfy Assumption (A), $\omega \in A_{\infty}^{\rm loc}(\mathbb{R}^n)$, $q_\omega$, $p_\Phi$ and $N_{\Phi,\,\omega}$ be respectively as in (2.4), (2.6) and (3.25), $N \geq N_{\Phi,\,\omega}$ an integer and $s_0 \equiv \lfloor n(q_\omega/p_\Phi - 1)\rfloor$. *Throughout this section, let*

$$f \in h^{\Phi}_{\omega,\,N}(\mathbb{R}^n).$$

We take $k_0 \in \mathbb{Z}$ such that $2^{k_0-1} \leq \inf_{x\in\mathbb{R}^n} \mathcal{G}_N(f)(x) < 2^{k_0}$ when

$$\inf_{x\in\mathbb{R}^n} \mathcal{G}_N(f)(x) > 0,$$

and when $\inf_{x\in\mathbb{R}^n} \mathcal{G}_N(f)(x) = 0$, let $k_0 \equiv -\infty$. *Throughout this section, we always assume that* $k \geq k_0$. For each integer $k \geq k_0$, consider the Calderón–Zygmund decomposition of $f$ of degree $s$ and height $\lambda = 2^k$ associated to $\mathcal{G}_N(f)$. Namely, for any $k \geq k_0$, by taking $\lambda \equiv 2^k$ in (4.1), we now write the Calderón–Zygmund decomposition in (4.7) as

$$f = g^k + \sum_i b_i^k; \tag{5.1}$$

here and in what follows in this section, we write $\{Q_i\}_i$ in (4.2), $\{\zeta_i\}_i$ in (4.3), $\{P_i\}_i$ in (4.5) and $\{b_i\}_i$ in (4.6), respectively, as $\{Q_i^k\}_i$, $\{\zeta_i^k\}_i$, $\{P_i^k\}_i$ and $\{b_i^k\}_i$. Now, the center and the sidelength of $Q_i^k$ are respectively denoted by $x_i^k$ and $l_i^k$. Recall that for all $i$ and $k$,

$$\sum_i \zeta_i^k = \chi_{\Omega_{2^k}}, \qquad \mathrm{supp}(b_i^k) \subset \mathrm{supp}(\zeta_i^k) \subset Q_i^{k*}, \tag{5.2}$$

$\{Q_i^{k*}\}_i$ has the bounded interior property, and for all $P \in \mathcal{P}_s(\mathbb{R}^n)$,

$$\langle f, P\zeta_i^k\rangle = \langle P_i^k, P\zeta_i^k\rangle. \tag{5.3}$$

For each integer $k \ge k_0$ and $i, j \in \mathbb{N}$, let $P_{i,j}^{k+1}$ be the orthogonal projection of $(f - P_j^{k+1})\zeta_i^k$ on $\mathcal{P}_s(\mathbb{R}^n)$ with respect to the norm

$$\|P\|_j^2 \equiv \frac{1}{\int_{\mathbb{R}^n} \zeta_j^{k+1}(y)\,dy} \int_{\mathbb{R}^n} |P(x)|^2 \zeta_j^{k+1}(x)\,dx,$$

namely, $P_{i,j}^{k+1}$ is the unique polynomial of $\mathcal{P}_s(\mathbb{R}^n)$ such that for any $P \in \mathcal{P}_s(\mathbb{R}^n)$,

$$\langle (f - P_j^{k+1})\zeta_i^k, P\zeta_j^{k+1}\rangle = \int_{\mathbb{R}^n} P_{i,j}^{k+1}(x)P(x)\zeta_j^{k+1}(x)\,dx. \tag{5.4}$$

Recall that $a \equiv 1 + 2^{-(11+n)}$. In what follows, let $Q_i^{k*} \equiv aQ_i^k$,

$$E_1^k \equiv \{i \in \mathbb{N} :\ |Q_i^k| \ge 1/(2^4 n)\}, \quad F_1^k \equiv \{i \in \mathbb{N} :\ |Q_i^k| \ge 1\},$$
$$E_2^k \equiv \{i \in \mathbb{N} :\ |Q_i^k| < 1/(2^4 n)\}, \quad F_2^k \equiv \{i \in \mathbb{N} :\ |Q_i^k| < 1\}.$$

Observe that

$$P_{i,j}^{k+1} \neq 0 \quad \text{if and only if} \quad Q_i^{k*} \cap Q_j^{(k+1)*} \neq \emptyset. \tag{5.5}$$

Indeed, this follows directly from the definition of $P_{i,j}^{k+1}$. Lemmas 5.1 through 5.3 below are just Lemmas 5.1 through 5.3 in [49].

Lemma 5.1. *Let $\Omega_{2^k}$ be as in (4.1) with $\lambda = 2^k$, and $Q_i^{k*}$ and $l_i^k$ be as above.*

(i) *If $Q_i^{k*} \cap Q_j^{(k+1)*} \neq \emptyset$, then $l_j^{k+1} \le 2^4\sqrt{n}\, l_i^k$ and $Q_j^{(k+1)*} \subset 2^6 n Q_i^{k*} \subset \Omega_{2^k}$.*
(ii) *There exists a positive integer $L$ such that for each $i \in \mathbb{N}$, the cardinality of $\{j \in \mathbb{N} : Q_i^{k*} \cap Q_j^{(k+1)*} \neq \emptyset\}$ is bounded by $L$.*

Lemma 5.2. *There exists a positive constant $C$ such that for all $i, j \in \mathbb{N}$ and integer $k \ge k_0$ with $l_j^{k+1} \in (0,1)$,*

$$\sup_{y \in \mathbb{R}^n} |P_{i,j}^{k+1}(y)\zeta_j^{k+1}(y)| \le C2^{k+1}. \tag{5.6}$$

Lemma 5.3. *For any $k \in \mathbb{Z}$ with $k \ge k_0$,*

$$\sum_{i \in \mathbb{N}} \Big( \sum_{j \in F_2^{k+1}} P_{i,j}^{k+1}\zeta_j^{k+1} \Big) = 0,$$

*where the series converges both in $\mathcal{D}'(\mathbb{R}^n)$ and pointwise.*

The following lemma gives the weighted atomic decomposition for a dense subspace of $h_{\omega,N}^{\Phi}(\mathbb{R}^n)$.

Lemma 5.4. *Let $\Phi$ satisfy Assumption (A), $\omega \in A_\infty^{\mathrm{loc}}(\mathbb{R}^n)$, $q_\omega$, $p_\Phi$ and $N_{\Phi,\omega}$ be respectively as in (2.4), (2.6) and (3.25). If $q \in (q_\omega, \infty)$, $N \ge N_{\Phi,\omega}$ is an integer, $s \ge \lfloor n(q_\omega/p_\Phi - 1)\rfloor$ and $N > s$, then for any $f \in L_\omega^q(\mathbb{R}^n) \cap h_{\omega,N}^{\Phi}(\mathbb{R}^n)$, there exist $\lambda_0 \in \mathbb{C}$, $\{\lambda_i^k\}_{k \ge k_0, i} \subset \mathbb{C}$, a $(\rho,\infty)_\omega$-single-atom $a_0$ and $(\rho,\infty,s)_\omega$-atoms $\{a_i^k\}_{k \ge k_0, i}$ such that*

$$f = \sum_{k \ge k_0} \sum_i \lambda_i^k a_i^k + \lambda_0 a_0, \tag{5.7}$$

*where the series converges both in $\mathcal{D}'(\mathbb{R}^n)$ and almost everywhere. Moreover, there exists a positive constant $C$, independent of $f$, such that*

$$\Lambda(\{\lambda_i^k a_i^k\}_{k \ge k_0, i} \cup \{\lambda_0 a_0\}) \le C\|f\|_{h_{\omega,N}^{\Phi}(\mathbb{R}^n)}. \tag{5.8}$$

*Proof.* Let $f \in (L^q_\omega(\mathbb{R}^n) \cap h^\Phi_{\omega,\,N}(\mathbb{R}^n))$. We first consider the case $k_0 = -\infty$. As above, for each $k \in \mathbb{Z}$, $f$ has a Calderón–Zygmund decomposition of degree $s$ and height $\lambda = 2^k$ associated to $\mathcal{G}_N(f)$ as in (5.1), namely,

$$f = g^k + \sum_i b^k_i.$$

By Corollary 4.8 and Proposition 3.2, $g^k \to f$ in both $h^\Phi_{\omega,\,N}(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^n)$ as $k \to \infty$. By Lemma 4.7(i), $\|g^k\|_{L^q_\omega(\mathbb{R}^n)} \to 0$ as $k \to -\infty$, and furthermore, by Lemma 2.6(ii), $g^k \to 0$ in $\mathcal{D}'(\mathbb{R}^n)$ as $k \to -\infty$. Therefore,

$$f = \sum_{k=-\infty}^{\infty} (g^{k+1} - g^k) \tag{5.9}$$

in $\mathcal{D}'(\mathbb{R}^n)$. Moreover, since $\mathrm{supp}(\sum_i b^k_i) \subset \Omega_{2^k}$ and $\omega(\Omega_{2^k}) \to 0$ as $k \to \infty$, it follows that $g^k \to f$ almost everywhere as $k \to \infty$. Thus, (5.9) also holds almost everywhere. By Lemma 5.3 and (5.2) with $\Omega_{2^{k+1}} \subset \Omega_{2^k}$,

$$\begin{aligned} g^{k+1} - g^k &= \Big(f - \sum_j b^{k+1}_j\Big) - \Big(f - \sum_i b^k_i\Big) \\ &= \sum_i b^k_i - \sum_j b^{k+1}_j + \sum_i \Big( \sum_{j \in F^{k+1}_2} P^{k+1}_{i,\,j} \zeta^{k+1}_j \Big) \\ &= \sum_i \Big[ b^k_i - \sum_j b^{k+1}_j \zeta^k_i + \sum_{j \in F^{k+1}_2} P^{k+1}_{i,\,j} \zeta^{k+1}_j \Big] \equiv \sum_i h^k_i, \end{aligned} \tag{5.10}$$

where all the series converge in both $\mathcal{D}'(\mathbb{R}^n)$ and almost everywhere. Furthermore, from the definitions of $b^k_j$ and $b^{k+1}_j$ as in (4.3), we infer that when $l^k_i \in (0, 1)$,

$$h^k_i = f\chi_{\Omega^\complement_{2^{k+1}}} \zeta^k_i - P^k_i \zeta^k_i + \sum_{j \in F^{k+1}_2} P^{k+1}_j \zeta^k_i \zeta^{k+1}_j + \sum_{j \in F^{k+1}_2} P^{k+1}_{i,\,j} \zeta^{k+1}_j, \tag{5.11}$$

and when $l^k_i \in [1, \infty)$,

$$h^k_i = f\chi_{\Omega^\complement_{2^{k+1}}} \zeta^k_i + \sum_{j \in F^{k+1}_2} P^{k+1}_j \zeta^k_i \zeta^{k+1}_j + \sum_{j \in F^{k+1}_2} P^{k+1}_{i,\,j} \zeta^{k+1}_j. \tag{5.12}$$

By Proposition 3.2(i), we know that for almost every $x \in \Omega^\complement_{2^{k+1}}$,

$$|f(x)| \le \mathcal{G}_N(f)(x) \le 2^{k+1},$$

which, together with Lemma 4.1, Lemma 5.1(ii), (5.5), Lemma 5.2, (5.11) and (5.12), implies that there exists a positive constant $C_{10}$ such that for all $i \in \mathbb{N}$,

$$\|h^k_i\|_{L^\infty_\omega(\mathbb{R}^n)} \le C_{10} 2^k. \tag{5.13}$$

Next, we show that for each $i$ and $k$, $h^k_i$ is a multiple of a $(\rho,\,\infty,\,s)_\omega$-atom by considering the following two cases for $i$.

*Case 1: $i \in E^k_1$.* In this case, from the fact that $l^{k+1}_j < 1$ for $j \in F^{k+1}_2$, we deduce that $Q^{(k+1)*}_j \subset Q(x^k_i, a(l^k_i + 2))$ for $j$ satisfying $Q^{k*}_i \cap Q^{(k+1)*}_j \ne \emptyset$. Let $\gamma \equiv 1 + 2^{-12-n}$. Thus, when $l_i^k \ge 2/(\gamma-1)$, if we let $\widetilde{Q}_i^k \equiv Q(x_i^k, a(l_i^k+2))$, then

$$\mathrm{supp}(h_i^k) \subset \widetilde{Q}_i^k \subset \gamma Q_i^{k*} \subset \Omega_{2^k}.$$

When $l_i^k < 2/(\gamma-1)$, if we let $\widetilde{Q}_i^k \equiv 2^6 n Q_i^{k*}$, then by Lemma 5.1(i), we have

$$\mathrm{supp}(h_i^k) \subset \widetilde{Q}_i^k \subset \Omega_{2^k}.$$

From the definition of $\widetilde{Q}_i^k$, Lemma 2.3(v) and Remark 2.4 with $\widetilde{C} \equiv 2/(\gamma-1)$, we infer that there exists a positive constant $C_{11}$ such that

$$\omega(\widetilde{Q}_i^k) \le C_{11}\omega(Q_i^{k*}). \tag{5.14}$$

Let $\widetilde{A}_1 \equiv \max\{C_{10},\, C_{11}\}$,

$$\lambda_i^k \equiv \widetilde{A}_1 2^k \omega(\widetilde{Q}_i^k)\rho(\omega(\widetilde{Q}_i^k)) \tag{5.15}$$

and $a_i^k \equiv (\lambda_i^k)^{-1}h_i^k$. From (5.13) and $\mathrm{supp}(h_i^k) \subset \widetilde{Q}_i^k$ with $l(\widetilde{Q}_i^k) \ge 2a > 1$, it follows that $a_i^k$ is a $(\rho,\,\infty,\,s)_\omega$-atom.

*Case 2: $i \in E_2^k$.* In this case, if $j \in F_1^{k+1}$, then $l_i^k < l_j^{k+1}/(2^4 n)$. By Lemma 5.1(i), we know that $Q_i^{k*} \cap Q_j^{(k+1)*} = \emptyset$ for $j \in F_1^{k+1}$. From this, (5.2) and (5.10), we conclude that

$$\begin{aligned} h_i^k &= (f-P_i^k)\zeta_i^k - \sum_{j\in F_1^{k+1}} f\zeta_j^{k+1}\zeta_i^k - \sum_{j\in F_2^{k+1}} (f-P_j^{k+1})\zeta_j^{k+1}\zeta_i^k \\ &\quad + \sum_{j\in F_2^{k+1}} P_{i,\,j}^{k+1}\zeta_j^{k+1} \\ &= (f-P_i^k)\zeta_i^k - \sum_{j\in F_2^{k+1}} \{(f-P_j^{k+1})\zeta_j^{k+1}\zeta_i^k - P_{i,\,j}^{k+1}\zeta_j^{k+1}\}. \end{aligned} \tag{5.16}$$

Let $\widetilde{Q}_i^k \equiv 2^6 n Q_i^{k*}$. Then $\mathrm{supp}(h_i^k) \subset \widetilde{Q}_i^k$. From $l_i^k < 1/(2^4 n)$, Lemma 2.3(v) and Remark 2.4 with $\widetilde{C} \equiv 4a$, we know that there exists a positive constant $C_{12}$ such that

$$\omega(\widetilde{Q}_i^k) \le C_{12}\omega(Q_i^{k*}). \tag{5.17}$$

Moveover, $h_i^k$ satisfies the desired moment conditions, which are deduced from the moment conditions of $(f-P_i^k)\zeta_i^k$ (see (5.3)) and $(f-P_j^{k+1})\zeta_j^{k+1}\zeta_i^k - P_{i,\,j}^{k+1}\zeta_j^{k+1}$ (see (5.4)). Let $\widetilde{A}_2 \equiv \max\{C_{10},\, C_{12}\}$,

$$\lambda_i^k \equiv \widetilde{A}_2 2^k \omega(\widetilde{Q}_i^k)\rho(\omega(\widetilde{Q}_i^k)) \tag{5.18}$$

and $a_i^k \equiv (\lambda_i^k)^{-1}h_i^k$. From this, (5.13), $\mathrm{supp}(h_i^k) \subset \widetilde{Q}_i^k$ and the moment conditions of $h_i^k$, we know that $a_i^k$ is a $(\rho,\,\infty,\,s)_\omega$-atom.

Thus, from (5.9), (5.10), and Cases 1 and 2, we infer that

$$f = \sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}} \lambda_i^k a_i^k$$

holds in both $\mathcal{D}'(\mathbb{R}^n)$ and almost everywhere, where for every $k$ and $i$, $\lambda_i^k \in \mathbb{C}$ and $a_i^k$ is a $(\rho,\,\infty,\,s)_\omega$-atom, which shows (5.7) in the case that $k_0 = -\infty$ by letting $\lambda_0 = 0$. Furthermore, from the fact that $\Phi(t) \sim \int_0^t \frac{\Phi(s)}{s}\,ds$ for all $t \in (0,\infty)$, (5.15), (5.18), (5.14), (5.17), the upper type 1 property of $\Phi$, Fubini's theorem and the bounded interior property of $\{Q_i^{k*}\}$, we know that for any $\lambda \in (0,\infty)$,

$$\begin{aligned}
&\sum_{k,\,i} \omega(\widetilde{Q}_i^k)\Phi\bigg(\frac{|\lambda_i^k|}{\lambda\rho(\omega(\widetilde{Q}_i^k))\omega(\widetilde{Q}_i^k)}\bigg)\\
&\quad\lesssim \sum_{k,\,i}\omega(\widetilde{Q}_i^k)\Phi\bigg(\frac{2^k}{\lambda}\bigg) \lesssim \sum_{k,i}\omega(Q_i^{k*})\Phi\bigg(\frac{2^k}{\lambda}\bigg)\\
&\quad\lesssim \sum_k \omega(\Omega_{2^k})\Phi\bigg(\frac{2^k}{\lambda}\bigg) \sim \sum_k \int_{\Omega_{2^k}} \Phi\bigg(\frac{2^k}{\lambda}\bigg)\omega(x)\,dx\\
&\quad\lesssim \int_{\mathbb{R}^n} \sum_{k<\log[\mathcal{G}_N(f)(x)]} \Phi\bigg(\frac{2^k}{\lambda}\bigg)\omega(x)\,dx \lesssim \int_{\mathbb{R}^n} \sum_{k<\log[\mathcal{G}_N(f)(x)]} \int_{2^k}^{2^{k+1}} \Phi\bigg(\frac{t}{\lambda}\bigg)\frac{dt}{t}\omega(x)\,dx\\
&\quad\lesssim \int_{\mathbb{R}^n}\int_0^{2\mathcal{G}_N(f)(x)/\lambda} \Phi(t)\frac{dt}{t}\omega(x)\,dx \lesssim \int_{\mathbb{R}^n} \Phi\bigg(\frac{\mathcal{G}_N(f)(x)}{\lambda}\bigg)\omega(x)\,dx,
\end{aligned}$$

which implies (5.8) in the case $k_0 = -\infty$.

Finally, we consider the case $k_0 > -\infty$. In this case, as $f \in h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)$, we see that $\omega(\mathbb{R}^n) < \infty$. Adapting the previous arguments, we conclude that

$$f = \sum_{k=k_0}^{\infty} (g^{k+1} - g^k) + g^{k_0} \equiv \widetilde{f} + g^{k_0}, \tag{5.19}$$

and for the function $\widetilde{f}$, we have the same $(\rho,\,\infty,\,s)_\omega$-atomic decomposition as above,

$$\widetilde{f} = \sum_{k\ge k_0,\,i} \lambda_i^k a_i^k \tag{5.20}$$

and

$$\Lambda(\{\lambda_i^k a_i^k\}_{k\ge k_0,\,i}) \lesssim \|f\|_{h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)}. \tag{5.21}$$

From Lemma 4.7(ii), it follows that

$$\|g^{k_0}\|_{L^\infty_\omega(\mathbb{R}^n)} \le C_9 2^{k_0} \le 2C_9 \inf_{x\in\mathbb{R}^n} \mathcal{G}_N(f)(x), \tag{5.22}$$

where $C_9$ is as in Lemma 4.7(ii). Let $\lambda_0 \equiv 2C_9 2^{k_0}\omega(\mathbb{R}^n)\rho(\omega(\mathbb{R}^n))$ and

$$a_0 \equiv \lambda_0^{-1} g^{k_0}.$$

Then

$$\|a_0\|_{L^\infty_\omega(\mathbb{R}^n)} \le [\omega(\mathbb{R}^n)\rho(\omega(\mathbb{R}^n))]^{-1}.$$

Thus, $a_0$ is a $(\rho,\,\infty)_\omega$-single-atom and $g^{k_0} = \lambda_0 a_0$, which together with (5.19) and (5.20) implies (5.7) in the case $k_0 > -\infty$. Moreover, from (5.22), we deduce that for any $\lambda \in (0,\infty)$,

$$\omega(\mathbb{R}^n)\Phi\bigg(\frac{|\lambda_0|}{\lambda\omega(\mathbb{R}^n)\rho(\omega(\mathbb{R}^n))}\bigg) = \omega(\mathbb{R}^n)\Phi\bigg(\frac{C_9 2^{k_0}}{\lambda}\bigg) \lesssim \int_{\mathbb{R}^n}\Phi\bigg(\frac{\mathcal{G}_N(f)(x)}{\lambda}\bigg)\omega(x)\,dx,$$

which together with (5.21) implies (5.8) in the case $k_0 > -\infty$. This finishes the proof of Lemma 5.4. ■

REMARK 5.5. By its proof, all $(\rho,\infty,s)_\omega$-atoms in Lemma 5.4 can be taken to have supports $Q$ satisfying $l(Q)\in(0,2]$. Indeed, for any $(\rho,\infty,s)_\omega$-atom $a$ supported in a cube $Q_0$ with $l(Q_0)>2$, there exist $N_0\in\mathbb{N}$, depending on $l(Q_0)$ and $n$, and cubes $\{Q_i\}_{i=1}^{N_0}$ satisfying $l(Q_i)\in[1,2]$ with $i\in\{1,\dots,N_0\}$ such that $\bigcup_{i=1}^{N_0}Q_i=Q_0$, for any $x\in Q_0$, $1\le\sum_{i=1}^{N_0}\chi_{Q_i}(x)\le C(n)$, and

$$a=\frac{1}{\sum_{j=1}^{N_0}\chi_{Q_j}}\sum_{i=1}^{N_0}a\chi_{Q_i},$$

where $C(n)$ is a positive integer, only depending on $n$. For any given $\lambda_0\in\mathbb{C}$ and $i\in\{1,\dots,N_0\}$, let

$$\gamma_i\equiv\frac{\lambda_0\omega(Q_i)\rho(\omega(Q_i))}{\omega(Q_0)\rho(\omega(Q_0))},\qquad b_i\equiv\frac{\omega(Q_0)\rho(\omega(Q_0))a\chi_{Q_i}}{\omega(Q_i)\rho(\omega(Q_i))\sum_{i=1}^{N_0}\chi_{Q_i}}.$$

Then for any $i\in\{1,\dots,N_0\}$, $b_i$ is a $(\rho,\infty,s)_\omega$-atom supported in the cube $Q_i$ and

$$\lambda_0a=\sum_{i=1}^{N_0}\gamma_ib_i. \tag{5.23}$$

From the definitions of $\gamma_i$ and $b_i$, $\bigcup_{i=1}^{N_0}Q_i=Q_0$, and for any $x\in Q_0$,

$$1\le\sum_{i=1}^{N_0}\chi_{Q_i}(x)\le C(n),$$

we also conclude that for all $\lambda\in(0,\infty)$,

$$\sum_{i=1}^{N_0}\omega(Q_i)\Phi\bigg(\frac{|\gamma_i|}{\lambda\omega(Q_i)\rho(\omega(Q_i))}\bigg)\le C(n)\omega(Q_0)\Phi\bigg(\frac{|\lambda_0|}{\lambda\omega(Q_0)\rho(\omega(Q_0))}\bigg). \tag{5.24}$$

Thus, by the proof of Lemma 5.4, (5.23) and (5.24), we see that the claim holds.

Now we state the weighted atomic decompositions of $h^\Phi_{\omega,N}(\mathbb{R}^n)$.

THEOREM 5.6. *Let $\Phi$ satisfy Assumption* (A), *$\omega\in A^{\mathrm{loc}}_\infty(\mathbb{R}^n)$, and $q_\omega$ and $N_{\Phi,\omega}$ be respectively as in* (2.4) *and* (3.25). *If $q\in(q_\omega,\infty]$, and integers $s$ and $N$ satisfy $N\ge N_{\Phi,\omega}$ and $N>s\ge\lfloor n(q_\omega/p_\Phi-1)\rfloor$, then*

$$h^{\rho,q,s}_\omega(\mathbb{R}^n)=h^\Phi_{\omega,N}(\mathbb{R}^n)=h^\Phi_{\omega,N_{\Phi,\omega}}(\mathbb{R}^n)$$

*with equivalent norms.*

*Proof.* It is easy to see that

$$h^{\rho,\infty,s_1}_\omega(\mathbb{R}^n)\subset h^{\rho,q,s}_\omega(\mathbb{R}^n)\subset h^\Phi_{\omega,N_{\Phi,\omega}}(\mathbb{R}^n)\subset h^\Phi_{\omega,N}(\mathbb{R}^n)\subset h^\Phi_{\omega,N_1}(\mathbb{R}^n),$$

where the integers $s_1$ and $N_1$ are respectively no less than $s$ and $N$, and the inclusions are continuous. Thus, to prove Theorem 5.6, it suffices to prove that for any integers $N$, $s$ satisfying $N>s\ge\lfloor n(q_\omega/p_\Phi-1)\rfloor$ we have $h^\Phi_{\omega,N}(\mathbb{R}^n)\subset h^{\rho,\infty,s}_\omega(\mathbb{R}^n)$, and for all $f\in h^\Phi_{\omega,N}(\mathbb{R}^n)$,

$$\|f\|_{h^{\rho,\infty,s}_\omega(\mathbb{R}^n)}\lesssim\|f\|_{h^\Phi_{\omega,N}(\mathbb{R}^n)}.$$

Let $f \in h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)$. By Corollary 4.8, there exists a sequence $\{f_m\}_{m\in\mathbb{N}} \subset h^{\Phi}_{\omega,\,N}(\mathbb{R}^n) \cap L^q_\omega(\mathbb{R}^n)$ such that for all $m \in \mathbb{N}$,

$$\|f_m\|_{h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)} \le 2^{-m}\|f\|_{h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)} \tag{5.25}$$

and $f = \sum_{m\in\mathbb{N}} f_m$ in $h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)$. By Lemma 5.4, for each $m \in \mathbb{N}$, $f_m$ has an atomic decomposition

$$f = \sum_{i\in\mathbb{Z}_+} \lambda_i^m a_i^m$$

in $\mathcal{D}'(\mathbb{R}^n)$ with

$$\Lambda(\{\lambda_i^m a_i^m\}_i) \lesssim \|f_m\|_{h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)},$$

where $\{\lambda_i^m\}_{i\in\mathbb{Z}_+} \subset \mathbb{C}$, $\{a_i^m\}_{i\in\mathbb{N}}$ are $(\rho,\,\infty,\,s)_\omega$-atoms and $a_0^m$ is a $(\rho,\,\infty)_\omega$-single-atom.

