1. Introduction

From the Author. The first question is: "What is the use of this theorem?" I can only answer: "Probably no use, at the immediate present". Indeed (see also the Introduction of [9]) the basis problem, after its birth in Banach's book in 1932, was strongly studied till 1973–75, when Enflo gave a negative answer and Ovsepian–Pełczyński a particular kind of positive answer. However the more general basis problem, "Does a good kind of basis exist in every infinite-dimensional separable Banach space?" was practically still unsolved. But after 1975 this problem was forsaken; probably the researchers were tired to study it, since it continued to appear difficult, while other interesting questions seemed to be more attractive and able to give gratification. When a rock stops the flux of a river, the current simply forks the rock and the river continues to flow; however the rock remains with all the unknown closed doors hidden in the inside.

The second question is: "Is it really worth trying to improve the basis with fixed blocks and uniformly controlled permutations of [9], by means of a basis with only uniformly controlled permutations?" I can answer that the use of blocks is a rather rough representation of the elements of the space, since only a subsequence of the sequence of partial sums converges to the given element. If we renounce the facility of the blocks and we wish to be concerned only with actual series, we pass to a mathematical problem more sophisticated and of a quite different order of difficulty. Only a small part of the ideas involved in this work were already present in [9], indeed I have been compelled to spend a time twice that long; also because, in an unkown field, only the second time the landscape begins to emerge and it is possible to recognize nice short cuts. Moreover I was not interested in the proof of the simple existence of this kind of basis, my actual aim was to give a direct construction in the following sense. Let X be a general separable Banach space. By definition, it is always possible to get a sequence of elements whose linear span is dense in X; then my construction directly works on this sequence. Therefore I did not look for shorter proofs of the mere existence.

On the other hand this basis seems to be the best possible kind of basis for the general separable Banach space; moreover, regarding the actual applications, this basis is comfortable enough, since the permutations cannot be quite arbitrary (their use would then be impractical) but they are uniformly finitely controlled; this basis is also equipped with a local alternative principle which much simplifies the representation of elements and practically, inside each block, we can express the norm in terms only of the l_∞-norm, l_1-norm and euclidean norm.

Coming back to the first question, apart from an intense irrational attraction, I have been involved in this problem due to my personal conviction that in the future there could be a greater connection between physics and spaces of infinite dimensions. The current models with finite—not all linear—dimensions have difficulty in explaining many questions which continuosly arise (the unknown black energy cause of recession of galaxies, absence of anti-matter; in particular inside the nuclei, in the black holes and in the first instant of the Universe, many physical phenomena go towards the infinite), moreover it seems impossible to immerse our Universe in a euclidean space; just this fact has been the reason of looking for a direct construction.

After a hard effort, I wish to dedicate this work to the whole mathematical community born after the Banach book, in particular to the great researchers of the first fifty years, the eagles that flew in the infinite, towards new horizons of science and of the future of humanity.

1.1. Aim of the work. The aim of this work is to determine a system of coordinate axes which characterizes separable Banach space, just as an orthogonal basis characterizes a Hilbert space. Section 2 deals with biorthogonal systems, in particular the use of the Walsh matrix both in l^n_∞ and in l^n_1, and finite transformability; we also introduce the "generating form" of a finite biorthogonal system. Section 3 concerns the actual construction of a basis with permutations in spaces of type 1, starting from any sequence complete in the space. Section 4 concerns the main properties of this construction. Section 5 describes the properties of the basis with permutations in spaces of type 1. Section 6 deals with the special case of spaces of type > 1, since in this case the structure of the space is much more regular, hence the construction is much simpler. An extension of this work to the complex case appears in [11] and the only changes are in the generating form of a biorthogonal system.

1.2. Definitions and recalls. A Banach space Y is said to be *finitely represented* in another Banach space X if, for each $\varepsilon > 0$ and for each finite-dimensional subspace $Y_\circ$ of Y, there exist a subspace $X_\circ$ of X and an isomorphism $T : Y_\circ \to X_\circ$ with $\|T\|\|T^{-1}\| < 1 + \varepsilon$. Moreover two sequences $\{y_n\}$ and $\{z_n\}$ are said to be $(1+\varepsilon)$-*equivalent* if there exist two positive numbers H and K with $HK < 1 + \varepsilon$ such that, for each sequence $\{a_n\}_{n=1}^m$ of numbers,

$$\frac{1}{H}\Big\|\sum_{n=1}^m a_n y_n\Big\| \le \Big\|\sum_{n=1}^m a_n z_n\Big\| \le K\Big\|\sum_{n=1}^m a_n y_n\Big\|.$$

We say that X has *type* 1 if l_1 is finitely represented in X, otherwise we say that X has *type* > 1; analogously we say that X has *cotype* ∞ if c_0 is finitely represented in X, otherwise we say that X has *finite cotype*. Moreover X is said to have *well complemented subspaces* if, for each m, X has an m-dimensional subspace X_m with a projection $P : X \to X_m$ with $\|P\| < K$ where K does not depend on m. A well known and important case of a space with well complemented subspaces is provided by the following theorem of Pisier ([5], see also [3, p. 112]):

Theorem I*. *If l_1 is not finitely represented in X, then there exists $C > 0$ such that, for each positive integer m and for each $\varepsilon > 0$, there exists another integer $N = N(m, \varepsilon)$*

such that every N-dimensional subspace of X has a sequence $\{e_n\}_{n=1}^m$ which is $(1+\varepsilon)$-equivalent to the natural basis of l_2^m and there exists a projection $P : X \to \operatorname{span}\{e_n\}_{n=1}^m$ with $\|P\| < C$.

But in general X does not have well complemented subspaces; we only recall an example, also due to Pisier [6], of a space $\widehat{X}$ such that, for each m-dimensional subspace X_m of $\widehat{X}$ and for each projection $P : \widehat{X} \to X_m$, we have $\|P\| > \delta\sqrt{m}$ where δ does not depend on X_m.

Let now $\{x_n, x_n^*\}$ be a *biorthogonal system* of a Banach space X, that is, $\{x_n\} \subset X$ and $\{x_n^*\} \subset X^*$ (the dual space) with $x_m^*(x_n) = \delta_{m,n}$ for all m and n. We say that $\{x_n\}$ is *uniformly minimal* if $\{x_n\}$ and $\{x_n^*\}$ are both bounded; we say that $\{x_n\}$ is a *basis with fixed brackets* of X if there exists an increasing sequence $\{p(m)\}$ of integers such that, for each $\overline{x} \in X$, setting $p(0) = 0$, we have

$$\overline{x} = \sum_{m=0}^{\infty} \Big(\sum_{n=p(m)+1}^{p(m+1)} x_n^*(\overline{x}) x_n \Big).$$

If $p(m) = m$ for each m, then $\{x_n\}$ is called a *Schauder basis* of X.

All the separable Banach spaces known till 1973 had a Schauder basis, but after the paper of Enflo [1] the panorama quite changed and the lack of a Schauder basis turned out to be a very common fact: for instance c_0 and l_p for $1 \le p\ (\ne 2) < \infty$ have subspaces without a basis with fixed brackets. Therefore we need another kind of basis, with the best possible properties, such that the existence is guaranteed in every separable Banach space; hence the most natural way is to weaken the definition of the basis with fixed brackets, and precisely: either to add permutations, or to renounce fixed brackets. In [9] and [10] we proved the existence of a *basis with fixed brackets and quasi fixed permutations*: that is, for a biorthogonal system $\{x_n, x_n^*\}$ as above, there exist two increasing sequences $\{q(m)\}$ and $\{p(m)\}$ of integers such that, to each $\overline{x} \in X$, there corresponds another sequence $\{\overline{q}(m)\}$ of integers such that, setting $p(0) = q(0) = \overline{q}(0) = 0$,

$$\overline{x} = \sum_{m=0}^{\infty} \Big(\sum_{n=p(m)+1}^{p(m+1)} x_{\overline{\pi}(n)}^*(\overline{x}) x_{\overline{\pi}(n)} \Big),$$

where, for each m, $\{\overline{\pi}(n)\}_{n=\overline{q}(m)+1}^{\overline{q}(m+1)}$ is permutation of $\{n\}_{n=\overline{q}(m)+1}^{\overline{q}(m+1)}$ with $q(m) \le \overline{q}(m) < q(m+1)$. This kind of basis, joined with the negative answer for the basis with fixed brackets, was an already sufficient approximation of the border between existence and nonexistence. The aim of this paper is to get this border into focus, in order to find the kind of basis which characterizes the general separable Banach space, just as an orthogonal basis is characteristic of a Hilbert space. With this perspective we have only two possibilities of further improvements of the basis with fixed brackets and quasi fixed permutations: either to improve the brackets by passing to a series, or to eliminate the permutations by passing to individual brackets; we will consider these two ways in the next subsection.

1.3. The basis characteristic of the general separable Banach space. Let $\{x_n, x_n^*\}$ be a biorthogonal system of a Banach space X. We say that $\{x_n\}$ is a *basis with permutations* of X if, for for each $\overline{x} \in X$, there exists a permutation $\{\overline{\pi}(n)\}$ of

$\{n\}$ such that

$$\overline{x} = \sum_{n=1}^{\infty} x^*_{\overline{\pi}(n)}(\overline{x})x_{\overline{\pi}(n)}. \tag{1}$$

But this simple definition is probably not practical for research in Banach spaces, since the permutations $\{\overline{\pi}(n)\}$ could be quite arbitrary. Therefore let us look for a more useful definition: we can associate to every $\{\overline{\pi}(n)\}$ a sequence $\{\overline{p}(m)\}$ of positive integers which measures its degree of permutation with respect to $\{n\}$, that is, for each m, $\{n\}_{n=1}^{m} \subseteq \{\overline{\pi}(n)\}_{n=1}^{\overline{p}(m)}$. Then, proceeding as for the simple convergence and uniform convergence of a series of functions, we can say that $\{x_n\}$ is a *basis with uniform permutations* if there exists an increasing sequence $\{p(m)\}$ of positive integers, independent of $\overline{x}$, such that the $\{\overline{\pi}(n)\}$ in (1) can be chosen such that, for each m, $\{n\}_{n=1}^{m} \subseteq \{\overline{\pi}(n)\}_{n=1}^{p(m)}$. In particular we will say that a basis with uniform permutations is a *basis with uniformly controlled permutations* if in the previous definition $\{\overline{\pi}(n)\}$ can be chosen so that, for each m,

$$\{n\}_{n=1}^{m} \subseteq \{\overline{\pi}(n)\}_{n=1}^{p(m)} \subseteq \{n\}_{n=1}^{p(m+1)}. \tag{2}$$

We point out, for instance, that for a basis with uniform permutations the set of all the possible permutations $\{\overline{\pi}(n)\}$ in (1) has in general the cardinality $2^{\aleph_0}$ (where $\aleph_0$ is the countable cardinal number); while for a basis with uniformly controlled permutations this cardinality is $\leq \aleph_0$. We prove in this work the following

THEOREM 1. *Every separable Banach space X has a basis with uniformly controlled permutations.*

Moreover we point out that, by the technique of [9], it is possible to prove the following extension property:

THEOREM 2. *Every infinite-dimensional subspace Y of a separable Banach space X has a basis with uniformly controlled permutations which can be extended to a basis of X with uniformly controlled permutations.*

We point out that in the previous theorems the control sequence $\{p(m)\}$ is universal (see the end of Subsection 1.6), that is, independent of the space. We remark that the proof of Theorem 1 (independent of the constructions of [9] and [10]) is in fact a direct construction of this kind of basis, starting from an arbitrary sequence whose closed span fills the space. We also remark that, only concerning the permutations, the definition of a basis with fixed brackets and quasi fixed permutations is a bit better than (2); however, proceeding as in the proof of [10], it would be possible by means of a small modification to turn a basis with uniformly controlled permutations into a basis with fixed brackets and quasi fixed permutations. On the other hand we do not see further possible improvements of the permutations because a *basis with block permutations* (that is, (1) with a fixed sequence $\{q(m)\}$ of increasing positive integers such that $\{\overline{\pi}(n)\}_{n=q(m)+1}^{q(m+1)}$ is a permutation of $\{n\}_{n=q(m)+1}^{q(m+1)}$ for each m and for each $\overline{x} \in X$) in general does not exist, since it would be a particular basis with fixed brackets.

By a *basis with individual brackets* we mean that, in the definition of a basis with fixed brackets, the sequence $(p(m))$ depends on $\overline{x}$; like a basis with individual permutations, this definition does not appear very useful, since these brackets could be quite arbitrary; hence again we will consider a better definition and we will call $\{x_n\}$, with $\{x_n, x_n^*\}$ biorthogonal, a *basis with quasi fixed brackets* if there exists an increasing sequence $(p(m))$ of positive integers such that for each $\overline{x} \in X$, setting $\overline{p}(0) = 0$, we have

$$\overline{x} = \sum_{m=0}^{\infty} \Big(\sum_{n=\overline{p}(m+1)}^{\overline{p}(m+1)} x_n^*(\overline{x}) x_n \Big) \quad \text{with } p(m)+1 \le \overline{p}(m) \le p(m+1)$$

for each m. Hence, by applying the proof of [8], one can also prove the following

THEOREM II*. *If a separable Banach space X has infinite cotype, then there exists a basis with uniformly controlled permutations which is also a basis with quasi fixed brackets, for the same sequence $(p(m))$ of positive integers.*

For spaces of finite cotype the question of existence of this kind of basis is still open. But we expect a negative answer in general for these spaces; for instance we expect a negative answer for the space $\widehat{X}$ of Pisier of Subsection 1.2, even for a basis with individual brackets. Indeed, if $\{x_n\}$, with $\{x_n, x_n^*\}$ biorthogonal, is some kind of basis for $\widehat{X}$, for instance a basis with permutations, we know that, for each m and for each projection $P : X \to \operatorname{span}\{x_n\}_{n=1}^m$, $\|P\| > \delta\sqrt{m}$; this fact suggests the possible existence of an increasing sequence $\{\overline{q}(m)\}$ of integers and of an element $\overline{x} = \sum_{m=1}^{\infty} \overline{x}_m$ with

$$\overline{x}_m = \sum_{n=1}^{\overline{q}(m+1)} \overline{a}_{m,n} x_n, \qquad \Big\| \sum_{n=1}^{\overline{q}(m)} \overline{a}_{m,n} x_n \Big\| > 2^m,$$

$$\|\overline{x}_m\| < \frac{1}{2^m} \ \text{ for each } m, \qquad \Big\| \sum_{n=1}^{m} x_n^*(\overline{x}) x_n \Big\| \to \infty.$$

A consequence of this conjecture is that a basis with uniformly controlled permutations is the kind of basis characteristic of the general separable Banach space.

1.4. Techniques of uniform minimalization. If $\{x_n, x_n^*\}$ is biorthogonal with $\overline{\operatorname{span}}\{x_n\} = X$, a well known technique to pass from $\{x_n, x_n^*\}$ to $\{y_n, y_n^*\}$ *uniformly minimal* (that is, with $\{y_n\}$ and $\{y_n^*\}$ both bounded), always with $\overline{\operatorname{span}}\{y_n\} = X$, is the construction of [4], modification of a lemma of Olevskii; but in this work we prefer to use the following very simple method (see also the first part of Subsection 1.5):

PROPOSITION 3. *Let $\{u_n, u_n^*\}_{n=1}^A$ be biorthogonal in a Banach space X, with $\|u_n\| = 1$ and $\|u_n^*\| < B$ for $1 \le n \le A$. We can extend this finite biorthogonal system to another biorthogonal system $(u_n, u_n^*)_{n=1}^A \cup ((e_{n,k}, e_{n,k}^*)_{k=1}^{2^{4B}})_{n=1}^A$ such that:*

(3.1) *If X has type 1 then, for each n with $1 \le n \le A$, $\|\sum_{i=1}^k e_{n,i}\| = 1$ and $\|e_{n,k}^*\| \le 2$ for $1 \le k \le 2^{4B}$ with $\|e_{n,k}\| = 2$ for $2 \le k \le 2^{4B}$; in this case we set*

$$x_{n,0} = \sum_{f=1}^{2^{4B}} e_{n,f}, \qquad x_{n,0}^* = \frac{1}{2^{4B}} \sum_{f=1}^{2^{4B}} e_{n,f}^* - \frac{u_n^*}{2^{2B}};$$

moreover, for $1 \le k \le 2^{4B}$, we set

$$x_{n,k} = e_{n,k} + \frac{u_n}{2^{2B}}, \qquad x^*_{n,k} = e^*_{n,k} - x^*_{n,0} = e^*_{n,k} - \frac{1}{2^{4B}} \sum_{f=1}^{2^{4B}} e^*_{n,f} + \frac{u^*_n}{2^{2B}};$$

then $((x_{n,k}, x^*_{n,k})_{k=0}^{2^{4B}})_{n=1}^{A}$ is biorthogonal with $\|x_{n,k}\| < 3$ and $\|x^*_{n,k}\| < 5$ for $0 \le k \le 2^{4B}$ and $1 \le n \le A$.

(3.2) *If X has type > 1 then, for each n with $1 \le n \le A$, $(e_{n,k})_{k=1}^{2^{4B}}$ is 1-equivalent to the natural basis of $l_2^{2^{4B}}$ and $(e^*_{n,k})_{k=1}^{2^{4B}}$ is K-equivalent to the natural basis of $l_2^{2^{4B}}$, where K depends only on X; in this case we set*

$$x_{n,0} = \frac{1}{2^{2B}} \sum_{f=1}^{2^{4B}} e_{n,f}, \qquad x^*_{n,0} = \frac{1}{2^{2B}} \sum_{f=1}^{2^{4B}} e^*_{n,f} - \frac{u^*_n}{2^B},$$

moreover, for each k with $1 \le k \le 2^{4B}$, we set

$$x_{n,k} = e_{n,k} + \frac{u_n}{2^B}, \qquad x^*_{n,k} = e^*_{n,k} - \frac{x^*_{n,0}}{2^{2B}} = e^*_{n,k} - \frac{1}{2^{4B}} \sum_{f=1}^{2^{4B}} e^*_{n,f} + \frac{u^*_n}{2^{3B}};$$

then $((x_{n,k}, x^*_{n,k})_{k=0}^{2^{4B}})_{n=1}^{A}$ is biorthogonal with $\|x_{n,k}\| < 2$ and $\|x^*_{n,k}\| < 2K + 1$ for $0 \le k \le 2^{4B}$ and $1 \le n \le A$.

This proposition follows from Theorem 11 of Subsection 2.3, and from (49.1) and (49.2) of the proof of Theorem 22 of Section 6.

We point out that a basis with fixed brackets is not in general uniformly minimal; however every basis with permutations is obviously uniformly minimal (since (1) implies that $x^*_{\overline{\pi}(n)}(\overline{x})x_{\overline{\pi}(n)} \to 0$).

1.5. Organization of the proof. In order to take flesh of heavy formalism off the ideas, in our proofs we will proceed with the following method: For instance if l_1 is finitely represented in X, when we will use a finite sequence $\{e_n\}_{n=1}^m$ in X which is $(1+\varepsilon)$-equivalent to the natural basis of l_1^m, we will always suppose $\varepsilon = 0$. Analogously, if X_m is a finite-dimensional subspace of X and we need a finite-codimensional subspace W of X such that X_m is $(1+\varepsilon)$-orthogonal to W, then, if $\{x_{m,n}\}_{n=1}^{p_m}$ is an ε-net of the unit sphere of X_m and if $\{x^*_{m,n}\}_{n=1}^{p_m} \subset X^*$ with $\|x^*_{m,n}\| = 1 = \|x_{m,n}\| = x^*_{m,n}(x_{m,n})$ for $1 \le n \le p_m$, then setting $W = X \cap \bigcap_{n=1}^{p_m} x^*_{m,n,\perp}$ we say that $\|x + w\| \ge \max\{\|x\|, \|w\|\backslash 2\}$ for each $x \in X_m$ and $w \in W$, that is, again we suppose $\varepsilon = 0$. More precisely, setting $\varepsilon = 0$ we only mean that the effect of ε is absorbed in other approximations which already appear in the proofs.

Since the proof in the real case needs less formalism, we develop the proof in this case; however the construction and the proofs work also in the complex case, apart from the generating biorthogonal systems, which are explained for the complex case separately in [11]. Moreover, owing to Theorem I* of Subsection 1.2 which enormously simplifies the structure of the space when the space has type > 1, our main effort for the proof has been in the spaces where l_1 is finitely represented and only Section 6 is devoted to space of type > 1; may we suggest reading this second case first, since it could help the understanding of the first case. Therefore in the next subsection we will be concerned only with the spaces where l_1 is finitely represented and, in order to stress just the ideas, we will only give a simplified description of the construction.

1.6. Description of the main ideas of the construction. To illustrate the general idea of the construction let us view this idea like a dodecahedron, and we are now going to enlighten all its faces; at the end we will briefly explain the fact that the sequence $(q(m))_{m=1}^{\infty}$ is universal. The construction of the basis with permutations

$$(x_n)_{n=1}^{\infty} = ((x_n)_{n=q(3m)+1}^{q(3(m+1))})_{m=0}^{\infty} \qquad (q(0)=0),$$

with $(x_n, x_n^*)_{n=1}^{\infty}$ biorthogonal, proceeds by induction through infinite stages where each stage concerns the block $(x_n, x_n^*)_{n=q(3m)+1}^{q(3(m+1))}$. Then it is sufficient to emphasize the construction of this block and its action for a general element $\overline{x} \in X$ with $\|\overline{x}\| = 1$.

FACE 1. Setting for each natural number p,

$$(n)_{n=q(3p)+1}^{q(3(p+1))} = ((p,0,n'))_{n=1}^{Q'(p)} \cup ((p,0,n''))_{n=1}^{Q''(p)},$$

suppose, for our fixed $m \geq 1$, to have already constructed a permutation $(\overline{\pi}(n))_{n=1}^{\overline{q}(m)}$ of $(n)_{n=1}^{q(3m)} \cup ((m,0,n'))_{n=1}^{Q'(m)}$ such that

$$(*) \qquad \Big\|\overline{x} - \sum_{n=1}^{\overline{q}(m)} x_{\overline{\pi}(n)}^*(\overline{x}) x_{\overline{\pi}(n)}\Big\| < \varepsilon_m \quad \text{and} \quad \Big\|\sum_{n=\overline{q}(m-1)+1}^{q} x_{\overline{\pi}(n)}^*(\overline{x}) x_{\overline{\pi}(n)}\Big\| < \varepsilon_{m-1}$$

for $\overline{q}(m-1)+1 \leq q \leq \overline{q}(m)$. Our aim is to find a subsequence $((m+1,0,n'))_{n=1}^{Q'(m+1)}$ of $(n)_{n=q(3m+1)+1}^{q(3m+3)}$ and a permutation $(\overline{\pi}(n))_{n=\overline{q}(m)+1}^{\overline{q}(m+1)}$ of

$$((m,0,n''))_{n=1}^{Q''(m)} \cup ((m+1,0,n'))_{n=1}^{Q'(m+1)}$$

such that $(*)$ is also true for m replaced by $m+1$; here (ε_m) is a nonincreasing sequence of positive numbers with $\varepsilon_m \to 0$. This is the first natural idea that we will follow in the construction of a basis with permutations.

FACE 2. The second natural idea is to use, for the construction of each

$$(x_n, x_n^*)_{n=q(p)+1}^{q(p+1)} = (((x_{p,n,k,l})_{l=1}^{2^{Q_p}})_{k=1}^{2^{M_p}})_{n=1}^{P_p},$$

two kinds of sequences: $(u'_{p,n}, u'^*_{p,n})_{n=1}^{A_p}$ (the *connection sequence*), which has all the information on the connection between $(x_n, x_n^*)_{n=1}^{q(p)}$ and $(x_n, x_n^*)_{n=q(p)+1}^{\infty}$; and $(\widehat{e}_{p,n}, \widehat{e}_{p,n}^*)_{n=1}^{\widehat{M}_p} \cup (e_{0,p,n}, e_{0,p,n}^*)_{n=1}^{M_{0,p}}$ (with $\|e_{p,n}^*\| \leq 2$ for $1 \leq n \leq \widehat{M_p}$) where $(\widehat{e}_{p,n}, \widehat{e}_{p,n}^*)_{n=1}^{\widehat{M}_p}$ (the *support sequence*) is a union of subsequences with a very special task for each of them: one of these subsequences has just the task to be the "support" for the elements of the connection sequence, while $(e_{0,p,n}, e_{0,p,n}^*)_{n=1}^{M_{0,p}}$ (the *insulating sequence*) is such that if we set $E_{0,p} = \operatorname{span}(e_{0,p,n})_{n=1}^{M_{0,p}}$, then $(\widehat{e}_{p,n} + E_{0,p})_{n=1}^{\widehat{M}_p}$ is 1-equivalent to the natural basis of $l_\infty^{\widehat{M}_p}$. To make the construction slender and in order to render more visible the tasks of the subsequences of the support sequence, for our fixed m the block $(x_n, x_n^*)_{n=q(3m)+1}^{q(3m+3)}$ in its turn is partitioned into three sub-blocks: $(x_n, x_n^*)_{n=q(3m)+1}^{q(3m+1)}$(the *completeness block*),

$(x_n,x_n^*)_{n=q(3m+1)+1}^{q(3m+2)}$ (the *regularization block*) and $(x_n,x_n^*)_{n=q(3m+2)+1}^{q(3m+3)}$ (the *free block*) (we will see better this fact in Faces 10 and 11).

Now we pass to more details. We can see that the connection sequence contains always the whole insulating sequence of the previous sub-block and also elements of the support sequence of the previous sub-block (this fact will be very important for Faces 8 and 9); indeed

$$(x_n)_{n=q(3m)+1}^{q(3m+1)} \subset \operatorname{span}(u'_{3m,n})_{n=1}^{A_{3m}} + \operatorname{span}(\widehat{e}_{3m,n})_{n=1}^{\widehat{M}_{3m}},$$
$$(u'_{3m,n})_{n=1}^{M_{0,3m-1}} = (e_{0,3m-1,n})_{n=1}^{M_{0,3m-1}},$$
$$(\widehat{e}_{3m,n})_{n=1}^{\widehat{M}_{3m}} = (e_{3m,\mathrm{aux},s})_{s=M_{0,3m-1}+1}^{A_{3m}} \cup ((e_{3m,\mathrm{aux},s,t})_{t=1}^{2^{2B_{3m}}})_{s=1}^{A_{3m}} \cup (e_{\mathrm{carr},n})_{n=q(3m)+1}^{q(3m+1)}$$
$$\cup\,(e_{\mathrm{arm},n})_{n=q(3m)+1}^{q(3m+1)} \cup ((e_{3m,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}};$$

analogously

$$(x_n)_{n=q(3m+1)+1}^{q(3m+2)} \subset \operatorname{span}(u'_{3m+1,n})_{n=1}^{A_{3m+1}} + \operatorname{span}(\widehat{e}_{3m+1,n})_{n=1}^{\widehat{M}_{3m+1}},$$
$$(u'_{3m+1,n})_{n=1}^{M_{0,3m}} = (e_{0,3m,n})_{n=1}^{M_{0,3m}},$$
$$(u'_{3m+1,n})_{n=A_{3m+1}-A'_{3m+1}+1}^{A_{3m+1}} = (e_{\mathrm{arm},n})_{n=q(3m)+1}^{q(3m+1)} \cup ((e_{3m,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}},$$
$$(\widehat{e}_{3m+1,n})_{n=1}^{\widehat{M}_{3m+1}} = (e_{3m+1,\mathrm{aux},s})_{s=M_{0,3m}+1}^{A_{3m+1}-A'_{3m+1}} \cup ((e_{3m+1,\mathrm{aux},s,t})_{t=1}^{2^{2B_{3m+1}}})_{s=1}^{A_{3m+1}}$$
$$\cup\,(e_{\mathrm{carr},n})_{n=q(3m+1)+1}^{q(3m+2)} \cup ((e_{3m+1,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m+1}}})_{n=1}^{P_{3m+1}};$$

analogously

$$(x_n)_{n=q(3m+2)+1}^{q(3m+3)} \subset \operatorname{span}(u'_{3m+2,n})_{n=1}^{A_{3m+2}} + \operatorname{span}(\widehat{e}_{3m+2,n})_{n=1}^{\widehat{M}_{3m+2}},$$
$$(u'_{3m+2,n})_{n=1}^{M_{0,3m+1}} = (e_{0,3m+1,n})_{n=1}^{M_{0,3m+1}},$$
$$(u'_{3m+2,n})_{n=A_{3m+2}-A'_{3m+2}+1}^{A_{3m+2}} = ((e_{3m+1,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m+1}}})_{n=1}^{P_{3m+1}},$$
$$(\widehat{e}_{3m+2,n})_{n=1}^{\widehat{M}_{3m+2}} = (e_{3m+2,\mathrm{aux},s})_{s=M_{0,3m+1}+1}^{A_{3m+2}-A'_{3m+2}} \cup ((e_{3m+2,\mathrm{aux},s,t})_{t=1}^{2^{2B_{3m+2}}})_{s=1}^{A_{3m+2}}$$
$$\cup\,(e_{\mathrm{carr},n})_{n=q(3m+2)+1}^{q(3m+3)}.$$

FACE 3. The third natural idea concerns the support sequence. We will make this sequence independent of all the previous sequences, that is, we wish that, for each p, $(\widehat{e}_{p,n},\widehat{e}^*_{p,n})_{n=1}^{\widehat{M}_p} \cup (e_{0,p,n},e^*_{0,p,n})_{n=1}^{M_{0,p}}$ (and in particular $(\widehat{e}_{p,n},\widehat{e}^*_{p,n})_{n=1}^{\widehat{M}_p}$) acts like an island as regards $(x_n,x_n^*)_{n=1}^{q(p)} \cup (u'_{p,n},u'^*_{p,n})_{n=1}^{M_{0,p-1}} \cup (u'_{p,n},u'^*_{p,n})_{n=A_p-A'_p+1}^{A_p}$. Therefore, for $p=3m$ (that is, for the completeness block) the following properties hold:

(i) $\|x+e\| \ge \max(\|x\|,\|e\|/2)$ for each $x \in \operatorname{span}((x_n)_{n=1}^{q(3m)} \cup (u'_{3m,n})_{n=1}^{M_{0,3m-1}})$ and $e \in \operatorname{span}((\widehat{e}_{3m,n})_{n=1}^{\widehat{M}_{3m}} \cup (e_{0,3m,n})_{n=1}^{M_{0,3m}})$;

(ii) $(\widehat{e}_{3m,n})_{n=1}^{\widehat{M}_{3m}}$ is 1-equivalent to the natural basis of $l_1^{\widehat{M}_{3m}}$;

(iii) $(\widehat{e}_{3m,n}+E_{0,3m})_{n=1}^{\widehat{M}_{3m}}$ is 1-equivalent to the natural basis of $l_\infty^{\widehat{M}_{3m}}$.

Then

$$\|x+e+E_{0,3m}\| \ge \|e+E_{0,3m}\|/2 \quad \text{for } e \in \operatorname{span}(\widehat{e}_{3m,n})_{n=1}^{\widehat{M}_{3m}}$$

and

$$x \in \operatorname{span}((x_n)_{n=1}^{q(3m)} \cup (u'_{3m,n})_{n=1}^{M_{0,3m-1}}) + X \cap \bigcap_{n=1}^{q(3m)} x^*_{n\perp}$$
$$\cap \bigcap_{n=1}^{M_{0,3m-1}} u'^*_{(3m,n)\perp} \cap \bigcap_{n=1}^{\widehat{M}_{3m}} \widehat{e}^*_{(3m,n)\perp} \cap \bigcap_{n=1}^{M_{0,3m}} \widehat{e}^*_{(0,3m,n)\perp}.$$

We emphasize the fact that the subspaces $\operatorname{span}(\widehat{e}_{3m,n}+E_{0,3m})_{n=1}^{\widehat{M}_{3m}}$ are *well complemented* in the space $X/E_{0,3m}$; hence we are in the same conditions of the spaces of type > 1 where, by Theorem I* of Subsection 1.2, there are well complemented (almost euclidean) subspaces; the only difference is that we now work in quotient spaces.

Analogously the same properties hold for $p = 3m+2$ (that is, for the free block); the only difference is that now $(u'_{3m,n})_{n=1}^{M_{0,3m-1}}$ is replaced by

$$(u'_{3m+2,n})_{n=1}^{M_{0,3m+1}} \cup (u'_{3m+2,n})_{n=A_{3m+2}-A'_{3m+2}+1}^{A_{3m+2}}.$$

Face 4. Now we consider the sequences $(e_{\mathrm{carr},n})_{n=q(3m)+1}^{q(3m+1)}$, $(e_{\mathrm{carr},n})_{n=q(3m+1)+1}^{q(3m+2)}$ and $(e_{\mathrm{carr},n})_{n=q(3m+2)+1}^{q(3m+3)}$, which are called *carrier sequences* because the task of their elements is to be the support of the elements of the connection sequences (we will see clearly this fact in the description of Face 8). Then for the support sequence of the regularization block we have the same properties of Face 3 for the completeness block, in particular (i) and (iii) continue to hold, apart from the fact that, like the free block, $(u'_{3m,n})_{n=1}^{M_{0,3m-1}}$ is now replaced by $(u'_{3m+1,n})_{n=1}^{M_{0,3m}} \cup (u'_{3m+1,n})_{n=A_{3m+1}-A'_{3m+1}+1}^{A_{3m+1}}$. The main difference concerns (ii) and only for the carrier sequence $(e_{\mathrm{carr},n})_{n=q(3m+1)+1}^{q(3m+2)}$. Because, when we need to express some element u of $\operatorname{span}(u'_{3m+1,n})_{n=1}^{A_{3m+1}}$ by means of subsums of $\sum_{n=q(3m+1)+1}^{q(3m+2)} x^*_n(\overline{x})x_n$, we are facing the following situation: each element $u'_{3m+1,n'}$ appears "pulverized" (that is, in very small quantities) in very many terms $x^*_n(\overline{x})x_n$ where $e_{\mathrm{carr},n}$ are the carriers, analogously in each $x^*_n(\overline{x})$ there is $e^*_{\mathrm{carr},n}(\overline{x})$ and in a very small quantity $u'^*_{3m+1,n'}(\overline{x})$; therefore we have to pick out a subsum such that the effect of the presence of the carriers $e_{\mathrm{carr},n}$ practically disappears while the presence of the element $u'_{3m+1,n'}$ is emphasized.

Therefore the fourth idea is that also (ii) of Face 3 continues to hold but for another sequence $(e'_{\mathrm{carr},n}, e'^*_{\mathrm{carr},n})_{n=q(3m+1)+1}^{q(3m+2)}$, that is, $(e'_{\mathrm{carr},n})_{n=q(3m+1)+1}^{q(3m+2)}$ is 1-equivalent to the natural basis of $l_1^{q(3m+2)-q(3m+1)}$, while property (ii) of Face 3, for

$$(e_{\mathrm{carr},n}, e^*_{\mathrm{carr},n})_{n=q(3m+1)+1}^{q(3m+2)} = (((e_{3m+1,\mathrm{carr},n,k,l}, e^*_{3m+1,\mathrm{carr},n,k,l})_{l=1}^{2^{Q_{3m+1}}})_{k=1}^{2^{M_{3m+1}}})_{n=1}^{P_{3m+1}},$$

is replaced by the following further properties, for each n with $1 \le n \le P_{3m+1}$:

(iv) for each k with $1 \le k \le 2^{M_{3m+1}}$,

$$e_{3m+1,\mathrm{carr},n,k,1} = e'_{3m+1,\mathrm{carr},n,k,1},$$
$$e_{3m+1,\mathrm{carr},n,k,l} = e'_{3m+1,\mathrm{carr},n,k,l} - e'_{3m+1,\mathrm{carr},n,k,l-1} \quad \text{for } 2 \le l \le 2^{Q_{3m+1}},$$

hence

$$\Big\|\sum_{l=1}^{L} e_{3m+1,\mathrm{carr},n,k,l}\Big\| = 1 \qquad \text{for } 1 \le L \le 2^{Q_{3m+1}};$$

(v) setting $3m+1, \mathrm{carr}, n = s$,

$$1 = \|\overline{x}\| \ge \Big\|\sum_{k=1}^{2^{M_{3m+1}}}\sum_{l=1}^{2^{Q_{3m+1}}} e^*_{s,k,l}(\overline{x})e_{s,k,l} + X \cap \bigcap_{k=1}^{2^{M_{3m+1}}}\bigcap_{l=1}^{2^{Q_{3m+1}}} e^*_{(s,k,l)\perp}\Big\|$$
$$\ge \max\Big(\sum_{k=1}^{2^{M_{3m+1}}} |e^*_{s,k,l}(\overline{x})| : 1 \le l \le 2^{Q_{3m+1}}\Big)/2^{1+M_{3m+1}/2}.$$

Then, since $\|e^*_{\mathrm{carr},n}\| \le 2$ for $q(3m+1)+1 \le n \le q(3m+2)$ and since M_{3m+1} is much larger than Q_{3m+1}, there is always $\overline{k}$ with $1 \le \overline{k} \le 2^{M_{3m+1}}$ so that

$$\sum_{l=1}^{2^{Q_{3m+1}}} |e^*_{3m+1,\mathrm{carr},n,\overline{k},l}(\overline{x})| < \frac{1}{2^{Q_{3m+1}}}.$$

Therefore, in order to emphasize the approximation of $u'^*_{3m+1,n}(\overline{x})u'_{3m+1,n}$ by means of a subsum of $\sum_{n=q(3m+1)+1}^{q(3m+2)} x^*_n(\overline{x})x_n$, setting $3m+1, \mathrm{carr}, n, \overline{k} = \overline{s}$ let us schematize (actually in Face 8 we will be more precise)

$$x^*_{\overline{s},l}(\overline{x})x_{\overline{s},l} = \Big(e^*_{\overline{s},l}(\overline{x}) + \frac{u'^*_{3m+1,n}(\overline{x})}{2^{Q_{3m+1}/2}}\Big)\Big(e_{\overline{s},l} + \frac{u'_{3m+1,n}}{2^{Q_{3m+1}/2}}\Big) \qquad \text{for } 1 \le l \le 2^{Q_{3m+1}},$$

with $\|u'_{3m+1,n}\| = 1$ and $\|u'^*_{3m+1,n}\| < 2^{Q_{3m+1}/4}$. By the above and by the last relation of (iv) it just follows that

$$\Big\|\sum_{l=1}^{2^{Q_{3m+1}}} x^*_{\overline{s},l}(\overline{x})x_{\overline{s},l} - u'^*_{3m+1,n}(\overline{x})u'_{3m+1,n}\Big\|$$
$$= \Big\|\sum_{l=1}^{2^{Q_{3m+1}}}\Big(e^*_{\overline{s},l}(\overline{x}) + \frac{u'^*_{3m+1,n}(\overline{x})}{2^{Q_{3m+1}/2}}\Big)\Big(e_{\overline{s},l} + \frac{u'_{3m+1,n}}{2^{Q_{3m+1}/2}}\Big) - u'^*_{3m+1,n}(\overline{x})u'_{3m+1,n}\Big\|$$
$$= \Big\|\sum_{l=1}^{2^{Q_{3m+1}}} e^*_{\overline{s},l}(\overline{x})\Big(e_{\overline{s},l} + \frac{u'_{3m+1,n}}{2^{Q_{3m+1}/2}}\Big) + \frac{u'^*_{3m+1,n}(\overline{x})}{2^{Q_{3m+1}/2}}\sum_{l=1}^{2^{Q_{3m+1}}} e_{\overline{s},l}\Big\|$$
$$\le \Big(\sum_{l=1}^{2^{Q_{3m+1}}} |e^*_{\overline{s},l}(\overline{x})|\Big)\Big(2 + \frac{\|u'_{3m+1,n}\|}{2^{Q_{3m+1}/2}}\Big) + \frac{|u'^*_{3m+1,n}(\overline{x})|}{2^{Q_{3m+1}/2}}\Big\|\sum_{l=1}^{2^{Q_{3m+1}}} e_{\overline{s},l}\Big\| < \frac{3}{2^{Q_{3m+1}}} + \frac{1}{2^{Q_{3m+1}/4}}.$$

FACE 5. Let us turn to a better description of the properties of the connection sequence. Another idea is that the main aim of $(u'_{3m+1,n})_{n=1}^{A_{3m+1}}$ is to connect the sub-blocks $(x_n, x^*_n)_{n=q(3m)+1}^{q(3m+1)}$ and $(x_n, x^*_n)_{n=q(3m+1)+1}^{q(3m+2)}$ not only by the presence of $(u'_{3m+1,n})_{n=1}^{M_{0,3m}} = (e_{0,3m,n})_{n=1}^{M_{0,3m}}$, but also by the presence of $(u'_{3m+1,n})_{n=M_{0,3m}+1}^{A_{3m+1}-A'_{3m+1}}$ in the following sense:

if $X = X_o + U + Y$ where

$$X_0 = \mathrm{span}(x_n)_{n=1}^{q(3m+1)} + \mathrm{span}((u'_{3m+1,n})_{n=1}^{M_{0,3m}} \cup (u'_{3m+1,n})_{n=A_{3m+1}-A'_{3m+1}+1}^{A_{3m+1}}),$$

$$Y = \mathrm{span}((\widehat{e}'_{3m+1,n})_{n=1}^{\widehat{M}'_{3m+1}} \cup (e_{0,3m+1,n})_{n=1}^{M_{0,3m+1}}) + \mathrm{span}(x_n)_{n\geq q(3m+2)+1}$$

and where $(\widehat{e}'_{3m+1,n})_{n=1}^{\widehat{M}'_{3m+1}}$ denotes the subsequence of $(\widehat{e}_{3m+1,n})_{n=1}^{\widehat{M}_{3m+1}}$ complementary to $(e_{3m+1,\mathrm{aux},s})_{s=M_{0,3m}+1}^{A_{3m+1}-A'_{3m+1}}$ and hence $U = \mathrm{span}(u'_{3m+1,n})_{n=M_{0,3m}+1}^{A_{3m+1}-A'_{3m+1}}$, the following property holds: $\|x+y\| \geq \|x\|$ for each $x \in X_0$ and $y \in Y$, that is, U contains all the connections between the two subspaces X_0 and Y. Therefore, if for instance

$$\sum_{n=1}^{A_{3m}} |u'^*_{3m,n}(\overline{x})| + \sum_{n=1}^{A_{3m+2}} |u'^*_{3m+2,n}(\overline{x})| < \varepsilon_m \to 0,$$

then, if we apply the previous reasoning to the first and third blocks, it follows that the block $\sum_{n=q(3m+1)+1}^{q(3m+2)} x^*_n(\overline{x})x_n$ is practically isolated and it cannot come under any influence from the set of subsums of

$$\sum_{n=1}^{q(3m+1)} x^*_n(\overline{x})x_n + \sum_{n\geq q(3m+2)+1} x^*_n(\overline{x})x_n$$

(indeed, setting $\varepsilon_m = 0$, for any $x \in \mathrm{span}(x_n)_{n=1}^{q(3m+1)}$, $y \in \mathrm{span}(x_n)_{n=q(3m+1)+1}^{q(3m+2)}$ and $z \in \mathrm{span}(x_n)_{n\geq q(3m+2)+1}$, it follows that $\|x+y+z\| \geq \|x+y\| \geq \|y\|/2$).

Analogously for the connection sequences $(u'_{3m,n})_{n=1}^{A_{3m}}$ and $(u'_{3m+2,n})_{n=1}^{A_{3m+2}}$.

Face 6. We now illustrate one of the main ideas of this work.

Let us consider the sub-block $\sum_{n=q(3m)+1}^{q(3m+1)} x^*_n(\overline{x})x_n$ (the same procedure will work for the sub-blocks $\sum_{n=q(3m+1)+1}^{q(3m+2)} x^*_n(\overline{x})x_n$ and $\sum_{n=q(3m+2)+1}^{q(3m+3)} x^*_n(\overline{x})x_n$). In general not only the sequence of the norms of its partial sums will be like a switchback also for each possible permutation of terms, but when we add the subsequent sub-block there could be retroactions by the elements of the connection sequence $(u'_{3m+1,n})_{n=1}^{A_{3m+1}}$ with consequent collapses of the norms of the partial sums of $\sum_{n=q(3m)+1}^{q(3m+1)} x^*_n(\overline{x})x_n$. Hence, in order to have an actual series for the representation of the element $\overline{x}$, not only do we have to put in advance and gradually these retroactions, but also if these retroactions were absent, it could be necessary to smooth out the summits of the switchback above. On the other hand, also if we can construct the elements $u \in \mathrm{span}(u'_{3m+1,n})_{n=1}^{A_{3m+1}}$ necessary for these operations by means of the method of Face 4, it is necessary that these $u'_{3m+1,n}$ are really present in the sub-block $\sum_{n=q(3m+1)+1}^{q(3m+2)} x^*_n(\overline{x})x_n$; moreover, these elements may be actually present but not in the quantity that we need to settle the whole sub-block $\sum_{n=q(3m)+1}^{q(3m+1)} x^*_n(\overline{x})x_n$. Therefore our idea is to organize the elements of $(u'^*_{3m+1,n}(\overline{x})u'_{3m+1,n})_{n=1}^{A_{3m+1}}$ in the sub-block $\sum_{n=q(3m+1)+1}^{q(3m+2)} x^*_n(\overline{x})x_n$ so as to have the following three characteristics:

(i) if, for some $\overline{n}'$ with $1 \leq \overline{n}'+1 \leq A_{3m+1}$, $|u'^*_{3m+1,\overline{n}'+1}(\overline{x})| > \varepsilon'_{3m+1}$ (for a suitable sequence (ε'_m) of positive numbers with $\varepsilon'_m \to 0$), we are sure that, by means of the method of Face 4, we can construct (more precisely, we can approximate) all the elements $u \in \mathrm{span}(u'_{3m+1,n})_{n=1}^{\overline{n}'}$ that we need;

(ii) in order to have the possibility to approximate each $u \in \operatorname{span}(u'_{3m+1,n})_{n=1}^{\overline{n}'}$ that we need (of course $\|u\| < 2\sum_{n=1}^{q(3m+1)} \|x_n^*\| \cdot \|x_n\|$) it is necessary not only that the elements $u'_{3m+1,n}$ are really at our disposal by means of the terms $x_n^*(\overline{x})x_n$ of the second sub-block, but also with both positive and negative signs;

(iii) in this operation of smoothing of $\sum_{n=q(3m)+1}^{q(3m+1)} x_n^*(\overline{x})x_n$, we will consume always only a very small quantity of terms $x_n^*(\overline{x})x_n$ of the sub-block $\sum_{n=q(3m+1)+1}^{q(3m+2)} x_n^*(\overline{x})x_n$, that is, in this sub-block all the elements of $(u'_{3m+1,n})_{n=1}^{\overline{n}'}$ have to be always present in a very large quantity.

At this point we are ready to understand the aim of the following definition.

We say that $(w_{m,n}, w_{m,n}^*)_{n=1}^{P_m}$ is an ε_m-*generating form* of a biorthogonal system

$$(u_{m,n}, u_{m,n}^*)_{n=1}^{P_m}$$

of a Banach space X if

$$\operatorname{span}(w_{m,n}^*)_{n=1}^{P_m} = \operatorname{span}(u_{m,n}^*)_{n=1}^{P_m} \quad \text{and} \quad \operatorname{span}(w_{m,k})_{k=1}^{n} = \operatorname{span}(u_{m,k})_{k=1}^{n}$$

for $1 \le n \le P_m$; moreover for each $\overline{x} \in X$ with $\|\overline{x}\| = 1$, if $\overline{n}$ is an index with $1 \le \overline{n} \le P_m - 1$ such that $|u_{m,\overline{n}+1}^*(\overline{x})| > \varepsilon_m$, there exists a sequence $(g(n))_{n=1}^{\overline{n}+1}$ of positive integers with $g(\overline{n}+1) \le P_m$ and $n \le g(n) < g(n+1)$ for $1 \le n \le \overline{n}$ such that, for each ε_m-net $(\overline{u}_{m,i})_{i=1}^{I_m}$ of the unit ball of $\operatorname{span}(u_{m,n})_{n=1}^{\overline{n}}$, there exist sequences $((\varepsilon_{m,i,n})_{n=1}^{\overline{n}})_{i=1}^{I_m}$ of positive numbers and $((f(i,n))_{n=1}^{\overline{n}})_{i=1}^{I_m}$ of integers, with, for each i with $1 \le i \le I_m$, either $f(i,n) = g(n+1)$ for $1 \le n \le \overline{n}$ or $f(i,n) = g(n)$ for $1 \le n \le \overline{n}$, such that, setting

$$\overline{w}_{m,i} = \sum_{n=1}^{\overline{n}} \varepsilon_{m,i,n} w_{m,f(i,n)}^*(\overline{x}) w_{m,f(i,n)},$$

we have

$$\|\overline{w}_{m,i} - \overline{u}_{m,i}\| < \frac{\varepsilon_m}{I_m}, \qquad \sum_{n=1}^{\overline{n}} \varepsilon_{m,i,n} < \varepsilon_m / I_m;$$

so

$$\sum_{i=1}^{I_m} \|\overline{w}_{m,i} - \overline{u}_{m,i}\| < \varepsilon_m, \qquad \sum_{i=1}^{I_m} \sum_{n=1}^{\overline{n}} \varepsilon_{m,i,n} < \varepsilon_m.$$

The main aim of these generating forms is *to approximate* all the elements of the unit ball of $\operatorname{span}(u_{m,n})_{n=1}^{\overline{n}}$ *by means of finite subsums of the series* $\sum x_n^*(\overline{x})x_n$; in particular the fact that $\sum_{i=1}^{I_m} \sum_{n=1}^{\overline{n}} \varepsilon_{m,i,n} < \varepsilon_m$ allows us to consume for these approximations only a relatively small number of terms for each block.

The word "generating" means that this transformation of the biorthogonal system just generates a large quantity of subsums of $\sum_{n=q(m)+1}^{q(m+1)} x_n^*(\overline{x})x_n$ such that we can approximate all the elements of $\operatorname{span}(u_{m,n})_{n=1}^{\overline{n}}$ that we need.

FACE 7. From Faces 4 and 6 it naturally follows that we could pass from $(u'_{3m+1,n})_{n=1}^{A_{3m+1}}$ directly to its generating form; but, if we are in the hypothesis of (i) of Face 6, with $|u'^*_{3m+1,n}(\overline{x})| \le \varepsilon'_m$ for $\overline{n}' + 2 \le n \le A_{2m}$, passing to the generating form, from the procedure of the second part of Face 6 it would follow that we will be able to approximate only all the elements of $\operatorname{span}(u'_{3m+1,n})_{n=1}^{\overline{n}'}$, that is, we lose the contribution of $u'^{*}_{3m+1,\overline{n}'}(\overline{x})u'_{3m+1,\overline{n}'}$ which could be essential.

Therefore our idea is that, before passing to the generating form,

$$(u'_{3m+1,n}, u'^{*}_{3m+1,n})_{n=1}^{A_{3m+1}}$$

has to undergo a preventive operation of uniform minimalization (see (3.1) of Proposition 3 of Subsection 1.4) by means of the *auxiliary sequence*

$$((e_{3m+1,\mathrm{aux},s,t}, e^{*}_{3m+1,\mathrm{aux},s,t})_{t=1}^{2^{2B_{3m+1}}})_{s=1}^{A_{3m+1}}$$

(which now justifies its name) and we pass to

$$(u_{3m+1,n}, u^{*}_{3m+1,n})_{n=1}^{P_{3m+1}} = ((u_{3m+1,s,t}, u^{*}_{3m+1,s,t})_{t=0}^{2^{2B_{3m+1}}})_{s=1}^{A_{3m+1}}$$

where, for each s and t with $1 \le s \le A_{3m+1}$ and $1 \le t \le 2^{2B_{3m+1}}$,

$$u_{3m+1,s,0} = \sum_{j=1}^{2^{2B_{3m+1}}} e_{3m+1,\mathrm{aux},s,j}, \qquad u^{*}_{3m+1,s,0} = \sum_{j=1}^{2^{2B_{3m+1}}} \frac{e^{*}_{3m+1,\mathrm{aux},s,j}}{2^{2B_{3m+1}}} - \frac{u'^{*}_{3m+1,s}}{2^{B_{3m+1}}},$$

$$u_{3m+1,s,t} = e_{3m+1,\mathrm{aux},s,t} + \frac{u'_{3m+1,s}}{2^{B_{3m+1}}}, u^{*}_{3m+1,s,t}$$

$$= \left(e^{*}_{3m+1,\mathrm{aux},s,t} - \sum_{j=1}^{2^{2B_{3m+1}}} \frac{e^{*}_{3m+1,\mathrm{aux},s,j}}{2^{2B_{3m+1}}}\right) + \frac{u'^{*}_{3m+1,s}}{2^{B_{3m+1}}}.$$

Therefore the first reason for uniform minimalization is that, when we pass to the associated generating form $(w_{3m+1,n}, w^{*}_{3m+1,n})_{n=1}^{P_{3m+1}}$, if we are in the condition of the second part of Face 6 with m replaced by $3m+1$, for $\overline{n}+1 = (1+2^{2B_{3m+1}})(\overline{s}-1)+\overline{t}$ with $1 \le \overline{s} \le A_{3m+1}$ and $0 \le \overline{t} \le 2^{2B_{3m+1}}$, by means of the procedure of the second part of Face 6 we will lose the contribution of $u^{*}_{3m+1,\overline{n}+1}(\overline{x})u_{3m+1,\overline{n}+1}$ where, if $1 \le \overline{t} \le 2^{2B_{3m+1}}$, $u'_{3m+1,\overline{s}}$ appears in the form $u^{*}_{3m+1,\overline{n}+1}(\overline{x})u'_{3m+1,\overline{s}}/2^{B_{3m+1}}$, hence this loss does not disturb the approximations since $\|u^{*}_{3m+1,\overline{n}+1}\| < 5$ and B_{3m+1} is very large, while, if $\overline{t}=0$, $u'_{3m+1,\overline{s}}$ does not even appear. There is also another more important reason for this uniform minimalization, which will be explained in the last face.

We proceed analogously for $(u'_{3m,n}, u'^{*}_{3m,n})_{n=1}^{A_{3m}}$ and $(u'_{3m+2,n}, u'^{*}_{3m+2,n})_{n=1}^{A_{3m+2}}$ of the first and third sub-block respectively.

FACE 8. Finally, we turn to our (simplified, that is not exact!) construction of the first sub-block

$$(x_n, x^{*}_n)_{n=q(3m)+1}^{q(3m+1)} = (((x_{3m,n,k,l}, x^{*}_{3m,n,k,l})_{l=1}^{2^{Q_{3m}}})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}},$$

$$x_{3m,n,k,l} = e_{3m,\mathrm{carr},n,k,l} + e_{3m,\mathrm{arm},n,k,l}(+\, e_{3m,\mathrm{brd},n,k} \text{ if } l = 2^{Q_{3m}}) + \frac{w_{3m,n}}{2^{M_{3m}}},$$

$$x^{*}_{3m,n,k,l} = e^{*}_{3m,\mathrm{carr},n,k,l} + \frac{w^{*}_{3m,n}}{2^{Q_{3m}}}$$

for $1 \le n \le P_{3m}$, $1 \le k \le 2^{M_{3m}}$ and $1 \le l \le 2^{Q_{3m}}$; analogously

$$(x_n)_{n=q(3m+1)+1}^{q(3m+2)} = (((x_{3m+1,n,k,l})_{l=1}^{2^{Q_{3m+1}}})_{k=1}^{2^{M_{3m+1}}})_{n=1}^{P_{3m+1}},$$

$$x_{3m+1,n,k,l} = e_{3m+1,\mathrm{carr},n,k,l}(+\, e_{3m+1,\mathrm{brd},n,k} \text{ if } l = 2^{Q_{3m+1}}) + \frac{w_{3m+1,n}}{2^{Q_{3m+1}}},$$

$$x^*_{3m+1,n,k,l} = e^*_{3m+1,\mathrm{carr},n,k,l} + \frac{w^*_{3m+1,n}}{2^{M_{3m+1}}}$$

for $1 \le n \le P_{3m+1}$, $1 \le k \le 2^{M_{3m+1}}$ and $1 \le l \le 2^{Q_{3m+1}}$; analogously

$$(x_n)_{n=q(3m+2)+1}^{q(3m+3)} = ((x_{3m+2,n,k})_{k=1}^{2^{M_{3m+2}}})_{n=1}^{P_{3m+2}}, \quad x_{3m+2,n,k} = e_{3m+2,\mathrm{carr},n,k} + w_{3m+2,n},$$

$$x^*_{3m+2,n,k} = e^*_{3m+2,\mathrm{carr},n,k} + \frac{w^*_{3m+2,n}}{2^{M_{3m+2}}}$$

(actually in Subsection 3.2, $e^*_{3m,\mathrm{carr},n,k,l}$ is replaced by

$$e^*_{3m1,\mathrm{carr},n,k,l} - \sum_{g=1}^{2^{Q_{3m}}} e^*_{3m,\mathrm{carr},n,k,g}/2^{Q_{3m}}$$

and analogously for $e^*_{3m+1,\mathrm{carr},n,k,l}$ and $e^*_{3m+2,\mathrm{carr},n,k}$). This clarifies the role of the carrier sequences which we already explained in Faces 3 and 4.

But our aim in this face is to draw attention to the fact that, from Face 2 and from the above, it follows that

$$\begin{aligned}
&(u'_{3m+1,n}, u'^*_{3m+1,n})_{n=A_{3m+1}-A'_{3m+1}+1}^{A_{3m+1}} \\
&\qquad = (e_{\mathrm{arm},n}, e^*_{\mathrm{arm},n} - x^*_n)_{n=q(3m)+1}^{q(3m+1)} \\
&\qquad\quad \cup ((e_{3m,\mathrm{brd},n,k}, e^*_{3m,\mathrm{brd},n,k} - x^*_{3m,\mathrm{brd},n,k,2^{Q_{3m}}})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}}, \\
&(u'_{3m+2,n}, u'^*_{3m+2,n})_{n=A_{3m+2}-A'_{3m+2}+1}^{A_{3m+2}} \\
&\qquad = ((e_{3m+1,\mathrm{brd},n,k}, e^*_{3m+1,\mathrm{brd},n,k} - x^*_{3m+1,\mathrm{brd},n,k,2^{Q_{3m+1}}})_{k=1}^{2^{M_{3m+1}}})_{n=1}^{P_{3m+1}}.
\end{aligned}$$

Therefore for instance $((e_{3m,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}}$ appears in both the first and the second sub-block, also if $((e^*_{3m,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}}$ appears only in the second sub-block; for this reason these sequences have been called *bridge sequences*. Our idea has been to use these bridge sequences in order to achieve the following two results (we consider now only the bridge sequence of the first sub-block, the same properties hold also for the bridge sequence of the second sub-block):

(i) if, for some n with $1 \le n \le P_{3m}$, $|u^*_{3m,n}(\overline{x})| > \varepsilon_{3m}$ (always for a suitable sequence (ε_m) of positive numbers with $\varepsilon_m \to 0$), then the calibration of the construction of $((x_{3m,n,k,l}, x^*_{3m,n,k,l})_{l=1}^{2^{Q_{3m}}})_{k=1}^{2^{M_{3m}}}$ is such that there always exists $k(n)$, with $1 \le k(n) \le 2^{M_{3m}}$, such that, if by the above n' is the index with $u'_{3m+1,n'} = e_{3m,\mathrm{brd},n,k(n)}$, then $A_{3m+1} - A'_{3m+1} + 1 \le n' \le A_{3m+1}$ (we remark that $((e_{3m,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}}$ appears at the end of $(u'_{3m+1,n})_{n=1}^{A_{3m+1}}$), with $|u'^*_{3m+1,n'+1}(\overline{x})| > \varepsilon'_{3m+1}$;

(ii) there also exists n'' with

$$P_{3m+1} - (2^{2B_{3m+1}} + 1)A'_{3m+1} + 1 \le n'' \le P_{3m+1}, \quad |u^*_{3m+1,n''}(\overline{x})| > \varepsilon_{3m+1};$$

and in particular all the properties of (i) of Face 6 hold.

FACE 9. The conclusion of the previous face allows us to illustrate another main idea of this (we point out that, for each p, A'_p is much smaller than A_p): either

(i) (the *disconnected chain condition*)

$$|u'^{*}_{3m+1,n}(\overline{x})| < \varepsilon_{3m+1} \to 0 \quad \text{for } 1 \le n \le A_{3m+1};$$

or

(ii) (the *operating chain condition*) (i) does not hold; in this case there exist $n(3m+1)$ and $n(3m+2)$, with $A_{3m+1} - A'_{3m+1} + 1 \le n(3m+1) \le A_{3m+1}$ and $A_{3m+2} - A'_{3m+2} + 1 \le n(3m+2) \le A_{3m+2}$, such that $|u'^{*}_{3m+1,n(3m+1)}(\overline{x})| \ge \varepsilon_{3m+1}$ and $|u'^{*}_{3m+2,n(3m+2)}(\overline{x})| \ge \varepsilon_{3m+2}$ (now the reason of the name "free block" for the third sub-block becomes clear: "free" means that this sub-block does not have the bridge sequence, hence the aim of this sub-block is to interrupt the chain, because with a global chain it would be difficult to keep the completeness).

This property is not necessary for a basis with uniformly controlled permutations, but we equip our basis also with this property because it strongly simplifies both the construction and its explanation.

FACE 10. In Face 6 there is already the general idea of the passage from the associated series to the actual series which represents the general element $\overline{x}$ of the unit sphere of X, just by means of permutation of the terms of the associated series. Now we will deepen this idea and we will also explain the reason for the name "regularizing block" for the second sub-block $(x_n, x_n^*)_{n=q(3m+1)+1}^{q(3m+2)}$. We say that a sequence $(a_n)_{n=1}^{Q}$ of numbers is $(0,\varepsilon)$-*monotone* if, for each q with $1 \le q \le Q$,

$$\Big|\sum_{n=1}^{q} a_n\Big| \le \Big|\sum_{n=1}^{Q} a_n\Big| + \varepsilon.$$

Then by *regularization* of $\sum_{n=q(3m)+1}^{q(3m+1)} x_n^*(\overline{x})x_n$ (by Face 1, to make things simpler, suppose $((m,0,n''))_{n=1}^{Q''(m)} = (n)_{n=q(3m)+1}^{q(3m+3)}$) we mean to find a permutation $(\pi(n))_{n=q(3m)+1}^{q(3m+1)}$ of $(n)_{n=q(3m)+1}^{q(3m+1)}$ and a sequence $(\widetilde{u}_n)_{n=q(3m)+1}^{q(3m+1)}$ of elements such that

$$\Big(\Big\|\sum_{n=q(3m)+1}^{Q} (x^*_{\pi(n)}(\overline{x})x_{\pi(n)} + \widetilde{u}_n)\Big\|\Big)_{Q=q(3m)+1}^{q(3m+1)} \quad \text{is } (0,\varepsilon_m)\text{-monotone}$$

with $\varepsilon_m \to 0$. By the definition of $(u'_{3m+1,n})_{n=1}^{A_{3m+1}}$ in Faces 2 and 4, we already know that it is just $(\widetilde{u}_n)_{n=q(3m)+1}^{q(3m+1)} \subset \operatorname{span}(u'_{3m+1,n})_{n=1}^{A_{3m+1}}$; now, under the operating chain condition, by Faces 4 and 6–8, it is always possible to construct a sequence $(\overline{u}_n)_{n=q(3m)+1}^{q(3m+1)}$ with $\sum_{n=q(3m)+1}^{q(3m+1)} \|\overline{u}_n - \widetilde{u}_n\| < \varepsilon_m$, where the elements $\overline{u}_n$ are all disjoint subsums of $\sum_{n=q(3m+1)+1}^{q(3m+2)} x_n^*(\overline{x})x_n$ such that, for each $\overline{u}_n$, the sequence of the norms of its partial subsums is also $(0,\varepsilon_m)$-monotone.

Therefore, if

$$\overline{u}_n = \sum_{k=Q(3m,n-1)+1}^{Q(3m,n)} x^*_{\pi(k)}(\overline{x})x_{\pi(k)} \quad \text{for } q(3m)+1 \le n \le q(3m+1),$$

where $((\pi(k))_{k=Q(3m,n-1)+1}^{Q(3m,n)})_{n=q(3m)+1}^{q(3m+1)} = (\pi(k))_{k=Q(3m,q(3m))+1}^{Q(3m,q(3m+1))}$ is a permutation of a subsequence of $(n)_{n=q(3m+1)+1}^{q(3m+2)}$, it follows that the sequence

$$\Big(\Big(\Big\| \sum_{n=q(3m)+1}^{N-1} (x^*_{\pi(n)}(\overline{x})x_{\pi(n)} + \sum_{k=Q(3m,n-1)+1}^{Q(3m,n)} x^*_{\pi(k)}(\overline{x})x_{\pi(k)}) + (x^*_{\pi(N)}(\overline{x})x_{\pi(N)} + \sum_{k=Q(3m,N-1)+1}^{K} x^*_{\pi(k)}(\overline{x})x_{\pi(k)})\Big\|\Big)_{K=Q(3m,N-1)+1}^{Q(3m,N)}\Big)_{N=q(3m)+1}^{q(3m+1)}$$

is $(0, 2\varepsilon_m)$-monotone, that is, $\sum_{n=q(3m)+1}^{q(3m+1)} x^*_n(\overline{x})x_n$ has been regularized.