Let

$$\widetilde{\lambda}_0 \equiv \omega(\mathbb{R}^n)\rho(\omega(\mathbb{R}^n)) \sum_{m=1}^{\infty} |\lambda_0^m|\, \|a_0^m\|_{L^\infty_\omega(\mathbb{R}^n)}, \qquad \widetilde{a}_0 \equiv (\widetilde{\lambda}_0)^{-1} \sum_{m=1}^{\infty} \lambda_0^m a_0^m.$$

Then

$$\widetilde{\lambda}_0 \widetilde{a}_0 = \sum_{m=1}^{\infty} \lambda_0^m a_0^m.$$

It is easy to see that

$$\|\widetilde{a}_0\|_{L^\infty_\omega(\mathbb{R}^n)} \le [\omega(\mathbb{R}^n)\rho(\omega(\mathbb{R}^n))]^{-1},$$

which implies that $\widetilde{a}_0$ is a $(\rho,\infty)_\omega$-single-atom. Since $\Phi$ is increasing, by (5.8), we know that for any $m \in \mathbb{N}$,

$$\omega(\mathbb{R}^n)\Phi\left(\frac{|\lambda_0^m|}{C\|f_m\|_{h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)}\omega(\mathbb{R}^n)\rho(\omega(\mathbb{R}^n))}\right) \le 1, \tag{5.26}$$

where $C$ is as in (5.8). Let

$$\widetilde{\gamma} \equiv C\Big(\sum_{i=m}^{\infty} \|f_m\|^{p_\Phi}_{h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)}\Big)^{1/p_\Phi},$$

where $C$ is as in (5.8). Then, from the continuity, subadditivity and the strictly lower type $p_\Phi$ property of $\Phi$, and (5.26), it follows that

$$\begin{aligned}
&\omega(\mathbb{R}^n)\Phi\left(\frac{|\widetilde{\lambda}_0|}{\widetilde{\gamma}\omega(\mathbb{R}^n)\rho(\omega(\mathbb{R}^n))}\right)\\
&\quad= \omega(\mathbb{R}^n)\Phi\left(\frac{\sum_{m=1}^{\infty} |\lambda_0^m|\,\|a_0^m\|_{L^\infty_\omega(\mathbb{R}^n)}}{\widetilde{\gamma}}\right)\\
&\quad\le \omega(\mathbb{R}^n)\sum_{m=1}^{\infty} \frac{\|f_m\|^{p_\Phi}_{h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)}}{\widetilde{\gamma}^{p_\Phi}}\Phi\left(\frac{|\lambda_0^m|}{C\|f_m\|_{h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)}\omega(\mathbb{R}^n)\rho(\omega(\mathbb{R}^n))}\right) \le 1,
\end{aligned}$$

which together with (5.25) implies that

$$\Lambda(\{\widetilde{\lambda}_0\widetilde{a}_0\}) \le \widetilde{\gamma} \lesssim \|f\|_{h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)}.$$

Thus, we see that

$$f = \sum_{m\in\mathbb{N}}\sum_{i\in\mathbb{N}} \lambda_i^m a_i^m + \widetilde{\lambda}_0\widetilde{a}_0 \in h^{\rho,\,\infty,\,s}_\omega(\mathbb{R}^n) \quad \text{and} \quad \|f\|_{h^{\rho,\,\infty,\,s}_\omega(\mathbb{R}^n)} \lesssim \|f\|_{h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)}.$$

This finishes the proof of Theorem 5.6. ∎

REMARK 5.7. Let $p \in (0,1]$. Theorem 5.1 when $\Phi(t) \equiv t^p$ for all $t \in (0,\infty)$ was obtained by Tang [49, Theorem 5.1].

For simplicity, from now on, we denote by $h^{\Phi}_{\omega}(\mathbb{R}^n)$ the *weighted local Orlicz–Hardy space* $h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)$ when $N \geq N_{\Phi,\,\omega}$.

## 6. Finite atomic decompositions

In this section, we prove that for any given finite linear combination of weighted atoms when $q < \infty$ (or continuous $(\rho,\,q,\,s)_\omega$-atoms when $q = \infty$), its norm in $h^{\Phi}_{\omega,\,N}(\mathbb{R}^n)$ can be achieved via all its finite weighted atomic decompositions. This extends the main results in [35, 57] to the setting of weighted local Orlicz–Hardy spaces. As applications, we see that for a given admissible triplet $(\rho,\,q,\,s)_\omega$ and a $\beta$-quasi-Banach space $\mathcal{B}_\beta$ with $\beta \in (0,1]$, if $T$ is a $\mathcal{B}_\beta$-sublinear operator, and maps all $(\rho,\,q,\,s)_\omega$-atoms and $(\rho,\,q)_\omega$-single-atoms with $q < \infty$ (or all continuous $(\rho,\,q,\,s)_\omega$-atoms with $q = \infty$) into uniformly bounded elements of $\mathcal{B}_\beta$, then $T$ uniquely extends to a bounded $\mathcal{B}_\beta$-sublinear operator from $h^{\Phi}_{\omega}(\mathbb{R}^n)$ to $\mathcal{B}_\beta$.

DEFINITION 6.1. Let $\Phi$ satisfy Assumption (A), $\omega \in A^{\mathrm{loc}}_{\infty}(\mathbb{R}^n)$ and $(\rho,\,q,\,s)_\omega$ be admissible as in Definition 3.4. Then $h^{\rho,\,q,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)$ is defined to be the vector space of all finite linear combinations of $(\rho,\,q,\,s)_\omega$-atoms and a $(\rho,\,q)_\omega$-single-atom, and the norm of $f$ in $h^{\rho,\,q,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)$ is defined by

$$\|f\|_{h^{\rho,\,q,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)} \equiv \inf\Big\{\Lambda(\{\lambda_i a_i\}_i) : f = \sum_{i=0}^{k} \lambda_i a_i,\ k \in \mathbb{Z}_+,\ \{\lambda_i\}_{i=0}^k \subset \mathbb{C},\ \{a_i\}_{i=1}^k \text{ are } (\rho,\,q,\,s)_\omega\text{-atoms and } a_0 \text{ is a } (\rho,\,q)_\omega\text{-single-atom}\Big\}.$$

Obviously, for any admissible triplet $(\rho,\,q,\,s)_\omega$, $h^{\rho,\,q,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)$ is dense in $h^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)$ with respect to the quasi-norm $\|\cdot\|_{h^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)}$.

THEOREM 6.2. *Let $\Phi$ satisfy Assumption* (A), *$\omega \in A^{\mathrm{loc}}_{\infty}(\mathbb{R}^n)$, $q_\omega$ be as in* (2.4) *and $(\rho,\,q,\,s)_\omega$ be admissible as in Definition* 3.4.

(i) *If $q \in (q_\omega,\infty)$, then $\|\cdot\|_{h^{\rho,\,q,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)}$ and $\|\cdot\|_{h^{\Phi}_{\omega}(\mathbb{R}^n)}$ are equivalent quasi-norms on $h^{\rho,\,q,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)$.*
(ii) *Let $h^{\rho,\,\infty,\,s}_{\omega,\,\mathrm{fin},\,c}(\mathbb{R}^n)$ denote the set of all $f \in h^{\rho,\,\infty,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)$ with compact support. Then $\|\cdot\|_{h^{\rho,\,\infty,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)}$ and $\|\cdot\|_{h^{\Phi}_{\omega}(\mathbb{R}^n)}$ are equivalent quasi-norms on $h^{\rho,\,\infty,\,s}_{\omega,\,\mathrm{fin},\,c}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$.*

*Proof.* We first show (i). Let $q \in (q_\omega,\infty)$ and $(\rho,\,q,\,s)_\omega$ be admissible. Obviously, from Theorem 5.6, we infer that $h^{\rho,\,q,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n) \subset h^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n) = h^{\Phi}_{\omega}(\mathbb{R}^n)$ and for all $f \in h^{\rho,\,q,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)$,

$$\|f\|_{h^{\Phi}_{\omega}(\mathbb{R}^n)} \lesssim \|f\|_{h^{\rho,\,q,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)}.$$

Thus, we only need to show that for all $f \in h^{\rho,\,q,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)$,

$$\|f\|_{h^{\rho,\,q,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)} \lesssim \|f\|_{h^{\Phi}_{\omega}(\mathbb{R}^n)}. \tag{6.1}$$

By homogeneity, without loss of generality, we may assume that $f \in h^{\rho,\,q,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)$ with $\|f\|_{h^{\Phi}_{\omega}(\mathbb{R}^n)} = 1$. In the rest of this section, for any $f \in h^{\rho,\,q,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)$, let $k_0$ be as in Section 5 and $\Omega_{2^k}$ with $k \geq k_0$ as in (4.1) with $\lambda = 2^k$. Since $f \in (h^{\Phi}_{\omega,\,N}(\mathbb{R}^n) \cap L^q_\omega(\mathbb{R}^n))$, by Lemma 5.4, there exist $\lambda_0 \in \mathbb{C}$, $\{\lambda_i^k\}_{k\geq k_0,\,i} \subset \mathbb{C}$, a $(\rho,\,\infty)_\omega$-single-atom $a_0$ and $(\rho,\,\infty,\,s)_\omega$-atoms $\{a_i^k\}_{k\geq k_0,\,i}$ such that

$$f = \sum_{k\geq k_0} \sum_i \lambda_i^k a_i^k + \lambda_0 a_0 \tag{6.2}$$

both in $\mathcal{D}'(\mathbb{R}^n)$ and almost everywhere. First, we claim that (6.2) also holds in $L^q_\omega(\mathbb{R}^n)$. For any $x \in \mathbb{R}^n$, from $\mathbb{R}^n = \bigcup_{k\geq k_0}(\Omega_{2^k} \setminus \Omega_{2^{k+1}})$, we see that there exists $j \in \mathbb{Z}$ such that $x \in \Omega_{2^j} \setminus \Omega_{2^{j+1}}$. By the proof of Lemma 5.4, we know that for all $k > j$, $\mathrm{supp}(a_i^k) \subset \widetilde{Q}_i^k \subset \Omega_{2^k} \subset \Omega_{2^{j+1}}$; then from (5.13) and (5.22), we conclude that

$$\Big|\sum_{k\geq k_0} \sum_i \lambda_i^k a_i^k(x)\Big| + |\lambda_0 a_0(x)| \lesssim \sum_{k_0\leq k\leq j} 2^k + 2^{k_0} \lesssim 2^j \lesssim \mathcal{G}_N(f)(x).$$

Since $f \in L^q_\omega(\mathbb{R}^n)$, from Proposition 3.2(ii), we infer that $\mathcal{G}_N(f)(x) \in L^q_\omega(\mathbb{R}^n)$. This combined with the Lebesgue dominated convergence theorem implies that

$$\sum_{k\geq k_0} \sum_i \lambda_i^k a_i^k + \lambda_0 a_0$$

converges to $f$ in $L^q_\omega(\mathbb{R}^n)$, which completes the proof of the claim.

Next, we show (6.1) by considering the following two cases for $\omega$.

*Case 1:* $\omega(\mathbb{R}^n) = \infty$. In this case, as $f \in L^q_\omega(\mathbb{R}^n)$, we know that $k_0 = -\infty$ and $a_0(x) = 0$ for almost every $x \in \mathbb{R}^n$ in (6.2). Thus, in this case, (6.2) has the version

$$f = \sum_{k\in\mathbb{Z}} \sum_i \lambda_i^k a_i^k.$$

Since, when $\omega(\mathbb{R}^n) = \infty$, all $(\rho,\,q)_\omega$-single-atoms are 0, if $f \in h^{\rho,\,q,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)$, then $f$ has compact support. Assume that $\mathrm{supp}(f) \subset Q_0 \equiv Q(x_0, r_0)$ and

$$\widetilde{Q}_0 \equiv Q(x_0, \sqrt{n}r_0 + 2^{3(10+n)+1}).$$

Then for any $\psi \in \mathcal{D}_N(\mathbb{R}^n)$, $x \in \mathbb{R}^n \setminus \widetilde{Q}_0$ and $t \in (0,1)$, we have

$$\psi_t * f(x) = \int_{Q(x_0,r_0)} \psi_t(x-y) f(y)\,dy = \int_{B(x,2^{3(10+n)})\cap Q(x_0,r_0)} \psi_t(x-y) f(y)\,dy = 0.$$

Thus, for any $k \in \mathbb{Z}$, $\Omega_{2^k} \subset \widetilde{Q}_0$, which implies that $\mathrm{supp}(\sum_{k\in\mathbb{Z}} \sum_i \lambda_i^k a_i^k) \subset \widetilde{Q}_0$. For each positive integer $K$, let

$$F_K \equiv \{(i,k) : k \in \mathbb{Z},\, k \geq k_0,\, i \in \mathbb{N},\, |k| + i \leq K\} \quad \text{and} \quad f_K \equiv \sum_{(k,i)\in F_K} \lambda_i^k a_i^k.$$

Then, by the above claim, $f_K$ converges to $f$ in $L^q_\omega(\mathbb{R}^n)$. Thus, for any given $\epsilon \in (0,1)$, there exists $K_0 \in \mathbb{N}$ large enough such that

$$\|(f - f_{K_0})/\epsilon\|_{L^q_\omega(\mathbb{R}^n)} \leq [\rho(\omega(\widetilde{Q}_0))]^{-1}[\omega(\widetilde{Q}_0)]^{1/q-1},$$

which together with $\mathrm{supp}(f - f_{K_0})/\epsilon \subset \widetilde{Q}_0$ implies that $(f - f_{K_0})/\epsilon$ is a $(\rho,\,q,\,s)_\omega$-atom. Moreover, we equivalently divide $\widetilde{Q}_0$ into the union of some cubes $\{Q_i\}_{i=1}^{N_0}$ with disjoint interior and sidelengths satisfying $l_i \in (1,2]$, where $N_0$ depends only on $r_0$ and $n$. It is clear that

$$\|(f - f_{K_0})\chi_{Q_i}/\epsilon\|_{L^q_\omega(\mathbb{R}^n)} \le [\rho(\omega(\widetilde{Q}_0))]^{-1}[\omega(\widetilde{Q}_0)]^{1/q-1} \le [\rho(\omega(Q_i))]^{-1}[\omega(Q_i)]^{1/q-1},$$

which together with $\mathrm{supp}((f-f_{K_0})\chi_{Q_i}/\epsilon) \subset Q_i$ implies that $(f-f_{K_0})\chi_{Q_i}/\epsilon$ is a $(\rho,\, q,\, s)_\omega$-atom for $i = 1, \ldots, N_0$. Thus,

$$f = f_{K_0} + \sum_{i=1}^{N_0}(f - f_{K_0})\chi_{Q_i}$$

almost everywhere is a finite linear weighted atom combination of $f$. Let

$$b_i \equiv (f - f_{K_0})\chi_{Q_i}/\epsilon$$

and take $\epsilon \equiv N_0^{-1/p_\Phi}$. Then, by (2.8) with $t \equiv \omega(Q_i)$, Remark 3.6(ii) and the lower type $p_\Phi$ property of $\Phi$,

$$\begin{aligned}\|f\|_{h^{\rho,\,q,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)} &\lesssim \Lambda(\{\lambda_i^k a_i^k\}_{(i,k)\in F_{K_0}}) + \Lambda(\{\epsilon b_i\}_{i=1}^{N_0})\\ &\lesssim \|f\|_{h^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)} + \inf\left\{\lambda\in(0,\infty) : \sum_{i=1}^{N_0}\omega(Q_i)\Phi\left(\frac{\epsilon}{\lambda\omega(Q_i)\rho(\omega(Q_i))}\right)\le 1\right\} \lesssim 1,\end{aligned}$$

which implies (6.1) in Case 1.

*Case 2:* $\omega(\mathbb{R}^n) < \infty$. In this case, $f$ may not have compact support. Similarly to Case 1, for any positive integer $K$, let

$$f_K \equiv \sum_{(k,i)\in F_K} \lambda_i^k a_i^k + \lambda_0 a_0$$

and $b_K \equiv f - f_K$, where $F_K$ is as in Case 1. From the above claim, $f_K$ converges to $f$ in $L^q_\omega(\mathbb{R}^n)$. Thus, there exists a positive integer $K_1 \in \mathbb{N}$ large enough such that

$$\|b_{K_1}\|_{L^q_\omega(\mathbb{R}^n)} \le [\rho(\omega(\mathbb{R}^n))]^{-1}[\omega(\mathbb{R}^n)]^{1/q-1}.$$

Thus, $b_{K_1}$ is a $(\rho,\, q)_\omega$-single-atom and $f = f_{K_1} + b_{K_1}$ is a finite linear weighted atom combination of $f$. Moreover, by Remark 3.6(ii) and (2.8) with $t \equiv \omega(\mathbb{R}^n)$,

$$\begin{aligned}\|f\|_{h^{\rho,\,q,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)} &\lesssim \Lambda(\{\lambda_i^k a_i^k\}_{(i,k)\in F_{K_1}}) + \Lambda(\{b_{K_1}\})\\ &\lesssim \|f\|_{h^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)} + \inf\left\{\lambda\in(0,\infty) : \omega(\mathbb{R}^n)\Phi\left(\frac{1}{\lambda\omega(\mathbb{R}^n)\rho(\omega(\mathbb{R}^n))}\right)\le 1\right\} \lesssim 1,\end{aligned}$$

which implies (6.1) in Case 2. This finishes the proof of (i).

We now prove (ii). In this case, similarly to the proof of (i), we only need to prove that for all $f \in h^{\rho,\,\infty,\,s}_{\omega,\,\mathrm{fin},\,c}(\mathbb{R}^n)$,

$$\|f\|_{h^{\rho,\,\infty,\,s}_{\omega}(\mathbb{R}^n)} \lesssim \|f\|_{h^{\Phi}_{\omega}(\mathbb{R}^n)}.$$

Again, by homogeneity, without loss of generality, we may assume that $\|f\|_{h^\Phi_\omega(\mathbb{R}^n)} = 1$. Since $f$ has compact support, by the definition of $\mathcal{G}_N(f)$, it is easy to see that $\mathcal{G}_N(f)$ also has compact support. Assume that $\mathrm{supp}(\mathcal{G}_N(f)) \subset B(0, R_0)$ for some $R_0 \in (0,\infty)$. As $f \in L^\infty_\omega(\mathbb{R}^n)$, we have $\mathcal{G}_N f \in L^\infty_\omega(\mathbb{R}^n)$. Thus, there exists $k_1 \in \mathbb{Z}$ such that $\Omega_{2^k} = \emptyset$ for any $k \in \mathbb{Z}$ with $k \ge k_1 + 1$. By Lemma 5.4, there exist $\lambda_0 \in \mathbb{C}$, $\{\lambda_i^k\}_{k_1\ge k\ge k_0,\, i} \subset \mathbb{C}$, a

$(\rho, \infty)_\omega$-single-atom $a_0$ and $(\rho, \infty, s)_\omega$-atoms $\{a_i^k\}_{k_1 \geq k \geq k_0, i}$ such that

$$f = \sum_{k=k_0}^{k_1} \sum_i \lambda_i^k a_i^k + \lambda_0 a_0$$

holds both in $\mathcal{D}'(\mathbb{R}^n)$ and almost everywhere. Since $f$ is uniformly continuous, for any given $\varepsilon \in (0, \infty)$ there exists a $\delta \in (0, \infty)$ such that if

$$|x - y| < \sqrt{n}\delta/2,$$

then $|f(x) - f(y)| < \varepsilon$. We may assume that $\delta < 1$. Write $f = f_1^\varepsilon + f_2^\varepsilon$ with

$$f_1^\varepsilon \equiv \sum_{(i,k)\in G_1} \lambda_i^k a_i^k + \lambda_0 a_0 \quad \text{and} \quad f_2^\varepsilon \equiv \sum_{(i,k)\in G_2} \lambda_i^k a_i^k,$$

where

$$G_1 \equiv \{(i,k) : l(\widetilde{Q}_i^k) \geq \delta,\, k_0 \leq k \leq k_1\}, \qquad G_2 \equiv \{(i,k) : l(\widetilde{Q}_i^k) < \delta,\, k_0 \leq k \leq k_1\},$$

and $\widetilde{Q}_i^k$ is the support of $a_i^k$ (see the proof of Lemma 5.4). For any fixed integer $k \in [k_0, k_1]$, by Lemma 5.1(ii) and $\Omega_{2^k} \subset B(0, R_0)$, we see that $G_1$ is a finite set.

For any $(i,k) \in G_2$ and $x \in \widetilde{Q}_i^k$, $|f(x) - f(x_i^k)| < \varepsilon$. For all $x \in \mathbb{R}^n$, let

$$\widetilde{f}(x) \equiv [f(x) - f(x_i^k)]\chi_{\widetilde{Q}_i^k}(x)$$

and $\widetilde{P}_i^k(x) \equiv P_i^k(x) - f(x_i^k)$. By the definition of $P_i^k$, for all $P \in \mathcal{P}_s(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} [\widetilde{f}(x) - \widetilde{P}_i^k(x)]P(x)\zeta_i^k(x)\,dx = 0.$$

Since $|\widetilde{f}(x)| < \varepsilon$ for all $x \in \mathbb{R}^n$ implies that $\mathcal{G}_N(\widetilde{f})(x) \lesssim \varepsilon$ for all $x \in \mathbb{R}^n$, by Lemma 4.1 we see that

$$\sup_{y\in\mathbb{R}^n} |\widetilde{P}_i^k(y)\zeta_i^k(y)| \lesssim \sup_{y\in\mathbb{R}^n} |\mathcal{G}_N(\widetilde{f})(y)| \lesssim \varepsilon. \tag{6.3}$$

Let $\widetilde{P}_{i,j}^k \in \mathcal{P}_s(\mathbb{R}^n)$ be such that for any $P \in \mathcal{P}_s(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} [\widetilde{f}(x) - \widetilde{P}_i^k(x)]\zeta_i^k(x)P(x)\zeta_i^{k+1}(x)\,dx = \int_{\mathbb{R}^n} \widetilde{P}_{i,j}^{k+1}(x)P(x)\zeta_j^{k+1}(x)\,dx.$$

Since $(\widetilde{f} - \widetilde{P}_i^k)\zeta_i^k = (f - P_i^k)\zeta_i^k$, from $\mathrm{supp}(\zeta_i^k) \subset \widetilde{Q}_i^k$ we have $\widetilde{P}_{i,j}^k = P_{i,j}^k$. Then from Lemma 5.2, we deduce that

$$\sup_{y\in\mathbb{R}^n} |\widetilde{P}_{i,j}^k(y)\zeta_i^{k+1}(y)| \lesssim \sup_{y\in\mathbb{R}^n} |\mathcal{G}_N(\widetilde{f})(y)| \lesssim \varepsilon. \tag{6.4}$$

Thus, from the definition of $\lambda_i^k a_i^k$, $\sum_j \zeta_j^{k+1} = \chi_{\Omega_{2^{k+1}}}$ and (5.11), we know that

$$\begin{aligned}\lambda_i^k a_i^k &= f\chi_{\Omega^{\complement}_{2^{k+1}}}\zeta_i^k - P_i^k\zeta_i^k + \sum_{j\in F_2^{k+1}} P_j^{k+1}\zeta_i^k\zeta_j^{k+1} + \sum_{j\in F_2^{k+1}} P_{i,j}^{k+1}\zeta_j^{k+1}\\ &= \widetilde{f}\chi_{\Omega^{\complement}_{2^{k+1}}}\zeta_i^k - \widetilde{P}_i^k\zeta_i^k + \sum_{j\in F_2^{k+1}} \widetilde{P}_j^{k+1}\zeta_i^k\zeta_j^{k+1} + \sum_{j\in F_2^{k+1}} \widetilde{P}_{i,j}^{k+1}\zeta_j^{k+1}.\end{aligned}$$

From this together with (6.3), (6.4) and Lemma 5.1(ii), it follows that $|\lambda_i^k a_i^k| \lesssim \varepsilon$ for all $x \in \widetilde{Q}_i^k$ with $(i,k) \in G_2$. Moreover, using Lemma 5.1(ii) again, we conclude that

$$|f_2^\varepsilon| \lesssim \sum_{k=k_0}^{k_1} \varepsilon \lesssim (k_1 - k_0)\varepsilon.$$

From the arbitrariness of $\varepsilon$, $\operatorname{supp}(f_2^\varepsilon) \subset B(0, R_0)$ and $|f_2^\varepsilon| \lesssim (k_1 - k_0)\varepsilon$, we choose $\varepsilon$ small enough such that $f_2^\varepsilon$ is an arbitrarily small multiple of a $(\rho,\, \infty,\, s)_\omega$-atom. In particular, we choose $\varepsilon_0 \in (0,\infty)$ such that $f_2^{\varepsilon_0} = \widetilde{\lambda}\widetilde{a}$ with $|\widetilde{\lambda}| \le 1$ and $\widetilde{a}$ is a $(\rho,\, \infty,\, s)_\omega$-atom. Then

$$f = \sum_{(i,k)\in G_1} \lambda_i^k a_i^k + \lambda_0 a_0 + \widetilde{\lambda}\widetilde{a}$$

is a finite weighted atomic decomposition of $f$, and

$$\|f\|_{h_\omega^{\rho,\,\infty,\,s}(\mathbb{R}^n)} \lesssim \|f\|_{h_\omega^\Phi(\mathbb{R}^n)} + 1 \lesssim 1,$$

which completes the proof of Theorem 6.2. ■

REMARK 6.3. (i) From the proof of Theorem 6.2, for any $f \in h_{\omega,\,\mathrm{fin}}^{\rho,\,q,\,s}(\mathbb{R}^n)$ with $q \in (q_\omega, \infty)$, there exist $\{\lambda_j\}_{j=0}^k \subset \mathbb{C}$, a $(\rho,\, q)_\omega$-single-atom $a_0$ and $(\rho,\, q,\, s)_\omega$-atoms $\{a_j\}_{j=1}^k$ satisfying $\operatorname{supp}(a_j) \subset Q_j$ with $l(Q_j) \in (0,2]$ such that $f = \sum_{j=0}^k \lambda_j a_j$ in both $L_\omega^q(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^n)$. Moreover, for all $f \in h_{\omega,\,\mathrm{fin}}^{\rho,\,q,\,s}(\mathbb{R}^n)$,

$$\begin{aligned}\|f\|_{h_{\omega,\,\mathrm{fin}}^{\rho,\,q,\,s}(\mathbb{R}^n)} &\sim \|f\|_{h_\omega^\Phi(\mathbb{R}^n)} \\ &\sim \inf\Big\{\Lambda\{\lambda_i a_i\}_i :\, f = \sum_{i=0}^k \lambda_i a_i,\ k \in \mathbb{Z}_+,\ \{a_i\}_{i=1}^k \text{ are } (\rho,\, q,\, s)_\omega\text{-atoms} \\ &\qquad \text{satisfying } \operatorname{supp}(a_j) \subset Q_j,\ l(Q_j) \in (0,2] \\ &\qquad \text{and } a_0 \text{ is a } (\rho,\, q)_\omega\text{-single-atom}\Big\}.\end{aligned}$$

(ii) Obviously, when $\omega(\mathbb{R}^n) = \infty$,

$$h_{\omega,\,\mathrm{fin},\,c}^{\rho,\,\infty,\,s}(\mathbb{R}^n) \cap C(\mathbb{R}^n) = h_{\omega,\,\mathrm{fin}}^{\rho,\,\infty,\,s}(\mathbb{R}^n) \cap C(\mathbb{R}^n).$$

As an application of Theorem 6.2, we establish the boundedness on $h_\omega^\Phi(\mathbb{R}^n)$ of quasi-Banach-valued sublinear operators.

Recall that a *quasi-Banach space* $\mathcal{B}$ is a vector space endowed with a quasi-norm $\|\cdot\|_{\mathcal{B}}$ which is nonnegative, nondegenerate (i. e., $\|f\|_{\mathcal{B}} = 0$ if and only if $f = 0$), homogeneous, and obeys the quasi-triangle inequality, i. e., there exists a positive constant $K$ no less than 1 such that for all $f,\, g \in \mathcal{B}$,

$$\|f + g\|_{\mathcal{B}} \le K(\|f\|_{\mathcal{B}} + \|g\|_{\mathcal{B}}).$$

Let $\beta \in (0,1]$. As in [56, 57], a quasi-Banach space $\mathcal{B}_\beta$ with the quasi-norm $\|\cdot\|_{\mathcal{B}_\beta}$ is called a *$\beta$-quasi-Banach space* if

$$\|f + g\|_{\mathcal{B}_\beta}^\beta \le \|f\|_{\mathcal{B}_\beta}^\beta + \|g\|_{\mathcal{B}_\beta}^\beta$$

for all $f, g \in \mathcal{B}_\beta$.

Notice that any Banach space is a 1-quasi-Banach space, and the quasi-Banach spaces $l^\beta$ and $L_\omega^\beta(\mathbb{R}^n)$ are typical $\beta$-quasi-Banach spaces. Let $\Phi$ satisfy Assumption (A). By the subadditivity of $\Phi$ and (2.6), we know that $h_\omega^\Phi(\mathbb{R}^n)$ is a $p_\Phi$-quasi-Banach space.

For any given $\beta$-quasi-Banach space $\mathcal{B}_\beta$ with $\beta \in (0,1]$ and a linear space $\mathcal{Y}$, an operator $T$ from $\mathcal{Y}$ to $\mathcal{B}_\beta$ is called $\mathcal{B}_\beta$*-sublinear* if for any $f, g \in \mathcal{B}_\beta$ and $\lambda, \nu \in \mathbb{C}$,

$$\|T(\lambda f + \nu g)\|_{\mathcal{B}_\beta} \le (|\lambda|^\beta \|T(f)\|_{\mathcal{B}_\beta}^\beta + |\nu|^\beta \|T(g)\|_{\mathcal{B}_\beta}^\beta)^{1/\beta}$$

and

$$\|T(f) - T(g)\|_{\mathcal{B}_\beta} \le \|T(f-g)\|_{\mathcal{B}_\beta}$$

(see [56, 57]).

We remark that if $T$ is linear, then it is $\mathcal{B}_\beta$-sublinear. Moreover, if $\mathcal{B}_\beta$ is a space of functions, and $T$ is nonnegative and sublinear in the classical sense, then $T$ is also $\mathcal{B}_\beta$-sublinear.

THEOREM 6.4. *Let $\Phi$ satisfy Assumption* (A), *$\omega \in A_\infty^{\mathrm{loc}}(\mathbb{R}^n)$, $q_\omega$ be as in* (2.4) *and $(\rho,\, q,\, s)_\omega$ be admissible. Let $\mathcal{B}_\beta$ be a $\beta$-quasi-Banach space with $\beta \in (0,1]$ and $\widetilde{p}$ be an upper type of $\Phi$ satisfying $\widetilde{p} \in (0, \beta]$. Suppose that one of the following holds:*

(i) *$q \in (q_\omega, \infty)$ and $T : h_{\omega,\,\mathrm{fin}}^{\rho,\,q,\,s}(\mathbb{R}^n) \to \mathcal{B}_\beta$ is a $\mathcal{B}_\beta$-sublinear operator such that*

$$S \equiv \sup\{\|T(a)\|_{\mathcal{B}_\beta} : \ a \textit{ is } a\,(\rho,\, q,\, s)_\omega\textit{-atom with } \mathrm{supp}(a) \subset Q \textit{ and } l(Q) \in (0,2] \textit{ or } (\rho,\, q)_\omega\textit{-single-atom}\} < \infty.$$

(ii) *$T$ is a $\mathcal{B}_\beta$-sublinear operator defined on continuous $(\rho,\, \infty,\, s)_\omega$-atoms such that*

$$S \equiv \sup\{\|T(a)\|_{\mathcal{B}_\beta} : a \textit{ is a continuous } (\rho,\, \infty,\, s)_\omega\textit{-atom}\} < \infty.$$

*Then there exists a unique bounded $\mathcal{B}_\beta$-sublinear operator $\widetilde{T}$ from $h_\omega^\Phi(\mathbb{R}^n)$ to $\mathcal{B}_\beta$ which extends $T$.*

*Proof.* We first show the conclusion under assumption (i). For any $f \in h_{\omega,\,\mathrm{fin}}^{\rho,\,q,\,s}(\mathbb{R}^n)$, by Theorem 6.2(i) and Remark 6.3(i), there exist a sequence $\{\lambda_j\}_{j=0}^l \subset \mathbb{C}$ with some $l \in \mathbb{N}$, a $(\rho,\, q)_\omega$-single-atom $a_0$ and $(\rho,\, q,\, s)_\omega$-atoms $\{a_j\}_{j=1}^l$ satisfying $\mathrm{supp}(a_j) \subset Q_j$ and $l(Q_j) \in (0,2]$ for $j \in \{1,\dots,l\}$ such that $f = \sum_{j=0}^l \lambda_j a_j$ pointwise and

$$\Lambda(\{\lambda_j a_j\}_{j=0}^l) \lesssim \|f\|_{h_\omega^\Phi(\mathbb{R}^n)}. \tag{6.5}$$

Then by the assumptions,

$$\|T(f)\|_{\mathcal{B}_\beta} \le \Big\{\sum_{i=0}^l |\lambda_i|^\beta \|T(a)\|_{\mathcal{B}_\beta}^\beta\Big\}^{1/\beta} \le \Big\{\sum_{i=0}^l |\lambda_i|^{\widetilde{p}} \|T(a)\|_{\mathcal{B}_\beta}^{\widetilde{p}}\Big\}^{1/\widetilde{p}} \lesssim \Big\{\sum_{i=0}^l |\lambda_i|^{\widetilde{p}}\Big\}^{1/\widetilde{p}}. \tag{6.6}$$

Since $\Phi$ is of upper type $\widetilde{p}$, for any $t \in (0,1]$ and $s \in (0,\infty)$ we have $\Phi(st) \gtrsim t^{\widetilde{p}}\Phi(s)$. Let $\widetilde{\lambda}_0 \equiv \{\sum_{i=0}^l |\lambda_i|^{\widetilde{p}}\}^{1/\widetilde{p}}$. Then

$$\sum_{i=0}^l \omega(Q_i)\Phi\bigg(\frac{|\lambda_i|}{\widetilde{\lambda}_0 \omega(Q_i)\rho(\omega(Q_i))}\bigg) \gtrsim \sum_{i=0}^l \omega(Q_i)\bigg(\frac{|\lambda_i|}{\widetilde{\lambda}_0}\bigg)^{\widetilde{p}} \frac{1}{\omega(Q_i)} \sim 1.$$

From this we deduce that $\widetilde{\lambda}_0 \lesssim \Lambda(\{\lambda_i a_i\}_{i=0}^l)$, which together with (6.5) and (6.6) implies that

$$\|T(f)\|_{\mathcal{B}_\beta} \lesssim \widetilde{\lambda}_0 \lesssim \Lambda(\{\lambda_i a_i\}_{i=0}^l) \lesssim \|f\|_{h_\omega^\Phi(\mathbb{R}^n)}.$$

Since $h_{\omega,\,\mathrm{fin}}^{\rho,\,q,\,s}(\mathbb{R}^n)$ is dense in $h_\omega^\Phi(\mathbb{R}^n)$, a density argument gives the desired conclusion in this case.

Now we prove the conclusion under assumption (ii) by considering the following two cases for $\omega$.

*Case 1:* $\omega(\mathbb{R}^n)=\infty$. In this case, similarly to the proof of (i), using Theorem 6.2(ii) and Remark 6.3(ii), we see that for all $f\in h^{\rho,\,\infty,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)\cap C(\mathbb{R}^n)$,

$$\|T(f)\|_{\mathcal{B}_\beta}\lesssim\|f\|_{h^\Phi_\omega(\mathbb{R}^n)}.$$

To extend $T$ to the whole $h^\Phi_\omega(\mathbb{R}^n)$, we only need to prove that $h^{\rho,\,\infty,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)\cap C(\mathbb{R}^n)$ is dense in $h^\Phi_\omega(\mathbb{R}^n)$. Since $h^{\rho,\,\infty,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)$ is dense in $h^\Phi_\omega(\mathbb{R}^n)$, it suffices to prove that $h^{\rho,\,\infty,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)\cap C(\mathbb{R}^n)$ is dense in $h^{\rho,\,\infty,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)$ with respect to the quasi-norm $\|\cdot\|_{h^\Phi_\omega(\mathbb{R}^n)}$.

To see this, let $f\in h^{\rho,\,\infty,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)$. In this case, for any $(\rho,\,\infty)_\omega$-single-atom $b$, $b(x)=0$ for almost every $x\in\mathbb{R}^n$. Thus, $f$ is a finite linear combination of $(\rho,\,\infty,\,s)_\omega$-atoms. Then there exists a cube $Q_0\equiv Q(x_0,r_0)$ such that $\mathrm{supp}(f)\subset Q_0$. Take $\phi\in\mathcal{D}(\mathbb{R}^n)$ such that $\mathrm{supp}(\phi)\subset Q(0,1)$ and $\int_{\mathbb{R}^n}\phi(x)\,dx=1$. Then it is easy to see that for any $k\in\mathbb{N}$, $\mathrm{supp}(\phi_k*f)\subset Q(x_0,r_0+1)$ and $\phi_k*f\in\mathcal{D}(\mathbb{R}^n)$. Assume that $f=\sum_{i=1}^N\lambda_ia_i$ with some $N\in\mathbb{N}$, $\{\lambda_i\}_{i=1}^N\subset\mathbb{C}$ and $\{a_i\}_{i=1}^N$ being $(\rho,\,\infty,\,s)_\omega$-atoms. Then for any $k\in\mathbb{N}$,

$$\phi_k*f=\sum_{i=1}^N\lambda_i\phi_k*a_i.$$

For any $k\in\mathbb{N}$ and $i\in\{1,\dots,N\}$, we now prove that $\phi_k*a_i$ is a multiple of some continuous $(\rho,\,\infty,\,s)_\omega$-atom, which implies that for any $k\in\mathbb{N}$,

$$\phi_k*f\in h^{\rho,\,\infty,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)\cap C(\mathbb{R}^n).\tag{6.7}$$

For $i\in\{1,\dots,N\}$, assume that $\mathrm{supp}(a_i)\subset Q_i\equiv Q(x_i,r_i)$. Then

$$\mathrm{supp}(\phi_k*a_i)\subset\widetilde{Q}_{i,\,k}\equiv Q(x_i,r_i+1/2^k).$$

Moreover,

$$\|\phi_k*a_i\|_{L^\infty_\omega(\mathbb{R}^n)}\le\|a_i\|_{L^\infty_\omega(\mathbb{R}^n)}\le\frac{1}{\omega(Q_i)\rho(\omega(Q_i))}.$$

Furthermore, for any $\alpha\in\mathbb{Z}^n_+$, $\int_{\mathbb{R}^n}a_i(x)x^\alpha\,dx=0$ implies that

$$\int_{\mathbb{R}^n}\phi_k*a_i(x)x^\alpha\,dx=0.$$

Thus, $\frac{\omega(Q_i)\rho(\omega(Q_i))}{\omega(\widetilde{Q}_{i,\,k})\rho(\omega(\widetilde{Q}_{i,\,k}))}\phi_k*a_i$ is a $(\rho,\,\infty,\,s)_\omega$-atom.

Likewise, $\mathrm{supp}(f-\phi_k*f)\subset Q(x_0,r_0+1)$ and $f-\phi_k*f$ has the same vanishing moments as $f$. Take $q\in(q_\omega,\infty)$. By Lemma 2.6(iii),

$$\|f-\phi_k*f\|_{L^q_\omega(\mathbb{R}^n)}\to0\quad\text{as }k\to\infty.\tag{6.8}$$

Without loss of generality, we may assume that when $k$ is large enough,

$$\|f-\phi_k*f\|_{L^q_\omega(\mathbb{R}^n)}>0.$$

Let

$$c_k\equiv\|f-\phi_k*f\|_{L^q_\omega(\mathbb{R}^n)}[\omega(Q(x_0,r_0+1))]^{1/q-1}\rho(\omega(Q(x_0,r_0+1)))$$

and $a_k \equiv (f - \phi_k * f)/c_k$. Then $a_k$ is a $(\rho,\, q,\, s)_\omega$-atom, $f - \phi_k * f = c_k a_k$, and $c_k \to 0$ as $k \to \infty$. Thus, from (2.8) with $t \equiv \omega(Q(x_0, r_0 + 1))$, and Theorem 5.6, we infer that

$$\|f - \phi_k * f\|_{h^\Phi_\omega(\mathbb{R}^n)} \lesssim \Lambda(\{c_k a_k\}) \lesssim |c_k| \to 0 \tag{6.9}$$

as $k \to \infty$, which together with (6.7) shows the desired conclusion in this case.

*Case 2:* $\omega(\mathbb{R}^n) < \infty$. In this case, similarly to the proof of Case 1, by Theorem 6.2(ii), to finish the proof of (ii), it suffices to prove that $h^{\rho,\,\infty,\,s}_{\omega,\,\mathrm{fin},\,c}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ is dense in $h^{\rho,\,\infty,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)$ in the quasi-norm $\|\cdot\|_{h^\Phi_\omega(\mathbb{R}^n)}$.

For any $f \in h^{\rho,\,\infty,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)$, assume that

$$f \equiv \sum_{i=1}^{N_1} \lambda_i a_i + \lambda_0 a_0,$$

where $N_1 \in \mathbb{N}$, $\{\lambda_i\}_{i=1}^{N_1} \subset \mathbb{C}$ and $a_0$ is a $(\rho,\, \infty)_\omega$-single-atom and $\{a_i\}_{i=1}^{N_1}$ are $(\rho,\, \infty,\, s)_\omega$-atoms. Let $\{\psi_k\}_{k\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^n)$ satisfy $0 \le \psi_k \le 1$, $\psi_k \equiv 1$ on the cube $Q(0, 2^k)$ and

$$\mathrm{supp}(\psi_k) \subset Q(0, 2^{k+1}).$$

We assume that $\mathrm{supp}(\sum_{i=1}^{N_1} \lambda_i a_i) \subset Q(0, R_0)$ for some $R_0 \in (0, \infty)$ and $k_0$ is the smallest integer such that $2^{k_0} \ge R_0$. For any integer $k \ge k_0$, let $f_k \equiv f\psi_k$. Then $f_k \in h^{\rho,\,\infty,\,s}_{\omega,\,\mathrm{fin},\,c}(\mathbb{R}^n)$. Indeed, by the choice of $\psi_k$,

$$f_k = \sum_{i=1}^{N_1} \lambda_i a_i + \lambda_0 a_0 \psi_k$$

and $\mathrm{supp}(f_k) \subset Q(0, 2^{k+1})$. Furthermore, from $\mathrm{supp}(a_0\psi_k) \subset Q(0, 2^{k+1})$ and

$$\|a_0\psi_k\|_{L^\infty_\omega(\mathbb{R}^n)} \le \|a_0\|_{L^\infty_\omega(\mathbb{R}^n)} \le \frac{1}{\omega(\mathbb{R}^n)\rho(\omega(\mathbb{R}^n))} \le \frac{1}{\omega(Q(0, 2^{k+1}))\rho(\omega(Q(0, 2^{k+1})))},$$

we deduce that $a_0\psi_k$ is a $(\rho,\, \infty,\, s)_\omega$-atom. Thus, $f_k \in h^{\rho,\,\infty,\,s}_{\omega,\,\mathrm{fin},\,c}(\mathbb{R}^n)$. For any fixed integer $k \ge k_0$ and any $i \in \mathbb{N}$, let $\widetilde{f}_{k,\,i} \equiv f_k * \phi_i$, where $\phi$ is as in Case 1. Similarly to the proof of (6.7), we have $\widetilde{f}_{k,\,i} \in h^{\rho,\,\infty,\,s}_{\omega,\,\mathrm{fin},\,c}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$. For any $q \in (q_\omega, \infty)$, from the choice of $f_k$ and $\omega(\mathbb{R}^n) < \infty$, we conclude that

$$\begin{aligned}\|f - f_k\|_{L^q_\omega(\mathbb{R}^n)} &\le \left\{\int_{Q(0,2^k)^\complement} |f(x)|^q \omega(x)\,dx\right\}^{1/q}\\ &\le \|\lambda_0 a_0\|_{L^\infty_\omega(\mathbb{R}^n)} \left\{\int_{Q(0,2^k)^\complement} \omega(x)\,dx\right\}^{1/q} \to 0 \end{aligned}\tag{6.10}$$

as $k \to \infty$. Furthermore, for any fixed $k \in \mathbb{Z}$ with $k \ge k_0$, similarly to the proof of (6.8), we see that $\|f_k - \widetilde{f}_{k,\,i}\|_{L^q_\omega(\mathbb{R}^n)} \to 0$ as $i \to \infty$, which together with (6.10) implies that

$$\|f - \widetilde{f}_{k,\,i}\|_{L^q_\omega(\mathbb{R}^n)} \to 0$$

as $k, i \to \infty$. Without loss of generality, we may assume that when $k$ and $i$ are large enough, $\|f - \widetilde{f}_{k,\,i}\|_{L^q_\omega(\mathbb{R}^n)} > 0$. Let

$$c_{k,\,i} \equiv \|f - \widetilde{f}_{k,\,i}\|_{L^q_\omega(\mathbb{R}^n)} [\omega(\mathbb{R}^n)]^{1/q-1} \rho(\omega(\mathbb{R}^n))$$

and $a_{k,\,i} \equiv (f - \widetilde{f}_{k,\,i})/c_{k,\,i}$. Then $f - \widetilde{f}_{k,\,i} = c_{k,\,i} a_{k,\,i}$, $a_{k,\,i}$ is a $(\rho,\, q)_\omega$-single-atom and

$c_{k,i} \to 0$ as $k, i \to \infty$. Then, similarly to the proof of (6.9), $\|f - \widetilde{f}_{k,i}\|_{h^{\Phi}_{\omega}(\mathbb{R}^n)} \to 0$ as $k, i \to \infty$, which completes the proof of Case 2 and hence of Theorem 6.4. ∎

REMARK 6.5. Let $p \in (0, 1]$. We point out that Theorems 6.2(i) and 6.4(i) when $\Phi(t) \equiv t^p$ for all $t \in (0, \infty)$ were obtained by Tang [49, Theorems 6.1 and 6.2]. Theorems 6.2(ii) and 6.4(ii) are new even when $\omega \equiv 1$ and $\Phi(t) \equiv t^p$ for all $t \in (0, \infty)$.

## 7. Dual spaces

In this section, we introduce the BMO-type space $\mathrm{bmo}^q_{\rho,\omega}(\mathbb{R}^n)$ and establish the duality between $h^{\rho,q,s}_{\omega}(\mathbb{R}^n)$ and $\mathrm{bmo}^{q'}_{\rho,\omega}(\mathbb{R}^n)$; here and in what follows, $1/q + 1/q' = 1$. From this and Theorem 5.6, we deduce the duality between $h^{\Phi}_{\omega}(\mathbb{R}^n)$ and $\mathrm{bmo}_{\rho,\omega}(\mathbb{R}^n)$, and that for $q \in [1, q_\omega/(q_\omega - 1))$, $\mathrm{bmo}^q_{\rho,\omega}(\mathbb{R}^n) = \mathrm{bmo}_{\rho,\omega}(\mathbb{R}^n)$ with equivalent norms, where $\mathrm{bmo}_{\rho,\omega}(\mathbb{R}^n)$ denotes $\mathrm{bmo}^1_{\rho,\omega}(\mathbb{R}^n)$. We begin with some definitions.

For any locally integrable function $f$ on $\mathbb{R}^n$, we denote the *minimizing polynomial* of $f$ on the cube $Q$ with degree at most $s$ by $P^s_Q f$, namely, for all multi-indices $\theta \in \mathbb{Z}^n_+$ with $0 \le |\theta| \le s$,

$$\int_Q [f(x) - P^s_Q f(x)] x^\theta \, dx = 0. \tag{7.1}$$

It is well known that if $f$ is locally integrable, then $P^s_Q f$ uniquely exists; see, for example, [48]. Now, we introduce the BMO-type space $\mathrm{bmo}^q_{\rho,\omega}(\mathbb{R}^n)$.

DEFINITION 7.1. Let $\Phi$ satisfy Assumption (A), $\omega \in A^{\mathrm{loc}}_\infty(\mathbb{R}^n)$, and $q_\omega$, $p_\Phi$ and $\rho$ be respectively as in (2.4), (2.6) and (2.7). Let $q \in [1, q_\omega/(q_\omega - 1))$ and $s \in \mathbb{Z}_+$ with $s \ge \lfloor n(q_\omega/p_\Phi - 1)\rfloor$. When $\omega(\mathbb{R}^n) = \infty$, a locally integrable function $f$ on $\mathbb{R}^n$ is said to belong to the *space* $\mathrm{bmo}^q_{\rho,\omega}(\mathbb{R}^n)$ if

$$\begin{aligned}\|f\|_{\mathrm{bmo}^q_{\rho,\omega}(\mathbb{R}^n)} \equiv & \sup_{Q \subset \mathbb{R}^n, |Q|<1} \frac{1}{\rho(\omega(Q))} \left\{ \frac{1}{\omega(Q)} \int_Q |f(x) - P^s_Q f(x)|^q [\omega(x)]^{1-q} \, dx \right\}^{1/q} \\ & + \sup_{Q \subset \mathbb{R}^n, |Q|\ge 1} \frac{1}{\rho(\omega(Q))} \left\{ \frac{1}{\omega(Q)} \int_Q |f(x)|^q [\omega(x)]^{1-q} \, dx \right\}^{1/q} < \infty,\end{aligned}$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ and $P^s_Q f$ is as in (7.1). When $\omega(\mathbb{R}^n) < \infty$, a function $f$ on $\mathbb{R}^n$ is said to belong to the *space* $\mathrm{bmo}^q_{\rho,\omega}(\mathbb{R}^n)$ if

$$\begin{aligned}\|f\|_{\mathrm{bmo}^q_{\rho,\omega}(\mathbb{R}^n)} \equiv & \sup_{Q \subset \mathbb{R}^n, |Q|<1} \frac{1}{\rho(\omega(Q))} \left\{ \frac{1}{\omega(Q)} \int_Q |f(x) - P^s_Q f(x)|^q [\omega(x)]^{1-q} \, dx \right\}^{1/q} \\ & + \sup_{Q \subset \mathbb{R}^n, |Q|\ge 1} \frac{1}{\rho(\omega(Q))} \left\{ \frac{1}{\omega(Q)} \int_Q |f(x)|^q [\omega(x)]^{1-q} \, dx \right\}^{1/q} \\ & + \frac{1}{\rho(\omega(\mathbb{R}^n))} \left\{ \frac{1}{\omega(\mathbb{R}^n)} \int_{\mathbb{R}^n} |f(x)|^q [\omega(x)]^{1-q} \, dx \right\}^{1/q} < \infty,\end{aligned}$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ and $P^s_Q f$ is as in (7.1).

When $\omega\equiv 1$, $\Phi\equiv t$ for all $t\in(0,\infty)$ and $q=1$, the space $\mathrm{bmo}^q_{\rho,\,\omega}(\mathbb{R}^n)$ is just the space $\mathrm{bmo}(\mathbb{R}^n)$ introduced in [18].

Now, we establish the duality between $h^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)$ and $\mathrm{bmo}^{q'}_{\rho,\,\omega}(\mathbb{R}^n)$. We begin with the notion of the weighted atomic Orlicz–Hardy space $H^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)$.

Definition 7.2. Let $\Phi$ satisfy Assumption (A), $\omega\in A^{\mathrm{loc}}_{\infty}(\mathbb{R}^n)$, $\rho$ be as in (2.7) and $(\rho,\,q,\,s)_\omega$ be admissible. A function $a$ on $\mathbb{R}^n$ is called an $H^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)$-*atom* if there exists a cube $Q\subset\mathbb{R}^n$ such that

(i) $\mathrm{supp}(a)\subset Q$;
(ii) $\|a\|_{L^q_\omega(\mathbb{R}^n)}\le[\omega(Q)]^{1/q-1}[\rho(\omega(Q))]^{-1}$;
(iii) $\int_{\mathbb{R}^n}a(x)x^\alpha\,dx=0$ for all multi-indices $\alpha\in\mathbb{Z}^n_+$ with $|\alpha|\le s$.