Face 11. We are now concerned with the first sub-block $(x_n, x^*_n)_{n=q(3m)+1}^{q(3m+1)}$, that is, with the completeness block, and we will also explain the reason for this name. By *completeness* we mean that, for each $\overline{x} \in X$ with $\|\overline{x}\| = 1$, not only does there exist, for each p,

$$\widetilde{u}_p \in \operatorname{span}(u'_{3p+1,n})_{n=1}^{A_{3p+1}}, \qquad \Big\|\overline{x} - \Big(\sum_{n=1}^{q(3p)} x^*_n(\overline{x})x_n + \widetilde{u}_p\Big)\Big\| < \eta_p \to 0,$$

but also there exists a subsum $\overline{u}_p$ of $\sum_{n=q(3p)+1}^{q(3p+2)} x^*_n(\overline{x})x_n$ with

$$\|\overline{u}_p - \widetilde{u}_p\| < \eta_p, \quad \text{hence} \quad \Big\|\overline{x} - \Big(\sum_{n=1}^{q(3p)} x^*_n(\overline{x})x_n + \overline{u}_p\Big)\Big\| < 2\eta_p.$$

To have simultaneously completeness and the chain effect is already a delicate fact, because the two things fight each other; but also the capability to regularize of Face 10 and completeness fight each other; hence to simplify the construction our idea has been to partition in two separate sub-blocks the settlement of completeness and of regularization. Therefore the construction of the first sub-block has been calibrated towards the two goals: of completeness and of preservation of the chain effect, hence the capability to regularize has been reduced. This means that, owing to the presence of the connection sequence $(u'_{3m,n})_{n=1}^{A_{3m}}$, the completeness block $\sum_{n=q(3m)+1}^{q(3m+1)} x^*_n(\overline{x})x_n$ continues to contribute to the regularization of the previous sub-block $\sum_{n=q(3m-1)+1}^{q(3m)} x^*_n(\overline{x})x_n$, but it is always necessary to have also the help of the sub-block $\sum_{n=q(3m+1)+1}^{q(3m+2)} x^*_n(\overline{x})x_n$. On the other hand, the construction of the second sub-block has been calibrated towards the two goals: of regularization and of preservation of the chain effect. Therefore, in the operating chain condition, the regularization of the whole previous block is possible.

Face 12. This last idea deals with the simpler case of the disconnected chain condition, moreover with the *armouring sequence* $(e_{\mathrm{arm},n})_{n=q(3m)+1}^{q(3m+1)}$, which will now justify its name. Indeed, in the disconnected chain condition this sequence practically appears only in the completeness block $\sum_{n=q(3m)+1}^{q(3m+1)} x^*_n(\overline{x})x_n$ (since, see Face 8,

$$(u'_{3m+1,n})_{n=A_{3m+1}-A'_{3m+1}+1}^{A_{3m+1}-A'_{3m+1}+q(3m+1)-q(3m)} = (e_{\mathrm{arm},n})_{n=q(3m)+1}^{q(3m+1)}$$

and the coefficients of the sequence $(u'_{3m+1,n})_{n=A_{3m+1}-A'_{3m+1}+1}^{A_{3m+1}}$ are negligible) and moreover we will prove that it cannot be influenced by other sequences; on the other hand,

$(e_{\mathrm{arm},n})_{n=q(3m)+1}^{q(3m+1)}$ is 1-equivalent to the natural basis of $l_1^{q(3m+1)-q(3m)}$ and $\|x_n\| < 4$ for each n, hence, for each Q with $q(3m)+1 \le Q \le q(3m+1)$ and for each permutation $(\pi(n))_{n=q(3m)+1}^{q(3m+1)}$ of $(n)_{n=q(3m)+1}^{q(3m+1)}$,

$$\begin{aligned} 4 \sum_{n=q(3m)+1}^{Q} |x_{\pi(n)}^*(\overline{x})| &\ge \Big\| \sum_{n=q(3m)+1}^{Q} x_{\pi(n)}^*(\overline{x}) x_{\pi(n)} \Big\| \\ &\ge \Big\| \sum_{n=q(2m-1)+1}^{Q} x_{\pi(n)}^*(\overline{x}) e_{\mathrm{arm},\pi(n)} \Big\| = \sum_{n=q(2m-1)+1}^{Q} |x_{\pi(n)}^*(\overline{x})| \end{aligned}$$

and the completeness block is automatically regularized. For the regularization block there are two possibilities: either the coefficients $u_{3m+2,n}'^*(\overline{x})$ are not all negligible, hence a partial (but sufficient) regularization is always possible by means of subsums of the third sub-block, or all these coefficients are negligible and we are in the situation at the end of Face 5, that is, this sub-block is practically isolated, hence a suitable permutation of its terms is sufficient. At this point we can partly explain the meaning of “suitable” and at the same time we can also complete the explanations of the reasons of the idea of Face 7. We will consider the second sub-block, but the same reasoning holds for the other two sub-blocks; indeed, it is very important for the regularization in the operating chain condition.

From Face 8 we have, for $1 \le n \le P_{3m+1}$, $1 \le k \le 2^{M_{3m+1}}$ and $1 \le l \le 2^{Q_{3m+1}}$, $x_{3m+1,n,k,l} = x'_{3m+1,n,k,l} + x''_{3m+1,n,k,l}$ with $x''_{3m+1,n,k,l} = w_{3m+1,n}/2^{Q_{3m+1}}$ and we also know that

$$\sum_{k=1}^{2^{M_{3m+1}}} \sum_{l=1}^{2^{Q_{3m+1}}} \left(x_{3m+1,n,k,l}^*(\overline{x}) - \frac{w_{3m+1,n}^*(\overline{x})}{2^{M_{3m+1}}} \right) = 0.$$

Moreover we know that (see the second part of Face 8)

$$\sum_{k=1}^{2^{M_{3m+1}}} \sum_{l=1}^{2^{Q_{3m+1}}} x_{3m+1,n,k,l}^*(\overline{x}) x''_{3m+1,n,k,l} = w_{3m+1,n}^*(\overline{x}) w_{3m+1,n},$$

$$\sum_{n=1}^{P_{3m+1}} w_{3m+1,n}^*(\overline{x}) w_{3m+1,n} = \sum_{n=1}^{P_{3m+1}} u_{3m+1,n}^*(\overline{x}) u_{3m+1,n},$$

where $(\| \sum_{n=q(3m+1)+1}^{q} x_{3m+1,n}^*(\overline{x}) x''_{3m+1,n} \|)_{q=q(3m+1)+1}^{q(3m+2)}$ in general is never $(0,\varepsilon)$-monotone for ε independent of m. Setting, for $1 \le n \le P_{3m+1}$,

$$((x_{3m+1,n,k,l})_{l=1}^{2^{Q_{3m+1}}})_{k=1}^{2^{M_{3m+1}}} = (x_{3m+1,n,g})_{g=1}^{2^{Q_{3m+1}+M_{3m+1}}},$$

there exists a permutation $(\pi(n,g))_{g=1}^{2^{Q_{3m+1}+M_{3m+1}}}$ of $(g)_{g=1}^{2^{Q_{3m+1}+M_{3m+1}}}$ with

$$\left(\Big| \sum_{g=1}^{G} x_{3m+1,n,\pi(n,g)}^*(\overline{x}) \Big| \right)_{G=1}^{2^{Q_{3m+1}+M_{3m+1}}}$$

$(0, 1/2^{Q_{3m+1}})$-monotone, moreover there is a partition $((g)_{t(3m+1,n,s-1)+1}^{t(3m+1,n,s)})_{s=1}^{S_{3m+1}}$ of $(g)_{g=1}^{2^{Q_{3m+1}+M_{3m+1}}}$ such that, for $1 \le s \le S_{3m+1}$,

$$\Big\| \sum_{g=t(3m+1,n,s-1)+1}^{t(3m+1,n,s)} x^*_{3m+1,n,\pi(n,g)}(\overline{x}) x''_{3m+1,n,\pi(n,g)} - \frac{w^*_{3m+1,n}(\overline{x}) w_{3m+1,n}}{S_{3m+1}} \Big\| < \frac{1}{2^{Q_{3m+1}}},$$

$$\Big\| \sum_{g=1}^{t(3m+1,n,s)} x^*_{3m+1,n,\pi(n,g)}(\overline{x}) x''_{3m+1,n,\pi(n,g)} - \frac{s}{S_{3m+1}} w^*_{3m+1,n}(\overline{x}) w_{3m+1,n} \Big\| < \frac{1}{2^{Q_{3m+1}}},$$

where Q_{3m+1} is much larger than $\sum_{n=1}^{P_{3m+1}} \|w^*_{3m+1,n}\|$ and S_{3m+1} is much larger than Q_{3m+1}. It follows that

$$\Big\| \sum_{n=1}^{P_{3m+1}} \sum_{g=t(3m+1,n,s-1)+1}^{t(3m+1,n,s)} x^*_{3m+1,n,\pi(n,g)}(\overline{x}) x''_{3m+1,n,\pi(n,g)} - \frac{1}{S_{3m+1}} \sum_{n=1}^{P_{3m+1}} u^*_{3m+1,n}(\overline{x}) u_{3m+1,n} \Big\| < \frac{P_{3m+1}}{2^{Q_{3m+1}}},$$

$$\Big\| \sum_{n=1}^{P_{3m+1}} \sum_{g=1}^{t(3m+1,n,s)} x^*_{3m+1,n,\pi(n,g)}(\overline{x}) x''_{3m+1,n,\pi(n,g)} - \frac{s}{S_{3m+1}} \sum_{n=1}^{P_{3m+1}} u^*_{3m+1,n}(\overline{x}) u_{3m+1,n} \Big\| < \frac{P_{3m+1}}{2^{Q_{3m+1}}}.$$

It is now sufficient to set $(\pi(n))_{n=q(3m+1)+1}^{q(3m+2)} = (((g)_{t(3m+1,n,s-1)+1}^{t(3m+1,n,s)})_{n=1}^{P_{3m+1}})_{s=1}^{S_{3m+1}}$ and $(\| \sum_{n=q(3m+1)+1}^{q} x^*_{\pi(n)}(\overline{x}) x''_{\pi(n)} \|)_{q=q(3m+1)+1}^{q(3m+2)}$ is $(0, 2/2^{Q_{3m+1}})$-monotone, in particular $(\sum_{n=q(3m+1)+1}^{q} x^*_{\pi(n)}(\overline{x}) x''_{\pi(n)})_{q=q(3m+1)+1}^{q(3m+2)}$ becomes a progressive enlargement of the $1/S_{3m+1}$-miniature $\frac{1}{S_{3m+1}} \sum_{n=1}^{P_{3m+1}} u^*_{3m+1,n}(\overline{x}) u_{3m+1,n}$ of $\sum_{n=1}^{P_{3m+1}} u^*_{3m+1,n}(\overline{x}) u_{3m+1,n}$.

Universality of the sequence $(q(m))_{m=1}^{\infty}$. The sequence $(q(m))_{m=1}^{\infty}$ is universal for all spaces of type 1, since its construction is essentially volumetric and it does not depend on the space. In general this sequence depends on the sequences $(A_m)_{m=1}^{\infty}$, $(B_m)_{m=1}^{\infty}$, $(Q_{0,m})_{m=1}^{\infty}$ (T_m in the spaces of type > 1), $(Q_m)_{m=1}^{\infty}$ and $(M_m)_{m=1}^{\infty}$; in spaces of type > 1 there is also a number K which depends on the space and influences the previous sequences; however (see, in Subsection 3.2, the beginning and Step 5 of SC III.1, the beginning and Step 2 of SC III.2 and the beginning of SC III.3, see moreover Substeps 1 and 5 of the proof of Theorem 24 in Section 6) in our construction these sequences are defined, in spaces of type 1, with a growth more rapid than in spaces of type > 1 (it is sufficient to check this fact for $(A_m)_{m=1}^{\infty}$ and $(B_m)_{m=1}^{\infty}$), hence we could use these sequences also for spaces of type > 1.

2. Theory of biorthogonal systems

The generating form is, for the basis with permutations of (1), the key which allows approximating the element $\overline{x}$ by finite subsums of the series $\sum_{n=1}^{\infty} x^*_n(\overline{x}) x_n$.

2.1. The generating form of a biorthogonal system in the real case. The next theorem provides a procedure to pass to the generating form in the real case; the complex case has been described in [11].

THEOREM 4 (Generating Biorthogonal System Theorem, GBST). *Let* $\{u_{m,n}, u^*_{m,n}\}_{n=1}^{P_m}$ *be biorthogonal in* X *with*

$$\|u_{m,n}\| = 1 \quad \textit{and} \quad \|u^*_{m,n}\| < K \quad \textit{for } 1 \le n \le P_m \tag{4}$$

(*hence* $K > 1$). *Fix two positive integers* M *and* Q. *For each positive number* a *and for any positive integers* q *and* m, *write*

$$EX[a,1,q] = a^{Q\cdot q}, \qquad EX[a,m+1,q] = EX[a,1,EX[a,m,q]] = a^{Q\cdot EX[a,m,q]}. \tag{5}$$

Then let $\{w_{m,n},\ w^*_{m,n}\}_{n=1}^{P_m}$ *be biorthogonal with* $w_{m,1} = u_{m,1}$ *and, for* $2 \le n \le P_m$,

$$w_{m,n} = \sum_{k=1}^{n} \frac{u_{m,k}}{A_{m,n,k}}, \tag{6}$$

where $A_{m,n,1} = 1$ *and, for* $2 \le k \le n$,

$$A_{m,n,k} = EX[2K, 2^{2P_m^2(P_m-n+1)} + 2(k-2), P_m],$$

in particular

$$A_{m,2,2} = EX[2K, 2^{2P_m^2(P_m-1)}, P_m], \qquad A_{m,P_m,k} = EX[2K, 2^{2P_m^2} + 2(k-2), P_m].$$

Fix $\overline{x} \in X$ *with* $\|\overline{x}\| = 1$ *and* $\overline{n}$ *with* $1 \le \overline{n} \le P_m - 1$ *such that*

$$|u^*_{m,\overline{n}+1}(\overline{x})| > \frac{1}{2^{MP_m}}. \tag{7}$$

Then for Q *sufficiently large* (*depending also on* M) *there exists a sequence* $\{g(n)\}_{n=1}^{\overline{n}+1}$ *of positive integers with* $g(\overline{n}+1) \le P_m$ *and* $n \le g(n) < g(n+1)$ *for* $1 \le n \le \overline{n}$ *such that, if we fix* $\overline{u} \in \operatorname{span}\{u_{m,n}\}_{n=1}^{\overline{n}}$ *with* $\|\overline{u}\| = 1$, *then*

$$\overline{u} = \sum_{n=1}^{\overline{n}} \overline{a}_n u_{m,n} \qquad \textit{with } |\overline{a}_n| < K \textit{ for each } n \textit{ with } 1 \le n \le \overline{n}, \tag{8}$$

in particular with $|\overline{a}_{\widetilde{n}}| \ge 1/2^{MP_m}$ *and* $|\overline{a}_n| < 1/2^{MP_m}$ *for* $\widetilde{n}+1 \le n \le \overline{n}$, *for some* $\widetilde{n}$ *with* $1 \le \widetilde{n} \le \overline{n}$; *then, setting* $\{f(n)\}_{n=1}^{\widetilde{n}} = \{g(n)\}_{n=2}^{\widetilde{n}+1}$ *if* $\overline{a}_{\widetilde{n}}$ *and* $w^*_{m,g((\widetilde{n}+1))}(\overline{x})$ *have the same sign, otherwise* $\{f(n)\}_{n=1}^{\widetilde{n}} = \{g(n)\}_{n=1}^{\widetilde{n}}$, *there exists a sequence* $\{\overline{b}_n\}_{n=1}^{\overline{n}}$ *of numbers with* $\overline{b}_n = 0$ *for* $\widetilde{n}+1 \le n \le \overline{n}$ *such that*

$$0 < \frac{\overline{b}_n}{w^*_{m,f(n)}(\overline{x})} < \frac{1}{2^{MP_m}} \quad \textit{for } 1 \le n \le \overline{n}, \tag{9}$$

moreover

$$\|\overline{w} - \overline{u}\| < \frac{1}{2^{MP_m}} \quad \textit{where} \quad \overline{w} = \sum_{n=1}^{\overline{n}} \overline{b}_n w_{m,f(n)}.$$

In particular from the proof it easily follows that, if $\overline{u}' \in \operatorname{span}(u_{m,n'})_{n=1}^{\overline{n}'}$ *with* $(n')_{n=1}^{\overline{n}'} \subset (n)_{n=1}^{\overline{n}}$ *and* $\|\overline{u}'\| = 1$, *if* $(g(n''))_{n=1}^{\overline{n}'+1}$ *is any subsequence of* $(g(n))_{n=1}^{\overline{n}+1}$ *such that the sequence* $(w^*_{m,g(n'')}(\overline{x}))_{n=1}^{\overline{n}'+1}$ *has alternate signs, then again there exist as above* $(f(n'))_{n=1}^{\overline{n}'} \subset (g(n''))_{n=1}^{\overline{n}'+1}$ *and numbers* $(\overline{b}'_n)_{n=1}^{\overline{n}'}$ *such that, for* $1 \le n \le \overline{n}'$,

$$0 < \frac{\overline{b}'_n}{w^*_{m,f(n')}(\overline{x})} < \frac{1}{2^{MP_m}}, \qquad \Big\| \sum_{n=1}^{\overline{n}'} \overline{b}'_n w_{m,f(n')} - \overline{u}' \Big\| < \frac{1}{2^{MP_m}}.$$

Proof. First part. By (6) we have

$$w^*_{m,P_m} = A_{m,P_m,P_m}.u^*_{m,P_m}, \qquad u^*_{m,P_m} = \frac{w^*_{m,P_m}}{A_{m,P_m,P_m}},$$
$$w^*_{m,P_m-1} = A_{m,P_m-1,P_m-1}\left(u^*_{m,P_m-1} - \frac{w^*_{m,P_m}}{A_{m,P_m,P_m-1}}\right),$$
$$u^*_{m,P_m-1} = \frac{w^*_{m,P_m-1}}{A_{m,P_m-1,P_m-1}} + \frac{w^*_{m,P_m}}{A_{m,P_m,P_m-1}}$$

and so on (in particular $w^*_{m,1} = u^*_{m,1} - \sum_{k=2}^{P_m} w^*_{m,k}$); let us prove that, for $1 \le n \le P_m$,

$$u^*_{m,n} = \sum_{k=n}^{P_m} \frac{w^*_{m,k}}{A_{m,k,n}}, \tag{10.1}$$

$$\|w^*_{m,n}\| < EX[2K, 2^{2P_m^2(P_m-n+1)} + 2(n-2)+1, P_m]. \tag{10.2}$$

Since (10.1) is obvious, we check (10.2).

If $n = P_m$, we see by (10.1) and (4)–(6) that

$$\begin{aligned}\|w^*_{m,P_m}\| &= \|A_{m,P_m,P_m}.u^*_{m,P_m}\| < K \cdot A_{m,P_m,P_m}\\ &= K \cdot EX[2K, 2^{2P_m^2} + 2(P_m-2), P_m] < EX[2K, 2^{2P_m^2} + 2(P_m-2)+1, P_m].\end{aligned}$$

Fix n with $1 \le n \le P_m - 1$ and suppose the assertion is true for each k with $n+1 \le k \le P_m$. By (10.1) and (4)–(6) we have

$$\begin{aligned}\|w^*_{m,n}\| &= \left\|A_{m,n,n}\left(u^*_{m,n} - \sum_{j=n+1}^{P_m} \frac{w^*_{m,j}}{A_{m,j,n}}\right)\right\| \le A_{m,n,n}\left(K + \sum_{j=n+1}^{P_m} \frac{\|w^*_{m,j}\|}{A_{m,j,n}}\right)\\
&< A_{m,n,n}\left(K + \sum_{j=n+1}^{P_m} \frac{EX[2K, 2^{2P_m^2(P_m-j+1)} + 2(j-2)+1, P_m]}{EX[2K, 2^{2P_m^2(P_m-j+1)} + 2(n-2), P_m]}\right)\\
&< A_{m,n,n}\Big(K + \sum_{j=n+1}^{P_m} EX[2K, 2^{2P_m^2(P_m-j+1)} + 2(j-2)+1, P_m]\Big)\\
&< 2A_{m,n,n}\Big(\sum_{j=n+1}^{P_m} EX[2K, 2^{2P_m^2(P_m-j+1)} + 2(j-2)+1, P_m]\Big)\\
&< 2P_m A_{m,n,n} EX[2K, 2^{2P_m^2(P_m-n)} + 2(n-1)+1, P_m]\\
&= 2P_m EX[2K, 2^{2P_m^2(P_m-n+1)} + 2(n-2), P_m]\\
&\quad \cdot EX[2K, 2^{2P_m^2(P_m-n)} + 2(n-1)+1, P_m]\\
&< (EX[2K, 2^{2P_m^2(P_m-n+1)} + 2(n-2), P_m])^2\\
&= ((2K)^{QEX[2K,2^{2P_m^2(P_m-n+1)}+2(n-2)-1,P_m]})^2\\
&< (2K)^{QEX[2K,2^{2P_m^2(P_m-n+1)}+2(n-2),P_m]}\\
&= EX[2K, 2^{2P_m^2(P_m-n+1)} + 2(n-2)+1, P_m],\end{aligned}$$

which completes the proof of (10.2).

Now by (7) and by (10.1) there exists $g(\overline{n}+1)$ such that

$$\overline{n}+1 \le g(\overline{n}+1) \le P_m \quad \text{and} \quad \frac{|w^*_{m,g(\overline{n}+1)}(\overline{x})|}{A_{m,g(\overline{n}+1),\overline{n}+1}} > \frac{1}{2^{(M+1)P_m}}; \tag{11}$$

indeed, otherwise

$$\begin{aligned}\frac{1}{2^{MP_m}} &< |u^*_{m,\overline{n}+1}(\overline{x})| = \left| \sum_{k=\overline{n}+1}^{P_m} \frac{w^*_{m,k}(\overline{x})}{A_{m,k,\overline{n}+1}} \right| \\ &\le \sum_{k=\overline{n}+1}^{P_m} \frac{|w^*_{m,k}(\overline{x})|}{A_{m,k,\overline{n}+1}} \le \sum_{k=\overline{n}+1}^{P_m} \frac{1}{2^{(M+1)P_m}} = \frac{P_m-\overline{n}}{2^{(M+1)P_m}} < \frac{1}{2^{MP_m}}.\end{aligned}$$

Next, let us prove that

$$\frac{|w^*_{m,g(\overline{n}+1)}(\overline{x})|}{A_{m,g(\overline{n}+1),\overline{n}}} > 2^{2P_m} EX[2K, 2^{2P_m^2(P_m-g(\overline{n}+1)+1)} + 2((\overline{n}+1)-2)-1, P_m], \tag{12.1}$$

$$\sum_{n=g(\overline{n}+1)+1}^{P_m} \frac{|w^*_{m,n}(\overline{x})|}{A_{m,n,\overline{n}}} < EX[2K, 2^{2P_m^2(P_m-g(\overline{n}+1))} + 2g(\overline{n}+1)-1, P_m]. \tag{12.2}$$

Indeed, by (11) and (6) we have

$$\begin{aligned}\frac{|w^*_{m,g(\overline{n}+1)}(\overline{x})|}{A_{m,g(\overline{n}+1),\overline{n}}} &= \frac{A_{m,g(\overline{n}+1),\overline{n}+1}}{A_{m,g(\overline{n}+1),\overline{n}}} \cdot \frac{|w^*_{m,g(\overline{n}+1)}(\overline{x})|}{A_{m,g(\overline{n}+1),\overline{n}+1}} > \frac{1}{2^{(M+1)P_m}} \cdot \frac{A_{m,g(\overline{n}+1),\overline{n}+1}}{A_{m,g(\overline{n}+1),\overline{n}}} \\ &= \frac{1}{2^{(M+1)P_m}} \cdot \frac{EX[2K, 2^{2P_m^2(P_m-g(\overline{n}+1)+1)} + 2(\overline{n}-1), P_m]}{EX[2K, 2^{2P_m^2(P_m-g(\overline{n}+1)+1)} + 2(\overline{n}-2), P_m]} \\ &> 2^{MP_m} EX[2K, 2^{2P_m^2(P_m-g(\overline{n}+1)+1)} + 2(\overline{n}-1)-1, P_m],\end{aligned}$$

while by (10.2) and by (6) we have

$$\begin{aligned}\sum_{n=g(\overline{n}+1)+1}^{P_m} \frac{|w^*_{m,n}(\overline{x})|}{A_{m,n,\overline{n}}} &\le \sum_{n=g(\overline{n}+1)+1}^{P_m} \frac{\|w^*_{m,n}\|}{A_{m,n,\overline{n}}} \\ &< \sum_{n=g(\overline{n}+1)+1}^{P_m} \frac{EX[2K, 2^{2P_m^2(P_m-n+1)} + 2(n-2)+1, P_m]}{EX[2K, 2^{2P_m^2(P_m-n+1)} + 2(\overline{n}-2), P_m]} \\ &< \sum_{n=g(\overline{n}+1)+1}^{P_m} EX[2K, 2^{2P_m^2(P_m-n+1)} + 2(n-2)+1, P_m]/P_m \\ &< EX[2K, 2^{2P_m^2(P_m-g(\overline{n}+1))} + 2g(\overline{n}+1)-1, P_m],\end{aligned}$$

which completes the proof of (12.2).

Let us prove that there is $g(\overline{n})$ with $\overline{n} \le g(\overline{n}) \le g(\overline{n}+1)-1$ so that

$$w^*_{m,g(\overline{n})}(\overline{x}) \text{ and } w^*_{m,g(\overline{n}+1)}(\overline{x}) \text{ have opposite signs;} \tag{13.1}$$

$$\frac{|w^*_{m,g(\overline{n})}(\overline{x})|}{A_{m,g(\overline{n}),\overline{n}}} > \frac{1}{2^{MP_m}} \frac{|w^*_{m,g(\overline{n}+1)}(\overline{x})|}{A_{m,g(\overline{n}+1),\overline{n}}}; \tag{13.2}$$

$$\frac{|w^*_{m,g(\overline{n})}(\overline{x})|}{A_{m,g(\overline{n}),\overline{n}-1}} > 2^{MP_m} EX[2K, 2^{2P_m^2(P_m-g(\overline{n})+1)} + 2(\overline{n}-2)-1, P_m]; \tag{13.3}$$

$$\sum_{n=g(\overline{n})+1}^{P_m} \frac{|w^*_{m,n}(\overline{x})|}{A_{m,n,\overline{n}-1}} < EX[2K, 2^{2P_m^2(P_m-g(\overline{n}))} + 2g(\overline{n}) - 1, P_m]. \tag{13.4}$$

Let us prove (13.1) and (13.2): By (10.1) and by (4), $\sum_{k=\overline{n}}^{P_m} w^*_{m,k}(\overline{x})/A_{m,k,\overline{n}} = u^*_{m,\overline{n}}(\overline{x})$ with $|u^*_{m,\overline{n}}(\overline{x})| < K$. Then by (12.2) and (12.1),

$$\begin{aligned}
\Big|\sum_{k=\overline{n}}^{g(\overline{n}+1)} \frac{w^*_{m,k}(\overline{x})}{A_{m,k,\overline{n}}}\Big| &= \Big|u^*_{m,\overline{n}}(\overline{x}) - \sum_{k=g(\overline{n}+1)+1}^{P_m} \frac{w^*_{m,k}(\overline{x})}{A_{m,k,\overline{n}}}\Big| \le |u^*_{m,\overline{n}}(\overline{x})| + \sum_{k=g(\overline{n}+1)+1}^{P_m} \frac{|w^*_{m,k}(\overline{x})|}{A_{m,k,\overline{n}}} \\
&< K + \sum_{n=g(\overline{n}+1)+1}^{P_m} \frac{|w^*_{m,n}(\overline{x})|}{A_{m,n,\overline{n}}} < K + EX[2K, 2^{2P_m^2(P_m-g(\overline{n}+1))} + 2g(\overline{n}+1) - 1, P_m] \\
&< 2\cdot EX[2K, 2^{2P_m^2(P_m-g(\overline{n}+1))} + 2g(\overline{n}+1) - 1, P_m] \\
&< 2\cdot EX[2K, 2^{2P_m^2(P_m-g(\overline{n}+1))} + 2g(\overline{n}+1) - 1, P_m] \\
&\quad \cdot \Big(\frac{|w^*_{m,g(\overline{n}+1)}(\overline{x})|}{A_{m,g(\overline{n}+1),\overline{n}}}\cdot\frac{1}{2^{MP_m}EX[2K, 2^{2P_m^2(P_m-g(\overline{n}+1)+1)} + 2(\overline{n}-1) - 1, P_m]}\Big) \\
&= \frac{|w^*_{m,g(\overline{n}+1)}(\overline{x})|}{A_{m,g(\overline{n}+1),\overline{n}}}\cdot\frac{1}{EX[2K, 2^{2P_m^2(P_m-g(\overline{n}+1)+1)} + 2(\overline{n}-2), P_m]}\cdot \\
&\quad \cdot \Big(\frac{EX[2K, 2^{2P_m^2(P_m-g(\overline{n}+1)+1)} + 2(\overline{n}-2), P_m]}{2^{MP_m}}\cdot \\
&\quad \cdot\frac{2EX[2K, 2^{2P_m^2(P_m-g(\overline{n}+1))} + 2g(\overline{n}+1) - 1, P_m]}{EX[2K, 2^{2P_m^2(P_m-g(\overline{n}+1)+1)} + 2(\overline{n}-1) - 1, P_m]}\Big)
\end{aligned}$$

with $(\dots) < 1$. Thus $\sum_{k=\overline{n}}^{g(\overline{n}+1)-1} w^*_{m,k}(\overline{x})/A_{m,k,\overline{n}}$ and $w^*_{m,g(\overline{n}+1)}(\overline{x})/A_{m,g(\overline{n}+1),\overline{n}}$ have opposite signs and

$$\begin{aligned}
\Big|\sum_{k=\overline{n}}^{g(\overline{n}+1)-1} \frac{w^*_{m,k}(\overline{x})}{A_{m,k,\overline{n}}}\Big| &> \frac{|w^*_{m,g(\overline{n}+1)}(\overline{x})|}{A_{m,g(\overline{n}+1),\overline{n}}} \\
&\quad \cdot (1 - 1/EX[2K, 2^{2P_m^2(P_m-g(\overline{n}+1)+1)} + 2(\overline{n}-2), P_m]);
\end{aligned}$$

hence, since $g(\overline{n}+1) - \overline{n} < P_m$, there also exists $g(\overline{n})$ with $\overline{n} \le g(\overline{n}) \le g(\overline{n}+1) - 1$ such that (13.1) and (13.2) are satisfied. On the other hand, the proofs of (13.3) and (13.4), starting from (13.2), are analogous to the proof of (12.1) and (12.2) starting from (11): indeed, by (12.1),

$$\begin{aligned}
\frac{|w^*_{m,g(\overline{n})}(\overline{x})|}{A_{m,g(\overline{n}),\overline{n}-1}} &= \frac{A_{m,g(\overline{n}),\overline{n}}}{A_{m,g(\overline{n}),\overline{n}-1}}\frac{|w^*_{m,g(\overline{n})}(\overline{x})|}{A_{m,g(\overline{n}),\overline{n}}} \\
&> \frac{A_{m,g(\overline{n}),\overline{n}}}{A_{m,g(\overline{n}),\overline{n}-1}}\frac{1}{2^{MP_m}}\frac{|w^*_{m,g(\overline{n}+1)}(\overline{x})|}{A_{m,g(\overline{n}+1),\overline{n}}} \\
&> \frac{2EX[2K, 2^{2P_m^2(P_m-g(\overline{n})+1)} + 2(\overline{n}-2), P_m]}{EX[2K, 2^{2P_m^2(P_m-g(\overline{n})+1)} + 2(\overline{n}-3), P_m]}
\end{aligned}$$

$$\cdot EX[2K, 2^{2P_m^2(P_m - g(\overline{n}+1)+1)} + 2(\overline{n}-1) - 1, P_m]$$
$$> 2^{2P_m} EX[2K, 2^{2P_m^2(P_m - g(\overline{n})+1)} + 2(\overline{n}-2) - 1, P_m],$$

$$\sum_{n=g(\overline{n})+1}^{P_m} \frac{|w^*_{m,n}(\overline{x})|}{A_{m,n,\overline{n}-1}} \le \sum_{n=g(\overline{n})+1}^{P_m} \frac{\|w^*_{m,n}\|}{A_{m,n,\overline{n}-1}} < \sum_{n=g(\overline{n})+1}^{P_m} \frac{\|w^*_{m,g(\overline{n})+1}\|}{P_m}$$
$$< \|w^*_{m,g(\overline{n})+1}\| < EX[2K, 2^{2P_m^2(P_m - g(\overline{n}))} + 2(g(\overline{n})-1) + 1, P_m].$$

By the same procedure we can determine $g(\overline{n}-1)$ and so on; that is, we get $(g(n))_{n=1}^{\overline{n}+1}$ such that $n \le g(n) < g(n+1)$ for $1 \le n \le \overline{n}$. Moreover

(14.1) $\quad w^*_{m,g(n)}(\overline{x})$ and $w^*_{m,g(n+1)}(\overline{x})$ have opposite signs;

(14.2) $$\frac{|w^*_{m,g(n)}(\overline{x})|}{A_{m,g(n),n}} > \frac{1}{2^{MP_m}} \frac{|w^*_{m,g(n+1)}(\overline{x})|}{A_{m,g(n+1),n}};$$

(14.3) $$\frac{|w^*_{m,g(n)}(\overline{x})|}{A_{m,g(n),n-1}} > 2^{MP_m} EX[2K, 2^{2P_m^2(P_m - g(n)+1)} + 2(n-2) - 1, P_m]$$

for $2 \le n \le \overline{n}$, where we have only to check (14.3), which we already know verified for $n = \overline{n}+1$ by (12.1) and for $n = \overline{n}$ by (13.3). Then fix n with $2 \le n \le \overline{n}-1$ and suppose that (14.3) is verified for each k with $n+1 \le k \le \overline{n}$. By (14.2) which is always satisfied and by (14.3) which we know to hold for n+1 we have

$$\frac{|w^*_{m,g(n)}(\overline{x})|}{A_{m,g(n),n-1}} = \frac{A_{m,g(n),n}}{A_{m,g(n),n-1}} \cdot \frac{|w^*_{m,g(n)}(\overline{x})|}{A_{m,g(n),n}} > \frac{A_{m,g(n),n}}{A_{m,g(n),n-1}} \cdot \frac{1}{2^{MP_m}} \frac{|w^*_{m,g(n+1)}(\overline{x})|}{A_{m,g(n+1),n}}$$
$$> \frac{A_{m,g(n),n}}{A_{m,g(n),n-1}} \cdot EX[2K, 2^{2P_m^2(P_m - g(n+1)+1)} + 2((n+1)-2) - 1, P_m]$$
$$= \frac{EX[2K, 2^{2P_m^2(P_m - g(n)+1)} + 2(n-2), P_m]}{EX[2K, 2^{2P_m^2(P_m - g(n)+1)} + 2(n-3), P_m]}$$
$$\cdot EX[2K, 2^{2P_m^2(P_m - g(n+1)+1)} + 2(n-1) - 1, P_m]$$
$$> \frac{EX[2K, 2^{2P_m^2(P_m - g(n)+1)} + 2(n-2), P_m]}{EX[2K, 2^{2P_m^2(P_m - g(n)+1)} + 2(n-3), P_m]}$$
$$> 2^{MP_m} EX[2K, 2^{2P_m^2(P_m - g(n)+1)} + 2(n-2) - 1, P_m],$$

that is, (iii) is verified also for n, which completes the proof of (14.3).

Our aim now, in order to prove (9), is to approximate the terms $\overline{a}_n u_{m,n}$ of $\overline{u}$ in (8), for $1 \le n \le \overline{n}$, and we can suppose that

(15) $$\widetilde{n} = \overline{n}, \quad \text{hence} \quad |\overline{a}_{\overline{n}}| > \frac{1}{2^{(M+1)P_m}} \text{ and } \{g(n')\}_{n=1}^{\widetilde{n}+1} = \{g(n)\}_{n=1}^{\overline{n}+1}$$

(if $|\overline{a}_{\overline{n}}| \le 1/2^{(M+1)P_m}$, we simply turn to considering, instead of $\overline{u}$, the new element $\widetilde{u} = \sum_{n=1}^{\widetilde{n}} \overline{a}_n u_{m,n}$ where $|\overline{a}_{\widetilde{n}}| > 1/2^{(M+1)P_m}$ while $|\overline{a}_n| \le 1/2^{(M+1)P_m}$ for $\widetilde{n}+1 \le n \le \overline{n}$, hence $\|\widetilde{u} - \overline{u}\| \le P_m/2^{(M+1)P_m}$; then the procedure will be the same for each subsequence $(g(n'))_{n=1}^{\widetilde{n}+1}$ of $(g(n))_{n=1}^{\overline{n}+1}$ with alternate signs).

Then we have two possibilities; the first one is:

(16) $\quad w^*_{m,g(\overline{n}+1)}(\overline{x})$ and $\overline{a}_{\overline{n}}$ have the same sign.

In this case we claim that, setting

$$\overline{b}_{\overline{n}} = \overline{a}_{\overline{n}} A_{m,g(\overline{n}+1),\overline{n}} \tag{17.1}$$

and moreover

$$\overline{w}_{m,g(\overline{n}+1)} = w_{m,g(\overline{n}+1)} - \sum_{k=1}^{\overline{n}-1} \frac{u_{m,k}}{A_{m,g(\overline{n}+1),k}} = \sum_{k=\overline{n}}^{g(\overline{n}+1)} \frac{u_{m,k}}{A_{m,g(\overline{n}+1),k}}, \tag{17.2}$$

we have

$$\|\overline{b}_{\overline{n}}\overline{w}_{m,g(\overline{n}+1)} - \overline{a}_{\overline{n}} u_{m,\overline{n}}\| < \frac{1}{2^{(M+1)P_m}}, \qquad 0 < \frac{\overline{b}_{\overline{n}}}{w^*_{m,g(\overline{n}+1)}(\overline{x})} < \frac{1}{2^{MP_m}}. \tag{17.3}$$

Indeed, the first part of (17.3) follows from (4), (5), (6) and (8) (for Q sufficiently large, we already said that Q depends on M) since (by (17.1) and (17.2))

$$\begin{aligned}
\|\overline{b}_{\overline{n}}\overline{w}_{m,g(\overline{n}+1)} - \overline{a}_{\overline{n}} u_{m,\overline{n}}\| &= \left\| \sum_{k=\overline{n}}^{g(\overline{n}+1)} \frac{\overline{b}_{\overline{n}} u_{m,k}}{A_{m,g(\overline{n}+1),k}} - \frac{\overline{b}_{\overline{n}} u_{m,\overline{n}}}{A_{m,g(\overline{n}+1),\overline{n}}} \right\| = \left\| \sum_{k=\overline{n}+1}^{g(\overline{n}+1)} \frac{\overline{b}_{\overline{n}} u_{m,k}}{A_{m,g(\overline{n}+1),k}} \right\| \\
&\le |\overline{b}_{\overline{n}}| \sum_{k=\overline{n}+1}^{g(\overline{n}+1)} \frac{1}{A_{m,g(\overline{n}+1),k}} \le |\overline{b}_{\overline{n}}| \frac{g(\overline{n}+1) - \overline{n}}{A_{m,g(\overline{n}+1),\overline{n}+1}} = \overline{a}_{\overline{n}}(g(\overline{n}+1) - \overline{n}) \frac{A_{m,g(\overline{n}+1),\overline{n}}}{A_{m,g(\overline{n}+1),\overline{n}+1}} \\
&< KP_m A_{m,g(\overline{n}+1),\overline{n}} / A_{m,g(\overline{n}+1),\overline{n}+1} \\
&= KP_m \frac{EX[2K, 2^{2P_m^2(P_m - g(\overline{n}+1)+1)} + 2(\overline{n}-2), P_m]}{EX[2K, 2^{2P_m^2(P_m - g(\overline{n}+1)+1)} + 2(\overline{n}-1), P_m]} \\
&< 1/EX[2K, 2^{2P_m^2(P_m - g(\overline{n}+1)+1)} + 2(\overline{n}-1) - 1, P_m] < 1/2^{(M+1)P_m}
\end{aligned}$$

for Q enough large (see (5)).

For the second part of (17.3), we start by pointing out that (16) and the definition of $\overline{b}_{\overline{n}}$ give the same sign for $\overline{b}_{\overline{n}}$ and $w^*_{m,g(\overline{n}+1)}(\overline{x})$; on the other hand by (12.1), and (8), (4), (6) and (5), we have

$$\begin{aligned}
\frac{\overline{b}_{\overline{n}}}{w^*_{m,g(\overline{n}+1)}(\overline{x})} &= \frac{|\overline{a}_{\overline{n}}| A_{m,g(\overline{n}+1),\overline{n}}}{|w^*_{m,g(\overline{n}+1)}(\overline{x})|} < K \Big/ \frac{|w^*_{m,g(\overline{n}+1)}(\overline{x})|}{A_{m,g(\overline{n}+1),\overline{n}}} \\
&< \frac{K}{2^{MP_m} EX[2K, 2^{2P_m^2(P_m - g(\overline{n}+1)+1)} + 2(\overline{n}-1) - 1, P_m]} < \frac{1}{2^{MP_m}}
\end{aligned}$$

(always for Q sufficiently large), which completes the proof of (13).

Second part. Our next step concerns the index $\overline{n}-1$ and we are going to prove that if we set

$$\overline{b}_{\overline{n}-1} = A_{m,g(\overline{n}),\overline{n}-1} \left(\overline{a}_{\overline{n}-1} - \frac{\overline{b}_{\overline{n}}}{A_{m,g(\overline{n}+1),\overline{n}-1}} \right) \tag{i}$$

and

$$\overline{w}_{m,g(\overline{n})} = w_{m,g(\overline{n})} - \sum_{j=1}^{\overline{n}-2} \frac{u_{m,j}}{A_{m,g(\overline{n}),j}} = \sum_{j=\overline{n}-1}^{g(\overline{n})} \frac{u_{m,j}}{A_{m,g(\overline{n}),j}}, \tag{ii}$$

then

(iii) $$\left\|\left(\frac{\overline{b}_{\overline{n}}}{A_{m,g(\overline{n}+1),\overline{n}-1}}u_{m,\overline{n}-1}+\overline{b}_{\overline{n}-1}\overline{w}_{m,g(\overline{n})}\right)-\overline{a}_{\overline{n}-1}u_{m,\overline{n}-1}\right\|<\frac{1}{2^{(M+1)P_m}},$$

(iv) $$|\overline{b}_{\overline{n}-1}|>\frac{|\overline{b}_{\overline{n}}|}{2}\frac{A_{m,g(\overline{n}),\overline{n}-1}}{A_{m,g(\overline{n}+1),\overline{n}-1}}$$
$$>A_{m,g(\overline{n}),\overline{n}-1}EX[2K,2^{2P_m^2(P_m-g(\overline{n}+1)+1)}+2(\overline{n}-2)-1,P_m],$$

(v) $$\frac{|\overline{b}_{\overline{n}-1}|}{A_{m,g(\overline{n}),\overline{n}-2}}>EX[2K,2^{2P_m^2(P_m-g(\overline{n})+1)}+2(\overline{n}-3)-1,P_m]\frac{\overline{b}_{\overline{n}}}{A_{m,g(\overline{n}+1),\overline{n}-2}},$$

(vi) $$0<\frac{\overline{b}_{\overline{n}-1}}{w^*_{m,g(\overline{n})}(\overline{x})}<\frac{1}{2^{MP_m}}.$$

To prove (iii), note that

$$\left\|\left(\frac{\overline{b}_{\overline{n}}}{A_{m,g(\overline{n}+1),\overline{n}-1}}u_{m,\overline{n}-1}+\overline{b}_{\overline{n}-1}\overline{w}_{m,g(\overline{n})}\right)-\overline{a}_{\overline{n}-1}u_{m,\overline{n}-1}\right\|$$
$$=\left\|\left(\frac{\overline{b}_{\overline{n}}}{A_{m,g(\overline{n}+1),\overline{n}-1}}u_{m,\overline{n}-1}+\overline{b}_{\overline{n}-1}\frac{u_{m,\overline{n}-1}}{A_{m,g(\overline{n}),\overline{n}-1}}+\sum_{j=\overline{n}}^{g(\overline{n})}\frac{\overline{b}_{\overline{n}-1}u_{m,j}}{A_{m,g(\overline{n}),j}}\right)-\overline{a}_{\overline{n}-1}u_{m,\overline{n}-1}\right\| \quad \text{(by (ii))}$$
$$=\Bigg\|\Bigg(\frac{\overline{b}_{\overline{n}}}{A_{m,g(\overline{n}+1),\overline{n}-1}}u_{m,\overline{n}-1}+A_{m,g(\overline{n}),\overline{n}-1}\left(\overline{a}_{\overline{n}-1}-\frac{\overline{b}_{\overline{n}}}{A_{m,g(\overline{n}+1),\overline{n}-1}}\right)\frac{u_{m,\overline{n}-1}}{A_{m,g(\overline{n}),\overline{n}-1}}$$
$$+\sum_{j=\overline{n}}^{g(\overline{n})}\frac{\overline{b}_{\overline{n}-1}u_{m,j}}{A_{m,g(\overline{n}),j}}\Bigg)-\overline{a}_{\overline{n}-1}u_{m,\overline{n}-1}\Bigg\| \quad \text{(by (i))}$$
$$=\left\|\sum_{j=\overline{n}}^{g(\overline{n})}\frac{\overline{b}_{\overline{n}-1}u_{m,j}}{A_{m,g(\overline{n}),j}}\right\|\le|\overline{b}_{\overline{n}-1}|\sum_{j=\overline{n}}^{g(\overline{n})}\frac{1}{A_{m,g(\overline{n}),j}}\le|\overline{b}_{\overline{n}-1}|\frac{g(\overline{n})-\overline{n}+1}{A_{m,g(\overline{n}),\overline{n}}}$$
$$<\frac{P_m|\overline{b}_{\overline{n}-1}|}{A_{m,g(\overline{n}),\overline{n}}}=\frac{P_mA_{m,g(\overline{n}),\overline{n}-1}}{A_{m,g(\overline{n}),\overline{n}}}\left|\overline{a}_{\overline{n}-1}-\frac{\overline{b}_{\overline{n}}}{A_{m,g(\overline{n}+1),\overline{n}-1}}\right| \quad \text{(by (i))}$$
$$<P_m\frac{A_{m,g(\overline{n}),\overline{n}-1}}{A_{m,g(\overline{n}),\overline{n}}}2\left|\frac{\overline{b}_{\overline{n}}}{A_{m,g(\overline{n}+1),\overline{n}-1}}\right|$$

since by (17.1), (15), (5) and for Q sufficiently large, by (8),

$$\frac{|\overline{b}_{\overline{n}}|}{A_{m,g(\overline{n}+1),\overline{n}-1}}=\frac{|\overline{b}_{\overline{n}}|}{A_{m,g(\overline{n}+1),\overline{n}}}\frac{A_{m,g(\overline{n}+1),\overline{n}}}{A_{m,g(\overline{n}+1),\overline{n}-1}}$$
$$=|\overline{a}_{\overline{n}}|\frac{A_{m,g(\overline{n}+1),\overline{n}}}{A_{m,g(\overline{n}+1),\overline{n}-1}}>\frac{1}{2^{(M+1)P_m}}\frac{A_{m,g(\overline{n}+1),\overline{n}}}{A_{m,g(\overline{n}+1),\overline{n}-1}}$$
$$=\frac{1}{2^{(M+1)P_m}}\frac{EX[2K,2^{2P_m^2(P_m-g(\overline{n}+1)+1)}+2(\overline{n}-2),P_m]}{EX[2K,2^{2P_m^2(P_m-g(\overline{n}+1)+1)}+2(\overline{n}-3),P_m]}>2K>2|\overline{a}_{\overline{n}-1}|.$$

In particular in what follows we will use the fact

(∗) $$\frac{|\overline{b}_{\overline{n}}|}{A_{m,g(\overline{n}+1),\overline{n}-1}}>\frac{1}{2^{(M+1)P_m}}\frac{A_{m,g(\overline{n}+1),\overline{n}}}{A_{m,g(\overline{n}+1),\overline{n}-1}}>2K.$$

Now

$$\begin{aligned}
&2P_m \frac{A_{m,g(\overline{n}),\overline{n}-1}}{A_{m,g(\overline{n}),\overline{n}}} \frac{|\overline{b}_{\overline{n}}|}{|w^*_{m,g(\overline{n}+1)}(\overline{x})|} \frac{|w^*_{m,g(\overline{n}+1)}(\overline{x})|}{A_{m,g(\overline{n}+1),\overline{n}-1}} \\
&< \frac{2P_m}{2^{MP_m}} \frac{A_{m,g(\overline{n}),\overline{n}-1}}{A_{m,g(\overline{n}),\overline{n}}} \frac{|w^*_{m,g(\overline{n}+1)}(\overline{x})|}{A_{m,g(\overline{n}+1),\overline{n}-1}} \leq \frac{2P_m}{2^{MP_m}} \frac{A_{m,g(\overline{n}),\overline{n}-1}}{A_{m,g(\overline{n}),\overline{n}}} \frac{\|w^*_{m,g(\overline{n}+1)}\|}{A_{m,g(\overline{n}+1),\overline{n}-1}} \\
&\hspace{20em} \text{(by the last part of (17.3))} \\
&< \frac{2P_m}{2^{MP_m}} \frac{A_{m,g(\overline{n}),\overline{n}-1}}{A_{m,g(\overline{n}),\overline{n}}} \frac{EX[2K, 2^{2P_m^2(P_m - g(\overline{n}+1)+1)} + 2(g(\overline{n}+1)-2)+1, P_m]}{A_{m,g(\overline{n}+1),\overline{n}-1}} \quad \text{(by (10.2))} \\
&= \frac{2P_m}{2^{MP_m}} \cdot \frac{EX[2K, 2^{2P_m^2(P_m - g(\overline{n})+1)} + 2(\overline{n}-3), P_m]}{EX[2K, 2^{2P_m^2(P_m - g(\overline{n})+1)} + 2(\overline{n}-2), P_m]} \\
&\quad \cdot \frac{EX[2K, 2^{2P_m^2(P_m - g(\overline{n}+1)+1)} + 2(g(\overline{n}+1)-2)+1, P_m]}{EX[2K, 2^{2P_m^2(P_m - g(\overline{n}+1)+1)} + 2(\overline{n}-3), P_m]} \quad \text{(by (6))} \\
&< \frac{1}{2^{(M+1)P_m}} \frac{1}{EX[2K, 2^{2P_m^2(P_m - g(\overline{n})+1)} + 2(\overline{n}-2)-1, P_m]} < \frac{1}{2^{(M+1)P_m}} \quad \text{(by (5))}.
\end{aligned}$$

By $(*)$, $|\overline{b}_{\overline{n}}|/A_{m,g(\overline{n}+1),\overline{n}-1} > 2K > 2|\overline{a}_{\overline{n}-1}|$, hence by (i), $\overline{b}_{\overline{n}-1}$ has the sign of $-\overline{b}_{\overline{n}}$, that is, by (17.3) $\overline{b}_{\overline{n}-1}$ has the sign of $-w^*_{m,g(\overline{n}+1)}(\overline{x})$, that is, by (14.1), $\overline{b}_{\overline{n}-1}$ has the sign of $w^*_{m,g(\overline{n})}(\overline{x})$, therefore also the first part of (vi) has been proved.

Turning to (iv), by (i) and $(*)$ we have

$$\begin{aligned}
|\overline{b}_{\overline{n}-1}| &> \frac{A_{m,g(\overline{n}),\overline{n}-1}}{2} \frac{|\overline{b}_{\overline{n}}|}{A_{m,g(\overline{n}+1),\overline{n}-1}} > \frac{A_{m,g(\overline{n}),\overline{n}-1}}{2 \cdot 2^{(M+1)P_m}} \frac{A_{m,g(\overline{n}+1),\overline{n}}}{A_{m,g(\overline{n}+1),\overline{n}-1}} \\
&= \frac{A_{m,g(\overline{n}),\overline{n}-1}}{2 \cdot 2^{(M+1)P_m}} \frac{EX[2K, 2^{2P_m^2(P_m - g(\overline{n}+1)+1)} + 2(\overline{n}-2), P_m]}{EX[2K, 2^{2P_m^2(P_m - g(\overline{n}+1)+1)} + 2(\overline{n}-3), P_m]} \\
&> A_{m,g(\overline{n}),\overline{n}-1} EX[2K, 2^{2P_m^2(P_m - g(\overline{n}+1)+1)} + 2(\overline{n}-2)-1, P_m].
\end{aligned}$$

To prove (v), by (iv) it also follows that

$$\begin{aligned}
\frac{|\overline{b}_{\overline{n}-1}|}{A_{m,g(\overline{n}),\overline{n}-2}} &= \frac{A_{m,g(\overline{n}),\overline{n}-1}}{A_{m,g(\overline{n}),\overline{n}-2}} \frac{|\overline{b}_{\overline{n}-1}|}{A_{m,g(\overline{n}),\overline{n}-1}} > \frac{1}{2} \frac{A_{m,g(\overline{n}),\overline{n}-1}}{A_{m,g(\overline{n}),\overline{n}-2}} \frac{|\overline{b}_{\overline{n}}|}{A_{m,g(\overline{n}+1),\overline{n}-1}} \\
&= \frac{1}{2} \frac{A_{m,g(\overline{n}),\overline{n}-1}}{A_{m,g(\overline{n}),\overline{n}-2}} \frac{A_{m,g(\overline{n}+1),\overline{n}-2}}{A_{m,g(\overline{n}+1),\overline{n}-1}} \frac{|\overline{b}_{\overline{n}}|}{A_{m,g(\overline{n}+1),\overline{n}-2}} \\
&= \frac{1}{2} \frac{EX[2K, 2^{2P_m^2(P_m - g(\overline{n})+1)} + 2(\overline{n}-3), P_m]}{EX[2K, 2^{2P_m^2(P_m - g(\overline{n})+1)} + 2(\overline{n}-4), P_m]} \\
&\quad \cdot \frac{EX[2K, 2^{2P_m^2(P_m - g(\overline{n}+1)+1)} + 2(\overline{n}-4), P_m]}{EX[2K, 2^{2P_m^2(P_m - g(\overline{n}+1)+1)} + 2(\overline{n}-3), P_m]} \cdot \frac{|\overline{b}_{\overline{n}}|}{A_{m,g(\overline{n}+1),\overline{n}-2}} \\
&> EX[2K, 2^{2P_m^2(P_m - g(\overline{n})+1)} + 2(\overline{n}-3)-1, P_m] \frac{|\overline{b}_{\overline{n}}|}{A_{m,g(\overline{n}+1),\overline{n}-2}}.
\end{aligned}$$

Finally, for the second part of (vi), by (i), $(*)$, (17.3) and (14.3) for $n = \overline{n}$, we have

$$\frac{\overline{b}_{\overline{n}-1}}{w^*_{m,g(\overline{n})}(\overline{x})} = \frac{|\overline{b}_{\overline{n}-1}|}{|w^*_{m,g(\overline{n})}(\overline{x})|} = \frac{|\overline{b}_{\overline{n}-1}|}{A_{m,g(\overline{n}),\overline{n}-1}} \Big/ \frac{|w^*_{m,g(\overline{n})}(\overline{x})|}{A_{m,g(\overline{n}),\overline{n}-1}} < 2 \frac{|\overline{b}_{\overline{n}}|}{A_{m,g(\overline{n}+1),\overline{n}-1}} \Big/ \frac{|w^*_{m,g(\overline{n})}(\overline{x})|}{A_{m,g(\overline{n}),\overline{n}-1}}$$

$$
\begin{aligned}
&= 2\frac{|w^*_{m,g(\overline{n}+1)}(\overline{x})|}{A_{m,g(\overline{n}+1),\overline{n}-1}}\frac{|\overline{b}_{\overline{n}}|}{|w^*_{m,g(\overline{n}+1)}(\overline{x})|}\Big/\frac{|w^*_{m,g(\overline{n})}(\overline{x})|}{A_{m,g(\overline{n}),\overline{n}-1}}\\
&< \frac{2}{2^{MP_m}}\frac{|w^*_{m,g(\overline{n}+1)}(\overline{x})|}{A_{m,g(\overline{n}+1),\overline{n}-1}}\Big/\frac{|w^*_{m,g(\overline{n})}(\overline{x})|}{A_{m,g(\overline{n}),\overline{n}-1}}\\
&< \frac{2}{2^{MP_m+2P_m}}\cdot\frac{|w^*_{m,g(\overline{n}+1)}(\overline{x})|}{A_{m,g(\overline{n}+1),\overline{n}-1}}\frac{1}{EX[2K,2^{2P_m^2(P_m-g(\overline{n})+1)}+2(\overline{n}-2)-1,P_m]}\\
&\leq \frac{2}{2^{(M+2)P_m}}\frac{\|w^*_{m,g(\overline{n}+1)}\|}{A_{m,g(\overline{n}+1),\overline{n}-1}}\frac{1}{EX[2K,2^{2P_m^2(P_m-g(\overline{n})+1)}+2(\overline{n}-2)-1,P_m]}\\
&< \frac{2}{2^{(M+2)P_m}}\cdot\frac{EX[2K,2^{2P_m^2(P_m-g(\overline{n}+1)+1)}+2(g(\overline{n}+1)-2)+1,P_m]}{EX[2K,2^{2P_m^2(P_m-g(\overline{n}+1)+1)}+2(\overline{n}-3),P_m]}\\
&\quad\cdot\frac{1}{EX[2K,2^{2P_m^2(P_m-g(\overline{n})+1)}+2(\overline{n}-2)-1,P_m]}\\
&< \frac{2}{2^{(M+2)P_m}}\frac{1}{EX[2K,2^{2P_m^2(P_m-g(\overline{n})+1)}+2(\overline{n}-3),P_m]} < \frac{1}{2^{MP_m}}.
\end{aligned}
$$

Now we can proceed by induction and we will exactly follow the procedure of the last step, with small modifications; however for completeness we will give all the details. Hence we fix n with $1\leq n\leq\overline{n}-2$ and we suppose to have found numbers $\{\overline{b}_k\}_{k=n+1}^{\overline{n}}$ such that, for each k with $n+1\leq k\leq\overline{n}-1$ (since we already considered the case of $k=\overline{n}-1$ in the preceding formula), we have

$$\overline{b}_k = A_{m,g(k+1),k}\left(\overline{a}_k-\sum_{j=k+1}^{\overline{n}}\frac{\overline{b}_j}{A_{m,g(j+1),k}}\right); \tag{18.1}$$

$$\overline{w}_{m,g(k+1)} = w_{m,g(k+1)}-\sum_{j=1}^{k-1}\frac{u_{m,j}}{A_{m,g(k+1),j}} = \sum_{j=k}^{g(k+1)}\frac{u_{m,j}}{A_{m,g(k+1),j}}; \tag{18.2}$$

$$\left\|\left(\left(\sum_{j=k+1}^{\overline{n}}\frac{\overline{b}_j}{A_{m,g(j+1),k}}\right)u_{m,k}+\overline{b}_k\overline{w}_{m,g(k+1)}\right)-\overline{a}_k u_{m,k}\right\| < \frac{1}{2^{(M+1)P_m}}; \tag{18.3}$$

$$
\begin{aligned}
\|\overline{b}_k| &> \frac{|\overline{b}_{k+1}|}{2}\frac{A_{m,g(k+1),k}}{A_{m,g(k+2),k}}\\
&> A_{m,g(k+1),k}\prod_{j=k}^{\overline{n}-1}EX[2K,2^{2P_m^2(P_m-g(j+2)+1)}+2(j-1)-1,P_m];
\end{aligned}\tag{18.4}
$$

$$
\begin{aligned}
&\frac{|\overline{b}_k|}{A_{m,g(k+1),k-1}}\\
&\qquad > EX[2K,2^{2P_m^2(P_m-g(k+1)+1)}+2(k-2)-1,P_m]\sum_{j=k+1}^{\overline{n}}\frac{|\overline{b}_j|}{A_{m,g(j+1),k-1}};
\end{aligned}\tag{18.5}
$$

$$0 < \frac{\overline{b}_k}{w^*_{m,g(k+1)}(\overline{x})} < \frac{1}{2^{MP_m}}. \tag{18.6}$$

We are going to prove (18.1)–(18.6) also if $n+1$ is replaced by n, hence we deduce that (18.1)–(18.6) are true for $1\leq k\leq\overline{n}-1$. Then setting

$$\overline{b}_n = A_{m,g(n+1),n}\Big(\overline{a}_n - \sum_{j=n+1}^{\overline{n}} \frac{\overline{b}_j}{A_{m,g(j+1),n}}\Big), \tag{19.1}$$

$$\overline{w}_{m,g(n+1)} = w_{m,g(n+1)} - \sum_{j=1}^{n-1} \frac{u_{m,j}}{A_{m,g(n+1),j}} = \sum_{j=n}^{g(n+1)} \frac{u_{m,j}}{A_{m,g(n+1),j}}, \tag{19.2}$$

by (18.5) for $k = n+1$ we know that

$$\frac{|\overline{b}_{n+1}|}{A_{m,g(n+2),n}} > EX[2K, 2^{2P_m^2(P_m-g(n+2)+1)} + 2(n-1)-1, P_m] \sum_{j=n+2}^{\overline{n}} \frac{|\overline{b}_j|}{A_{m,g(j+1),n}}, \tag{20}$$

which implies, by (19.1), (8) and (18.6) for $k = n+1$, that

$$\begin{aligned}|\overline{b}_n| &< 2A_{m,g(n+1),n}\frac{|\overline{b}_{n+1}|}{A_{m,g(n+2),n}} = \frac{2A_{m,g(n+1),n}|\overline{b}_{n+1}|}{|w^*_{m,g(n+2)}(\overline{x})|}\frac{|w^*_{m,g(n+2)}(\overline{x})|}{A_{m,g(n+2),n}} \\ &< \frac{2}{2^{MP_m}}\frac{A_{m,g(n+1),n}}{A_{m,g(n+2),n}}\|w^*_{m,g(n+2)}\|;\end{aligned} \tag{21}$$

consequently by (19.1), (19.2), (6), (10.2), (21) and again by (10.2), we have

$$\begin{aligned}
&\Big\|\Big(\Big(\sum_{j=n+1}^{\overline{n}} \frac{\overline{b}_j}{A_{m,g(j+1),n}}\Big)u_{m,n} + \overline{b}_n\overline{w}_{m,g(n+1)}\Big) - \overline{a}_n u_{m,n}\Big\| \\
&\quad = \Big\|\Big(\Big(\sum_{j=n+1}^{\overline{n}} \frac{\overline{b}_j}{A_{m,g(j+1),n}}\Big)u_{m,n} + \overline{b}_n\frac{u_{m,n}}{A_{m,g(n+1),n}} + \overline{b}_n\sum_{j=n+1}^{g(n+1)}\frac{u_{m,j}}{A_{m,g(n+1),j}}\Big) - \overline{a}_n u_{m,n}\Big\| \\
&\quad = \Big\|\Big(\Big(\sum_{j=n+1}^{\overline{n}} \frac{\overline{b}_j}{A_{m,g(j+1),n}}\Big)u_{m,n} + A_{m,g(n+1),n} \\
&\qquad \cdot\Big(\overline{a}_n - \sum_{j=n+1}^{\overline{n}} \frac{\overline{b}_j}{A_{m,g(j+1),n}}\Big)\frac{u_{m,n}}{A_{m,g(n+1),n}} + \overline{b}_n\sum_{j=n+1}^{g(n+1)}\frac{u_{m,j}}{A_{m,g(n+1),j}}\Big) - \overline{a}_n u_{m,n}\Big\| \\
&\quad = \Big\|\overline{b}_n\sum_{j=n+1}^{g(n+1)}\frac{u_{m,j}}{A_{m,g(n+1),j}}\Big\| \le |\overline{b}_n|\sum_{j=n+1}^{g(n+1)}\frac{1}{A_{m,g(n+1),j}} < \frac{P_m|\overline{b}_n|}{A_{m,g(n+1),n+1}} \\
&\quad < \frac{P_m}{A_{m,g(n+1),n+1}}\frac{2}{2^{MP_m}}\frac{A_{m,g(n+1),n}}{A_{m,g(n+2),n}}\|w^*_{m,g(n+2)}\| \\
&\quad < \frac{2P_m}{2^{MP_m}}\frac{A_{m,g(n+1),n}}{A_{m,g(n+1),n+1}}\cdot\frac{EX[2K, 2^{2P_m^2(P_m-g(n+2)+1)} + 2(g(n+2)-2)+1, P_m]}{A_{m,g(n+2),n}} \\
&\quad = \frac{2P_m}{2^{MP_m}}\frac{EX[2K, 2^{2P_m^2(P_m-g(n+1)+1)} + 2(n-2), P_m]}{EX[2K, 2^{2P_m^2(P_m-g(n+1)+1)} + 2(n-1), P_m]} \\
&\qquad \cdot\frac{EX[2K, 2^{2P_m^2(P_m-g(n+2)+1)} + 2(g(n+2)-2)+1, P_m]}{EX[2K, 2^{2P_m^2(P_m-g(n+2)+1)} + 2(n-2), P_m]} \\
&\quad < \frac{2P_m}{2^{MP_m}}\cdot\frac{1}{EX[2K, 2^{2P_m^2(P_m-g(n+1)+1)} + 2(n-1)-1, P_m]} < \frac{1}{2^{(M+1)P_m}};
\end{aligned}$$

hence we have (18.1), (18.2) and (18.3) for $k = n$.