The *weighted atomic Orlicz–Hardy space* $H^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)$ is defined to be the space of all $f\in\mathcal{D}'(\mathbb{R}^n)$ such that $f=\sum_{i=1}^\infty\lambda_i a_i$ in $\mathcal{D}'(\mathbb{R}^n)$, where $\{a_i\}_{i\in\mathbb{N}}$ are $H^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)$-atoms with $\mathrm{supp}(a_i)\subset Q_i$, and $\{\lambda_i\}_{i\in\mathbb{N}}\subset\mathbb{C}$, satisfying

$$\sum_{i=1}^\infty\omega(Q_i)\Phi\left(\frac{|\lambda_i|}{\omega(Q_i)\rho(\omega(Q_i))}\right)<\infty.$$

Moreover, the quasi-norm of $f\in H^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)$ is defined by

$$\|f\|_{H^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)}\equiv\inf\{\Lambda(\{\lambda_i a_i\}_{i=1}^\infty)\},$$

where the infimum is taken over all the decompositions of $f$ as above and

$$\Lambda(\{\lambda_i a_i\}_{i=1}^\infty)\equiv\inf\left\{\lambda\in(0,\infty):\sum_{i=1}^\infty\omega(Q_i)\Phi\left(\frac{|\lambda_i|}{\lambda\omega(Q_i)\rho(\omega(Q_i))}\right)\le 1\right\}.$$

Furthermore, $H^{\rho,\,q,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)$ is defined to be the set of all finite linear combinations of $H^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)$-atoms.

Obviously, $H^{\rho,\,q,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)$ is dense in the space $H^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)$ with respect to the quasi-norm $\|\cdot\|_{H^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)}$.

Definition 7.3. Let $\Phi$ satisfy Assumption (A), $\omega\in A^{\mathrm{loc}}_{\infty}(\mathbb{R}^n)$, $q_\omega$, $p_\Phi$ and $\rho$ be respectively as in (2.4), (2.6) and (2.7). Let $q\in[1,q_\omega/(q_\omega-1))$ and $s\in\mathbb{Z}_+$ with $s\ge\lfloor n(q_\omega/p_\Phi-1)\rfloor$. A locally integrable function $f$ on $\mathbb{R}^n$ is said to belong to the *space* $\mathrm{BMO}^q_{\rho,\,\omega}(\mathbb{R}^n)$ if

$$\|f\|_{\mathrm{BMO}^q_{\rho,\,\omega}(\mathbb{R}^n)}\equiv\sup_{Q\subset\mathbb{R}^n}\frac{1}{\rho(\omega(Q))}\left\{\frac{1}{\omega(Q)}\int_Q|f(x)-P^s_Qf(x)|^q[\omega(x)]^{1-q}\,dx\right\}^{1/q}<\infty,$$

where the supremum is taken over all cubes $Q\subset\mathbb{R}^n$ and $P^s_Qf$ is as in (7.1).

Now, we establish the duality between $H^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)$ and $\mathrm{BMO}^{q'}_{\rho,\,\omega}(\mathbb{R}^n)$.

Lemma 7.4. *Let $\Phi$ satisfy Assumption* (A), *$\omega\in A^{\mathrm{loc}}_{\infty}(\mathbb{R}^n)$, $q_\omega$ and $\rho$ be respectively as in* (2.4) *and* (2.7), *and $(\rho,\,q,\,s)_\omega$ be admissible. Then $[H^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)]^*$, the dual space of $H^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)$, coincides with $\mathrm{BMO}^{q'}_{\rho,\,\omega}(\mathbb{R}^n)$ in the following sense.*

(i) *Let $g\in\mathrm{BMO}^{q'}_{\rho,\,\omega}(\mathbb{R}^n)$. Then the linear functional $L$, which is initially defined on $H^{\rho,\,q,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)$ by*

$$L(f)=\langle g,f\rangle,\tag{7.2}$$

*has a unique extension to* $H^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)$ *with*

$$\|L\|_{[H^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)]^*} \le C\|g\|_{\mathrm{BMO}^{q'}_{\rho,\,\omega}(\mathbb{R}^n)},$$

*where* $C$ *is a positive constant independent of* $g$.

(ii) *Conversely, for any* $L \in [H^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)]^*$, *there exists* $g \in \mathrm{BMO}^{q'}_{\rho,\,\omega}(\mathbb{R}^n)$ *such that* (7.2) *holds for all* $f \in H^{\rho,\,q,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)$ *and*

$$\|g\|_{\mathrm{BMO}^{q'}_{\rho,\,\omega}(\mathbb{R}^n)} \le C\|L\|_{[H^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)]^*},$$

*where* $C$ *is a positive constant independent of* $L$.

*Proof.* We borrow some ideas from [48] and [33, Theorem 4.1]. Let $(\rho,\,q,\,s)_\omega$ be an admissible triplet. First, we prove (i). Let $a$ be an $H^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)$-atom with $\mathrm{supp}(a) \subset Q$ and $g \in \mathrm{BMO}^{q'}_{\rho,\,\omega}(\mathbb{R}^n)$. Then by the vanishing condition of $a$ and Hölder's inequality, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^n} a(x)g(x)\,dx\right| &= \left|\int_{\mathbb{R}^n} a(x)[g(x) - P^s_Q g(x)]\,dx\right|\\ &\le \|a\|_{L^q_\omega(\mathbb{R}^n)}\left\{\int_Q |g(x) - P^s_Q g(x)|^{q'}[\omega(x)]^{1-q'}\,dx\right\}^{1/q'}\\ &\le \|g\|_{\mathrm{BMO}^{q'}_{\rho,\,\omega}(\mathbb{R}^n)},\end{aligned} \tag{7.3}$$

where $P^s_Q g$ is as in (7.1). Let

$$f = \sum_{i=1}^{k_0} \lambda_i a_i \in H^{\rho,\,q,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n),$$

where $k_0 \in \mathbb{N}$, $\{\lambda_i\}_{i=1}^{k_0} \subset \mathbb{C}$ and for $i \in \{1, \ldots, k_0\}$, $a_i$ is an $H^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)$-atom with $\mathrm{supp}(a_i) \subset Q_i$. Since $\Phi$ is concave and has upper type 1, by Remark 3.6(iii), we know that $\sum_i |\lambda_i| \lesssim \Lambda(\{\lambda_i a_i\}_{i=1}^{k_0})$, which together with (7.3) implies that

$$\begin{aligned}\left|\int_{\mathbb{R}^n} f(x)g(x)\,dx\right| &\le \sum_{i=1}^{k_0}|\lambda_i|\left|\int_{Q_i} a_i(x)g(x)\,dx\right|\\ &\le \left\{\sum_{i=1}^{k_0}|\lambda_i|\right\}\|g\|_{\mathrm{BMO}^{q'}_{\rho,\omega}(\mathbb{R}^n)} \lesssim \Lambda(\{\lambda_i a_i\}_{i=1}^{k_0})\|g\|_{\mathrm{BMO}^{q'}_{\rho,\omega}(\mathbb{R}^n)}.\end{aligned}$$

Thus, by the above estimate and the fact that $H^{\rho,\,q,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)$ is dense in $H^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)$ with respect to the quasi-norm $\|\cdot\|_{H^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)}$, we find that (i) holds.

To prove (ii), assume that $L \in [H^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)]^*$. Let $Q \subset \mathbb{R}^n$ be a closed cube and

$$L^q_{\omega,\,s}(Q) \equiv \left\{f \in L^q_\omega(Q) :\ \int_Q f(x)x^\alpha\,dx = 0,\ \alpha \in \mathbb{Z}^n_+,\ |\alpha| \le s\right\},$$

where $f \in L^q_\omega(Q)$ means that $f \in L^q_\omega(\mathbb{R}^n)$ and $\mathrm{supp}(f) \subset Q$. We first prove that

$$[H^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)]^* \subset [L^q_{\omega,\,s}(Q)]^*. \tag{7.4}$$

Obviously, for any given $f \in L^q_{\omega,\,s}(Q)$,

$$a \equiv [\omega(Q)]^{1/q-1}[\rho(\omega(Q))]^{-1}\|f\|^{-1}_{L^q_\omega(Q)} f$$

is an $H^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)$-atom. Thus, $f\in H^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)$ and

$$\|f\|_{H^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)}\le[\omega(Q)]^{1/q'}\rho(\omega(Q))\|f\|_{L^q_\omega(Q)},$$

which implies that for all $f\in L^q_{\omega,\,s}(Q)$,

$$|Lf|\le\|L\|\,\|f\|_{H^{\rho,\,q,\,s}_{\omega}(\mathbb{R}^n)}\le[\omega(Q)]^{1/q'}\rho(\omega(Q))\|L\|\,\|f\|_{L^q_\omega(Q)}.$$

That is, $L\in[L^q_{\omega,\,s}(Q)]^*$. Thus, (7.4) holds.

From (7.4), the Hahn–Banach theorem and the Riesz representation theorem, it follows that there exists a $\widetilde{g}\in L^{q'}_{\omega}(Q)$ such that for all $f\in L^q_{\omega,\,s}(Q)$,

$$Lf=\int_Q f(x)\widetilde{g}(x)\omega(x)\,dx, \tag{7.5}$$

where when $q=\infty$, we used the fact that $L^\infty_{\omega,\,s}(Q)\subset L^\gamma_{\omega,\,s}(Q)$ for any $\gamma\in[1,\infty)$ and $L^{\gamma'}_{\omega,\,s}(Q)\subset L^1_{\omega,\,s}(Q)$. Taking a sequence $\{Q_j\}_{j\in\mathbb{N}}$ of cubes such that for any $j\in\mathbb{N}$, $Q_j\subset Q_{j+1}$ and $\lim_{j\to\infty}Q_j=\mathbb{R}^n$. From the above result, it follows that for each $Q_j$, there exists a $\widetilde{g}_j\in L^{q'}_{\omega}(Q_j)$ such that for all $f\in L^q_{\omega,\,s}(Q_j)$,

$$Lf=\int_{Q_j} f(x)\widetilde{g}_j(x)\omega(x)\,dx. \tag{7.6}$$

Now, we construct a function $g$ such that

$$Lf=\int_{Q_j} f(x)g(x)\,dx$$

for all $f\in L^q_{\omega,\,s}(Q_j)$ and all $j\in\mathbb{N}$. First, assume that $f\in L^q_{\omega,\,s}(Q_1)$. By (7.6), we know that there exists a $\widetilde{g}_1\in L^{q'}_{\omega}(Q_1)$ such that

$$Lf=\int_{Q_1} f(x)\widetilde{g}_1(x)\omega(x)\,dx.$$

Notice that $f\in L^q_{\omega,\,s}(Q_1)\subset L^q_{\omega,\,s}(Q_2)$. By (7.6) again, there exists a $\widetilde{g}_2\in L^{q'}_{\omega}(Q_2)$ such that

$$Lf=\int_{Q_1} f(x)\widetilde{g}_1(x)\omega(x)\,dx=\int_{Q_2} f(x)\widetilde{g}_2(x)\omega(x)\,dx,$$

which implies that for all $f\in L^q_{\omega,\,s}(Q_1)$,

$$\int_{Q_1} f(x)[\widetilde{g}_1(x)-\widetilde{g}_2(x)]\omega(x)\,dx=0. \tag{7.7}$$

For any given $h\in L^q_\omega(Q_1)$, let $f_1\equiv h-P^s_{Q_1}h$. Then by (7.1), we know that $f_1\in L^q_{\omega,\,s}(Q_1)$. For $f_1$, by (7.7), we have

$$\int_{Q_1}[h(x)-P^s_{Q_1}h(x)][\widetilde{g}_1(x)-\widetilde{g}_2(x)]\omega(x)\,dx=0,$$

which combined with the well-known fact that

$$\int_{Q_1}P^s_{Q_1}h(x)[\widetilde{g}_1(x)-\widetilde{g}_2(x)]\omega(x)\,dx=\int_{Q_1}h(x)P^s_{Q_1}((\widetilde{g}_1-\widetilde{g}_2)\omega)(x)\,dx$$

implies that

$$\int_{Q_1}h(x)\{[\widetilde{g}_1(x)-\widetilde{g}_2(x)]\omega(x)-P^s_{Q_1}((\widetilde{g}_1-\widetilde{g}_2)\omega)(x)\}\,dx=0. \tag{7.8}$$

For $j = 1, 2$, let $g_j \equiv \widetilde{g}_j \omega$. By (7.8), we know that for all $h \in L^q_\omega(Q_1)$,

$$\int_{Q_1} h(x)\bigg\{\frac{[g_1(x) - g_2(x)] - P^s_{Q_1}(g_1 - g_2)(x)}{\omega(x)}\bigg\}\omega(x)\,dx = 0,$$

which implies that for almost every $x \in Q_1$,

$$g_1(x) - g_2(x) = P^s_{Q_1}(g_1 - g_2)(x).$$

Let

$$g(x) \equiv \begin{cases} g_1(x) & \text{when } x \in Q_1, \\ g_1(x) + P^s_{Q_1}(g_1 - g_2)(x) & \text{when } x \in Q_2 \setminus Q_1. \end{cases}$$

It is easy to see that for any $f \in L^q_{\omega,\,s}(Q_j)$ with $j \in \{1, 2\}$,

$$Lf = \int_{Q_j} f(x)g(x)\,dx. \tag{7.9}$$

In this way, we obtain a function $g$ on $\mathbb{R}^n$ such that (7.9) holds for any $j \in \mathbb{N}$.

Finally, we show that $g \in \mathrm{BMO}^{q'}_{\rho,\,\omega}(\mathbb{R}^n)$ and for all $f \in H^{\rho,\,q,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)$,

$$Lf = \int_{\mathbb{R}^n} f(x)g(x)\,dx. \tag{7.10}$$

Indeed, for any $H^{\rho,\,q,\,s}_\omega(\mathbb{R}^n)$-atom $a$, there exists a $j_0 \in \mathbb{N}$ such that $a \in L^q_{\omega,\,s}(Q_{j_0})$. From this and the fact that (7.9) holds for any $j \in \mathbb{N}$, we see that (7.10) holds for any $f \in H^{\rho,\,q,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)$. It remains to prove that $g \in \mathrm{BMO}^{q'}_{\rho,\,\omega}(\mathbb{R}^n)$. Take any cube $Q \subset \mathbb{R}^n$ as well as any $f \in L^q_\omega(Q)$ satisfying $\|f\|_{L^q_\omega(Q)} \le 1$ and $\mathrm{supp}(f) \subset Q$. Let

$$a \equiv \widetilde{C}^{-1}[\omega(Q)]^{1/q'}[\rho(\omega(Q))]^{-1}(f - P^s_Q f)\chi_Q, \tag{7.11}$$

where $\widetilde{C}$ is a positive constant. Obviously, $\mathrm{supp}(a) \subset Q$. We choose $\widetilde{C}$ such that $a$ becomes an $H^{\rho,\,q,\,s}_\omega(\mathbb{R}^n)$-atom. From the equality

$$La = \int_Q a(x)g(x)\,dx$$

and $L \in [H^{\rho,\,q,\,s}_\omega(\mathbb{R}^n)]^*$, it follows that

$$|La| = \left|\int_Q a(x)[g(x) - P^s_Q g(x)]\,dx\right| \le \|L\|_{[H^{\rho,\,q,\,s}_\omega(\mathbb{R}^n)]^*}. \tag{7.12}$$

By (7.11), (7.12) and (7.1), for all $f \in L^q_\omega(Q)$ with $\|f\|_{L^q_\omega(Q)} \le 1$, we see that

$$[\omega(Q)]^{1/q'}[\rho(\omega(Q))]^{-1}\left|\int_Q f(x)[g(x) - P^s_Q g(x)]\,dx\right| \lesssim \|L\|_{[H^{\rho,\,q,\,s}_\omega(\mathbb{R}^n)]^*},$$

which implies that

$$[\omega(Q)]^{1/q'}[\rho(\omega(Q))]^{-1}\bigg\{\int_Q |g(x) - P^s_Q g(x)|^{q'}[\omega(x)]^{1-q'}\,dx\bigg\}^{1/q'} \lesssim \|L\|_{[H^{\rho,\,q,\,s}_\omega(\mathbb{R}^n)]^*}.$$

Thus, $g \in \mathrm{BMO}^{q'}_{\rho,\,\omega}(\mathbb{R}^n)$ and $\|g\|_{\mathrm{BMO}^{q'}_{\rho,\,\omega}(\mathbb{R}^n)} \lesssim \|L\|_{[H^{\rho,\,q,\,s}_\omega(\mathbb{R}^n)]^*}$. This finishes the proof of Lemma 7.4. ∎

Now, we give the duality between $h^{\rho,\,q,\,s}_\omega(\mathbb{R}^n)$ and $\mathrm{bmo}^{q'}_{\rho,\,\omega}(\mathbb{R}^n)$ by invoking Lemma 7.4.

Theorem 7.5. *Let $\Phi$ satisfy Assumption* (A), $\omega \in A_\infty^{\mathrm{loc}}(\mathbb{R}^n)$, $q_\omega$ *and* $\rho$ *be respectively as in* (2.4) *and* (2.7), *and* $(\rho,\, q,\, s)_\omega$ *be admissible. Then* $[h_\omega^{\rho,\, q,\, s}(\mathbb{R}^n)]^*$, *the dual space of* $h_\omega^{\rho,\, q,\, s}(\mathbb{R}^n)$, *coincides with* $\mathrm{bmo}_{\rho,\,\omega}^{q'}(\mathbb{R}^n)$ *in the following sense.*

(i) *Let* $g \in \mathrm{bmo}_{\rho,\,\omega}^{q'}(\mathbb{R}^n)$. *Then the linear functional* $L$, *which is initially defined on* $h_{\omega,\,\mathrm{fin}}^{\rho,\, q,\, s}(\mathbb{R}^n)$ *by*

$$L(f) = \langle g, f\rangle, \tag{7.13}$$

*has a unique extension to* $h_\omega^{\rho,\, q,\, s}(\mathbb{R}^n)$ *with*

$$\|L\|_{[h_\omega^{\rho,\, q,\, s}(\mathbb{R}^n)]^*} \le C\|g\|_{\mathrm{bmo}_{\rho,\,\omega}^{q'}(\mathbb{R}^n)},$$

*where* $C$ *is a positive constant independent of* $g$.

(ii) *Conversely, for any* $L \in [h_\omega^{\rho,\, q,\, s}(\mathbb{R}^n)]^*$, *there exists* $g \in \mathrm{bmo}_{\rho,\,\omega}^{q'}(\mathbb{R}^n)$ *such that* (7.13) *holds for all* $f \in h_{\omega,\,\mathrm{fin}}^{\rho,\, q,\, s}(\mathbb{R}^n)$ *and*

$$\|g\|_{\mathrm{bmo}_{\rho,\,\omega}^{q'}(\mathbb{R}^n)} \le C\|L\|_{[h_\omega^{\rho,\, q,\, s}(\mathbb{R}^n)]^*},$$

*where* $C$ *is a positive constant independent of* $L$.

*Proof.* Let $(\rho,\, q,\, s)_\omega$ be an admissible triplet. Obviously, the proof of (i) is similar to the proof of Lemma 7.4(i). We omit the details.

Now, we prove (ii) by considering the following two cases for $\omega$.

*Case I:* $\omega(\mathbb{R}^n) = \infty$. In this case, let $Q \subset \mathbb{R}^n$ be a cube with $l(Q) \in [1,\infty)$. We first prove that

$$[h_\omega^{\rho,\, q,\, s}(\mathbb{R}^n)]^* \subset [L_\omega^q(Q)]^*. \tag{7.14}$$

Obviously, for any given $f \in L_\omega^q(Q)$,

$$a \equiv [\omega(Q)]^{1/q-1}[\rho(\omega(Q))]^{-1}\|f\|_{L_\omega^q(Q)}^{-1} f\chi_Q$$

is a $(\rho,\, q,\, s)_\omega$-atom. Thus, $f \in h_\omega^{\rho,\, q,\, s}(\mathbb{R}^n)$ and

$$\|f\|_{h_\omega^{\rho,\, q,\, s}(\mathbb{R}^n)} \le [\omega(Q)]^{1/q'}\rho(\omega(Q))\|f\|_{L_\omega^q(Q)},$$

which implies that for any $L \in [h_\omega^{\rho,\, q,\, s}(\mathbb{R}^n)]^*$,

$$|Lf| \le \|L\|_{[h_\omega^{\rho,\, q,\, s}(\mathbb{R}^n)]^*}\|f\|_{h_\omega^{\rho,\, q,\, s}(\mathbb{R}^n)} \le [\omega(Q)]^{1/q'}\rho(\omega(Q))\|f\|_{L_\omega^q(Q)}\|L\|_{[h_\omega^{\rho,\, q,\, s}(\mathbb{R}^n)]^*}.$$

That is $L \in [L_\omega^q(Q)]^*$. Thus, (7.14) holds.

Now, assume that $L \in [h_\omega^{\rho,\, q,\, s}(\mathbb{R}^n)]^*$. Similarly to the proof of (7.5), we know that there exists a $\widetilde{g} \in L_\omega^{q'}(Q)$ such that for all $f \in L_\omega^q(Q)$,

$$Lf = \int_Q f(x)\widetilde{g}(x)\omega(x)\,dx.$$

Take a sequence $\{Q_j\}_{j\in\mathbb{N}}$ of cubes such that for any $j \in \mathbb{N}$, $Q_j \subset Q_{j+1}$, $\lim_{j\to\infty} Q_j = \mathbb{R}^n$ and $l(Q_1) \in [1,\infty)$. From the above result, it follows that for each $Q_j$, there exists a $\widetilde{g}_j \in L_\omega^{q'}(Q_j)$ such that for all $f \in L_\omega^q(Q_j)$,

$$Lf = \int_{Q_j} f(x)\widetilde{g}_j(x)\omega(x)\,dx. \tag{7.15}$$

Now, we construct a function $g$ on $\mathbb{R}^n$ such that

$$Lf = \int_{Q_j} f(x)g(x)\,dx$$

for all $f \in L^q_\omega(Q_j)$ and $j \in \mathbb{N}$. First, assume that $f \in L^q_\omega(Q_1)$. By (7.15), we know that there exists a $\widetilde{g}_1 \in L^{q'}_\omega(Q_1)$ such that

$$Lf = \int_{Q_1} f(x)\widetilde{g}_1(x)\omega(x)\,dx.$$

Notice that $f \in L^q_\omega(Q_1) \subset L^q_\omega(Q_2)$. By (7.15) again, there exists a $\widetilde{g}_2 \in L^{q'}_\omega(Q_2)$ such that

$$Lf = \int_{Q_1} f(x)\widetilde{g}_1(x)\omega(x)\,dx = \int_{Q_2} f(x)\widetilde{g}_2(x)\omega(x)\,dx,$$

which implies that for all $f \in L^q_\omega(Q_1)$,

$$\int_{Q_1} f(x)[\widetilde{g}_1(x) - \widetilde{g}_2(x)]\omega(x)\,dx = 0.$$

Thus, for almost every $x \in Q_1$, $\widetilde{g}_1(x) = \widetilde{g}_2(x)$. For $j = 1,\ 2$, let $g_j \equiv \widetilde{g}_j\omega$ and

$$g(x) \equiv \begin{cases} g_1(x) & \text{when } x \in Q_1, \\ g_2(x) & \text{when } x \in Q_2 \setminus Q_1. \end{cases}$$

It is easy to see that for all $f \in L^q_\omega(Q_j)$ with $j \in \{1, 2\}$,

$$Lf = \int_{Q_j} f(x)g_j(x)\,dx. \tag{7.16}$$

Continuing in this way, we obtain a function $g$ on $\mathbb{R}^n$ such that (7.16) holds for all $j \in \mathbb{N}$.

Finally, we show that $g \in \mathrm{bmo}^{q'}_{\rho,\,\omega}(\mathbb{R}^n)$ and for all $f \in h^{\rho,\,q,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)$,

$$Lf = \int_{\mathbb{R}^n} f(x)g(x)\,dx. \tag{7.17}$$

Indeed, since $\omega(\mathbb{R}^n) = \infty$, all $(\rho,\ q)_\omega$-single-atoms are 0, and for any $(\rho,\ q,\ s)_\omega$-atom $a$, there exists a $j_0 \in \mathbb{N}$ such that $a \in L^q_\omega(Q_{j_0})$. From this and the fact that (7.16) holds for all $j \in \mathbb{N}$, we see that (7.17) holds.

Now, we prove that $g \in \mathrm{bmo}^{q'}_{\rho,\,\omega}(\mathbb{R}^n)$. Take any cube $Q \subset \mathbb{R}^n$ with $l(Q) \in [1, \infty)$ as well as any $f \in L^q_\omega(Q)$ with $\|f\|_{L^q_\omega(Q)} \le 1$. Let

$$a \equiv [\omega(Q)]^{-1/q'}[\rho(\omega(Q))]^{-1} f\chi_Q.$$

Then $a$ is a $(\rho,\ q,\ s)_\omega$-atom and $\mathrm{supp}(a) \subset Q$. From the equality

$$La = \int_Q a(x)g(x)\,dx$$

and $L \in [h^{\rho,\,q,\,s}_\omega(\mathbb{R}^n)]^*$, we deduce that

$$|La| = \left|\int_Q a(x)g(x)\,dx\right| \le \|L\|_{[h^{\rho,\,q,\,s}_\omega(\mathbb{R}^n)]^*}.$$

Thus, for any $f \in L^q_\omega(Q)$ with $\|f\|_{L^q_\omega(Q)} \le 1$, we have

$$[\omega(Q)]^{-1/q'}[\rho(\omega(Q))]^{-1}\left|\int_Q f(x)g(x)\,dx\right| \le \|L\|_{[h^{\rho,\,q,\,s}_\omega(\mathbb{R}^n)]^*},$$

which implies that

$$[\omega(Q)]^{-1/q'}[\rho(\omega(Q))]^{-1}\left\{\int_Q |g(x)|^{q'}[\omega(x)]^{1-q'}\,dx\right\}^{1/q'} \le \|L\|_{[h^{\rho,\,q,\,s}_\omega(\mathbb{R}^n)]^*}. \quad (7.18)$$

Furthermore, from $h^{\rho,\,q,\,s}_\omega(\mathbb{R}^n) \supset H^{\rho,\,q,\,s}_\omega(\mathbb{R}^n)$ and

$$\|f\|_{h^{\rho,\,q,\,s}_\omega(\mathbb{R}^n)} \le \|f\|_{H^{\rho,\,q,\,s}_\omega(\mathbb{R}^n)}$$

for all $f \in H^{\rho,\,q,\,s}_\omega(\mathbb{R}^n)$, we deduce that

$$[h^{\rho,\,q,\,s}_\omega(\mathbb{R}^n)]^* \subset [H^{\rho,\,q,\,s}_\omega(\mathbb{R}^n)]^*$$

and $L|_{H^{\rho,\,q,\,s}_\omega(\mathbb{R}^n)} \in [H^{\rho,\,q,\,s}_\omega(\mathbb{R}^n)]^*$. Since (7.17) holds for all $f \in H^{\rho,\,q,\,s}_{\omega,\,\mathrm{fin}}(\mathbb{R}^n)$, by Lemma 7.4(ii) we know that $g \in \mathrm{BMO}^{q'}_{\rho,\,\omega}(\mathbb{R}^n)$ and

$$\|g\|_{\mathrm{BMO}^{q'}_{\rho,\,\omega}(\mathbb{R}^n)} \lesssim \|L\,|_{H^{\rho,\,q,\,s}_\omega(\mathbb{R}^n)}\,\|_{[H^{\rho,\,q,\,s}_\omega(\mathbb{R}^n)]^*} \lesssim \|L\|_{[h^{\rho,\,q,\,s}_\omega(\mathbb{R}^n)]^*}.$$

Thus, this estimate together with (7.18) implies that $g \in \mathrm{bmo}^{q'}_{\rho,\,\omega}(\mathbb{R}^n)$ and

$$\|g\|_{\mathrm{bmo}^{q'}_{\rho,\,\omega}(\mathbb{R}^n)} \lesssim \|L\|_{[h^{\rho,\,q,\,s}_\omega(\mathbb{R}^n)]^*},$$

which completes the proof of Theorem 7.5(ii) in Case I.

*Case II:* $\omega(\mathbb{R}^n) < \infty$. In this case, let

$$\widetilde{h^{\rho,\,q,\,s}_\omega}(\mathbb{R}^n) \equiv \left\{f = \sum_{i=1}^\infty \lambda_i a_i \text{ in } \mathcal{D}'(\mathbb{R}^n):\ \text{for } i \in \mathbb{N},\ a_i \text{ is a } (\rho,\,q,\,s)_\omega\text{-atom}, \right.$$
$$\left. \mathrm{supp}(a_i) \subset Q_i,\ \lambda_i \in \mathbb{C} \text{ and } \sum_{i=1}^\infty \omega(Q_i)\Phi\left(\frac{|\lambda_i|}{\omega(Q_i)\rho(\omega(Q_i))}\right) < \infty\right\}$$

and for all $f \in \widetilde{h^{\rho,\,q,\,s}_\omega}(\mathbb{R}^n)$,

$$\|f\|_{\widetilde{h^{\rho,\,q,\,s}_\omega}(\mathbb{R}^n)} \equiv \inf\{\Lambda(\{\lambda_i a_i\}_{i=1}^\infty)\},$$

where the infimum is taken over all the decompositions of $f$ as above. For any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, let

$$\|f\|_{\widetilde{\mathrm{bmo}^{q'}_{\rho,\,\omega}}(\mathbb{R}^n)} \equiv \sup_{Q\subset\mathbb{R}^n,\,|Q|<1} \frac{1}{\rho(\omega(Q))}\left\{\frac{1}{\omega(Q)}\int_Q |f(x) - P^s_Q f(x)|^{q'}[\omega(x)]^{1-q'}\,dx\right\}^{1/q'}$$
$$+ \sup_{Q\subset\mathbb{R}^n,\,|Q|\ge 1} \frac{1}{\rho(\omega(Q))}\left\{\frac{1}{\omega(Q)}\int_Q |f(x)|^{q'}[\omega(x)]^{1-q'}\,dx\right\}^{1/q'}$$

and

$$\widetilde{\mathrm{bmo}^{q'}_{\rho,\,\omega}}(\mathbb{R}^n) \equiv \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n):\ \|f\|_{\widetilde{\mathrm{bmo}^{q'}_{\rho,\,\omega}}(\mathbb{R}^n)} < \infty\}.$$

Similarly to the proofs of (i) and Case I, we conclude that

$$[\widetilde{h^{\rho,\,q,\,s}_\omega}(\mathbb{R}^n)]^* = \widetilde{\mathrm{bmo}^{q'}_{\rho,\,\omega}}(\mathbb{R}^n). \quad (7.19)$$

Now we claim that

$$[h_\omega^{\rho,\,q,\,s}(\mathbb{R}^n)]^* \subset [L_\omega^q(\mathbb{R}^n)]^*. \tag{7.20}$$

Indeed, for any $f \in L_\omega^q(\mathbb{R}^n)$, let

$$a \equiv [\omega(\mathbb{R}^n)]^{1/q-1}[\rho(\omega(\mathbb{R}^n))]^{-1}\|f\|_{L_\omega^q(\mathbb{R}^n)}^{-1} f.$$

Then $a$ is a $(\rho,\,q)_\omega$-single-atom, which implies that $f \in h_\omega^{\rho,\,q,\,s}(\mathbb{R}^n)$ and

$$\|f\|_{h_\omega^{\rho,\,q,\,s}(\mathbb{R}^n)} \le [\omega(\mathbb{R}^n)]^{1/q'}\rho(\omega(\mathbb{R}^n))\|f\|_{L_\omega^q(\mathbb{R}^n)}.$$

Thus, for any given $L \in [h_\omega^{\rho,\,q,\,s}(\mathbb{R}^n)]^*$ and all $f \in L_\omega^q(\mathbb{R}^n)$, we have

$$|Lf| \le \|L\|_{[h_\omega^{\rho,\,q,\,s}(\mathbb{R}^n)]^*}\|f\|_{h_\omega^{\rho,\,q,\,s}(\mathbb{R}^n)} \le [\omega(\mathbb{R}^n)]^{1/q'}\rho(\omega(\mathbb{R}^n))\|f\|_{L_\omega^q(\mathbb{R}^n)}\|L\|_{[h_\omega^{\rho,\,q,\,s}(\mathbb{R}^n)]^*}.$$

That is, $L \in [L_\omega^q(\mathbb{R}^n)]^*$. Thus, (7.20) holds.

Now, assume that $L \in [h_\omega^{\rho,\,q,\,s}(\mathbb{R}^n)]^*$. From $\omega(\mathbb{R}^n) < \infty$, it follows that

$$L_\omega^\infty(\mathbb{R}^n) \subset L_\omega^\gamma(\mathbb{R}^n)$$

for any $\gamma \in [1,\infty)$ and $L_\omega^{\gamma'}(\mathbb{R}^n) \subset L_\omega^1(\mathbb{R}^n)$. From this, (7.20), the Hahn–Banach theorem and the Riesz representation theorem, we conclude that there exists a $\widetilde{g} \in L_\omega^{q'}(\mathbb{R}^n)$ such that for all $f \in L_\omega^q(\mathbb{R}^n)$ with $q \in (q_\omega,\infty]$,

$$Lf = \int_{\mathbb{R}^n} f(x)\widetilde{g}(x)\omega(x)\,dx.$$

Let $g \equiv \widetilde{g}\omega$. Then for all $f \in L_\omega^q(\mathbb{R}^n)$,

$$Lf = \int_{\mathbb{R}^n} f(x)g(x)\,dx. \tag{7.21}$$

Finally, we prove that $g \in \mathrm{bmo}_{\rho,\,\omega}^{q'}(\mathbb{R}^n)$ and

$$\|g\|_{\mathrm{bmo}_{\rho,\,\omega}^{q'}(\mathbb{R}^n)} \lesssim \|L\|_{[h_\omega^{\rho,\,q,\,s}(\mathbb{R}^n)]^*}.$$

Obviously, (7.21) holds for all $f \in h_{\omega,\,\mathrm{fin}}^{\rho,\,q,\,s}(\mathbb{R}^n)$. For any $f \in L_\omega^q(\mathbb{R}^n)$ with $\|f\|_{L_\omega^q(\mathbb{R}^n)} \le 1$, let

$$a \equiv [\omega(\mathbb{R}^n)]^{-1/q'}[\rho(\omega(\mathbb{R}^n))]^{-1} f.$$

Then $a$ is a $(\rho,\,q)_\omega$-single-atom. From (7.21) with $f \equiv a$ and $L \in [h_\omega^{\rho,\,q,\,s}(\mathbb{R}^n)]^*$, we deduce that

$$|La| = \left|\int_{\mathbb{R}^n} a(x)g(x)\,dx\right| \le \|L\|_{[h_\omega^{\rho,\,q,\,s}(\mathbb{R}^n)]^*}.$$

That is,

$$[\omega(\mathbb{R}^n)]^{-1/q'}[\rho(\omega(\mathbb{R}^n))]^{-1}\left|\int_{\mathbb{R}^n} f(x)g(x)\,dx\right| \le \|L\|_{[h_\omega^{\rho,\,q,\,s}(\mathbb{R}^n)]^*},$$

which together with $\|f\|_{L_\omega^q(\mathbb{R}^n)} \le 1$ implies that

$$[\omega(\mathbb{R}^n)]^{-1/q'}[\rho(\omega(\mathbb{R}^n))]^{-1}\left\{\int_{\mathbb{R}^n} |g(x)|^{q'}[\omega(x)]^{1-q'}\,dx\right\}^{1/q'} \le \|L\|_{[h_\omega^{\rho,\,q,\,s}(\mathbb{R}^n)]^*}. \tag{7.22}$$

Moveover, from $h_\omega^{\rho,\,q,\,s}(\mathbb{R}^n) \supset \widetilde{h_\omega^{\rho,\,q,\,s}}(\mathbb{R}^n)$ and

$$\|f\|_{h_\omega^{\rho,\,q,\,s}(\mathbb{R}^n)} \le \|f\|_{\widetilde{h_\omega^{\rho,\,q,\,s}}(\mathbb{R}^n)}$$

for all $f \in \widetilde{h^{\rho,\,q,\,s}_\omega}(\mathbb{R}^n)$, we conclude that $[h^{\rho,\,q,\,s}_\omega(\mathbb{R}^n)]^* \subset [\widetilde{h^{\rho,\,q,\,s}_\omega}(\mathbb{R}^n)]^*$ and

$$L\,|_{\widetilde{h^{\rho,\,q,\,s}_\omega}(\mathbb{R}^n)} \in [\widetilde{h^{\rho,\,q,\,s}_\omega}(\mathbb{R}^n)]^*.$$

Thus, by (7.19) and (7.21), we know that $g \in \widetilde{\mathrm{bmo}^{q'}_{\rho,\,\omega}}(\mathbb{R}^n)$ and

$$\|g\|_{\widetilde{\mathrm{bmo}^{q'}_{\rho,\,\omega}}(\mathbb{R}^n)} \lesssim \|L\,|_{\widetilde{h^{\rho,\,q,\,s}_\omega}(\mathbb{R}^n)}\|_{[\widetilde{h^{\rho,\,q,\,s}_\omega}(\mathbb{R}^n)]^*} \lesssim \|L\|_{[h^{\rho,\,q,\,s}_\omega(\mathbb{R}^n)]^*},$$

which together with (7.22) implies that $g \in \mathrm{bmo}^{q'}_{\rho,\,\omega}(\mathbb{R}^n)$ and

$$\|g\|_{\mathrm{bmo}^{q'}_{\rho,\,\omega}(\mathbb{R}^n)} \lesssim \|L\|_{[h^{\rho,\,q,\,s}_\omega(\mathbb{R}^n)]^*}.$$

This finishes the proof of Theorem 7.5. ■

When $q=1$, we denote $\mathrm{bmo}^q_{\rho,\,\omega}(\mathbb{R}^n)$ simply by $\mathrm{bmo}_{\rho,\,\omega}(\mathbb{R}^n)$. By Theorems 5.6 and 7.5, we have the following conclusions.

Corollary 7.6. *Let $\Phi$ satisfy Assumption* (A), $\omega \in A^{\mathrm{loc}}_\infty(\mathbb{R}^n)$, *and $q_\omega$ and $\rho$ be respectively as in* (2.4) *and* (2.7). *Then for $q \in [1, q_\omega/(q_\omega - 1))$, $\mathrm{bmo}^q_{\rho,\,\omega}(\mathbb{R}^n) = \mathrm{bmo}_{\rho,\,\omega}(\mathbb{R}^n)$ with equivalent norms.*

Corollary 7.7. *Let $\Phi$ satisfy Assumption* (A), $\omega \in A^{\mathrm{loc}}_\infty(\mathbb{R}^n)$ *and $\rho$ be as in* (2.7). *Then* $[h^\Phi_\omega(\mathbb{R}^n)]^* = \mathrm{bmo}_{\rho,\,\omega}(\mathbb{R}^n)$.

## 8. Some applications

In this section, we first show that local Riesz transforms are bounded on $h^\Phi_\omega(\mathbb{R}^n)$. Moreover, we introduce local fractional integrals and show that they are bounded from $h^p_{\omega^p}(\mathbb{R}^n)$ to $L^q_{\omega^q}(\mathbb{R}^n)$ when $q \in [1,\infty)$, and from $h^p_{\omega^p}(\mathbb{R}^n)$ to $h^q_{\omega^q}(\mathbb{R}^n)$ when $q \in (0,1]$. Finally, we prove that some pseudo-differential operators are bounded on $h^\Phi_\omega(\mathbb{R}^n)$, where $\omega \in A_p(\phi)$, a space introduced by Tang [50] (see also Definition 8.13 below) and contained in $A^{\mathrm{loc}}_p(\mathbb{R}^n)$ for $p \in [1,\infty)$.

Now, we recall the notion of local Riesz transforms introduced by Goldberg [18]. In what follows, $\mathcal{S}(\mathbb{R}^n)$ denotes the space of all Schwartz functions on $\mathbb{R}^n$.

Definition 8.1. Let $\phi_0 \in \mathcal{D}(\mathbb{R}^n)$ be such that $\phi_0 \equiv 1$ on $Q(0,1)$ and $\mathrm{supp}(\phi_0) \subset Q(0,2)$. For $j \in \{1,\dots,n\}$ and $x \in \mathbb{R}^n$, let

$$k_j(x) \equiv \frac{x_j}{|x|^{n+1}}\phi_0(x).$$

For $f \in \mathcal{S}(\mathbb{R}^n)$, the *local Riesz transform* $r_j(f)$ of $f$ is defined by $r_j(f) \equiv k_j * f$.

We remark that in [18] it was assumed that $\phi_0 \equiv 1$ in a neighborhood of the origin and $\phi_0 \in \mathcal{D}(\mathbb{R}^n)$. In this paper, for convenience, we assume $\phi_0 \equiv 1$ on $Q(0,1)$ and $\mathrm{supp}(\phi_0) \subset Q(0,2)$. We prove the boundedness on $h^\Phi_\omega(\mathbb{R}^n)$ of local Riesz transforms $\{r_j\}_j$.

Theorem 8.2. *Let $\Phi$ satisfy Assumption* (A), $\omega \in A^{\mathrm{loc}}_\infty(\mathbb{R}^n)$ *and $p_\Phi$ be as in* (2.6). *For $j \in \{1,\dots,n\}$, let $r_j$ be the local Riesz operator as in Definition* 8.1. *If $p_\Phi = p^+_\Phi$ and $\Phi$ is* *of upper type* $p_\Phi^+$, *then there exists a positive constant* $C_0 \equiv C_0(\Phi, \omega, n)$, *depending only on* $\Phi$, $q_\omega$, *the weight constant of* $\omega$ *and* $n$, *such that for all* $f \in h_\omega^\Phi(\mathbb{R}^n)$,

$$\|r_j(f)\|_{h_\omega^\Phi(\mathbb{R}^n)} \le C_0 \|f\|_{h_\omega^\Phi(\mathbb{R}^n)}.$$

To prove Theorem 8.2, we need the following lemma established in [49, Lemma 8.2].

LEMMA 8.3. *For* $j \in \{1, \dots, n\}$, *let* $r_j$ *be the local Riesz operator as in Definition* 8.1.

(i) *For* $\omega \in A_p^{\mathrm{loc}}(\mathbb{R}^n)$ *with* $p \in (1, \infty)$, *there exists a positive constant*

$$C_1 \equiv C_1(p, \omega, n),$$

*depending only on* $p$, *the weight constant of* $\omega$, *and* $n$, *such that for all* $f \in L_\omega^p(\mathbb{R}^n)$,

$$\|r_j(f)\|_{L_\omega^p(\mathbb{R}^n)} \le C_1 \|f\|_{L_\omega^p(\mathbb{R}^n)}.$$

(ii) *For* $\omega \in A_1^{\mathrm{loc}}(\mathbb{R}^n)$, *there exists a positive constant* $C_2 \equiv C_2(\omega, n)$, *depending only on the weight constant of* $\omega$, *and* $n$, *such that for all* $f \in L_\omega^1(\mathbb{R}^n)$,

$$\|r_j(f)\|_{L_\omega^{1,\infty}(\mathbb{R}^n)} \le C_2 \|f\|_{L_\omega^1(\mathbb{R}^n)}.$$

Now, we prove Theorem 8.2 by using Theorem 6.2 and Lemma 8.3.

*Proof of Theorem 8.2.* Let $s \equiv \lfloor n(q_\omega/p_\Phi - 1)\rfloor$, where $q_\omega$ and $p_\Phi$ are respectively as in (2.4) and (2.6). Then $(n+s+1)p_\Phi > nq_\omega$, which implies that there exists $q \in (q_\omega, \infty)$ such that $(n+s+1)p_\Phi > nq$ and $\omega \in A_q^{\mathrm{loc}}(\mathbb{R}^n)$. To show Theorem 8.2, by Theorem 6.4(i) and Theorem 3.2, it suffices to show that for any $(\rho, q)_\omega$-single-atom $a$ or $(\rho, q, s)_\omega$-atom $a$ supported in $Q(x_0, R_0)$ with $R_0 \in (0, 2]$,

$$\|\mathcal{G}_N^0(r_j(a))\|_{L_\omega^\Phi(\mathbb{R}^n)} \lesssim 1. \tag{8.1}$$

First, we prove (8.1) for any $(\rho, q)_\omega$-single-atom $a \ne 0$. In this case, $\omega(\mathbb{R}^n) < \infty$. Since $\Phi$ is concave, by Jensen's inequality, Hölder's inequality, Proposition 3.2(ii), Lemma 8.3(i) and (2.8) with $t \equiv \omega(\mathbb{R}^n)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Phi(\mathcal{G}_N^0(r_j(a))(x))\omega(x)\,dx \\
&\quad \le \omega(\mathbb{R}^n)\Phi\bigg(\frac{1}{\omega(\mathbb{R}^n)}\int_{\mathbb{R}^n} \mathcal{G}_N^0(r_j(a))(x)\omega(x)\,dx\bigg) \\
&\quad \le \omega(\mathbb{R}^n)\Phi\bigg(\frac{1}{[\omega(\mathbb{R}^n)]^{1/q}}\bigg\{\int_{\mathbb{R}^n} [\mathcal{G}_N^0(r_j(a))(x)]^q\omega(x)\,dx\bigg\}^{1/q}\bigg) \\
&\quad \lesssim \omega(\mathbb{R}^n)\Phi\bigg(\frac{1}{[\omega(\mathbb{R}^n)]^{1/q}}\|r_j(a)\|_{L_\omega^q(\mathbb{R}^n)}\bigg) \lesssim \omega(\mathbb{R}^n)\Phi\bigg(\frac{1}{[\omega(\mathbb{R}^n)]^{1/q}}\|a\|_{L_\omega^q(\mathbb{R}^n)}\bigg) \\
&\quad \lesssim \omega(\mathbb{R}^n)\Phi\bigg(\frac{1}{\omega(\mathbb{R}^n)\rho(\omega(\mathbb{R}^n))}\bigg) \sim 1,
\end{aligned}$$

which implies (8.1) in this case.

Now, let $a$ be any $(\rho, q, s)_\omega$-atom supported in $Q_0 \equiv Q(x_0, R_0)$ with $R_0 \in (0, 2]$. We prove (8.1) for $a$ by considering the following two cases for $R_0$.

*Case 1:* $R_0 \in [1, 2]$. In this case, by the definitions of $r_j(a)$ and $\mathcal{G}_N^0(r_j(a))$, we see that

$$\mathrm{supp}(\mathcal{G}_N^0(r_j(a))) \subset Q_0^* \equiv Q(x_0, R_0 + 8).$$

From this, Jensen's inequality, Hölder's inequality, Proposition 3.2(ii), Lemmas 8.3 and 2.3(v), Remark 2.4 with $\widetilde{C} \equiv 2$ and (2.8) with $t \equiv \omega(Q_0)$, we infer that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Phi(\mathcal{G}_N^0(r_j(a))(x))\omega(x)\,dx \\
&\qquad \le \omega(Q_0^*)\Phi\bigg(\frac{1}{\omega(Q_0^*)}\int_{Q_0^*} \mathcal{G}_N^0(r_j(a))(x)\omega(x)\,dx\bigg) \\
&\qquad \le \omega(Q_0^*)\Phi\bigg(\frac{1}{[\omega(Q_0^*)]^{1/q}}\bigg\{\int_{Q_0^*} [\mathcal{G}_N^0(r_j(a))(x)]^q\omega(x)\,dx\bigg\}^{1/q}\bigg) \\
&\qquad \lesssim \omega(Q_0^*)\Phi\bigg(\frac{1}{[\omega(Q_0)]^{1/q}}\|r_j(a)\|_{L^q_\omega(\mathbb{R}^n)}\bigg) \lesssim \omega(Q_0)\Phi\bigg(\frac{1}{[\omega(Q_0)]^{1/q}}\|a\|_{L^q_\omega(\mathbb{R}^n)}\bigg) \\
&\qquad \lesssim \omega(Q_0)\Phi\bigg(\frac{1}{\omega(Q_0)\rho(\omega(Q_0))}\bigg) \sim 1,
\end{aligned}$$

which implies (8.1) in Case 1.

*Case 2:* $R_0 \in (0,1)$. In this case, let $\widetilde{Q}_0 \equiv 8nQ_0$. Then

$$\begin{aligned}
\int_{\mathbb{R}^n} \Phi(\mathcal{G}_N^0(r_j(a))(x))\omega(x)\,dx &= \int_{\widetilde{Q}_0} \Phi(\mathcal{G}_N^0(r_j(a))(x))\omega(x)\,dx + \int_{(\widetilde{Q}_0)^\complement} \cdots \\
&\equiv \mathrm{I}_1 + \mathrm{I}_2. \qquad (8.2)
\end{aligned}$$

For $\mathrm{I}_1$, similarly to the proof of Case 1, we have

$$\begin{aligned}
\mathrm{I}_1 &\le \omega(\widetilde{Q}_0)\Phi\bigg(\frac{1}{[\omega(\widetilde{Q}_0)]^{1/q}}\bigg\{\int_{\widetilde{Q}_0} [\mathcal{G}_N^0(r_j(a))(x)]^q\omega(x)\,dx\bigg\}^{1/q}\bigg) \\
&\lesssim \omega(\widetilde{Q}_0)\Phi\bigg(\frac{1}{[\omega(Q_0)]^{1/q}}\|r_j(a)\|_{L^q_\omega(\mathbb{R}^n)}\bigg) \lesssim \omega(Q_0)\Phi\bigg(\frac{1}{\omega(Q_0)\rho(\omega(Q_0))}\bigg) \sim 1. \qquad (8.3)
\end{aligned}$$

To estimate $\mathrm{I}_2$, let $x \in (\widetilde{Q}_0)^\complement$, $t \in (0,1)$, $\psi \in \mathcal{D}_N^0(\mathbb{R}^n)$ and $P_\psi^s$ be the Taylor expansion of $\psi$ about $(x-x_0)/t$ with degree $s$. Then by the vanishing condition on $a$, we see that

$$\begin{aligned}
|r_j(a) * \psi_t(x)| &= \frac{1}{t^n}\bigg|\int_{\mathbb{R}^n} r_j(a)(y)\psi\Big(\frac{x-y}{t}\Big)\,dy\bigg| \\
&= \frac{1}{t^n}\bigg|\int_{\mathbb{R}^n} r_j(a)(y)\Big\{\psi\Big(\frac{x-y}{t}\Big) - P_\psi^s\Big(\frac{x-y}{t}\Big)\Big\}\,dy\bigg| \\
&\le \frac{1}{t^n}\int_{2\sqrt{n}Q_0} |r_j(a)(y)|\bigg|\psi\Big(\frac{x-y}{t}\Big) - P_\psi^s\Big(\frac{x-y}{t}\Big)\bigg|\,dy \\
&\quad + \frac{1}{t^n}\int_{Q(x_0, \frac{|x-x_0|}{2\sqrt{n}})\setminus(2\sqrt{n}Q_0)} \cdots + \frac{1}{t^n}\int_{Q(x_0,\frac{|x-x_0|}{2\sqrt{n}})^\complement} \cdots \\
&\equiv \mathrm{G}_1 + \mathrm{G}_2 + \mathrm{G}_3. \qquad (8.4)
\end{aligned}$$

To estimate $\mathrm{G}_1$, as $t \in (0,1)$ and $x \in (\widetilde{Q}_0)^\complement$, we see that $\mathrm{G}_1 \neq 0$ implies that $t > 3|x-x_0|/4$. From this, Taylor's remainder theorem, Hölder's inequality, Lemma 8.3(i), (2.1) and Remark 2.2(ii), we deduce that

$$\begin{aligned}
\mathrm{G}_1 &\lesssim \frac{1}{t^{n+s+1}}\|r_j(a)\|_{L^q_\omega(\mathbb{R}^n)}\Bigg\{\sum_{\alpha\in\mathbb{Z}^n_+,\,|\alpha|=s+1}\int_{2\sqrt{n}Q_0}\left|\left(\partial^\alpha\psi\right)\left(\frac{\xi}{t}\right)\right|^{q'}\\
&\qquad\times|y-x_0|^{(s+1)q'}[\omega(y)]^{-q'/q}\,dy\Bigg\}^{1/q'}\\
&\lesssim \frac{R_0^{s+1}}{|x-x_0|^{n+s+1}}\|a\|_{L^q_\omega(\mathbb{R}^n)}\left\{\int_{2\sqrt{n}Q_0}[\omega(y)]^{-q'/q}\,dy\right\}^{1/q'}\\
&\lesssim \frac{1}{\omega(Q_0)\rho(\omega(Q_0))}\frac{R_0^{n+s+1}}{|x-x_0|^{n+s+1}}, \qquad (8.5)
\end{aligned}$$

where $\theta\in(0,1)$, $\xi\equiv\theta(x-y)+(1-\theta)(x-x_0)$ and $1/q+1/q'=1$.