By (20), $\overline{a}_n - \sum_{j=n+1}^{\overline{n}} \overline{b}_j/A_{m,g(j+1),n}$ has the sign of $-\overline{b}_{n+1}$; hence, by (18.6) for $k = n+1$, $\overline{b}_{n+1}$ has the sign of $w^*_{m,g(n+2)}(\overline{x})$; therefore by (19.1), $\overline{b}_n$ has the sign of

$-w^*_{m,g(n+2)}(\overline{x})$; hence by (14.1), $\overline{b}_n$ has the sign of $w^*_{m,g(n+1)}(\overline{x})$; therefore also the first part of (18.6) is proved for $k = n$. Turning to (18.4) for $k = n$, by (19.1), (20), and the second part of (18.4) for $k = n+1$, we have

$$\begin{aligned}
|\overline{b}_n| &> \frac{A_{m,g(n+1),n}}{2} \frac{|\overline{b}_{n+1}|}{A_{m,g(n+2),n}} = \frac{A_{m,g(n+1),n}}{2} \frac{A_{m,g(n+2),n+1}}{A_{m,g(n+2),n}} \cdot \frac{|\overline{b}_{n+1}|}{A_{m,g(n+2),n+1}} \\
&> \frac{A_{m,g(n+1),n}}{2} \frac{A_{m,g(n+2),n+1}}{A_{m,g(n+2),n}} \\
&\quad \cdot \prod_{j=n+1}^{\overline{n}-1} EX[2K, 2^{2P_m^2(P_m-g(j+2)+1)} + 2(j-1)-1, P_m] \\
&= \frac{A_{m,g(n+1),n}}{2} \frac{EX[2K, 2^{2P_m^2(P_m-g(n+2)+1)} + 2(n-1), P_m]}{EX[2K, 2^{2P_m^2(P_m-g(n+2)+1)} + 2(n-2), P_m]} \\
&\quad \cdot \prod_{j=n+1}^{\overline{n}-1} EX[2K, 2^{2P_m^2(P_m-g(j+2)+1)} + 2(j-1)-1, P_m] \\
&> A_{m,g(n+1),n} EX[2K, 2^{2P_m^2(P_m-g(n+2)+1)} + 2(n-1)-1, P_m] \\
&\quad \cdot \prod_{j=n+1}^{\overline{n}-1} EX[2K, 2^{2P_m^2(P_m-g(j+2)+1)} + 2(j-1)-1, P_m];
\end{aligned}$$

hence (18.4) is verified also for $k = n$. Concerning (18.5) for $k = n$ it is only sufficient, by (18.6) for $n+1 \le k \le \overline{n}$ and by (10.2), to point out that

$$\begin{aligned}
\sum_{j=n+1}^{\overline{n}} \frac{|\overline{b}_j|}{A_{m,g(j+1),n-1}} &= \sum_{j=n+1}^{\overline{n}} \frac{|\overline{b}_j|}{|w^*_{m,g(j+1)}(\overline{x})|} \frac{|w^*_{m,g(j+1)}(\overline{x})|}{A_{m,g(j+1),n-1}} < \sum_{j=n+1}^{\overline{n}} \frac{|w^*_{m,g(j+1)}(\overline{x})|}{2^{MP_m} A_{m,g(j+1),n-1}} \\
&< \frac{1}{2^{MP_m}} \sum_{j=n+1}^{\overline{n}} |w^*_{m,g(j+1)}(\overline{x})| \le \frac{1}{2^{MP_m}} \sum_{j=n+1}^{\overline{n}} \|w^*_{m,g(j+1)}\| \\
&\le \frac{1}{2^{MP_m}} \sum_{j=n+1}^{\overline{n}} EX[2K, 2^{2P_m^2(P_m-g(j+1)+1)} + 2(g(j+1)-2)+1, P_m] \\
&< \frac{P_m}{2^{MP_m}} EX[2K, 2^{2P_m^2(P_m-g(n+2)+1)} + 2(g(n+2)-2)+1, P_m];
\end{aligned}$$

that is,

$$\begin{aligned}
&EX[2K, 2^{2P_m^2(P_m-g(n+2)+1)} + 2(g(n+2)-2)+1, P_m] \\
&\qquad > \frac{2^{MP_m}}{P_m} \sum_{j=n+1}^{\overline{n}} \frac{|\overline{b}_j|}{A_{m,g(j+1),n-1}} > \sum_{j=n+1}^{\overline{n}} \frac{|\overline{b}_j|}{A_{m,g(j+1),n-1}};
\end{aligned}$$

while by the second part of (18.4) for $k = n$ (already verified), and by (6) and (5), we have

$$\begin{aligned}
\frac{|\overline{b}_n|}{A_{m,g(n+1),n-1}} &= \frac{A_{m,g(n+1),n}}{A_{m,g(n+1),n-1}} \frac{|\overline{b}_n|}{A_{m,g(n+1),n}} > \frac{A_{m,g(n+1),n}}{A_{m,g(n+1),n-1}} \cdot 1 \\
&= \frac{EX[2K, 2^{2(P_m-g(n+1)+1)} + 2(n-2), P_m]}{EX[2K, 2^{2(P_m-g(n+1)+1)} + 2(n-3), P_m]}
\end{aligned}$$

$$
\begin{aligned}
&> EX[2K, 2^{2P_m^2(P_m-g(n+1)+1)} + 2(n-2) - 1, P_m] \\
&\quad \cdot EX[2K, 2^{2P_m^2(P_m-g(n+2)+1)} + 2(g(n+2)-2) + 1, P_m],
\end{aligned}
$$

that is, by what we just proved above about

$$
EX[2K, 2^{2P_m^2(P_m-g(n+2)+1)} + 2(g(n+2)-2) + 1, P_m],
$$

also (18.5) for $k = n$ has been verified. To complete the proof of (18.6) for $k = n$, by the last part of (21), (10.2) and (14.3) with n replaced by $n+1$, we have

$$
\begin{aligned}
\frac{\overline{b}_n}{w^*_{m,g(n+1)}(\overline{x})} &= \frac{|\overline{b}_n|}{|w^*_{m,g(n+1)}(\overline{x})|} = \frac{|\overline{b}_n|}{A_{m,g(n+1),n}} \Big/ \frac{|w^*_{m,g(n+1)}(\overline{x})|}{A_{m,g(n+1),n}} \\
&< \frac{2}{2^{MP_m}} \frac{\|w^*_{m,g(n+2)}\|}{A_{m,g(n+2),n}} \Big/ \frac{|w^*_{m,g(n+1)}(\overline{x})|}{A_{m,g(n+1),n}} \\
&< \frac{2}{2^{MP_m}} \frac{EX[2K, 2^{2P_m^2(P_m-g(n+2)+1)} + 2(g(n+2)-2) + 1, P_m]}{EX[2K, 2^{2P_m^2(P_m-g(n+2)+1)} + 2(n-2), P_m]} \\
&\quad \cdot \frac{1}{2^{MP_m}} \frac{1}{EX[2K, 2^{2P_m^2(P_m-g(n+1)+1)} + 2(n-1) - 1, P_m]} \\
&< \frac{2}{2^{2MP_m}} \frac{1}{EX[2K, 2^{2P_m^2(P_m-g(n+1)+1)} + 2(n-2), P_m]} < \frac{1}{2^{MP_m}},
\end{aligned}
$$

which completes the proof of (18.1)–(18.6).

At this point, setting $\overline{w} = \sum_{n=1}^{\overline{n}} \overline{b}_n w_{m,g(n+1)}$, since by (18.2), $\overline{w}_{m,g(2)} = w_{m,g(2)}$, by (8) and (18.3) with k replaced by n and by (17.3), we have

$$
\begin{aligned}
\|\overline{w} - \overline{u}\| &= \Big\| \sum_{n=1}^{\overline{n}} (\overline{b}_n w_{m,g(n+1)} - \overline{a}_n u_{m,n}) \Big\| \\
&= \Big\| \sum_{n=2}^{\overline{n}} (\overline{b}_n (w_{m,g(n+1)} - \overline{w}_{m,g(n+1)})) + \sum_{n=1}^{\overline{n}} (\overline{b}_n \overline{w}_{m,g(n+1)} - \overline{a}_n u_{m,n}) \Big\| \\
&= \Big\| \sum_{n=2}^{\overline{n}} \Big(\overline{b}_n \sum_{k=1}^{n-1} \frac{u_{m,k}}{A_{m,g(n+1),k}} \Big) + \sum_{n=1}^{\overline{n}} (\overline{b}_n \overline{w}_{m,g(n+1)} - \overline{a}_n u_{m,n}) \Big\| \\
&= \Big\| \sum_{k=1}^{\overline{n}-1} \Big(\sum_{n=k+1}^{\overline{n}} \frac{\overline{b}_n}{A_{m,g(n+1),k}} \Big) u_{m,k} + \sum_{n=1}^{\overline{n}} (\overline{b}_n \overline{w}_{m,g(n+1)} - \overline{a}_n u_{m,n}) \Big\| \\
&= \Big\| \sum_{n=1}^{\overline{n}-1} \Big(\Big[\Big(\sum_{j=n+1}^{\overline{n}} \frac{\overline{b}_j}{A_{m,g(j+1),n}} \Big) u_{m,n} + \overline{b}_n \overline{w}_{m,g(n+1)} \Big] - \overline{a}_n u_{m,n} \Big) \\
&\quad + (\overline{b}_{\overline{n}} \overline{w}_{m,g(\overline{n}+1)} - \overline{a}_{\overline{n}} u_{m,\overline{n}}) \Big\| \\
&\le \sum_{n=1}^{\overline{n}-1} \Big\| \Big(\Big(\sum_{j=n+1}^{\overline{n}} \frac{\overline{b}_j}{A_{m,g(j+1),n}} \Big) u_{m,n} + \overline{b}_n \overline{w}_{m,g(n+1)} \Big) - \overline{a}_n u_{m,n} \Big\| \\
&\quad + \|\overline{b}_{\overline{n}} \overline{w}_{m,g(\overline{n}+1)} - \overline{a}_{\overline{n}} u_{m,\overline{n}}\| \\
&< \sum_{n=1}^{\overline{n}-1} \frac{1}{2^{(M+1)P_m}} + \frac{1}{2^{(M+1)P_m}} = \frac{\overline{n}}{2^{(M+1)P_m}} < \frac{1}{2^{MP_m}};
\end{aligned}
$$

that is, (9) is verified with $f(n) = g(n+1)$ for $1 \le n \le \overline{n}$; this completes the proof of Theorem 4 under the hypothesis (16).

If (16) is not satisfied, we have the second possibility: $w^*_{m,g(\overline{n}+1)}(\overline{x})$ and $\overline{a}_{\overline{n}}$ have opposite signs. Then by (14.1), $w^*_{m,g(\overline{n})}(\overline{x})$ and $\overline{a}_{\overline{n}}$ have the same sign; therefore we can repeat the whole procedure of the proof under the hypothesis (16), only we now have to replace $g(n+1)$ by $g(n)$ for $1 \le n \le \overline{n}$, with the conclusion that (9) is again satisfied with $f(n) = g(n)$ for $1 \le n \le \overline{n}$. This completes the proof of Theorem 4. ∎

REMARK 5 (Modified Generating Biorthogonal System, MGBS). In Theorem 4 we can replace $(w_{m,n}, w^*_{m,n})_{n=1}^{P_m}$ by $(v_{m,n}, v^*_{m,n})_{n=1}^{P_m}$ where, for each n with $1 \le n \le P_m/2$,

$$v_{m,2n-1} = w_{m,2n} + \frac{w_{m,2n-1}}{D'_m}, \qquad v^*_{m,2n-1} = \frac{w^*_{m,2n}}{2} + \frac{w^*_{m,2n-1}}{2} D'_m,$$
$$v_{m,2n} = w_{m,2n} - \frac{w_{m,2n-1}}{D'_m}, \qquad v^*_{m,2n} = \frac{w^*_{m,2n}}{2} - \frac{w^*_{m,2n-1}}{2} D'_m,$$

where D'_m is an integer $\ge 2^{\sum_{n=1}^{P_m} \|w^*_{m,n}\|}$. Suppose that (7)–(9) of Theorem 4 continue to hold. Then there exists an integer $\widetilde{n}$ with $\overline{n} \le \widetilde{n} \le P_m$ and a strictly increasing sequence $(h(n))_{n=1}^{\widetilde{n}}$ of integers and a sequence $(\widetilde{b}_n)_{n=1}^{\widetilde{n}}$ of numbers such that again we have $f(\overline{n}) - 1 \le h(\widetilde{n}) \le f(\overline{n})$ and $n \le h(n) < h(n+1)$ for $1 \le n \le \widetilde{n} - 1$,

$$0 < \frac{\widetilde{b}_n}{v^*_{m,h(n)}(\overline{x})} < \frac{2}{2^{MP_m}} \quad \text{for } 1 \le n \le \widetilde{n}$$

and

$$\|\widetilde{w} - \overline{u}\| < \frac{2}{2^{MP_m}} \quad \text{where} \quad \widetilde{w} = \sum_{n=1}^{\widetilde{n}} \widetilde{b}_n v_{m,h(n)}.$$

Proof. Consider an integer n' with $1 \le n' \le \overline{n}$ and let n'' be an integer such that we can suppose to have already defined $\{h(n)\}_{n=n''+1}^{\widetilde{n}}$ by means of $\{f(n)\}_{n=n'+1}^{\overline{n}}$ (in the first step $n' = \overline{n}$, hence $\{f(n)\}_{n=n'+1}^{\overline{n}}$ and $\{h(n)\}_{n=n''+1}^{\widetilde{n}}$ do not appear). There are four possibilities:

(a) There exists n''' such that $f(n'-1) = 2n'''-1$ and $f(n') = 2n'''$ (in this case in the first step $n''' = f(\overline{n})/2$). We set $h(n''-1) = 2n'''-1$ $(= f(n'-1))$ and $h(n'') = 2n'''$ $(= f(n'))$ (hence $n'' = \overline{n}$ if we are in the first step and $h(\widetilde{n}) = f(\overline{n})$). Moreover

$$\widetilde{b}_{n''-1} = \frac{\overline{b}_{n'}}{2} + \frac{\overline{b}_{n'-1}}{2} D'_m, \qquad \widetilde{b}_{n''} = \frac{\overline{b}_{n'}}{2} - \frac{\overline{b}_{n'-1}}{2} D'_m;$$

hence

$$\begin{aligned}
\widetilde{b}_{n''-1} v_{m,h(n''-1)} + \widetilde{b}_{n''} v_{m,h(n'')} &= \widetilde{b}_{n''-1} v_{m,2n'''-1} + \widetilde{b}_{n''} v_{m,2n'''} \\
&= \left(\frac{\overline{b}_{n'}}{2} + \frac{\overline{b}_{n'-1}}{2} D'_m\right)\left(w_{m,2n'''} + \frac{w_{m,2n'''-1}}{D'_m}\right) \\
&\quad + \left(\frac{\overline{b}_{n'}}{2} - \frac{\overline{b}_{n'-1}}{2} D'_m\right)\left(w_{m,2n'''} - \frac{w_{m,2n'''-1}}{D'_m}\right) \\
&= \overline{b}_{n'-1} w_{m,2n'''-1} + \overline{b}_{n'} w_{m,2n'''} = \overline{b}_{n'-1} w_{m,f(n'-1)} + \overline{b}_{n'} w_{m,f(n')}.
\end{aligned}$$

By (14.2) (where we can suppose $f(n) = g(n+1)$ for $1 \le n \le \overline{n}$ since if $f(n) = g(n)$ for $1 \le n \le \overline{n}$ the procedure would be the same, hence we can use (14.2) for n replaced by n', $g(n)$ replaced by $g(n') = f(n'-1)$ and $g(n+1)$ replaced by $g(n'+1) = f(n')$, that is,

$$\frac{|w^*_{m,g(n')}(\overline{x})|}{A_{m,g(n'),n'}} = \frac{|w^*_{m,f(n'-1)}(\overline{x})|}{A_{m,f(n'-1),n'}} > \frac{1}{2^{2P_m}} \frac{|w^*_{m,f(n')}(\overline{x})|}{A_{m,f(n'),n'}})$$

we know that

$$\begin{aligned}|w^*_{m,2n'''-1}(\overline{x})| = |w^*_{m,f(n'-1)}(\overline{x})| &> \frac{|w^*_{m,f(n')}(\overline{x})|}{2^{2P_m}} \frac{A_{m,f(n'-1),n'}}{A_{m,f(n'),n'}} \\ &= \frac{|w^*_{m,2n'''}(\overline{x})|}{2^{2P_m}} \frac{EX[2K, 2^{2P_m^2(P_m - f(n'-1)+1)} + 2(n'-2), P_m]}{EX[2K, 2^{2P_m^2(P_m - f(n')+1)} + 2(n'-2), P_m]} \\ &> EX[2K, 2^{2P_m^2(P_m - f(n'-1)+1)} + 2(n'-2) - 1, P_m] |w^*_{m,2n'''}(\overline{x})|;\end{aligned}$$

analogously by (18.4) with k replaced by $n'-1$, hence $g(k+1)$ replaced by $g(n') = f(n'-1)$ and $g(k+2)$ replaced by $g(n'+1) = f(n')$, we know that

$$|\overline{b}_{n'-1}| > \frac{1}{2} \frac{A_{m,f(n'-1),n'-1}}{A_{m,f(n'),n'-1}} |\overline{b}_{n'}| > EX[2K, 2^{2P_m^2(P_m - f(n'-1)+1)} + 2(n'-3) - 1, P_m] |\overline{b}_{n'}|;$$

therefore $\widetilde{b}_{n''-1}$ has the sign of $\overline{b}_{n'-1}$, that is (by (18.6) for $k = n'-1$), $\widetilde{b}_{n''-1}$ has the sign of $w^*_{m,f(n'-1)}(\overline{x}) = w^*_{m,2n'''-1}(\overline{x})$, hence the sign of $v^*_{m,2n'''-1}(\overline{x}) = v^*_{m,h(n''-1)}(\overline{x})$, while $\widetilde{b}_{n''}$ has the sign of $-\overline{b}_{n'-1}$, that is (by (18.6) for $k = n'-1$), $\widetilde{b}_{n''}$ has the sign of $-w^*_{m,f(n'-1)}(\overline{x}) = -w^*_{m,2n'''-1}(\overline{x})$, hence the sign of $v^*_{m,2n'''}(\overline{x}) = v^*_{m,h(n'')}(\overline{x})$; moreover, since by (14.1) and by (18.6), $\overline{b}_{n'}$ and $\overline{b}_{n'-1}$ have opposite signs (and analogously for $w^*_{m,2n'''}(\overline{x})$ and $w^*_{m,2n'''-1}(\overline{x})$), it follows that

$$\begin{aligned}0 &< \frac{\widetilde{b}_{n''-1}}{v^*_{m,h(n''-1)}(\overline{x})} \frac{\widetilde{b}_{n''-1}}{v^*_{m,2n'''-1}(\overline{x})} \\ &= \left(\frac{\overline{b}_{n'}}{2} + \frac{\overline{b}_{n'-1}}{2} D'_m\right) \Big/ \left(\frac{w^*_{m,2n'''}(\overline{x})}{2} + \frac{w^*_{m,2n'''-1}(\overline{x})}{2} D'_m\right) \\ &< \left(\frac{\overline{b}_{n'-1}}{2} D'_m\right) \Big/ \left(\frac{1}{2} \frac{w^*_{m,2n'''-1}(\overline{x})}{2} D'_m\right) \\ &= 2\frac{\overline{b}_{n'-1}}{w^*_{m,2n'''-1}(\overline{x})} = 2\frac{\overline{b}_{n'-1}}{w^*_{m,f(n'-1)}(\overline{x})} < \frac{2}{2^{MP_m}}\end{aligned}$$

(by (18.6)). Analogously we have

$$\begin{aligned}0 &< \frac{\widetilde{b}_{n''}}{v^*_{m,h(n'')}(\overline{x})} = \frac{\widetilde{b}_{n''}}{v^*_{m,2n'''}(\overline{x})} \\ &= \left(\frac{\overline{b}_{n'}}{2} - \frac{\overline{b}_{n'-1}}{2} D'_m\right) \Big/ \left(\frac{w^*_{m,2n'''}(\overline{x})}{2} - \frac{w^*_{m,2n'''-1}(\overline{x})}{2} D'_m\right) \\ &< \left(-2\frac{\overline{b}_{n'-1}}{2} D'_m\right) \Big/ \left(-\frac{w^*_{m,2n'''-1}(\overline{x})}{2} D'_m\right) = 2\frac{\overline{b}_{n'-1}}{w^*_{m,f(n'-1)}(\overline{x})} < \frac{2}{2^{MP_m}}.\end{aligned}$$

(b) There exists n''' with $1 \le n''' < n'$ such that $f(n') = 2n''' - 1$ and $f(n'+1) > 2n'''$ (hence in this case we cannot be in the first step, since the procedure of this construction would imply that we already used $f(n'+1)$, therefore it would not be possible to be in the first step). By (14.2) (for $n = n'+1$, hence with $g(n)$ replaced by $g(n'+1) = f(n')$ and $g(n+1)$ replaced by $g(n'+2) = f(n'+1)$) and by (12.1) (with $\overline{n}$ replaced by $n'+1$, hence $g(\overline{n}+1)$ replaced by $g(n'+2) = f(n'+1)$), moreover by (10.2), we have

$$\begin{aligned}
|w^*_{m,f(n')}(\overline{x})| &> \frac{|w^*_{m,f(n'+1)}(\overline{x})|}{2^{2P_m}} \frac{A_{m,f(n'),n'+1}}{A_{m,f(n'+1),n'+1}} \\
&> A_{m,f(n'),n'+1} EX[2K, 2^{2P_m^2(P_m - f(n'+1)+1)} + 2n' - 1, P_m] \\
&= EX[2K, 2^{2P_m^2(P_m - f(n')+1)} + 2(n'-1), P_m] \\
&\quad \cdot EX[2K, 2^{2P_m^2(P_m - f(n'+1)+1)} + 2n' - 1, P_m], \\
\sum_{n=f(n')+1}^{P_m} \|w^*_{m,n}\| &< \sum_{n=f(n')+1}^{P_m} EX[2K, 2^{2P_m^2(P_m - n+1)} + 2(n-2)+1, P_m] \\
&< P_m EX[2K, 2^{2P_m^2(P_m - f(n'))} + 2(f(n')-1)+1, P_m] \\
&< EX[2K, 2^{2P_m^2(P_m - f(n')+1)} + 2(n'-1), P_m].
\end{aligned}$$

If we read the last sequence of inequalities from top to bottom, since

$$\sum_{n=f(n')+1}^{P_m} \|w^*_{m,n}\| > \|w^*_{m,f(n')+1}\| \ge |w^*_{m,f(n')+1}(\overline{x})| = |w^*_{m,2n'''}(\overline{x})|,$$

we find that

$$\begin{aligned}
|w^*_{m,f(n')}(\overline{x})| &> \Big(\sum_{n=f(n')+1}^{P_m} \|w^*_{m,n}\| \Big) EX[2K, 2^{2P_m^2(P_m - f(n'+1)+1)} + 2n' - 1, P_m] \\
&> EX[2K, 2^{2P_m^2(P_m - f(n'+1)+1)} + 2n' - 1, P_m] |w^*_{m,2n'''}(\overline{x})|.
\end{aligned}$$

We set $h(n''-1) = 2n''' - 1$ $(= f(n'))$ and $h(n'') = 2n'''$ $(= f(n')+1 < f(n'+1))$, in particular

$$v^*_{m,h(n''-1)}(\overline{x}) = \frac{w^*_{m,2n'''}(\overline{x})}{2} + \frac{w^*_{m,2n'''-1}(\overline{x})}{2} D'_m.$$

Hence $v^*_{m,h(n''-1)}(\overline{x})$ has the sign of $w^*_{m,2n'''-1}(\overline{x}) = w^*_{m,f(n')}(\overline{x})$, while

$$v^*_{m,h(n'')}(\overline{x}) = \frac{w^*_{m,2n'''}(\overline{x})}{2} - \frac{w^*_{m,2n'''-1}(\overline{x})}{2} D'_m$$

and $-w^*_{m,f(n')}(\overline{x})$ have the same sign. Then, if $\widetilde{b}_{n''-1} = \frac{\overline{b}_{n'}}{2} D'_m$ and $\widetilde{b}_{n''} = -\frac{\overline{b}_{n'}}{2} D'_m$,

$$\begin{aligned}
&\widetilde{b}_{n''-1} v_{m,h(n''-1)} + \widetilde{b}_{n''} v_{m,h(n'')} \\
&\quad = \widetilde{b}_{n''-1} v_{m,2n'''-1} + \widetilde{b}_{n''} v_{m,2n'''} \\
&\quad = \frac{\overline{b}_{n'}}{2} D'_m \left(w_{m,2n'''} + \frac{w_{m,2n'''-1}}{D'_m} \right) - \frac{\overline{b}_{n'}}{2} D'_m \left(w_{m,2n'''} - \frac{w_{m,2n'''-1}}{D'_m} \right) \\
&\quad = \overline{b}_{n'} w_{m,2n'''-1} = \overline{b}_{n'} w_{m,f(n')}.
\end{aligned}$$

Moreover $\widetilde{b}_{n''-1}$ has the sign of $\overline{b}_{n'}$ and hence of $w^*_{m,f(n')}(\overline{x}) = w^*_{m,2n'''-1}(\overline{x})$ and hence of $v^*_{m,h(n''-1)}(\overline{x})$, while $\widetilde{b}_{n''}$ has the sign of $-\overline{b}_{n'}$ and hence of $-w^*_{m,2n'''-1}(\overline{x})$, hence of $v^*_{m,2n'''}(\overline{x}) = v^*_{m,h(n'')}(\overline{x})$ by the above. Finally,

$$\begin{aligned}\frac{\widetilde{b}_{n''-1}}{v^*_{m,h(n''-1)}(\overline{x})} &= \frac{\widetilde{b}_{n''-1}}{v^*_{m,2n'''-1}(\overline{x})} = \Big(\frac{\overline{b}_{n'}}{2}D'_m\Big)\Big/\Big(\frac{w^*_{m,2n'''}(\overline{x})}{2} + \frac{w^*_{m,2n'''-1}(\overline{x})}{2}D'_m\Big)\\ &< \Big(\frac{\overline{b}_{n'}}{2}D'_m\Big)\Big/\Big(\frac{1}{2}\frac{w^*_{m,2n'''-1}(\overline{x})}{2}D'_m\Big) = 2\frac{\overline{b}_{n'}}{w^*_{m,2n'''-1}(\overline{x})} = 2\frac{\overline{b}_{n'}}{w^*_{m,f(n')}(\overline{x})} < \frac{2}{2^{MP_m}}.\end{aligned}$$

Analogously (we do not know the sign of $w^*_{m,2n'''}(\overline{x})$, but $|w^*_{m,2n'''-1}(\overline{x})| > |w^*_{m,2n'''}(\overline{x})|$ by the first part of (b))

$$\begin{aligned}\frac{\widetilde{b}_{n''}}{v^*_{m,h(n'')}(\overline{x})} &= \frac{\widetilde{b}_{n''}}{v^*_{m,2n'''}(\overline{x})} = \Big(-\frac{\overline{b}_{n'}}{2}D'_m\Big)\Big/\Big(\frac{w^*_{m,2n'''}(\overline{x})}{2} - \frac{w^*_{m,2n'''-1}(\overline{x})}{2}D'_m\Big)\\ &< \Big(-\frac{\overline{b}_{n'}}{2}D'_m\Big)\Big/\Big(-\frac{1}{2}\frac{w^*_{m,2n'''-1}(\overline{x})}{2}D'_m\Big) = \frac{2\overline{b}_n}{w^*_{m,f(n')}(\overline{x})} < \frac{2}{2^{MP_m}}.\end{aligned}$$

(c) There exists n''' with $1 \le n''' < n'$ such that $f(n') = 2n'''$ while $f(n'-1) < 2n'''-1$ (hence $n''' = f(\overline{n})/2$ if we are in the first step), moreover either $w^*_{m,2n'''-1}(\overline{x})$ has the sign of $w^*_{m,2n'''}(\overline{x})$ or $w^*_{m,2n'''-1}(\overline{x}) = 0$.

Then we set $h(n'') = 2n''' - 1$ (hence $h(n'') = h(\widetilde{n}) = 2\frac{f(\overline{n})}{2} - 1 = f(\overline{n}) - 1$ if we are in the first step) and $\widetilde{b}_{n''} = \overline{b}_{n'}$; hence (we recall (18.6) for $k = n'$ hence $g(k+1) = g(n'+1) = f(n')$)

$$\begin{aligned}\widetilde{b}_{n''}v_{m,h(n'')} &= \widetilde{b}_{n''}v_{m,2n'''-1} = \overline{b}_{n'}\Big(w_{m,2n'''} + \frac{w_{m,2n'''-1}}{2^{D'_m}}\Big),\\ \|\widetilde{b}_{n''}v_{m,h(n'')} - \overline{b}_{n'}w_{m,f(n')}\| &= \|\widetilde{b}_{n''}v_{m,h(n'')} - \overline{b}_{n'}w_{m,2n'''}\|\\ &= |\overline{b}_{n'}|\frac{\|w_{m,2n'''-1}\|}{D'_m} < 2\frac{|\overline{b}_{n'}|}{D'_m} < 2\frac{|w^*_{m,f(n')}(\overline{x})|}{2^{MP_m}D'_m}\\ &\le 2\frac{\|w^*_{m,f(n')}\|}{2^{MP_m}D'_m} \le 2\|w^*_{m,f(n')}\|/(2^{MP_m}2^{MP_m^2\sum_{f=1}^{P_m}\|w^*_{m,f}\|})\\ &< \frac{1}{P_m 2^{MP_m}};\end{aligned}$$

$\widetilde{b}_{n''}$ has the sign of $\overline{b}_{n'}$ hence of $w^*_{m,f(n')}(\overline{x}) = w^*_{m,2n'''}(\overline{x})$, hence of

$$\begin{aligned}\frac{w^*_{m,2n'''}(\overline{x})}{2} + \frac{w^*_{m,2n'''-1}(\overline{x})}{2}D'_m &= v^*_{m,2n'''-1}(\overline{x}) = v^*_{m,h(n'')}(\overline{x});\\ \frac{\widetilde{b}_{n''}}{v^*_{m,h(n'')}(\overline{x})} = \frac{\widetilde{b}_{n''}}{v^*_{m,2n'''-1}(\overline{x})} &= \overline{b}_{n'}\Big/\Big(\frac{w^*_{m,2n'''}(\overline{x})}{2} + \frac{w^*_{m,2n'''-1}(\overline{x})}{2}D'_m\Big)\\ &< \frac{\overline{b}_{n'}}{w^*_{m,2n'''}(\overline{x})/2} = 2\frac{\overline{b}_{n'}}{w^*_{m,f(n')}(\overline{x})} < \frac{2}{2^{MP_m}}.\end{aligned}$$

(d) As in (c) as regards n''', $f(n')$, $f(n'-1)$ and n'', but with $w^*_{m,2n'''-1}(\overline{x})$ and $w^*_{m,2n'''}(\overline{x})$ of opposite sign. We set $h(n'') = 2n'''$ (hence $h(\widetilde{n}) = f(\overline{n})$ if we are in the first step) and $\widetilde{b}_{n''} = \overline{b}_{n'}$; hence, by (18.6) as above,

$$\widetilde{b}_{n''} v_{m,h(n'')} = \widetilde{b}_{n''} v_{m,2n'''} = \overline{b}_{n'} \left(w_{m,2n'''} - \frac{w_{m,2n'''-1}}{D'_m} \right),$$

$$\begin{aligned} \|\widetilde{b}_{n''} v_{m,h(n'')} - \overline{b}_{n'} w_{m,f(n')}\| &= \|\overline{b}_{n'} v_{m,2n'''} - \overline{b}_{n'} w_{m,2n'''}\| = |\overline{b}_{n'}| \frac{\|w_{m,2n'''-1}\|}{D'_m} \\ &< 2\frac{|\overline{b}_{n'}|}{D'_m} < 2\frac{|w^*_{m,f(n')}(\overline{x})|}{2^{MP_m} D'_m} \le 2\frac{\|w^*_{m,f(n')}\|}{2^{MP_m} D'_m} < \frac{1}{P_m 2^{MP_m}}; \end{aligned}$$

$\widetilde{b}_{n''}$ has the sign of $\overline{b}_{n'}$, hence of $w^*_{m,f(n')}(\overline{x}) = w^*_{m,2n'''}(\overline{x})$ and of

$$\frac{w^*_{m,2n'''}(\overline{x})}{2} - \frac{w^*_{m,2n'''-1}(\overline{x})}{2} D'_m = v^*_{m,2n'''}(\overline{x}) = v^*_{m,h(n'')}(\overline{x});$$

$$\begin{aligned} \frac{\widetilde{b}_{n''}}{v^*_{m,h(n'')}(\overline{x})} &= \frac{\widetilde{b}_{n''}}{v^*_{m,2n'''}(\overline{x})} = \overline{b}_{n'} \Big/ \left(\frac{w^*_{m,2n'''}(\overline{x})}{2} - \frac{w^*_{m,2n'''-1}(\overline{x})}{2} D'_m \right) \\ &< \overline{b}_{n'} \Big/ \left(\frac{w^*_{m,2n'''}(\overline{x})}{2} \right) = 2\frac{\overline{b}_{n'}}{w^*_{m,f(n')}(\overline{x})} < \frac{2}{2^{MP_m}}. \end{aligned}$$

We conclude that, by the above (in particular by cases (c) and (d), since in (a) and (b) we have only equalities), it follows that

$$\begin{aligned} \|\widetilde{w} - \overline{u}\| &\le \Big\| \sum_{n=1}^{\overline{n}} \overline{b}_n w_{m,f(n)} - \overline{u} \Big\| + \Big\| \sum_{n=1}^{\overline{n}} \overline{b}_n w_{m,f(n)} - \sum_{n=1}^{\widetilde{n}} \widetilde{b}_n v_{m,h(n)} \Big\| \\ &< \frac{1}{2^{MP_m}} + \Big\| \sum_{n=1}^{\overline{n}} \overline{b}_n w_{m,f(n)} - \sum_{n=1}^{\widetilde{n}} \widetilde{b}_n v_{m,h(n)} \Big\| \\ &< \frac{1}{2^{MP_m}} + P_m \frac{1}{P_m 2^{MP_m}} = \frac{2}{2^{MP_m}}, \end{aligned}$$

which completes the proof of Remark 5.

2.2. Some properties of the Walsh matrix in l^n_∞. One of the main tools in this work is the Walsh matrix (for the definition see for instance [2, p. 104], or [7, p. 398], or [9, p. 70]) when applied to the natural basis of l^n_∞. We introduce the following notations:

(22) $\{o_n\}_{n=1}^{2^S}$ is the natural basis of $E = l_\infty^{2^S}$, with $\{o_n, o^*_n\}_{n=1}^{2^S}$ biorthogonal; $\{\widehat{o}_n, \widehat{o}^*_n\}_{n=1}^{2^S}$ is the biorthogonal system derived from $\{o_n, o^*_n\}_{n=1}^{2^S}$ by means of the Walsh matrix, hence again $\|\widehat{o}_n\| = \|\widehat{o}^*_n\| = 1$ for $1 \le n \le 2^S$.

If $\{a_k\}_{k=1}^P$ is a sequence of numbers, we will say that $\{\sum_{k=1}^K a_k\}_{K=1}^P$ is (H, M, ε)-*monotone* (in particular (M, ε)-*monotone* if $H = 1$) if, for each K and Q with $1 \le K \le Q \le P$,

$$\Big| \sum_{k=1}^K a_k \Big| \le H \Big| \sum_{k=1}^Q a_k \Big| + M \max\{|a_k| : 1 \le k \le P\} + \varepsilon.$$

Moreover we recall the following known definitions: we say that a sequence $\{x_n\}_{n=1}^P$ $(1 \le P \le +\infty)$ in a Banach space is:

(i) H-*monotone* if, for each sequence $\{a_n\}_{n=1}^P$ of numbers, $(\|\sum_{n=1}^N a_n x_n\|)_{N=1}^P$ is $(H,0,0)$-monotone;

(ii) H-*unconditional* if, for each sequence $\{a_n\}_{n=1}^P$ of numbers and for each partition $\{n\}_{n=1}^P = \{n_k\}_{k=1}^K \cup \{n'_k\}_{k=1}^{K'}$,

$$\Big\|\sum_{k=1}^K a_{n_k} x_{n_k}\Big\| \le H \Big\|\sum_{n=1}^P a_n x_n\Big\|$$

(that is, each permutation of $\{x_n\}_{n=1}^P$ is H-monotone);

(iii) H-*indiscernible* (*indiscernible* if $H=1$) if, for the general subset $\{n_k\}_{k=1}^K$ of (ii), $\{x_{n_k}\}_{k=1}^K$ and $\{x_k\}_{k=1}^K$ are H^2-equivalent, that is,

$$\frac{1}{H}\Big\|\sum_{n=1}^K a_n x_n\Big\| \le \Big\|\sum_{k=1}^K a_k x_{n_k}\Big\| \le H \Big\|\sum_{n=1}^K a_n x_n\Big\|$$

(that is, if $\{x_{n_k}\}_{k=1}^K$ and $\{x_k\}_{k=1}^K$ are H^2-equivalent, hence in particular

$$\Big\|\sum_{k=1}^K a_k x_{n_k}\Big\| = \Big\|\sum_{n=1}^K a_n x_n\Big\|$$

if $\{x_n\}_{n=1}^P$ is indiscernible).

We point out some properties of the Walsh matrix (V. Kadets pointed out to us property (v), which allowed a simplification):

REMARK 6 (Properties of the Walsh matrix). By the definition of the Walsh matrix

(23.1) for each n, $\widehat{o}_n = \sum_{k=1}^{2^S} \varepsilon_{S,n,k} o_k$ where, for $1 \le k \le 2^S$, $\varepsilon_{S,n,k} \in \{-1,+1\}$; in particular $\widehat{o}_1 = \sum_{k=1}^{2^S} o_k$ and, for $0 \le m \le S-1$,

$$\widehat{o}_{2^m+1} = \sum_{j=1}^{2^{m+1}} (-1)^{j+1} \sum_{k=(j-1)2^{S-m-1}+1}^{j.2^{S-m-1}} o_k;$$

(23.2) $(\sum_{n=1}^{2^S} a_n o_n, \sum_{n=1}^{2^S} b_n o_n) = \sum_{k=1}^{2^S} a_n b_n$ for each sequence $\{a_n, b_n\}_{n=1}^{2^S}$ of numbers is a scalar product in $\mathrm{span}\{o_n\}_{n=1}^{2^S}$, which we call E_2 when equipped with the norm $\|\cdot\|_2$ of this scalar product (hence $\{o_n\}_{n=1}^{2^S}$ is 1-equivalent to the natural basis of $l_2^{2^S}$);

$$\widehat{o}_n^* = \frac{1}{2^S}\sum_{k=1}^{2^S} \varepsilon_{S,n,k} o_k^* \quad \text{for } 1 \le n \le 2^S; \tag{23.3}$$

$$(\widehat{o}_n/2^{S/2})_{n=1}^{2^S} \text{ is an orthogonal basis of } E_2; \tag{23.4}$$

(23.5) for each $A \subset \{1,\dots,2^S\}$ and $x = \sum_{n\in A} a_n \widehat{o}_n$ and $y \in \mathrm{span}\{\widehat{o}_n : n \in \{1,\dots,2^S\}\setminus A\}$, we have

$$\|x+y\| \ge \Big(\sum_{n\in A} |a_n|\Big)/\sqrt{A}.$$

Proof. For (23.3) and (23.4): From (23.2) and (23.1) it follows that

$$(\widehat{o}_m, \widehat{o}_n) = \sum_{k=1}^{2^S} \varepsilon_{S,m,k}\varepsilon_{S,n,k}$$

for $1 \leq m, n \leq 2^S$. We claim that

$$\sum_{k=1}^{2^S} \varepsilon_{S,m,k}\varepsilon_{S,n,k} = 2^S \delta_{m,n} \qquad \text{for } 1 \leq m, n \leq 2^S. \tag{24}$$

Indeed, it is obvious that (24) holds for $1 \leq m, n \leq 2^1$. Then we proceed by induction and suppose (24) true for $1 \leq m, n \leq 2^p$ for some integer p with $1 \leq p \leq S-1$. To check (24) for $1 \leq m, n \leq 2^{p+1}$, set

$$\widetilde{o}_k = \sum_{j=(k-1)2^{S-p-1}+1}^{k.2^{S-p-1}} o_j \qquad \text{for } 1 \leq k \leq 2^{p+1};$$

hence, for $1 \leq k \leq 2^p$,

$$\widetilde{o}_{2k-1} + \widetilde{o}_{2k} = \sum_{j=(2k-2)2^{S-p-1}+1}^{(2k-1)\cdot 2^{S-p-1}} o_j + \sum_{j=(2k-1)2^{S-p-1}+1}^{2k\cdot 2^{S-p-1}} o_j = \sum_{j=(k-1)2^{S-p}+1}^{k\cdot 2^{S-p}} o_j.$$

By (23.1), for each fixed m and n with $1 \leq m \leq 2^p$ and $2^p + 1 \leq n \leq 2^{p+1}$ (hence $1 \leq n - 2^p \leq 2^p$),

$$\widehat{o}_m = \sum_{k=1}^{2^p} \varepsilon_{p,m,k} \sum_{j=(k-1)2^{S-p}+1}^{k.2^{S-p}} o_j = \sum_{k=1}^{2^p} \varepsilon_{p,m,k}(\widetilde{o}_{2k-1} + \widetilde{o}_{2k}) = \sum_{k=1}^{2^{p+1}} \varepsilon_{p+1,m,k}\widetilde{o}_k$$

so that, for each k with $1 \leq k \leq 2^p$, $\varepsilon_{p+1,m,2k-1} = +\varepsilon_{p,m,k}$ and $\varepsilon_{p+1,m,2k} = +\varepsilon_{p,m,k}$. Analogously, by the second part of (23.1) for $m = p$,

$$\widehat{o}_{2^p+1} = \sum_{k=1}^{2^{p+1}} (-1)^{k+1} \sum_{j=(k-1)2^{S-p-1}+1}^{k.2^{S-p-1}} o_j = \sum_{k=1}^{2^{p+1}} (-1)^{k+1}\widetilde{o}_k,$$

hence in our case

$$\widehat{o}_n = \sum_{k=1}^{2^{p+1}} \varepsilon_{p+1,n,k}\widetilde{o}_k = \sum_{k=1}^{2^p} \varepsilon_{p,n,k}(\widetilde{o}_{2k-1} - \widetilde{o}_{2k}),$$

that is, $\varepsilon_{p+1,n,2k-1} = +\varepsilon_{p,n-2^p,k}$ and $\varepsilon_{p+1,n,2k} = -\varepsilon_{p,n-2^p,k}$ for each k with $1 \leq k \leq 2^p$, whence

$$\begin{aligned}(\widehat{o}_m, \widehat{o}_n) &= \sum_{k=1}^{2^{p+1}} \varepsilon_{p+1,m,k}\varepsilon_{p+1,n,k} = \sum_{k=1}^{2^p}(\varepsilon_{p+1,m,2k-1}\varepsilon_{p+1,n,2k-1} + \varepsilon_{p+1,m,2k}\varepsilon_{p+1,n,2k}) \\ &= \sum_{k=1}^{2^p}(\varepsilon_{p,m,k}\varepsilon_{p,m,k} - \varepsilon_{p,m,k}\varepsilon_{p,m,k}) = 0.\end{aligned}$$

On the other hand, it also follows that $\{\widehat{o}_n\}_{n=2^p+1}^{2^{p+1}}$ has the same behaviour as $\{\widehat{o}_n\}_{n=1}^{2^p}$ (since it is sufficient to replace $\widetilde{o}_{2k-1}+\widetilde{o}_{2k}$ which appears in the expression of the elements of $\{\widehat{o}_n\}_{n=1}^{2^p}$ by $\widetilde{o}_{2k-1}-\widetilde{o}_{2k}$ which appears in the expression of the elements of $\{\widehat{o}_n\}_{n=2^p+1}^{2^{p+1}}$), that is, (24) holds also for $2^p+1 \le m,n \le 2^{p+1}$, hence also for $1 \le m,n \le 2^{p+1}$; therefore (24) holds; hence $(\widehat{o}_m/2^{S/2}, \widehat{o}_n/2^{S/2}) = \delta_{m,n}$ for $1 \le m,n \le 2^S$; that is, (23.4) has been verified and also (23.3) follows from (24) and from (22), since, for each fixed m and n with $1 \le m,n \le 2^S$,

$$\Big(\frac{1}{2^S}\sum_{k=1}^{2^S}\varepsilon_{S,m,k}o_k^*\Big)(\widehat{o}_n) = \Big(\frac{1}{2^S}\sum_{k=1}^{2^S}\varepsilon_{S,m,k}o_k^*\Big)\Big(\sum_{i=1}^{2^S}\varepsilon_{S,n,i}o_i\Big)$$
$$= \frac{1}{2^S}\sum_{k=1}^{2^S}\sum_{i=1}^{2^S}\varepsilon_{S,m,k}\varepsilon_{S,n,i}o_k^*(o_i) = \frac{1}{2^S}\sum_{k=1}^{2^S}\varepsilon_{S,m,k}\varepsilon_{S,n,k} = \delta_{m,n}.$$

For (23.5): We know that

$$\sqrt{Q}\max\{|b_n| : 1 \le n \le Q\} \ge \sqrt{\sum_{n=1}^{Q} b_n^2} \ge \frac{\sum_{n=1}^{Q}|b_n|}{\sqrt{Q}}$$

for each sequence $\{b_n\}_{n=1}^{Q}$ of numbers (from a theorem of John on the Banach–Mazur distances $d(l_\infty^Q, l_2^Q)$ and $d(l_1^Q, l_2^Q)$, but also the direct verification is easy).

Therefore by (23.4), for each $A \subset \{1,\dots,2^S\}$ and for any $x = \sum_{n\in A} a_n\widehat{o}_n$ and $y \in \operatorname{span}\{\widehat{o}_n : n \in \{1,\dots,2^S\}\setminus A\}$, it follows that

$$\|x+y\|_\infty \ge \frac{\|x+y\|_2}{2^{S/2}} \ge \frac{1}{2^{S/2}}\|x\|_2 = \frac{1}{2^{S/2}}\Big\|\sum_{n\in A} a_n\widehat{o}_n\Big\|_2$$
$$= \Big\|\sum_{n\in A} a_n\Big(\frac{\widehat{o}_n}{2^{S/2}}\Big)\Big\|_2 = \sqrt{\sum_{n\in A} a_n^2} \ge \frac{\sum_{n\in A}|a_n|}{\sqrt{A}},$$

which completes the proof of Remark 6. ∎

The next proposition concerns the property of the most important (for the paper) subsequence of the $\{\widehat{o}_n\}_{n=1}^{2^S}$ of (22) (it is a particular subcase of (23.5)):

PROPOSITION 7 (Special sequences by means of the Walsh matrix). *Setting in* (22) $e_1 = \widehat{o}_1$, $e_1^* = \widehat{o}_1^*, e_m = \widehat{o}_{2^{m-1}+1}$, $e_m^* = \widehat{o}_{2^{m-1}+1}^*$ *for* $2 \le m \le S$, $E_0 = \operatorname{span}(\widehat{o}_n : 2 \le n \le 2^S, n \notin (2^{m-1}+1)_{m=2}^{S})$ *we have*:

(i) $\{e_m\}_{m=1}^{S}$ *is* 1-*equivalent to the natural basis of* l_1^S *and* $\{e_m^*\}_{m=1}^{S}$ *is indiscernible*;
(ii) $\{e_m + E_0\}_{m=1}^{S}$ *is indiscernible and* 1-*unconditional.*

Proof. (i) The fact that $\{e_m\}_{m=1}^{S}$ is 1-equivalent to the natural basis of l_1^S is well known, therefore we turn to considering $\{e_m^*\}_{m=1}^{S}$ where, by (23.1) and (23.3),

$$e_1^* = \widehat{o}_1^* = \frac{1}{2^S}\sum_{k=1}^{2^S} o_k^*, \qquad e_m^* = \widehat{o}_{2^{m-1}+1}^* = \frac{1}{2^S}\sum_{j=1}^{2^m}(-1)^{j+1}\sum_{k=(j-1)2^{S-m}+1}^{j\cdot 2^{S-m}} o_k^*$$

for $2 \le m \le S$. So let $\{n_k\}_{k=1}^{K}$ (where without loss of generality we can suppose $n_1 = 1$) be a subsequence of $\{m\}_{m=1}^{S}$ and let $\{a_k\}_{k=1}^{K}$ be a sequence of numbers. Since $\{o_k^*\}_{k=1}^{2^S}$

is 1-equivalent to the natural basis of $l_1^{2^S}$ we have

$$\begin{aligned}
\Big\|\sum_{k=1}^{2} a_k e_k^*\Big\| &= \Big\|\frac{1}{2^S}\Big(a_1\sum_{k=1}^{2^S} o_k^* + a_2\sum_{j=1}^{2^2}(-1)^{j+1}\sum_{k=(j-1)2^{S-2}+1}^{j.2^{S-2}} o_k^*\Big)\Big\| \\
&= \frac{1}{2^S}\Big\|\sum_{j=1}^{2^2}(a_1+(-1)^{j+1}a_2)\sum_{k=(j-1)2^{S-2}+1}^{j\cdot 2^{S-2}} o_k^*\Big\| \\
&= \frac{2^{S-2}}{2^S}\sum_{j=1}^{2^2}|a_1+(-1)^{j+1}a_2| = \frac{|a_1+a_2|+|a_1-a_2|}{2^1} \\
&= \frac{2^{S-n_2}}{2^S}(2^{n_2-1}|a_1+a_2| + 2^{n_2-1}|a_1-a_2|) \\
&= \frac{2^{S-n_2}}{2^S}\sum_{j=1}^{2^{n_2}}|a_1+(-1)^{j+1}a_2| = \frac{1}{2^S}\Big\|\sum_{j=1}^{2^{n_2}}(a_1+(-1)^{j+1}a_2)\sum_{k=(j-1)2^{S-n_2}+1}^{j\cdot 2^{S-n_2}} o_k^*\Big\| \\
&= \Big\|\frac{1}{2^S}\Big(a_1\sum_{k=1}^{2^S} o_k^* + a_2\sum_{j=1}^{2^{n_2}}(-1)^{j+1}\sum_{k=(j-1)2^{S-n_2}+1}^{j\cdot 2^{S-n_2}} o_k^*\Big)\Big\| = \Big\|\sum_{k=1}^{2} a_k e_{n_k}^*\Big\|.
\end{aligned}$$

Analogously, setting $A_j = \sum_{k=(j-1)2^{S-n_3}+1}^{j2^{S-n_3}} o_k^*$ for $1 \le j \le 2^{n_3}$, we have

$$\begin{aligned}
\Big\|\sum_{k=1}^{3} a_k e_{n_k}^*\Big\| &= \frac{1}{2^S}\Big\|a_1\sum_{k=1}^{2^S} o_k^* + a_2\sum_{j=1}^{2^{n_2}}(-1)^{j+1}\sum_{k=(j-1)2^{S-n_2}+1}^{j.2^{S-n_2}} o_k^* \\
&\qquad + a_3\sum_{j=1}^{2^{n_3}}(-1)^{j+1}\sum_{k=(j-1)2^{S-n_3}+1}^{j\cdot 2^{S-n_3}} o_k^*\Big\| \\
&= \frac{1}{2^S}\Big\|\sum_{i=1}^{2^{n_2}}\sum_{j=(i-1)2^{n_3-n_2}+1}^{i2^{n_3-n_2}}(a_1+(-1)^{i+1}a_2+(-1)^{j+1}a_3)A_j\Big\| \\
&= \frac{2^{S-n_3}}{2^S}\sum_{i=1}^{2^{n_2}}\sum_{j=(i-1)2^{n_3-n_2}+1}^{i2^{n_3-n_2}}|a_1+(-1)^{i+1}a_2+(-1)^{j+1}a_3| \\
&= \frac{2^{S-n_3}}{2^S}(2^{n_2-1}(2^{n_3-n_2-1}|a_1+a_2+a_3| + 2^{n_3-n_2-1}|a_1+a_2-a_3|) \\
&\qquad + 2^{n_2-1}(2^{n_3-n_2-1}|a_1-a_2+a_3| + 2^{n_3-n_2-1}|a_1-a_2-a_3|)) \\
&= \frac{1}{2^2}(|a_1+a_2+a_3| + |a_1+a_2-a_3| + |a_1-a_2+a_3| + |a_1-a_2-a_3|)
\end{aligned}$$

(since $\frac{2^{S-n_3}}{2^S}2^{n_2-1}2^{n_3-n_2-1} = \frac{1}{2^2}$) which does not depend on n_2 and n_3. Then

$$\Big\|\sum_{k=1}^{3} a_k e_{n_k}^*\Big\| = \Big\|\sum_{k=1}^{3} a_k e_k^*\Big\|$$

and so on, till we get

$$\Big\|\sum_{k=1}^{K} a_k e_k^*\Big\| = \Big\|\sum_{k=1}^{K} a_k e_{n_k}^*\Big\|.$$

(ii) We can immediately see that $\{e_m + E_0\}_{m=1}^S$ is 1-monotone (indeed, for each sequence $c_1 \cup (c_{2^{m-1}+1})_{m=2}^S$ of numbers let $(c_n)_{n(\notin(2^{m-1}+1)_{m=2}^S)=2}^S$ be another sequence of numbers such that

$$\Big\|c_1\widehat{o}_1 + \sum_{m=2}^{S} c_{2^{m-1}+1}\widehat{o}_{2^{m-1}+1} + E_0\Big\| = \Big\|\sum_{n=1}^{2^S} c_n\widehat{o}_n\Big\|;$$

then obviously

$$\begin{aligned}\Big\|\sum_{n=1}^{2^S} c_n\widehat{o}_n\Big\| \geq \Big\|\sum_{n=1}^{2^{S-1}} c_n\widehat{o}_n\Big\| \\ &\geq \Big\|c_1\widehat{o}_1 + \sum_{m=2}^{S-1} c_{2^{m-1}+1}\widehat{o}_{2^{m-1}+1} + \operatorname{span}\{\widehat{o}_n\}_{n(\notin(2^{m-1}+1)_{m=2}^S)=2}^S\Big\| \\ &= \Big\|c_1\widehat{o}_1 + \sum_{m=2}^{S-1} c_{2^{m-1}+1}\widehat{o}_{2^{m-1}+1} + E_0\Big\|,\end{aligned}$$

and so on). Therefore let $\{a_k\}_{k=1}^K \cup \{a'_k\}_{k=1}^{K'}$ be a sequence of numbers and let $\{n_k\}_{k=1}^K \cup \{n'_k\}_{k=1}^{K'}$ (where without loss of generality we can suppose $n_1 = 1$) be a partition of $\{m\}_{m=1}^S$. Setting $\widehat{o} = \sum_{k=1}^K a_k e_k$ and $\widehat{o}' = \sum_{k=1}^K a_k e_{n_k}$ we can see that $\|\widehat{o} + E_0\| \geq \|\widehat{o}' + E_0\|$. Indeed, let $b_2 \cup ((b_i)_{i=2^{m-1}+2}^{2^m})_{m=2}^K$ be a sequence of numbers such that

$$\begin{aligned}\|\widehat{o} + E_0\| &= \Big\|a_1\widehat{o}_1 + \sum_{m=2}^{K} a_m\widehat{o}_{2^{m-1}+1} + E_0\Big\| \\ &= \Big\|a_1\widehat{o}_1 + \sum_{m=2}^{K} a_m\widehat{o}_{2^{m-1}+1} + \operatorname{span}(\widehat{o}_2 \cup ((\widehat{o}_i)_{i=2^{m-1}+2}^{2^m})_{m=2}^K)\Big\| \\ &= \Big\|a_1\widehat{o}_1 + \sum_{m=2}^{K} a_m\widehat{o}_{2^{m-1}+1} + b_2\widehat{o}_2 + \sum_{m=2}^{K}\sum_{i=2^{m-1}+2}^{2^m} b_i\widehat{o}_i\Big\|;\end{aligned}$$

since it is immediate to see that

$$(\widehat{o}_i)_{i=1}^{2^K} = (\widehat{o}_i)_{i=1}^{2^1} \cup ((\widehat{o}_i)_{i=2^{m-1}+1}^{2^{m-1}+2^{m-1}})_{m=2}^K \quad \text{and} \quad (\widehat{o}_i)_{i=1}^{2^1} \cup ((\widehat{o}_i)_{i=2^{n_k-1}+1}^{2^{n_k-1}+2^{m-1}})_{k=2}^K$$

are 1-equivalent (for instance by induction), it follows that

$$\begin{aligned}\|\widehat{o}' + E_0\| &\leq \Big\|a_1\widehat{o}_1 + \sum_{k=2}^{K} a_k\widehat{o}_{2^{n_k-1}+1} + b_2\widehat{o}_2 + \sum_{k=2}^{K}\sum_{i=2^{n_k-1}+1}^{2^{n_k-1}+2^{m-1}} b_i\widehat{o}_i\Big\| \\ &= \Big\|a_1\widehat{o}_1 + \sum_{m=2}^{K} a_k\widehat{o}_{2^{m-1}+1} + b_2\widehat{o}_2 + \sum_{m=2}^{K}\sum_{i=2^{m-1}+2}^{2^m} b_i\widehat{o}_k\Big\| = \|\widehat{o} + E_0\|.\end{aligned}$$

Therefore we are now going to prove that $\|\widehat{o}' + E_0\| \geq \|\widehat{o} + E_0\|$ (hence $\|\widehat{o}' + E_0\| = \|\widehat{o} + E_0\|$, that is, $(e_m + E_0)_{m=1}^S$ is indiscernible) and also that

$$\Big\|\sum_{k=1}^{K} a_k e_{n_k} + \sum_{k=1}^{K'} a'_k e_{n'_k} + E_0\Big\| \geq \Big\|\sum_{k=1}^{K} a_k e_{n_k} + E_0\Big\|$$

(hence $(e_m + E_0)_{m=1}^S$ is 1-unconditional).