To estimate $\mathrm{G}_2$, by the definition of $k_j$ with $j\in\{1,\dots,n\}$, we have

$$\sum_{\alpha\in\mathbb{Z}^n_+,\,|\alpha|=s+1}|(\partial^\alpha k_j)(z)|\lesssim\frac{1}{|z|^{n+s+1}} \qquad (8.6)$$

for all $z\in\mathbb{R}^n\setminus\{0\}$. For any fixed $y\in(Q(x_0,\frac{|x-x_0|}{2\sqrt{n}})\setminus 2\sqrt{n}Q_0)$, let $K_j^s$ be the Taylor expansion of $k_j(\cdot)$ at the point $y-x_0$ with degree $s$. Moreover, it is easy to see that $\mathrm{G}_2\neq 0$ implies that $t>|x-x_0|/2$. From this, Taylor's remainder theorem, (8.6), Hölder's inequality, Lemma 8.3(i) and (2.1), we conclude that

$$\begin{aligned}
\mathrm{G}_2 &\le \frac{1}{t^n}\int_{Q(x_0,\frac{|x-x_0|}{2\sqrt{n}})\setminus(2\sqrt{n}Q_0)}\left\{\int_{Q_0}|a(z)|\,|k_j(y-z)-K_j^s(y-z)|\,dz\right\}\\
&\qquad\times\left|\psi\left(\frac{x-y}{t}\right)-P_\psi^s\left(\frac{x-y}{t}\right)\right|dy\\
&\lesssim \frac{1}{t^{n+s+1}}\int_{Q(x_0,\frac{|x-x_0|}{2\sqrt{n}})\setminus(2\sqrt{n}Q_0)}\left\{\int_{Q_0}|a(z)|\frac{|z-x_0|^{s+1}}{|\xi|^{n+s+1}}\,dz\right\}\\
&\qquad\times|y-x_0|^{s+1}\,dy\\
&\lesssim \frac{1}{|x-x_0|^{n+s+1}}\int_{Q(x_0,\frac{|x-x_0|}{2\sqrt{n}})\setminus(2\sqrt{n}Q_0)}\frac{1}{|y-x_0|^n}\left\{\int_{Q_0}|a(z)|\,|z-x_0|^{s+1}\,dz\right\}dy\\
&\lesssim \frac{R_0^{s+1}}{|x-x_0|^{n+s+1}}\|a\|_{L^q_\omega(\mathbb{R}^n)}\frac{|Q_0|}{[\omega(Q_0)]^{1/q}}\int_{Q(x_0,\frac{|x-x_0|}{2\sqrt{n}})\setminus(2\sqrt{n}Q_0)}\frac{1}{|y-x_0|^n}\,dy\\
&\lesssim \frac{1}{\omega(Q_0)\rho(\omega(Q_0))}\frac{R_0^{n+s+1}}{|x-x_0|^{n+s+1}}\int_{\sqrt{n}R_0}^{\frac{|x-x_0|}{2\sqrt{n}}}z^{-1}\,dz\\
&\lesssim \frac{1}{\omega(Q_0)\rho(\omega(Q_0))}\frac{R_0^{n+s+1-\delta}}{|x-x_0|^{n+s+1-\delta}}, \qquad (8.7)
\end{aligned}$$

where $\xi\equiv\gamma(y-z)+(1-\gamma)(y-x_0)$ for some $\gamma\in(0,1)$, $\delta$ is a small positive constant which is determined later, and in the third inequality we used the fact that for any $y\in Q(x_0,\frac{|x-x_0|}{2\sqrt{n}})\setminus(2\sqrt{n}Q_0)$ and $z\in Q_0$,

$$|(y-x_0)-\gamma(z-x_0)|\ge|y-x_0|-|y-x_0|/2=|y-x_0|/2.$$

Finally, we estimate $\mathrm{G}_3$. For any $y \in [Q(x_0, \frac{|x-x_0|}{2\sqrt{n}})]^\complement$, by the definition of $P_\psi^s$ and the support condition on $\psi$, we have

$$\frac{1}{t^n}\left|P_\psi^s\left(\frac{x-y}{t}\right)\right| \lesssim \frac{|y-x_0|^s}{|x-x_0|^{n+s}}. \tag{8.8}$$

Thus, from the vanishing condition of $a$, Taylor's remainder theorem, (8.6), Hölder's inequality, (2.1) and (8.8), we deduce that

$$\begin{aligned}
\mathrm{G}_3 &\lesssim \frac{1}{t^n}\int_{Q(x_0, \frac{|x-x_0|}{2\sqrt{n}})^\complement}\left\{\int_{Q_0}|a(z)|\,|k_j(y-z)-K_j^s(y-z)|\,dz\right\}\\
&\quad\times\left\{\left|\psi\left(\frac{x-y}{t}\right)\right|+\left|P_\psi^s\left(\frac{x-y}{t}\right)\right|\right\}dy\\
&\lesssim \frac{1}{t^n}\int_{Q(x_0, \frac{|x-x_0|}{2\sqrt{n}})^\complement}\left\{\int_{Q_0}|a(z)|\frac{|z-x_0|^{s+1}}{|\xi|^{n+s+1}}\,dz\right\}\\
&\quad\times\left\{\left|\psi\left(\frac{x-y}{t}\right)\right|+\left|P_\psi^s\left(\frac{x-y}{t}\right)\right|\right\}dy\\
&\lesssim \|a\|_{L^q_\omega(\mathbb{R}^n)}\frac{R_0^{s+n+1}}{[\omega(Q_0)]^{1/q}}\frac{1}{t^n}\int_{Q(x_0, \frac{|x-x_0|}{2\sqrt{n}})^\complement}\frac{1}{|y-x_0|^{n+s+1}}\\
&\quad\times\left\{\left|\psi\left(\frac{x-y}{t}\right)\right|+\left|P_\psi^s\left(\frac{x-y}{t}\right)\right|\right\}dy\\
&\lesssim \|a\|_{L^q_\omega(\mathbb{R}^n)}\frac{R_0^{s+n+1}}{[\omega(Q_0)]^{1/q}}\left\{\frac{1}{|x-x_0|^{n+s+1}}\frac{1}{t^n}\int_{Q(x_0, \frac{|x-x_0|}{2\sqrt{n}})^\complement}\left|\psi\left(\frac{x-y}{t}\right)\right|dy\right.\\
&\quad\left.+\frac{1}{t^n}\int_{Q(x_0, \frac{|x-x_0|}{2\sqrt{n}})^\complement}\frac{1}{|y-x_0|^{n+s+1}}\left|P_\psi^s\left(\frac{x-y}{t}\right)\right|dy\right\}\\
&\lesssim \frac{1}{\omega(Q_0)\rho(\omega(Q_0))}\left\{\frac{R_0^{n+s+1}}{|x-x_0|^{n+s+1}}\right.\\
&\quad\left.+\frac{R_0^{n+s+1}}{|x-x_0|^{n+s}}\int_{Q(x_0, \frac{|x-x_0|}{2\sqrt{n}})^\complement}\frac{1}{|y-x_0|^{n+1}}\,dy\right\}\\
&\lesssim \frac{1}{\omega(Q_0)\rho(\omega(Q_0))}\frac{R_0^{n+s+1}}{|x-x_0|^{n+s+1}},
\end{aligned} \tag{8.9}$$

where $\xi \equiv \gamma(y-z)+(1-\gamma)(y-x_0)$ for some $\gamma \in (0,1)$ and in the third inequality we used the fact that for any $y \in [Q(x_0, \frac{|x-x_0|}{2\sqrt{n}})]^\complement$ and $z \in Q_0$,

$$|(y-x_0)-\gamma(z-x_0)| \gtrsim |y-x_0|.$$

Thus, from (8.4), (8.5), (8.7), (8.9) and $|x-x_0| \ge 4nR_0$, we know that

$$\begin{aligned}
|r_j(a)*\psi_t(x)| &\lesssim \frac{1}{\omega(Q_0)\rho(\omega(Q_0))}\left\{\frac{R_0^{(n+s+1)}}{|x-x_0|^{n+s+1}}+\frac{R_0^{(n+s+1-\delta)}}{|x-x_0|^{n+s+1-\delta}}\right\}\\
&\lesssim \frac{1}{\omega(Q_0)\rho(\omega(Q_0))}\frac{R_0^{(n+s+1-\delta)}}{|x-x_0|^{n+s+1-\delta}},
\end{aligned}$$

which together with the arbitrariness of $\psi \in \mathcal{D}_N^0(\mathbb{R}^n)$ implies that for all $x \in (\widetilde{Q}_0)^{\complement}$,

$$\mathcal{G}_N^0(r_j(a))(x) \lesssim \frac{1}{\omega(Q_0)\rho(\omega(Q_0))} \frac{R_0^{(n+s+1-\delta)}}{|x-x_0|^{n+s+1-\delta}}. \tag{8.10}$$

Take $\delta \in (0,\infty)$ small enough such that $p_\Phi(n+s+1-\delta) > nq$. By the fact that

$$\mathrm{supp}(\mathcal{G}_N^0(r_j(a))) \subset Q(x_0, R_0+8) \subset Q(x_0, 9)$$

and Lemma 2.3(i), we know that there exists an $\widetilde{\omega} \in A_q(\mathbb{R}^n)$ such that $\widetilde{\omega} = \omega$ on $Q(x_0, 9)$. Let $m_0$ be the integer such that $2^{m_0-1}nR_0 \le 9 < 2^{m_0}nR_0$. From (8.10), the lower type $p_\Phi$ property of $\Phi$, Lemma 2.3(viii) and $p_\Phi(n+s+1-\delta) > nq$, we conclude that

$$\begin{aligned}
\mathrm{I}_2 &\lesssim \int_{Q(x_0,9)\setminus\widetilde{Q}_0} \Phi(\mathcal{G}_N^0(r_j(a))(x))\widetilde{\omega}(x)\,dx \\
&\lesssim \sum_{j=3}^{m_0} \int_{2^{j+1}nQ_0\setminus 2^jnQ_0} \Phi\bigg(\frac{1}{\omega(Q_0)\rho(\omega(Q_0))} \frac{R_0^{(n+s+1-\delta)}}{|x-x_0|^{n+s+1-\delta}}\bigg)\widetilde{\omega}(x)\,dx \\
&\lesssim \frac{1}{\omega(Q_0)} \sum_{j=3}^{m_0} \int_{2^{j+1}nQ_0\setminus 2^jnQ_0} \bigg(\frac{R_0^{n+s+1-\delta}}{|x-x_0|^{n+s+1-\delta}}\bigg)^{p_\Phi} \widetilde{\omega}(x)\,dx \\
&\lesssim \sum_{j=3}^{m_0} 2^{k[(n+s+1-\delta)p_\Phi - nq]} \lesssim 1,
\end{aligned}$$

which together with (8.2) and (8.3) implies (8.1) in Case 2. This finishes the proof of Theorem 8.2. ∎

Remark 8.4. Theorem 8.2 when $\omega \in A_1^{\mathrm{loc}}(\mathbb{R}^n)$ and $\Phi(t) \equiv t$ for all $t \in (0,\infty)$ was obtained by Tang [49, Lemma 8.3].

Next, we introduce the local fractional integral and, using Theorem 6.4, prove that they are boundedness from $h^p_{\omega^p}(\mathbb{R}^n)$ to $L^q_{\omega^q}(\mathbb{R}^n)$ when $q \in [1,\infty)$, and from $h^p_{\omega^p}(\mathbb{R}^n)$ to $h^q_{\omega^q}(\mathbb{R}^n)$ when $q \in (0,1]$, provided that $\omega$ satisfies $\omega^{\frac{nr}{nr-n-r\alpha}} \in A_1^{\mathrm{loc}}(\mathbb{R}^n)$ for some $r \in (n/(n-\alpha),\infty)$ and $\int_{\mathbb{R}^n}[\omega(x)]^p\,dx = \infty$. We begin with some notions.

Definition 8.5. Let $\alpha \in [0,n)$ and $\phi_0$ be as in Definition 8.1. For any $f \in \mathcal{S}(\mathbb{R}^n)$ and all $x \in \mathbb{R}^n$, the *local fractional integral* $I_\alpha^{\mathrm{loc}}(f)$ of $f$ is defined by

$$I_\alpha^{\mathrm{loc}}(f)(x) \equiv \int_{\mathbb{R}^n} \frac{\phi_0(y)}{|y|^{n-\alpha}} f(x-y)\,dy.$$

Definition 8.6. (i) If there exist $r \in (1,\infty)$ and a positive constant $C$ such that for all cubes $Q \subset \mathbb{R}^n$ with sidelength $l(Q) \in (0,1]$,

$$\bigg(\frac{1}{|Q|}\int_Q [\omega(x)]^r\,dx\bigg)^{1/r} \le \frac{C}{|Q|}\int_Q \omega(x)\,dx, \tag{8.11}$$

then $\omega$ is said to satisfy the *local reverse Hölder inequality of order* $r$, which is denoted by $\omega \in RH_r^{\mathrm{loc}}(\mathbb{R}^n)$. Furthermore, let $RH_r^{\mathrm{loc}}(\omega) \equiv \inf\{C\}$, where the infimum is taken over all the positive constants $C$ satisfying (8.11).

(ii) Let $p, q \in (1,\infty)$. A locally integrable nonnegative function $\omega$ on $\mathbb{R}^n$ is said to belong to the *class* $A^{\mathrm{loc}}(p,\,q)$, if there exists a positive constant $C$ such that for all cubes

$Q \subset \mathbb{R}^n$ with sidelength $l(Q) \in (0,1]$,

$$\left(\frac{1}{|Q|}\int_Q [\omega(x)]^q\,dx\right)^{1/q}\left(\frac{1}{|Q|}\int_Q [\omega(x)]^{-p'}\,dx\right)^{1/p'} \le C; \tag{8.12}$$

here and in what follows, $1/p+1/p'=1$. Furthermore, let $A^{\mathrm{loc}}(p,\,q)(\omega) \equiv \inf\{C\}$, where the infimum is taken over all the positive constants $C$ satisfying (8.12).

Remark 8.7. (i) Let $r$ be as in Definition 8.6(i). If (8.11) holds for all cubes $Q \subset \mathbb{R}^n$, then $\omega$ is said to satisfy the *reverse Hölder inequality of order* $r$, which is denoted by $\omega \in RH_r(\mathbb{R}^n)$ (see, for example, [17]). Let $p$, $q$ be as in Definition 8.6(ii). If (8.12) holds for all cubes $Q \subset \mathbb{R}^n$, then $\omega$ is said to belong to the *class* $A(p,\,q)$.

(ii) For any given positive constant $A_1$, let the cube $Q$ satisfy $l(Q) = A_1$. Similarly to the proof of Lemma 2.3(i), for any $\omega \in RH_r^{\mathrm{loc}}(\mathbb{R}^n)$, there exists an $\widetilde{\omega} \in RH_r(\mathbb{R}^n)$ such that $\omega = \widetilde{\omega}$ on $Q$ and $RH_r(\widetilde{\omega}) \lesssim RH_r^{\mathrm{loc}}(\omega)$, where $RH_r(\widetilde{\omega})$ is defined similarly to $RH_r^{\mathrm{loc}}(\omega)$ and the implicit constant depends only on $A_1$ and $n$.

(iii) Similarly to Remark 2.2(ii), for any given constant $A_2 \in (0,\infty)$, the condition $l(Q) \in (0,1]$ in (8.11) can be replaced by $l(Q) \in (0,A_2]$ with the positive constant $C$ in (8.11) depending on $A_2$.

About the relations of $A_\infty^{\mathrm{loc}}(\mathbb{R}^n)$, $RH_r^{\mathrm{loc}}(\mathbb{R}^n)$ and $A^{\mathrm{loc}}(p,\,q)$, we have the following conclusions.

Lemma 8.8.

(i) *Let $r \in (1,\,\infty)$. Then $\omega^r \in A_\infty^{\mathrm{loc}}(\mathbb{R}^n)$ if and only if $\omega \in RH_r^{\mathrm{loc}}(\mathbb{R}^n)$.*

(ii) *Let $\alpha \in (0,\,n)$, $p \in (1,\,n/\alpha)$ and $1/q = 1/p - \alpha/n$. Then $\omega \in A^{\mathrm{loc}}(p,\,q)$ if and only if $\omega^{-p'} \in A_{1+p'/q}^{\mathrm{loc}}(\mathbb{R}^n)$.*

*Proof.* We first prove (i). Let $\omega^r \in A_\infty^{\mathrm{loc}}(\mathbb{R}^n)$. Then by Lemma 2.3(i), we know that for any cube $Q \equiv Q(x_0, l(Q))$ with $l(Q) \in (0,1]$, there exists a function $\widetilde{\omega}$ on $\mathbb{R}^n$ such that

$$\widetilde{\omega}^r \in A_\infty(\mathbb{R}^n) \quad \text{and} \quad \widetilde{\omega} = \omega \text{ on } Q(x_0,1). \tag{8.13}$$

Moreover, by [12, Lemma A], we know that

$$\widetilde{\omega}^r \in A_\infty(\mathbb{R}^n) \quad \text{if and only if} \quad \widetilde{\omega} \in RH_r(\mathbb{R}^n). \tag{8.14}$$

Thus, for any cube $Q(x_0, l(Q))$ with $l(Q) \in (0,1]$, by (8.13) and (8.14), we have

$$\left(\frac{1}{|Q|}\int_Q [\omega(x)]^r\,dx\right)^{1/r} = \left(\frac{1}{|Q|}\int_Q [\widetilde{\omega}(x)]^r\,dx\right)^{1/r} \lesssim \frac{1}{|Q|}\int_Q \widetilde{\omega}(x)\,dx \sim \frac{1}{|Q|}\int_Q \omega(x)\,dx,$$

which together with the arbitrariness of the cube $Q(x_0, l(Q))$ implies that $\omega \in RH_r^{\mathrm{loc}}(\mathbb{R}^n)$.

Conversely, let $\omega \in RH_r^{\mathrm{loc}}(\mathbb{R}^n)$. Then by Remark 8.7(ii), we know that for any cube $Q(x_0, l(Q))$ with $l(Q) \in (0,1]$, there exists a function $\widetilde{\omega}$ on $\mathbb{R}^n$ such that $\widetilde{\omega} \in RH_r(\mathbb{R}^n)$ and $\widetilde{\omega} = \omega$ on $Q(x_0,1)$, which together with (8.14) and the arbitrariness of the cube $Q(x_0, l(Q))$ implies that $\omega \in A_\infty^{\mathrm{loc}}(\mathbb{R}^n)$. This finishes the proof of (i).

By the definitions of $A^{\mathrm{loc}}(p,\,q)$ and $A_{1+p'/q}^{\mathrm{loc}}(\mathbb{R}^n)$, we see that (ii) holds, which completes the proof of Lemma 8.8. ∎

To establish the boundedness of local fractional integrals, we need the following technical lemma.

LEMMA 8.9. *Let* $\alpha \in (0,n)$, $p \in (1, n/\alpha)$ *and* $1/q = 1/p - \alpha/n$. *For some* $r \in (q,\infty)$, *if*

$$\omega^{-r'} \in A^{\mathrm{loc}}(q'/r',\, p'/r'),$$

*then there exists a positive constant* $C$ *such that for all* $f \in L^p_{\omega^p}(\mathbb{R}^n)$,

$$\|I^{\mathrm{loc}}_\alpha(f)\|_{L^q_{\omega^q}(\mathbb{R}^n)} \le C\|f\|_{L^p_{\omega^p}(\mathbb{R}^n)}, \tag{8.15}$$

*where* $p'$, $q'$ *and* $r'$ *respectively denote the conjugate indices of* $p$, $q$ *and* $r$.

*Proof.* Let $\omega^{-r'} \in A^{\mathrm{loc}}(q'/r',\, p'/r')$. For any unit cube $Q \subset \mathbb{R}^n$, from Lemmas 8.8(ii) and 2.3(i), and Remark 2.4, we deduce that there exists a function $\widetilde{\omega}$ on $\mathbb{R}^n$ such that $\widetilde{\omega}^{-r'} \in A(q'/r',\, p'/r')$ and $\widetilde{\omega} = \omega$ on $5Q$. For $\widetilde{\omega}^{-r'} \in A(q'/r',\, p'/r')$, similarly to the proof of [13, Theorem 2], we know that for all $f \in L^p_{\widetilde{\omega}^p}(\mathbb{R}^n)$,

$$\|I^{\mathrm{loc}}_\alpha(f)\|_{L^q_{\widetilde{\omega}^q}(\mathbb{R}^n)} \lesssim \|f\|_{L^p_{\widetilde{\omega}^p}(\mathbb{R}^n)},$$

which combined with the definition of $I^{\mathrm{loc}}_\alpha(f)$ implies that

$$\|I^{\mathrm{loc}}_\alpha(f)\|_{L^q_{\omega^q}(Q)} = \|I^{\mathrm{loc}}_\alpha(f\chi_{5Q})\|_{L^q_{\widetilde{\omega}^q}(Q)} \lesssim \|f\chi_{5Q}\|_{L^p_{\widetilde{\omega}^p}(\mathbb{R}^n)} \sim \|f\|_{L^p_{\omega^p}(5Q)}. \tag{8.16}$$

Take unit cubes $\{Q_i\}_{i=1}^\infty$ with disjoint interiors such that $\bigcup_{i=1}^\infty Q_i = \mathbb{R}^n$, and

$$\sum_{i=1}^\infty \chi_{5Q_i} \le M,$$

where $M$ is a positive integer depending only on $n$. From this and (8.16), we infer that

$$\|I^{\mathrm{loc}}_\alpha(f)\|^q_{L^q_{\omega^q}(\mathbb{R}^n)} = \sum_{i=1}^\infty \|I^{\mathrm{loc}}_\alpha(f)\|^q_{L^q_{\omega^q}(Q_i)} \lesssim \sum_{i=1}^\infty \|f\|^q_{L^p_{\omega^p}(5Q_i)} \lesssim \|f\|^q_{L^p_{\omega^p}(\mathbb{R}^n)},$$

which implies (8.15). This finishes the proof of Lemma 8.9. ∎

THEOREM 8.10. *Let* $\alpha \in (0,n)$, $p \in [n/(n+\alpha), 1]$ *and* $1/q = 1/p - \alpha/n$. *For some* $r \in (n/(n-\alpha), \infty)$, *if the weight* $\omega$ *satisfies* $\omega^{nr/(nr-n-r\alpha)} \in A^{\mathrm{loc}}_1(\mathbb{R}^n)$ *and* $\int_{\mathbb{R}^n} [\omega(x)]^p\,dx = \infty$, *then there exists a positive constant* $C$ *such that for all* $f \in h^p_{\omega^p}(\mathbb{R}^n)$,

$$\|I^{\mathrm{loc}}_\alpha(f)\|_{L^q_{\omega^q}(\mathbb{R}^n)} \le C\|f\|_{h^p_{\omega^p}(\mathbb{R}^n)}.$$

*Proof.* Let $r$ and $\omega$ be as in the assumption. Then by Lemma 2.3(ii), we know that there exists an $\eta_1 \in (0,\infty)$ such that

$$\omega^{\frac{nr(1+\eta_1)}{nr-n-r\alpha}} \in A^{\mathrm{loc}}_1(\mathbb{R}^n). \tag{8.17}$$

Let

$$\frac{1}{p_1} \equiv \frac{1}{r} + \frac{\alpha}{n} + \left(1 - \frac{1}{r} - \frac{\alpha}{n}\right)/(1+\eta_1) \quad \text{and} \quad \frac{1}{q_1} \equiv \frac{1}{p_1} - \frac{\alpha}{n}. \tag{8.18}$$

Then from $r \in (n/(n-\alpha), \infty)$, we know that

$$p_1 \in (1, n/\alpha), \quad r > q_1 \quad \text{and} \quad \omega^{-r'} \in A^{\mathrm{loc}}(q_1'/r',\, p_1'/r'). \tag{8.19}$$

Furthermore, from (8.17), the fact that $p_1 < \frac{nr(1+\eta_1)}{nr-n-r\alpha}$ and Hölder's inequality, we infer that

$$\omega^{p_1} \in A^{\mathrm{loc}}_1(\mathbb{R}^n), \tag{8.20}$$

which together with Lemma 2.3(ii) implies that there exists an $\eta_2 \in (0,\infty)$ such that

$\omega^{p_1(1+\eta_2)} \in A_1^{\mathrm{loc}}(\mathbb{R}^n)$. Let

$$\widetilde{q} \equiv p_1(1+\eta_2). \tag{8.21}$$

From $\frac{nr}{nr-n-r\alpha} > p$ and Hölder's inequality, we see that $\omega^p \in A_1^{\mathrm{loc}}(\mathbb{R}^n)$. Let $s \equiv \lfloor n(1/p-1)\rfloor$. To show Theorem 8.10, as $h^p_{\omega^p}(\mathbb{R}^n)$ and $L^q_{\omega^q}(\mathbb{R}^n)$ are respectively a $p$-quasi-Banach space and a 1-quasi-Banach space, Theorem 6.4(i) with $\Phi(t) \equiv t^p$ for all $t \in (0,\infty)$ implies that it suffices to show that for any $(p,\, \widetilde{q},\, s)_{\omega^p}$-atom $a$ supported in $Q_0 \equiv Q(x_0, R_0)$ with $R_0 \in (0, 2]$,

$$\|I_\alpha^{\mathrm{loc}}(a)\|_{L^q_{\omega^q}(\mathbb{R}^n)} \lesssim 1. \tag{8.22}$$

From $\mathrm{supp}(a) \subset Q_0$ and the definition of $I_\alpha^{\mathrm{loc}}(a)$, we see that

$$\mathrm{supp}(I_\alpha^{\mathrm{loc}}(a)) \subset Q(x_0, R_0+4). \tag{8.23}$$

Now, we prove (8.22) by considering the following two cases for $R_0$.