Since by the above $(e_m + E_0)_{m=1}^S$ is 1-monotone, we have

$$\|\widehat{o} + E_0\| = \Big\|\sum_{k=1}^K a_k e_k + E_0\Big\| = \Big\|a_1\widehat{o}_1 + \sum_{m=2}^K a_m \widehat{o}_{2^{m-1}+1} + E_0\Big\|$$
$$= \Big\|a_1\widehat{o}_1 + \sum_{m=2}^K a_m \widehat{o}_{2^{m-1}+1} + E_0 + \operatorname{span}(\widehat{o}_{2^{m-1}+1})_{m=K+1}^S\Big\|;$$

hence, if $(e_m + E_0, F_m)_{m=1}^S$ is biorthogonal, there exists $F \in (\operatorname{span}(e_m + E_0)_{m=K+1}^S)^\top = \operatorname{span}\{F_m\}_{m=1}^K$, $F = \sum_{m=1}^K b_m F_m$, such that $F(\widehat{o} + E_0) = \|\widehat{o} + E_0\|$ (hence $\|\widehat{o} + E_0\| = \sum_{m=1}^K a_m b_m$) with $\|F\| = 1$. On the other hand, $(E/E_0)^*$ is linearly isometric to

$$(E_0)^\top = \operatorname{span}(\widehat{o}_1^* \cup \operatorname{span}(\widehat{o}_{2^{m-1}+1}^*)_{m=2}^S)$$

and $\{F_m\}_{m=1}^S$ and $\widehat{o}_1^* \cup \{\widehat{o}_{2^{m-1}+1}^*\}_{m=2}^S$ are 1-equivalent; hence, setting $F' = \sum_{k=1}^K b_k F_{n_k}$, since by (i), $\widehat{o}_1^* \cup \{\widehat{o}_{2^{m-1}+1}^*\}_{m=2}^S = \{e_m^*\}_{m=1}^S$ is indiscernible, it follows that

$$1 = \|F\| = \Big\|\sum_{m=1}^K b_m F_m\Big\| = \Big\|b_1\widehat{o}_1^* + \sum_{m=2}^K b_m \widehat{o}_{2^{m-1}+1}^*\Big\|$$
$$= \Big\|b_1\widehat{o}_1^* + \sum_{k=2}^K b_k \widehat{o}_{2^{n_k-1}+1}^*\Big\| = \Big\|\sum_{k=1}^K b_k F_{n_k}\Big\| = \|F'\|.$$

At this point we notice that

$$F'(\widehat{o}' + E_0) = \Big(\sum_{k=1}^K b_k F_{n_k}\Big)\Big(\sum_{k=1}^K a_k e_{n_k} + E_0\Big) = \sum_{k=1}^K a_k b_k = \|\widehat{o} + E_0\|$$

with $\|F'\| = 1$ and $F'_\perp = \operatorname{span}(e_{n'_k})_{k=1}^{K'}$, hence

$$\Big\|\sum_{k=1}^K a_k e_{n_k} + E_0\Big\| = \|\widehat{o}' + E_0\| \le \Big\|\widehat{o}' + \sum_{k=1}^{K'} a'_k e_{n'_k} + E_0\Big\|.$$

Therefore we conclude that both

$$\|\widehat{o}' + E_0\| = \sup(\widetilde{F}(\widehat{o}' + E_0) : \|\widetilde{F}\| = 1) \ge F'(\widehat{o}' + E_0) = \|\widehat{o} + E_0\|$$

(which was our aim) and

$$\|\widehat{o}' + E_0\| = \Big\|\sum_{k=1}^K a_k e_{n_k} + E_0\Big\| \le \Big\|\sum_{k=1}^K a_k e_{n_k} + \sum_{k=1}^{K'} a'_k e_{n'_k} + E_0\Big\|$$

(by the above), that is, $\{e_m + E_0\}_{m=1}^S$ is also 1-unconditional. This completes the proof of (ii) and hence of Proposition 7. ∎

Construction I. We start from the biorthogonal system $\{\widehat{o}_n, \widehat{o}_n^*\}_{n=1}^{2^S}$ of (22) for $S = M + R$. For each

$$o = \sum_{n=1}^{2^{M+R}} a_n e_n \in \widehat{E} = l_\infty^{2^{M+R}} = \operatorname{span}\{\widehat{o}_n\}_{n=1}^{2^{M+R}} = \operatorname{span}\{o_n\}_{n=1}^{2^{M+R}}$$

we define the *support* of o to be the set

$$\operatorname{supp}(o) = \{n : 1 \le n \le 2^{M+R}; a_n \ne 0\}.$$

First step. We can write

(25) $\{\widehat{o}_n,\widehat{o}^*_n\}_{n=1}^{2^{M+R}}=\{\widehat{o}_{1,m},\widehat{o}^*_{1,m}\}_{m=1}^{2^{M+1}}\cup\{\{\widehat{o}_{r,m},\widehat{o}^*_{r,m}\}_{m=1}^{2^{M+r-1}}\}_{r=2}^{R}$ with $\{\widehat{o}_{1,m},\widehat{o}^*_{1,m}\}_{m=1}^{2^{M+1}}=\{\widehat{o}_n,\widehat{o}^*_n\}_{n=1}^{2^{M+1}}$ and we set $\widehat{E}_1=\operatorname{span}\{\widehat{o}_{1,m}\}_{m=1}^{2^{M+1}}$, moreover, for $2\le r\le R$, $\{\widehat{o}_{r,m},\widehat{o}^*_{r,m}\}_{m=1}^{2^{M+r-1}}=\{\widehat{o}_n,\widehat{o}^*_n\}_{n=2^{M+r-1}+1}^{2^{M+r}}$ and we set $\widehat{E}_r=\operatorname{span}\{\widehat{o}_{r,m}\}_{m=1}^{2^{M+r-1}}$, hence $\sum_{r=1}^R\widehat{E}_r=\operatorname{span}\{o_n\}_{n=1}^{2^{M+R}}=\widehat{E}$; finally we set, for $1\le r\le R$, $\widehat{E}_r=\widehat{E}'_r+\widehat{E}''_r$, where $\widehat{E}_1=\widehat{E}'_1$ and $\widehat{E}_2=\widehat{E}'_2$, while, for $3\le r\le R$, $\widehat{E}'_r=\operatorname{span}\{\widehat{o}_{r,m}\}_{m=1}^{2^{M+1}}$ and $\widehat{E}''_r=\operatorname{span}\{\widehat{o}_{r,m}\}_{m=2^{M+1}+1}^{2^{M+r-1}}$, moreover $\widehat{E}=\widehat{E}'+\widehat{E}''$ with $\widehat{E}'=\sum_{r=1}^R\widehat{E}'_r$ and $\widehat{E}''=\sum_{r=3}^R\widehat{E}''_r$

(in particular, for $r=2$, $\{\widehat{o}_{2,m},\widehat{o}^*_{2,m}\}_{m=1}^{2^{M+1}}=\{\widehat{o}_n\}_{n=2^{M+1}+1}^{2^{M+2}}$). By (23.1) and (23.2) we can set (see also the explanations after the formula)

(26.1) for $1\le r\le R$,

$$\widetilde{o}_{0,r,j}=\sum_{t=(j-1)2^{R-r}+1}^{j.2^{R-r}}o_t\quad\text{and}\quad\widetilde{o}^*_{0,r,j}=\sum_{t=(j-1)2^{R-r}+1}^{j.2^{R-r}}o^*_t$$

for $1\le j\le 2^{M+r}$ (in particular $\widetilde{o}_{0,R,j}=o_j$ for $1\le j\le 2^{M+R}$ and $\widetilde{o}_{0,1,j}=\sum_{t=(j-1)2^{R-1}+1}^{j\cdot 2^{R-1}}o_t$ for $1\le j\le 2^{M+1}$, finally $\widetilde{o}_{0,r',j}=\sum_{l=1}^{2^{r''-r'}}\widetilde{o}_{0,r'',(j-1)2^{r''-r'}+l}$ for $1\le j\le 2^{M+r'}$ and $1\le r'\le r''\le R$); we point out that, for $1\le m\le 2^{M+r-1}$,

$$\widehat{o}_{r,m}=\widehat{o}_{2^{M+r-1}+m}=\sum_{k=1}^{2^{M+r}}\varepsilon_{M+r,2^{M+r-1}+m,k}\widetilde{o}_{0,r,k};$$

(26.2) for $1\le k\le 2^{M+1}$ we set

$$\widetilde{o}_{1,k}=\widetilde{o}_{0,1,k}=\sum_{t=(k-1)2^{R-1}+1}^{k\cdot 2^{R-1}}o_t\quad\text{and}\quad\widetilde{o}^*_{1,k}=\widetilde{o}^*_{0,1,k},$$

analogously

$$\widetilde{o}_{2,k}=\sum_{j=1}^{2^1}(-1)^{j+1}\sum_{t=(k-1)2^{R-1}+(j-1)2^{R-2}+1}^{(k-1)2^{R-1}+j.2^{R-2}}o_t=\sum_{j=1}^{2^1}(-1)^{j+1}\widetilde{o}_{0,2,(k-1)2^1+j}$$

and

$$\widetilde{o}^*_{2,k}=\sum_{j=1}^{2^1}(-1)^{j+1}\widetilde{o}^*_{0,2,(k-1)2^1+j};$$

it follows that, for $1\le r\le 2, \widehat{o}_{r,1}=\sum_{k=1}^{2^{M+1}}\widetilde{o}_{r,k}$ and in general, for $1\le m\le 2^{M+1}, \widehat{o}_{r,m}=\sum_{k=1}^{2^{M+1}}\varepsilon_{M+1,m,k}\widetilde{o}_{r,k}$;

(26.3) fix now r with $3\le r\le R$; we know that

$$\widehat{o}_{r,n}(=\widehat{o}_{2^{M+r-1}+n})=\sum_{j=1}^{2^{M+r}}\varepsilon_{M+r,2^{M+r-1}+n,j}\widetilde{o}_{0,r,j}$$

for $1 \le n \le 2^{M+r-1}$, hence

$$\widehat{o}_{r,1} = \widehat{o}_{2^{M+r-1}+1} = \sum_{j=1}^{2^{M+r}} (-1)^{j+1}\widetilde{o}_{0,r,j} = \sum_{k=1}^{2^{M+1}} \sum_{j=(k-1)2^{r-1}+1}^{k.2^{r-1}} (-1)^{j+1}\widetilde{o}_{0,r,j}$$
$$= \sum_{k=1}^{2^{M+1}} \widetilde{o}_{r,k}$$

where, for $1 \le k \le 2^{M+1}$,

$$\widetilde{o}_{r,k} = \sum_{j=(k-1)2^{r-1}+1}^{k.2^{r-1}} (-1)^{j+1}\widetilde{o}_{0,r,j} = \sum_{j=1}^{2^{r-1}} (-1)^{j+1}\widetilde{o}_{0,r,(k-1)2^{r-1}+j}$$
$$= \sum_{j=1}^{2^{r-1}} (-1)^{j+1} \sum_{t=(k-1)2^{R-1}+(j-1)2^{R-r}+1}^{(k-1)2^{R-1}+j\cdot 2^{R-r}} o_t$$

(and we set $\widetilde{o}^*_{r,k} = \sum_{j=(k-1)2^{r-1}+1}^{k.2^{r-1}} (-1)^{j+1}\widetilde{o}^*_{0,r,j}$); it follows that

$$\widehat{o}_{r,m} = \widehat{o}_{2^{M+r-1}+m} = \sum_{k=1}^{2^{M+1}} \varepsilon_{M+1,m,k}\widetilde{o}_{r,k}$$

for $1 \le m \le 2^{M+1}$;

(26.4) for $1 \le r \le R$ and $1 \le k \le 2^{M+1}$,

$$\operatorname{supp}(\widetilde{o}_{r,k}) = \{n : (k-1)2^{R-1} + 1 \le n \le k \cdot 2^{R-1}\}.$$

(26.1) concerns, for each r with $1 \le r \le R$, the representation of the elements of $\{\widehat{o}_{r,m}\}_{m=1}^{2^{M+r-1}}$, where the "bricks" of the construction are the elements $\widetilde{o}_{0,r,j}$, in particular we use $2^{M+R}/2^{M+r} = 2^{R-r}$, $2^{M+r''}/2^{M+r'} = 2^{r''-r'}$ and

$$\widetilde{o}_{0,r',j} = \sum_{t=(j-1)2^{R-r'}+1}^{j.2^{R-r'}} o_t = \sum_{l=1}^{2^{r''-r'}} \sum_{t=(j-1)2^{R-r'}+(l-1)2^{R-r''}+1}^{(j-1)2^{R-r'}+l.2^{R-r''}} o_t$$
$$= \sum_{l=1}^{2^{r''-r'}} \sum_{t=((j-1)2^{r''-r'}+(l-1))2^{R-r''}+1}^{((j-1)2^{r''-r'}+l)\cdot 2^{R-r''}} o_t = \sum_{l=1}^{2^{r''-r'}} \widetilde{o}_{0,r'',(j-1)2^{r''-r'}+(l-1)+1}$$
$$= \sum_{l=1}^{2^{r''-r'}} \widetilde{o}_{0,r'',(j-1)2^{r''-r'}+l}.$$

(26.2) and (26.3) concern the representation of the first 2^{M+1} elements of $\{\widehat{o}_{r,m}\}_{m=1}^{2^{M+r-1}}$; we call $\widetilde{o}_{r,k}$ the "bricks" of this representation. Hence we consider separately the case of $1 \le r \le 2$ (where $\widehat{E}''_r$ does not appear) and of $3 \le r \le R$. For (26.2) we have

$$\widetilde{o}_{2,k} = \sum_{j=1}^{2^1} (-1)^{j+1} \sum_{t=(k-1)2^{R-1}+(j-1)2^{R-2}+1}^{(k-1)2^{R-1}+j.2^{R-2}} o_t$$

$$= \sum_{j=1}^{2^1} (-1)^{j+1} \sum_{t=((k-1)2^1+(j-1))2^{R-2}+1}^{((k-1)2^1+j)2^{R-2}} o_t = \sum_{j=1}^{2^1} (-1)^{j+1} \widetilde{o}_{0,2,(k-1)2^1+(j-1)+1}$$

$$= \sum_{j=1}^{2^1} (-1)^{j+1} \widetilde{o}_{0,2,(k-1)2^1+j} = \widetilde{o}_{0,2,2k-1} - \widetilde{o}_{0,2,2k}.$$

On the other hand, by (23.1) and by the definition of the Walsh matrix,

$$\widehat{o}_{2,1} = \widehat{o}_{2^{M+1}+1} = \sum_{k=1}^{2^{M+2}} \varepsilon_{M+2,2^{M+1}+1,k} \widetilde{o}_{0,2,k} = \sum_{k=1}^{2^{M+2}} (-1)^{k+1} \widetilde{o}_{0,2,k}$$

$$= \sum_{k=1}^{2^{M+1}} (\widetilde{o}_{0,2,2k-1} - \widetilde{o}_{0,2,2k}) = \sum_{k=1}^{2^{M+1}} \widetilde{o}_{2,k}.$$

Hence, for $1 \le m \le 2^{M+1}$,

$$\widehat{o}_{2,m} = \widehat{o}_{2^{M+1}+m} \left(= \sum_{k=1}^{2^{M+2}} \varepsilon_{M+2,2^{M+1}+m,k} \widetilde{o}_{0,2,k} \right) = \sum_{k=1}^{2^{M+1}} \varepsilon_{M+1,m,k} \widetilde{o}_{2,k}.$$

For (26.3), analogously, the fact that $\varepsilon_{M+r,2^{M+r-1}+1,j} = (-1)^{j+1}$ for $1 \le j \le 2^{M+r}$ comes from (23.1) and from the definition of the Walsh matrix).

Second step. We are now going to define another Hamel basis

$$(v_n)_{n=1}^{2^{M+R}} = (v_{1,m})_{m=1}^{2^{M+1}} \cup ((v_{r,m})_{m=1}^{2^{M+r-1}})_{r=2}^{R}$$

of $E = \operatorname{span}(\widehat{o}_n)_{n=1}^{2^{M+R}}$. We start with $r = 1$ and we will define (we suggest making a picture)

$$(v_{1,n})_{n=1}^{2^{M+1}} = ((v_{1,m,g})_{g=1}^{2^{M-m+1}})_{m=1}^{M} \cup (v_{1,M+1,g})_{g=1}^{2}.$$

We start with $m = 1$ and, by (26.2) and by (25), moreover by the definition of the Walsh matrix, for $1 \le g \le 2^M$ (taking into account the fact that, for these g, $\varepsilon_{M+1,2(g-1)+1,k} = \varepsilon_{M,g,k}$ for $1 \le k \le 2^M$) we set

$$v_{1,1,g} = \frac{(\widehat{o}_{1,2(g-1)+1} + \widehat{o}_{1,2g})}{2^1} = \frac{1}{2}\Big(\sum_{k=1}^{2^{M+1}} \varepsilon_{M+1,2(g-1)+1,k} \widetilde{o}_{1,k} + \sum_{k=1}^{2^{M+1}} \varepsilon_{M+1,2g,k} \widetilde{o}_{1,k} \Big)$$

$$= \sum_{k=1}^{2^{M+1}} \frac{\varepsilon_{M+1,2(g-1)+1,k} + \varepsilon_{M+1,2g,k}}{2} \widetilde{o}_{1,k} = \sum_{k=1}^{2^M} \varepsilon_{M+1,2(g-1)+1,k} \widetilde{o}_{1,k} = \sum_{k=1}^{2^M} \varepsilon_{M,g,k} \widetilde{o}_{1,k}.$$

For $m = 2$ we set analogously for $1 \le g \le 2^{M-1}$,

$$v_{1,2,g} = \frac{(\widehat{o}_{1,2^2(g-1)+1} - \widehat{o}_{1,2^2(g-1)+2}) + (\widehat{o}_{1,2^2(g-1)+3} - \widehat{o}_{1,2^2 g})}{2^2}$$

$$= \sum_{k=1}^{2^{M+1}} \frac{(\varepsilon_{M+1,2^2(g-1)+1,k} - \varepsilon_{M+1,2^2(g-1)+2,k}) + (\varepsilon_{M+1,2^2(g-1)+3,k} - \varepsilon_{M+1,2^2 g,k})}{2^2} \widetilde{o}_{1,k}$$

$$= \sum_{k=2^M+1}^{2^M+2^{M-1}} \varepsilon_{M+1,2^2(g-1)+1,k} \widetilde{o}_{1,k} = \sum_{k=2^M+1}^{2^M+2^{M-1}} \varepsilon_{M-1,g,k-2^M} \widetilde{o}_{1,k}.$$

Continuing we get, for $2 \le m \le M-1$, $1 \le g \le 2^{M-m+1}$,

$$v_{1,m,g}\ (\in \operatorname{span}(\widehat{o}_{1,n})_{n=1}^{2^{M+1}}) = \sum_{k=2^M+\cdots+2^{M-m+2}+1}^{2^M+\cdots+2^{M-m+1}} \varepsilon_{M-m+1,g,k-(2^M+\cdots+2^{M-m+2})}\widetilde{o}_{1,k},\ v_{1,M,g}$$

$$= \sum_{k=2^M+\cdots+2^2+1}^{2^M+\cdots+2^1} \varepsilon_{1,g,k-(2^M+\cdots+2^2)}\widetilde{o}_{1,k} = \sum_{k=2^{M+1}-2^2+1}^{2^{M+1}-2^1} \varepsilon_{1,g,k-(2^{M+1}-2^2)}\widetilde{o}_{1,k},$$

$$v_{1,M+1,g} = \sum_{k=2^{M+1}-2+1}^{2^{M+1}} \varepsilon_{1,g,k-(2^{M+1}-2)}\widetilde{o}_{1,k}\ (\in \operatorname{span}\{\widehat{o}_{1,n}\}_{n=1}^{2^{M+1}}).$$

At this point we set, for $1 \le m \le M$, $w_{m,1} = v_{1,m,1}$. Since $(v_{1,n})_{n=1}^{2^{M+1}}$ is a Hamel basis of $\operatorname{span}(\widehat{o}_{1,n})_{n=1}^{2^{M+1}}$, there exists $(v^*_{1,n})_{n=1}^{2^{M+1}} \subset \operatorname{span}(\widehat{o}^*_{1,n})_{n=1}^{2^{M+1}}$ such that $(v_{1,n}, v^*_{1,n})_{n=1}^{2^{M+1}}$ is biorthogonal. In particular by the above and by (26.2) and (23.3), it is easy to deduce that

$$v^*_{1,1,g} = \widehat{o}^*_{1,2(g-1)+1} + \widehat{o}^*_{1,2g} = \frac{1}{2^{M+R-1}} \sum_{k=1}^{2^M} \varepsilon_{M,g,k}\widetilde{o}^*_{1,k}$$

(since by (26.4) the cardinality of $\operatorname{supp}(\widetilde{o}_{r,k})$ is 2^{R-1} for $1 \le r \le R$ and $1 \le k \le 2^{M+1}$) for $1 \le g \le 2^M$. Analogously for $2 \le m \le M$, since $2^{M-m+1}2^{R-1} = 2^{M+R-m}$, for $1 \le g \le 2^{M-m+1}$ we have

$$v^*_{1,m,g} = \frac{\sum_{k=2^M+\cdots+2^{M-m+2}+1}^{2^M+\cdots+2^{M-m+1}} \varepsilon_{M-m+1,g,k-(2^M+\cdots+2^{M-m+2})}\widetilde{o}^*_{1,k}}{2^{M+R-m}},$$

$$v^*_{1,M+1,g} = \frac{1}{2^R} \sum_{k=2^{M+1}-2+1}^{2^{M+1}} \varepsilon_{1,g,k-(2^{M+1}-2)}\widetilde{o}^*_{1,k} \quad \text{for } 1 \le g \le 2.$$

In particular by the above, by (23.1) and (23.3) and by the definition of the Walsh matrix,

$$w_{1,1} = v_{1,1,1} = \sum_{k=1}^{2^M} \varepsilon_{M,1,k}\widetilde{o}_{1,k} = \sum_{k=1}^{2^M} \widetilde{o}_{1,k},\ w^*_{1,1} = \frac{1}{2^{M+R-1}} \sum_{k=1}^{2^M} \widetilde{o}^*_{1,k}$$

and, for $2 \le m \le M$,

$$w_{m,1} = v_{1,m,1} = \sum_{k=2^M+\cdots+2^{M-m+2}+1}^{2^M+\cdots+2^{M-m+1}} \varepsilon_{M-m+1,1,k-(2^M+\cdots+2^{M-m+2})}\widetilde{o}_{1,k}$$

$$= \sum_{k=2^M+\cdots+2^{M-m+2}+1}^{2^M+\cdots+2^{M-m+1}} \widetilde{o}_{1,k},\ w^*_{m,1} = \frac{1}{2^{M+R-m}} \sum_{k=2^M+\cdots+2^{M-m+2}+1}^{2^M+\cdots+2^{M-m+1}} \widetilde{o}^*_{1,k}.$$

We set $\widehat{E}_{0,1} = \widehat{E}'_{0,1} = \sum_{m=1}^{M+1} \widehat{E}_{0,1,m}$ where $\widehat{E}_{0,1,m} = \operatorname{span}\{v_{1,m,g}\}_{g=2}^{2^{M-m+1}}$ for $1 \le m \le M$, while $\widehat{E}_{0,1,M+1} = \operatorname{span}\{v_{1,M+1,g}\}_{g=1}^{2}$. By (25) it follows that $\widehat{E}_1 = \widehat{E}'_1 = \operatorname{span}\{w_{m,1}\}_{m=1}^{M} + \widehat{E}'_{0,1}$.

For $r = 2$ the same procedure works: if we replace $\{\widetilde{o}_{1,k}, \widetilde{o}^*_{1,k}\}_{k=1}^{2^{M+1}}$ by $\{\widetilde{o}_{2,k}, \widetilde{o}^*_{2,k}\}_{k=1}^{2^{M+1}}$, then we get

$$w_{1,2} = v_{2,1,1} = \sum_{k=1}^{2^M} \widetilde{o}_{2,k}, \qquad w^*_{1,2} = \frac{1}{2^{M+R-1}} \sum_{k=1}^{2^M} \widetilde{o}^*_{2,k},$$
$$v_{2,1,g} = \sum_{k=1}^{2^M} \varepsilon_{M,g,k} \widetilde{o}_{2,k}, \qquad v^*_{2,1,g} = \frac{1}{2^{M+R-1}} \sum_{k=1}^{2^M} \varepsilon_{M,g,k} \widetilde{o}^*_{2,k}$$

for $2 \le g \le 2^M$. Moreover, for $2 \le m \le M$,

$$\begin{aligned}
w_{m,2} = v_{2,m,1} &= \sum_{k=2^M+\dots+2^{M-m+2}+1}^{2^M+\dots+2^{M-m+1}} \widetilde{o}_{2,k},\\
w^*_{m,2} &= \frac{1}{2^{M+R-m}} \sum_{k=2^M+\dots+2^{M-m+2}+1}^{2^M+\dots+2^{M-m+1}} \widetilde{o}^*_{2,k} \quad \text{for } 2 \le g \le 2^{M-m+1},\\
v_{2,m,g} &= \sum_{k=2^M+\dots+2^{M-m+2}+1}^{2^M+\dots+2^{M-m+1}} \varepsilon_{M-m+1,g,k-(2^M+\dots+2^{M-m+2})} \widetilde{o}_{2,k},\\
v^*_{2,m,g} &= \frac{\sum_{k=2^M+\dots+2^{M-m+2}+1}^{2^M+\dots+2^{M-m+1}} \varepsilon_{M-m+1,g,k-(2^M+\dots+2^{M-m+2})} \widetilde{o}^*_{2,k}}{2^{M+R-m}},\\
v_{2,M+1,g} &= \sum_{k=2^{M+1}-2+1}^{2^{M+1}} \varepsilon_{1,g,k-(2^{M+1}-2)} \widetilde{o}_{2,k},\\
v^*_{2,M+1,g} &= \frac{1}{2^R} \sum_{k=2^{M+1}-2+1}^{2^{M+1}} \varepsilon_{1,g,k-(2^{M+1}-2)} \widetilde{o}^*_{2,k} \quad \text{for } 1 \le g \le 2.
\end{aligned}$$

Again we set $\widehat{E}_{0,2} = \widehat{E}'_{0,2} = \sum_{m=1}^{M+1} \widehat{E}_{0,2,m}$ where $\widehat{E}_{0,2,m} = \operatorname{span}\{v_{2,m,g}\}_{g=2}^{2^{M-m+1}}$ for $1 \le m \le M$, while $\widehat{E}_{0,2,M+1} = \operatorname{span}\{v_{2,M+1,g}\}_{g=1}^{2}$. By (25), $\widehat{E}_2 = \widehat{E}'_2 = \operatorname{span}\{w_{m,2}\}_{m=1}^{M} + \widehat{E}'_{0,2}$.

Third step. Now we fix r with $3 \le r \le R$. Compared to the previous two constructions, there is a difference, since now the Hamel basis $(\widehat{o}_{r,n})_{n=1}^{2^{M+r-1}}$ of $\operatorname{span}(\widehat{o}_{r,n})_{n=1}^{2^{M+r-1}}$ has $2^{M+r-1} > 2^{M+1}$ elements; again we will define another Hamel basis

$$(v_{r,n})_{n=1}^{2^{M+r-1}} = ((v_{r,m,g})_{g=1}^{2^{M+r-1-m}})_{m=1}^{M} \cup (v_{r,M+1,g})_{g=1}^{2^{r-1}}$$

of $\operatorname{span}(\widehat{o}_{r,n})_{n=1}^{2^{M+r-1}}$ and we will use the expression of $\widehat{o}_{r,n}$ of (26.3), for each n with $1 \le n \le 2^{M+r-1}$. We start with $m = 1$; by (26.3) and (25), and by the definition of the Walsh matrix, for $1 \le g \le 2^{M+r-2}$ we set

$$\begin{aligned}
v_{r,1,g} &= \frac{1}{2^1}(\widehat{o}_{r,2(g-1)+1} + \widehat{o}_{r,2g})\\
&= \frac{1}{2^1}\Big(\sum_{j=1}^{2^{M+r}} \varepsilon_{M+r,2^{M+r-1}+2(g-1)+1,j} \widetilde{o}_{0,r,j} + \sum_{j=1}^{2^{M+r}} \varepsilon_{M+r,2^{M+r-1}+2g,j} \widetilde{o}_{0,r,j}\Big)\\
&= \sum_{j=1}^{2^{M+r}} \frac{1}{2^1}(\varepsilon_{M+r,2^{M+r-1}+2(g-1)+1,j} + \varepsilon_{M+r,2^{M+r-1}+2g,j}) \widetilde{o}_{0,r,j}\\
&= \sum_{j=1}^{2^{M+r-1}} \varepsilon_{M+r,2^{M+r-1}+2(g-1)+1,j} \widetilde{o}_{0,r,j} = \sum_{j=1}^{2^{M+r-1}} \varepsilon_{M+r-1,2^{M+r-2}+g,j} \widetilde{o}_{0,r,j}
\end{aligned}$$

(by the construction of the Walsh matrix, always taking into account that, for $1 \leq g \leq 2^{M+r-2}$ and $1 \leq j \leq 2^{M+r-1}$,

$$\varepsilon_{M+r,2^{M+r-1}+2(g-1)+1,j} = \varepsilon_{M+r-1,2^{M+r-2}+g,j}).$$

Continuing, we get $v_{r,m,g}$ for $2 \leq m \leq M+1$ and $1 \leq g \leq 2^{M+r-m-1}$ (see the fourth step for the precise expression) and in particular

$$\begin{aligned} v_{r,M,g} &= \sum_{j=2^{M+r-1}+\cdots+2^{r+1}+1}^{2^{M+r-1}+\cdots+2^{r}} \varepsilon_{r,2^{r-1}+g,j-(2^{M+r-1}+\cdots+2^{r+1})} \widetilde{o}_{0,r,j} \\ &= \sum_{j=2^{M+r}-2^{r+1}+1}^{2^{M+r}-2^{r}} \varepsilon_{r,2^{r-1}+g,j-(2^{M+r}-2^{r+1})} \widetilde{o}_{0,r,j} \quad \text{for } 1 \leq g \leq 2^{r-1}. \end{aligned}$$

Concerning the functionals, again we have the biorthogonal system

$$(v_{r,n}, v^*_{r,n})_{n=1}^{2^{M+r-1}} = ((v_{r,m,g}, v^*_{r,m,g})_{g=1}^{2^{M+r-1-m}})_{m=1}^{M} \cup (v_{r,M+1,g}, v^*_{r,M+1,g})_{g=1}^{2^{r-1}}$$

where, since by (26.1) the cardinality of $\operatorname{supp}(\widetilde{o}_{0,r,j})$ is 2^{R-r} for $1 \leq r \leq R$ and $1 \leq j \leq 2^{M+r}$,

$$v^*_{r,1,g} = \widehat{o}^*_{r,2(g-1)+1} + \widehat{o}^*_{r,2g} = \frac{1}{2^{M+R-1}} \sum_{j=1}^{2^{M+r-1}} \varepsilon_{M+r-1,2^{M+r-2}+g,j} \widetilde{o}^*_{0,r,j}$$

for $1 \leq g \leq 2^{M+r-2}$; analogously for $2 \leq m \leq M+1$ and $1 \leq g \leq 2^{M+r-m-1}$, we have $v^*_{r,m,g}$ (see the fourth step). On the other hand, if we consider only the first part $(\widehat{o}_{r,m})_{m=1}^{2^{M+1}}$ of $(\widehat{o}_{r,m})_{m=1}^{2^{M+r-1}}$, since by the end of (26.3), $\widehat{o}_{r,m} = \sum_{k=1}^{2^{M+1}} \varepsilon_{M+1,m,k} \widetilde{o}_{r,k}$ for $1 \leq m \leq 2^{M+1}$, we can repeat the same construction of $r = 1, 2$ and obtain another Hamel basis $(v'_{r,m})_{m=1}^{2^{M+1}}$ of $\operatorname{span}(\widehat{o}_{r,m})_{m=1}^{2^{M+1}}$ with

$$(v'_{r,n} v'^*_{r,n})_{n=1}^{2^{M+1}} = ((v'_{r,m,g}, v'^*_{r,m,g})_{g=1}^{2^{M-m+1}})_{m=1}^{M} \cup (v'_{r,M+1,g}, v'^*_{r,M+1,g})_{g=1}^{2^1}$$

biorthogonal. Precisely, for $1 \leq g \leq 2^M$,

$$v'_{r,1,g} = \sum_{k=1}^{2^M} \varepsilon_{M,g,k} \widetilde{o}_{r,k},\, v'^*_{r,1,g} = \frac{1}{2^{M+R-1}} \sum_{k=1}^{2^M} \varepsilon_{M,g,k} \widetilde{o}^*_{r,k}.$$

Continuing, we get, for $2 \leq m \leq M$ and $1 \leq g \leq 2^{M-m+1}$,

$$\begin{aligned} v'_{r,m,g} &= \sum_{k=2^M+\cdots+2^{M-m+2}+1}^{2^M+\cdots+2^{M-m+1}} \varepsilon_{M-m+1,g,k-(2^M+\cdots+2^{M-m+2})} \widetilde{o}_{r,k}, \\ v'^*_{r,m,g} &= \frac{1}{2^{M+R-m}} \sum_{k=2^M+\cdots+2^{M-m+2}+1}^{2^M+\cdots+2^{M-m+1}} \varepsilon_{M-m+1,g,k-(2^M+\cdots+2^{M-m+2})} \widetilde{o}^*_{r,k}, \\ v'_{r,M+1,g} &= \sum_{k=2^{M+1}-2+1}^{2^{M+1}} \varepsilon_{1,g,k-(2^{M+1}-2)} \widetilde{o}_{r,k}, \\ v'^*_{r,M+1,g} &= \frac{1}{2^R} \sum_{k=2^{M+1}-2+1}^{2^{M+1}} \varepsilon_{1,g,k-(2^{M+1}-2)} \widetilde{o}^*_{r,k} \quad \text{for } 1 \leq g \leq 2. \end{aligned}$$

At this point, if we compare with the fourth step, it is easy to see that

$$((v'_{r,m,g}, v'^*_{r,m,g})_{g=1}^{2^{M-m+1}})_{m=1}^{M} \cup (v'_{r,M+1,g}, v'^*_{r,M+1,g})_{g=1}^{2^1} = ((v_{r,m,g}, v^*_{r,m,g})_{g=1}^{2^{M-m+1}})_{m=1}^{M} \cup (v_{r,M+1,g}, v^*_{r,M+1,g})_{g=1}^{2^1}.$$

Indeed, for $2 \le m \le M$ and $1 \le g \le 2^{M-m+1}$, and for $1 \le g \le 2^1$ if $m = M+1$, if we use $\{\widetilde{o}_{r,m}\}_{m=1}^{2^{M+1}}$ instead of $\{\widetilde{o}_{0,r,j}\}_{j=1}^{2^{M+r}}$ in the construction of $v_{r,m,g}$, we have exactly the definition of $v'_{r,m,g}$. Now we set $w_{m,r} = v_{r,m,1}$ and $w^*_{m,r} = v^*_{r,m,1} = v'^*_{r,m,1}$ for $1 \le m \le M$ and $1 \le r \le R$.

We can also replace $((w_{m,r}, w^*_{m,r})_{r=1}^{R})_{m=1}^{M}$ by the biorthogonal system

$$((u_{m,r}, u^*_{m,r})_{r=1}^{R})_{m=1}^{M},$$

where, for $1 \le r \le R$,

$$u_{m,r} = \sum_{s=1}^{m} w_{s,r} = w_{1,r} + \sum_{s=2}^{m} w_{s,r} = \sum_{k=1}^{2^M} \widetilde{o}_{r,k} + \sum_{s=2}^{m} \sum_{k=2^M+\dots+2^{M-s+2}+1}^{2^M+\dots+2^{M-s+1}} \widetilde{o}_{r,k} = \sum_{k=1}^{2^M+\dots+2^{M-m+1}} \widetilde{o}_{r,k}$$

for $1 \le m \le M$. It is easy to check that

$$u^*_{M,r} = w^*_{M,r} = \frac{1}{2^R} \sum_{k=2^M+\dots+2^2+1}^{2^M+\dots+2^1} \widetilde{o}^*_{r,k},$$

$$u^*_{m,r} = w^*_{m,r} - w^*_{m+1,r} = \frac{1}{2^{M+R-m}} \cdot \sum_{k=2^M+\dots+2^{M-m+2}+1}^{2^M+\dots+2^{M-m+1}} \widetilde{o}^*_{r,k} - \frac{1}{2^{M+R-m-1}} \sum_{k=2^M+\dots+2^{M-m+1}+1}^{2^M+\dots+2^{M-m}} \widetilde{o}^*_{r,k}$$

for $2 \le m \le M-1$, $u^*_{1,r} = w^*_{1,r} - w^*_{2,r}$. It is easy to see that $w_{1,r} = u_{1,r}$ while $w_{m,r} = u_{m,r} - u_{m-1,r}$ for $2 \le m \le M$.

Fourth step. The next formula summarizes the definitions of the main previous sequences:

(27) for each r with $1 \le r \le R$ we have

$$w_{1,r} = \sum_{k=1}^{2^M} \varepsilon_{M,1,k} \widetilde{o}_{r,k} = \sum_{j=1}^{2^{M+r-1}} \varepsilon_{M+r-1,2^{M+r-2}+1,j} \widetilde{o}_{0,r,j}$$

with

$$w^*_{1,r} = \frac{1}{2^{M+R-1}} \sum_{j=1}^{2^{M+r-1}} \varepsilon_{M+r-1,2^{M+r-2}+1,j} \widetilde{o}^*_{0,r,j} = \frac{1}{2^{M+R-1}} \sum_{k=1}^{2^M} \widetilde{o}^*_{r,k},$$

moreover, for $2 \le g \le 2^{M+r-2}$ (2^M if $r = 1$),

$$v_{r,1,g} = \sum_{j=1}^{2^{M+r-1}} \varepsilon_{M+r-1,2^{M+r-2}+g,j} \widetilde{o}_{0,r,j}$$

($= \sum_{j=1}^{2^M} \varepsilon_{M,g,j}\widetilde{o}_{0,1,j}$ if $r = 1$, and $= \sum_{k=1}^{2^M} \varepsilon_{M,g,k}\widetilde{o}_{r,k}$ for $2 \le g \le 2^M$) with

$$v^*_{r,1,g} = \frac{\sum_{j=1}^{2^{M+r-1}} \varepsilon_{M+r-1,2^{M+r-2}+g,j}\widetilde{o}^*_{0,r,j}}{2^{M+R-1}}$$

($= \sum_{j=1}^{2^M} \varepsilon_{M,g,j}\widetilde{o}^*_{0,1,j}/2^{M+R-1}$ if $r = 1$, and $= \sum_{k=1}^{2^M} \varepsilon_{M,g,k}\widetilde{o}^*_{r,k}/2^{M+R-1}$ for $2 \le g \le 2^M$); while, for $2 \le m \le M$,

$$\begin{aligned} w_{m,r} &= \sum_{k=2^M+\cdots+2^{M-m+2}+1}^{2^M+\cdots+2^{M-m+1}} \widetilde{o}_{r,k} \\ &= \sum_{j=2^{M+r-1}+\cdots+2^{M+r-m+1}+1}^{2^{M+r-1}+\cdots+2^{M+r-m}} \varepsilon_{M+r-m,2^{M+r-m-1}+1,j-(2^{M+r-1}+\cdots+2^{M+r-m+1})}\widetilde{o}_{0,r,j} \end{aligned}$$

with

$$\begin{aligned} w^*_{m,r} &= \sum_{j=2^{M+r-1}+\cdots+2^{M+r-m+1}+1}^{2^{M+r-1}+\cdots+2^{M+r-m}} \frac{\varepsilon_{M+r-m,2^{M+r-m-1}+1,j-(2^{M+r-1}+\cdots+2^{M+r-m+1})}\widetilde{o}^*_{0,r,j}}{2^{M+R-m}} \\ &= \frac{1}{2^{M+R-m}} \sum_{k=2^M+\cdots+2^{M-m+2}+1}^{2^M+\cdots+2^{M-m+1}} \widetilde{o}^*_{r,k}, \end{aligned}$$

moreover, for $2 \le g \le 2^{M+r-m-1}$ (2^{M+1-m} if $r = 1$),

$$\begin{aligned} &v_{r,m,g} \\ &\quad= \sum_{j=2^{M+r-1}+\cdots+2^{M+r-m+1}+1}^{2^{M+r-1}+\cdots+2^{M+r-m}} \varepsilon_{M+r-m,2^{M+r-m-1}+g,j-(2^{M+r-1}+\cdots+2^{M+r-m+1})}\widetilde{o}_{0,r,j} \end{aligned}$$

(also $= \sum_{k=2^M+\cdots+2^{M-m+2}+1}^{2^M+\cdots+2^{M-m+1}} \varepsilon_{M-m+1,g,k-(2^M+\cdots+2^{M-m+2})}\widetilde{o}_{r,k}$ for $2 \le g \le 2^{M-m+1}$) with

$$\begin{aligned} &v^*_{r,m,g} \\ &\quad= \frac{\sum_{j=2^{M+r-1}+\cdots+2^{M+r-m+1}+1}^{2^{M+r-1}+\cdots+2^{M+r-m}} \varepsilon_{M+r-m,2^{M+r-m-1}+g,j-(2^{M+r-1}+\cdots+2^{M+r-m+1})}\widetilde{o}^*_{0,r,j}}{2^{M+R-m}} \end{aligned}$$

($= \frac{1}{2^{M+R-m}} \sum_{k=2^M+\cdots+2^{M-m+2}+2}^{2^M+\cdots+2^{M-m+1}} \varepsilon_{M-m+1,g,k-(2^M+\cdots+2^{M-m+2})}\widetilde{o}^*_{r,k}$ for $2 \le g \le 2^{M-m+1}$); finally

$$v_{r,M+1,g} = \sum_{j=2^{M+r}-2^r+1}^{2^{M+r}} \varepsilon_{r,2^{r-1}+g,j-(2^{M+r}-2^r)}\widetilde{o}_{0,r,j}$$

with

$$v^*_{r,M+1,g} = \frac{1}{2^R} \sum_{j=2^{M+r}-2^r+1}^{2^{M+r}} \varepsilon_{r,2^{r-1}+g,j-(2^{M+r}-2^r)}\widetilde{o}^*_{0,r,j}$$

for $1 \le g \le 2^{r-1}$ (2 if $r = 1$) (also $v_{r,M+1,g} = \sum_{k=2^{M+1}-2+1}^{2^{M+1}} \varepsilon_{1,g,k-(2^{M+1}-2)}\widetilde{o}_{r,k}$ for $1 \le g \le 2$).

FIFTH STEP. The next formula sheds more light on the preceding steps (see (27)):

(28.1) $((w_{m,r},w^*_{m,r})_{r=1}^R)_{m=1}^M \cup ((v_{1,m,g},v^*_{1,m,g})_{g=2}^{2^{M+1-m}})_{m=1}^M \cup (v_{1,M+1,g},v^*_{1,M+1,g})_{g=1}^{2^1} \cup (((v_{r,m,g},v^*_{r,m,g})_{g=2}^{2^{M+r-1-m}})_{m=1}^M \cup (v_{r,M+1,g},v^*_{r,M+1,g})_{g=1}^{2^{r-1}})_{r=2}^R$ is biorthogonal with $\mathrm{span}(w_{m,1}\cup(v_{1,m,g})_{g=2}^{2^{M+1-m}})_{m=1}^M=\widehat{E}_1$ and $\mathrm{span}(w_{m,r}\cup(v_{r,m,g})_{g=2}^{2^{M+r-1-m}})_{m=1}^M=\widehat{E}_r$ for $2\le r\le R$;

(28.2) we set $\widehat{E}_0=\sum_{r=1}^R\widehat{E}_{0,r}=\sum_{m=1}^{M+1}\widetilde{E}_{0,m}$, $\widehat{E}_{0,r}=\sum_{m=1}^{M+1}\widehat{E}_{0,r,m}$ for $1\le r\le R$ and, for $1\le m\le M+1$, $\widetilde{E}_{0,m}=\sum_{r=1}^R\widehat{E}_{0,r,m}$ where $\widehat{E}_{0,1,m}=\mathrm{span}\{v_{1,m,g}\}_{g=2}^{2^{M+1-m}}$ for $1\le m\le M$ and $\widehat{E}_{0,1,M+1}=\mathrm{span}\{v_{1,M+1,g}\}_{g=2}^{2^1}$, while, for $2\le r\le R$, $\widehat{E}_{0,r,m}=\mathrm{span}\{v_{r,m,g}\}_{g=2}^{2^{M+r-1-m}}$ for $1\le m\le M$ and $\widehat{E}_{0,r,M+1}=\mathrm{span}\{v_{r,M+1,g}\}_{g=1}^{2^{r-1}}$; hence, for $1\le r\le R$, $\widehat{E}_r=\widehat{E}_{0,r}+W_r$ with $W_r=\mathrm{span}\{w_{m,r}\}_{m=1}^M$; moreover $\widehat{E}=\widehat{E}_0+W$ with $W=\sum_{r=1}^R W_r$, $\widehat{E}_0=\widehat{E}\cap\{\cap_{r=1}^R\cap_{m=1}^M w^*_{(m,r)\perp}\}$; $\widehat{E}=\widehat{E}'+\widehat{E}''$ with

$$\begin{aligned}\widehat{E}'&=W+\sum_{r=1}^R\Big(\sum_{m=1}^M\mathrm{span}\{v_{r,m,g}\}_{g=2}^{2^{M+1-m}}+\mathrm{span}\{v_{r,M+1,g}\}_{g=1}^{2^1}\Big)\\&=\sum_{r=1}^R\mathrm{span}\{\widehat{o}_{r,m}\}_{m=1}^{2^{M+1}}\end{aligned}$$

and

$$\begin{aligned}\widehat{E}''&=\sum_{r=3}^R\mathrm{span}\{\widehat{o}_{r,m}\}_{m=2^{M+1}+1}^{2^{M+r-1}}\\&=\sum_{r=3}^R\Big(\sum_{m=1}^M\mathrm{span}\{v_{r,m,g}\}_{g=2^{M+1-m}+1}^{2^{M+r-1-m}}+\mathrm{span}\{v_{r,M+1,g}\}_{g=2^1+1}^{2^{r-1}}\Big);\end{aligned}$$

then

$$\widehat{E}_0=\sum_{r=1}^R\Big(\sum_{m=1}^M\mathrm{span}\{v_{r,m,g}\}_{g=2}^{2^{M+1-m}}+\mathrm{span}\{v_{r,M+1,g}\}_{g=1}^{2^1}\Big)+\widehat{E}'';$$

(28.3) $((u_{m,r},u^*_{m,r})_{r=1}^R)_{m=1}^M$ is biorthogonal with

$$u_{m,r}=\sum_{s=1}^m w_{s,r}=\sum_{k=1}^{2^M+\cdots+2^{M-m+1}}\widetilde{o}_{r,k}\quad\text{for } 1\le m\le M \text{ and } 1\le r\le R;$$

$$u^*_{M,r}=w^*_{M,r}=\frac{1}{2^R}\sum_{k=2^M+\cdots+2^2+1}^{2^M+\cdots+2^1}\widetilde{o}^*_{r,k}\quad\text{for } 1\le r\le R,$$

for $2\le m\le M-1$,

$$u^*_{m,r}=w^*_{m,r}-w^*_{m+1,r}=\frac{\sum_{k=2^M+\cdots+2^{M-m+2}+1}^{2^M+\cdots+2^{M-m+1}}\widetilde{o}^*_{r,k}-2\sum_{k=2^M+\cdots+2^{M-m+1}+1}^{2^M+\cdots+2^{M-m}}\widetilde{o}^*_{r,k}}{2^{M+R-m}}$$

and

$$u^*_{1,r}=w^*_{1,r}-w^*_{2,r}=\frac{\sum_{k=1}^{2^M}\widetilde{o}^*_{r,k}-2\sum_{k=2^M+1}^{2^M+2^{M-1}}\widetilde{o}^*_{r,k}}{2^{M+R-1}};$$

$w_{1,r}=u_{1,r}$ while $w_{m,r}=u_{m,r}-u_{m-1,r}$ for $2\le m\le M$;

(28.4) for each fixed r with $1 \le r \le R$, $\operatorname{supp}(v_{r,1,g}) = \{t\}_{t=1}^{2^{M+R-1}}$ for $1 \le g \le 2^{M+r-2}$ ($1 \le g \le 2^M$ if $r = 1$), while for $2 \le m \le M$,

$$\operatorname{supp}(v_{r,m,g}) = \{t\}_{t=2^{M+R-1}+\dots+2^{M+R-m+1}+1}^{2^{M+R-1}+\dots+2^{M+R-m}}$$

for $1 \le g \le 2^{M+r-1-m}$ ($1 \le g \le 2^{M+1-m}$ if $r = 1$); therefore it follows that, for any m, m', m'', r, r', r'' with $1 \le m, m' \ne m'' \le M$ and $1 \le r, r' \ne r'' \le R$, $\operatorname{supp}(w_{m',r}) \cap \operatorname{supp}(w_{m'',r}) = \emptyset$ while $\operatorname{supp}(w_{m,r'}) = \operatorname{supp}(w_{m,r''})$ with cardinality $= 2^{M+R-m}$.

For (28.4) we recall that for $2 \le m \le M$, by (26.1) and (27), $\operatorname{supp}(v_{r,1,g}) = \{t\}_{t=1}^{2^{M+r-1}2^{R-r}} = \{t\}_{t=1}^{2^{M+R-1}}$, while, for $2 \le m \le M$,

$$\operatorname{supp}(v_{r,1,g}) = \{t\}_{t=1}^{2^{M+r-1}2^{R-r}} = \{t\}_{t=1}^{2^{M+R-1}},$$

$$\operatorname{supp}(v_{r,m,g}) = \{t\}_{t=(2^{M+r-1}+\dots+2^{M+r-m+1})2^{R-r}+1}^{(2^{M+r-1}+\dots+2^{M+r-m})2^{R-r}} = (t)_{t=2^{M+R-1}+\dots+2^{M+R-m+1}+1}^{2^{M+R-1}+\dots+2^{M+R-m}}.$$

Sixth step. By the previous formula for each sequence $((a_{m,r})_{r=1}^R)_{m=1}^M$ of numbers we infer that (see the beginning of the second and third steps)

(29) if $(\widehat{o}_{0,1,n})_{n=1}^{2^{M+R-1}}$ is the sequence derived from $(o_n)_{n=1}^{2^{M+R-1}}$ by means of the Walsh matrix and $(\widehat{o}_{0,m,n})_{n=1}^{2^{M+R-m}}$ the sequence derived from $(o_{0,m,n})_{n=1}^{2^{M+R-m}}$ for $2 \le m \le M$, then

$$(v_{1,1,g})_{g=1}^{2^M} \cup ((v_{s,1,g})_{g=1}^{2^{M+s-2}})_{s=2}^R = (\widehat{o}_{0,1,n})_{n=1}^{2^{M+R-1}}$$

and

$$(v_{1,m,g})_{g=1}^{2^{M+1-m}} \cup ((v_{s,m,g})_{g=1}^{2^{M+s-m-1}})_{s=2}^R = (\widehat{o}_{0,m,n})_{n=1}^{2^{M+R-m}}$$

for $2 \le m \le M$.

Indeed, it is sufficient to consider only the case of $m = 1$, since for the other cases the reasoning is exactly the same. Moreover in what follows we use (26.1) and (26.2) and (28.4); we start from the expressions of $(\widehat{o}_n)_{n=1}^{2^{M+R}}$ of (25), of $(v_{1,n})_{n=1}^{2^{M+1}}$ of the second step and, for $2 \le r \le R$, of the $(v_{r,n})_{n=1}^{2^{M+r-1}}$ of the second and third steps, where $(v_{r,n})_{n=1}^{2^{M+r-1}}$ is a Hamel basis of $\operatorname{span}(\widehat{o}_{r,m})_{m=1}^{2^{M+r-1}} = \operatorname{span}(\widehat{o}_n)_{n=2^{M+r-1}+1}^{2^{M+r}}$. Therefore now (for $m = 1$) we consider only $(v_{1,1,g})_{g=1}^{2^M} \cup ((v_{r,1,g})_{g=1}^{2^{M+r-2}})_{r=2}^R$ and we know that these sequences concern only the first part $(o_n)_{n=1}^{2^{M+R-1}}$ of $(o_n)_{n=1}^{2^{M+R}}$. Moreover we denote by $(\widehat{o}_{0,1,n})_{n=1}^{2^{M+R-1}}$ the sequence derived from $(o_n)_{n=1}^{2^{M+R-1}}$ by means of the Walsh matrix. Then we know by (27) and (26.1) that, for $1 \le g \le 2^M$,

$$v_{1,1,g} = \sum_{k=1}^{2^M} \varepsilon_{M,g,k}\widetilde{o}_{1,k} = \sum_{j=1}^{2^M} \varepsilon_{M,g,j}\widetilde{o}_{0,1,j} = \sum_{j=1}^{2^M} \varepsilon_{M,g,j} \sum_{t=(j-1)2^{R-1}+1}^{j.2^{R-1}} o_t,$$

$$v_{2,1,g} = \sum_{k=1}^{2^M} \varepsilon_{M,g,k}\widetilde{o}_{2,k} = \sum_{j=1}^{2^{M+1}} \varepsilon_{M+1,2^M+g,j}\widetilde{o}_{0,2,j} = \sum_{j=1}^{2^{M+1}} \varepsilon_{M+1,2^M+g,j} \sum_{t=(j-1)2^{R-2}+1}^{j.2^{R-2}} o_t;$$

hence also

$$\begin{aligned} v_{1,1,g} &= \sum_{j=1}^{2^M} \varepsilon_{M,g,j} \sum_{t=(j-1)2^{R-1}+1}^{j.2^{R-1}} o_t \\ &= \sum_{j=1}^{2^M} \varepsilon_{M,g,j} \Big(\sum_{t=((2(j-1)+1)-1)2^{R-2}+1}^{(2(j-1)+1)2^{R-2}} o_t + \sum_{t=(2j-1)2^{R-2}+1}^{2j.2^{R-2}} o_t \Big) \\ &= \sum_{j=1}^{2^M} \varepsilon_{M,g,j} (\widetilde{o}_{0,2,2j-1} + \widetilde{o}_{0,2,2j}) \end{aligned}$$

while (see the proof of (26.2)) $v_{2,1,g} = \sum_{j=1}^{2^M} \varepsilon_{M,g,j}(\widetilde{o}_{0,2,2j-1} - \widetilde{o}_{0,2,2j})$. Hence it is easy to see that

$$(v_{1,1,g})_{g=1}^{2^M} \cup (v_{2,1,g})_{g=1}^{2^M} = \Big(\sum_{j=1}^{2^{M+1}} \varepsilon_{M+1,g,j} \widetilde{o}_{0,2,j} \Big)_{g=1}^{2^{M+1}} = (\widehat{o}_{0,1,n})_{n=1}^{2^{M+1}}.$$

Now fix r with $3 \le r \le R$ and suppose we have already verified that

$$(v_{1,1,g})_{g=1}^{2^M} \cup ((v_{s,1,g})_{g=1}^{2^{M+s-2}})_{s=2}^{r-1} = (\widehat{o}_{0,1,n})_{n=1}^{2^{M+r-2}}.$$

By (26.1) for $r' = r-1$ and $r'' = r$, since $2^{M+R-1}/2^{M+r-2} = 2^{R-(r-1)}$, for $1 \le g \le 2^{M+r-2}$ we have

$$\begin{aligned} \widehat{o}_{0,1,g} &= \sum_{j=1}^{2^{M+r-2}} \varepsilon_{M+r-2,g,j} \widetilde{o}_{0,r-1,j} = \sum_{j=1}^{2^{M+r-2}} \varepsilon_{M+r-2,g,j} \sum_{l=1}^{2^1} \widetilde{o}_{0,r,(j-1)2^1+l} \\ &= \sum_{j=1}^{2^{M+r-2}} \varepsilon_{M+r-2,g,j} (\widetilde{o}_{0,r,(j-1)2^1+1} + \widetilde{o}_{0,r,j2^1}), \end{aligned}$$

while, by the same procedure, we know (by (27)) that

$$v_{r,1,g} = \sum_{j=1}^{2^{M+r-1}} \varepsilon_{M+r-1,2^{M+r-2}+g,j} \widetilde{o}_{0,r,j} = \sum_{j=1}^{2^{M+r-2}} \varepsilon_{M+r-2,g,j} (\widetilde{o}_{0,r,(j-1)2^1+1} - \widetilde{o}_{0,r,j2^1}).$$

Hence, proceeding as above we get

$$(v_{1,1,g})_{g=1}^{2^M} \cup ((v_{s,1,g})_{g=1}^{2^{M+s-1}})_{s=2}^{r} = (\widehat{o}_{0,1,n})_{n=1}^{2^{M+r-1}}$$

and so on. This last step completes Construction I.

THEOREM 8 (Special sequences in spaces of infinite cotype). *Let* $((w_{m,r}, w^*_{m,r})_{r=1}^R)_{m=1}^M$, $((u_{m,r}, u^*_{m,r})_{r=1}^R)_{m=1}^M$ *and* $\widehat{E}_0$ *be the biorthogonal systems and the subspace* (*of* $\widehat{E} = l_\infty^{2^{M+R}}$) *of Construction I* (see in particular (28)). *Let* $((a_{m,r})_{r=1}^R)_{m=1}^M$ *be a sequence of numbers and set* $R = 2^{\widetilde{R}}$ *and* $M = 2^{\widetilde{M}}$. *Then the following properties hold*:

(30.1) *for each fixed* r *with* $1 \le r \le M$, *we have* $u_{m,r} = \sum_{s=1}^m w_{s,r}$ *for* $1 \le m \le M$, $u^*_{M,r} = w^*_{M,r}$ *while* $u^*_{m,r} = w^*_{m,r} - w^*_{m+1,r}$ *for* $1 \le m \le M-1$, $w_{1,r} = u_{1,r}$ *while* $w_{m,r} = u_{m,r} - u_{m-1,r}$ *for* $2 \le m \le M$;

(30.2) $\|w_{m,r}\| = \|w^*_{m,r}\| = \|u_{m,r}\| = 1$ *for* $1 \le r \le R$ *and* $1 \le m \le M$ (*in particular*, *for each fixed* r *with* $1 \le r \le R$, $\|\sum_{s=1}^m w_{s,r}\| = 1$ *for* $1 \le m \le M$, $\|u^*_{M,r}\| = 1$ *and* $\|u^*_{m,r}\| = 2$ *for* $1 \le m \le M-1$);

(30.3) $$\Big\|\sum_{m=1}^{M}\sum_{r=1}^{R}a_{m,r}w_{m,r}\Big\| = \max\Big(\Big\|\sum_{r=1}^{R}a_{m,r}w_{m,r}\Big\| : 1\le m\le M\Big),$$

$$\Big\|\sum_{m=1}^{M}\sum_{r=1}^{R}a_{m,r}w_{m,r}+\widehat{E}_0\Big\| = \max\Big(\Big\|\sum_{r=1}^{R}a_{m,r}w_{m,r}+\widehat{E}_0\Big\| : 1\le m\le M\Big) = \max\Big(\Big\|\sum_{r=1}^{R}a_{m,r}w_{m,r}+\widehat{E}_{0,m}\Big\| : 1\le m\le M\Big);$$

(30.4) $((w_{m,r}+\widehat{E}_0)_{r=1}^R)_{m=1}^M$ *is* 1*-unconditional and in particular* $(w_{m,r}+\widehat{E}_0)_{r=1}^R$ *is indiscernible for* $1\le m\le M$, *moreover*

$$\widehat{E}_0=\widehat{E}\cap\bigcap_{r=1}^{R}\bigcap_{m=1}^{M}w^*_{(m,r)\perp}=\widehat{E}\cap\bigcap_{r=1}^{R}\bigcap_{m=1}^{M}u^*_{(m,r)\perp};$$

(30.5) *for each* $\varepsilon>0$, *if* $\widetilde{R}\ge\widetilde{R}(\widetilde{M},\varepsilon)=2\log_2(2^{2\widetilde{M}}\sqrt{2^{2^{\widetilde{M}}+1}}/\varepsilon)$, *then for each* $\overline{x}\in E$ *with* $\|\overline{x}\|=1$ *there exists an index* $\overline{r}$ *with* $1\le\overline{r}\le 2^{\widetilde{R}}$ *such that* $\sum_{m=1}^{2^{\widetilde{M}}}|w^*_{m,\overline{r}}(\overline{x})|<\varepsilon$.

Proof. (30.1) comes from (28.1) and (28.3).

(30.2) follows from (30.1), (26.4), (28.1) and (28.3) since $\{o_n^*\}_{n=1}^{2^{M+R}}$ is the natural basis of $E=l_1^{2^{M+R}}$; hence, for $1\le r\le R$, $\|\widetilde{o}^*_{0,r,j}\|=2^{R-r}$ for $1\le j\le 2^{M+r}$, and $\|\widetilde{o}^*_{r,k}\|=2^{R-1}$ for $1\le k\le 2^{M+1}$, in particular by (28.3) it also follows that, for $1\le r\le R$ and $1\le m\le M-1$,

$$\|u^*_{m,r}\|=\frac{2^{M-m+1}+2.2^{M-m}}{2^{M+R-m}}2^{R-1}=2.$$

(30.3) comes from (27), (28.2) and (28.4).

(30.4) comes from (30.3), from the end of (28.2), and from the fact that, by Proposition 7(ii) and by (29), for $1\le m\le M$, $(w_{m,r}+\widehat{E}_0)_{r=1}^R=(w_{m,r}+\widetilde{E}_{0,m})_{r=1}^R$ is indiscernible and 1-unconditional.

We now prove (30.5). By (28.2) and by (23.5) of Remark 6, for each sequence $((b_{r,m})_{m=1}^{2^{2^{\widetilde{M}}+1}})_{r=1}^{2^{\widetilde{R}}}$ of numbers,

$$\Big\|\sum_{r=1}^{2^{\widetilde{R}}}\sum_{m=1}^{2^{2^{\widetilde{M}}+1}}b_{r,m}\widehat{o}_{r,m}+\widehat{E}''\Big\|\ge\Big(\sum_{r=1}^{2^{\widetilde{R}}}\sum_{m=1}^{2^{2^{\widetilde{M}}+1}}|b_{r,m}|\Big)/\sqrt{2^{\widetilde{R}}2^{2^{\widetilde{M}}+1}}.$$

Hence in our case, if $\overline{x}\in\widehat{E}$ with $\|\overline{x}\|=1$ we have

$$1=\|\overline{x}\|\ge\|\overline{x}+\widehat{E}''\|=\Big\|\sum_{r=1}^{2^{\widetilde{R}}}\sum_{m=1}^{2^{2^{\widetilde{M}}+1}}\widehat{o}^*_{r,m}(\overline{x})\widehat{o}_{r,m}+\widehat{E}''\Big\|$$
$$\ge\Big(\sum_{r=1}^{2^{\widetilde{R}}}\sum_{m=1}^{2^{2^{\widetilde{M}}+1}}|\widehat{o}^*_{r,m}(\overline{x})|\Big)/\sqrt{2^{\widetilde{R}}2^{2^{\widetilde{M}}+1}}.$$

Thus there is $\overline{r}$ with $1 \le \overline{r} \le 2^{\widetilde{R}}$ so that

$$\sum_{m=1}^{2^{2^{\widetilde{M}}+1}} |\widehat{o}^*_{\overline{r},m}(\overline{x})| \le \frac{\sum_{r=1}^{2^{\widetilde{R}}} \sum_{m=1}^{2^{2^{\widetilde{M}}+1}} |\widehat{o}^*_{r,m}(\overline{x})|}{2^{\widetilde{R}}} = \frac{\sqrt{2^{2^{\widetilde{M}}+1}}}{\sqrt{2^{\widetilde{R}}}} \frac{\sum_{r=1}^{2^{\widetilde{R}}} \sum_{m=1}^{2^{2^{\widetilde{M}}+1}} |\widehat{o}^*_{r,m}(\overline{x})|}{\sqrt{2^{\widetilde{R}} 2^{2^{\widetilde{M}}+1}}}.$$

Therefore

$$\sum_{m=1}^{2^{2^{\widetilde{M}}+1}} |\widehat{o}^*_{\overline{r},m}(\overline{x})| \le \sqrt{2^{2^{\widetilde{M}}+1}}/\sqrt{2^{\widetilde{R}}}. \tag{31}$$

Hence, since by (30.2), $\|w^*_{m,\overline{r}}\| = 1$ for $1 \le m \le 2^{\widetilde{M}}$ and moreover

$$\widehat{E}_0 = \widehat{E} \cap \bigcap_{r=1}^{R} \bigcap_{m=1}^{M} w^*_{(m,r)\perp}$$

by (30.4), moreover by the last relations of (28.2) and (22), finally by (31), we have (we recall that

$$\mathrm{span}(w_{m,\overline{r}})_{m=1}^{2^{\widetilde{M}}} + \mathrm{span}(((v_{\overline{r},m,g})_{g=2}^{2^{2^{\widetilde{M}}+1-m}})_{m=1}^{2^{\widetilde{M}}} \cup (v_{\overline{r},2^{\widetilde{M}}+1,g})_{g=1}^{2^1}) = \mathrm{span}(\widehat{o}_{\overline{r},m})_{m=1}^{2^{2^{\widetilde{M}}+1}}$$

by the second part of the third step)

$$\begin{aligned}
\sum_{m=1}^{2^{\widetilde{M}}} |w^*_{m,\overline{r}}(\overline{x})| &\le 2^{\widetilde{M}} \max(|w^*_{m,\overline{r}}(\overline{x})| : 1 \le m \le 2^{\widetilde{M}}) \\
&\le 2^{\widetilde{M}} \Big\| \sum_{m=1}^{2^{\widetilde{M}}} w^*_{m,\overline{r}}(\overline{x}) w_{m,\overline{r}} + \sum_{r(\ne\overline{r})=1}^{R} \mathrm{span}(w_{m,r})_{m=1}^{2^{2^{\widetilde{M}}}} + \widehat{E}_0 \Big\| \\
&\le 2^{\widetilde{M}} \Big\| \sum_{m=1}^{2^{\widetilde{M}}} w^*_{m,\overline{r}}(\overline{x}) w_{m,\overline{r}} + \sum_{m=1}^{2^{\widetilde{M}}} \sum_{g=2}^{2^{2^{\widetilde{M}}+1-m}} v^*_{\overline{r},m,g}(\overline{x}) v_{\overline{r},m,g} \\
&\quad + \sum_{g=1}^{2^1} v^*_{\overline{r},2^{\widetilde{M}}+1,g}(\overline{x}) v_{\overline{r},2^{\widetilde{M}}+1,g} \\
&\quad + \sum_{r(\ne\overline{r})=1}^{R} \Big(\sum_{m=1}^{2^{\widetilde{M}}} \mathrm{span}\{v_{r,m,g}\}_{g=1}^{2^{M+1-m}} + \mathrm{span}\{v_{r,M+1,g}\}_{g=1}^{2^1} \Big) + \widehat{E}'' \Big\| \\
&= 2^{\widetilde{M}} \Big\| \sum_{m=1}^{2^{2^{\widetilde{M}}+1}} \widehat{o}^*_{\overline{r},m}(\overline{x}) \widehat{o}_{\overline{r},m} \\
&\quad + \sum_{r(\ne\overline{r})=1}^{R} \Big(\sum_{m=1}^{2^{\widetilde{M}}} \mathrm{span}\{v_{r,m,g}\}_{g=2}^{2^{M+1-m}} + \mathrm{span}\{v_{r,M+1,g}\}_{g=1}^{2^1} \Big) + \widehat{E}'' \Big\| \\
&\le 2^{\widetilde{M}} \Big\| \sum_{m=1}^{2^{2^{\widetilde{M}}+1}} \widehat{o}^*_{\overline{r},m}(\overline{x}) \widehat{o}_{\overline{r},m} + \widehat{E}'' \Big\| \le 2^{\widetilde{M}} \sum_{m=1}^{2^{2^{\widetilde{M}}+1}} |\widehat{o}^*_{\overline{r},m}(\overline{x})| \le 2^{\widetilde{M}} \frac{\sqrt{2^{2^{\widetilde{M}}+1}}}{\sqrt{2^{\widetilde{R}}}}.
\end{aligned}$$

Hence, in order to have (30.5), we can set $R \geq R(N,\varepsilon)$ so that

$$2^{\widetilde{M}}\sqrt{2^{2^{\widetilde{M}}+1}}/\sqrt{2^{R(N,\varepsilon)}} = \varepsilon, \quad \text{then} \quad R(N,\varepsilon) = 2\log_2(2^{\widetilde{M}}\sqrt{2^{2^{\widetilde{M}}+1}}/\varepsilon).$$

This completes the proof of Theorem 8. ∎

2.3. Finite transformability and the Walsh matrix in l_1^n. The next lemma concerns the following question. If $\{e'_n\}_{n=1}^{2^Q}$ is the natural basis of $l_1^{2^Q}$, if $e_1 = e'_1$ and $e_n = e'_n - e'_{n-1}$ for $2 \leq n \leq 2^Q$, moreover if w is another element with $\|w+e\| \geq \max(\|w\|, \|e\|/2)$ for each $e \in \operatorname{span}\{e'_n\}_{n=1}^{2^Q}$, finally if $\{a_n\}_{n=1}^{2^Q}$ is a sequence of numbers, does there exist a permutation $(\pi(n))_{n=1}^{2^Q}$ of $\{n\}_{n=1}^{2^Q}$ such that $(\|\sum_{n=1}^{q} a_{\pi(n)}(e_{\pi(n)} + w)\|)_{q=1}^{2^Q}$ becomes $(1,0)$-monotone?

Lemma 9 (Particular sequences in l_1^n). *Let $\{e'_n\}_{n=1}^{2^Q}$ be the natural basis of $l_1^{2^Q}$ and set $e_1 = e'_1$ and $e_n = e'_n - e'_{n-1}$ for $2 \leq n \leq 2^Q$. Then for each sequence $\{a_n\}_{n=1}^{2^Q}$ of numbers there exists a permutation $\{\pi(n)\}_{n=1}^{2^Q}$ of $\{n\}_{n=1}^{2^Q}$ such that:*

(32.1) $(|\sum_{n=1}^{q} a_{\pi(n)}|)_{q=1}^{2^Q}$ *is $(1,0)$-monotone;*

(32.2) $(\|\sum_{n=1}^{q} a_{\pi(n)}e_{\pi(n)}\|)_{q=1}^{2^Q}$ *is $(0,0)$-monotone.*

Proof. We point out that

$$\Big\|\sum_{n=1}^{2^Q} a_n e_n\Big\| = \sum_{n=1}^{2^Q-1} |a_n - a_{n+1}| + |a_{2^Q}| \tag{33.1}$$

(hence $(\|\sum_{n=1}^{q} a_n e_n\|)_{q=1}^{2^Q}$ is $(0,0)$-monotone). We set

$$A = \sum_{n=1}^{2^Q} a_n \quad \text{and} \quad a = \max\{|a_n| : 1 \leq n \leq 2^Q\}. \tag{33.2}$$

We can suppose $A \geq 0$ (if $A < 0$ the procedure does not change). We set $\pi(1) = 1$ and we proceed by induction, that is, we fix a positive integer m with $1 \leq m \leq 2^Q - 1$ and we suppose that

(34.1) $\{n\}_{n=1}^{2^Q} = \{s(n)\}_{n=1}^{m} \cup \{r(n)\}_{n=1}^{2^Q-m}$ and $\{\pi(n)\}_{n=1}^{m}$ is a permutation of $\{s(n)\}_{n=1}^{m}$;

(34.2) $-a \leq \sum_{n=1}^{p} a_{\pi(n)} \leq a$ for $1 \leq p \leq m$;

(34.3) $(\|\sum_{n=1}^{q} a_{\pi(n)}e_{\pi(n)}\|)_{q=1}^{m}$ is $(0,0)$-monotone.

Then we have three possibilities:

(I) $-a \leq \sum_{n=1}^{m} a_{\pi(n)} + a_{r(1)} \leq a$. In this case we set $\pi(m+1) = r(1)$ and hence (34.2) continues to hold for m replaced by $m+1$. Now we verify (34.3). We notice that $r(1) - 1 = \pi(m+1) - 1 \in \{\pi(n)\}_{n=1}^{m}$ (by (34.1) since $\{r(n)\}_{n=1}^{2^Q-m}$ is a subsequence of $\{n\}_{n=1}^{2^Q}$ according to the natural order, hence $r(1)$ is the first index according to the natural order such that $r(1) \notin \{\pi(n)\}_{n=1}^{m}$), that is, there is n' with $1 \leq n' \leq m$ such that $\pi(n') = r(1) - 1 = \pi(m+1) - 1$. The index $\pi(m+1)+1$ can belong to $\{\pi(n)\}_{n=1}^{m}$ only if the sign of $a_{\pi(m+1)+1}$ is opposite to the sign of $a_{\pi(m+1)}$ (because, as we will see better in (II) and (III), when we need some a_n for example positive, we will pick up the first free index n'' such that $a_{n''} > 0$; therefore it is impossible for $\pi(m+1)+1$ to belong to

$\{\pi(n)\}_{n=1}^m$ if $a_{\pi(m+1)+1}$ has the same sign of $a_{\pi(m+1)}$ where $\pi(m+1) = r(1)$ and hence $\pi(m+1)+1 \notin \{\pi(n)\}_{n=1}^m$). Therefore we have the following subcases:

(I_1) $\pi(m+1)+1 \notin \{\pi(n)\}_{n=1}^m$. Hence, since $\pi(m+1)-1 = \pi(n')$, from (33.1) it follows that

$$\Big\|\sum_{n=1}^{m} a_{\pi(n)}e_{\pi(n)}\Big\| = \|B_{n'} + a_{\pi(n')}e'_{\pi(n')}\| = A_{n'} + |a_{\pi(m+1)-1}|,$$

where $B_{n'}$ cannot contain any term involving $e'_{\pi(n')}$, since $B_{n'}$ contains $e'_{\pi(n')}$ only if $\pi(n')+1 = \pi(m+1) \in \{\pi(n)\}_{n=1}^m$, because

$$a_{\pi(m+1)}e_{\pi(m+1)} = a_{\pi(n')+1}e_{\pi(n')+1} = a_{\pi(n')+1}e'_{\pi(n')+1} - a_{\pi(n')+1}e'_{\pi(n')},$$

which is impossible since $\pi(m+1) = r(1)$. Moreover $A_{n'} = \|B_{n'}\|$ where $A_{n'}$ could contain also the term $|a_{\pi(n')-1} - a_{\pi(n')}|$ if $\pi(n')-1 \in \{\pi(n)\}_{n=1}^m$ but this fact does not influence what follows; finally $a_{\pi(n')} = a_{\pi(m+1)-1}$ by the above. Hence

$$\begin{aligned}\Big\|\sum_{n=1}^{m+1} a_{\pi(n)}e_{\pi(n)}\Big\| &= \|B_{n'} + a_{\pi(m+1)-1}e'_{\pi(m+1)-1} + a_{\pi(m+1)}(e'_{\pi(m+1)} - e'_{\pi(m+1)-1})\| \\ &= A_{n'} + |a_{\pi(m+1)-1} - a_{\pi(m+1)}| + |a_{\pi(m+1)}| \\ &\geq A_{n'} + |a_{\pi(m+1)-1}| = \Big\|\sum_{n=1}^{m} a_{\pi(n)}e_{\pi(n)}\Big\|,\end{aligned}$$

that is, (32.2) continues to hold.

(I_2) $\pi(m+1)+1 \in \{\pi(n)\}_{n=1}^m$. Then there is n'' with $1 \leq n'' \leq m$ and $\pi(n'') = \pi(m+1)+1$ (then by the above $a_{\pi(m+1)}$ and $a_{\pi(m+1)+1}$ have opposite signs, moreover we can suppose $\pi(n'')+1 \notin \{\pi(n)\}_{n=1}^m$ since this fact does not influence the procedure that follows). Hence from (33.1) we have (in what follows $A_{(n',n'')} = \|B_{(n',n'')}\|$ where $B_{(n',n'')}$ contains no term involving elements of $(e'_{\pi(n')}, e'_{\pi(n'')-1}, e'_{\pi(n'')})$, indeed $\pi(n'') - 1 = (\pi(m+1)+1) - 1 = \pi(m+1) = r(1) \notin \{\pi(n)\}_{n=1}^m$)

$$\begin{aligned}\Big\|\sum_{n=1}^{m} a_{\pi(n)}e_{\pi(n)}\Big\| &= \|B_{(n',n'')} + a_{\pi(n')}e'_{\pi(n')} + a_{\pi(n'')}e_{\pi(n'')}\| \\ &= \|B_{(n',n'')} + a_{\pi(n')}e'_{\pi(n')} - a_{\pi(n'')}e'_{\pi(n'')-1} + a_{\pi(n'')}e'_{\pi(n'')}\| \\ &= \|B_{(n',n'')} + a_{\pi(m+1)-1}e'_{\pi(m+1)-1} - a_{\pi(m+1)+1}e'_{\pi(m+1)} \\ &\quad + a_{\pi(m+1)+1}e'_{\pi(m+1)+1}\| \\ &= A_{(n',n'')} + |a_{\pi(m+1)-1}| + 2|a_{\pi(m+1)+1}|, \\ \Big\|\sum_{n=1}^{m+1} a_{\pi(n)}e_{\pi(n)}\Big\| &= \Big\|\sum_{n=1}^{m} a_{\pi(n)}e_{\pi(n)} + a_{\pi(m+1)}(e'_{\pi(m+1)} - e'_{\pi(m+1)-1})\Big\| \\ &= \|B_{(n',n'')} + (a_{\pi(m+1)-1} - a_{\pi(m+1)})e'_{\pi(m+1)-1} \\ &\quad + (a_{\pi(m+1)} - a_{\pi(m+1)+1})e'_{\pi(m+1)} + a_{\pi(m+1)+1}e'_{\pi(m+1)+1}\| \\ &= A_{(n',n'')} + |a_{\pi(m+1)-1} - a_{\pi(m+1)}| \\ &\quad + |a_{\pi(m+1)} - a_{\pi(m+1)+1}| + |a_{\pi(m+1)+1}|\end{aligned}$$

$$
\begin{aligned}
&= A_{(n',n'')} + |a_{\pi(m+1)-1} - a_{\pi(m+1)}| + |a_{\pi(m+1)}| + 2|a_{\pi(m+1)+1}| \\
&\geq A_{(n',n'')} + |a_{\pi(m+1)-1}| + 2|a_{\pi(m+1)+1}| = \Big\| \sum_{n=1}^{m} a_{\pi(n)} e_{\pi(n)} \Big\|,
\end{aligned}
$$

that is, (32.2) continues to hold and hence we can repeat the whole procedure, starting from (34) with m replaced by $m+1$.

(II) $\sum_{n=1}^{m} a_{\pi(n)} + a_{r(1)} < -a$ (hence $\sum_{n=1}^{m} a_{\pi(n)} < 0$ and $a_{r(1)} < 0$). Let n''' be the first index with $2 \leq n''' \leq 2^Q - m$ such that $a_{r(n''')} \geq 0$ and set $\pi(m+1) = r(n''')$; then (34.2) continues to hold with m replaced by $m+1$. Now we verify (34.3). We have the following subcases:

(II$_1$) $\pi(m+1) - 1 \in \{\pi(n)\}_{n=1}^{m}$. Then the whole procedure of (I) works through (I$_1$) and (I$_2$) and hence (32.2) continues to hold and we can repeat the whole procedure, starting from (34) with m replaced by $m+1$.

(II$_2$) $\pi(m+1) - 1 \notin \{\pi(n)\}_{n=1}^{m}$ (hence $a_{\pi(m+1)-1} < 0$ since n''' is the first index with $2 \leq n''' \leq 2^Q - m$ such that $a_{r(n''')} \geq 0$). Suppose that $\pi(m+1) + 1 \in \{\pi(n)\}_{n=1}^{m}$, that is, there is n'' with $1 \leq n'' \leq m$ and $\pi(n'') = \pi(m+1) + 1$ (consequently again $a_{\pi(m+1)+1} < 0$ by the same reason of (I) since in this case $a_{\pi(m+1)}$ and $a_{\pi(m+1)+1}$ have opposite signs). Then from (33.1) it follows that (we can suppose $\pi(n'') + 1 \notin \{\pi(n)\}_{n=1}^{m}$ since it does not influence the procedure of the proof, moreover we set $A_{n''} = \|B_{n''}\|$ where $B_{n''}$ does not contain elements of $(e'_{\pi(m+1)-1}, e'_{\pi(m+1)}, e'_{\pi(m+1)+1})$)

$$
\begin{aligned}
\Big\| \sum_{n=1}^{m} a_{\pi(n)} e_{\pi(n)} \Big\| &= \|B_{n''} + a_{\pi(m+1)+1} e_{\pi(m+1)+1}\| \\
&= \|B_{n''} - a_{\pi(m+1)+1} e'_{\pi(m+1)} + a_{\pi(m+1)+1} e'_{\pi(m+1)+1}\| = A_{n''} + 2|a_{\pi(m+1)+1}|.
\end{aligned}
$$

Hence, since $a_{\pi(m+1)}$ and $a_{\pi(m+1)+1}$ have opposite signs,

$$
\begin{aligned}
\Big\| \sum_{n=1}^{m+1} a_{\pi(n)} e_{\pi(n)} \Big\| &= \Big\| \sum_{n=1}^{m} a_{\pi(n)} e_{\pi(n)} + a_{\pi(m+1)} (e'_{\pi(m+1)} - e'_{\pi(m+1)-1}) \Big\| \\
&= \|B_{n''} - a_{\pi(m+1)} e'_{\pi(m+1)-1} + a_{\pi(m+1)} e'_{\pi(m+1)} \\
&\quad - a_{\pi(m+1)+1} e'_{\pi(m+1)} + a_{\pi(m+1)+1} e'_{\pi(m+1)+1}\| \\
&= \|B_{n''} - a_{\pi(m+1)} e'_{\pi(m+1)-1} + (a_{\pi(m+1)} - a_{\pi(m+1)+1}) e'_{\pi(m+1)} \\
&\quad + a_{\pi(m+1)+1} e'_{\pi(m+1)+1}\| \\
&= A_{n''} + |a_{\pi(m+1)}| + |a_{\pi(m+1)} - a_{\pi(m+1)+1}| + |a_{\pi(m+1)+1}| \\
&= A_{n''} + 2|a_{\pi(m+1)}| + 2|a_{\pi(m+1)+1}| \\
&\geq A_{n''} + 2|a_{\pi(m+1)+1}| = \Big\| \sum_{n=1}^{m} a_{\pi(n)} e_{\pi(n)} \Big\|,
\end{aligned}
$$

that is, (32.2) continues to hold and hence we can repeat the whole procedure, starting from (34) with m replaced by $m+1$.

(II$_3$) $\pi(m+1) - 1 \notin \{\pi(n)\}_{n=1}^{m}$ and $\pi(m+1) + 1 \notin \{\pi(n)\}_{n=1}^{m}$ too (hence again $a_{\pi(m+1)-1} < 0$). From (33.1) it directly follows that

$$
\Big\| \sum_{n=1}^{m+1} a_{\pi(n)} e_{\pi(n)} \Big\| = \Big\| \sum_{n=1}^{m} a_{\pi(n)} e_{\pi(n)} \Big\| + \|a_{\pi(m+1)} e_{\pi(m+1)}\|
$$

$$= \Big\| \sum_{n=1}^{m} a_{\pi(n)} e_{\pi(n)} \Big\| + \| a_{\pi(m+1)} e'_{\pi(m+1)} - a_{\pi(m+1)} e'_{\pi(m+1)-1} \|$$
$$= \Big\| \sum_{n=1}^{m} a_{\pi(n)} e_{\pi(n)} \Big\| + 2|a_{\pi(m+1)}|$$

and hence (32.2) continues to hold and we can repeat the whole procedure, starting from (34) with m replaced by $m+1$.

(III) $\sum_{n=1}^{m} a_{\pi(n)} + a_{r(1)} > a$ (hence $\sum_{n=1}^{m} a_{\pi(n)} > 0$ and $a_{r(1)} > 0$). If there are some indices n with $a_{r(n)} \le 0$ we let n''' be the first index n such that $2 \le n''' \le 2^Q - m$ and $a_{r(n''')} \le 0$ and we set $\pi(m+1) = r(n''')$. Hence (34.2) continues to hold with m replaced by $m+1$, moreover also the whole procedure of (II) (with the opposite signs) through $(\mathrm{II}_1) \ldots (\mathrm{II}_3)$ continues to work, hence also (33.1) continues to hold and we can repeat the whole procedure, starting from (34) with m replaced by $m+1$. If $a_{r(n)} > 0$ for $2 \le n \le 2^Q - m$, we set $\pi(m+n) = r(n)$ for $1 \le n \le 2^Q - m$. Then obviously (32.1) holds and we pass to check (32.2). For $\pi(m+1) = r(1)$ we have only to repeat the whole procedure of (I), while, for $m+2 \le k \le 2^Q$, again we repeat for $\pi(k) = r(k-m)$ the whole procedure of (I) except that we start from $\sum_{n=1}^{k-1} a_{\pi(n)} e_{\pi(n)}$ instead of from $\sum_{n=1}^{m} a_{\pi(n)} e_{\pi(n)}$ as in (I). This completes the proof of Lemma 9. ■

Even if we do not now use the results of [9] and [10], the theory of finite transformability of [9, pp. 63–71] is fundamental in this work, only now every time we directly prove what we need. We recall that a Banach space Y is *finitely transformable* into another Banach space X if, for each finite-dimensional subspace Y_0 of Y and for each $\varepsilon > 0$, there exists a subspace X_0 of X and an isomorphism $T : X/X_0 \to Y_0$ with $\|T\|\|T^{-1}\| < 1+\varepsilon$; then from [9, Prop. 1.1, p. 64] it follows that the finite representability of l_1 in X and the finite transformability of X in c_0 are the same thing. We will frequently use this fact in this work, even if we shall not explicitly mention finite transformability.

The aim of the next lemma and the next theorem is to provide a sequence with properties analogous to the properties of the sequences of Theorem 8, but in l_1^n instead of in l_∞^n. In the next lemma we will not follow the simplification of Subsection 1.5 only in order to show the difference between the two procedures (but in what follows, when we will use this lemma, we will always suppose $\varepsilon = 0$).

LEMMA 10 (Finite transformability into l_1^n). *Let X be a Banach space where l_1 is finitely represented, fix $\varepsilon > 0$ and a positive integer S.*

(35.1) *There exist $\{e'_n, e'^*_n\}_{n=1}^{2^S} \cup \{e_{0,n}, e'^*_{0,n}\}_{n=1}^{S_0}$ biorthogonal in X, with $\{e'_n\}_{n=1}^{2^S} \cup \{e_{0,n}\}_{n=1}^{S_0}$ $(1+\varepsilon')$-equivalent to the natural basis of $l_1^{2^S+S_0}$, $\|e'_n\| = 1$ for $1 \le n \le 2^S$ and $\|e_{0,n}\| = 1$ for $1 \le n \le S_0$; we set*

$$E = \operatorname{span}((e'_n)_{n=1}^{2^S} \cup (e_{0,n})_{n=1}^{S_0}), \qquad U = X \cap \bigcap_{n=1}^{2^S} e'^*_{n\perp} \cap \bigcap_{n=1}^{S_0} e'^*_{(0,n)\perp};$$

(35.2) *there exist $\{e'_n, e''^*_n\}_{n=1}^{2^S} \cup \{e_{0,n} - v_{0,n},\, e'^*_{0,n}\}_{n=1}^{S_0}$ biorthogonal such that if we set $E_0 = \operatorname{span}\{e_{0,n} - v_{0,n}\}_{n=1}^{S_0}$, then $\{e'_n + E_0\}_{n=1}^{2^S}$ and $\{e'_n + E_0 + U\}_{n=1}^{2^S}$ are $(1+\varepsilon)$-equivalent to $\{o_n\}_{n=1}^{2^S}$ of (22);*

(35.3) *there exist* $\{e_{0,n}, \widehat{e}_n^*\}_{n=1}^{2^S} \cup \{e_{0,n} - v_{0,n}, e_{0,n}^* - \widehat{e}_n^*\}_{n=1}^{2^S} \cup \{e_{0,n} - v_{0,n}, e_{0,n}^*\}_{n=2^S+1}^{S_0}$ *biorthogonal such that* $\{e_{0,n} + E_0\}_{n=1}^{2^S}$ *(basis of* E/E_0*) and* $\{e_{0,n} + E_0 + U\}_{n=1}^{2^S}$ *(basis of* $X/(E_0+U)$*) are* $(1+\varepsilon)$*-equivalent to* $\{\widehat{o}_n\}_{n=1}^{2^S}$ *of* (22).

Proof. We only have to prove (35.2) and (35.3). Coming back to (22) we can suppose the integer S_0 is such that there exists $(v'_{0,n})_{n=1}^{S_0} \subset l_\infty^{2^S}$ so that:

(36) $(v'_{0,n})_{n=1}^{S_0}$ is ε''-dense in the unit ball of $l_\infty^{2^S}$, with $\varepsilon'' = \varepsilon'/2^S$ and $\varepsilon' = \sqrt{1+\varepsilon} - 1$, $v'_{0,n} = \sum_{k=1}^{2^S} a_{n,k} o_k$ for $1 \le n \le S_0$, with $v'_{0,n} = \widehat{o}_n$ for $1 \le n \le 2^S$.

Moreover, since l_1 is finitely represented in X, there exists $(e'_n, e'^*_n)_{n=1}^{2^S} \cup (e_{0,n}, e'^*_{0,n})_{n=1}^{S_0}$ biorthogonal in X with the properties of (i) and we set $E = \operatorname{span}((e'_n)_{n=1}^{2^S} \cup (e_{0,n})_{n=1}^{S_0})$. We are going to prove that

(37) if we set $v_{0,n} = \sum_{k=1}^{2^S} a_{n,k} e'_k$ for $1 \le n \le S_0$ (hence $v_{0,k} = \sum_{n=1}^{2^S} a_{k,n} e'_n$ for $1 \le k \le S_0$) and $e''^*_n = e'^*_n - \sum_{k=1}^{S_0} a_{k,n} e'^*_{0,k}$ for $1 \le n \le 2^S$, and $E_0 = \operatorname{span}\{e_{0,n} - v_{0,n}\}_{n=1}^{S_0}$, then $\{e'_n, e''^*_n\}_{n=1}^{2^S} \cup \{e_{0,n} - v_{0,n}, e'^*_{0,n}\}_{n=1}^{S_0}$ is biorthogonal with $\{e'_n + E_0\}_{n=1}^{2^S}$ $(1+\varepsilon)$-equivalent to $\{o_n\}_{n=1}^{2^S}$.

Indeed, by (35.1), $\{e'^*_n|_E\}_{n=1}^{2^S} \cup \{e'^*_{0,n}|_E\}_{n=1}^{S_0}$ is $(1+\varepsilon')$-equivalent to the natural basis of $l_\infty^{2^S+S_0}$; moreover by (36), $|a_{n,k}| \le 1$ for $1 \le k \le 2^S$ and $1 \le n \le S_0$; hence in (37), $\|e''^*_n|_E\| \le 1 + \varepsilon'$ for $1 \le n \le 2^S$. Moreover, for each sequence $\{a'_n\}_{n=1}^{2^S}$ of numbers with $a' = \max\{|a'_n| \colon 1 \le n \le 2^S\}$, hence with $\sum_{n=1}^{2^S} (a'_n/a') o_n$ in the unit ball of $l_\infty^{2^S}$, by (36) there is n' with $1 \le n' \le S_0$ so that

$$\Big\| \sum_{n=1}^{2^S} \frac{a'_n}{a'} o_n - v'_{0,n'} \Big\| = \Big\| \sum_{k=1}^{2^S} \frac{a'_k}{a'} o_k - \sum_{k=1}^{2^S} a_{n',k} o_k \Big\| = \max\bigg(\Big| \frac{a'_k}{a'} - a_{n',k} \Big| : 1 \le k \le 2^S \bigg) < \varepsilon''.$$

Hence by (36) and (37),

$$\Big\| \sum_{n=1}^{2^S} \frac{a'_n}{a'} e'_n - v_{0,n'} \Big\| \le \sum_{k=1}^{2^S} \Big| \frac{a'_k}{a'} - a_{n',k} \Big| < \varepsilon'' 2^S = \varepsilon'.$$

Then by (36) and by the definition of a' it follows that

$$\begin{aligned}
\frac{a'}{1+\varepsilon'} &\le \frac{a'}{\max\{\|e''^*_n|_E\| : 1 \le n \le 2^S\}} \le a' \Big\| \sum_{n=1}^{2^S} \frac{a'_n}{a'} e'_n + E_0 \Big\| \\
&= \Big\| \sum_{n=1}^{2^S} a'_n e'_n + E_0 \Big\| = a' \bigg\| \bigg(\Big(\sum_{n=1}^{2^S} \frac{a'_n}{a'} e'_n + E_0 \Big) - (v_{0,n'} + E_0) \bigg) + (v_{0,n'} + E_0) \bigg\| \\
&\le a' \bigg\| \Big(\sum_{n=1}^{2^S} \frac{a'_n}{a'} e'_n - v_{0,n'} \Big) + E_0 \bigg\| + a' \|v_{0,n'} + E_0\| \\
&\le a' \Big\| \sum_{n=1}^{2^S} \frac{a'_n}{a'} e'_n - v_{0,n'} \Big\| + a' \|(v_{0,n'} + E_0) + (e_{0,n'} - v_{0,n'} + E_0)\| \\
&< a'\varepsilon' + a' \|e_{0,n'} + E_0\| \le a'\varepsilon' + a' \|e_{0,n'}\| = a'(1+\varepsilon');
\end{aligned}$$

that is, $\{e'_n + E_0\}_{n=1}^{2^S}$ is $(1+\varepsilon')^2$-equivalent to the natural basis of $l_\infty^{2^{S+1}}$, on the other hand $(1+\varepsilon')^2 = 1+\varepsilon$. This completes the proof of (37) and of the beginning of (35.2). Now we will prove that

(38.1) if $e_n'''^*$ is the Hahn–Banach extension of $e_n''^*|_E$ to X for $1 \le n \le 2^S$, $e_{0,n}^*$ the Hahn–Banach extension of $e_{0,n}'^*|_E$ to X for $1 \le n \le S_0$, then $\{e'_n, e_n'''^*\}_{n=1}^{2^S} \cup \{e_{0,n} - v_{0,n}, e_{0,n}^*\}_{n=1}^{S_0}$ is biorthogonal;

(38.2) $\{e'_n + E_0 + U\}_{n=1}^{2^S}$ (basis of $X/(E_0+U)$) is $(1+\varepsilon)$-equivalent to the natural basis of $l_\infty^{2^S}$;

(38.3) $\{e_n'''^*\}_{n=1}^{2^S}$ is $(1+\varepsilon)$-equivalent to the natural basis of $l_1^{2^S}$.

Indeed, for $1 \le n \le 2^S$, in the proof of (37) we noticed that $\|e_n''^*|_E\| \le 1+\varepsilon'$, hence by the definition of (38.1), $\|e_n'''^*\| \le 1+\varepsilon'$; consequently, for each sequence $(a_n)_{n=1}^{2^S}$ of numbers by the last assertion of (37) we have

$$\begin{aligned}\max(|a_n| : 1 \le n \le 2^S)/(1+\varepsilon') &\le \Big\| \sum_{n=1}^{2^S} a_n e'_n + E_0 + U \Big\| \\ &\le \Big\| \sum_{n=1}^{2^S} a_n e'_n + E_0 \Big\| \le (1+\varepsilon') \max\{|a_n| : 1 \le n \le 2^S\}\end{aligned}$$

(by the central part of the proof of (37)), which, since $(1+\varepsilon')^2 = 1+\varepsilon$, proves (38.2) and also (38.3) since it is consequence of (38.2). Now, if $\{\widehat{e}_n, \widehat{e}_n^*\}_{n=1}^{2^S}$ is derived from $\{e'_n, e_n'''^*\}_{n=1}^{2^S}$ by means of the Walsh matrix, by (36) and (37) it follows that $v_{0,n} = \widehat{e}_n$ for $1 \le n \le 2^S$, hence also

$$\widehat{e}_n + E_0 = \widehat{e}_n + (e_{0,n} - v_{0,n}) + E_0 = \widehat{e}_n + (e_{0,n} - \widehat{e}_n) + E_0 = e_{0,n} + E_0.$$

Therefore the biorthogonal system $(\widehat{e}_n, \widehat{e}_n^*)_{n=1}^{2^S} \cup (e_{0,n} - v_{0,n}, e_{0,n}^*)_{n=1}^{S_0}$ becomes

$$(e_{0,n}, \widehat{e}_n^*)_{n=1}^{2^S} \cup (e_{0,n} - v_{0,n}, e_{0,n}^* - \widehat{e}_n^*)_{n=1}^{2^S} \cup (e_{0,n} - v_{0,n}, e_{0,n}^*)_{n=2^S+1}^{S_0};$$

hence also (35.3) has been proved. This completes the proof of Lemma 10. ∎

We point out that in what follows we will always use the simplifications of Subsection 1.5 of the introduction.

THEOREM 11 (Special sequences in spaces of type 1). *Let X be a Banach space where l_1 is finitely represented. Let $(x_n, x_n^*)_{n=1}^{Q}$ be biorthogonal in X and fix two positive integers N and R. There exist $(e_n)_{n=1}^{2^{N+R}} \cup (e_{0,n})_{n=1}^{N_0} \cup (e'_n)_{n=1}^{2^{N+R}}$ in X and $(e_n^*)_{n=1}^{2^{N+R}} \cup (e_n'^*)_{n=1}^{2^{N+R}} \cup (e_{0,n}^*)_{n=1}^{N_0}$ in X^* such that*

(39.1) *$(x_n, x_n^*)_{n=1}^{Q} \cup (e_n, e_n^*)_{n=1}^{2^{N+R}} \cup (e_{0,n}, e_{0,n}^*)_{n=1}^{N_0}$ is biorthogonal and $X = X_0 + E + U$ with $E = E' + E'_0, X_0 = \mathrm{span}(x_n)_{n=1}^{Q}$, $E' = \mathrm{span}(e_n)_{n=1}^{2^{N+R}}$, $E'_0 = \mathrm{span}(e_{0,n})_{n=1}^{N_0}$ and $U = X \cap \bigcap_{n=1}^{Q} x_{n\perp}^* \cap \bigcap_{n=1}^{2^{N+R}} e_{n\perp}^* \cap \bigcap_{n=1}^{M_0} e_{(o,n)\perp}^*$, moreover $(e_n, e_n^*)_{n=1}^{2^{N+R}} = ((e_{r,n}, e_{r,n}^*)_{n=1}^{2^N})_{r=1}^{2^R}$ where we suppose $R = 4^N$;*

(39.2) *$\|x+e\| \ge \max(\|x\|, \|e\|/2)$ for each $x \in X_0$ and $e \in E$;*

(39.3) $((e_{r,n}+E_0')_{n=1}^{2^N})_{r=1}^{2^R}$ *is* 1-*equivalent* (*and* $((e_{r,n}+E_0'+X_0+U)_{n=1}^{2^N})_{r=1}^{2^R}$ *is* 2-*equivalent*) *to* $((w_{m,r}+E_0)_{r=1}^{R})_{m=1}^{M}$ *of Theorem* 8 *with* $(m)_{m=1}^{M}$ *and* $(r)_{r=1}^{R}$ *replaced by* $(n)_{n=1}^{2^N}$ *and* $(r)_{n=1}^{2^R}$ *respectively*; *in particular* $((e_{r,n}+E_0'+X_0+U)_{n=1}^{2^N})_{r=1}^{2^R}$ *is a* 2-*unconditional basis of* $X/(E_0'+X_0+U)$;

(39.4) $(x_n,x_n^*)_{n=1}^{Q}\cup(e_n',e_n'^*)_{n=1}^{2^{N+R}}\cup(e_{0,n},e_{0,n}^*)_{n=1}^{N_0}$ *is biorthogonal and* $(e_n')_{n=1}^{2^{N+R}}$ *is* 1-*equivalent to the natural basis of* $l_1^{2^{N+R}}$, *with* $((e_{r,n}'+E_0')_{n=1}^{2^N})_{r=1}^{2^R}$ 1-*equivalent to* $((u_{m,r}+E_0)_{r=1}^{R})_{m=1}^{M}$ *of Theorem* 8 *with* $(m)_{m=1}^{M}$ *and* $(r)_{r=1}^{R}$ *replaced by* $(n)_{n=1}^{2^N}$ *and* $(r)_{r=1}^{2^R}$ *respectively*; *in particular*, *for* $1\le r\le 2^R$, $e_{r,1}=e_{r,1}'$ *and* $e_{r,n}=e_{r,n}'-e_{r,n-1}'$ *for* $2\le n\le 2^N$, *hence* $\|e_{r,m}\|=2$ *for* $2\le m\le 2^N$ *and* $\|\sum_{n=1}^{m}e_{r,n}\|=1$ *for* $1\le m\le 2^N$, *moreover* $\|e_{r,n}^*\|\le 2$ *for* $1\le n\le 2^N$;

(39.5) *for each* $\overline{x}\in X$ *with* $\|\overline{x}\|=1$ *there exists* $\overline{r}$ *with* $1\le\overline{r}\le 2^R$ *such that* $\sum_{n=1}^{2^N}|e_{\overline{r},n}^*(\overline{x})| < 1/2^N$;

(39.6) *for each* $e\in E'$, $\|e+E_0'+X_0+U\|\ge\frac12\|e+E_0'\|$.

Proof. Following the simplification of Subsection 1.5, there exists a finite-codimensional subspace V of X such that

$$(*)\qquad \|x+v\|\ge\max\{\|x\|,\|v\|/2\}\quad\text{for each } x\in X_0 \text{ and } v\in V$$

(since $\|x+v\|\ge\|x\|$ implies that $\|x+v\|\ge\|v\|/2$ because $\|x+v\|<\|v\|/2$ would imply $\|x\|>\|v\|/2$, hence $\|x+v\|\ge\|x\|>\|v\|/2$) and we can suppose $V\subset\bigcap_{n=1}^{Q}x_{n\perp}^*$. Hence, setting $S=2^{N+R}$, we can take $(e_n'')_{n=1}^{2^S}\cup(e_{0,n}')_{n=1}^{M_0}$ in V and $(e_n'''^*)_{n=1}^{2^S}\cup(e_{0,n}'''^*)_{n=1}^{M_0}$ in X^* such that the following step holds:

STEP 1. $(e_n'',e_n'''^*)_{n=1}^{2^S}\cup(e_{0,n}',e_{0,n}'''^*)_{n=1}^{M_0}$ is biorthogonal, with $(e_n'')_{n=1}^{2^S}\cup(e_{0,n}')_{n=1}^{M_0}$ 1-equivalent to the natural basis of $l_1^{2^S+M_0}$. Now we will follow the procedure of the proof of (35.2) of Lemma 10 (in what follows we always suppose $\varepsilon=0$). This will put us in the situation of Subsection 2.2, that is, we will work with sequences 1-equivalent to the natural basis of l_∞^P for some P, with the difference that we now work in quotient spaces.

STEP 2. We pass to $(e_n'',\widetilde{e}_n'''^*)_{n=1}^{2^S}\cup(e_{0,n}'-v_{0,n},\widetilde{e}_{0,n}'''^*)_{n=1}^{M_0}$ biorthogonal so that, if we set

$$E_0''=\operatorname{span}(e_{0,n}'-v_{0,n})_{n=1}^{M_0},\quad E''=\operatorname{span}(e_n'')_{n=1}^{2^S},\quad E=E''+E_0'',$$

then $(e_n''+E_0'')_{n=1}^{2^S}$ is 1-equivalent to the natural basis of $l_\infty^{2^S}$, that is, to $(o_n)_{n=1}^{2^S}$ of (22), hence $(\widetilde{e}_{n|_E}'''^*)_{n=1}^{2^S}$ is 1-equivalent to the natural basis of $l_1^{2^S}$; according to the end of the proof of Lemma 10, in the choice of $(v_{0,n})_{n=1}^{M_0}$, there is a condition which will be explained in Step 3.

STEP 3. By $(*)$ we can pass to the biorthogonal system $(x_n,x_n^*)_{n=1}^{Q}\cup(e_n'',\widetilde{e}_n''^*)_{n=1}^{2^S}\cup(e_{0,n}'-v_{0,n},\widetilde{e}_{0,n}''^*)_{n=1}^{M_0}$ with $(e_n''+E_0''+X_0)_{n=1}^{2^S}$ 2-equivalent to the natural basis of $l_\infty^{2^S}$ (hence $(\widetilde{e}_{n|_{E+X_0}}''^*)_{n=1}^{2^S}$ is 2-equivalent to the natural basis of $l_1^{2^S}$). At this point, by means of the Hahn–Banach theorem, we pass from $(\widetilde{e}_{n|_{E+X_0}}''^*)_{n=1}^{2^S}\cup(\widetilde{e}_{(0,n)|_{E+X_0}}''^*)_{n=1}^{M_0}$ to $(e_n''^*)_{n=1}^{2^S}\cup$

$(e''^*_{0,n})_{n=1}^{M_0}$. Set

$$U = X \cap \bigcap_{n=1}^{Q} x^*_{n\perp} \cap \bigcap_{n=1}^{2^S} e''^*_{n\perp} \cap \bigcap_{n=1}^{M_0} e''^*_{(0,n)\perp}.$$

Then $(e''_n + E''_0 + X_0 + U)_{n=1}^{2^S}$ is 2-equivalent to the natural basis of $l_\infty^{2^S}$, that is,

$$\frac{1}{2}\Big\|\sum_{n=1}^{2^S} a_n e''_n + E''_0\Big\| = \frac{1}{2}\max(|a_n| : 1 \le n \le 2^S) \le \Big\|\sum_{n=1}^{2^S} a_n(e''_n + E''_0) + (X_0 + U)/E''_0\Big\|$$
$$= \Big\|\sum_{n=1}^{2^S} a_n e''_n + E''_0 + X_0 + U\Big\| \le \max(|a_n| : 1 \le n \le 2^S) = \Big\|\sum_{n=1}^{2^S} a_n e''_n + E''_0\Big\|$$

for each sequence $(a_n)_{n=1}^{2^S}$ of numbers; hence, for each $e'' \in E''$,

$$\|e'' + E''_0 + X_0 + U\| \ge \frac{1}{2}\|e'' + E''_0\|.$$

Moreover, if $(\widehat{e}''_n)_{n=1}^{2^S}$ comes from $(e''_n)_{n=1}^{2^S}$ by means of the Walsh matrix, and if $(\widehat{e}'_n)_{n=1}^{2^{N+R}} = ((\widehat{e}'_{r,n})_{n=1}^{2^N})_{n=1}^{2^R}$ comes from $(\widehat{e}''_n)_{n=1}^{2^S}$ by the same definition of (28.3) of Construction I which gives $((u_{m,r})_{r=1}^{R})_{m=1}^{M}$ starting from $(\widehat{o}_n)_{n=1}^{2^S}$ of (22), we specify that the condition on the choice of $(v_{0,n})_{n=1}^{M_0}$ (in the previous step) is $v_{0,n} = \widehat{e}'_n$ for $1 \le n \le 2^{N+R}$.

Step 4. Step 3 completes the passage of Step 2; now we specify the passage of Step 3. By means of the Walsh matrix we passed from $(e''_n, e''^*_n)_{n=1}^{2^S}$ to $(\widehat{e}''_n, \widehat{e}''^*_n)_{n=1}^{2^S}$, then we passed from $(\widehat{e}''_n, \widehat{e}''^*_n)_{n=1}^{2^S}$ to $((\widehat{e}'_{r,n}, \widehat{e}'^*_{r,n})_{n=1}^{2^N})_{n=1}^{2^R} \cup (e'_{0,0,n}, e'^*_{0,0,n})_{n=1}^{2^S-2^{N+R}}$ biorthogonal, with, if E''_0 is the subspace of Step 2,

$$\operatorname{span}(((\widehat{e}'_{r,n})_{n=1}^{2^N})_{r=1}^{2^R} \cup (e'_{0,0,n})_{n=1}^{2^S-2^{N+R}}) = \operatorname{span}(\widehat{e}''_n)_{n=1}^{2^S},$$
$$\operatorname{span}(((\widehat{e}'^*_{r,n})_{n=1}^{2^N})_{r=1}^{2^R} \cup (e'^*_{0,0,n})_{n=1}^{2^S-2^{N+R}}) = \operatorname{span}(\widehat{e}''^*_n)_{n=1}^{2^S}.$$

Set $E'''_0 = \operatorname{span}(e'_{0,0,n})_{n=1}^{2^S-2^{N+R}}$ and $E'_0 = E''_0 + E'''_0$. Then $((\widehat{e}'_{r,n} + E'_0)_{n=1}^{2^N})_{r=1}^{2^R}$ is 1-equivalent to $((u_{m,r} + E_0)_{r=1}^{R})_{m=1}^{M}$ of (28.3) of Construction I with $(m)_{m=1}^{M}$ and $(r)_{r=1}^{R}$ replaced by $(n)_{n=1}^{2^N}$ and $(r)_{r=1}^{2^R}$.

Step 5. Now we have the biorthogonal system

$$((\widehat{e}'_{r,n}, \widehat{e}'^*_{r,n})_{n=1}^{2^N})_{r=1}^{2^R} \cup (e'_{0,0,n}, e'^*_{0,0,n})_{n=1}^{2^S-2^{N+R}} \cup (e'_{0,n} - v_{0,n}, \widetilde{e}''^*_{0,n})_{n=1}^{M_0}$$

with $\widehat{e}'_n = v_{0,n}$ for $1 \le n \le 2^{N+R}$ (by the end of Step 3). Hence by the definition of E'_0 it also follows, for $1 \le n \le 2^{N+R}$, that

$$\widehat{e}'_n + E'_0 = \widehat{e}'_n + (e'_{0,n} - v_{0,n}) + E'_0 = \widehat{e}'_n + (e'_{0,n} - \widehat{e}'_n) + E'_0 = e'_{0,n} + E'_0.$$

Set $e'_{0,n} = e'_n$ and $\widetilde{e}''^*_{0,n} - \widehat{e}'^*_n = \widetilde{e}'^*_{0,n}$ for $1 \le n \le 2^{N+R}$. Then the following facts hold:

(i) $(x_n, x^*_n)_{n=1}^{Q} \cup ((e'_{r,n}, e'^*_{r,n})_{n=1}^{2^N})_{r=1}^{2^R} \cup (e_{0,n}, e^*_{0,n})_{n=1}^{N_0}$ is biorthogonal where

$$((e'_{r,n}, e'^*_{r,n})_{n=1}^{2^N})_{r=1}^{2^R} = (e'_n, e'^*_n)_{n=1}^{2^{N+R}},$$
$$(e_{0,n}, e^*_{0,n})_{n=1}^{N_0} = (e'_{0,0,n}, e'^*_{0,0,n})_{n=1}^{2^S-2^{N+R}} \cup (e'_{0,n} - v_{0,n}, \widetilde{e}'^*_{0,n})_{n=1}^{2^{N+R}}$$
$$\cup\, (e'_{0,n} - v_{0,n}, \widetilde{e}''^*_{0,n})_{n=2^{N+R}+1}^{M_0}$$

(then $E'_0 = \operatorname{span}(e_{0,n})_{n=1}^{N_0}$ and $E = E' + E'_0 = E'' + E''_0$ where $E' = \operatorname{span}((e'_{r,n},)_{n=1}^{2^N})_{r=1}^{2^R})$;

(ii) $((e'_{r,n}+E'_0)_{n=1}^{2^N})_{r=1}^{2^R}$ is 1-equivalent to $((u_{m,r}+E_0)_{r=1}^{R})_{m=1}^{M}$ of (28.3) of Construction I, moreover $((e'_{r,n}+E'_0+U+X_0)_{n=1}^{2^N})_{r=1}^{2^R}$ (basis of $X/(E'_0+U+X_0)$) is 2-equivalent to $((u_{m,r}+E_0)_{r=1}^{R})_{m=1}^{M}$, for $\{m\}_{m=1}^{M}$ replaced by $(n)_{n=1}^{2^N}$ and $(r)_{r=1}^{R}$ by $(r)_{r=1}^{2^R}$;

(iii) $(e'_n)_{n=1}^{2^{N+R}}$ is 1-equivalent to the natural basis of $l_1^{2^{N+R}}$.

STEP 6. Now we set, for each r with $1\le r\le 2^R$, $e_{r,1}=e'_{r,1}$ and $e_{r,n}=e'_{r,n}-e'_{r,n-1}$ for $2\le n\le 2^N$. Hence, by the end of (28.3) of Construction I, (39.4) is satisfied and in particular the fact that $\|e^*_{r,n}\|\le 2$ for $1\le n\le 2^N$ comes from the beginning of (30.2) of Theorem 8 and from the fact that $((e_{r,n}+E'_0+X_0+U)_{n=1}^{2^N})_{n=1}^{2^R}$ is 2-equivalent and not 1-equivalent to $((w_{m,r}+E_0)_{r=1}^{R})_{m=1}^{M}$ of Theorem 8. Also the end of (39.3) follows from (30.4) of Theorem 8. Analogously (39.5) follows from the end of (39.1) and from (30.5) (see also its proof) of Theorem 8; in particular we now have

$$\widetilde{R}(\widetilde{M},\varepsilon)=2\log_2(2^{2\widetilde{M}}\sqrt{2^{2^{\widetilde{M}}+1}}/\varepsilon),$$

so $\widetilde{M}=N$, $\widetilde{R}=R$, $\varepsilon=1/2^N$)

$$R\Big(N,\frac{1}{2^N}\Big)=2\log_2\Big(2^{2N}\sqrt{2^{2^N+1}}/\frac{1}{2^N}\Big)=\log_2(2^{3N}\sqrt{2^{2^N+1}})^2=6N+2^N+1<4^N.$$

Finally, (39.6) has been proved in Step 3 (since $E''+E''_0=E'+E'_0$). This completes the proof of Theorem 11. ∎

We point out that (i) of Proposition 3 of the Introduction follows from (39.4) of Theorem 11; while (ii) of that proposition will follow from the proof of (49.1) and (49.2) of Theorem 22 of Section 6.

3. Construction of a basis with permutations

3.1. Construction of connections among blocks. Our aim now is to guarantee the completeness of our construction, that is, we wish to be sure that our basis can represent each element of the space and not only the elements of some proper subspace.

CONSTRUCTION II. The next construction concerns each separable Banach space.

Let X be a separable Banach space. We fix a positive integer m and a sequence $(\overline{x}_k)$ in X and we suppose to have the following situation:

(40.1) $(y_n,y^*_n)_{n=1}^{Q(m)}\cup(v'_{m+1,n},v'^*_{m+1,n})_{n=1}^{Q''_{m+1}}$ biorthogonal in X with $\|y_n\|<6$ and $\|y^*_n\|<13$ for $1\le n\le Q(m)$, $\|v'_{m+1,n}\|=1$ and $\|v'^*_{m+1,n}\|<5$ for $1\le n\le Q''_{m+1}$;

(40.2) $X=X''_m+U''_m$ where $X''_m=\operatorname{span}((y_n)_{n=1}^{Q(m)}\cup(v'_{m+1,n})_{n=1}^{Q''_{m+1}})$ and $U''_m=X\cap\bigcap_{n=1}^{Q(m)}y^*_{n\perp}\cap\bigcap_{n=1}^{Q''_{m+1}}v'^*_{(m+1,n)\perp}$;

(40.3) $\operatorname{dist}(\overline{x}_k,X''_m)<1/2^m$ for $1\le k\le m$, where $\overline{\operatorname{span}}(\overline{x}_k)=X$ with $\|\overline{x}_k\|=1$ for each k.

Fix $\eta_{m+1} > 0$; we can set $\eta_{m+1} = 1/2^{2Q(m)+1}$. By (40.3) there exist

$$(v'_{m+1,n})_{n=Q''_{m+1}+1}^{Q'_{m+1}} \cup ((v'_{m+1,n,k})_{k=1}^{Q'_{m+1,n}} \cup (v''_{m+1,n,k})_{k=1}^{Q''_{m+1,n}})_{n=Q''_{m+1}+1}^{Q'_{m+1}} \cup (v'_{m+2,n})_{n=1}^{Q''_{m+2}} \subset X,\ (v'^*_{m+1,n})_{n=Q''_{m+1}+1}^{Q'_{m+1}} \cup ((\widetilde{v}'^*_{m+1,n,k})_{k=1}^{Q'_{m+1,n}} \cup (v''^*_{m+1,n,k})_{k=1}^{Q''_{m+1,n}})_{n=Q''_{m+1}+1}^{Q'_{m+1}} \cup (v'^*_{m+2,n})_{n=1}^{Q''_{m+2}} \subset X^*$$

so that

(41.1) $(y_n, y_n^*)_{n=1}^{Q(m)} \cup (v'_{m+1,n}, v'^*_{m+1,n})_{n=1}^{Q'_{m+1}} \cup ((v'_{m+1,n,k}, \widetilde{v}'^*_{m+1,n,k})_{k=1}^{Q'_{m+1,n}} \cup (v''_{m+1,n,k}, v''^*_{m+1,n,k})_{k=1}^{Q''_{m+1,n}})_{n=Q''_{m+1}+1}^{Q'_{m+1}} \cup (v'_{m+2,n}, v'^*_{m+2,n})_{n=1}^{Q''_{m+2}}$

is biorthogonal, with, for $Q''_{m+1}+1 \le n \le Q'_{m+1}$, $\|v'_{m+1,n}\| = 1$ and $\|v'^*_{m+1,n}\| < 5$, $\|v'_{m+1,n,k}\| = 1$ for $1 \le k \le Q'_{m+1,n}$ and $\|v''_{m+1,n,k}\| = 1$ for $1 \le k \le Q''_{m+1,n}$, $\|v'_{m+2,n}\| = 1$ and $\|v'^*_{m+2,n}\| < 5$ for $1 \le n \le Q''_{m+2}$.

Set moreover

(41.2) $$U'''_{m+1} = U''_m \cap \bigcap_{n=Q''_{m+1}+1}^{Q'_{m+1}} v'^*_{(m+1,n)\perp} \cap \bigcap_{k=1}^{Q'_{m+1,n}} \widetilde{v}'^*_{(m+1,n,k)\perp} \cap \bigcap_{k=1}^{Q''_{m+1,n}} v''^*_{(m+1,n,k)\perp},$$

$$X'''_{m+1} = X''_m + \operatorname{span}(v'_{m+1,n} \cup (v'_{m+1,n,k})_{k=1}^{Q'_{m+1,n}} \cup (v''_{m+1,n,k})_{k=1}^{Q''_{m+1,n}})_{n=Q''_{m+1}+1}^{Q'_{m+1}},$$

$$U''_{m+1} = U'''_{m+1} \cap \bigcap_{n=1}^{Q''_{m+2}} v'^*_{(m+2,n)\perp}, \qquad X''_{m+1} = X'''_{m+1} + \operatorname{span}(v'_{m+2,n})_{n=1}^{Q''_{m+2}}.$$

Then the following properties hold:

(42.1) $\operatorname{dist}(\overline{x}_k, X'''_m + \operatorname{span}(v'_{m+1,n})_{n=Q''_{m+1}+1}^{Q'_{m+1}}) < 1/2^{m+1}$ for $1 \le k \le m+1$;

(42.2) for each $x \in X''_m$, with $\|x\| < 78Q(m) + 5Q''_{m+1}$,

$$\operatorname{dist}(x, \operatorname{span}((v'_{m+1,n,k})_{k=1}^{Q'_{m+1,n}} \cup (v''_{m+1,n,k})_{k=1}^{Q''_{m+1,n}})_{n=Q''_{m+1}+1}^{Q'_{m+1}} + U'''_{m+1}) > \|x\| - \eta_{m+1}/2$$

while

$$\operatorname{dist}(x, \operatorname{span}(v'_{m+1,n})_{n=Q''_{m+1}+1}^{Q'_{m+1}}) < \operatorname{dist}(x, \operatorname{span}(v'_{m+1,n} \cup (v'_{m+1,n,k})_{k=1}^{Q'_{m+1,n}} \cup (v''_{m+1,n,k})_{k=1}^{Q''_{m+1,n}})_{n=Q''_{m+1}+1}^{Q'_{m+1}} + U'''_{m+1}) + \eta_{m+1}/2;$$

(42.3) for each n with $Q''_{m+1} + 1 \le n \le Q'_{m+1}$ and for each

$$x \in X''_m + \operatorname{span}((v'_{m+1,k})_{k=Q''_{m+1}+1}^{Q'_{m+1}} \cup ((v'_{m+1,f,k})_{k=1}^{Q'_{m+1,f}} \cup (v''_{m+1,f,k})_{k=1}^{Q''_{m+1,f}})_{f=Q''_{m+1}+1}^{n-1})$$

with

$$\|x\| < 78Q(m) + 5Q'_{m+1} + \sum_{f=Q''_{m+1}+1}^{n-1} \Big(\sum_{k=1}^{Q'_{m+1,f}} \|\widetilde{v}'^*_{m+1,f,k}\| + \sum_{k=1}^{Q''_{m+1,f}} \|v''^*_{m+1,n,k}\| \Big),$$

we have

$$\operatorname{dist}(x, \operatorname{span}((v''_{m+1,n,k})_{k=1}^{Q''_{m+1,n}} \cup ((v'_{m+1,f,k})_{k=1}^{Q'_{m+1,f}} \cup (v''_{m+1,f,k})_{k=1}^{Q''_{m+1,f}})_{f=n+1}^{Q'_{m+1}}) + U'''_{m+1}) > \|x\| - \eta_{m+1}/2$$

while

$$\operatorname{dist}(x, \operatorname{span}(v'_{m+1,n,k})_{k=1}^{Q'_{m+1,n}}) < \eta_{m+1}/2 + \operatorname{dist}(x, \operatorname{span}((v'_{m+1,f,k})_{k=1}^{Q'_{m+1,f}} \cup (v''_{m+1,f,k})_{k=1}^{Q''_{m+1,f}})_{f=n}^{Q'_{m+1}} + U'''_{m+1});$$

(42.4) for each $x \in X'''_m$ with

$$\|x\| < 78Q(m) + 5Q'_{m+1} + \sum_{f=Q''_{m+1}+1}^{Q'_{m+1}} \Big(\sum_{k=1}^{Q'_{m+1,f}} \|\widetilde{v}'^{*}_{m+1,f,k}\| + \sum_{k=1}^{Q''_{m+1,f}} \|v''^{*}_{m+1,n,k}\|\Big),$$

we have $\operatorname{dist}(x, U''_{m+1}) > \|x\| - \eta_{m+1}/2$ while

$$\operatorname{dist}(x, \operatorname{span}(v'_{m+2,n})_{n=1}^{Q''_{m+2}}) < \operatorname{dist}(x, \operatorname{span}(v'_{m+2,n})_{n=1}^{Q''_{m+2}} + U''_{m+1}) + \eta_{m+1}/2;$$

(42.5) for each n with $Q''_{m+1} + 1 \le n \le Q'_{m+1}$ there is $(w'_{m+1,n,k})_{k=1}^{Q''_{m+1,n}}$ $\eta_{m+1}/2$-dense in the ball of $\operatorname{span}(v'_{m+1,n,k})_{k=1}^{Q'_{m+1,n}}$ with radius

$$78Q(m) + 5Q'_{m+1} + \sum_{f=Q''_{m+1}+1}^{n-1} \Big(\sum_{k=1}^{Q'_{m+1,f}} \|\widetilde{v}'^{*}_{m+1,f,k}\| + \sum_{k=1}^{Q''_{m+1,f}} \|v''^{*}_{m+1,n,k}\|\Big)$$

and with $w'_{m+1,n,k} = \sum_{f=1}^{Q'_{m+1,n}} a_{m+1,n,k,f} v'_{m+1,n,f}$ for $1 \le k \le Q''_{m+1,n}$

(which we will also write

$$w'_{m+1,n,f} = \sum_{k=1}^{Q'_{m+1,n}} a_{m+1,n,f,k} v'_{m+1,n,k} \quad \text{for } 1 \le f \le Q''_{m+1,n};$$

we also point out that $(w'_{m+1,n,k})_{k=1}^{Q''_{m+1,n}}$ of (42.5) defines the integer $Q''_{m+1,n}$ which appears in (41.1)); indeed, we only have to specfy that in (41.1) it is possible to get $\|v'^{*}_{m+1,n}\| < 5$ for $Q''_{m+1} + 1 \le n \le Q'_{m+1}$ and $\|v'^{*}_{m+2,n}\| < 5$ for $1 \le n \le Q''_{m+2}$ for instance by Proposition 3(i) of the introduction; moreover, in order to get (42.2) and (42.3), see the procedure of Subsection 1.5 of the Introduction to get W, starting from a finite-dimensional subspace X_m of X such that the following properties hold: $X = X_m + W + W_0$, X_m is $(1+\varepsilon)$-orthogonal to W and W_0 has finite dimension.

In the next lemma we use the fact that, for each biorthogonal system $(x_n, x_n^*)_{n=1}^{P}$, setting $\sum_{n=1}^{P}(x_n \|x_n^*\|) = \widehat{x}$, we never have $\|\widehat{x}\| < 1$ since $(x_n^*/\|x_n^*\|)(\widehat{x}) = 1$ for $1 \le n \le P$.

Lemma 12 (Properties of the connection sequence). *In the setting of Construction II (for the end of* (43.1) *we use the notation after formula* (42.5))

(43.1) $(u''_{m+1,n}, u''^*_{m+1,n})_{n=1}^{Q''_{m+1}} = (v'_{m+1,n}, v'^*_{m+1,n})_{n=1}^{Q''_{m+1}}$, $(u''_{m+1,n}, u''^*_{m+1,n})_{n=Q''_{m+1}+1}^{P''_{m+1}} = (u'''_{m+1,n}/\|u'''_{m+1,n}\|, \|u'''_{m+1,n}\|u'''^*_{m+1,n})_{n=Q''_{m+1}+1}^{P''_{m+1}}$ *where* $(u'''_{m+1,n}, u'''^*_{m+1,n})_{n=Q''_{m+1}+1}^{P''_{m+1}}$ $= ((v'_{m+1,n,k}, v'^*_{m+1,n,k})_{k=1}^{Q'_{m+1,n}} \cup (u'''_{m+1,n,k}, u'''^*_{m+1,n,k})_{k=0}^{Q''_{m+1,n}})_{n=Q''_{m+1}+1}^{Q'_{m+1}}$ *where, for* $Q''_{m+1}+1 \le n \le Q'_{m+1}$,

$$u'''_{m+1,n,0} = \frac{\eta_{m+1} v'_{m+1,n}}{2Q'_{m+1}} - 3 \sum_{k=1}^{Q''_{m+1,n}} v''_{m+1,n,k} \|v''^*_{m+1,n,k}\|$$

(then $\|u'''_{m+1,n,0}\| > 2$*) and* $u'''^*_{m+1,n,0} = (2Q'_{m+1}/\eta_{m+1}) v'^*_{m+1,n}$*; moreover, for* $1 \le k \le Q''_{m+1,n}$, $u'''_{m+1,n,k} = v''_{m+1,n,k}\eta_{m+1}/2 + w'_{m+1,n,k}$ *and*

$$\begin{aligned} u'''^*_{m+1,n,k} &= \frac{2}{\eta_{m+1}} v''^*_{m+1,n,k} + \frac{2.3}{\eta_{m+1}} \|v''^*_{m+1,n,k}\| u'''^*_{m+1,n,0} \\ &= \frac{2}{\eta_{m+1}} v''^*_{m+1,n,k} + 3\frac{2^2 Q'_{m+1}}{\eta^2_{m+1}} \|v''^*_{m+1,n,k}\| v'^*_{m+1,n}; \end{aligned}$$

setting moreover, for

$$1 \le k \le Q'_{m+1,n},\ v'^*_{m+1,n,k} = \widetilde{v}'^*_{m+1,n,k} - \sum_{f=1}^{Q''_{m+1,n}} a_{m+1,n,f,k} u''^*_{m+1,n,f},$$

the following properties hold:

(43.2) $(y_n, y_n^*)_{n=1}^{Q(m)} \cup (u''_{m+1,n}, u''^*_{m+1,n})_{n=1}^{P''_{m+1}} \cup (v'_{m+2,n}, v'^*_{m+2,n})_{n=1}^{Q''_{m+2}}$ *is biorthogonal;*

(43.3) *for each* $x' \in X$ *with* $\|x'\| = 1$*, if* $|v^*_{m+1,n'}(x')| > \eta_{m+1}/(2Q'_{m+1})$ *for some* n' *with* $Q''_{m+1}+1 \le n' \le Q'_{m+1}$*, while* $|v^*_{m+1,n}(x')| \le \eta_{m+1}/(2Q'_{m+1})$ *for* $n'+1 \le n \le Q'_{m+1}$*, then* $(au''^*_{m+1,n',k}(x')u''_{m+1,n',k} : 0 < |a| < 1)_{k=1}^{Q''_{m+1,n'}}$ *is* η_{m+1}*-dense in the ball of* $\operatorname{span}(v'_{m+1,n',k})_{k=1}^{Q'_{m+1,n'}}$ *of radius*

$$78Q(m) + 5Q'_{m+1} + \sum_{f=Q''_{m+1}+1}^{n'-1} \Big(\sum_{k=1}^{Q'_{m+1,f}} \|\widetilde{v}'^*_{m+1,f,k}\| + \sum_{k=1}^{Q''_{m+1,f}} \|v''^*_{m+1,n,k}\|\Big).$$

Proof. Indeed, let w be in the ball of $\operatorname{span}(v'_{m+1,n',k})_{k=1}^{Q'_{m+1,n'}}$ of radius as in (43.3), and let k' be an integer with $1 \le k' \le Q''_{m+1,n'}$ so that (by (42.5))

$$\|w - w'_{m+1,n',k'}\| < \eta_{m+1}/2. \tag{44}$$

Set $a = 1/u'''^*_{m+1,n',k'}(x')$. By the hypothesis of (43.3),

$$\begin{aligned} |u'''^*_{m+1,n',k'}(x')| &= \frac{2}{\eta_{m+1}} \left| v''^*_{m+1,n',k'}(x') + \frac{6Q'_{m+1}}{\eta_{m+1}} \|v''^*_{m+1,n',k'}\| v'^*_{m+1,n'}(x') \right| \\ &\ge \frac{2}{\eta_{m+1}} \left(\frac{6Q'_{m+1}}{\eta_{m+1}} \|v''^*_{m+1,n',k'}\| |v'^*_{m+1,n'}(x')| - |v''^*_{m+1,n',k'}(x')| \right) \\ &\ge \frac{2}{\eta_{m+1}} \left(\frac{6Q'_{m+1}}{\eta_{m+1}} \|v''^*_{m+1,n',k'}\| |v'^*_{m+1,n'}(x')| - \|v''^*_{m+1,n',k'}\| \right) \end{aligned}$$

$$= \frac{2}{\eta_{m+1}} \|v''^*_{m+1,n',k'}\| \left(\frac{6Q'_{m+1}}{\eta_{m+1}} |v'^*_{m+1,n'}(x')| - 1 \right)$$
$$> \frac{2}{\eta_{m+1}} \|v''^*_{m+1,n',k'}\| \left(\frac{6Q'_{m+1}}{\eta_{m+1}} \frac{\eta_{m+1}}{2Q'_{m+1}} - 1 \right)$$
$$= \frac{4\|v''^*_{m+1,n',k'}\|}{\eta_{m+1}} = 2^{2Q(m)+3} \|v''^*_{m+1,n',k'}\| \geq 2^{2Q(m)+3}.$$

Then $0 < |a| < 1/2^{2Q(m)+3}$ and

$$\begin{aligned} \|w - au''^*_{m+1,n',k'}(x')u''_{m+1,n',k'}\| &= \|w - au'''^*_{m+1,n',k'}(x')u'''_{m+1,n',k'}\| = \|w - u'''_{m+1,n',k'}\| \\ &\leq \|w - w'_{m+1,n',k'}\| + \|w'_{m+1,n',k'} - u'''_{m+1,n',k'}\| \\ &= \|w - w'_{m+1,n',k'}\| + \|w'_{m+1,n',k'} - (v''_{m+1,n',k'}\eta_{m+1}/2 + w'_{m+1,n',k'})\| \\ &= \|w - w'_{m+1,n',k'}\| + \|v''_{m+1,n',k'}\eta_{m+1}/2\| = \|w - w'_{m+1,n',k'}\| + \eta_{m+1}/2 < \eta_{m+1} \end{aligned}$$

by (44). This completes the proof of Lemma 12. ■

3.2. Construction of each block. This subsection is the heart of the whole construction of a basis with permutations.

Construction III. Our aim now is the construction of each block; that is, we suppose to have already constructed

$$(x_n, x_n^*)_{n=1}^{q(3m)} \cup (u'_{3m,s}, u'^*_{3m,s})_{s=1}^{M_{3m-1,o}+P''_{3m}} \cup (v'_{3m,n}, v'^*_{3m,n})_{n=1}^{Q''_{3m}}$$

and we turn to the construction of

$$(x_n, x_n^*)_{n=1}^{q(3(m+1))} \cup (u'_{3(m+1),s}, u'^*_{3(m+1),s})_{s=1}^{M_{3m+2,o}+P''_{3(m+1)}} \cup (v'_{3(m+1),n}, v'^*_{3(m+1),n})_{n=1}^{Q''_{3(m+1)}}.$$

We give separately the construction of $(x_n, x_n^*)_{n=q(3m)+1}^{q(3m+1)}$ (the *completeness block*), of $(x_n, x_n^*)_{n=q(3m+1)+1}^{q(3m+2)}$ (the *regularization block*) and of $(x_n, x_n^*)_{n=q(3m+2)+1}^{q(3(m+1))}$ (the *free block*). We start with the completeness block.

Subconstruction III.1 (SC III.1, construction of the completeness block). We start from the biorthogonal system

$$\begin{gathered} B_1(3m) = (x_n, x_n^*)_{n=1}^{q(3m)} \cup (u'_{3m,s}, u'^*_{3m,s})_{s=1}^{A''_{3m}} \cup (v'_{3m+1,n}, v'^*_{3m+1,n})_{n=1}^{Q''_{3m+1}}, \\ A''_{3m} = M_{3m-1,0} + P''_{3m} = A_{3m}, \ K_{3m} = \max(\|u'^*_{3m,s}\| : 1 \leq s \leq A_{3m}), \\ B_{3m} > A_{3m} 2^{3K_{3m}A_{3m}}, \qquad P_{3m} = A_{3m}(1 + 2^{2B_{3m}}). \end{gathered}$$

We will also use an integer M_{3m} which will be defined in Step 5. The construction develops through six steps:

Step 1. By means of the procedure of Lemma 10 we pass to the biorthogonal system

$$B'_2(3m) = B_1(3m) \cup (\widehat{e}_{3m,n}, \widehat{e}^*_{3m,n})_{n=1}^{S'_{3m}+M'_{3m,0}}$$

with the following properties: $(\widehat{e}_{3m,n})_{n=1}^{S'_{3m}+M'_{3m,0}}$ is 1-equivalent to the natural basis of $l_1^{S'_{3m}+M'_{3m,0}}$; moreover, setting $(y_{3m,n})_{n=1}^{Q(3m)} = (x_n)_{n=1}^{q(3m)} \cup (u'_{3m,s})_{s=1}^{A''_{3m}} \cup (v'_{3m+1,n})_{n=1}^{Q''_{3m+1}}$,

we have

$$\|y+e\| \geq \max(\|y\|, \|e\|/2), \qquad y \in \operatorname{span}(y_{3m,n})_{n=1}^{Q(3m)},\ e \in \operatorname{span}(\widehat{e}_{3m,n})_{n=1}^{S'_{3m}+M'_{3m,0}};$$

moreover we suppose S'_{3m} and $M'_{3m,0}$ are such that there are $(\widehat{v}_{0,3m,n})_{n=S'_{3m}+1}^{S'_{3m}+M'_{3m,0}}$ in $\operatorname{span}(\widehat{e}_{3m,n})_{n=1}^{S'_{3m}}$ and $(\widehat{e}''^*_{3m,n})_{n=1}^{S'_{3m}}$ in X^* so that, if we set

$$(\widehat{e}_{3m,n} - \widehat{v}_{0,3m,n}, \widehat{e}^*_{3m,n})_{n=S'_{3m}+1}^{S'_{3m}+M'_{3m,0}} = (e_{3m,0,n}, e^*_{3m,0,n})_{n=1}^{M'_{3m,0}}, E'_{3m,0} = \operatorname{span}(e_{3m,0,n})_{n=1}^{M'_{3m,0}},$$

then $(\widehat{e}_{3m,n}, \widehat{e}''^*_{3m,n})_{n=1}^{S'_{3m}} \cup (e_{3m,0,n}, e^*_{3m,0,n})_{n=1}^{M'_{3m,0}}$ is biorthogonal and $(\widehat{e}_{3m,n} + E'_{3m,0})_{n=1}^{S'_{3m}}$ is 1-equivalent to the natural basis of $l_\infty^{S'_{3m}}$. We now specify that

$$S'_{3m} = P_{3m}(2+2^{M_{3m}}) + 2^{M'_{3m}}, \qquad M'_{3m} = A_{3m}2^{2B_{3m}} + P_{3m}(2\cdot 2^{M_{3m}} + 2^{2M_{3m}}),$$
$$\begin{aligned}(\widehat{e}_{3m,n}, \widehat{e}''^*_{3m,n})_{n=1}^{P_{3m}(2+2^{M_{3m}})} &= (e'''_{3m,n}, e'''^*_{3m,n})_{n=1}^{P_{3m}(1+2^{M_{3m}})}\\ &= ((\omega_{3m,n}, \omega^*_{3m,n}) \cup (\omega_{3m,n,k}, \omega^*_{3m,n,k})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}}.\end{aligned}$$

At the end of this step we point out the following useful fact:

With a little change as regards Lemma 10 we can set above

$$(\widehat{v}_{0,3m,n})_{n=S'_{3m}+1}^{S'_{3m}+M'_{3m,0}} = ((\widehat{v}_{0,3m,k})_{k=M'_{3m,0,n-1}+1}^{M'_{3m,0,n}})_{n=1}^{S'_{3m}}$$

such that, for $1 \leq n \leq S'_{3m}$, $(\widehat{v}_{0,3m,k})_{k=M'_{3m,0,n-1}+1}^{M'_{3m,0,n}} \subset \operatorname{span}(\widehat{e}_{3m,n'})_{n'=1}^{n}$ and we can suppose that $(\widehat{e}_{3m,n'} + \operatorname{span}(\widehat{v}_{0,3m,k})_{k=1}^{M'_{3m,0,n}})_{n'=1}^{n}$ is 1-equivalent to the natural basis of $l_\infty^{S'_{3m}}$.