*Case 1:* $R_0 \in [1,2]$. In this case, from (8.23), Hölder's inequality, (8.19), Lemma 8.3, $R_0 \in [1,2]$ and $1/q - 1/q_1 = 1/p - 1/p_1$, we deduce that

$$\begin{aligned}\left\{\int_{\mathbb{R}^n} |I_\alpha^{\mathrm{loc}}(a)(x)|^q[\omega(x)]^q\,dx\right\}^{1/q}&\\ &\lesssim \left\{\int_{Q(x_0,R_0+4)} |I_\alpha^{\mathrm{loc}}(a)(x)|^{q_1}[\omega(x)]^{q_1}\,dx\right\}^{1/q_1}|Q_0|^{1/q-1/q_1}\\ &\lesssim \|a\|_{L^{p_1}_{\omega^{p_1}}(\mathbb{R}^n)}|Q_0|^{1/p-1/p_1}.\end{aligned} \tag{8.24}$$

By (8.20) and the definition of $A_1^{\mathrm{loc}}(\mathbb{R}^n)$, we know that $\omega^{\frac{p_1(\widetilde{q}-p)}{(\widetilde{q}-p_1)}} \in A_1^{\mathrm{loc}}(\mathbb{R}^n)$. From this and Lemma 8.2(i), we infer that $\omega^p \in RH^{\mathrm{loc}}_{\frac{p_1(\widetilde{q}-p)}{p(\widetilde{q}-p_1)}}(\mathbb{R}^n)$, which implies that

$$\left\{\int_{Q_0}[\omega(x)]^p\,dx\right\}^{\frac{1}{\widetilde{q}}-\frac{1}{p}}\left\{\int_{Q_0}[\omega(x)]^{\frac{p_1(\widetilde{q}-p)}{(\widetilde{q}-p_1)}}\,dx\right\}^{\frac{1}{p_1}-\frac{1}{\widetilde{q}}} \lesssim |Q_0|^{\frac{1}{p_1}-\frac{1}{p}}. \tag{8.25}$$

This, combined with (8.24), Hölder's inequality and the fact that $a$ is a $(p,\, \widetilde{q},\, s)_{\omega^p}$-atom, yields

$$\begin{aligned}\|I_\alpha^{\mathrm{loc}}(a)\|_{L^q_{\omega^q}(\mathbb{R}^n)} &\lesssim \|a\|_{L^{p_1}_{\omega^{p_1}}(\mathbb{R}^n)}|Q_0|^{\frac{1}{p}-\frac{1}{p_1}}\\ &\lesssim \left\{\int_{Q_0}|a(x)|^{\widetilde{q}}[\omega(x)]^p\,dx\right\}^{1/\widetilde{q}}\left\{\int_{Q_0}[\omega(x)]^{\frac{p_1(\widetilde{q}-p)}{(\widetilde{q}-p_1)}}\,dx\right\}^{\frac{1}{p_1}-\frac{1}{\widetilde{q}}}|Q_0|^{\frac{1}{p}-\frac{1}{p_1}}\\ &\lesssim \left\{\int_{Q_0}[\omega(x)]^p\,dx\right\}^{\frac{1}{\widetilde{q}}-\frac{1}{p}}\left\{\int_{Q_0}[\omega(x)]^{\frac{p_1(\widetilde{q}-p)}{(\widetilde{q}-p_1)}}\,dx\right\}^{\frac{1}{p_1}-\frac{1}{\widetilde{q}}}|Q_0|^{\frac{1}{p}-\frac{1}{p_1}} \lesssim 1.\end{aligned}$$

This shows (8.22) in Case 1.

*Case 2:* $R_0 \in (0,1)$. In this case, let $\widetilde{Q}_0 \equiv 4nQ_0$. From (8.23), it follows that

$$\begin{aligned}\|I_\alpha^{\mathrm{loc}}(a)\|_{L^q_{\omega^q}(\mathbb{R}^n)} &\le \left\{\int_{\widetilde{Q}_0}|I_\alpha^{\mathrm{loc}}(a)(x)|^q[\omega(x)]^q\,dx\right\}^{1/q}\\ &\quad+\left\{\int_{Q(x_0,R_0+4)\setminus\widetilde{Q}_0}\cdots\right\}^{1/q}\\ &\equiv \mathrm{I}_1+\mathrm{I}_2.\end{aligned} \tag{8.26}$$

To estimate $\mathrm{I}_1$, by Hölder's inequality, (8.15) and (8.25), we conclude that

$$\begin{aligned}
\mathrm{I}_1 &\le \left(\int_{Q_1} |I_\alpha^{\mathrm{loc}}(a)(x)|^{q_1}[\omega(x)]^{q_1}\,dx\right)^{1/q_1} |Q_0|^{\frac{1}{p}-\frac{1}{p_1}} \\
&\lesssim \|a\|_{L^{p_1}_{\omega^{p_1}}(\mathbb{R}^n)} |Q_0|^{\frac{1}{p}-\frac{1}{p_1}} \\
&\lesssim \left\{\int_{Q_0}[\omega(x)]^p\,dx\right\}^{\frac{1}{\widetilde{q}}-\frac{1}{p}} \left\{\int_{Q_0}[\omega(x)]^{\frac{p_1(\widetilde{q}-p)}{(\widetilde{q}-p_1)}}\,dx\right\}^{\frac{1}{p_1}-\frac{1}{\widetilde{q}}} |Q_0|^{\frac{1}{p}-\frac{1}{p_1}} \lesssim 1. \qquad (8.27)
\end{aligned}$$

To estimate $\mathrm{I}_2$, for any fixed $x \in Q(x_0, R_0+4) \setminus \widetilde{Q}_0$, let $E^s$ be the Taylor expansion of $\phi_0(\cdot)/|\cdot|^{n-\alpha}$ about $x-x_0$ with degree $s$. Let $m_0$ be the integer such that

$$2^{m_0-1} nR_0 \le R_0 + 4 < 2^{m_0} nR_0.$$

Since $\omega^p \in A_1^{\mathrm{loc}}(\mathbb{R}^n) \subset A_{\widetilde{q}}^{\mathrm{loc}}(\mathbb{R}^n)$, by (2.1), we have

$$\left(\int_{Q_0}[\omega(x)]^p\,dx\right)^{\frac{1}{\widetilde{q}}-\frac{1}{p}} \left(\int_{Q_0}[\omega(x)]^{-p\widetilde{q}'/\widetilde{q}}\,dx\right)^{1/\widetilde{q}'} \lesssim \left(\int_{Q_0}[\omega(x)]^p\,dx\right)^{1/p} |Q_0|.$$

From this, the vanishing condition of $a$, Minkowski's inequality, Taylor's remainder theorem, the fact that

$$\sum_{\alpha\in\mathbb{Z}_+^n,\,|\alpha|=s+1} \left|\partial^\alpha\left(\frac{\phi_0(\cdot)}{|\cdot|^{n-\alpha}}\right)(z)\right| \lesssim \frac{1}{|z|^{n+s+1-\alpha}}$$

for all $z \in \mathbb{R}^n \setminus \{0\}$, and Hölder's inequality, we deduce that

$$\begin{aligned}
\mathrm{I}_2 &\le \left(\sum_{k=2}^{m_0}\int_{2^{k+1}nQ_0\setminus 2^k nQ_0}\left\{\int_{Q_0}\left|\frac{\phi_0(x-y)}{|x-y|^{n-\alpha}} - E^s(x-y)\right||a(y)|\,dy\right\}^q [\omega(x)]^q\,dx\right)^{1/q} \\
&\le \sum_{k=2}^{m_0}\int_{Q_0}|a(y)|\left\{\int_{2^{k+1}nQ_0\setminus 2^k nQ_0}\left|\frac{\phi_0(x-y)}{|x-y|^{n-\alpha}} - E^s(x-y)\right|^q [\omega(x)]^q\,dx\right\}^{1/q} dy \\
&\lesssim \sum_{k=2}^{m_0}\int_{Q_0}|a(y)|\left\{\int_{2^{k+1}nQ_0\setminus 2^k nQ_0}\left(\frac{|y-x_0|^{s+1}}{|\theta(x-y)-(1-\theta)(x-x_0)|^{n+s+1-\alpha}}\right)^q \right. \\
&\qquad \left. \times [\omega(x)]^q\,dx\right\}^{1/q} dy \\
&\lesssim \sum_{k=2}^{m_0}\int_{Q_0}|a(y)|\left\{\int_{2^{k+1}nQ_0\setminus 2^k nQ_0}\left(\frac{|y-x_0|^{s+1}}{|x-x_0|^{n+s+1-\alpha}}\right)^q [\omega(x)]^q\,dx\right\}^{1/q} dy \\
&\lesssim \sum_{k=2}^{m_0}\frac{R_0^{\alpha-n}}{2^{k(n+s+1-\alpha)}}\left\{\int_{Q_0}|a(y)|\,dy\right\}\left\{\int_{2^{k+1}nQ_0}[\omega(x)]^q\,dx\right\}^{1/q} \\
&\lesssim \sum_{k=2}^{m_0}\frac{R_0^{\alpha-n}}{2^{k(n+s+1-\alpha)}}\left\{\int_{Q_0}|a(y)|^{\widetilde{q}}[\omega(y)]^p\,dy\right\}^{1/\widetilde{q}} \\
&\qquad \times\left\{\int_{Q_0}[\omega(y)]^{-p\widetilde{q}'/\widetilde{q}}\,dy\right\}^{1/\widetilde{q}'}\left\{\int_{2^{k+1}nQ_0}[\omega(x)]^q\,dx\right\}^{1/q} \\
&\lesssim \sum_{k=2}^{m_0}\frac{R_0^{\alpha}}{2^{k(n+s+1-\alpha)}}\left\{\int_{Q_0}[\omega(x)]^p\,dx\right\}^{-1/p}\left\{\int_{2^{k+1}nQ_0}[\omega(x)]^q\,dx\right\}^{1/q}, \qquad (8.28)
\end{aligned}$$

where $\theta \in (0,1)$ and in the fourth inequality we used the fact that for any $y \in Q_0$ and $x \in 2^{k+1}nQ_0 \setminus 2^k nQ_0$ with $k \in \{2, \dots, m_0\}$, $|(x-x_0) - \theta(y-x_0)| \gtrsim |x-x_0|$.

From $\frac{nr(1+\eta_1)}{nr-n-r\alpha} = \frac{rq_1}{r-q_1} > \frac{rq}{r-q}$, (8.17) and Hölder's inequality, it follows that

$$\omega^{\frac{rq}{r-q}} \in A_1^{\mathrm{loc}}(\mathbb{R}^n). \tag{8.29}$$

By Lemma 2.3(i) and Remark 2.4 with $\widetilde{C} \equiv 20n$, we know that there exists a function $\widetilde{\omega}$ on $\mathbb{R}^n$ such that $\widetilde{\omega}^{rq/(r-q)} \in A_1(\mathbb{R}^n)$ such that $\widetilde{\omega} = \omega$ on $Q(x_0, 20n)$, which together with $2^{m_0+1}nQ_0 \subset Q(x_0, 20n)$ and Lemma 2.3(viii) implies that for any $k \in \{1, \dots, m_0\}$,

$$\int_{2^k nQ_0} [\omega(x)]^{\frac{rq}{r-q}}\, dx = \int_{2^k nQ_0} [\widetilde{\omega}(x)]^{\frac{rq}{r-q}}\, dx \lesssim 2^{kn} \int_{Q_0} [\widetilde{\omega}(x)]^{\frac{rq}{r-q}}\, dx \lesssim 2^{kn} \int_{Q_0} [\omega(x)]^{\frac{rq}{r-q}}\, dx.$$

By this estimate and Hölder's inequality, we have

$$\left\{ \int_{2^{k+1}nQ_0} [\omega(x)]^q\, dx \right\}^{1/q} \lesssim R_0^{n/r} 2^{kn/q} \left\{ \int_{Q_0} [\omega(x)]^{\frac{rq}{r-q}}\, dx \right\}^{\frac{1}{q}-\frac{1}{r}}. \tag{8.30}$$

Moreover, by (8.29) and Lemma 8.8(i), we know that $\omega^p \in RH^{\mathrm{loc}}_{\frac{rq}{p(r-q)}}(\mathbb{R}^n)$. Thus, we have

$$\left\{ \int_{Q_0} [\omega(x)]^p\, dx \right\}^{-1/p} \left\{ \int_{Q_0} [\omega(x)]^{\frac{rq}{r-q}}\, dx \right\}^{\frac{1}{q}-\frac{1}{r}} \lesssim R_0^{-\frac{n}{r}-\alpha},$$

which together with (8.28) and (8.30) implies that

$$\mathrm{I}_2 \lesssim \sum_{k=2}^{m_0} 2^{-k(n+s+1-\alpha-n/q)}.$$

From $1/q = 1/p - \alpha/n$ and $r > n/(n-\alpha)$, we deduce that $n+s+1-\alpha-n/q > n+s+1-n/p$, which together with $s = \lfloor n(1/p-1) \rfloor$ implies that $n+s+1-\alpha-n/q > 0$. Thus,

$$\mathrm{I}_2 \lesssim \sum_{k=2}^{m_0} 2^{-k(n+s+1-\alpha-n/q)} \lesssim 1.$$

This combined with (8.26) and (8.27) proves (8.22) in Case 2, which completes the proof of Theorem 8.10. ∎

Theorem 8.11. *Let $\alpha \in (0,1)$, $p \in (0, n/(n+\alpha)]$ and $1/q = 1/p - \alpha/n$. For some $r \in (n/(n-\alpha), \infty)$, if the weight $\omega$ satisfies $\omega^{nr/(nr-n-r\alpha)} \in A_1^{\mathrm{loc}}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} [\omega(x)]^p\, dx = \infty$, then there exists a positive constant $C$ such that for all $f \in h^p_{\omega^p}(\mathbb{R}^n)$,*

$$\|I_\alpha^{\mathrm{loc}}(f)\|_{h^q_{\omega^q}(\mathbb{R}^n)} \le C \|f\|_{h^p_{\omega^p}(\mathbb{R}^n)}.$$

*Proof.* Let $p_1$, $q_1$ and $\widetilde{q}$ be respectively as in (8.18) and (8.21). To show Theorem 8.11, since $h^p_{\omega^p}(\mathbb{R}^n)$ and $h^q_{\omega^q}(\mathbb{R}^n)$ are respectively a $p$-quasi-Banach space and a $q$-quasi-Banach space, Theorem 6.4(i) with $\Phi(t) \equiv t^p$ for all $t \in (0,\infty)$ and Theorem 3.14 imply that it suffices to show that for any $(p, \widetilde{q}, s)_{\omega^p}$-atom $a$ supported in $Q_0 \equiv Q(x_0, R_0)$ with $R_0 \in (0, 2]$,

$$\|\mathcal{G}_N^0(I_\alpha^{\mathrm{loc}}(a))\|_{L^\Phi_{\omega^q}(\mathbb{R}^n)} \lesssim 1. \tag{8.31}$$

From $\mathrm{supp}(a) \subset Q_0$ and the definitions of $I_\alpha^{\mathrm{loc}}(a)$ and $\mathcal{G}_N^0(I_\alpha^{\mathrm{loc}}(a))$, we know that

$$\mathrm{supp}(\mathcal{G}_N^0(I_\alpha^{\mathrm{loc}}(a))) \subset Q(x_0, R_0+8). \tag{8.32}$$

Now, we prove (8.31) by considering the following two cases for $R_0$.

*Case 1:* $R_0 \in [1,2]$. In this case, by (8.17), the fact that $\frac{nr(1+\eta_1)}{nr-n-r\alpha} > q_1$ and Hölder's inequality, we know that $\omega^{q_1} \in A_1^{\rm loc}(\mathbb{R}^n)$, where $\eta_1$ is as in (8.17). From this, (8.32), Hölder's inequality, Proposition 3.2(ii), Lemma 8.9 and (8.25), it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\mathcal{G}_N^0(I_\alpha^{\rm loc}(a))(x)|^q[\omega(x)]^q\,dx\\
&\qquad\lesssim |Q_0|^{1-\frac{q}{q_1}}\left\{\int_{Q(x_0,R_0+8)} |\mathcal{G}_N^0(I_\alpha^{\rm loc}(a))(x)|^{q_1}[\omega(x)]^{q_1}\,dx\right\}^{q/q_1}\\
&\qquad\lesssim |Q_0|^{1-\frac{q}{q_1}}\left\{\int_{Q(x_0,R_0+8)} |I_\alpha^{\rm loc}(a)(x)|^{q_1}[\omega(x)]^{q_1}\,dx\right\}^{q/q_1}\\
&\qquad\lesssim |Q_0|^{1-\frac{q}{q_1}}\left\{\int_{Q_0} |a(x)|^{p_1}[\omega(x)]^{p_1}\,dx\right\}^{q/p_1}\\
&\qquad\lesssim |Q_0|^{1-\frac{q}{q_1}}\left\{\int_{Q_0} |a(x)|^{\widetilde q}[\omega(x)]^{p}\,dx\right\}^{q/\widetilde q}\left\{\int_{Q_0} [\omega(x)]^{\frac{p_1(\widetilde q-p)}{\widetilde q-p_1}}\,dx\right\}^{(\frac{1}{p_1}-\frac{1}{\widetilde q})q}\\
&\qquad\lesssim \left\{|Q_0|^{\frac1q-\frac1{q_1}}\left(\int_{Q_0}[\omega(x)]^p\,dx\right)^{\frac1{\widetilde q}-\frac1p}\left(\int_{Q_0}[\omega(x)]^{\frac{p_1(\widetilde q-p)}{\widetilde q-p_1}}\,dx\right)^{\frac1{p_1}-\frac1{\widetilde q}}\right\}^q \lesssim 1,
\end{aligned}$$

which proves (8.31) in Case 1.

*Case 2:* $R_0 \in (0,1)$. In this case, let $\widetilde Q_0 \equiv 8nQ_0$. Then from (8.32), we conclude that

$$\begin{aligned}
\int_{\mathbb{R}^n}[\mathcal{G}_N^0(I_\alpha^{\rm loc}a)(x)]^q[\omega(x)]^q\,dx &= \int_{\widetilde Q_0}[\mathcal{G}_N^0(I_\alpha^{\rm loc}a)(x)]^q[\omega(x)]^q\,dx\\
&\quad+\int_{Q(x_0,R_0+8)\setminus\widetilde Q_0}\cdots \equiv \mathrm{I}_1+\mathrm{I}_2. \qquad (8.33)
\end{aligned}$$

For $\mathrm{I}_1$, similarly to the proof of Case 1, we have

$$\begin{aligned}
\mathrm{I}_1 &\lesssim |Q_0|^{1-\frac{q}{q_1}}\left\{\int_{\widetilde Q_0} [\mathcal{G}_N^0(I_\alpha^{\rm loc}(a))(x)]^{q_1}[\omega(x)]^{q_1}\,dx\right\}^{q/q_1}\\
&\lesssim |Q_0|^{1-\frac{q}{q_1}}\left\{\int_{\mathbb{R}^n} |I_\alpha^{\rm loc}(a)(x)|^{q_1}[\omega(x)]^{q_1}\,dx\right\}^{q/q_1}\\
&\lesssim |Q_0|^{1-\frac{q}{q_1}}\left\{\int_{Q_0} |a(x)|^{p_1}[\omega(x)]^{p_1}\,dx\right\}^{q/p_1}\\
&\lesssim |Q_0|^{1-\frac{q}{q_1}}\left\{\int_{Q_0} |a(x)|^{\widetilde q}[\omega(x)]^{p}\,dx\right\}^{q/\widetilde q}\left\{\int_{Q_0} [\omega(x)]^{\frac{p_1(\widetilde q-p)}{\widetilde q-p_1}}\,dx\right\}^{(\frac{1}{p_1}-\frac{1}{\widetilde q})q}\\
&\lesssim \left\{|Q_0|^{\frac1q-\frac1{q_1}}\left(\int_{Q_0}[\omega(x)]^p\,dx\right)^{\frac1{\widetilde q}-\frac1p}\left(\int_{Q_0}[\omega(x)]^{\frac{p_1(\widetilde q-p)}{\widetilde q-p_1}}\,dx\right)^{\frac1{p_1}-\frac1{\widetilde q}}\right\}^q \lesssim 1. \qquad (8.34)
\end{aligned}$$

To estimate $\mathrm{I}_2$, let $x \in (\widetilde Q_0)^\complement$, $\psi \in \mathcal{D}_N^0(\mathbb{R}^n)$, $t \in (0,1)$ and $P_\psi^s$ be the Taylor expansion of $\psi$ about $(x-x_0)/t$ with degree $s$, where $s \equiv \lfloor n(1/p-1)\rfloor$. Then we have

$$\begin{aligned}
|I_\alpha^{\rm loc}(a)*\psi_t(x)| &= \frac{1}{t^n}\left|\int_{\mathbb{R}^n} I_\alpha^{\rm loc}(a)(y)\psi\left(\frac{x-y}{t}\right)dy\right|\\
&= \frac{1}{t^n}\left|\int_{\mathbb{R}^n} I_\alpha^{\rm loc}(a)(y)\left[\psi\left(\frac{x-y}{t}\right)-P_\psi^s\left(\frac{x-y}{t}\right)\right]dy\right|
\end{aligned}$$

$$\begin{aligned}
&\le \frac{1}{t^n}\int_{2\sqrt{n}Q_0} |I_\alpha^{\rm loc}(a)(y)|\left|\psi\left(\frac{x-y}{t}\right)-P_\psi^s\left(\frac{x-y}{t}\right)\right|dy\\
&\quad+\frac{1}{t^n}\int_{Q(x_0,\frac{|x-x_0|}{2\sqrt{n}})\setminus 2\sqrt{n}Q_0}\cdots+\frac{1}{t^n}\int_{[Q(x_0,\frac{|x-x_0|}{2\sqrt{n}})]^\complement}\cdots\\
&\equiv \mathrm{E}_1+\mathrm{E}_2+\mathrm{E}_3. \qquad (8.35)
\end{aligned}$$

To estimate $\mathrm{E}_1$, as $x\in(\widetilde{Q}_0)^\complement$ and $t\in(0,1)$, we see that $\mathrm{E}_1\ne 0$ implies that $t>|x-x_0|/2$. From

$$\omega^{\frac{nr}{nr-n-r\alpha}}\in A_1^{\rm loc}(\mathbb{R}^n)$$

and the definition of $A_1^{\rm loc}(\mathbb{R}^n)$, it follows that $\omega\in A_1^{\rm loc}(\mathbb{R}^n)$. Let $q_2\equiv(2q_1-1)/q_1$. Then since $\omega\in A_1^{\rm loc}(\mathbb{R}^n)\subset A_{q_2}^{\rm loc}(\mathbb{R}^n)$, Lemma 2.3(iv) implies that $\omega^{-q_1'}=\omega^{1-q_2'}\in A_{q_2'}^{\rm loc}(\mathbb{R}^n)$. From these facts, Taylor's remainder theorem, Hölder's inequality, Lemmas 8.9 and 2.3(v), Remark 2.4 with $\widetilde{C}=2\sqrt{n}$, (8.25) and (2.2), we infer that

$$\begin{aligned}
\mathrm{E}_1&\lesssim \frac{1}{t^{n+s+1}}\|I_\alpha^{\rm loc}(a)\|_{L^{q_1}_{\omega^{q_1}}(\mathbb{R}^n)}\left\{\sum_{\alpha\in\mathbb{Z}_+^n,\,|\alpha|=s+1}\int_{2\sqrt{n}Q_0}\left|\partial^\alpha\psi\left(\frac{\xi}{t}\right)\right|^{q_1'}\right.\\
&\quad\left.\times|y-x_0|^{(s+1)q_1'}[\omega(y)]^{-q_1'}\,dy\right\}^{1/q_1'}\\
&\lesssim \frac{R_0^{s+1}}{|x-x_0|^{n+s+1}}\|a\|_{L^{p_1}_{\omega^{p_1}}(\mathbb{R}^n)}\left\{\int_{2\sqrt{n}Q_0}[\omega(y)]^{-q_1'}\,dy\right\}^{1/q_1'}\\
&\lesssim \frac{R_0^{s+1}}{|x-x_0|^{n+s+1}}\left\{\int_{Q_0}|a(x)|^{\widetilde{q}}[\omega(x)]^p\,dx\right\}^{1/\widetilde{q}}\\
&\quad\times\left\{\int_{Q_0}[\omega(x)]^{\frac{p_1(\widetilde{q}-p)}{\widetilde{q}-p_1}}\,dx\right\}^{\frac{1}{p_1}-\frac{1}{\widetilde{q}}}\left\{\int_{Q_0}[\omega(y)]^{-q_1'}\,dy\right\}^{1/q_1'}\\
&\lesssim \frac{R_0^{s+1}}{|x-x_0|^{n+s+1}}|Q_0|^{\frac{1}{q_1}-\frac{1}{q}}\left\{\int_{Q_0}[\omega(y)]^{-q_1'}\,dy\right\}^{1/q_1'}\\
&\lesssim \frac{R_0^{s+1}}{|x-x_0|^{n+s+1}}|Q_0|^{1-\frac{1}{q}}\left[\operatorname*{ess\,inf}_{z\in Q_0}\omega(z)\right]^{-1}, \qquad (8.36)
\end{aligned}$$

where $\gamma\in(0,1)$ and $\xi\equiv\gamma(x-y)+(1-\gamma)(x-x_0)$. Similarly to the estimates of $G_2$ and $G_3$ in the proof of Theorem 8.2, we have

$$\max\{\mathrm{E}_2,\,\mathrm{E}_3\}\lesssim\frac{R_0^{n+s+1}}{|x-x_0|^{n+s+1-\alpha}}|Q_0|^{-1/p}\left[\operatorname*{ess\,inf}_{z\in Q_0}\omega(z)\right]^{-1}. \qquad (8.37)$$

Thus, from (8.35), (8.36), (8.37) and the facts that $|x-x_0|\ge 4nR_0$ and $1/q=1/p-\alpha/n$, we deduce that

$$\begin{aligned}
|[I_\alpha^{\rm loc}(a)]*\psi_t(x)|&\lesssim\frac{R_0^{s+1}}{|x-x_0|^{n+s+1}}|Q_0|^{1-\frac{1}{q}}\left[\operatorname*{ess\,inf}_{z\in Q_0}\omega(z)\right]^{-1}\\
&\quad+\frac{R_0^{n+s+1}}{|x-x_0|^{n+s+1-\alpha}}|Q_0|^{-1/p}\left[\operatorname*{ess\,inf}_{z\in Q_0}\omega(z)\right]^{-1}\\
&\lesssim\frac{R_0^{n+s+1}}{|x-x_0|^{n+s+1-\alpha}}|Q_0|^{-1/p}\left[\operatorname*{ess\,inf}_{z\in Q_0}\omega(z)\right]^{-1},
\end{aligned}$$

which together with the arbitrariness of $\psi \in \mathcal{D}_N^0(\mathbb{R}^n)$ implies that for any $x \in (\widetilde{Q}_0)^\complement$,

$$\mathcal{G}_N^0(I_\alpha^{\rm loc}(a))(x) \lesssim \frac{R_0^{n+s+1}}{|x-x_0|^{n+s+1-\alpha}}|Q_0|^{-1/p}\Big[\operatorname*{ess\,inf}_{z\in Q_0} \omega(z)\Big]^{-1}. \tag{8.38}$$

As $s = \lfloor n(1/p-1)\rfloor$ and $1/q = 1/p - \alpha/n$, we know that $(n+s+1-\alpha)q - n > 0$. Let $m_0$ be the integer such that $2^{m_0}nR_0 \le R_0 + 8 < 2^{m_0+1}nR_0$. From

$$\omega^{\frac{nr}{nr-n-r\alpha}} \in A_1^{\rm loc}(\mathbb{R}^n)$$

and the definition of $A_1^{\rm loc}(\mathbb{R}^n)$, we see that $\omega^q \in A_1^{\rm loc}(\mathbb{R}^n)$, which together Lemma 2.3(i) implies that there exists a function $\widetilde{\omega}$ on $\mathbb{R}^n$ such that $\widetilde{\omega} = \omega$ on $Q(x_0, R_0+8)$ and $\widetilde{\omega}^q \in A_1(\mathbb{R}^n)$. From this, $1/q = 1/p - \alpha/n$, (8.38), (i) and (viii) of Lemma 2.3, the definition of $A_1^{\rm loc}(\mathbb{R}^n)$ and $(n+s+1-\alpha)q - n > 0$, we infer that

$$\begin{aligned}
\mathrm{I}_2 &\lesssim \int_{Q(x_0,R_0+8)\setminus\widetilde{Q}_0} \left\{\frac{R_0^{n+s+1}}{|x-x_0|^{n+s+1-\alpha}}|Q_0|^{-1/p}\Big[\operatorname*{ess\,inf}_{z\in Q_0}\omega(z)\Big]^{-1}\right\}^q [\widetilde{\omega}(x)]^q\,dx\\
&\lesssim \sum_{k=3}^{m_0}\int_{2^{k+1}nQ_0\setminus 2^k nQ_0}\left\{\frac{R_0^{n+s+1-n/q-\alpha}}{(2^kR_0)^{n+s+1-\alpha}}\Big[\operatorname*{ess\,inf}_{z\in Q_0}\omega(z)\Big]^{-1}\right\}^q[\widetilde{\omega}(x)]^q\,dx\\
&\lesssim \sum_{k=3}^{m_0}\frac{R_0^{-n}}{2^{k[(n+s+1-\alpha)q-n]}}\Big[\operatorname*{ess\,inf}_{z\in Q_0}\omega(z)\Big]^{-q}\int_{Q_0}[\omega(x)]^q\,dx\\
&\lesssim \sum_{k=3}^{m_0}\frac{R_0^{-n}}{2^{k[(n+s+1-\alpha)q-n]}}|Q_0|\Big[\operatorname*{ess\,inf}_{z\in Q_0}\omega(z)\Big]^{-q}\operatorname*{ess\,inf}_{z\in Q_0}[\omega(z)]^q\\
&\lesssim \sum_{k=3}^{m_0}2^{-k[(n+s+1-\alpha)q-n]}\lesssim 1,
\end{aligned}$$

which together with (8.33) and (8.34) implies (8.31) in Case 2. This finishes the proof of Theorem 8.11. ■

Pseudo-differential operators have been extensively studied in the literature, and they are important in partial differential equations and harmonic analysis; see, for example, [46, 51, 45, 50]. Now, we recall the notion of pseudo-differential operators of order zero.