STEP 2. Starting from $(\widehat{e}_{3m,n}, \widehat{e}''^*_{3m,n})_{n=P_{3m}(2+2^{M_{3m}})+1}^{S'_{3m}}$ and $E'_{3m,0}$ we get, from the procedure of Theorem 11, the biorthogonal system

$$\begin{aligned}&((e'_{3m,\mathrm{aux},s,t}, e'^*_{3m,\mathrm{aux},s,t})_{t=1}^{2^{2B_{3m}}})_{s=1}^{A_{3m}} \cup ((e'_{3m,n,k}, e'^*_{3m,n,k})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}}\\ &\quad \cup (((e'_{3m,n,k,l}, e'^*_{3m,n,k,l})_{l=1}^{2^{M_{3m}}})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}}\\ &\quad \cup (((e'_{3m,n,k,0,l}, e'^*_{3m,n,k,0,l})_{l=1}^{2^{M_{3m}}})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}} \cup (e_{3m,0,n}, e^*_{3m,0,n})_{n=M'_{3m,0}+1}^{M_{3m,0}},\end{aligned}$$
$$E_{3m,0} = \operatorname{span}(e_{3m,0,n})_{n=1}^{M_{3m,0}}, \qquad M_{3m,0} = M'_{3m,0} + (2^{M'_{3m}} - M'_{3m}).$$

For $1 \leq s \leq A_{3m}$, we set $e_{3m,\mathrm{aux},s,1} = e'_{3m,\mathrm{aux},s,1}$ and $e_{3m,\mathrm{aux},s,t} = e'_{3m,\mathrm{aux},s,t} - e'_{3m,\mathrm{aux},s,t-1}$ for $2 \leq t \leq 2^{2B_{3m}}$; for $1 \leq n \leq P_{3m}$ we set $e_{3m,n,1} = e'_{3m,n,1}$ and $e_{3m,n,k} = e'_{3m,n,k} - e'_{3m,n,k-1}$ for $2 \leq k \leq 2^{M_{3m}}$, $e_{3m,n,1,2^{M_{3m}}} = e'_{3m,n,1,2^{M_{3m}}}$ and $e_{3m,n,k,2^{M_{3m}}} = e'_{3m,n,k,2^{M_{3m}}} - e'_{3m,n,k-1,2^{M_{3m}}}$ for $2 \leq k \leq 2^{M_{3m}}$; moreover for each n and k with $1 \leq n \leq P_{3m}$ and $1 \leq k \leq 2^{M_{3m}}$ we set $e_{3m,n,k,1} = e'_{3m,n,k,1}$ and $e_{3m,n,k,l} = e'_{3m,n,k,l} - e'_{3m,n,k,l-1}$ for $2 \leq l \leq 2^{M_{3m}} - 1$; $e_{3m,n,k,0,1} = e'_{3m,n,k,0,1}$ and $e_{3m,n,k,0,l} = e'_{3m,n,k,0,l} - e'_{3m,n,k,0,l-1}$ for $2 \leq l \leq 2^{M_{3m}}$. Then $(e'_{3m,\mathrm{aux},s,t})_{t=1}^{2^{2B_{3m}}}$, $(e_{3m,\mathrm{aux},s,t})_{t=1}^{2^{2B_{3m}}}$ and $E_{3m,0}$ have properties analogous to the properties of $((e'_{r,n})_{n=1}^{2^N})_{n=1}^{2^R}$, $((e_{r,n})_{n=1}^{2^N})_{n=1}^{2^R}$ and E'_0 of (39.3) and (39.4) of Theorem 11, with $\{n\}_{n=1}^{2^N}$ and $\{r\}_{r=1}^{2^R}$ replaced respectively by $(t)_{t=1}^{2^{2B_{3m}}}$ and $\{1\}$; and analogously for $(e'_{3m,n,k})_{k=1}^{2^{M_{3m}}}$ and $(e_{3m,n,k})_{k=1}^{2^{M_{3m}}}$, for $(e'_{3m,n,k,l})_{l=1}^{2^{M_{3m}}-1}$ and $(e_{3m,n,k,l})_{l=1}^{2^{M_{3m}}-1}$, for $(e'_{3m,n,k,0,l})_{l=1}^{2^{M_{3m}}}$ and $(e_{3m,n,k,0,l})_{l=1}^{2^{M_{3m}}}$, for $(e'_{3m,n,k,2^{M_{3m}}})_{k=1}^{2^{M_{3m}}}$ and $(e_{3m,n,k,2^{M_{3m}}})_{k=1}^{2^{M_{3m}}}$.

Next we pass to the biorthogonal system

$$B_2(3m) = B_1(3m) \cup (e_{3m,0,n}, e^*_{3m,0,n})_{n=1}^{M_{3m,0}} \cup (e'''_{3m,n}, e'''^*_{3m,n})_{n=1}^{S_{3m}},$$

$$S_{3m} = S'_{3m} - (2^{M'_{3m}} - M'_{3m}),$$

$$\begin{aligned}(e'''_{3m,n}, e'''^*_{3m,n})_{n=P_{3m}(2+2^{M_{3m}})+1}^{S_{3m}} &= ((e_{3m,\mathrm{aux},s,t}, e^*_{3m,\mathrm{aux},s,t})_{t=1}^{2^{2B_{3m}}})_{s=1}^{A_{3m}} \\ &\cup ((e_{3m,n,k}, e^*_{3m,n,k})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}} \\ &\cup (((e_{3m,n,k,l}, e^*_{3m,n,k,l})_{l=1}^{2^{M_{3m}}})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}} \\ &\cup (((e_{3m,n,k,0,l}, e^*_{3m,n,k,0,l})_{l=1}^{2^{M_{3m}}})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}}.\end{aligned}$$

In particular, by Lemma 10 and Theorem 11 and since $\|y+e\| \geq \max(\|y\|, \|e\|/2)$ at the beginning of Step 1, we have $\|e'''_{3m,n}\| \leq 2$ and $\|e'''^*_{3m,n}\| \leq 2$ for $1 \leq n \leq S_{3m}$.

STEP 3. We pass, by the procedure of Lemma 10, to the biorthogonal system

$$\begin{aligned}B'_3(3m) = B_2(3m) &\cup ((e_{3m,\mathrm{arm},n,j}, e^*_{3m,\mathrm{arm},n,j})_{j=1}^{J_{3m,\mathrm{arm}}})_{n=1}^{P_{3m}} \\ &\cup (e_{3m,\mathrm{arm},0,n}, e^*_{3m,\mathrm{arm},0,n})_{n=1}^{M_{3m,\mathrm{arm},0}}\end{aligned}$$

where, setting for $1 \leq n \leq P_{3m}$,

$$(e_{3m,\mathrm{arm},n,j})_{j=1}^{J_{3m,\mathrm{arm}}} = ((e_{3m,\mathrm{arm},n,k,l})_{l=1}^{2^{M_{3m}}})_{k=1}^{2^{M_{3m}}},$$

$$E'_{3m,\mathrm{arm},0} = \mathrm{span}(e_{3m,\mathrm{arm},0,n})_{n=1}^{M'_{3m,\mathrm{arm},0}},$$

$$E_{3m,\mathrm{arm},0} = \mathrm{span}(e_{3m,\mathrm{arm},0,n})_{n=1}^{M_{3m,\mathrm{arm},0}} \ (M'_{3m,\mathrm{arm},0} < M_{3m,\mathrm{arm},0}),$$

$((e_{3m,\mathrm{arm},n,j})_{j=1}^{J_{3m,\mathrm{arm}}})_{n=1}^{P_{3m}}$ is 1-equivalent to the natural basis of $l_1^{P_{3m}J_{3m,\mathrm{arm}}}$ and

$$((e_{3m,\mathrm{arm},n,j} + E'_{3m,\mathrm{arm},0})_{j=1}^{J_{3m,\mathrm{arm}}})_{n=1}^{P_{3m}}$$

(hence also $((e_{3m,\mathrm{arm},n,j} + E_{3m,\mathrm{arm},0})_{j=1}^{J_{3m,\mathrm{arm}}})_{n=1}^{P_{3m}}$) is 1-equivalent to the natural basis of $l_\infty^{P_{3m}I_{3m,\mathrm{arm}}}$; while $E_{3m,\mathrm{arm},0}$ is so that, setting

$$\begin{aligned}W' = X &\cap \bigcap_{n=1}^{Q(3m)} y^*_{(3m,n)\perp} \cap \bigcap_{n=1}^{S_{3m}} e'''^*_{(3m,n)\perp} \\ &\cap \bigcap_{n=1}^{M_{3m,0}} e^*_{(3m,0,n)\perp} \cap \bigcap_{n=1}^{P_{3m}} \bigcap_{j=1}^{J_{3m,\mathrm{arm}}} e^*_{(3m,\mathrm{arm},n,j)\perp} \cap \bigcap_{n=1}^{M'_{3m,\mathrm{arm},0}} e^*_{(3m,\mathrm{arm},0,n)\perp},\end{aligned}$$

$$U_{3m,\mathrm{arm}} = W' \cap \bigcap_{n=M'_{3m,\mathrm{arm},0}+1}^{M_{3m,\mathrm{arm},0}} e^*_{(3m,\mathrm{arm},0,n)\perp},$$

$$X' = \mathrm{span}((y_{3m,n})_{n=1}^{Q(3m)} \cup (e'''_{3m,n})_{n=1}^{S_{3m}} \cup (e_{3m,0,n})_{n=1}^{M_{3m,0}}),$$

we have $\|x+e\| \geq \|e\|/2$ for each $x \in X' + U_{3m,\mathrm{arm}}$ and

$$e \in \mathrm{span}(((e_{3m,\mathrm{arm},n,j})_{j=1}^{J_{3m,\mathrm{arm}}})_{n=1}^{P_{3m}} \cup (e_{3m,\mathrm{arm},0,n})_{n=1}^{M'_{3m,\mathrm{arm},0}})$$

(we apply twice successively the procedure to get the subspace W of Subsection 1.5, first to get $((e_{3m,\mathrm{arm},n,j})_{j=1}^{J_{3m,\mathrm{arm}}})_{n=1}^{P_{3m}} \cup (e_{3m,\mathrm{arm},0,n})_{n=1}^{M'_{3m,\mathrm{arm},0}}$ with$\|x+e\| \geq \max\{\|x\|, \|e\|/2\}$

for each $x \in X'$ and $e \in \operatorname{span}((((e_{3m,\mathrm{arm},n,j})_{j=1}^{J_{3m,\mathrm{arm}}})_{n=1}^{P_{3m}} \cup (e_{3m,\mathrm{arm},0,n})_{n=1}^{M'_{3m,\mathrm{arm},0}})$ with the properties of above; and second to get a finite-codimensional subspace W_0 of W' such that $\|x+w_0\| \geq \max(\|x\|, \|w_0\|/2)$ for $w_0 \in W_0$ and $x \in X' + \operatorname{span}((((e_{3m,\mathrm{arm},n,j})_{j=1}^{J_{3m,\mathrm{arm}}})_{n=1}^{P_{3m}} \cup (e_{3m,\mathrm{arm},0,n})_{n=1}^{M'_{3m,\mathrm{arm},0}})$; then we can find $(e_{3m,\mathrm{arm},0,n}, e^*_{3m,\mathrm{arm},0,n})_{n=M'_{3m,\mathrm{arm},0}+1}^{M_{3m,\mathrm{arm},0}}$ so that, setting

$$W_1 = \operatorname{span}(e_{3m,\mathrm{arm},0,n})_{n=M'_{3m,\mathrm{arm},0}+1}^{M_{3m,\mathrm{arm},0}}, \qquad W' = W_0 + W_1,$$

we have $\|x+e+w_0\| \geq \|x+e\| \geq \|e\|/2$ for $x \in X'$, $w_0 \in W_0$, and

$$e \in \operatorname{span}((((e_{3m,\mathrm{arm},n,j})_{j=1}^{J_{3m,\mathrm{arm}}})_{n=1}^{P_{3m}} \cup (e_{3m,\mathrm{arm},0,n})_{n=1}^{M'_{3m,\mathrm{arm},0}})).$$

Analogously to the end of Step 2 we show that, for $1 \leq n \leq P_{3m}$, $\|e^*_{3m,\mathrm{arm},n,j}\| \leq 2$ for $1 \leq j \leq J_{3m,\mathrm{arm}}$.

Step 4. We pass from $(u'_{3m,s}, u'^*_{3m,s})_{s=1}^{A_{3m}}$ to $(u'_{3m,s}/\|u'_{3m,s}\|, \|u'_{3m,s}\|u'^*_{3m,s})_{s=1}^{A_{3m}}$, which we call $(u'_{3m,s}, u'^*_{3m,s})_{s=1}^{A_{3m}}$ again; then from

$$(u'_{3m,s}, u'^*_{3m,s})_{s=1}^{A_{3m}} \cup ((e_{3m,\mathrm{aux},s,t}, e^*_{3m,\mathrm{aux},s,t})_{t=1}^{2^{2B_{3m}}})_{s=1}^{A_{3m}}$$

we pass to

$$(u_{3m,n}, u^*_{3m,n})_{n=1}^{P_{3m}} = ((u_{3m,s,t}, u^*_{3m,s,t})_{t=0}^{2^{2B_{3m}}})_{s=1}^{A_{3m}}$$

where, for each s with $1 \leq s \leq A_{3m}$ and $1 \leq t \leq 2^{2B_{3m}}$,

$$u_{3m,s,0} = \sum_{j=1}^{2^{2B_{3m}}} e_{3m,\mathrm{aux},s,j}, \qquad u^*_{3m,s,0} = \frac{1}{2^{2B_{3m}}} \sum_{j=1}^{2^{2B_{3m}}} e^*_{3m,\mathrm{aux},s,j} - \frac{u'^*_{3m,s}}{2^{B_{3m}}},$$

$$u_{3m,s,t} = e_{3m,\mathrm{aux},s,t} + \frac{u'_{3m,s}}{2^{B_{3m}}},$$

$$u^*_{3m,s,t} = e^*_{3m,\mathrm{aux},s,t} - u^*_{3m,s,0} = \left(e^*_{3m,\mathrm{aux},s,t} - \frac{1}{2^{2B_{3m}}} \sum_{j=1}^{2^{2B_{3m}}} e^*_{3m,\mathrm{aux},s,j}\right) + \frac{u'^*_{3m,s}}{2^{B_{3m}}};$$

that is, $(u_{3m,n}, u^*_{3m,n})_{n=1}^{P_{3m}}$ is a "uniform minimalization" of $(u'_{3m,s}, u'^*_{3m,s})_{s=1}^{A_{3m}}$.

At this point we pass to the generating biorthogonal system

$$(w_{3m,n}, w^*_{3m,n})_{n=1}^{P_{3m}} = ((w_{3m,s,t}, w^*_{3m,s,t})_{t=0}^{2^{2B_{3m}}})_{s=1}^{A_{3m}}$$

and we specify that, setting $Q_{0,3m} = \max(\|w^*_{3m,n}\| : 1 \leq n \leq P_{3m})$, we choose $M_{3m} > 4^{P_{3m}Q_{0,3m}}$.

In what follows in SC III.1 we will use also a biorthogonal system

$$((\omega'_{3m,n}, \omega'^*_{3m,n}) \cup (e_{3m,\mathrm{brd},n,k}, e^*_{3m,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}}$$

which will be defined in SC III.2.

Now we fix n with $1 \leq n \leq P_{3m}$. Then we set, for $1 \leq k \leq 2^{M_{3m}}$ and $1 \leq l \leq 2^{M_{3m}}-1$,

$$\omega_{3m,n,0} = \sum_{f=1}^{2^{M_{3m}}} e_{3m,n,f,2^{M_{3m}}}, \quad \omega^*_{3m,n,0} = \frac{1}{2^{M_{3m}}} \sum_{f=1}^{2^{M_{3m}}} e^*_{3m,n,f,2^{M_{3m}}} - \frac{\omega^*_{3m,n}}{2^{M_{3m}}},$$

$$e''_{3m,n,k,2^{M_{3m}}} = e_{3m,n,k,2^{M_{3m}}} + \omega_{3m,n}, e''^*_{3m,n,k,2^{M_{3m}}}$$

$$= e^*_{3m,n,k,2^{M_{3m}}} - \omega^*_{3m,n,0} = e^*_{0,3m,n,k,2^{M_{3m}}} + \frac{\omega^*_{3m,n}}{2^{M_{3m}}},$$

$$e^*_{0,3m,n,k,2^{M_{3m}}} = e^*_{3m,n,k,2^{M_{3m}}} - \frac{1}{2^{M_{3m}}} \sum_{f=1}^{2^{M_{3m}}} e^*_{3m,n,f,2^{M_{3m}}};$$

$$\omega_{3m,n,k,0} = \sum_{g=1}^{2^{M_{3m}}-1} e_{3m,n,k,g}, \qquad \omega^*_{3m,n,k,0} = \sum_{g=1}^{2^{M_{3m}}-1} \frac{e^*_{3m,n,k,g}}{2^{M_{3m}}-1} - \frac{\omega^*_{3m,n,k}}{2^{M_{3m}}-1},$$

$$e''_{3m,n,k,l} = e_{3m,n,k,l} + \omega_{3m,n,k},$$

$$e''^*_{3m,n,k,l} = e^*_{3m,n,k,l} - \omega^*_{3m,n,k,0} = e^*_{0,3m,n,k,l} + \frac{\omega^*_{3m,n,k}}{2^{M_{3m}}-1},$$

$$e^*_{0,3m,n,k,l} = e^*_{3m,n,k,l} - \sum_{g=1}^{2^{M_{3m}}-1} \frac{e^*_{3m,n,k,g}}{2^{M_{3m}}-1},$$

hence

$$\sum_{k=1}^{2^{M_{3m}}} e^*_{0,3m,n,k} = \sum_{k=1}^{2^{M_{3m}}} e^*_{0,3m,n,k,2^{M_{3m}}} = \sum_{l=1}^{2^{M_{3m}}-1} e^*_{0,3m,n,k,l} = 0,$$

$$\sum_{l=1}^{2^{M_{3m}}-1} e''^*_{3m,n,k,l} = \omega^*_{3m,n,k}.$$

In the next steps we will work with the biorthogonal system

$$\begin{aligned} B_4(3m) = (x_n, x_n^*)_{n=1}^{q(3m)} &\cup ((w_{3m,n}, w^*_{3m,n}) \cup ((e_{3m,n,k}, e^*_{3m,n,k}) \\ &\cup (e''_{3m,n,k,l}, e''^*_{3m,n,k,l}))_{l=1}^{2^{M_{3m}}})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}} \\ &\cup ((\omega_{3m,n,0}, \omega^*_{3m,n,0}) \cup (\omega'_{3m,n,0}, \omega'^*_{3m,n,0}) \cup (\omega_{3m,n,k,0}, \omega^*_{3m,n,k,0})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}} \\ &\cup (((e_{3m,n,k,0,l}, e^*_{3m,n,k,0,l})_{l=1}^{2^{M_{3m}}})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}} \\ &\cup ((e_{3m,\mathrm{arm},n,j}, e^*_{3m,\mathrm{arm},n,j})_{j=1}^{J_{3m,\mathrm{arm}}})_{n=1}^{P_{3m}} \cup (e_{3m,0,n}, e^*_{3m,0,n})_{n=1}^{M_{3m,0}} \\ &\cup (e_{3m,\mathrm{arm},0,n}, e^*_{3m,\mathrm{arm},0,n})_{n=1}^{M_{3m,\mathrm{arm},0}} \cup (v'_{3m+1,n}, v'^*_{3m+1,n})_{n=1}^{Q''_{3m+1}}. \end{aligned}$$

STEP 5. Let us fix n with $1 \le n \le P_{3m}$. For each l and k with $1 \le l, k \le 2^{M_{3m}}$ we set

$$x_{3m,n,0} = \sum_{f=1}^{2^{M_{3m}}} e_{3m,n,f}, \qquad x^*_{3m,n,0} = \sum_{f=1}^{2^{M_{3m}}} \frac{e^*_{3m,n,f}}{2^{M_{3m}}} - \frac{w^*_{3m,n}}{2^{M_{3m}}};$$

$$x_{3m,n,k} = e_{3m,n,k} + w_{3m,n},$$

$$x^*_{3m,n,k} = e^*_{3m,n,k} - x^*_{3m,n,0} = x^*_{0,3m,n,k} + \frac{w^*_{3m,n}}{2^{M_{3m}}}, x^*_{0,3m,n,k} = e^*_{3m,n,k} - \sum_{f=1}^{2^{M_{3m}}} \frac{e^*_{3m,n,f}}{2^{M_{3m}}};$$

$$x_{3m,n,k,0} = \sum_{g=1}^{2^{M_{3m}}} e''_{3m,n,k,g} = \sum_{g=1}^{2^{M_{3m}}} e_{3m,n,k,g} + (2^{M_{3m}}-1)\omega_{3m,n,k} + \omega_{3m,n},$$

$$x^*_{3m,n,k,0} = \frac{1}{2^{M_{3m}}} \sum_{g=1}^{2^{M_{3m}}} e''^*_{3m,n,k,g} - x^*_{3m,n,k} = \frac{\omega^*_{3m,n,k} + e''^*_{3m,n,k,2^{M_{3m}}}}{2^{M_{3m}}} - x^*_{3m,n,k};$$

$$x_{3m,n,k,l} = e''_{3m,n,k,l} + e_{3m,\mathrm{arm},n,k,l} + x_{3m,n,k}/2^{M_{3m}}$$
$$= e_{3m,n,k,l} + \omega_{3m,n,k} + e_{3m,\mathrm{arm},n,k,l} + \frac{e_{3m,n,k}}{2^{M_{3m}}} + \frac{w_{3m,n}}{2^{M_{3m}}}$$

and, for $1 \le l \le 2^{M_{3m}} - 1$,

$$x^*_{3m,n,k,l} = e''^*_{3m,n,k,l} - x^*_{3m,n,k,0} = x^*_{0,3m,n,k,l} + x^*_{1,3m,n,k,l} + w^*_{3m,n}/2^{M_{3m}},$$
$$x^*_{0,3m,n,k,l} = e^*_{0,3m,n,k,l} - (e^*_{0,3m,n,k,2^{M_{3m}}} - x^*_{0,3m,n,k})/2^{M_{3m}},$$
$$x^*_{1,3m,n,k,l} = \omega^*_{3m,n,k}/(2^{M_{3m}}(2^{M_{3m}} - 1)) - \omega^*_{3m,n}/2^{2M_{3m}};$$
$$x_{3m,n,k,2^{M_{3m}}} = e''_{3m,n,k,2^{M_{3m}}} + x_{3m,n,k} + e''_{3m,\mathrm{brd},n,k}$$
$$= e_{3m,n,k,2^{M_{3m}}} + \omega_{3m,n} + e_{3m,\mathrm{arm},n,k,2^{M_{3m}}} + \frac{e_{3m,n,k}}{2^{M_{3m}}} + \frac{w_{3m,n}}{2^{M_{3m}}} + e_{3m,\mathrm{brd},n,k} + \omega'_{3m,n},$$
$$x^*_{3m,n,k,2^{M_{3m}}} = e''^*_{3m,n,k,2^{M_{3m}}} - x^*_{3m,n,k,0}$$
$$= e''^*_{3m,n,k,2^{M_{3m}}}\left(1 - \frac{1}{2^{M_{3m}}}\right) - \frac{\omega^*_{3m,n,k}}{2^{M_{3m}}} + x^*_{3m,n,k}$$
$$= x^*_{0,3m,n,k,2^{M_{3m}}} + x^*_{1,3m,n,k,2^{M_{3m}}} + \frac{w^*_{3m,n}}{2^{M_{3m}}},$$
$$x^*_{0,3m,n,k,2^{M_{3m}}} = e^*_{0,3m,n,k,2^{M_{3m}}}\left(1 - \frac{1}{2^{M_{3m}}}\right) + x^*_{0,3m,n,k}$$
$$x^*_{1,3m,n,k,2^{M_{3m}}} = \frac{\omega^*_{3m,n}}{2^{M_{3m}}}\left(1 - \frac{1}{2^{M_{3m}}}\right) - \frac{\omega^*_{3m,n,k}}{2^{M_{3m}}};$$
$$x_{3m,n,k,0,0} = \sum_{g=1}^{2^{M_{3m}}} e_{3m,n,k,0,g},\ x^*_{3m,n,k,0,0} = \frac{1}{2^{M_{3m}}}\sum_{g=1}^{2^{M_{3m}}} e^*_{3m,n,k,0,g} - x^*_{3m,n,k,0},$$
$$x_{3m,n,k,0,l} = e_{3m,n,k,0,l} + x_{3m,n,k,0}/2^{M_{3m}},$$
$$x^*_{3m,n,k,0,l} = e^*_{3m,n,k,0,l} - x^*_{3m,n,k,0,0};$$
$$e'''_{3m,\mathrm{brd},n,k} = e''_{3m,\mathrm{brd},n,k},$$
$$e'''^*_{3m,\mathrm{brd},n,k} = e''^*_{3m,\mathrm{brd},n,k} - x^*_{3m,n,k,2^{M_{3m}}} = e'''^*_{0,3m,\mathrm{brd},n,k} + e'''^*_{1,3m,\mathrm{brd},n,k} - \frac{w^*_{3m,n}}{2^{M_{3m}}},$$
$$e'''^*_{0,3m,\mathrm{brd},n,k} = e''^*_{0,3m,\mathrm{brd},n,k} - x^*_{0,3m,n,k,2^{M_{3m}}},$$
$$e'''^*_{1,3m,\mathrm{brd},n,k} = -\frac{\omega^*_{3m,n}}{2^{M_{3m}}}\left(1 - \frac{1}{2^{M_{3m}}}\right) + \frac{\omega^*_{3m,n,k}}{2^{M_{3m}}} + \frac{\omega'^*_{3m,n}}{2^{M_{3m}}}.$$

STEP 6. Now we set

$$(x_n, x^*_n)_{n=q(3m)+1}^{q(3m+1)} = (((x_{3m,n,k,l}, x^*_{3m,n,k,l})_{l=1}^{2^{M_{3m}}})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}} = (x_{3m,g}, x^*_{3m,g})_{g=1}^{G_{3m}};$$
$$((e'_{3m,\mathrm{arm},n,j}, e'^*_{3m,\mathrm{arm},n,j})_{j=1}^{J_{3m,\mathrm{arm}}})_{n=1}^{P_{3m}} = (e'_{3m,\mathrm{arm},g}, e'^*_{3m,\mathrm{arm},g})_{g=1}^{G_{3m}}$$
$$= (e_{3m,\mathrm{arm},g}, e^*_{3m,\mathrm{arm},g} - x^*_{3m,g})_{g=1}^{G_{3m}};$$
$$(x_{0,0,0,3m,g}, x^*_{0,0,0,3m,g})_{g=1}^{G_{0,0,0,3m}} = (e_{3m,0,n}, e^*_{3m,0,n})_{n=1}^{M_{3m,0}}$$
$$\cup\,(x_{0,0,3m,g}, x^*_{0,0,3m,g})_{g=1}^{G_{0,0,3m}} \cup (e'_{3m,\mathrm{arm},g}, e'^*_{3m,\mathrm{arm},g})_{g=1}^{G_{3m}};$$

$$\begin{aligned}&(x_{0,0,3m,g}, x^*_{0,0,3m,g})_{g=1}^{G_{0,0,3m}}\\ &\quad = ((\omega_{3m,n,0}, \omega^*_{3m,n,0}) \cup (\omega_{3m,n,k,0}, \omega^*_{3m,n,k,0})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}}\\ &\quad\quad \cup (x_{3m,n,0}, x^*_{3m,n,0})_{n=1}^{P_{3m}} \cup (((x_{3m,n,k,0,l}, x^*_{3m,n,k,0,l})_{l=0}^{2^{M_{3m}}})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}};\end{aligned}$$

$$\begin{aligned}&(u'_{3m+1,s}, u'^*_{3m+1,s})_{s=A_{3m+1}-A'_{3m+1}+1}^{A_{3m+1}}\\ &\quad = ((\omega'_{3m,n,0}, \omega'^*_{3m,n,0}) \cup (e'''_{3m,\mathrm{brd},n,k}, e'''^*_{3m,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}};\end{aligned}$$

$$(u'_{3m+1,s}, u'^*_{3m+1,s})_{s=A_{3m+1}/2+1}^{A_{3m+1}/2+M_{3m,\mathrm{arm},0}} = (e_{3m,\mathrm{arm},0,n}, e^*_{3m,\mathrm{arm},0,n})_{n=1}^{M_{3m,\mathrm{arm},0}};$$

$$(x_{0,3m,g}, x^*_{0,3m,g})_{g=1}^{G_{0,3m}} = (x_{3m,g}, x^*_{3m,g})_{g=1}^{G_{3m}} \cup (x_{0,0,3m,g}, x^*_{0,0,3m,g})_{g=1}^{G_{0,0,3m}};$$

$$\begin{aligned}B_5(3m) &= (x_n, x^*_n)_{n=1}^{q(3m)} \cup (x_{0,3m,g}, x^*_{0,3m,g})_{g=1}^{G_{0,3m}}\\ &\quad \cup (u'_{3m+1,s}, u'^*_{3m+1,s})_{s=A_{3m+1}/2+1}^{A_{3m+1}/2+M_{3m,\mathrm{arm},0}}\\ &\quad \cup (u'_{3m+1,s}, u'^*_{3m+1,s})_{s=A_{3m+1}-A'_{3m+1}+1}^{A_{3m+1}}.\end{aligned}$$

Till now we passed from $B_4(3m)$ of Step 5 to the biorthogonal system

$$B_5(3m) \cup (v'_{3m+1,n}, v'^*_{3m+1,n})_{n=1}^{Q''_{3m+1}}.$$

At this point, by means of the procedure of Construction II and of Lemma 12, where we replace $(y_n, y^*_n)_{n=1}^{Q(m)}$ by $\widetilde{B}_5(3m)$ (where $\widetilde{B}_5(3m)$ is the system $B_5(3m)$ when we remove all the elements of

$$((\omega'_{3m,n}, \omega'^*_{3m,n}) \cup (e_{3m,\mathrm{brd},n,k}, e^*_{3m,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}})$$

and $m+1$ by $3m$, we define

$$(u''_{3m+1,s}, u''^*_{3m+1,s})_{s=1}^{P''_{3m+1}} \cup (v'_{3m+2,n}, v'^*_{3m+2,n})_{n=1}^{Q''_{3m+2}}$$

(where $(u''_{3m+1,s}, u''^*_{3m+1,s})_{s=1}^{Q''_{3m+1}} = (v'_{3m+1,s}, v'^*_{3m+1,s})_{s=1}^{Q''_{3m+1}}$) where $(v'_{3m+2,n})_{n=1}^{Q''_{3m+2}}$ will be used only in the construction of $(x_n, x^*_n)_{n=q(3m+2)+1}^{q(3m+3)}$; finally we set

$$\begin{aligned}B_6(3m) &= (x_n, x^*_n)_{n=1}^{q(3m+1)} \cup (u'_{3m+1,s}, u'^*_{3m+1,s})_{s=1}^{A''_{3m+1}}\\ &\quad \cup (u'_{3m+1,s}, u'^*_{3m+1,s})_{s=A_{3m+1}/2+1}^{A_{3m+1}/2+M_{3m,\mathrm{arm},0}}\\ &\quad \cup (u'_{3m+1,s}, u'^*_{3m+1,s})_{s=A_{3m+1}-A'_{3m+1}+1}^{A_{3m+1}} \cup (v'_{3m+2,n}, v'^*_{3m+2,n})_{n=1}^{Q''_{3m+2}};\end{aligned}$$

$$(u'_{3m+1,s}, u'^*_{3m+1,s})_{s=1}^{A''_{3m+1}} = (x_{0,0,0,3m,g}, x^*_{0,0,0,3m,g})_{g=1}^{G_{0,0,0,3m}} \cup (u''_{3m+1,s}, u''^*_{3m+1,s})_{s=1}^{P''_{3m+1}}.$$

We point out that, by the end of Step 2 and of Step 3 and by Lemma 10, $\|x_n\| < 10$ and $\|x^*_n\| < 7$ for $q(3m)+1 \le n \le q(3m+1)$. We also set, for $q(3m)+1 \le n \le q(3m+1)$,

$$\begin{gathered}x_n = x'_n + \widetilde{x}_n + x_{\mathrm{arm},n} + x_{\mathrm{brd},n}, \qquad \widetilde{x}_n = x''_n + x'''_n,\\ x'_n \in \operatorname{span}(e'''_{3m,i} : 1 \le i \le S_{3m}, e'''_{3m,i} \notin \operatorname{span}((e_{3m,\mathrm{aux},s,t})_{t=1}^{2^{2B_{3m}}})_{s=1}^{A_{3m}}),\\ x''_n \in \operatorname{span}((e_{3m,\mathrm{aux},s,t})_{t=1}^{2^{2B_{3m}}})_{s=1}^{A_{3m}}), x'''_n \in \operatorname{span}(u'_{3m,s})_{s=1}^{A_{3m}},\\ x_{\mathrm{arm},n} \in \operatorname{span}((e_{3m,\mathrm{arm},r,j})_{j=1}^{J_{3m,\mathrm{arm}}})_{r=1}^{P_{3m}}),\\ x_{\mathrm{brd},n} \in \operatorname{span}(\omega'_{3m,r} \cup (e_{3m,\mathrm{brd},k,r})_{k=1}^{2^{M_{3m}}})_{r=1}^{P_{3m}}.\end{gathered}$$

We set, for each $\overline{x} \in X$ with $\|\overline{x}\| = 1$,

$$\overline{a}_{3m} = \overline{a}_{3m}(\overline{x}) = \max(\max(|x_n^*(\overline{x})| : q(3m)+1 \le n \le q(3m+1)), \max(|x_{0,3m,g}^*(\overline{x})| : 1 \le g \le G_{0,3m}), \max(|e_{3m,i}^{\prime\prime\prime *}(\overline{x})| : 1 \le i \le S_{3m})).$$

Finally, we set $B_1(3m+1) = B_6(3m)$ and we are ready for Subconstruction III.2.

Subconstruction III.2 (SC III.2, construction of the regularization block). We turn to the construction of $(x_n, x_n^*)_{n=q(3m+1)+1}^{q(3m+2)}$ and we start from the biorthogonal system $B_1(3m+1)$ defined in SC III.1; we will proceed through four steps. We set

$$A_{3m+1} = 4^{q(3m+1)(A'_{3m+1}+M_{3m,\mathrm{arm},0}+A''_{3m+1})},$$

$$K_{3m+1} = \max(\|u_{3m+1,s}^{\prime *}\| : 1 \le s \le A''_{3m+1}, A_{3m+1}/2+1 \le s \le A_{3m+1}/2+M_{3m,\mathrm{arm},0}, A_{3m+1}-A'_{3m+1}+1 \le s \le A_{3m+1}),$$

$$B_{3m+1} > A_{3m+1}2^{K_{3m+1}A_{3m+1}}, \qquad P_{3m+1} = A_{3m+1}(2^{2B_{3m+1}}+1).$$

Step 1. By the procedure of Lemma 10 we pass from $B_1(3m+1)$ to the biorthogonal system $B_1(3m+1) \cup (\widehat{e}_{3m+1,n}, \widehat{e}_{3m+1,n}^*)_{n=1}^{S'_{3m+1}+M'_{3m+1,0}}$ where $(\widehat{e}_{3m+1,n})_{n=1}^{S'_{3m+1}+M'_{3m+1,0}}$ is 1-equivalent to the natural basis of $l_1^{S'_{3m+1}+M'_{3m+1,0}}$, moreover with

$$\|y+e\| \ge \max(\|y\|, \|e\|/2)$$

for $y \in \mathrm{span}(y_{3m+1,n})_{n=1}^{Q(3m+1)}$, $e \in \mathrm{span}(\widehat{e}_{3m+1,n})_{n=1}^{S'_{3m+1}+M'_{3m+1,0}}$, and

$$(y_{3m+1,n})_{n=1}^{Q(3m+1)} = (x_n)_{n=1}^{q(3m+1)} \cup (u'_{3m+1,s})_{s=1}^{A''_{3m+1}} \cup (u'_{3m+1,s})_{s=A_{3m+1}/2+1}^{A_{3m+1}/2+M_{3m,\mathrm{arm},0}} \cup (u'_{3m+1,s})_{s=A_{3m+1}-A'_{3m+1}+1}^{A_{3m+1}} \cup (v'_{3m+2,n})_{n=1}^{Q''_{3m+2}}.$$

Then, according to the procedure of Lemma 10, there are $(\widehat{v}_{0,3m+1,n})_{n=S'_{3m+1}+1}^{S'_{3m+1}+M'_{3m+1,0}}$ in $\mathrm{span}(\widehat{e}_{3m+1,n})_{n=1}^{S'_{3m+1}}$ and $(\widehat{e}_{3m+1,n}^{\prime\prime *})_{n=1}^{S'_{3m+1}}$ in X^* so that we have the biorthogonal system

$$(\widehat{e}_{3m+1,n}, \widehat{e}_{3m+1,n}^{\prime\prime *})_{n=1}^{S'_{3m+1}} \cup (e_{3m+1,0,n}, e_{3m+1,0,n}^*)_{n=1}^{M'_{3m+1,0}},$$

$$(e_{3m+1,0,n}, e_{3m+1,0,n}^*)_{n=1}^{M'_{3m+1,0}} = (\widehat{e}_{3m+1,n} - \widehat{v}_{0,3m+1,n}, \widehat{e}_{3m+1,n}^*)_{n=S'_{3m+1}+1}^{S'_{3m+1}+M'_{3m+1,0}};$$

$$E'_{0,3m+1} = \mathrm{span}(e_{3m+1,0,n})_{n=1}^{M'_{3m+1,0}};$$

$$S'_{3m+1} = 2^{M'_{3m+1}} + (A_{3m+1} - (A'_{3m+1} + A''_{3m+1} + M_{3m,\mathrm{arm},0})) + P_{3m+1} + P_{3m};$$

$$M'_{3m+1} = A_{3m+1}2^{2B_{3m+1}} + P_{3m+1}2^{M_{3m+1}}(2^{4Q_{3m+1}}+1) + P_{3m}2^{M_{3m}}.$$

Then $(\widehat{e}_{3m+1,n} + E'_{0,3m+1})_{n=1}^{S'_{3m+1}}$ is 1-equivalent to the natural basis of $l_\infty^{S'_{3m+1}}$ and we can suppose that the method at the end of Step 1 of SC III.1 continues to work.

Now, by the procedures of the proof of Theorem 11 and of Lemma 10, we pass from $(\widehat{e}_{3m+1,n}, \widehat{e}_{3m+1,n}^{\prime\prime *})_{n=1}^{S'_{3m+1}}$ to the biorthogonal system

$$(\widehat{e}'_{3m+1,n}, \widehat{e}_{3m+1,n}^{\prime *})_{n=1}^{M'_{3m+1}} \cup (e_{3m+1,0,n}, e_{3m+1,0,n}^*)_{n=M'_{3m+1,0}+1}^{M_{3m+1,0}} \cup (e_{3m+1,n}^{\prime\prime\prime}, e_{3m+1,n}^{\prime\prime\prime *})_{n=M'_{3m+1}+1}^{S_{3m+1}};$$

$$M_{3m+1,0} = M'_{3m+1,0} + (2^{M'_{3m+1}} - M'_{3m+1}); \quad S_{3m+1} = S'_{3m+1} - (2^{M'_{3m+1}} - M'_{3m+1});$$

$$E_{3m+1,0} = \operatorname{span}(e_{3m+1,0,n})_{n=1}^{M_{3m+1,0}};$$

$$\begin{aligned}(\widehat{e}'_{3m+1,n}, \widehat{e}'^{*}_{3m+1,n})_{n=1}^{M'_{3m+1}} &= ((e'_{3m+1,\mathrm{aux},s,t}, e'^{*}_{3m+1,\mathrm{aux},s,t})_{t=1}^{2^{2B_{3m+1}}})_{s=1}^{A_{3m+1}} \\ &\cup ((e'_{3m+1,n,k}, e'^{*}_{3m+1,n,k})_{k=1}^{2^{M_{3m+1}}})_{n=1}^{P_{3m+1}} \\ &\cup (((e'_{3m+1,n,k,l}, e'^{*}_{3m+1,n,k,l})_{l=1}^{2^{4Q_{3m+1}}})_{k=1}^{2^{M_{3m+1}}})_{n=1}^{P_{3m+1}} \\ &\cup ((e'_{3m,\mathrm{brd},n,k}, e'^{*}_{3m,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}};\end{aligned}$$

$$\begin{aligned}(e'''_{3m+1,n}, e'''^{*}_{3m+1,n})_{n=M'_{3m+1}+1}^{S_{3m+1}} &= (\widehat{e}_{3m+1,n}, \widehat{e}''^{*}_{3m+1,n})_{n=2^{M'_{3m+1}}+1}^{S'_{3m+1}} \\ &= (e_{3m+1,\mathrm{aux},s}, e^{*}_{3m+1,\mathrm{aux},s})_{s=A''_{3m+1}+1}^{A_{3m+1}/2} \\ &\quad \cup (e_{3m+1,\mathrm{aux},s}, e^{*}_{3m+1,\mathrm{aux},s})_{s=A_{3m+1}/2+M_{3m,\mathrm{arm},0}+1}^{A_{3m+1}-A'_{3m+1}} \\ &\quad \cup (\omega'_{3m+1,n}, \omega'^{*}_{3m+1,n})_{n=1}^{P_{3m+1}} \cup (\omega'_{3m,n}, \omega'^{*}_{3m,n})_{n=1}^{P_{3m}};\end{aligned}$$

so that setting, for $1 \le s \le A_{3m+1}$,

$$e_{3m+1,\mathrm{aux},s,1} = e'_{3m+1,\mathrm{aux},s,1} \quad \text{and} \quad e_{3m+1,\mathrm{aux},s,t} = e'_{3m+1,\mathrm{aux},s,t} - e'_{3m+1,\mathrm{aux},s,t-1}$$

for $2 \le t \le 2^{2B_{3m+1}}$, $(e'_{3m+1,\mathrm{aux},s,t})_{t=1}^{2^{2B_{3m+1}}}$, $(e_{3m+1,\mathrm{aux},s,t})_{t=1}^{2^{2B_{3m+1}}}$ and $E_{3m+1,0}$ correspond to $((e'_{r,n})_{n=1}^{2^N})_{r=1}^{2^R}$, $((e_{r,n})_{n=1}^{2^N})_{r=1}^{2^R}$ and E'_0 of (39.3) and (39.4) of Theorem 11 with $\{n\}_{n=1}^{2^N}$ and $\{r\}_{r=1}^{2^R}$ replaced respectively by $(t)_{t=1}^{2^{2B_{3m+1}}}$ and $\{1\}$. Moreover for $1 \le n \le P_{3m+1}$, if we set for $1 \le k \le 2^{M_{3m+1}}$, $e_{3m+1,n,k,1} = e'_{3m+1,n,k,1}$ and $e_{3m+1,n,k,l} = e'_{3m+1,n,k,l} - e'_{3m+1,n,k,l-1}$ for $2 \le l \le 2^{4Q_{3m+1}} - 1$, then $((e'_{3m+1,n,k,l})_{l=1}^{2^{4Q_{3m+1}}-1})_{k=1}^{2^{M_{3m+1}}}$, $((e_{3m+1,n,k,l})_{l=1}^{2^{4Q_{3m+1}}-1})_{k=1}^{2^{M_{3m+1}}}$ and $E_{3m+1,0}$ correspond to $((e'_{r,n})_{n=1}^{2^N})_{r=1}^{2^R}$, $((e_{r,n})_{n=1}^{2^N})_{r=1}^{2^R}$ and E'_0 of (39.3) and (39.4) of Theorem 11 with $\{n\}_{n=1}^{2^N}$ and $\{r\}_{r=1}^{2^R}$ replaced respectively by $(l)_{l=1}^{2^{4Q_{3m+1}}-1}$ and $(k)_{k=1}^{2^{M_{3m+1}}}$. Moreover for $1 \le n \le P_{3m+1}$, if we set $e_{3m+1,n,1} = e'_{3m+1,n,1}$ and $e_{3m+1,n,1,2^{4Q_{3m+1}}} = e'_{3m+1,n,1,2^{4Q_{3m+1}}}$ and $e_{3m,\mathrm{brd},n,1} = e'_{3m,\mathrm{brd},n,1}$, moreover

$$\begin{gathered} e_{3m+1,n,k} = e'_{3m+1,n,k} - e'_{3m+1,n,k-1}, \\ e_{3m+1,n,k,2^{4Q_{3m+1}}} = e'_{3m+1,n,k,2^{4Q_{3m+1}}} - e'_{3m+1,n,k-1,2^{4Q_{3m+1}}} \qquad \text{for } 2 \le k \le 2^{M_{3m+1}}, \\ e_{3m,\mathrm{brd},n,k} = e'_{3m,\mathrm{brd},n,k} - e'_{3m,\mathrm{brd},n,k-1} \qquad \text{for } 2 \le k \le 2^{M_{3m}}, \end{gathered}$$

then

$$\begin{gathered} (e'_{3m+1,n,k})_{k=1}^{2^{M_{3m+1}}} \quad \text{and} \quad (e_{3m+1,n,k})_{k=1}^{2^{M_{3m+1}}}, \\ (e'_{3m+1,n,k,2^{4Q_{3m+1}}})_{k=1}^{2^{M_{3m+1}}} \quad \text{and} \quad (e_{3m+1,n,k,2^{4Q_{3m+1}}})_{k=1}^{2^{M_{3m+1}}}, \\ (e'_{3m+1,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m+1}}} \quad \text{and} \quad (e_{3m+1,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m+1}}} \end{gathered}$$

and $E_{3m+1,0}$ correspond to $((e'_{r,n})_{n=1}^{2^N})_{r=1}^{2^R}$, $((e_{r,n})_{n=1}^{2^N})_{r=1}^{2^R}$ and E'_0 of (39.3) and (39.4) of Theorem 11 with $\{n\}_{n=1}^{2^N}$ and $\{r\}_{r=1}^{2^R}$ replaced respectively by $(k)_{k=1}^{2^{M_{3m+1}}}$ $((k)_{k=1}^{2^{M_{3m}}}$ for the last one) and (1). Then we pass to the biorthogonal system

$$B_2(3m+1) = B_1(3m+1) \cup (e'''_{3m+1,n}, e'''^*_{3m+1,n})_{n=1}^{S_{3m+1}} \cup (e_{3m+1,0,n}, e^*_{3m+1,0,n})_{n=1}^{M_{3m+1,0}};$$

$$\begin{aligned}(e'''_{3m+1,n}, e'''^*_{3m+1,n})_{n=1}^{M'_{3m+1}} &= ((e_{3m+1,\mathrm{aux},s,t}, e^*_{3m+1,\mathrm{aux},s,t})_{t=1}^{2^{2B_{3m+1}}})_{s=1}^{A_{3m+1}} \\ &\cup ((e_{3m+1,n,k}, e^*_{3m+1,n,k})_{k=1}^{2^{M_{3m+1}}})_{n=1}^{P_{3m+1}} \\ &\cup (((e_{3m+1,n,k,l}, e^*_{3m+1,n,k,l})_{l=1}^{2^{4Q_{3m+1}}})_{k=1}^{2^{M_{3m+1}}})_{n=1}^{P_{3m+1}} \\ &\cup ((e_{3m,\mathrm{brd},n,k}, e^*_{3m,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}}.\end{aligned}$$

In particular, by (30.3) of Theorem 8 and by Theorem 11,

$$\begin{aligned}&\Big\| \sum_{n=1}^{P_{3m+1}} \sum_{k=1}^{2^{M_{3m+1}}} \sum_{l=1}^{2^{4Q_{3m+1}}-1} a_{n,k,l} e_{3m+1,n,k,l} + E_{3m+1,0} \Big\| \\ &\quad = \max\Big(\Big\| \sum_{k=1}^{2^{M_{3m+1}}} a_{n,k,l} e_{3m+1,n,k,l} + E_{3m+1,0} \Big\| : 1 \le n \le P_{3m+1}, 1 \le l \le 2^{4Q_{3m+1}} - 1 \Big)\end{aligned}$$

for each sequence $((a_{n,k,l})_{l=1}^{2^{4Q_{3m+1}}-1})_{k=1}^{2^{M_{3m+1}}})_{n=1}^{P_{3m+1}}$ of numbers. Analogously to the end of Step 2 of Subconstruction III.1, $\|e'''^*_{3m+1,n}\| \le 2$ for $1 \le n \le S_{3m+1}$.

STEP 2. Now we use a biorthogonal system

$$((\omega_{3m+1,n}, \omega^*_{3m+1,n}) \cup (e_{3m+1,\mathrm{brd},n,k}, e^*_{3m+1,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m+1}}})_{n=1}^{P_{3m+1}}$$

which will be defined in SC III.3, then we pass to

$$(\omega_{3m+1,n,0}, \omega^*_{3m+1,n,0}) \cup (e''_{3m+1,\mathrm{brd},n,k}, e''^*_{3m+1,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m+1}}}$$

where, for $1 \le k \le 2^{M_{3m+1}}$,

$$\omega_{3m+1,n,0} = \sum_{f=1}^{2^{M_{3m+1}}} e_{3m+1,\mathrm{brd},n,f},$$

$$\omega^*_{3m+1,n,0} = \frac{1}{2^{M_{3m+1}}} \sum_{f=1}^{2^{M_{3m+1}}} e^*_{3m+1,\mathrm{brd},n,f} - \omega^*_{3m+1,n}/2^{M_{3m+1}},$$

$$e''_{3m+1,\mathrm{brd},n,k} = e_{3m+1,\mathrm{brd},n,k} + \omega_{3m+1,n},$$

$$\begin{aligned}e''^*_{3m+1,\mathrm{brd},n,k} &= e^*_{3m+1,\mathrm{brd},n,k} - \omega^*_{3m+1,n,0} \\ &= \Big(e^*_{3m+1,\mathrm{brd},n,k} - \frac{1}{2^{M_{3m+1}}} \sum_{f=1}^{2^{M_{3m+1}}} e^*_{3m+1,\mathrm{brd},n,f} \Big) + \frac{\omega^*_{3m+1,n}}{2^{M_{3m+1}}};\end{aligned}$$

and the same from $(\omega'_{3m+1,n}, \omega'^*_{3m+1,n}) \cup (e_{3m+1,n,k,2^{4Q_{3m+1}}}, e^*_{3m+1,n,k,2^{4Q_{3m+1}}})_{k=1}^{2^{M_{3m+1}}}$ to $(\omega'_{3m+1,n,0}, \omega'^*_{3m+1,n,0}) \cup (e''_{3m+1,n,k,2^{4Q_{3m+1}}}, e''^*_{3m+1,n,k,2^{4Q_{3m+1}}})_{k=1}^{2^{M_{3m+1}}}$ where, for $1 \le k \le 2^{M_{3m+1}}$,

$$\omega'_{3m+1,n,0} = \sum_{f=1}^{2^{M_{3m+1}}} e_{3m+1,n,f,2^{4Q_{3m+1}}},$$

$$\omega'^*_{3m+1,n,0} = \frac{1}{2^{M_{3m+1}}} \sum_{f=1}^{2^{M_{3m+1}}} e^*_{3m+1,n,f,2^{4Q_{3m+1}}} - \omega'^*_{3m+1,n}/2^{M_{3m+1}},$$
$$e''_{3m+1,n,k,2^{4Q_{3m+1}}} = e_{3m+1,n,k,2^{4Q_{3m+1}}} + \omega'_{3m+1,n},$$
$$\begin{aligned} e''^*_{3m+1,n,k,2^{4Q_{3m+1}}} &= e^*_{3m+1,n,k,2^{4Q_{3m+1}}} - \omega'^*_{3m+1,n,0} \\ &= \Big(e^*_{3m+1,n,k,2^{4Q_{3m+1}}} - \frac{1}{2^{M_{3m+1}}} \sum_{f=1}^{2^{M_{3m+1}}} e^*_{3m+1,n,f,2^{4Q_{3m+1}}}\Big) + \frac{\omega'^*_{3m+1,n}}{2^{M_{3m+1}}}. \end{aligned}$$

Setting

$$(e_{3m+1,\mathrm{aux},s}, e^*_{3m+1,\mathrm{aux},s})_{s=A''_{3m+1}+1}^{A_{3m+1}/2} = (u'_{3m+1,s}, u'^*_{3m+1,s})_{s=A''_{3m+1}+1}^{A_{3m+1}/2}$$

and

$$\begin{aligned} &(e_{3m+1,\mathrm{aux},s}, e^*_{3m+1,\mathrm{aux},s})_{s=A_{3m+1}/2+M_{3m,\mathrm{arm},0}+1}^{A_{3m+1}-A'_{3m+1}} \\ &\qquad\qquad = (u'_{3m+1,s}, u'^*_{3m+1,s})_{s=A_{3m+1}/2+M_{3m,\mathrm{arm},0}+1}^{A_{3m+1}-A'_{3m+1}}, \end{aligned}$$

from the expression of $B_1(3m+1) = B_6(3m)$ of SC III.1 we get the biorthogonal system $(u'_{3m+1,s}, u'^*_{3m+1,s})_{s=1}^{A_{3m+1}} \cup ((e_{3m+1,\mathrm{aux},s,t}, e^*_{3m+1,\mathrm{aux},s,t})_{t=1}^{2^{2B_{3m+1}}})_{s=1}^{A_{3m+1}}$ from which at first we pass from $(u'_{3m+1,s}, u'^*_{3m+1,s})_{s=1}^{A_{3m+1}}$ to $(u'_{3m+1,s}/\|u'_{3m+1,s}\|, \|u'_{3m+1,s}\| u'^*_{3m+1,s})_{s=1}^{A_{3m+1}}$ which we call $(u'_{3m+1,s}, u'^*_{3m+1,s})_{s=1}^{A_{3m+1}}$ again; then we pass to $(u_{3m+1,n}, u^*_{3m+1,n})_{n=1}^{P_{3m+1}} = ((u_{3m+1,s,t}, u^*_{3m+1,s,t})_{t=0}^{2^{2B_{3m+1}}})_{s=1}^{A_{3m+1}}$ where, for $1 \le s \le A_{3m+1}$ and $1 \le t \le 2^{2B_{3m+1}}$,

$$u_{3m+1,s,0} = \sum_{j=1}^{2^{2B_{3m+1}}} e_{3m+1,\mathrm{aux},s,j}, \qquad u^*_{3m+1,s,0} = \sum_{j=1}^{2^{2B_{3m+1}}} \frac{e^*_{3m+1,\mathrm{aux},s,j}}{2^{2B_{3m+1}}} - \frac{u'^*_{3m+1,s}}{2^{B_{3m+1}}},$$
$$u_{3m+1,s,t} = e_{3m+1,\mathrm{aux},s,t} + \frac{u'_{3m+1,s}}{2^{B_{3m+1}}},$$
$$u^*_{3m+1,s,t} = e^*_{3m+1,\mathrm{aux},s,t} - u^*_{3m+1,s,0} = \Bigg(e^*_{3m+1,\mathrm{aux},s,t} - \sum_{j=1}^{2^{2B_{3m+1}}} \frac{e^*_{3m+1,\mathrm{aux},s,j}}{2^{2B_{3m+1}}}\Bigg) + \frac{u'^*_{3m+1,s}}{2^{B_{3m+1}}}.$$

Then we pass to the generating biorthogonal system

$$(w_{3m+1,n}, w^*_{3m+1,n})_{n=1}^{P_{3m+1}} = ((w_{3m+1,s,t}, w^*_{3m+1,s,t})_{t=0}^{2^{2B_{3m+1}}})_{s=1}^{A_{3m+1}}$$

and we specify that, if we set $Q_{0,3m+1} = \max(\|w^*_{3m+1,n}\| : 1 \le n \le P_{3m+1})$, then $Q_{3m+1} > 4^{2P_{3m+1}Q_{0,3m+1}}$ while $M_{3m+1} > 4^{2Q_{3m+1}+4\cdot 2^{4Q_{3m+1}}}$.

Finally, we pass to the modified generating biorthogonal system

$$(v_{3m+1,n}, v^*_{3m+1,n})_{n=1}^{P_{3m+1}}$$

where, for $1 \le n \le P_{3m+1}/2$,

$$\begin{aligned} v_{3m+1,2n-1} &= w_{3m+1,2n} + w_{3m+1,2n-1}/2^{M_{3m+1}+Q_{3m+1}}, \\ v^*_{3m+1,2n-1} &= (w^*_{3m+1,2n} + 2^{M_{3m+1}+Q_{3m+1}} w^*_{3m+1,2n-1})/2, \\ v_{3m+1,2n} &= w_{3m+1,2n} - w_{3m+1,2n-1}/2^{M_{3m+1}+Q_{3m+1}}, \\ v^*_{3m+1,2n} &= \frac{w^*_{3m+1,2n} - 2^{M_{3m+1}+Q_{3m+1}} w^*_{3m+1,2n-1}}{2}. \end{aligned}$$

In the next steps we will work with the following biorthogonal system:

$$
\begin{aligned}
B_3(3m+1) = {} & (x_n, x_n^*)_{n=1}^{q(3m+1)} \cup ((v_{3m+1,n}, v_{3m+1,n}^*)\\
& \cup ((e_{3m+1,n,k}, e_{3m+1,n,k}^*) \cup (e_{3m+1,n,k,l}, e_{3m+1,n,k,l}^*)_{l=1}^{2^{4Q_{3m+1}}-1}\\
& \cup (e''_{3m+1,n,k,2^{4Q_{3m+1}}}, e''^*_{3m+1,n,k,2^{4Q_{3m+1}}}))_{k=1}^{2^{M_{3m+1}}}\\
& \cup (e''_{3m+1,\mathrm{brd},n,k}, e''^*_{3m+1,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m+1}}})\\
& \cup ((\omega_{3m+1,n,0}, \omega_{3m+1,n,0}^*) \cup (\omega'_{3m+1,n,0}, \omega'^*_{3m+1,n,0})_{n=1}^{P_{3m+1}}\\
& \cup (e_{3m+1,0,n}, e_{3m+1,0,n}^*)_{n=1}^{M_{3m+1,0}} \cup (v'_{3m+2,n}, v'^*_{3m+2,n})_{n=1}^{Q''_{3m+2}}.
\end{aligned}
$$

STEP 3. Let us fix n with $1 \le n \le P_{3m+1}$; for $1 \le k \le 2^{M_{3m+1}}$ and $1 \le l \le 2^{4Q_{3m+1}} - 1$, we set

$$
\begin{aligned}
x_{3m+1,n,0} &= \sum_{f=1}^{2^{M_{3m+1}}} e_{3m+1,n,f},\\
x_{3m+1,n,0}^* &= \sum_{f=1}^{2^{M_{3m+1}}} \frac{e_{3m+1,n,f}^*}{2^{M_{3m+1}}} - \frac{v_{3m+1,n}^*}{2^{M_{3m+1}+2Q_{3m+1}}};\\
x_{3m+1,n,k} &= e_{3m+1,n,k} + 2^{2Q_{3m+1}} v_{3m+1,n},\\
x_{3m+1,n,k}^* &= e_{3m+1,n,k}^* - x_{3m+1,n,0}^*\\
&= \left(e_{3m+1,n,k}^* - \sum_{f=1}^{2^{M_{3m+1}}} \frac{e_{3m+1,n,f}^*}{2^{M_{3m+1}}}\right) + \frac{v_{3m+1,n}^*}{2^{M_{3m+1}+2Q_{3m+1}}};\\
x_{3m+1,n,k,0} &= \sum_{g=1}^{2^{4Q_{3m+1}}-1} e_{3m+1,n,k,g} + e''_{3m+1,n,k,2^{4Q_{3m+1}}},\\
x_{3m+1,n,k,0}^* &= \frac{1}{2^{4Q_{3m+1}}}\left(\sum_{g=1}^{2^{4Q_{3m+1}}-1} e_{3m+1,n,k,g}^* + e''^*_{3m+1,n,k,2^{4Q_{3m+1}}}\right)\\
&\quad - \frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{2Q_{3m+1}}} x_{3m+1,n,k}^*;\\
x_{3m+1,n,k,l} &= e_{3m+1,n,k,l} + \frac{x_{3m+1,n,k}}{2^{P_{3m+1}Q_{0,3m+1}} 2^{2Q_{3m+1}}}\\
&= e_{3m+1,n,k,l} + \frac{e_{3m+1,n,k}}{2^{P_{3m+1}Q_{0,3m+1}} 2^{2Q_{3m+1}}} + \frac{v_{3m+1,n}}{2^{P_{3m+1}Q_{0,3m+1}}},\\
x_{3m+1,n,k,l}^* &= e_{3m+1,n,k,l}^* - x_{3m+1,n,k,0}^*\\
&= \left(e_{3m+1,n,k,l}^* - \frac{1}{2^{4Q_{3m+1}}}\left(\sum_{g=1}^{2^{4Q_{3m+1}}-1} e_{3m+1,n,k,g}^*\right.\right.\\
&\quad \left.\left. + \left(\left(e_{3m+1,n,k,2^{4Q_{3m+1}}}^* - \sum_{f=1}^{2^{M_{3m+1}}} \frac{e_{3m+1,n,f,2^{4Q_{3m+1}}}^*}{2^{M_{3m+1}}}\right) + \frac{\omega'^*_{3m+1,n}}{2^{M_{3m+1}}}\right)\right)\\
&\quad + \frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{2Q_{3m+1}}}\left(e_{3m+1,n,k}^* - \sum_{f=1}^{2^{M_{3m+1}}} \frac{e_{3m+1,n,f}^*}{2^{M_{3m+1}}}\right)\\
&\quad + \frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{M_{3m+1}+4Q_{3m+1}}} v_{3m+1,n}^*,
\end{aligned}
$$

$$\begin{aligned}
x_{3m+1,n,k,2^{4Q_{3m+1}}} &= e''_{3m+1,n,k,2^{4Q_{3m+1}}} + \frac{x_{3m+1,n,k}}{2^{P_{3m+1}Q_{0,3m+1}}2^{2Q_{3m+1}}} + e''_{3m+1,\mathrm{brd},n,k} \\
&= e_{3m+1,n,k,2^{4Q_{3m+1}}} + \omega'_{3m+1,n} + \frac{e_{3m+1,n,k}}{2^{P_{3m+1}Q_{0,3m+1}}2^{2Q_{3m+1}}} \\
&\quad + \frac{v_{3m+1,n}}{2^{P_{3m+1}Q_{0,3m+1}}} + e_{3m+1,\mathrm{brd},n,k} + \omega_{3m+1,n}, x^*_{3m+1,n,k,2^{4Q_{3m+1}}} \\
&= e''^*_{3m+1,n,k,2^{4Q_{3m+1}}} - x^*_{3m+1,n,k,0} \\
&= \left(\left(e^*_{3m+1,n,k,2^{4Q_{3m+1}}} - \sum_{f=1}^{2^{M_{3m+1}}} \frac{e^*_{3m+1,n,f,2^{4Q_{3m+1}}}}{2^{M_{3m+1}}}\right) + \frac{\omega'^*_{3m+1,n}}{2^{M_{3m+1}}}\right) \\
&\quad - \frac{1}{2^{4Q_{3m+1}}}\left(\sum_{g=1}^{2^{4Q_{3m+1}}-1} e^*_{3m+1,n,k,g}\right. \\
&\quad + \left.\left(\left(e^*_{3m+1,n,k,2^{4Q_{3m+1}}} - \sum_{f=1}^{2^{M_{3m+1}}} \frac{e^*_{3m+1,n,f,2^{4Q_{3m+1}}}}{2^{M_{3m+1}}}\right) + \frac{\omega'^*_{3m+1,n}}{2^{M_{3m+1}}}\right)\right) \\
&\quad + \frac{2^{P_{3m+1}Q_{0,m}}}{2^{2Q_{3m+1}}}\left(e^*_{3m+1,n,k} - \sum_{f=1}^{2^{M_{3m+1}}} \frac{e^*_{3m+1,n,f}}{2^{M_{3m+1}}}\right) \\
&\quad + \frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{M_{3m+1}+4Q_{3m+1}}} v^*_{3m+1,n}.
\end{aligned}$$

After this construction, $((e''_{3m+1,\mathrm{brd},n,k}, e''^*_{3m+1,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m+1}}})_{n=1}^{P_{3m+1}}$ of Step 2 becomes $((e'''_{3m+1,\mathrm{brd},n,k}, e'''^*_{3m+1,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m+1}}})_{n=1}^{P_{3m+1}}$ where, for $1 \leq n \leq P_{3m+1}$ and $1 \leq k \leq 2^{M_{3m+1}}$,

$$\begin{aligned}
e'''_{3m+1,\mathrm{brd},n,k} &= e''_{3m+1,\mathrm{brd},n,k} = e_{3m+1,\mathrm{brd},n,k} + \omega_{3m+1,n}, \\
e'''^*_{3m+1,\mathrm{brd},n,k} &= e''^*_{3m+1,\mathrm{brd},n,k} - x^*_{3m+1,n,k,2^{4Q_{3m+1}}} \\
&= e''^*_{0,3m+1,\mathrm{brd},n,k} + e''''^*_{3m+1,\mathrm{brd},n,k} + \left(-\frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{M_{3m+1}+4Q_{3m+1}}} v^*_{3m+1,n}\right),
\end{aligned}$$

$$\begin{aligned}
e''^*_{0,3m+1,\mathrm{brd},n,k} &= \left(e^*_{3m+1,\mathrm{brd},n,k} - \frac{1}{2^{M_{3m+1}}} \sum_{f=1}^{2^{M_{3m+1}}} e^*_{3m+1,\mathrm{brd},n,f}\right) \\
&\quad - \left(\left(e^*_{3m+1,n,k,2^{4Q_{3m+1}}} - \sum_{f=1}^{2^{M_{3m+1}}} \frac{e^*_{3m+1,n,f,2^{4Q_{3m+1}}}}{2^{M_{3m+1}}}\right)\left(1 - \frac{1}{2^{4Q_{3m+1}}}\right)\right. \\
&\quad + \left.\frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{2Q_{3m+1}}}\left(e^*_{3m+1,n,k} - \sum_{f=1}^{2^{M_{3m+1}}} \frac{e^*_{3m+1,n,f}}{2^{M_{3m+1}}}\right)\right),
\end{aligned}$$

$$e''''^*_{3m+1,\mathrm{brd},n,k} = \frac{\omega^*_{3m+1,n}}{2^{M_{3m+1}}} - \left(\frac{\omega'^*_{3m+1,n}}{2^{M_{3m+1}}}\left(1 - \frac{1}{2^{4Q_{3m+1}}}\right) - \frac{1}{2^{4Q_{3m+1}}} \sum_{g=1}^{2^{4Q_{3m+1}}-1} e^*_{3m+1,n,k,g}\right).$$

Then $\sum_{k=1}^{2^{M_{3m+1}}} e''^*_{0,3m+1,\mathrm{brd},n,k} = 0$. Moreover, if we fix n with $1 \leq n \leq P_{3m+1}$, and set

$$x^*_{3m+1,n,k,l} = x^*_{0,3m+1,n,k,l} + \frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{M_{3m+1}+4Q_{3m+1}}} v^*_{3m+1,n}$$

for $1 \leq k \leq 2^{M_{3m+1}}$ and $1 \leq l \leq 2^{4Q_{3m+1}}$, then, recalling that

$$\sum_{k=1}^{2^{M_{3m+1}}} \left(e^*_{3m+1,n,k,2^{4Q_{3m+1}}} - \sum_{f=1}^{2^{M_{3m+1}}} \frac{e^*_{3m+1,n,f,2^{4Q_{3m+1}}}}{2^{M_{3m+1}}} \right) = 0$$

and

$$\sum_{k=1}^{2^{M_{3m+1}}} \left(e^*_{3m+1,n,k} - \frac{1}{2^{M_{3m+1}}} \sum_{f=1}^{2^{M_{3m+1}}} e^*_{3m+1,n,f} \right) = 0,$$

we have

$$\begin{aligned}
&\sum_{k=1}^{2^{M_{3m+1}}} \sum_{l=1}^{2^{4Q_{3m+1}}} x^*_{0,3m+1,n,k,l} \\
&\quad = \sum_{k=1}^{2^{M_{3m+1}}} \Bigg(\sum_{l=1}^{2^{4Q_{3m+1}}-1} \Bigg(e^*_{3m+1,n,k,l} - \frac{1}{2^{4Q_{3m+1}}} \Bigg(\sum_{g=1}^{2^{4Q_{3m+1}}-1} e^*_{3m+1,n,k,g} \\
&\qquad + \Bigg(\Bigg(e^*_{3m+1,n,k,2^{4Q_{3m+1}}} - \sum_{f=1}^{2^{M_{3m+1}}} \frac{e^*_{3m+1,n,f,2^{4Q_{3m+1}}}}{2^{M_{3m+1}}} \Bigg) + \frac{\omega'^*_{3m+1,n}}{2^{M_{3m+1}}} \Bigg) \Bigg) \Bigg) \\
&\qquad + \frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{2Q_{3m+1}}} \Bigg(e^*_{3m+1,n,k} - \frac{1}{2^{M_{3m+1}}} \sum_{f=1}^{2^{M_{3m+1}}} e^*_{3m+1,n,f} \Bigg) \\
&\qquad + \Bigg(\Bigg(e^*_{3m+1,n,k,2^{4Q_{3m+1}}} - \sum_{f=1}^{2^{M_{3m+1}}} \frac{e^*_{3m+1,n,f,2^{4Q_{3m+1}}}}{2^{M_{3m+1}}} \Bigg) + \frac{\omega'^*_{3m+1,n}}{2^{M_{3m+1}}} \Bigg) \\
&\qquad - \frac{1}{2^{4Q_{3m+1}}} \Bigg(\sum_{g=1}^{2^{4Q_{3m+1}}-1} e^*_{3m+1,n,k,g} + \Bigg(\Bigg(e^*_{3m+1,n,k,2^{4Q_{3m+1}}} \\
&\qquad - \sum_{f=1}^{2^{M_{3m+1}}} \frac{e^*_{3m+1,n,f,2^{4Q_{3m+1}}}}{2^{M_{3m+1}}} \Bigg) + \frac{\omega'^*_{3m+1,n}}{2^{M_{3m+1}}} \Bigg) \Bigg) \\
&\qquad + \frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{2Q_{3m+1}}} \Bigg(e^*_{3m+1,n,k} - \sum_{f=1}^{2^{M_{3m+1}}} \frac{e^*_{3m+1,n,f}}{2^{M_{3m+1}}} \Bigg) \Bigg) \\
&\quad = \sum_{k=1}^{2^{M_{3m+1}}} \Bigg(\sum_{l=1}^{2^{4Q_{3m+1}}-1} \Bigg(e^*_{3m+1,n,k,l} \\
&\qquad - \frac{1}{2^{4Q_{3m+1}}} \Bigg(\sum_{g=1}^{2^{4Q_{3m+1}}-1} e^*_{3m+1,n,k,g} + \frac{\omega'^*_{3m+1,n}}{2^{M_{3m+1}}} \Bigg) \Bigg) \Bigg) \\
&\qquad + \omega'^*_{3m+1,n} - \frac{2^{M_{3m+1}}}{2^{4Q_{3m+1}}} \Bigg(\sum_{g=1}^{2^{4Q_{3m+1}}-1} e^*_{3m+1,n,k,g} + \frac{\omega'^*_{3m+1,n}}{2^{M_{3m+1}}} \Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{2^{M_{3m+1}}} \Bigg(\sum_{l=1}^{2^{4Q_{3m+1}}-1} e^*_{3m+1,n,k,l} \\
&\quad - \frac{2^{4Q_{3m+1}}-1}{2^{4Q_{3m+1}}} \sum_{g=1}^{2^{4Q_{3m+1}}-1} e^*_{3m+1,n,k,g} - \sum_{g=1}^{2^{4Q_{3m+1}}-1} \frac{e^*_{3m+1,n,k,g}}{2^{4Q_{3m+1}}} \Bigg) \\
&\quad - \frac{2^{4Q_{3m+1}}-1}{2^{4Q_{3m+1}}} \omega'^*_{3m+1,n} + \omega'^*_{3m+1,n} - \frac{1}{2^{4Q_{3m+1}}} \omega'^*_{3m+1,n} \\
&= \sum_{k=1}^{2^{M_{3m+1}}} \Bigg(\sum_{l=1}^{2^{4Q_{3m+1}}-1} e^*_{3m+1,n,k,l} \Big(1 - \frac{2^{4Q_{3m+1}}-1}{2^{4Q_{3m+1}}} - \frac{1}{2^{4Q_{3m+1}}} \Big) \Bigg) \\
&\quad + \omega'^*_{3m+1,n} \Big(- \frac{2^{4Q_{3m+1}}-1}{2^{4Q_{3m+1}}} + 1 - \frac{1}{2^{4Q_{3m+1}}} \Big) = 0.
\end{aligned}
$$

STEP 4. We pass from $B_3(3m+1)$ of Step 2 to the biorthogonal system

$$
\begin{aligned}
&B'_3(3m+1) \cup (v'_{3m+2,n}, v'^*_{3m+2,n})_{n=1}^{Q''_{3m+2}}, \\
&B'_3(3m+1) = (x_n, x^*_n)_{n=1}^{q(3m+2)} \cup (e_{3m+1,0,n}, e^*_{3m+1,0,n})_{n=1}^{M_{3m+1,0}} \\
&\qquad \cup ((x_{3m+1,n,0}, x^*_{3m+1,n,0}) \\
&\qquad \cup (\omega'_{3m+1,n,0}, \omega'^*_{3m+1,n,0}) \cup (\omega_{3m+1,n,0}, \omega^*_{3m+1,n,0}) \\
&\qquad \cup (x_{3m+1,n,k,0}, x^*_{3m+1,n,k,0})_{k=1}^{2^{M_{3m+1}}})_{n=1}^{P_{3m+1}} \\
&\qquad \cup ((e'''_{3m+1,\mathrm{brd},n,k}, e'''^*_{3m+1,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m+1}}})_{n=1}^{P_{3m+1}}; \\
&(x_n, x^*_n)_{n=q(3m+1)+1}^{q(3m+2)} = (((x_{3m+1,n,k,l}, x^*_{3m+1,n,k,l})_{l=1}^{2^{4Q_{3m+1}}})_{k=1}^{2^{M_{3m+1}}})_{n=1}^{P_{3m+1}}.
\end{aligned}
$$

At this point, always by the procedure of Construction II and of Lemma 12, where we replace $(y_n, y^*_n)_{n=1}^{Q(m)}$ by $\widetilde{B}_3(3m+1)$ which comes from $B'_3(3m+1)$ when we remove all the elements of

$$
((\omega_{3m+1,n}, \omega^*_{3m+1,n}) \cup (e_{3m+1,\mathrm{brd},n,k}, e^*_{3m+1,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m+1}}})_{n=1}^{P_{3m+1}},
$$

we define

$$
\begin{aligned}
&(u''_{3m+2,s}, u''^*_{3m+2,s})_{s=1}^{P''_{3m+2}} \cup (v'_{3m+3,n}, v'^*_{3m+3,n})_{n=1}^{Q''_{3m+3}}, \\
&(u''_{3m+2,s}, u''^*_{3m+2,s})_{s=1}^{Q''_{3m+2}} = (v'_{3m+2,n}, v'^*_{3m+2,n})_{n=1}^{Q''_{3m+2}}; \\
&B_4(3m+1) = B'_4(3m+1) \cup (v'_{3m+3,n}, v'^*_{3m+3,n})_{n=1}^{Q''_{3m+3}}, \\
&B'_4(3m+1) = (x_n, x^*_n)_{n=1}^{q(3m+2)} \cup (u'_{3m+2,s}, u'^*_{3m+2,s})_{s=1}^{A''_{3m+2}} \\
&\qquad \cup (u'_{3m+2,s}, u'^*_{3m+2,s})_{s=A_{3m+2}-A'_{3m+2}+1}^{A_{3m+2}}; \\
&A''_{3m+2} = M_{3m+1,0} + P''_{3m+2} + (2 + 2^{M_{3m+1}}) P_{3m+1}, \qquad A'_{3m+2} = (1 + 2^{M_{3m+1}}) P_{3m+1}, \\
&(u'_{3m+2,s}, u'^*_{3m+2,s})_{s=1}^{A''_{3m+2}} = (e_{3m+1,0,n}, e^*_{3m+1,0,n})_{n=1}^{M_{3m+1,0}} \cup (u''_{3m+2,s}, u''^*_{3m+2,s})_{s=1}^{P''_{3m+2}} \\
&\qquad \cup ((x_{3m+1,n,0}, x^*_{3m+1,n,0}) \cup (\omega'_{3m+1,n,0}, \omega'^*_{3m+1,n,0}) \\
&\qquad \cup (x_{3m+1,n,k,0}, x^*_{3m+1,n,k,0})_{k=1}^{2^{M_{3m+1}}})_{n=1}^{P_{3m+1}},
\end{aligned}
$$

$$(u'_{3m+2,s}, u'^*_{3m+2,s})_{s=A_{3m+2}-A'_{3m+2}+1}^{A_{3m+2}} = ((\omega_{3m+1,n,0}, \omega^*_{3m+1,n,0}) \cup (e'''_{3m+1,\mathrm{brd},n,k}, e'''^*_{3m+1,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m+1}}})_{n=1}^{P_{3m+1}}.$$

We point out that, by (39.3) and (39.4) of Theorem 11 and by Steps 1 and 3, $\|x_n\| < 5$ and $\|x_n^*\| < 7$ for $q(3m+1)+1 \le n \le q(3m+2)$; moreover $\|\omega'_{3m+1,n,0}\| = \|\omega_{3m+1,n,0}\| = 1 = \|x_{3m+1,n,0}\|$ and $\|x_{3m+1,n,k,0}\| = 3$, while $\max(\|\omega'^*_{3m+1,n,0}\|, \|\omega^*_{3m+1,n,0}\|, \|x^*_{3m+1,n,0}\|, \|x^*_{3m+1,n,k,0}\|) < 3$ for $1 \le n \le P_{3m+1}$ and $1 \le k \le 2^{M_{3m+1}}$. We set $(x_n)_{n=q(3m+1)+1}^{q(3m+2)} = (x_{3m+1,n})_{n=1}^{G_{3m+1}}$ and, for $q(3m+1)+1 \le n \le q(3m+2)$,

$$x_n = x'_n + \widetilde{x}_n + x_{\mathrm{brd},n}, \quad \widetilde{x}_n = x''_n + x'''_n;$$

$$\begin{aligned}
x'_n \in \mathrm{span}(e'''_{3m+1,i} &: 1 \le i \le S_{3m+1}, e'''_{3m+1,i} \notin \mathrm{span}((e_{3m+1,\mathrm{aux},s})_{s=A''_{3m+1}+1}^{A_{3m+1}/2} \\
&\cup (e_{3m+1,\mathrm{aux},s})_{s=A_{3m+1}/2+M_{3m,\mathrm{arm},0}+1}^{A_{3m+1}-A'_{3m+1}} \\
&\cup ((e_{3m+1,\mathrm{aux},s,t})_{t=1}^{2^{2B_{3m+1}}})_{s=1}^{A_{3m+1}}) \cup (\omega_{3m+1,n} \cup (e_{3m+1,\mathrm{brd},r,k})_{k=1}^{2^{M_{3m+1}}})_{r=1}^{P_{3m+1}}), \\
x''_n \in \mathrm{span}(&(e_{3m+1,\mathrm{aux},s})_{s=A''_{3m+1}+1}^{A_{3m+1}/2} \\
&\cup (e_{3m+1,\mathrm{aux},s})_{s=A_{3m+1}/2+M_{3m,\mathrm{arm},0}+1}^{A_{3m+1}-A'_{3m+1}} \cup ((e_{3m+1,\mathrm{aux},s,t})_{t=1}^{2^{2B_{3m+1}}})_{s=1}^{A_{3m+1}}), \\
x'''_n \in \mathrm{span}(&(u'_{3m+1,s})_{s=1}^{A''_{3m+1}} \cup (u'_{3m+1,s})_{s=A_{3m+1}/2+1}^{A_{3m+1}/2+M_{3m,\mathrm{arm},0}} \\
&\cup (u'_{3m+1,s})_{s=A_{3m+1}-A'_{3m+1}+1}^{A_{3m+1}}), \\
x_{\mathrm{brd},n} \in \mathrm{span}(&\omega_{3m+1,n} \cup (e_{3m+1,\mathrm{brd},r,k})_{k=1}^{2^{M_{3m+1}}})_{r=1}^{P_{3m+1}}.
\end{aligned}$$

We set, for each $\overline{x} \in X$ with $\|\overline{x}\| = 1$,

$$\begin{aligned}
\overline{a}_{3m+1} = \overline{a}_{3m+1}(\overline{x}) = \ &\max(\max(|x_n^*(\overline{x})| : q(3m+1)+1 \le n \le q(3m+2)), \\
&\max(|e'''^*_{3m+1,n}(\overline{x})| : 1 \le n \le S_{3m+1}), \max(|\omega^*_{3m+1,n,0}(\overline{x})|, \\
&|x^*_{3m+1,n,0}(\overline{x})|, |x^*_{3m+1,n,k,0}(\overline{x})| : 1 \le n \le P_{3m+1}, 1 \le k \le 2^{M_{3m+1}})).
\end{aligned}$$

Finally, we set $B_1(3m+2) = B_4(3m+1)$ and we are ready for the construction of $(x_n, x_n^*)_{n=q(3m+2)+1}^{q(3m+3)}$.

STEP $4'$. In this sub-block the condition

$$\|\omega'_{3m+1,n,0}\| = \|\omega_{3m+1,n,0}\| = 1 = \|x_{3m+1,n,0}\|$$

is not strictly necessary, it is only useful to avoid more formalism, in particular in the proof of (A) of RL. Indeed, in Step 1 we could also set

$$\begin{aligned}
S'_{3m+1} &= 2^{M'_{3m+1}} + (A_{3m+1} - (A'_{3m+1} + A''_{3m+1} + M_{3m,\mathrm{arm},0} + M_{3m,\mathrm{brd},0})) \\
&\quad + 2P_{3m+1} + 3.2^{M_{3m+1}}; \\
M'_{3m+1} &= A_{3m+1} 2^{2B_{3m+1}} + P_{3m+1} 2^{M_{3m+1}} (2^{4Q_{3m+1}-1} + 3).
\end{aligned}$$

Again, by means of the procedures of the proof of Theorem 11 and of Lemma 10, we pass from $(\widehat{e}_{3m+1,n},\widehat{e}''^*_{3m+1,n})_{n=1}^{S'_{3m+1}}$ to the same biorthogonal system

$$(\widehat{e}'_{3m+1,n},\widehat{e}'^*_{3m+1,n})_{n=1}^{M'_{3m+1}}\cup(e_{3m+1,0,n},e^*_{3m+1,0,n})_{n=M'_{3m+1,0}+1}^{M_{3m+1,0}}$$
$$\cup(e'''_{3m+1,n},e'''^*_{3m+1,n})_{n=M'_{3m+1}+1}^{S_{3m+1}}$$

of Step 1, where now in $(\widehat{e}'_{3m+1,n},\widehat{e}'^*_{3m+1,n})_{n=1}^{M'_{3m+1}}$ we replace

$$((e'_{3m+1,n,k},e'^*_{3m+1,n,k})_{k=1}^{2^{M_{3m+1}}}\cup((e'_{3m+1,n,k,2^{4Q_{3m+1}}},e'^*_{3m+1,n,k,2^{4Q_{3m+1}}})_{k=1}^{2^{M_{3m+1}}}$$
$$\cup(e'_{3m+1,\mathrm{brd},n,k},e'^*_{3m+1,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m+1}}})_{n=1}^{P_{3m+1}}$$

by $(((e'_{i,3m+1,n,0,k},e'^*_{i,3m+1,n,0,k})_{k=1}^{2^{M_{3m+1}}})_{n=1}^{P_{3m+1}})_{i=1}^{3}$, which then become

$$(((e_{i,3m+1,n,0,k},e^*_{i,3m+1,n,0,k})_{k=1}^{2^{M_{3m+1}}})_{n=1}^{P_{3m+1}})_{i=1}^{3},$$

while we include directly in $(e'''_{3m+1,n},e'''^*_{3m+1,n})_{n=M'_{3m+1}+1}^{S_{3m+1}}$ also the part

$$((e_{3m+1,n,k},e^*_{3m+1,n,k})_{k=1}^{2^{M_{3m+1}}}\cup((e_{3m+1,n,k,2^{4Q_{3m+1}}},$$
$$e^*_{3m+1,n,k,2^{4Q_{3m+1}}})_{k=1}^{2^{M_{3m+1}}}\cup(e_{3m+1,\mathrm{brd},n,k},e'^*_{3m+1,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m+1}}})_{n=1}^{P_{3m+1}};$$

that is, now

$$((e_{3m+1,n,k})_{k=1}^{2^{M_{3m+1}}}\cup(e_{3m+1,n,k,2^{4Q_{3m+1}}})_{k=1}^{2^{M_{3m+1}}}\cup(e_{3m+1,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m+1}}})_{n=1}^{P_{3m+1}}$$

is 1-equivalent to the natural basis of $l_1^{3P_{3m+1}2^{M_{3m+1}}}$. It follows that in Step 3, through $(((e_{i,3m+1,n,0,k},e^*_{i,3m+1,n,0,k})_{k=1}^{2^{M_{3m+1}}})_{n=1}^{P_{3m+1}})_{i=1}^{3}$ and the procedure of uniform minimalization, we have to replace

$$(x_{3m+1,n,0},x^*_{3m+1,n,0})\cup(\omega'_{3m+1,n,0},\omega'^*_{3m+1,n,0})\cup(\omega_{3m+1,n,0},\omega^*_{3m+1,n,0})$$

by

$$((x_{3m+1,n,0,k},x^*_{3m+1,n,0,k})\cup(\omega'_{3m+1,n,0,k},\omega'^*_{3m+1,n,0,k})\cup(\omega_{3m+1,n,0,k},\omega^*_{3m+1,n,0,k}))_{k=0}^{2^{M_{3m+1}}}.$$

Subconstruction III.3 (construction of the free block). We turn to the construction of $(x_n,x_n^*)_{n=q(3m+2)+1}^{q(3m+3)}$ and we start from the biorthogonal system $B_1(3m+2)$ defined in the previous subconstruction; we will proceed through two steps. Again we set

$$A_{3m+2}=4^{q(3m+2)(A'_{3m+2}+A''_{3m+2})},$$
$$K_{3m+2}=\max(\|u'^*_{3m+2,s}\|:1\le s\le A''_{3m+2},A_{3m+2}-A'_{3m+2}+1\le s\le A_{3m+2}),$$
$$B_{3m+2}>A_{3m+2}2^{K_{3m+2}A_{3m+2}},\qquad P_{3m+2}=A_{3m+2}(2^{2B_{3m+2}}+1).$$

Step 1. By the procedure of Lemma 10 we pass from $B_1(3m+2)$ to the biorthogonal system $B_1(3m+2)\cup(\widehat{e}_{3m+2,n},\widehat{e}^*_{3m+2,n})_{n=1}^{S'_{3m+2}+M'_{3m+2,0}}$ where $(\widehat{e}_{3m+2,n})_{n=1}^{S'_{3m+2}+M'_{3m+2,0}}$ is 1-equivalent to the natural basis of $l_1^{S'_{3m+2}+M'_{3m+2,0}}$, moreover with

$$\|y+e\|\ge\max(\|y\|,\|e\|/2)$$

for $y\in\operatorname{span}(y_{3m+2,n})_{n=1}^{Q(3m+2)}$,

$$e\in\operatorname{span}(\widehat{e}_{3m+2,n})_{n=1}^{S'_{3m+2}+M'_{3m+2,0}},\ \text{and}$$

$$(y_{3m+2,n})_{n=1}^{Q(3m+2)} = (x_n)_{n=1}^{q(3m+2)} \cup (u'_{3m+2,s})_{s=1}^{A''_{3m+2}}$$
$$\cup (u'_{3m+2,s})_{s=A_{3m+2}-A'_{3m+2}+1}^{A_{3m+2}} \cup (v'_{3m+3,n})_{n=1}^{Q''_{3m+3}}.$$

Then we suppose that, according to the procedure of Lemma 10, there are

$$(\widehat{v}_{0,3m+2,n})_{n=S'_{3m+2}+1}^{S'_{3m+2}+M'_{3m+2,0}}$$

in $\operatorname{span}(\widehat{e}_{3m+2,n})_{n=1}^{S'_{3m+2}}$ and $(\widehat{e}''^{*}_{3m+2,n})_{n=1}^{S'_{3m+2}}$ in X^* so that we have the biorthogonal system

$$(\widehat{e}_{3m+2,n}, \widehat{e}''^{*}_{3m+2,n})_{n=1}^{S'_{3m+2}} \cup (e_{3m+2,0,n}, e^{*}_{3m+2,0,n})_{n=1}^{M'_{3m+2,0}},$$
$$(e_{3m+2,0,n}, e^{*}_{3m+2,0,n})_{n=1}^{M'_{3m+2,0}} = (\widehat{e}_{3m+2,n} - \widehat{v}_{0,3m+2,n}, \widehat{e}^{*}_{3m+2,n})_{n=S'_{3m+2}+1}^{S'_{3m+2}+M'_{3m+2,0}};$$
$$E'_{0,3m+2} = \operatorname{span}(e_{3m+2,0,n})_{n=1}^{M'_{3m+2,0}}; \quad S'_{3m+2} = 2^{M'_{3m+2}} + (A_{3m+2} - (A'_{3m+2} + A''_{3m+2})) + 1,$$
$$M'_{3m+2} = A_{3m+2} 2^{2B_{3m+2}} + P_{3m+2} 2^{M_{3m+2}}.$$

Then $(\widehat{e}_{3m+2,n} + E'_{0,3m+2})_{n=1}^{S'_{3m+2}}$ is 1-equivalent to the natural basis of $l_\infty^{S'_{3m+2}}$ and we can suppose that the method at the end of Step 1 of SC III.1 continues to work. Unlike the previous two subconstructions now the bridge sequence does not appear. By the procedures of the proof of Theorem 11 and of Lemma 10, we pass from $(\widehat{e}_{3m+2,n}, \widehat{e}''^{*}_{3m+2,n})_{n=1}^{S'_{3m+2}}$ to the biorthogonal system

$$((e'_{3m+2,\text{aux},s,t}, e'^{*}_{3m+2,\text{aux},s,t})_{t=1}^{2^{2B_{3m+2}}})_{s=1}^{A_{3m+2}} \cup ((e'_{3m+2,n,k}, e'^{*}_{3m+2,n,k})_{k=1}^{2^{M_{3m+2}}})_{n=1}^{P_{3m+2}}$$
$$\cup (e'_{3m+1,brd,n,k}, e'^{*}_{3m+1,brd,n,k})_{k=1}^{2^{M_{3m+1}}})_{n=1}^{P_{3m+1}}$$
$$\cup (e_{3m+2,0,n}, e^{*}_{3m+2,0,n})_{n=M'_{3m+2,0}+1}^{M_{3m+2,0}} \cup (e'''_{3m+2,n}, e'''^{*}_{3m+2,n})_{n=M'_{3m+2}+1}^{S_{3m+2}};$$
$$M_{3m+2,0} = M'_{3m+2,0} + (2^{M'_{3m+2}} - M'_{3m+2}); \quad S_{3m+2} = S'_{3m+2} - (2^{M'_{3m+2}} - M'_{3m+2});$$
$$E_{3m+2,0} = \operatorname{span}(e_{3m+2,0,n})_{n=1}^{M_{3m+2,0}};$$
$$(e'''_{3m+2,n}, e'''^{*}_{3m+2,n})_{n=M'_{3m+2}+1}^{S_{3m+2}} = (\widehat{e}_{3m+2,n}, \widehat{e}''^{*}_{3m+2,n})_{n=2^{M'_{3m+2}}+1}^{S'_{3m+2}}$$
$$= (e_{3m+2,\text{aux},s}, e^{*}_{3m+2,\text{aux},s})_{s=A''_{3m+2}+1}^{A_{3m+2}-A'_{3m+2}}$$
$$\cup (\omega_{3m+1,n}, \omega^{*}_{3m+1,n})_{n=1}^{P_{3m+1}};$$

so that if we set, for $1 \le s \le A_{3m+2}$,

$$e_{3m+2,\text{aux},s,1} = e'_{3m+2,\text{aux},s,1} \quad \text{and} \quad e_{3m+2,\text{aux},s,t} = e'_{3m+2,\text{aux},s,t} - e'_{3m+2,\text{aux},s,t-1}$$

for $2 \le t \le 2^{2B_{3m+2}}$, then $(e'_{3m+2,\text{aux},s,t})_{t=1}^{2^{2B_{3m+2}}}$, $(e_{3m+2,\text{aux},s,t})_{t=1}^{2^{2B_{3m+2}}}$ and $E_{3m+2,0}$ correspond to $((e'_{r,n})_{n=1}^{2^N})_{r=1}^{2^R}$, $((e_{r,n})_{n=1}^{2^N})_{r=1}^{2^R}$ and E'_0 of (39.3) and (39.4) of Theorem 11 with $\{n\}_{n=1}^{2^N}$ and $\{r\}_{r=1}^{2^R}$ replaced respectively by $(t)_{t=1}^{2^{2B_{3m+2}}}$ and $\{1\}$; analogously, if we set, for $1 \le n \le P_{3m+2}$, $e_{3m+2,n,1} = e'_{3m+2,n,1}$ and $e_{3m+2,n,k} = e'_{3m+2,n,k} - e'_{3m+2,n,k-1}$ for $2 \le k \le 2^{M_{3m+2}}$, $e_{3m+1,\text{brd},n,1} = e'_{3m+1,\text{brd},n,1}$ and

$$e_{3m+1,\text{brd},n,k} = e'_{3m+1,\text{brd},n,k} - e'_{3m+1,\text{brd},n,k-1} \quad \text{for } 2 \le k \le 2^{M_{3m+1}},$$

then $(e'_{3m+2,n,k})_{k=1}^{2^{M_{3m+2}}}$, $(e_{3m+2,n,k})_{k=1}^{2^{M_{3m+2}}}$ and $E_{3m+2,0}$, moreover $(e'_{3m+1,\text{brd},n,k})_{k=1}^{2^{M_{3m+1}}}$,

$(e_{3m+1,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m+1}}}$ and $E_{3m+2,0}$, correspond to $((e'_{r,n})_{n=1}^{2^N})_{r=1}^{2^R}$, $((e_{r,n})_{n=1}^{2^N})_{r=1}^{2^R}$ and E'_0 of (39.3) and (39.4) of Theorem 11 with $\{n\}_{n=1}^{2^N}$ and $\{r\}_{r=1}^{2^R}$ replaced respectively by $(k)_{k=1}^{2^{M_{3m+2}}}$ and $\{1\}$, moreover by $(k)_{k=1}^{2^{M_{3m+1}}}$ and $\{1\}$. Then we pass to the biorthogonal system

$$B_2(3m+2) = B_1(3m+2) \cup (e'''_{3m+2,n}, e'''^{*}_{3m+2,n})_{n=1}^{S_{3m+2}} \cup (e_{3m+2,0,n}, e^{*}_{3m+2,0,n})_{n=1}^{M_{3m+2,0}};$$

$$(e'''_{3m+2,n}, e'''^{*}_{3m+2,n})_{n=1}^{M'_{3m+2}} = ((e_{3m+2,\mathrm{aux},s,t}, e^{*}_{3m+2,\mathrm{aux},s,t})_{t=1}^{2^{2B_{3m+2}}})_{s=1}^{A_{3m+2}}$$
$$\cup\, ((e_{3m+2,n,k}, e^{*}_{3m+2,n,k})_{k=1}^{2^{M_{3m+2}}})_{n=1}^{P_{3m+2}}.$$

As for the previous subconstructions, $\|e'''^{*}_{3m+2,n}\| \le 2$ for $1 \le n \le S_{3m+2}$. Setting

$$(e_{3m+2,aux,s}, e^{*}_{3m+2,aux,s})_{s=A''_{3m+2}+1}^{A_{3m+2}-A'_{3m+2}} = (u'_{3m+2,s}, u'^{*}_{3m+2,s})_{s=A''_{3m+2}+1}^{A_{3m+2}-A'_{3m+2}},$$

from the expression of $B_1(3m+2) = B_4(3m+1)$ of SC III.2 we get the biorthogonal system

$$(u'_{3m+2,s}, u'^{*}_{3m+2,s})_{s=1}^{A_{3m+2}} \cup ((e_{3m+2,\mathrm{aux},s,t}, e^{*}_{3m+2,\mathrm{aux},s,t})_{t=1}^{2^{2B_{3m+2}}})_{s=1}^{A_{3m+2}}$$

from which, by the same procedure of Step 2 with $3m+1$ replaced by $3m+2$,we pass to $(u_{3m+2,n}, u^{*}_{3m+2,n})_{n=1}^{P_{3m+2}} = ((u_{3m+2,s,t}, u^{*}_{3m+2,s,t})_{t=0}^{2^{2B_{3m+2}}})_{s=1}^{A_{3m+2}}$ and then to the generating biorthogonal system $(w_{3m+2,n}, w^{*}_{3m+2,n})_{n=1}^{P_{3m+2}} = ((w_{3m+2,s,t}, w^{*}_{3m+2,s,t})_{t=0}^{2^{2B_{3m+2}}})_{s=1}^{A_{3m+2}}$ and we specify that, setting $Q_{0,3m+2} = \max(\|w^{*}_{3m+2,n}\| : 1 \le n \le P_{3m+2})$, $M_{3m+2} > 4^{2P_{3m+2}Q_{0,3m+2}}$. Then we have the biorthogonal system

$$B_3(3m+2) = (x_n, x_n^{*})_{n=1}^{q(3m+2)} \cup ((w_{3m+2,n}, w^{*}_{3m+2,n})$$
$$\cup\, (e_{3m+2,n,k}, e^{*}_{3m+2,n,k})_{k=1}^{2^{M_{3m+2}}})_{n=1}^{P_{3m+2}} \cup (v'_{3m+3,n}, v'^{*}_{3m+3,n})_{n=1}^{Q''_{3m+3}}.$$

STEP 2. Let us fix n with $1 \le n \le P_{3m+2}$. For $1 \le k \le 2^{M_{3m+2}}$ we set

$$x_{3m+2,n,0} = \sum_{f=1}^{2^{M_{3m+2}}} e_{3m+2,n,f}, \quad x^{*}_{3m+2,n,0} = \sum_{f=1}^{2^{M_{3m+2}}} \frac{e^{*}_{3m+2,n,f}}{2^{M_{3m+2}}} - \frac{w^{*}_{3m+2,n}}{2^{M_{3m+2}}};$$

$$x_{3m+2,n,k} = e_{3m+2,n,k} + w_{3m+2,n},$$

$$x^{*}_{3m+2,n,k} = e^{*}_{3m+2,n,k} - x^{*}_{3m+2,n,0} = \left(e^{*}_{3m+2,n,k} - \sum_{f=1}^{2^{M_{3m+2}}} \frac{e^{*}_{3m+2,n,f}}{2^{M_{3m+2}}}\right) + \frac{w^{*}_{3m+2,n}}{2^{M_{3m+2}}}.$$

We pass from $B_3(3m+2)$ of Step 1 to the biorthogonal system

$$B'_3(3m+2) \cup (v'_{3m+3,n}, v'^{*}_{3m+3,n})_{n=1}^{Q''_{3m+3}},$$

$$B'_3(3m+2) = (x_n, x_n^{*})_{n=1}^{q(3m+3)} \cup (e_{3m+2,0,n}, e^{*}_{3m+2,0,n})_{n=1}^{M_{3m+2,0}};$$

$$(x_n, x_n^{*})_{n=q(3m+2)+1}^{q(3m+3)} = (((x_{3m+2,n,k}, x^{*}_{3m+2,n,k})_{k=0}^{2^{M_{3m+2}}})_{n=1}^{P_{3m+2}}.$$

At this point, always by the procedure of Construction II and of Lemma 12, where we replace $(y_n, y_n^{*})_{n=1}^{Q(m)}$ by $B'_3(3m+2)$, we define

$$(u''_{3m+3,s}, u''^{*}_{3m+3,s})_{s=1}^{P''_{3m+3}} \cup (v'_{3m+4,n}, v'^{*}_{3m+4,n})_{n=1}^{Q''_{3m+4}},$$

$$(u''_{3m+3,s}, u''^{*}_{3m+3,s})_{s=1}^{Q''_{3m+3}} = (v'_{3m+3,n}, v'^{*}_{3m+3,n})_{n=1}^{Q''_{3m+3}};$$

$$B_4(3m+2)=(x_n,x_n^*)_{n=1}^{q(3m+3)}\cup(u'_{3m+3,s},u'^*_{3m+3,s})_{s=1}^{A''_{3m+3}}\cup(v'_{3m+4,n},v'^*_{3m+4,n})_{n=1}^{Q''_{3m+4}},$$
$$(u'_{3m+3,s},u'^*_{3m+3,s})_{s=1}^{A''_{3m+3}}=(e_{3m+2,0,n},e^*_{3m+2,0,n})_{n=1}^{M_{3m+2,0}}\cup(u''_{3m+3,s},u''^*_{3m+3,s})_{s=1}^{P''_{3m+3}}.$$

By (39.3) and (39.4) of Theorem 11 and by the above, $\|x_n\|<3$ and $\|x_n^*\|<5$ for $q(3m+2)+1\le n\le q(3m+3)$ (we recall that $\|x_{3m+2,n,0}\|=1$ for $1\le n\le P_{3m+2}$); we set $(x_n)_{n=q(3m+2)+1}^{q(3m+3)}=(x_{3m+2,n})_{n=1}^{G_{3m+2}}$ and, for $q(3m+2)+1\le n\le q(3m+3)$,

$$x_n=x'_n+\widetilde{x}_n;\quad \widetilde{x}_n=x''_n+x'''_n;$$
$$x'_n\in\operatorname{span}(e'''_{3m+2,i}:1\le i\le S_{3m+2},e'''_{3m+2,i}\notin\operatorname{span}((e_{3m+2,\mathrm{aux},s})_{s=A''_{3m+1}+1}^{A_{3m+2}}$$
$$\cup\,((e_{3m+2,\mathrm{aux},s,t})_{t=1}^{2^{2B_{3m+2}}})_{s=1}^{A_{3m+2}});$$
$$x''_n\in\operatorname{span}((e_{3m+2,\mathrm{aux},s})_{s=A''_{3m+1}+1}^{A_{3m+2}}\cup((e_{3m+2,\mathrm{aux},s,t})_{t=1}^{2^{2B_{3m+2}}})_{s=1}^{A_{3m+2}});$$
$$x'''_n\in\operatorname{span}(u'_{3m+2,s})_{s=1}^{A''_{3m+2}}.$$

We set, for each $\overline{x}\in X$ with $\|\overline{x}\|=1$,

$$\overline{a}_{3m+2}=\overline{a}_{3m+2}(\overline{x})=\max(\max(|x_n^*(\overline{x})|:q(3m+2)+1\le n\le q(3m+3)),$$
$$\max(|e'''^*_{3m+2,n}(\overline{x})|:1\le n\le S_{3m+2})).$$

Finally, we set $B_1(3m+3)=B_4(3m+2)$ and we are ready for the construction of $(x_n,x_n^*)_{n=q(3(m+1))+1}^{q(3(m+2))}$. This step completes SC III.3 and hence C III.

Construction IV (C IV, second version of the construction of each block). In this second version the block $(x_n,x_n^*)_{n=q(3m)+1}^{q(3m+3)}$ continues to be partitioned in three sub-blocks, however now the second sub-block in turn is parted in a set of sub-sub-blocks. Since C IV is an improvement of C III, the real construction of the blocks of the basis with permutations is C IV, but the advantage of C III is that it gives a better idea of the properties of the basis, hence all the proofs of the general properties will be given for C III since the procedures work also for C IV and only in Lemma 18 (RL) it will be necessary to specify a part of the proof also for C IV; the same for the proofs of the properties of the basis, where only in the proof of (ii) of Lemma 19 (FRCL) we use directly C IV. Hence, for what concerns the second sub-block, the proofs of Lemma 13 (RBL) and of (i) of Lemma 19 (FRCL) work also for SC IV.2.

Subconstruction IV.1 (SC IV.1, construction of the completeness block). The same construction of SC III.1 works, with the following simplification: In the analogue of Step 6 of SC III.1 we will not define now the system $(v'_{3m+2,s},v'^*_{3m+2,s})_{s=1}^{Q''_{3m+2}}$, hence $B_1(3m+1)$ becomes

$$B_{1,0}(3m+1)=B_{1,1}(3m+1)\cup(b'_{3m,i},b'^*_{3m,i})_{i=1}^{L'_{3m}},$$
$$B_{1,1}(3m+1)=(x_n,x_n^*)_{n=1}^{q(3m+1)}\cup(u'_{1,3m+1,s},u'^*_{1,3m+1,s})_{s=1}^{A''_{1,3m+1}},$$
$$(b'_{3m,i},b'^*_{3m,i})_{i=1}^{L'_{3m}}=((\omega'_{3m,n,0},\omega'^*_{3m,n,0})\cup(e'''_{3m,\mathrm{brd},n,k},e'''^*_{3m,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}},$$

where $(u'_{1,3m+1,s})_{s=1}^{A''_{1,3m+1}}$ is the sequence $(u'_{3m+1,s})_{s=1}^{A''_{3m+1}}$ of Step 6 of SC III.1. Moreover we recall that actually we defined only $(x_n - x_{\mathrm{brd},n})_{n=q(3m)+1}^{q(3m+1)}$, since we still have to define $((\omega'_{3m,n} \cup (e_{3m,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}}$, hence also $(b'_{3m,i}, b'^{*}_{3m,i})_{i=1}^{L'_{3m}}$ where $L'_{3m} = (2^{M_{3m}}+1)P_{3m}$.

SUBCONSTRUCTION IV.2 (SC IV.2, construction of the regularization block). This sub-block $(x_n, x_n^*)_{n=q(3m+1)+1}^{q(3m+2)}$ of the block $(x_n, x_n^*)_{n=q(3m)+1}^{q(3m+3)}$ will now be union of L_{3m} sub-sub-blocks (which in what follows we will call sub-blocks), where the construction of each sub-block will be a simplification of SC III.2, precisely the bridge sequence will not appear and the sequence $(v'_{3m+3,s})_{s=1}^{Q''_{3m+2}}$, together with the sequence $((\omega'_{3m,n} \cup (e_{3m,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}}$, will be defined only in the last (L_{3m})th sub-block (that is, also the sequences analogous to $(v'_{3m+3,s})_{s=1}^{Q''_{3m+2}}$ do not appear in the first $L_{3m}-1$ sub-blocks). We will proceed through 4 steps.

STEP 1 (starting point and point of arrival). The starting point is $B_{1,1}(3m+1)$ of (i) above, while the point of arrival will be

$$B_1(3m+2) = (x_n, x_n^*)_{n=1}^{q(3m+2)} \cup (u'_{3m+2,s}, u'^{*}_{3m+2,s})_{s=1}^{A''_{3m+2}} \cup (v'_{3m+3,s}, u'^{*}_{3m+3,s})_{s=1}^{Q''_{3m+3}},$$

$$(u'_{3m+2,s})_{s=1}^{A''_{3m+2}} = (e_{L_{3m},3m+1,0,n})_{n=1}^{L_{L_{3m},3m+1,0}} \cup (u''_{3m+2,s})_{s=1}^{P''_{3m+2}},$$

$$(x_n)_{n=q(3m+1)+1}^{q(3m+2)} = (x_{o,3m+1,g})_{g=1}^{G_{0,3m+1}} = ((x_{d,0,3m+1,g})_{g=1}^{G_{d,0,3m+1}})_{d=1}^{L_{3m}}$$

where, for $1 \le d \le L_{3m}$, $1 \le n \le P_{d,3m+1}$ and $1 \le k \le 2^{M_{d,3m+1}}$,

$$(x_{d,0,3m+1,g})_{g=1}^{G_{d,0,3m+1}} = ((x_{d,0,3m+1,n,g})_{g=1}^{G_{d,0,3m+1,0}})_{n=1}^{P_{d,3m+1}},$$

$$(x_{d,0,3m+1,n,g})_{g=1}^{G_{d,0,3m+1,0}} = (x_{d,0,0,3m+1,n,g})_{g=1}^{G_{d,0,0,3m+1,0}} \cup (x_{d,3m+1,n,g})_{g=1}^{G_{d,3m+1,0}},$$

$$(x_{d,0,0,3m+1,n,g})_{g=1}^{G_{d,0,0,3m+1,0}} = x_{d,3m+1,n,0} \cup (x_{d,3m+1,n,k,0})_{k=1}^{2^{M_{d,3m+1}}},$$

$$(x_{d,3m+1,n,g})_{g=1}^{G_{d,3m+1,0}} = ((x_{d,3m+1,n,k,l})_{l=1}^{2^{4Q_{d,3m+1}}})_{k=1}^{2^{M_{d,3m+1}}},$$

$$(((x_{d,3m+1,n,g})_{g=1}^{G_{d,3m+1,0}})_{n=1}^{P_{d,3m+1}})_{d=1}^{L_{3m}} = ((x_{d,3m+1,g})_{g=1}^{G_{d,3m+1}})_{d=1}^{L_{3m}} = (x_{3m+1,g})_{g=1}^{G_{3m+1}};$$

moreover, for $1 \le d \le L_{3m}$, $1 \le n \le P_{d,3m+1}$, $1 \le k \le 2^{M_{d,3m+1}}$ and $1 \le l \le 2^{4Q_{d,3m+1}}$,

$$x_{d,3m+1,n,0} = \sum_{f=1}^{2^{M_{d,3m+1}}} e_{d,3m+1,n,f},$$

$$x^*_{d,3m+1,n,0} = \sum_{f=1}^{2^{M_{d,3m+1}}} \frac{e^*_{d,3m+1,n,f}}{2^{M_{d,3m+1}}} - \frac{v^*_{d,3m+1,n}}{2^{M_{d,3m+1}+2Q_{d,3m+1}}},$$

$$x_{d,3m+1,n,k,0} = \sum_{g=1}^{2^{4Q_{d,3m+1}}} e_{d,3m+1,n,k,g}, x^*_{d,3m+1,n,k,0}$$

$$= \sum_{g=1}^{2^{4Q_{d,3m+1}}} \frac{e^*_{d,3m+1,n,k,g}}{2^{4Q_{d,3m+1}}} - \frac{2^{P_{d,3m+1}Q_{d,0,3m+1}}}{2^{2Q_{d,3m+1}}}(e^*_{d,3m+1,n,k} - x^*_{d,3m+1,n,0}),$$

$$x_{d,3m+1,n,k,l} = e_{d,3m+1,n,k,l} + \frac{e_{d,3m+1,n,k}}{2^{P_{d,3m+1}Q_{d,0,3m+1}+2Q_{d,3m+1}}} + \frac{v_{d,3m+1,n}}{2^{P_{d,3m+1}Q_{d,0,3m+1}}},$$

$$\begin{aligned}
x^*_{d,3m+1,n,k,l} &= e^*_{d,3m+1,n,k,l} - x^*_{d,3m+1,n,k,0} \\
&= \Big(e^*_{d,3m+1,n,k,l} - \sum_{g=1}^{2^{4Q_{d,3m+1}}} \frac{e^*_{d,3m+1,n,k,g}}{2^{4Q_{d,3m+1}}} \Big) \\
&\quad + \frac{2^{P_{d,3m+1}Q_{d,0,3m+1}}}{2^{2Q_{d,3m+1}}} \Big(e^*_{d,3m+1,n,k} - \sum_{f=1}^{2^{M_{d,3m+1}}} \frac{e^*_{d,3m+1,n,f}}{2^{M_{d,3m+1}}} \Big) \\
&\quad + \frac{2^{P_{d,3m+1}Q_{d,0,3m+1}}}{2^{M_{d,3m+1}+4Q_{d,3m+1}}} v^*_{d,3m+1,n}.
\end{aligned}$$

STEP 2 (description of the support sequence). First we define some integers and we point out that some elements which appear in these definitions will be defined in the next steps:

$$A_{3m+1} \geq 4^{q(3m+1)A''_{1,3m+1}}, \quad K_{3m+1} = \max(\|u'^*_{1,3m+1,s}\| : 1 \leq s \leq A''_{1,3m+1}),$$
$$B_{3m+1} \geq A_{3m+1} 2^{K_{3m+1}A_{3m+1}}, \quad L_{3m} = (2^{2B_{3m+1}} + 1)L'_{3m};$$

moreover for $1 \leq d \leq L_{3m}$, setting

$$q(1,0,3m+1) = q(3m+1), \quad q(d,0,3m+1) = q(3m+1) + \sum_{c=1}^{d-1} G_{c,0,3m+1}$$

for $2 \leq d \leq L_{3m}$, we define

$$\begin{gathered}
B_{0,3m+1} \geq 2^{L_{3m}}, \\
A_{d,3m+1} \geq 4^{q(d,0,3m+1)(A''_{d,3m+1}+B_{0,3m+1})}, \\
K_{d,3m+1} = \max(\|u'^*_{d,3m+1,s}\| : 1 \leq s \leq A''_{d,3m+1}), \\
B_{d,3m+1} \geq A_{d,3m+1} 2^{K_{3m+1}A_{d,3m+1}}, \\
P_{d,3m+1} = (2^{2B_{d,3m+1}} + 1)(A_{d,3m+1} - 1) + 2^{2B_{0,3m+1}} + 1, \\
Q_{d,0,3m+1} = \max(\|w^*_{d,3m+1,n}\| : 1 \leq n \leq P_{d,3m+1}), \\
Q_{d,3m+1} \geq 4^{2P_{d,3m+1}Q_{d,0,3m+1}}, \qquad M_{d,3m+1} \geq 4^{2Q_{d,3m+1}+4.2^{4Q_{d,3m+1}}}.
\end{gathered}$$

We start from a biorthogonal system (see Step 1)

$$(x_n, x^*_n)_{n=1}^{q(3m+1)} \cup (u'_{1,3m+1,s}, u'^*_{1,3m+1,s})_{s=1}^{A''_{1,3m+1}} \cup (\widehat{e}_{3m+1,n}, \widehat{e}^*_{3m+1,n})_{n=1}^{\widehat{L}_{3m+1}}$$

where $(\widehat{e}_{3m+1,n})_{n=1}^{\widehat{L}_{3m+1}}$ is 1-equivalent to the natural basis of $l_1^{\widehat{L}_{3m+1}}$ and $\|\widehat{e}^*_{3m+1,n}\| \leq 2$ for $1 \leq n \leq \widehat{L}_{3m+1}$. Our aim is to get a biorthogonal system

$$(\widetilde{e}_{3m+1,n}, \widetilde{e}^*_{3m+1,n})_{n=1}^{\widetilde{L}_{3m+1}} \cup (e_{3m+1,0,n}, e^*_{3m+1,0,n})_{n=1}^{\widetilde{L}_{3m+1,0}}$$

with the following properties:

$$\begin{aligned}
(\widetilde{e}_{3m+1,n})_{n=1}^{\widetilde{L}_{3m+1}} &= ((\widetilde{e}_{d,3m+1,n})_{n=1}^{\widetilde{L}_{d,3m+1}})_{d=1}^{L_{3m}} \cup (\omega'_{3m,n} \cup (e_{3m,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}} \\
&\quad \cup ((e_{3m+1,\mathrm{aux},i,j})_{j=1}^{2^{2B_{3m+1}}})_{i=1}^{L'_{3m}}, \\
(e_{3m+1,0,n})_{n=1}^{\widetilde{L}_{3m+1,0}} &= ((e_{d,3m+1,0,n})_{n=1}^{\widetilde{L}_{d,3m+1,0}})_{d=1}^{L_{3m}}, \\
(e_{d,3m+1,0,n})_{n=1}^{\widetilde{L}_{d,3m+1,0}} &= ((e_{c,d,3m+1,0,n})_{n=1}^{\widetilde{L}_{d,c,3m+1,0}})_{c=1}^{d},
\end{aligned}$$

$$\begin{aligned}(\widetilde{e}_{d,3m+1,n})_{n=1}^{\widetilde{L}_{d,3m+1}} &= (e_{d,3m+1,n,k} \cup (e_{d,3m+1,n,k,l})_{l=1}^{2^{4Q_{d,3m+1}}})_{k=1}^{2^{M_{d,3m+1}}} \cup (e_{d,3m+1,\mathrm{aux},s})_{s=A''_{d,3m+1}+1}^{A_{d,3m+1}-1} \\ &\quad \cup ((e_{d,3m+1,\mathrm{aux},s,t})_{t=1}^{2^{2B_{d,3m+1}}})_{s=1}^{A_{d,3m+1}-1} \cup (e_{d,3m+1,\mathrm{aux},A_{d,3m+1},t})_{t=1}^{2^{2B_{0,3m+1}}};\end{aligned}$$

moreover there is also a connected biorthogonal system $(\widetilde{e}'_{3m+1,n}, \widetilde{e}'^{*}_{3m+1,n})_{n=1}^{\widetilde{L}'_{3m+1}}$, with $(\widetilde{e}'_{3m+1,n})_{n=1}^{\widetilde{L}'_{3m+1}}$ 1-equivalent to the natural basis of $l_1^{\widetilde{L}'_{3m+1}}$ and

$$\begin{aligned}(\widetilde{e}'_{3m+1,g})_{g=1}^{\widetilde{L}'_{3m+1}} = (((&e'_{d,3m+1,n,k} \cup (e'_{d,3m+1,n,k,l})_{l=1}^{2^{4Q_{d,3m+1}}})_{k=1}^{2^{M_{d,3m+1}}})_{n=1}^{P_{d,3m+1}} \\ &\cup ((e'_{d,3m+1,\mathrm{aux},s,t})_{t=1}^{2^{2B_{d,3m+1}}})_{s=1}^{A_{d,3m+1}-1} \\ &\cup (e_{d,3m+1,\mathrm{aux},A_{d,3m+1},t})_{t=1}^{2^{2B_{0,3m+1}}})_{d=1}^{L_{3m}} \\ &\cup ((e'_{3m,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}} \cup ((e'_{3m+1,\mathrm{aux},i,j})_{j=1}^{2^{2B_{3m+1}}})_{i=1}^{L'_{3m}},\end{aligned}$$

such that the following properties hold:

$$e_{3m,\mathrm{brd},n,1} = e'_{3m,\mathrm{brd},n,1}, \qquad e_{3m,\mathrm{brd},n,k} = e'_{3m,\mathrm{brd},n,k} - e'_{3m,\mathrm{brd},n,k-1}$$

for $2 \le k \le 2^{M_{3m}}$ and $1 \le n \le P_{3m}$; moreover, for $2 \le j \le 2^{2B_{3m+1}}$ and $1 \le i \le L'_{3m}$,

$$e_{3m+1,\mathrm{aux},i,1} = e'_{3m+1,\mathrm{aux},i,1}, \qquad e_{3m+1,\mathrm{aux},i,j} = e'_{3m+1,\mathrm{aux},i,j} - e'_{3m+1,\mathrm{aux},i,j-1};$$

moreover, for $1 \le d \le L_{3m}$,

$$e_{d,3m+1,\mathrm{aux},s,1} = e'_{d,3m+1,\mathrm{aux},s,1}, \qquad e_{d,3m+1,\mathrm{aux},s,t} = e'_{d,3m+1,\mathrm{aux},s,t} - e'_{d,3m+1,\mathrm{aux},s,t-1}$$

for $2 \le t \le 2^{2B_{d,3m+1}}$ and $1 \le s \le A_{d,3m+1} - 1$; analogously, for $2 \le t \le 2^{2B_{0,3m+1}}$,

$$\begin{gathered} e_{d,3m+1,\mathrm{aux},A_{d,3m+1},1} = e'_{d,3m+1,\mathrm{aux},A_{d,3m+1},1}, \\ e_{d,3m+1,\mathrm{aux},A_{d,3m+1},t} = e'_{d,3m+1,\mathrm{aux},A_{d,3m+1},t} - e'_{d,3m+1,\mathrm{aux},A_{d,3m+1},t-1};\end{gathered}$$

moreover, for $1 \le n \le P_{d,3m+1}$,

$$e_{d,3m+1,n,1} = e'_{d,3m+1,n,1}, \qquad e_{d,3m+1,n,k} = e'_{d,3m+1,n,k} - e'_{d,3m+1,n,k-1}$$

for $2 \le k \le 2^{M_{d,3m+1}}$, moreover, for $2 \le l \le 2^{4Q_{d,3m+1}}$ and $1 \le k \le 2^{M_{d,3m+1}}$,

$$e_{d,3m+1,n,k,1} = e'_{d,3m+1,n,k,1}, \qquad e_{d,3m+1,n,k,l} = e'_{d,3m+1,n,k,l} - e'_{d,3m+1,n,k,l-1};$$

moreover, setting for $1 \le d \le L_{3m}$,

$$E_{d,3m+1,0} = \mathrm{span}(e_{d,3m+1,0,n})_{n=1}^{\widetilde{L}_{d,3m+1,0}}, \qquad E_{3m+1,0} = \mathrm{span}(e_{3m+1,0,n})_{n=1}^{\widetilde{L}_{3m+1,0}},$$

the following properties hold:

- $((e'_{3m,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}}$, $((e_{3m,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}}$ and $E_{3m+1,0}$,
- $((e'_{3m+1,\mathrm{aux},i,j})_{j=1}^{2^{2B_{3m+1}}})_{i=1}^{L'_{3m}}$, $((e_{3m+1,\mathrm{aux},i,j})_{j=1}^{2^{2B_{3m+1}}})_{i=1}^{L'_{3m}}$ and $E_{3m+1,0}$,

moreover, for $1 \le d \le L_{3m}$,

- $((e'_{d,3m+1,\mathrm{aux},s,t})_{t=1}^{2^{2B_{d,3m+1}}})_{s=1}^{A_{d,3m+1}-1}$, $((e_{d,3m+1,\mathrm{aux},s,t})_{t=1}^{2^{2B_{d,3m+1}}})_{s=1}^{A_{d,3m+1}-1}$ and $E_{d,3m+1,0}$,
- $(e'_{d,3m+1,\mathrm{aux},A_{d,3m+1},t})_{t=1}^{2^{2B_{0,3m+1}}}$, $(e_{d,3m+1,\mathrm{aux},A_{d,3m+1},t})_{t=1}^{2^{2B_{0,3m+1}}}$ and $E_{d,3m+1,0}$,
- $((e'_{d,3m+1,n,k})_{k=1}^{2^{M_{d,3m+1}}})_{n=1}^{P_{d,3m+1}}$, $((e_{d,3m+1,n,k})_{k=1}^{2^{M_{d,3m+1}}})_{n=1}^{P_{d,3m+1}}$ and $E_{d,3m+1,0}$,

- $(((e'_{d,3m+1,n,k,l})_{l=1}^{2^{4Q_{d,3m+1}}})_{k=1}^{2^{M_{d,3m+1}}})_{n=1}^{P_{d,3m+1}}$, $(((e_{d,3m+1,n,k,l})_{l=1}^{2^{4Q_{d,3m+1}}})_{k=1}^{2^{M_{d,3m+1}}})_{n=1}^{P_{d,3m+1}}$ and $E_{d,3m+1,0}$,

have the same properties of $((e'_{r,n})_{n=1}^{2^N})_{r=1}^{2^R}$, $((e_{r,n})_{n=1}^{2^N})_{r=1}^{2^R}$ and E'_0 of (39.3) and (39.4) of Th. 11, for $(n)_{n=1}^{2^N}$ and $(r)_{r=1}^{2^R}$ replaced respectively by

- $(((n,k))_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}}$ and $\{1\}$,
- $(((i,j))_{j=1}^{2^{2B_{3m+1}}})_{i=1}^{L'_{3m}}$ and $\{1\}$,
- $(((s,t))_{t=1}^{2^{2B_{d,3m+1}}})_{s=1}^{A_{d,3m+1}-1}$ and $\{1\}$,
- $(A_{d,3m+1},t)_{t=1}^{2^{2B_{0,3m+1}}}$ and $\{1\}$,
- $(((n,k))_{k=1}^{2^{M_{d,3m+1}}})_{n=1}^{P_{d,3m+1}}$ and $\{1\}$,
- $(((n,k,l))_{l=1}^{2^{4Q_{d,3m+1}}})_{n=1}^{P_{d,3m+1}}$ and $((n,k,l))_{k=1}^{2^{M_{d,3m+1}}}$;

in particular all the previous properties continue to hold also if, for each d with $1 \le d \le L_{3m}$, we replace $E_{d,3m+1,0}$ by $E_{d+i,3m+1,0}$ for $1 \le i \le L_{3m} - d$ or by $E_{3m+1,0}$; finally the following property also holds, for each sequence $(a_{3m+1,n})_{n=1}^{\widetilde{L}_{3m+1}}$ of numbers:

$$
\begin{aligned}
&\Big\| \sum_{n=1}^{\widetilde{L}_{3m+1}} a_{3m+1,n}\widetilde{e}_{3m+1,n} + E_{3m+1,0}\Big\| \\
&= \Big\| \sum_{n=1}^{P_{3m}} (a_{3m,n}\omega'_{3m,n} + \sum_{k=1}^{2^{M_{3m}}} a_{3m,\mathrm{brd},n,k}e_{3m,\mathrm{brd},n,k}) + \sum_{i=1}^{L'_{3m}} \sum_{j=1}^{2^{2B_{3m+1}}} a_{3m+1,\mathrm{aux},i,j}e_{3m+1,\mathrm{aux},i,j} \\
&\qquad + \sum_{d=1}^{L_{3m}} \Big(\sum_{s=A''_{d,3m+1}+1}^{A_{d,3m+1}-1} a_{d,3m+1,\mathrm{aux},s}e_{d,3m+1,\mathrm{aux},s} \\
&\qquad + \sum_{s=1}^{A_{d,3m+1}-1} \sum_{t=1}^{2^{2B_{d,3m+1}}} a_{d,3m+1,\mathrm{aux},s,t}e_{d,3m+1,\mathrm{aux},s,t} \\
&\qquad + \sum_{t=1}^{2^{2B_{0,3m+1}}} a_{d,3m+1,\mathrm{aux},A_{d,3m+1},t}e_{d,3m+1,\mathrm{aux},A_{d,3m+1},t} \\
&\qquad + \sum_{n=1}^{P_{d,3m+1}} \sum_{k=1}^{2^{M_{d,3m+1}}} (a_{d,3m+1,n,k}e_{d,3m+1,n,k} \\
&\qquad + \sum_{l=1}^{2^{4Q_{d,3m+1}}} a_{d,3m+1,n,k,l}e_{d,3m+1,n,k,l}\Big) + E_{3m+1,0}\Big\| \\
&= \max\Big(\max(|a_{3m,n}| : 1 \le n \le P_{3m}), \max(|a_{3m,\mathrm{brd},n,k}| : 1 \le k \le 2^{M_{3m}}, 1 \le n \le P_{3m}), \\
&\qquad \max(|a_{3m+1,\mathrm{aux},i,j}| : 1 \le j \le 2^{2B_{3m+1}}, 1 \le i \le L'_{3m}), \\
&\qquad \max(|a_{d,3m+1,\mathrm{aux},s}| : A''_{d,3m+1} + 1 \le s \le A_{d,3m+1} - 1, 1 \le d \le L_{3m}), \\
&\qquad \max(|a_{d,3m+1,\mathrm{aux},s,t}| : 1 \le s \le A_{d,3m+1} - 1, 1 \le t \le 2^{2B_{d,3m+1}}, 1 \le d \le L_{3m}) \\
&\qquad \max(|a_{d,3m+1,\mathrm{aux},A_{d,3m+1},t}| : 1 \le t \le 2^{2B_{0,3m+1}}, 1 \le d \le L_{3m}),
\end{aligned}
$$

$$\max(|a_{d,3m+1,n,k}| : 1 \le k \le 2^{M_{d,3m+1}}, 1 \le n \le P_{d,3m+1}, 1 \le d \le L_{3m}),$$

$$\max\Big(\Big\|\sum_{k=1}^{2^{M_{d,3m+1}}} a_{d,3m+1,n,k,l}e_{d,3m+1,n,k,l} + E_{3m+1,0}\Big\| : \\ 1 \le l \le 2^{4Q_{d,3m+1}}, 1 \le n \le P_{d,3m+1}, 1 \le d \le L_{3m}\Big)\Big).$$

STEP 3 (construction of the connection sequences). (i) First we pass from $(b'_{3m,i}, b'^{*}_{3m,i})_{i=1}^{L'_{3m}}$ of SC IV.1 to $(b_{3m,d}, b^{*}_{3m,d})_{d=1}^{L_{3m}} = ((b_{3m,i,j}, b^{*}_{3m,i,j})_{i=0}^{2^{2B_{3m+1}}})_{i=1}^{L'_{3m}}$ where, for $1 \le j \le 2^{2B_{3m+1}}$ and $1 \le i \le L'_{3m}$,

$$b_{3m,i,0} = \sum_{g=1}^{2^{2B_{3m+1}}} e_{3m+1,\mathrm{aux},i,g}, \qquad b^{*}_{3m,i,0} = \sum_{g=1}^{2^{2B_{3m+1}}} \frac{e^{*}_{3m+1,\mathrm{aux},i,g}}{2^{2B_{3m+1}}} - \frac{b'^{*}_{3m,i}}{2^{B_{3m+1}}},$$

$$b_{3m,i,j} = 7e_{3m+1,\mathrm{aux},i,j} + \frac{b'_{3m,i}}{2^{B_{3m+1}}},$$

$$\begin{aligned} b^{*}_{3m,i,j} &= e^{*}_{3m+1,\mathrm{aux},i,j} - b^{*}_{3m,i,0} \\ &= \Big(e^{*}_{3m+1,\mathrm{aux},i,j} - \sum_{g=1}^{2^{2B_{3m+1}}} \frac{e^{*}_{3m+1,\mathrm{aux},i,g}}{2^{2B_{3m+1}}}\Big) + \frac{b'^{*}_{3m,i}}{2^{B_{3m+1}}}. \end{aligned}$$