Definition 8.12. Let $\delta$ be a real number. A *symbol* in $S^0_{1,\delta}(\mathbb{R}^n)$ is a smooth function $\sigma(x,\xi)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ such that for all multi-indices $\alpha$ and $\beta$, the following estimate holds:

$$|\partial_x^\alpha\partial_\xi^\beta\sigma(x,\xi)| \le C(\alpha,\,\beta)(1+|\xi|)^{-|\beta|+\delta|\alpha|},$$

where $C(\alpha,\,\beta)$ is a positive constant independent of $x$ and $\xi$. Let $f$ be a Schwartz function and $\hat{f}$ denote its Fourier transform. The operator $T$ given by setting, for all $x \in \mathbb{R}^n$,

$$Tf(x) \equiv \int_{\mathbb{R}^n}\sigma(x,\xi)e^{2\pi i x\xi}\hat{f}(\xi)\,d\xi$$

is called a *pseudo-differential operator* with symbol $\sigma(x,\xi) \in S^0_{1,\delta}(\mathbb{R}^n)$.

In the rest of this section, let

$$\phi(t) \equiv (1+t)^\alpha \tag{8.39}$$

for all $\alpha \in (0,\infty)$ and $t \in (0,\infty)$. Recall that a *weight* always means a locally integrable function which is positive almost everywhere. The following weight class $A_p(\phi)$ was introduced by Tang [50].

Definition 8.13. A weight $\omega$ is said to belong to the *class* $A_p(\phi)$ for $p\in(1,\infty)$ if there exists a positive constant $C$ such that for all cubes $Q\equiv Q(x,r)$,

$$\left(\frac{1}{\phi(|Q|)|Q|}\int_Q \omega(y)\,dy\right)\left(\frac{1}{\phi(|Q|)|Q|}\int_Q [\omega(y)]^{-\frac{1}{p-1}}\,dy\right)^{p-1}\le C.$$

A weight $\omega$ is said to belong to the *class* $A_1(\phi)$ if there exists a positive constant $C$ such that for all cubes $Q\subset\mathbb{R}^n$ and almost every $x\in\mathbb{R}^n$, $M_\phi(\omega)(x)\le C\omega(x)$, where for all $x\in\mathbb{R}^n$,

$$M_\phi(\omega)(x)\equiv\sup_{Q\ni x}\frac{1}{\phi(|Q|)|Q|}\int_Q |f(y)|\,dy,$$

and the supremum is taken over all cubes $Q\subset\mathbb{R}^n$ and $Q\ni x$.

Remark 8.14. By the definition of $A_p(\phi)$, we see that $A_p(\phi)\subset A_p^{\rm loc}(\mathbb{R}^n)$, and that $\phi(t)\ge1$ for all $t\in(0,\infty)$ implies that $A_p(\mathbb{R}^n)\subset A_p(\phi)$ for all $p\in[1,\infty)$. Moreover, if $\omega\in A_p(\phi)$, then $\omega(x)\,dx$ may not be a doubling measure; see the remark of Section 7 in [49] for the details.

Similarly to the classical Muckenhoupt weights, we recall some properties of weights $\omega\in A_\infty(\phi)\equiv\bigcup_{1\le p<\infty}A_p(\phi)$. Lemmas 8.15 and 8.16 below are respectively Lemmas 7.3 and 7.4 in [49].

Lemma 8.15.

(i) *If $1\le p_1<p_2<\infty$, then $A_{p_1}(\phi)\subset A_{p_2}(\phi)$.*
(ii) *For $p\in(1,\infty)$, $\omega\in A_p(\phi)$ if and only if $\omega^{-1/(p-1)}\in A_{p'}(\phi)$, where $1/p+1/p'=1$.*
(iii) *If $\omega\in A_p(\phi)$ for $p\in[1,\infty)$, then there exists a positive constant $C$ such that for any cube $Q\subset\mathbb{R}^n$ and measurable set $E\subset Q$,*

$$\frac{|E|}{\phi(|Q|)|Q|}\le C\left(\frac{\omega(E)}{\omega(Q)}\right)^{1/p}.$$

Lemma 8.16. *Let $T$ be an $S^0_{1,0}(\mathbb{R}^n)$ pseudo-differential operator. Then for $\omega\in A_p(\phi)$ with $p\in(1,\infty)$, there exists a positive constant $C(p,\omega)$ such that for all $f\in L^p_\omega(\mathbb{R}^n)$,*

$$\|Tf\|_{L^p_\omega(\mathbb{R}^n)}\le C(p,\omega)\|f\|_{L^p_\omega(\mathbb{R}^n)}.$$

Lemma 8.17 below is just [18, Lemma 6].

Lemma 8.17. *Let $T$ be an $S^0_{1,0}(\mathbb{R}^n)$ pseudo-differential operator. If $\psi\in\mathcal{D}(\mathbb{R}^n)$, then $T_tf=\psi_t*Tf$ has a symbol $\sigma_t$ which satisfies that*

$$|\partial_x^\beta\partial_\xi^\alpha\sigma(x,\xi)|\le C(\alpha,\,\beta)(1+|\xi|)^{-|\alpha|},$$

*and a kernel $K_t(x,\,z)$ which satisfies that*

$$|\partial_x^\beta\partial_z^\alpha K_t(x,z)|\le C(\alpha,\,\beta)|z|^{-n-|\alpha|},$$

*where $C(\alpha,\,\beta)$ is independent of $t$ when $t\in(0,1)$.*

Now, we establish the boundedness on $h^\Phi_\omega(\mathbb{R}^n)$ of $S^0_{1,0}(\mathbb{R}^n)$ pseudo-differential operators as follows.

THEOREM 8.18. *Let $T$ be an $S^0_{1,0}(\mathbb{R}^n)$ pseudo-differential operator, $\Phi$ satisfy Assumption* (A), *$\omega \in A_\infty(\phi)$ and $p_\Phi$ be as in* (2.6)*. If $p_\Phi = p_\Phi^+$ and $\Phi$ is of upper type $p_\Phi^+$, then there exists a positive constant $C(\Phi, \omega)$, depending only on $\Phi$, $q_\omega$ and the weight constant of $A_\infty(\phi)$, such that for all $f \in h^\Phi_\omega(\mathbb{R}^n)$,*

$$\|Tf\|_{h^\Phi_\omega(\mathbb{R}^n)} \le C(\Phi, \omega)\|f\|_{h^\Phi_\omega(\mathbb{R}^n)}.$$

*Proof.* Since $\omega \in A_\infty(\phi)$, we have $\omega \in A_q(\phi)$ for some $q \in (1, \infty)$. To prove Theorem 8.18, as $\omega \in A_\infty(\phi) \subset A^{\rm loc}_\infty(\mathbb{R}^n)$, Theorems 6.4(i) and 3.14 imply that it suffices to show that for all $(\rho, q)_\omega$-single-atoms and $(\rho, q, s)_\omega$-atoms $a$ supported in $Q_0 \equiv Q(x_0, R_0)$ with $R_0 \in (0, 2]$,

$$\|\mathcal{G}^0_N(Ta)\|_{L^\Phi_\omega(\mathbb{R}^n)} \lesssim 1. \tag{8.40}$$

By Theorem 5.6, we may assume that $s$ satisfies $(n+s+1)p_\Phi > nq_\omega(1+\alpha)$, where $p_\Phi$, $q_\omega$ and $\alpha$ are respectively as in (2.6), (2.4) and (8.39).

First, we prove (8.40) for any $(\rho, q)_\omega$-single-atom $a \neq 0$. In this case, $\omega(\mathbb{R}^n) < \infty$. Since $\Phi$ is concave, by Jensen's inequality, Hölder's inequality, Proposition 3.2(ii) and Lemma 8.16 and (2.8) with $t \equiv \omega(\mathbb{R}^n)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Phi(\mathcal{G}^0_N(Ta)(x))\omega(x)\,dx\\
&\qquad \le \omega(\mathbb{R}^n)\Phi\bigg(\frac{1}{\omega(\mathbb{R}^n)}\int_{\mathbb{R}^n} \mathcal{G}^0_N(Ta)(x)\omega(x)\,dx\bigg)\\
&\qquad \le \omega(\mathbb{R}^n)\Phi\bigg(\frac{1}{[\omega(\mathbb{R}^n)]^{1/q}}\bigg\{\int_{\mathbb{R}^n} [\mathcal{G}^0_N(Ta)(x)]^q\omega(x)\,dx\bigg\}^{1/q}\bigg)\\
&\qquad \lesssim \omega(\mathbb{R}^n)\Phi\bigg(\frac{1}{[\omega(\mathbb{R}^n)]^{1/q}}\|Ta\|_{L^q_\omega(\mathbb{R}^n)}\bigg) \lesssim \omega(\mathbb{R}^n)\Phi\bigg(\frac{1}{[\omega(\mathbb{R}^n)]^{1/q}}\|a\|_{L^q_\omega(\mathbb{R}^n)}\bigg)\\
&\qquad \lesssim \omega(\mathbb{R}^n)\Phi\bigg(\frac{1}{\omega(\mathbb{R}^n)\rho(\omega(\mathbb{R}^n))}\bigg) \sim 1,
\end{aligned}$$

which shows (8.40) in this case.

Now, let $a$ be any $(\rho, q, s)_\omega$-atom supported in $Q_0 \equiv Q(x_0, R_0)$ with $R_0 \in (0, 2]$. Let $\widetilde{Q}_0 \equiv 2Q_0$. Then from Jensen's inequality, Hölder's inequality, Proposition 3.2(ii), Lemma 8.16, Lemma 8.15(iii) and (2.8) with $t \equiv \omega(\widetilde{Q}_0)$, it follows that

$$\begin{aligned}
&\int_{\widetilde{Q}_0} \Phi(\mathcal{G}^0_N(Ta)(x))\omega(x)\,dx\\
&\quad \le \omega(\widetilde{Q}_0)\Phi\bigg(\frac{1}{\omega(\widetilde{Q}_0)}\int_{\widetilde{Q}_0} \mathcal{G}^0_N(Ta)(x)\omega(x)\,dx\bigg)\\
&\quad \le \omega(\widetilde{Q}_0)\Phi\bigg(\frac{1}{[\omega(\widetilde{Q}_0)]^{1/q}}\bigg\{\int_{\widetilde{Q}_0} [\mathcal{G}^0_N(Ta)(x)]^q\omega(x)\,dx\bigg\}^{1/q}\bigg)\\
&\quad \lesssim \omega(\widetilde{Q}_0)\Phi\bigg(\frac{1}{[\omega(\widetilde{Q}_0)]^{1/q}}\|Ta\|_{L^q_\omega(\mathbb{R}^n)}\bigg) \lesssim \omega(\widetilde{Q}_0)\Phi\bigg(\frac{1}{[\omega(\widetilde{Q}_0)]^{1/q}}\|a\|_{L^q_\omega(\mathbb{R}^n)}\bigg)\\
&\quad \lesssim \omega(\widetilde{Q}_0)\Phi\bigg(\frac{1}{\omega(\widetilde{Q}_0)\rho(\omega(\widetilde{Q}_0))}\bigg) \sim 1.
\end{aligned} \tag{8.41}$$

For any $\psi \in \mathcal{D}_N^0(\mathbb{R}^n)$ and $t \in (0,1)$, let $K_t(x, x-z)$ be the kernel of $T_t a(x) \equiv \psi_t * Ta(x)$. To estimate $\int_{\mathbb{R}^n \setminus \widetilde{Q}_0} \Phi(\mathcal{G}_N^0(Ta)(x))\omega(x)\,dx$, we consider the following two cases for $R_0$.

*Case 1:* $R_0 \in (0,1)$. In this case, we expand $K_t(x, x-z)$ into a Taylor series about $z = x_0$ such that for any $x \in (\mathbb{R}^n \setminus \widetilde{Q}_0)$,

$$\psi_t * Ta(x) = \int_{\mathbb{R}^n} K_t(x, x-z)a(z)\,dz = \int_{Q_0} \sum_{\substack{\alpha \in \mathbb{Z}_+^n,\\ |\alpha| = s+1}} (\partial_z^\alpha K_t)(x, x-\xi)(z-x_0)^\alpha a(z)\,dz,$$

where $\xi \equiv \theta z + (1-\theta)x_0$ for some $\theta \in (0,1)$. As $z$, $x_0 \in Q_0$, we know that $\xi \in Q_0$. Thus, for any $x \in (\mathbb{R}^n \setminus \widetilde{Q}_0)$, $|x-\xi| \sim |x-x_0|$. From the above facts and Lemma 8.17, we deduce that

$$|\psi_t * Ta(x)| \lesssim |x-\xi|^{-(n+s+1)} R_0^{s+1} \|a\|_{L^1(\mathbb{R}^n)} \lesssim |x-x_0|^{-(n+s+1)} |Q_0|^{\frac{s+1}{n}} \|a\|_{L^1(\mathbb{R}^n)},$$

which together with the arbitrariness of $\psi \in \mathcal{D}_N^0(\mathbb{R}^n)$ implies that for all $x \in \mathbb{R}^n \setminus \widetilde{Q}_0$,

$$\mathcal{G}_N^0(Ta)(x) \lesssim |x-x_0|^{-(n+s+1)} |Q_0|^{\frac{s+1}{n}} \|a\|_{L^1(\mathbb{R}^n)}.$$

This, combined with Hölder's inequality, Lemma 8.15(iii) and the definition of $A_p(\phi)$, yields

$$\begin{aligned}
&\int_{\mathbb{R}^n \setminus \widetilde{Q}_0} \Phi(\mathcal{G}_N^0(Ta)(x))\omega(x)\,dx \\
&\quad \le C \int_{\mathbb{R}^n \setminus \widetilde{Q}_0} \Phi(|x-x_0|^{-(n+s+1)} |Q_0|^{\frac{s+1}{n}} \|a\|_{L^1(\mathbb{R}^n)})\omega(x)\,dx \\
&\quad \le C \int_{\mathbb{R}^n \setminus \widetilde{Q}_0} \Phi(|Q_0|^{\frac{s+1}{n}} |x-x_0|^{-(n+s+1)} \|a\|_{L^q_\omega(\mathbb{R}^n)} \phi(|Q_0|)|Q_0|[\omega(Q_0)]^{-1/q})\omega(x)\,dx \\
&\quad \le C \int_{\mathbb{R}^n \setminus \widetilde{Q}_0} \Phi\left(|Q_0|^{\frac{s+1}{n}} |x-x_0|^{-(n+s+1)} \frac{\phi(|Q_0|)|Q_0|}{\omega(Q_0)\rho(\omega(Q_0))}\right)\omega(x)\,dx \\
&\quad \le C \sum_{k=1}^\infty \int_{2^k Q_0} \Phi\left(|Q_0|^{\frac{s+1}{n}} (2^k R_0)^{-(n+s+1)} \frac{\phi(|Q_0|)|Q_0|}{\omega(Q_0)\rho(\omega(Q_0))}\right)\omega(x)\,dx \\
&\quad \le C \left\{ \sum_{k=1}^{m_0} \int_{2^k Q_0} \Phi\left(|Q_0|^{\frac{s+1}{n}} (2^k R_0)^{-(n+s+1)} \frac{|Q_0|}{\omega(Q_0)\rho(\omega(Q_0))}\right)\omega(x)\,dx \right. \\
&\qquad \left. + \sum_{k=m_0+1}^\infty \int_{2^k Q_0} \cdots \right\} \equiv C(\mathrm{I}_1 + \mathrm{I}_2), 
\end{aligned} \tag{8.42}$$

where the integer $m_0$ satisfies $2^{m_0-1} \le 1/R_0 < 2^{m_0}$.

To estimate $\mathrm{I}_1$, for any $k \in \{1, \ldots, m_0\}$, by the choice of $m_0$ and $R_0 \in (0,1)$, we know that $2^k R_0^n \le 1$, which, together with Jensen's inequality, the lower type $p_\Phi$ property of $\Phi$, Lemma 8.15(iii) and the fact that $(n+s+1)p_\Phi > nq(1+\alpha)$, implies that

$$\begin{aligned}
\mathrm{I}_1 &\lesssim \sum_{k=1}^{m_0} \omega(2^k Q_0)\Phi\left(\frac{1}{\omega(2^k Q_0)} \int_{2^k Q_0} 2^{-k(n+s+1)} \{\omega(Q_0)\rho(\omega(Q_0))\}^{-1}\omega(x)\,dx\right) \\
&\lesssim \sum_{k=1}^{m_0} 2^{-k(n+s+1)p_\Phi} \frac{\omega(2^k Q_0)}{\omega(Q_0)} \lesssim \sum_{k=1}^{m_0} 2^{knq}[\phi(|2^k Q_0|)]^q 2^{-k(n+s+1)p_\Phi} \\
&\lesssim \sum_{k=1}^{m_0} 2^{-k[(n+s+1)p_\Phi - nq]} \lesssim 1.
\end{aligned} \tag{8.43}$$

For $\mathrm{I}_2$, similarly to the estimate of $\mathrm{I}_1$, we have

$$\begin{aligned}\mathrm{I}_2 &\lesssim \sum_{k=m_0+1}^{\infty} \omega(2^kQ_0)\Phi\bigg(\frac{1}{\omega(2^kQ_0)}\int_{2^kQ_0} 2^{-k(n+s+1)}\{\omega(Q_0)\rho(\omega(Q_0))\}^{-1}\omega(x)\,dx\bigg)\\ &\lesssim \sum_{k=m_0+1}^{\infty} 2^{-k(n+s+1)p_\Phi}\frac{\omega(2^kQ_0)}{\omega(Q_0)} \lesssim \sum_{k=m_0+1}^{\infty} 2^{knq}[\phi(|2^kQ_0|)]^q 2^{-k(n+s+1)p_\Phi}\\ &\lesssim \sum_{k=m_0+1}^{\infty} 2^{-k[(n+s+1)p_\Phi-q(\alpha+1)]} \lesssim 1,\end{aligned}$$

which together with (8.43), (8.42) and (8.41) implies (8.40) in Case 1.

*Case 2:* $R_0\in[1,2]$. In this case, for any $x\in\mathbb{R}^n\setminus\widetilde{Q}_0$ and $z\in Q_0$, we have

$$|x-z|\sim|x-x_0|$$

and $|x-x_0|>1$. From this and [46, p. 235, (9)], we infer that for any positive integer $M$, there exists a positive constant $C(M)$ such that

$$|K_t(x,x-z)|\le C(M)|x-x_0|^{-M},$$

which implies that for any $x\in\mathbb{R}^n\setminus\widetilde{Q}_0$,

$$|\psi_t*Ta(x)|\le\int_{\mathbb{R}^n}|K_t(x,x-z)a(z)|\,dz\lesssim|x-x_0|^{-M}\|a\|_{L^1(\mathbb{R}^n)}.$$

This, combined with the arbitrariness of $\psi\in\mathcal{D}_N^0(\mathbb{R}^n)$, shows that for any $x\in\mathbb{R}^n\setminus\widetilde{Q}_0$,

$$\mathcal{G}_N^0(Ta)(x)\lesssim|x-x_0|^{-M}\|a\|_{L^1(\mathbb{R}^n)}. \tag{8.44}$$

Take $M>nq(1+\alpha)/p_\Phi$. By Jensen's inequality, (8.44), Hölder's inequality and Lemma 8.15(iii), we have

$$\begin{aligned}&\int_{\mathbb{R}^n\setminus\widetilde{Q}_0}\Phi(\mathcal{G}_N^0(Ta)(x))\omega(x)\,dx\\ &\quad\lesssim\int_{\mathbb{R}^n\setminus\widetilde{Q}_0}\Phi(|x-x_0|^{-M}\|a\|_{L^1(\mathbb{R}^n)})\omega(x)\,dx\\ &\quad\lesssim\sum_{k=1}^{\infty}\int_{2^kQ_0}\Phi\bigg((2^kR_0)^{-M}\frac{\phi(|Q_0|)|Q_0|}{\omega(Q_0)\rho(\omega(Q_0))}\bigg)\omega(x)\,dx\\ &\quad\lesssim\sum_{k=1}^{\infty}\omega(2^kQ_0)\Phi\bigg((2^kR_0)^{-M}\frac{1}{\omega(Q_0)\rho(\omega(Q_0))}\bigg)\\ &\quad\lesssim\sum_{k=1}^{\infty}2^{-kMp_\Phi}R_0^{-Mp_\Phi}\frac{\omega(2^kQ_0)}{\omega(Q_0)}\lesssim\sum_{k=1}^{\infty}2^{-k(Mp_\Phi-nq)}\phi(|2^kQ_0|)R_0^{-Mp_\Phi}\\ &\quad\lesssim\sum_{k=1}^{\infty}2^{-k[Mp_\Phi-nq(1+\alpha)]}R_0^{-(Mp_\Phi-nq\alpha)}\lesssim\sum_{k=1}^{\infty}2^{-k[Mp_\Phi-nq(1+\alpha)]}\lesssim 1,\end{aligned}$$

which together with (8.41) implies (8.40) in Case 2. This finishes the proof of Theorem 8.18. ∎

Remark 8.19. Let $p\in(0,1]$. Theorem 8.18 with $\omega\equiv1$ and $\Phi(t)\equiv t^p$ for all $t\in(0,\infty)$ was obtained by Goldberg [18, Theorem 4]; moreover, Theorem 8.18 with $\Phi(t)\equiv t^p$ for all

$t \in (0, \infty)$ was obtained by Tang [49, Theorem 7.3]. Also, Theorem 8.18 with $\omega \in A_1(\mathbb{R}^n)$ and $\Phi(t) \equiv t$ for all $t \in (0, \infty)$ was obtained by Lee, C.-C. Lin and Y.-C. Lin [32, Theorem 2].

By Theorems 8.18 and 7.5, [46, p. 233, (4)] and the proposition in [46, p. 259], we have the following result.

COROLLARY 8.20. *Let $T$ be an $S^0_{1,0}(\mathbb{R}^n)$ pseudo-differential operator, $\Phi$ satisfy Assumption* (A), *$\omega \in A_\infty(\phi)$ and $p_\Phi$ be as in* (2.6). *If $p_\Phi = p_\Phi^+$ and $\Phi$ is of upper type $p_\Phi^+$, then there exists a positive constant $C(\Phi, \omega)$ such that for all $f \in \mathrm{bmo}_{\rho,\omega}(\mathbb{R}^n)$,*

$$\|Tf\|_{\mathrm{bmo}_{\rho,\omega}(\mathbb{R}^n)} \le C(\Phi, \omega)\|f\|_{\mathrm{bmo}_{\rho,\omega}(\mathbb{R}^n)}.$$

## References


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