(ii) Now, for each d with $1 \le d \le L_{3m}$, we start from the system

$$\begin{aligned} &(u'_{d,3m+1,s}, u'^{*}_{d,3m+1,s})_{s=1}^{A''_{d,3m+1}} \cup (e_{d,3m+1,\mathrm{aux},s}, e^{*}_{d,3m+1,\mathrm{aux},s})_{s=A''_{d,3m+1}+1}^{A_{d,3m+1}-1} \\ &\quad \cup ((e_{d,3m+1,\mathrm{aux},s,t}, e^{*}_{d,3m+1,\mathrm{aux},s,t})_{t=1}^{2^{2B_{d,3m+1}}})_{s=1}^{A_{d,3m+1}-1} \\ &\quad \cup (e_{d,3m+1,\mathrm{aux},A_{d,3m+1},t}, e^{*}_{d,3m+1,\mathrm{aux},A_{d,3m+1},t})_{t=1}^{2^{2B_{0,3m+1}}} \end{aligned}$$

where, for $d = 1$, $(u'_{1,3m+1,s}, u'^{*}_{1,3m+1,s})_{s=1}^{A''_{1,3m+1}}$ is the system of SC IV.1, while for $2 \le d \le L_{3m}$,

$$(u'_{d,3m+1,s}, u'^{*}_{d,3m+1,s})_{s=1}^{A'''_{d,3m+1}} = (e_{d-1,3m+1,0,n}, e^{*}_{d-1,3m+1,0,n})_{n=1}^{\widetilde{L}_{d-1,3m+1,0}},$$

$$(u'_{d,3m+1,s}, u'^{*}_{d,3m+1,s})_{s=A'''_{d,3m+1}+1}^{A''_{d,3m+1}} = (u''_{d,3m+1,s}, u''^{*}_{d,3m+1,s})_{s=1}^{P''_{d,3m+1}}$$

(hence $A'''_{d,3m+1} = \widetilde{L}_{d-1,3m+1,0}$ and $A''_{d,3m+1} = A'''_{d,3m+1} + P''_{d,3m+1}$). Then we set (we recall that $P_{d,3m+1} = (2^{2B_{d,3m+1}} + 1)(A_{d,3m+1} - 1) + 2^{2B_{0,3m+1}} + 1$)

$$(u'_{d,3m+1,s}, u'^{*}_{d,3m+1,s})_{s=A''_{d,3m+1}+1}^{A_{d,3m+1}-1} = (e_{d,3m+1,\mathrm{aux},s}, e^{*}_{d,3m+1,\mathrm{aux},s})_{s=A''_{d,3m+1}+1}^{A_{d,3m+1}-1},$$

$$(u_{d,3m+1,n}, u^{*}_{d,3m+1,n})_{n=1}^{P_{d,3m+1}-2^{2B_{0,3m+1}}-1} = ((u_{d,3m+1,s,t}, u^{*}_{d,3m+1,s,t})_{t=0}^{2^{2B_{3m+1}}})_{s=1}^{A_{d,3m+1}-1},$$

$$(u_{d,3m+1,n}, u^{*}_{d,3m+1,n})_{n=P_{d,3m+1}-2^{2B_{0,3m+1}}}^{P_{d,3m+1}} \\ = (u_{d,3m+1,A_{d,3m+1},t}, u^{*}_{d,3m+1,A_{d,3m+1},t})_{t=0}^{2^{2B_{0,3m+1}}},$$

where, for $1 \le s \le A_{d,3m+1} - 1$ and $1 \le t \le 2^{2B_{d,3m+1}}$,

$$u_{d,3m+1,s,0} = \sum_{g=1}^{2^{2B_{d,3m+1}}} e_{d,3m+1,\mathrm{aux},s,g}, \; u^*_{d,3m+1,s,0} = \sum_{g=1}^{2^{2B_{d,3m+1}}} \frac{e^*_{d,3m+1,\mathrm{aux},s,g}}{2^{2B_{d,3m+1}}} - \frac{u'^*_{d,3m+1,s}}{2^{B_{d,3m+1}}},$$

$$\begin{aligned} u_{d,3m+1,s,t} &= e_{d,3m+1,\mathrm{aux},s,t} + \frac{u'_{d,3m+1,s}}{2^{B_{d,3m+1}}}, \\ u^*_{d,3m+1,s,t} &= e^*_{d,3m+1,\mathrm{aux},s,t} - u^*_{d,3m+1,s,0} \\ &= \left(e^*_{d,3m+1,\mathrm{aux},s,t} - \sum_{g=1}^{2^{2B_{d,3m+1}}} \frac{e^*_{d,3m+1,\mathrm{aux},s,g}}{2^{2B_{d,3m+1}}} \right) + \frac{u'^*_{d,3m+1,s}}{2^{B_{d,3m+1}}}, \end{aligned}$$

while, for $s = A_{d,3m+1}$ and $1 \le t \le 2^{2B_{0,3m+1}}$,

$$\begin{aligned} u_{d,3m+1,A_{d,3m+1},0} &= \sum_{g=1}^{2^{2B_{0,3m+1}}} e_{d,3m+1,\mathrm{aux},A_{d,3m+1},g}, \\ u^*_{d,3m+1,A_{d,3m+1},0} &= \sum_{g=1}^{2^{2B_{0,3m+1}}} \frac{e^*_{d,3m+1,\mathrm{aux},A_{d,3m+1},g}}{2^{2B_{0,3m+1}}} - \frac{b^*_{3m,d}}{2^{B_{0,3m+1}}}, \\ u_{d,3m+1,A_{d,3m+1},t} &= e_{d,3m+1,\mathrm{aux},A_{d,3m+1},t} + \frac{b_{3m,d}}{2^{B_{0,3m+1}}}, \\ u^*_{d,3m+1,A_{d,3m+1},t} &= e^*_{d,3m+1,\mathrm{aux},A_{d,3m+1},t} - u^*_{d,3m+1,A_{d,3m+1},0} \\ &= \left(e^*_{d,3m+1,\mathrm{aux},A_{d,3m+1},t} - \sum_{g=1}^{2^{2B_{0,3m+1}}} \frac{e^*_{d,3m+1,\mathrm{aux},A_{d,3m+1},g}}{2^{2B_{0,3m+1}}} \right) + \frac{b^*_{3m,d}}{2^{B_{0,3m+1}}}. \end{aligned}$$

Now, by means of the procedures of GBST and MGBS, we pass to the generating biorthogonal system $(w_{d,3m+1,n}, w^*_{d,3m+1,n})_{n=1}^{P_{d,3m+1}}$ and then to $(v_{d,3m+1,n}, v^*_{d,3m+1,n})_{n=1}^{P_{d,3m+1}}$, where

$$\begin{aligned} (v_{d,3m+1,n}, v^*_{d,3m+1,n})_{n=1}^{P_{d,3m+1}} &= ((v_{d,3m+1,2n-1}, v^*_{d,3m+1,2n-1}) \\ &\qquad \cup (v_{d,3m+1,2n}, v^*_{d,3m+1,2n}))_{n=1}^{P_{d,3m+1}/2}, \\ v_{d,3m+1,2n-1} &= w_{d,3m+1,2n} + w_{d,3m+1,2n-1}/2^{M_{d,3m+1}+Q_{d,3m+1}}, \\ v_{d,3m+1,2n} &= w_{d,3m+1,2n} - w_{d,3m+1,2n-1}/2^{M_{d,3m+1}+Q_{d,3m+1}}, \\ v^*_{d,3m+1,2n-1} &= (w^*_{d,3m+1,2n} + 2^{M_{d,3m+1}+Q_{d,3m+1}} w^*_{d,3m+1,2n-1})/2, \\ v^*_{d,3m+1,2n} &= (w^*_{d,3m+1,2n} - 2^{M_{d,3m+1}+Q_{d,3m+1}} w^*_{d,3m+1,2n-1})/2; \end{aligned}$$

finally (see Step 1), for $1 \le d \le L_{3m}$, $1 \le n \le P_{d,3m+1}$ and $1 \le g \le G_{d,0,3m+1,0}$,

$$\begin{gathered} x_{d,0,3m+1,n,g} = x'_{d,0,3m+1,n,g} + \widetilde{x}_{d,0,3m+1,n,g}, \\ (\widetilde{x}_{d,0,3m+1,n,g})_{g=1}^{G_{d,0,3m+1,0}} = (\widetilde{x}_{d,3m+1,n,g})_{g=1}^{G_{d,3m+1,0}} \end{gathered}$$

where, for $1 \le g \le G_{d,3m+1,0}$,

$$\begin{aligned} \widetilde{x}_{d,3m+1,n,g} &= \widetilde{x}''_{d,3m+1,n,g} + \widetilde{x}'''_{d,3m+1,n,g}, \\ \widetilde{x}''_{d,3m+1,n,g} &= v''_{d,3m+1,n}/2^{P_{d,3m+1}Q_{d,0,3m+1}}, \\ \widetilde{x}'''_{d,3m+1,n,g} &= v'''_{d,3m+1,n}/2^{P_{d,3m+1}Q_{d,0,3m+1}}, \end{aligned}$$

$$v_{d,3m+1,n} = v''_{d,3m+1,n} + v'''_{d,3m+1,n}, \ (\widetilde{x}''_{d,3m+1,n,g})_{g=1}^{G_{d,3m+1,0}} \subset \operatorname{span}(\widetilde{e}_{d,3m+1,n})_{n=1}^{\widetilde{L}_{d,3m+1}},$$

$$\sum_{n=1}^{P_{d,3m+1}} v'''^{*}_{d,3m+1,n}(\overline{x})v'''_{d,3m+1,n} = \sum_{s=1}^{A''_{d,3m+1}} u'^{*}_{d,3m+1,s}(\overline{x})u_{d,3m+1,s} + b^{*}_{3m,d}(\overline{x})b_{3m,d}.$$

STEP 4 (construction of the support sequence). Let us fix d with $2 \le d \le L_{3m}$ and suppose we have already defined (see (ii) of Step 3)

$$\begin{aligned} B_{d,1}(3m+1) = {} & (x_n, x_n^*)_{n=1}^{q(3m+1)} \\ & \cup ((x_{c,0,3m+1,g}, x^*_{c,0,3m+1,g})_{g=1}^{G_{c,0,3m+1}})_{c=1}^{d-1} \cup (u'_{d,3m+1,s}, u'^{*}_{d,3m+1,s})_{s=1}^{A'''_{d,3m+1}} \end{aligned}$$

where we recall that we still have to define $(x_{\mathrm{brd},n})_{n=q(3m)+1}^{q(3m+1)} \cup (b_{3m,d})_{d=1}^{L_{3m}}$; therefore let us set

$$\begin{aligned} X_{d,0,3m+1} = {} & \operatorname{span}((x_n)_{n=1}^{q(3m)} \cup (x_n - x_{\mathrm{brd},n})_{n=q(3m)+1}^{q(3m+1)} \\ & \cup ((\widehat{x}_{c,0,3m+1,g})_{g=1}^{G_{c,0,3m+1}})_{c=1}^{d-1} \cup (u'_{d,3m+1,s})_{s=1}^{A'''_{d,3m+1}} \end{aligned}$$

where, for $1 \le c \le d-1$,

$$\begin{aligned} (\widehat{x}_{c,0,3m+1,g})_{g=1}^{G_{c,0,3m+1}} = {} & (x_{c,3m+1,n,0} \cup (x_{c,3m+1,n,k,0})_{k=1}^{2^{M_{c,3m+1}}})_{n=1}^{P_{c,3m+1}} \\ & \cup (((x_{c,3m+1,n,k,l})_{l=1}^{2^{4Q_{c,3m+1}}})_{k=1}^{2^{M_{c,3m+1}}})_{n=1}^{P_{c,3m+1}-1} \\ & \cup ((x_{c,3m+1,P_{c,3m+1},k,l})_{l=1}^{2^{4Q_{c,3m+1}}-1})_{k=1}^{2^{M_{c,3m+1}}} \\ & \cup \left(x_{c,3m+1,P_{c,3m+1},k,2^{4Q_{c,3m+1}}} - \frac{v_{c,3m+1,P_{c,3m+1}}}{2^{P_{d,3m+1}Q_{d,0,3m+1}}} \right)_{k=1}^{2^{M_{c,3m+1}}}. \end{aligned}$$

Then, setting $\widehat{E}_{3m+1} = \operatorname{span}(\widehat{e}_{3m+1,n})_{n=1}^{\widehat{L}_{3m+1}}$, if $\widehat{L}_{3m+1}$ is sufficiently large there exists a subspace $\widehat{E}_{d,\mathrm{ort},3m+1}$ of $\widehat{E}_{3m+1}$ such that we can write $\|x+e\| \ge \|x\|$ for each $x \in X_{d,0,3m+1} + E_{d-1,3m+1,0}$ and $e \in \widehat{E}_{d,\mathrm{ort},3m+1}$. Moreover we can also suppose $\widehat{L}_{3m+1}$ sufficiently large such that there exists in $\widehat{E}_{d,\mathrm{ort},3m+1}$ a block sequence $(\widehat{e}_{d,3m+1,n})_{n=1}^{\widehat{L}_{d,3m+1}}$ of $(\widehat{e}_{3m+1,n})_{n=1}^{\widehat{L}_{3m+1}}$ (actually, by means of a standard procedure, we can only get a sequence of $\widehat{E}_{d,\mathrm{ort},3m+1}$ which can approximate a block sequence of $(\widehat{e}_{3m+1,n})_{n=1}^{\widehat{L}_{3m+1}}$, however we always follow the idea of the first part of Subsection 1.5). At this point we have only to follow the same procedure of the first sub-block (which is the procedure of Step 1 of SC III.2, precisely a more simplified version since now we do not need the analogue of the elements $\omega_{3m+1,n}$, $\omega'_{3m+1,n}$, $e_{3m+1,\mathrm{brd},n,k}$ of SC III.2), where we have to replace $(\widehat{e}_{3m+1,n})_{n=1}^{\widehat{L}_{3m+1}}$ by $(\widehat{e}_{d,3m+1,n})_{n=1}^{\widehat{L}_{d,3m+1}}$. Then, if $2 \le d \le L_{3m} - 1$, we can define the biorthogonal system

$$(\widetilde{e}_{d,3m+1,n}, \widetilde{e}^{*}_{d,3m+1,n})_{n=1}^{\widetilde{L}_{d,3m+1}} \cup (e_{0,d,3m+1,0,n}, e^{*}_{0,d,3m+1,0,n})_{n=1}^{\widetilde{L}_{o,d,3m+1,0}}$$

where $(\widetilde{e}_{d,3m+1,n}, \widetilde{e}^{*}_{d,3m+1,n})_{n=1}^{\widetilde{L}_{d,3m+1}}$ is the biorthogonal system of the first part of Step 2, moreover

$$(e_{0,d,3m+1,0,n})_{n=1}^{\widetilde{L}_{o,d,3m+1,0}} = ((e_{d,c,3m+1,0,n})_{n=1}^{\widetilde{L}_{d,c,3m+1,0}})_{c=d}^{L_{3m}}$$

so that, for each c with $d+1 \le c \le L_{3m}$, $(e_{d,c,3m+1,0,n}, e^{*}_{d,c,3m+1,0,n})_{n=1}^{\widetilde{L}_{d,c,3m+1,0}}$ has, for what concerns $(\widetilde{e}_{d,3m+1,n}, \widetilde{e}^*_{d,3m+1,n})_{n=1}^{\widetilde{L}_{d,3m+1}}$, the same properties of

$$(e_{3m+1,0,n}, e^*_{3m+1,0,n})_{n=1}^{M_{3m+1,0}}$$

as regards $(e'''_{3m+1,n}, e'''^*_{3m+1,n})_{n=1}^{S_{3m+1}}$ in Step 1 of SC III.2; therefore, if according to the definition of Step 2,

$$(e_{d,3m+1,0,n})_{n=1}^{\widetilde{L}_{d,3m+1,0}} = ((e_{c,d,3m+1,0,n})_{n=1}^{\widetilde{L}_{d,c,3m+1,0}})_{c=1}^{d},$$

we have the proof of the property of Step 2 that we can replace $E_{d,3m+1,0}$ by $E_{d+i,3m+1,0}$ for $1 \leq i \leq L_{3m} - d$. We point out that, for $d = 1$, $(\widetilde{e}_{1,3m+1,n})_{n=1}^{\widetilde{L}_{1,3m+1}}$ has to comprise also the sequence $((e_{3m+1,\mathrm{aux},i,j})_{j=1}^{2^{2^{B_{0,3m+1}}}})_{i=1}^{L'_{3m}}$. Finally, if $d = L_{3m}$, we will define

$$(\widetilde{e}_{L_{3m},3m+1,n}, \widetilde{e}^*_{L_{3m},3m+1,n})_{n=1}^{\widetilde{L}_{L_{3m},3m+1}} \cup (e_{L_{3m},3m+1,0,n}, e^*_{L_{3m},3m+1,0,n})_{n=1}^{\widetilde{L}_{L_{3m},3m+1,0}}$$
$$\cup\, ((\omega'_{3m,n}, \omega'^*_{3m,n}) \cup (e_{3m,\mathrm{brd},n,k}, e^*_{3m,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}}.$$

Step 5 (completion of the construction of the connection sequence). Suppose $1 \leq d \leq L_{3m} - 1$. By means of the procedure of C II and of Lemma 12, where now we replace $(y_n)_{n=1}^{Q(m)}$ by the sequence (see the first part of Step 4)

$$(x_n)_{n=1}^{q(3m)} \cup (x_n - x_{\mathrm{brd},n})_{n=q(3m)+1}^{q(3m+1)} \cup ((\widehat{x}_{c,0,3m+1,g})_{g=1}^{G_{c,0,3m+1}})_{c=1}^{d-1} \cup (u'_{d,3m+1,s})_{s=1}^{A'''_{d,3m+1}},$$
$$(u'_{d,3m+1,s})_{s=1}^{A'''_{d,3m+1}} = (e_{d-1,3m+1,0,n})_{n=1}^{\widetilde{L}_{d-1,3m+1,0}},$$

we can define

$$(u''_{d,3m+1,s}, u''^*_{d,3m+1,s})_{s=1}^{P''_{d,3m+1}} = (u'_{d,3m+1,s}, u'^*_{d,3m+1,s})_{s=A'''_{d,3m+1}+1}^{A''_{d,3m+1}}$$

and we are ready for the construction of the dth sub-block. For $d = L_{3m}$ suppose have also defined

$$(x_{L_{3m},0,3m+1,g}, x^*_{L_{3m},0,3m+1,g})_{g=1}^{G_{L_{3m},0,3m+1}} = (x_n, x^*_n)_{n=q(3m+2)-G_{L_{3m},0,3m+1}+1}^{q(3m+2)}.$$

If we replace $(y_n)_{n=1}^{Q(m)}$ by

$$(x_n)_{n=1}^{q(3m+2)} \cup (u'_{3m+2,s})_{s=1}^{A'''_{d,3m+1}}, (u'_{3m+2,s})_{s=1}^{A'''_{d,3m+1}} = (e_{L_{3m},3m+1,0,n})_{n=1}^{\widetilde{L}_{L_{3m},3m+1,0}},$$

by means of the procedure of C II and of Lemma 2 we get

$$(u''_{3m+2,s}, u''^*_{3m+2,s})_{s=1}^{P''_{3m+2}} \cup (v'_{3m+3,s}, v'^*_{3m+3,s})_{s=1}^{P''_{3m+2}}$$

and, setting $(u'_{3m+2,s})_{s=A'''_{d,3m+1}+1}^{A''_{d,3m+1}} = (u''_{3m+2,s})_{s=1}^{P''_{3m+2}}$, we are ready for the construction of SC IV.3, where we can see that the sequence $(u'_{3m+2,s})_{s=A_{3m+2}-A'_{3m+2}+1}^{A_{3m+2}}$ of SC III.3 does not appear.

Subconstruction IV.3 (SC IV.3, construction of the free block). Again the same construction of SC III.3 works, apart from the fact that now the procedure of SC IV.2 will cause that the analogue of the system $(u'_{3m+2,s}, u'^*_{3m+2,s})_{s=A_{3m+2}-A'_{3m+2}+1}^{A_{3m+2}}$ of SC III.3 will not appear.

This completes C IV.

4. Properties of each block

The next lemmas concern the properties of the block $(x_n, x_n^*)_{n=q(3m)+1}^{q(3m+3)}$.

The first lemma allows the construction of the "bricks" (precisely, particular subsums of $\sum_{n=q(3m+1)+1}^{q(3m+2)} x_n^*(\overline{x})x_n$) necessary to form the elements able to "regularize" the sum $\sum_{n=q(3m)+1}^{q(3m+1)} x_n^*(\overline{x})x_n$, that is, to transform it into another sum where the sequence of the norms of the partial subsums is $(0,\varepsilon_m)$-monotone for $\varepsilon_m \to 0$.

LEMMA 13 (Regularization Block Lemma, RBL). *Let $\overline{x} \in X$ with $\|\overline{x}\| = 1$ and fix n with $1 \le n \le P_{3m+1}$ so that $n = 2\overline{n} - 1$ for some $\overline{n}$ with $1 \le \overline{n} \le P_{3m+1}/2$. Suppose that $|w^*_{3m+1,2\overline{n}-1}(\overline{x})| > 1$. Then*

$$\text{(o)}\quad \frac{1}{4}\frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{3Q_{3m+1}}} < \frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{M_{3m+1}+4Q_{3m+1}}}|v^*_{3m+1,n}(\overline{x})| < Q_{0,3m+1}\frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{3Q_{3m+1}}};$$

moreover there exist two integers $\overline{k}$ and $\widetilde{k}$, with $1 \le \overline{k}, \widetilde{k} \le 2^{M_{3m+1}}$, so that

$$\text{(i)}\quad e''^{*}_{0,3m+1,\mathrm{brd},n,\overline{k}}(\overline{x}) < \frac{1}{2}\frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{M_{3m+1}+4Q_{3m+1}}}v^*_{3m+1,n}(\overline{x}) \quad \text{if } v^*_{3m+1,n}(\overline{x}) > 0,$$

$$e''^{*}_{0,3m+1,\mathrm{brd},n,\overline{k}}(\overline{x}) > -\frac{1}{2}\frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{M_{3m+1}+4Q_{3m+1}}}v^*_{3m+1,n}(\overline{x}) \quad \text{if } v^*_{3m+1,n}(\overline{x}) < 0;$$

$$\text{(ii)}\quad \frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{2Q_{3m+1}}}\Bigg(e^*_{3m+1,n,\widetilde{k}}(\overline{x}) - \frac{1}{2^{M_{3m+1}}}\sum_{f=1}^{2^{M_{3m+1}}} e^*_{3m+1,n,f}(\overline{x})\Bigg) > -\frac{1}{2}\frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{M_{3m+1}+4Q_{3m+1}}}v^*_{3m+1,n}(\overline{x}) \quad \text{if } v^*_{3m+1,n}(\overline{x}) > 0,$$

$$\frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{2Q_{3m+1}}}\Bigg(e^*_{3m+1,n,\widetilde{k}}(\overline{x}) - \frac{1}{2^{M_{3m+1}}}\sum_{f=1}^{2^{M_{3m+1}}} e^*_{3m+1,n,f}(\overline{x})\Bigg) < \frac{1}{2}\frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{M_{3m+1}+4Q_{3m+1}}}v^*_{3m+1,n}(\overline{x}) \quad \text{if } v^*_{3m+1,n}(\overline{x}) < 0;$$

$$\text{(iii)}\quad \sum_{g=1}^{2^{4Q_{3m+1}}-1} |e^*_{3m+1,n,\overline{k},g}(\overline{x})| + \sum_{g=1}^{2^{4Q_{3m+1}}-1} |e^*_{3m+1,n,\widetilde{k},g}(\overline{x})| < \frac{1}{2^{4Q_{3m+1}}};$$

$$\text{(iv) in particular } |e'''^{*}_{3m+1,\mathrm{brd},n,\overline{k}}(\overline{x})| > \frac{1}{9}\frac{1}{2^{3Q_{3m+1}}};$$

(v) *for each number a with $|a| < Q_{0,3m+1}$, there exists an integer $L(a)$ with $1 \le L(a) < Q_{0,3m+1}2^{3Q_{3m+1}}$ such that*

$$\Bigg\|\sum_{l=1}^{L(a)} x^*_{3m+1,n,\widetilde{k},l}(\overline{x})x_{3m+1,n,\widetilde{k},l} - aw_{3m+1,2\overline{n}}\Bigg\| < \frac{1}{2^{Q_{0,3m+1}2P_{3m+1}}};$$

in particular $n = 2\overline{n} - 1$ if a and $w^*_{3m+1,2\overline{n}-1}(\overline{x})$ have the same signs, while $n = 2\overline{n}$ if a and $w^*_{3m+1,2\overline{n}-1}(\overline{x})$ have opposite signs.

Proof. In this proof we refer to the steps of SC III.2.

By Step 2, in particular by the definition of $Q_{0,3m+1}$ and by the relations among $Q_{0,3m+1}$, Q_{3m+1} and M_{3m+1}, since $|w^*_{3m+1,2\overline{n}-1}(\overline{x})| > 1$ and $|w^*_{3m+1,2\overline{n}}(\overline{x})| \le \|w^*_{3m+1,2\overline{n}}\|$

$< Q_{0,3m+1} < Q_{3m+1}$, we have

$$\begin{aligned}
&\frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{M_{3m+1}+4Q_{3m+1}}}|v^*_{3m+1,n}(\overline{x})| \\
&\qquad = \frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{M_{3m+1}+4Q_{3m+1}}}\frac{1}{2}|w^*_{3m+1,2\overline{n}}(\overline{x}) + 2^{M_{3m+1}+Q_{3m+1}}w^*_{3m+1,2\overline{n}-1}(\overline{x})| \\
&\qquad > \frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{M_{3m+1}+4Q_{3m+1}}}\frac{1}{4}2^{M_{3m+1}+Q_{3m+1}}|w^*_{3m+1,2\overline{n}-1}(\overline{x})| \\
&\qquad = \frac{2^{P_{3m+1}Q_{0,3m+1}}}{4}\frac{|w^*_{3m+1,2\overline{n}-1}(\overline{x})|}{2^{3Q_{3m+1}}} > \frac{1}{4}\frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{3Q_{3m+1}}};
\end{aligned}$$

analogously

$$\begin{aligned}
&\frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{M_{3m+1}+4Q_{3m+1}}}|v^*_{3m+1,n}(\overline{x})| \\
&\qquad\qquad = \frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{M_{3m+1}+4Q_{3m+1}}}\frac{1}{2}|w^*_{3m+1,2\overline{n}}(\overline{x}) + 2^{M_{3m+1}+Q_{3m+1}}w^*_{3m+1,2\overline{n}-1}(\overline{x})| \\
&\qquad\qquad \le \frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{M_{3m+1}+4Q_{3m+1}}}\frac{1}{2}(|w^*_{3m+1,2\overline{n}}(\overline{x})| + 2^{M_{3m+1}+Q_{3m+1}}|w^*_{3m+1,2\overline{n}-1}(\overline{x})|) \\
&\qquad\qquad < \frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{M_{3m+1}+4Q_{3m+1}}}\frac{1}{2}(Q_{0,3m+1} + 2^{M_{3m+1}+Q_{3m+1}}Q_{0,3m+1}) \\
&\qquad\qquad = Q_{0,3m+1}\frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{M_{3m+1}+4Q_{3m+1}}}\frac{1+2^{M_{3m+1}+Q_{3m+1}}}{2} \\
&\qquad\qquad < Q_{0,3m+1}\frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{M_{3m+1}+4Q_{3m+1}}}2^{M_{3m+1}+Q_{3m+1}} = Q_{0,3m+1}\frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{3Q_{3m+1}}};
\end{aligned}$$

that is, (o) has been proved.

Now let us prove (ii) (since the proof of (i) is analogous). It is sufficient to consider only the case of $w^*_{3m+1,2\overline{n}-1}(\overline{x})$ positive hence $v^*_{3m+1,n}(\overline{x}) > 0$. By the end of Step 1 we know that, if we set, for $1 \le k \le 2^{M_{3m+1}}$,

$$a_k = \frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{2Q_{3m+1}}}\left(e^*_{3m+1,n,k}(\overline{x}) - \sum_{f=1}^{2^{M_{3m+1}}}\frac{e^*_{3m+1,n,f}(\overline{x})}{2^{M_{3m+1}}}\right),$$

then

$$|a_k| < 4\frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{2Q_{3m+1}}} \quad \text{and} \quad \sum_{k=1}^{2^{M_{3m+1}}} a_k = 0.$$

Now suppose we have a sequence $(a_i)_{i=1}^N$ of real numbers with

$$\sum_{i=1}^N a_i = 0, \qquad |a_i| < c = 42^{P_{3m+1}Q_{0,3m+1}}/2^{2Q_{3m+1}} \qquad \text{for } 1 \le i \le N,$$

and set

$$b = \frac{1}{4}\frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{3Q_{3m+1}}}.$$

Our aim is to estimate the lowest $\overline{N}$ such that always there exists $(i')_{i=1}^{N'} \subset (i)_{i=1}^{\overline{N}}$ with $a_{i'} > -b/2$ for $1 \le i \le N'$, with $N' > 2^{4\cdot 2^{4Q_{3m+1}}}$. In order to estimate $\overline{N}$ we consider the most unfavourable situation, which happens for instance for $a_i = c$ for $1 \le i \le N'$ and $a_i = -b/2$ for $N'+1 \le i \le \overline{N}$. Since $\sum_{i=1}^{\overline{N}} a_i = 0$ we have $cN' = (\overline{N} - N')b/2$, hence, since $M_{3m+1} > 4^{2Q_{3m+1}+4\cdot 2^{4Q_{3m+1}}}$ by Step 2, it follows that setting $N' = 2^{4\cdot 2^{4Q_{3m+1}}}$ we have

$$\begin{aligned}\overline{N} = (2c/b+1)N' &= \left(2\cdot 4\frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{2Q_{3m+1}}}\cdot\frac{4\cdot 2^{3Q_{3m+1}}}{2^{P_{3m+1}Q_{0,3m+1}}}+1\right)2^{4\cdot 2^{4Q_{3m+1}}}\\ &= (32\cdot 2^{Q_{3m+1}}+1)\cdot 2^{4\cdot 2^{4Q_{3m+1}}} < 2^{5\cdot 2^{4Q_{3m+1}}} < M_{3m+1} < 2^{M_{3m+1}}.\end{aligned}$$

Therefore we conclude that there exists $(k_i')_{i=1}^{K'} \cup (k_i'')_{i=1}^{K''} \subset (k)_{k=1}^{2^{M_{3m+1}}}$, with $\min(K', K'') > 2^{4.2^{4Q_{3m+1}}}$, such that $e''^{*}_{0,3m+1,\mathrm{brd},n,k_i'}(\overline{x}) < \frac{1}{2}b$ for $1 \le i \le K'$, and

$$\frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{2Q_{3m+1}}}\left(e^*_{3m+1,n,k_i''}(\overline{x}) - \frac{1}{2^{M_{3m+1}}}\sum_{f=1}^{2^{M_{3m+1}}} e^*_{3m+1,n,f}(\overline{x})\right) > -\frac{1}{2}b$$

for $1 \le i \le K''$. Then (iii) follows by the last properties of Step 1, by the end of (39.1) and by (39.5) of Theorem 11; therefore there exists $\widetilde{k} \in (k_i')_{i=1}^{K'}$ so that (ii) holds, and analogously there exists $\overline{k} \in (k_i'')_{i=1}^{K''}$ so that also (i) holds. By (i) and (iii), and by Step 3, it also follows that

$$\begin{aligned}&|e'''^{*}_{3m+1,\mathrm{brd},n,\overline{k}}(\overline{x})|\\ &\quad= \left|e''^{*}_{0,3m+1,\mathrm{brd},n,\overline{k}}(\overline{x}) + e''''^{*}_{3m+1,\mathrm{brd},n,\overline{k}}(\overline{x}) + \left(-\frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{M_{3m+1}+4Q_{3m+1}}}v^*_{3m+1,n}(\overline{x})\right)\right|\\ &\quad\ge \frac{1}{2}\frac{2^{P_{3m+1}Q_{0,3m+1}}|v^*_{3m+1,n}(\overline{x})|}{2^{M_{3m+1}+4Q_{3m+1}}} - |e''''^{*}_{3m+1,\mathrm{brd},n,\overline{k}}(\overline{x})|\\ &\quad\ge \frac{1}{8}\frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{3Q_{3m+1}}} - \left(\frac{|\omega^*_{3m+1,n}(\overline{x})|}{2^{4Q_{3m+1}}2^{M_{3m+1}}} + \frac{1}{2^{4Q_{3m+1}}}\sum_{g=1}^{2^{4Q_{3m+1}}-1}|e^*_{3m+1,n,\overline{k},g}(\overline{x})|\right)\\ &\quad> \frac{1}{8}\frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{3Q_{3m+1}}} - \frac{1}{2^{4Q_{3m+1}}}\left(\frac{2}{2^{M_{3m+1}}}+\frac{1}{2^{4Q_{3m+1}}}\right) > \frac{1}{9}\frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{3Q_{3m+1}}},\end{aligned}$$

hence also (iv) has been proved.

Finally, let us turn to proving (v): we can suppose that a and $w^*_{3m+1,2\overline{n}-1}(\overline{x})$, hence a and $v^*_{3m+1,n}(\overline{x})$, have the same sign (indeed, if the signs were opposite, it would be sufficient by Step 2 to use $n = 2\overline{n}$ instead of $n = 2\overline{n}-1$). Then, setting

$$\begin{aligned}V^*_{3m+1,n}(\overline{x}) = &\frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{2Q_{3m+1}}}\left(e^*_{3m+1,n,\widetilde{k}}(\overline{x}) - \frac{1}{2^{M_{3m+1}}}\sum_{f=1}^{2^{M_{3m+1}}} e^*_{3m+1,n,f}(\overline{x})\right)\\ &+\frac{2^{P_{3m+1}Q_{0,3m+1}}v^*_{3m+1,n}(\overline{x})}{2^{M_{3m+1}+4Q_{3m+1}}},\end{aligned}$$

we find, by (o), (ii), (iii) and by the beginning of this proof, that

$$(*)\qquad \frac{1}{8}\frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{3Q_{3m+1}}} < V^*_{3m+1,n}(\overline{x}) < 2Q_{0,3m+1}\frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{3Q_{3m+1}}},$$

$$(**)\qquad \frac{1}{8}\frac{1}{2^{3Q_{3m+1}}} < \frac{V^*_{3m+1,n}(\overline{x})}{2^{P_{3m+1}Q_{0,3m+1}}} < \frac{2Q_{0,3m+1}}{2^{3Q_{3m+1}}}.$$

Hence there is an integer $L(a)$ $(= 1$ if $a \le V^*_{3m+1,n}(\overline{x})/2^{P_{3m+1}Q_{0,3m+1}})$ so that

$$(L(a)-1)\frac{V^*_{3m+1,n}(\overline{x})}{2^{P_{3m+1}Q_{0,3m+1}}} < a \le L(a)\frac{V^*_{3m+1,n}(\overline{x})}{2^{P_{3m+1}Q_{0,3m+1}}}$$

whence, by the first inequality and by $(**)$,

$$\left|L(a)\frac{V^*_{3m+1,n}(\overline{x})}{2^{P_{3m+1}Q_{0,3m+1}}} - a\right| < \frac{V^*_{3m+1,n}(\overline{x})}{2^{P_{3m+1}Q_{0,3m+1}}} < \frac{2Q_{0,3m+1}}{2^{3Q_{3m+1}}},$$

$$L(a) < a\frac{2^{P_{3m+1}Q_{0,3m+1}}}{V^*_{3m+1,n}(\overline{x})} + 1 < a8\cdot 2^{3Q_{3m+1}} + 1 < 9Q_{0,3m+1}2^{3Q_{3m+1}}$$

(by the hypothesis on a), that is,

$$(***)\qquad \left|L(a)\frac{V^*_{3m+1,n}(\overline{x})}{2^{P_{3m+1}Q_{0,3m+1}}} - a\right| < \frac{2Q_{0,3m+1}}{2^{3Q_{3m+1}}}, \qquad L(a) < 9Q_{0,3m+1}2^{3Q_{3m+1}}.$$

Hence, by the definition of $v_{3m+1,n}$ and by $(***)$,

$$\begin{aligned}
&\left\|L(a)\frac{V^*_{3m+1,n}(\overline{x})}{2^{P_{3m+1}Q_{0,3m+1}}}v_{3m+1,n} - aw_{3m+1,2\overline{n}}\right\| \\
&\quad\le \left\|L(a)\frac{V^*_{3m+1,n}(\overline{x})}{2^{P_{3m+1}Q_{0,3m+1}}}v_{3m+1,n} - av_{3m+1,n}\right\| + \|av_{3m+1,n} - aw_{3m+1,2\overline{n}}\| \\
&\qquad = \left|L(a)\frac{V^*_{3m+1,n}(\overline{x})}{2^{P_{3m+1}Q_{0,3m+1}}} - a\right|\cdot\|v_{3m+1,n}\| + a\left\|\frac{w_{3m+1,2\overline{n}-1}}{2^{M_{3m+1}+Q_{3m+1}}}\right\| \\
&\qquad < \frac{2Q_{0,3m+1}}{2^{3Q_{3m+1}}}2 + \frac{2Q_{0,3m+1}}{2^{M_{3m+1}+Q_{3m+1}}} < \frac{5Q_{0,3m+1}}{2^{3Q_{3m+1}}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\left\|\sum_{l=1}^{L(a)} x^*_{3m+1,n,\widetilde{k},l}(\overline{x})x_{3m+1,n,\widetilde{k},l} - aw_{3m+1,2\overline{n}}\right\| \\
&\quad\le \left\|\sum_{l=1}^{L(a)} x^*_{3m+1,n,\widetilde{k},l}(\overline{x})x_{3m+1,n,\widetilde{k},l} - L(a)\frac{V^*_{3m+1,n}(\overline{x})}{2^{P_{3m+1}Q_{0,3m+1}}}v_{3m+1,n}\right\| \\
&\qquad + \left\|L(a)\frac{V^*_{3m+1,n}(\overline{x})}{2^{P_{3m+1}Q_{0,3m+1}}}v_{3m+1,n} - aw_{3m+1,2\overline{n}}\right\| \\
&\quad< \left\|\sum_{l=1}^{L(a)} x^*_{3m+1,n,\widetilde{k},l}(\overline{x})x_{3m+1,n,\widetilde{k},l} - L(a)\frac{V^*_{3m+1,n}(\overline{x})}{2^{P_{3m+1}Q_{0,3m+1}}}v_{3m+1,n}\right\| + \frac{5Q_{0,3m+1}}{2^{3Q_{3m+1}}}
\end{aligned}$$

(by Step 3 and by the definition of $V^*_{3m+1,n}(\overline{x})$)

$$\begin{aligned}
&= \Bigg\|\sum_{l=1}^{L(a)}\Bigg(e^*_{3m+1,n,\widetilde{k},l}(\overline{x}) - \frac{1}{2^{4Q_{3m+1}}}\Bigg(\sum_{g=1}^{2^{4Q_{3m+1}}-1} e^*_{3m+1,n,\widetilde{k},g}(\overline{x}) \\
&\quad + \Bigg(\Bigg(e^*_{3m+1,n,\widetilde{k},2^{4Q_{3m+1}}}(\overline{x}) - \frac{1}{2^{M_{3m+1}}}\sum_{f=1}^{2^{M_{3m+1}}} e^*_{3m+1,n,f,2^{4Q_{3m+1}}}(\overline{x})\Bigg) \\
&\quad + \frac{\omega'^*_{3m+1,n}(\overline{x})}{2^{M_{3m+1}}}\Bigg)\Bigg) + V^*_{3m+1,n}(\overline{x})\Bigg)\Bigg(e_{3m+1,n,\widetilde{k},l} + \frac{e_{3m+1,n,\widetilde{k}}}{2^{P_{3m+1}Q_{0,3m+1}}2^{2Q_{3m+1}}} \\
&\quad + \frac{v_{3m+1,n}}{2^{P_{3m+1}Q_{0,3m+1}}}\Bigg) - L(a)\frac{V^*_{3m+1,n}(\overline{x})}{2^{P_{3m+1}Q_{0,3m+1}}}v_{3m+1,n}\Bigg\| + \frac{5Q_{0,3m+1}}{2^{3Q_{3m+1}}}
\end{aligned}$$

$$
\begin{aligned}
&= \Big\| \sum_{l=1}^{L(a)} \Big(e^*_{3m+1,n,\widetilde{k},l}(\overline{x}) - \frac{1}{2^{4Q_{3m+1}}}\Big(\sum_{g=1}^{2^{4Q_{3m+1}}-1} e^*_{3m+1,n,\widetilde{k},g}(\overline{x}) \\
&\quad + \Big(\Big(e^*_{3m+1,n,\widetilde{k},2^{4Q_{3m+1}}}(\overline{x}) - \frac{1}{2^{M_{3m+1}}} \sum_{f=1}^{2^{M_{3m+1}}} e^*_{3m+1,n,f,2^{4Q_{3m+1}}}(\overline{x})\Big) \\
&\quad + \frac{\omega'^*_{3m+1,n}(\overline{x})}{2^{M_{3m+1}}}\Big)\Big) x_{3m+1,n,\widetilde{k},l} + \sum_{l=1}^{L(a)} V^*_{3m+1,n}(\overline{x})\Big(e_{3m+1,n,\widetilde{k},l} \\
&\quad + \frac{e_{3m+1,n,\widetilde{k}}}{2^{P_{3m+1}Q_{0,3m+1}}2^{2Q_{3m+1}}}\Big)\Big\| + \frac{5Q_{0,3m+1}}{2^{3Q_{3m+1}}} \\
&\le \sum_{l=1}^{L(a)} \Big(|e^*_{3m+1,n,\widetilde{k},l}(\overline{x})| \\
&\quad + \frac{1}{2^{4Q_{3m+1}}}\Big(\sum_{g=1}^{2^{4Q_{3m+1}}-1} |e^*_{3m+1,n,\widetilde{k},g}(\overline{x})| + |e^*_{3m+1,n,\widetilde{k},2^{4Q_{3m+1}}}(\overline{x})| \\
&\quad + \sum_{f=1}^{2^{M_{3m+1}}} \frac{|e^*_{3m+1,n,f,2^{4Q_{3m+1}}}(\overline{x})|}{2^{M_{3m+1}}} + \frac{|\omega'^*_{3m+1,n}(\overline{x})|}{2^{M_{3m+1}}}\Big)\Big) \|x_{3m+1,n,\widetilde{k},l}\| \\
&\quad + V^*_{3m+1,n}(\overline{x}) \Big\| \sum_{l=1}^{L(a)} e_{3m+1,n,\widetilde{k},l} \Big\| + \frac{L(a)V^*_{3m+1,n}(\overline{x})\|e_{3m+1,n,\widetilde{k}}\|}{2^{P_{3m+1}Q_{0,3m+1}}2^{2Q_{3m+1}}} \\
&\quad + \frac{5Q_{0,3m+1}}{2^{3Q_{3m+1}}} < (\text{by Step 4})\ 5 \sum_{l=1}^{L(a)} \Big(|e^*_{3m+1,n,\widetilde{k},l}(\overline{x})| \\
&\quad + \frac{1}{2^{4Q_{3m+1}}}\Big(\sum_{g=1}^{2^{4Q_{3m+1}}-1} |e^*_{3m+1,n,\widetilde{k},g}(\overline{x})| + |e^*_{3m+1,n,\widetilde{k},2^{4Q_{3m+1}}}(\overline{x})| \\
&\quad + \frac{1}{2^{M_{3m+1}}} \sum_{f=1}^{2^{M_{3m+1}}} |e^*_{3m+1,n,f,2^{4Q_{3m+1}}}(\overline{x})| + \frac{|\omega'^*_{3m+1,n}(\overline{x})|}{2^{M_{3m+1}}}\Big)\Big) +
\end{aligned}
$$

(by the definitions of $((e'_{3m+1,n,k,l})_{l=1}^{2^{4Q_{3m+1}}-1})_{k=1}^{2^{M_{3m+1}}}$ and $((e_{3m+1,n,k,l})_{l=1}^{2^{4Q_{3m+1}}-1})_{k=1}^{2^{M_{3m+1}}}$ in Step 1, and $\sum_{l=1}^{L(a)} e_{3m+1,n,\widetilde{k},l} = e'_{3m+1,n,\widetilde{k},L(a)}$)

$$
\begin{aligned}
&V^*_{3m+1,n}(\overline{x}) \cdot 1 + L(a)V^*_{3m+1,n}(\overline{x}) \frac{1}{2^{P_{3m+1}Q_{0,3m+1}}2^{2Q_{3m+1}}} + \frac{5Q_{0,3m+1}}{2^{3Q_{3m+1}}} \\
&\qquad = 5 \sum_{l=1}^{L(a)} |e^*_{3m+1,n,\widetilde{k},l}(\overline{x})| + \frac{5L(a)}{2^{4Q_{3m+1}}} \sum_{g=1}^{2^{4Q_{3m+1}}-1} |e^*_{3m+1,n,\widetilde{k},g}(\overline{x})| \\
&\qquad\quad + \frac{5L(a)}{2^{4Q_{3m+1}}}\Big(|e^*_{3m+1,n,\widetilde{k},2^{4Q_{3m+1}}}(\overline{x})| \\
&\qquad\quad + \frac{1}{2^{M_{3m+1}}} \sum_{f=1}^{2^{M_{3m+1}}} |e^*_{3m+1,n,f,2^{4Q_{3m+1}}}(\overline{x})| + \frac{|\omega'^*_{3m+1,n}(\overline{x})|}{2^{M_{3m+1}}}\Big) \\
&\qquad\quad + V^*_{3m+1,n}(\overline{x})\Big(1 + \frac{L(a)}{2^{P_{3m+1}Q_{0,3m+1}}2^{2Q_{3m+1}}}\Big) + \frac{5Q_{0,3m+1}}{2^{3Q_{3m+1}}}
\end{aligned}
$$

$<$ (by (iii)) $\frac{5}{2^{4Q_{3m+1}}}+$ (by (iii)) $\frac{5L(a)}{2^{4Q_{3m+1}}}\frac{1}{2^{4Q_{3m+1}}}+$ (by the end of Step 1) $\frac{5L(a)}{2^{4Q_{3m+1}}}(2+2+\frac{2}{2^{M_{3m+1}}})+$ (by the upper bound of $V^*_{3m+1,n}(\overline{x})$ of $(*)$) $2Q_{0,3m+1}\frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{3Q_{3m+1}}}(1+\frac{L(a)}{2^{P_{3m+1}Q_{0,3m+1}}2^{2Q_{3m+1}}})+\frac{5Q_{0,3m+1}}{2^{3Q_{3m+1}}}<$ (by the upper bound of $L(a)$ of $(***)$)

$$\begin{aligned}
&\frac{5}{2^{4Q_{3m+1}}}+\frac{5.9Q_{0,3m+1}2^{3Q_{3m+1}}}{2^{4Q_{3m+1}}}\frac{1}{2^{4Q_{3m+1}}}+\frac{5.9Q_{0,3m+1}2^{3Q_{3m+1}}}{2^{4Q_{3m+1}}}5\\
&\qquad+2Q_{0,3m+1}\frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{3Q_{3m+1}}}\left(1+\frac{9Q_{03m+1}2^{3Q_{3m+1}}}{2^{P_{3m+1}Q_{0,3m+1}}2^{2Q_{3m+1}}}\right)+\frac{5Q_{0,3m+1}}{2^{3Q_{3m+1}}}\\
&\qquad=\frac{5}{2^{4Q_{3m+1}}}+\frac{45Q_{0,3m+1}}{2^{5Q_{3m+1}}}+\frac{225Q_{0,3m+1}}{2^{Q_{3m+1}}}\\
&\qquad+2Q_{0,3m+1}\frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{3Q_{3m+1}}}+18\left(\frac{Q_{0,3m+1}}{2^{Q_{3m+1}}}\right)^2+\frac{5Q_{0,3m+1}}{2^{3Q_{3m+1}}}<
\end{aligned}$$

(by the relation between Q_{3m+1} and $Q_{0,3m+1}$ of Step 2)

$$\begin{aligned}
&\bigg(\frac{5}{2^{4Q_{3m+1}}}+\frac{45}{2^{4Q_{3m+1}}}\frac{Q_{0,3m+1}}{2^{2Q_{0,3m+1}P_{3m+1}}}+225\frac{Q_{0,3m+1}}{2^{2Q_{0,3m+1}P_{3m+1}}}\\
&\quad+\frac{2}{2^{Q_{3m+1}}}\frac{Q_{0,3m+1}}{2^{2Q_{0,3m+1}P_{3m+1}}}\frac{1}{2^{Q_{0,3m+1}P_{3m+1}}}+18\left(\frac{Q_{0,3m+1}}{2^{2Q_{0,3m+1}P_{3m+1}}}\right)^2\bigg)\\
&\quad+\frac{5Q_{0,3m+1}}{2^{2Q_{0,3m+1}P_{3m+1}}2^{Q_{3m+1}}}\\
&=\frac{5}{2^{4Q_{3m+1}}}+\frac{Q_{0,3m+1}}{2^{2Q_{0,3m+1}P_{3m+1}}}\bigg(\frac{45}{2^{4Q_{3m+1}}}+225\\
&\quad+\frac{2}{2^{Q_{3m+1}}}\frac{1}{2^{Q_{0,3m+1}P_{3m+1}}}+18\frac{Q_{0,3m+1}}{2^{2Q_{0,3m+1}P_{3m+1}}}+\frac{5}{2^{Q_{3m+1}}}\bigg)\\
&\quad<\frac{5}{2^{4Q_{3m+1}}}+226\frac{Q_{0,3m+1}}{2^{2Q_{0,3m+1}P_{3m+1}}}<227\frac{Q_{0,3m+1}}{2^{2Q_{0,3m+1}P_{3m+1}}}<\frac{1}{2^{Q_{0,3m+1}2P_{3m+1}}}.
\end{aligned}$$

This completes the proof of Lemma 13. ∎

We point out that the most important properties of Lemma 13 are (iv) (which renders possible the proof of the chain lemma) and (v) (which allows the construction of the elements of $\operatorname{span}(u'_{3m+1,s})_{s=1}^{A''_{3m+1}}$ necessary to *regularize*

$$\sum_{n=1}^{P_{3m+1-1}}\sum_{k=1}^{2^{M_{3m+1-1}}}\sum_{l=1}^{2^{Q_{3m+1-1}}-1}x^*_{3m+1-1,n,k,l}(\overline{x})x_{3m+1-1,n,k,l};$$

this fact is the reason for the name *regularization block*).

LEMMA 14 (Completeness Block Lemma, CBL). *Fix $\overline{x}\in X$ with $\|\overline{x}\|=1$ and n with $1\le n\le P_{3m}$. Then, if $|w^*_{3m,n}(\overline{x})|\ge 7$, there exists $\overline{k}$ with $1\le\overline{k}\le 2^{M_{3m}}$ such that $|e'''^*_{3m,\mathrm{brd},n,\overline{k}}(\overline{x})|\ge 1/2^{M_{3m}}$.*

Proof. By Steps 4 and 5 of SC III.1 we know that, for $1\le k\le 2^{M_{3m}}$,

$$e'''^*_{3m,\mathrm{brd},n,k}=e''^*_{0,3m,\mathrm{brd},n,k}-x^*_{0,3m,n,k,2^{M_{3m}}}+e'''^*_{1,3m,\mathrm{brd},n,k}-\frac{w^*_{3m,n}}{2^{M_{3m}}},$$

$$e'''^{*}_{1,3m,\mathrm{brd},n,k} = -\frac{\omega^*_{3m,n}}{2^{M_{3m}}}\left(1-\frac{1}{2^{M_{3m}}}\right)+\frac{\omega^*_{3m,n,k}}{2^{M_{3m}}}+\frac{\omega'^*_{3m,n}}{2^{M_{3m}}},$$

with

$$\sum_{k=1}^{2^{M_{3m}}}(e''^{*}_{0,3m,\mathrm{brd},n,k}-x^*_{0,3m,n,k,2^{M_{3m}}})=0, \qquad \|e'''^{*}_{1,3m,\mathrm{brd},n,k}\|<\frac{6}{2^{M_{3m}}}.$$

Therefore there exists $\overline{k}$ with $1\le \overline{k}\le 2^{M_{3m}}$ such that the sign of $e''^{*}_{0,3m,\mathrm{brd},n,k}(\overline{x}) - x^*_{0,3m,n,k,2^{M_{3m}}}(\overline{x})$ is opposite to the sign of $w^*_{3m,n}(\overline{x})$, hence

$$\begin{aligned}|e'''^{*}_{3m,\mathrm{brd},n,\overline{k}}(\overline{x})| &\ge \left|e''^{*}_{0,3m,\mathrm{brd},n,k}(\overline{x})-x^*_{0,3m,n,k,2^{M_{3m}}}(\overline{x})-\frac{w^*_{3m,n}(\overline{x})}{2^{M_{3m}}}\right|-|e'''^{*}_{1,3m,\mathrm{brd},n,k}(\overline{x})|\\ &\ge \frac{|w^*_{3m,n}(\overline{x})|}{2^{M_{3m}}}-|e'''^{*}_{1,3m,\mathrm{brd},n,k}(\overline{x})|>\frac{1}{2^{M_{3m}}},\end{aligned}$$

which completes the proof of Lemma 14. ∎

The next lemma, important for our construction, justifies the name *bridge sequence* for the sequences $(e_{3m,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m}}}$ and $(e_{3m+1,\mathrm{brd},n,k})_{k=1}^{2^{M_{3m+1}}}$. In what follows we will use the following notations, for each fixed integer m:

$$\varepsilon'_m=\frac{1}{A_m 2^{2q(m)A_m}} \quad\text{and}\quad \varepsilon_m=\frac{1}{P_m 2^{2q(m)P_m}}.$$

LEMMA 15 (Local Chain Lemma, LCL). *Each $\overline{x}\in X$ with $\|\overline{x}\|=1$ has, for each m, the following property: either*

(i) (*the* disconnected chain condition)

$$|u'^{*}_{3m,n}(\overline{x})|<\varepsilon'_{3m} \qquad \textit{for } 1\le n\le A_{3m};$$

or

(ii) (*the* operating chain condition) (i) *does not hold*; *in this case there exist* $n'(3m+1)$ *and* $n'(3m+2)$, *with* $A_{3m+1}-A'_{3m+1}+1\le n'(3m+1)\le A_{3m+1}$ *and* $A_{3m+2}-A'_{3m+2}+1\le n'(3m+2)\le A_{3m+2}$, *such that* $|u'^{*}_{3m+1,n'(3m+1)}(\overline{x})|\ge\varepsilon'_{3m+1}$ *and* $|u'^{*}_{3m+2,n'(3m+2)}(\overline{x})|\ge \varepsilon'_{3m+2}$.

Proof. Suppose that (i) does not hold. This means that there exists $n'(3m)$ with $1\le n'(3m)\le A_{3m}$ such that $|u'^{*}_{3m,n'(3m)}(\overline{x})|\ge\varepsilon'_{3m}$; we are going to prove that the properties of (ii) hold. Indeed, by Step 4 of SC III.1 we know that, for $1\le t\le 2^{2B_{3m}}$, $u^*_{3m,n'(3m),t}=u^*_{0,3m,n'(3m),t}+u'^{*}_{3m,n'(3m)}/2^{B_{2m}}$ with

$$u^*_{0,3m,n'(3m),t}=e^*_{3m,\mathrm{aux},n'(3m),t}-\frac{1}{2^{2B_{3m}}}\sum_{j=1}^{2^{2B_{3m}}}e^*_{3m,\mathrm{aux},n'(3m),j},$$

hence $\sum_{j=1}^{2^{2B_{3m}}}u^*_{0,3m,\mathrm{aux},n'(3m),j}=0$; therefore, there exists $t(n'(3m))$ with $1\le t(n'(3m))\le 2^{2B_{3m}}$ such that $u^*_{0,3m,n'(3m),t(n'(3m))}(\overline{x})$ either is 0 or has the same sign as $u'^{*}_{3m,n'(3m)}(\overline{x})$, whence, if we set

$$n(3m)=(n'(3m),t(n'(3m)))=(n'(3m)-1)(1+2^{2B_{3m}})+t(n'(3m)),$$

then

$$|u^*_{3m,n(3m)}(\overline{x})| = |u^*_{3m,n'(3m),t(n'(3m))}(\overline{x})| \ge \frac{|u'^*_{3m,n'(3m)}(\overline{x})|}{2^{B_{3m}}}$$
$$\ge \frac{\varepsilon'_{3m}}{2^{B_{3m}}} = \frac{1}{2^{B_{3m}}}\frac{1}{A_{3m}2^{2q(3m)A_{3m}}} > \frac{1}{P_{3m}2^{2q(3m)P_{3m}}} = \varepsilon_{3m},$$

with $2 \le n(3m) \le P_{3m}$. By (7) and (12.1) of GBST (= Theorem 4) for M replaced by $4q(3m)$ (hence $\varepsilon_{3m} > 1/2^{MP_{3m}}$), this implies that there exists $(g(3m,n))_{n=1}^{n(3m)}$ with $g(3m,n(3m)) \ge 2$ and $|w^*_{3m,g(3m,n(3m))}(\overline{x})| > 2^{4q(3m)P_{3m}}$, which, by (ii) of CBL, implies that $|e'''^*_{3m,\mathrm{brd},g(3m,n(3m)),\overline{k}(3m)}(\overline{x})| > 1/2^{M_{3m}}$ for some $1 \le \overline{k}(3m) \le 2^{M_{3m}}$. Consequently, by Step 6 of SC III.1, there exists $n'(3m+1)$ with $A_{3m+1} - A'_{3m+1} + 1 \le n'(3m+1) \le A_{3m+1}$ such that (recall that $2^{M_{3m}} < q(3m+1)$ by the definition of $q(3m+1)$ in Step 6 of SC III.1)

$$|u'^*_{3m+1,n'(3m+1)}(\overline{x})| > \frac{1}{2^{M_{3m}}} > \frac{1}{A_{3m+1}2^{2q(3m+1)A_{3m+1}}} = \varepsilon'_{3m+1}.$$

By the same procedure there exist $n(3m+1)$ with

$$P_{3m+1} - A'_{3m+1}(2^{2B_{3m+1}}+1) + 1 \le n(3m+1) \le P_{3m+1}$$

and $(g(3m+1,n))_{n=1}^{n(3m+1)}$ with $g(3m+1,n(3m+1)) \ge n(3m+1)$ such that

$$|u^*_{3m+1,n(3m+1)}(\overline{x})| > \varepsilon_{3m+1} \quad \text{and} \quad |w^*_{3m+1,g(3m+1,n(3m+1))}(\overline{x})| > 2^{4q(3m+1)P_{3m+1}}.$$

Then (see also Remark 5) by the relation between A_{3m+1} and A'_{3m+1} at the beginning of SC III.2 there exists $(n')_{n=1}^{A''_{3m+1}} \subset (n)_{n=1}^{P_{3m+1}/2}$ with $(2n'-1,2n')_{n=1}^{A''_{3m+1}} \subset (g(3m+1,n))_{n=1}^{n(3m+1)}$. Hence by the definition of $(v_{3m+1,n})_{n=1}^{P_{3m+1}}$ and by (iv) of RBL there exist $\widetilde{n}(3m+1)$ and $k(3m+1)$ with $A''_{3m+1} \le \widetilde{n}(3m+1) \le P_{3m+1}/2$ and $1 \le k(3m+1) \le 2^{M_{3m+1}}$ such that $|w^*_{3m+1,2\widetilde{n}(3m+1)-1}(\overline{x})| > 2^{4q(3m+1)P_{3m+1}}$ and hence

$$|e'''^*_{3m+1,\mathrm{brd},2\widetilde{n}(3m+1)-1,k(3m+1)}(\overline{x})| > \frac{1}{9}\frac{1}{2^{3Q_{3m+1}}}.$$

By Step 4 of SC III.2, this implies that there exists $n'(3m+2)$ with $A_{3m+2} - A'_{3m+2} + 1 \le n'(3m+2) \le A_{3m+2}$, such that (recall that $2^{3Q_{3m+1}} < q(3m+2)$ by the definition of $q(3m+2)$ in Step 4 of SC III.2)

$$|u'^*_{3m+2,n'(3m+2)}(\overline{x})| > \frac{1}{9}\frac{1}{2^{3Q_{3m+1}}} > \frac{1}{A_{3m+2}2^{2q(3m+2)A_{3m+2}}} = \varepsilon'_{3m+2}.$$

This completes the proof of Lemma 15. ■

Notations and partitions. In the next lemmas we will use the following partitions:

$$\begin{aligned}
(x_{\mathrm{brd},n})_{n=q(3m+1)+1}^{q(3m+2)} &= ((e_{3m+1,\mathrm{brd},n,k} + \omega_{3m+1,n})_{k=1}^{2^{M_{3m+1}}})_{n=1}^{P_{3m+1}} \\
&= (x_{\mathrm{brd},a,n})_{n=q(3m+1)+1}^{q(3m+2)} \cup (x_{\mathrm{brd},p,n})_{n=q(3m+1)+1}^{q(3m+2)},\ (x_{\mathrm{brd},a,n})_{n=q(3m+1)+1}^{q(3m+2)} \\
&= (x_{3m+1,\mathrm{brd},a,g})_{g=1}^{N_{3m+1,\mathrm{brd},a}} \\
&= ((e_{3m+1,\mathrm{brd},n,k} + \omega_{3m+1,n})_{k=1}^{2^{M_{3m+1}}})_{n=1}^{n''(3m+1,\mathrm{brd})-1} \\
&\quad \cup (e_{3m+1,\mathrm{brd},n''(3m+1,\mathrm{brd}),k} + \omega_{3m+1,n''(3m+1,\mathrm{brd})})_{k=1}^{k(3m+1,\mathrm{brd})} \\
&\subset \operatorname{span}(u'_{3m+2,s})_{s=1}^{n'(3m+2)},
\end{aligned}$$

$$\begin{aligned}(x_{\mathrm{brd},p,n})_{n=q(3m+1)+1}^{q(3m+2)} &= (x_{3m+1,\mathrm{brd},p,g})_{g=1}^{N_{3m+1,\mathrm{brd},p}}\\ &= (e_{3m+1,\mathrm{brd},n''(3m+1,\mathrm{brd}),k} + \omega_{3m+1,n''(3m+1,\mathrm{brd})})_{k=k(3m+1,\mathrm{brd})+1}^{2^{M_{3m+1}}}\\ &\quad \cup ((e_{3m+1,\mathrm{brd},n,k} + \omega_{3m+1,n})_{k=1}^{2^{M_{3m+1}}})_{n=n''(3m+1,\mathrm{brd})+1}^{P_{3m+1}}\\ &\subset \mathrm{span}(u'_{3m+2,s})_{s=n'(3m+2)+1}^{A_{3m+2}};\end{aligned}$$

$$\begin{aligned}(x_{\mathrm{brd},n})_{n=q(3m)+1}^{q(3m+1)} &= ((e_{3m,\mathrm{brd},n,k} + \omega'_{3m,n})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}} = (x_{3m,\mathrm{brd},a,g})_{g=1}^{N_{3m,\mathrm{brd},a}}\\ &\quad \cup (x_{3m,\mathrm{brd},p,g})_{g=1}^{N_{3m,\mathrm{brd},p}}, \ (x_{\mathrm{brd},a,n})_{n=q(3m)+1}^{q(3m+1)} = (x_{3m,\mathrm{brd},a,g})_{g=1}^{N_{3m,\mathrm{brd},a}}\\ &= ((e_{3m,\mathrm{brd},n,k} + \omega'_{3m,n})_{k=1}^{2^{M_{3m}}})_{n=1}^{n''(3m,\mathrm{brd})-1}\\ &\quad \cup (e_{3m,\mathrm{brd},n''(3m,\mathrm{brd}),k} + \omega'_{3m,n''(3m,\mathrm{brd})})_{k=1}^{k(3m,\mathrm{brd})}\\ &\subset \mathrm{span}(u'_{3m+1,s})_{s=1}^{n'(3m+1)},\end{aligned}$$

$$\begin{aligned}(x_{\mathrm{brd},p,n})_{n=q(3m)+1}^{q(3m+1)} &= (x_{3m,\mathrm{brd},p,g})_{g=1}^{N_{3m,\mathrm{brd},p}}\\ &= (e_{3m,\mathrm{brd},n''(3m,\mathrm{brd}),k} + \omega'_{3m,n''(3m,\mathrm{brd})})_{k=k(3m,\mathrm{brd})+1}^{2^{M_{3m}}}\\ &\quad \cup ((e_{3m,\mathrm{brd},n,k} + \omega'_{3m,n})_{k=1}^{2^{M_{3m}}})_{n=n''(3m,\mathrm{brd})+1}^{P_{3m}}\\ &\subset \mathrm{span}(u'_{3m+1,s})_{s=n'(3m+1)+1}^{A_{3m+1}}\end{aligned}$$

(where "a" means "absent" since they disappear in the regularization, while "p" means "present" since they continue to be also after the regularization; moreover obviously it is possible that $x_{\mathrm{brd},n} = 0$. Moreover we will use the notations (see Step 6 of SC III 1)

$$\begin{aligned}(x_{0,3m+1,g})_{g=1}^{G_{0,3m+1}} &= ((x_{0,3m+1,n,g})_{g=1}^{G_{0,3m+1,0}})_{n=1}^{P_{3m+1}},\\ (x_{0,3m+1,n,g})_{g=1}^{G_{0,3m+1,0}} &= (\omega'_{3m+1,n,0} \cup x_{3m+1,n,0} \cup (x_{3m+1,n,k,0})_{k=1}^{2^{M_{3m+1}}})\\ &\quad \cup (x_{3m+1,n,g})_{g=1}^{G_{3m+1,0}} \quad \text{for } 1 \le n \le P_{3m+1};\\ (x_{0,3m,g})_{g=1}^{G_{0,3m}} &= ((x_{0,3m,n,g})_{g=1}^{G_{0,3m,0}})_{n=1}^{P_{3m}},\\ (x_{0,3m,n,g})_{g=1}^{G_{0,3m,0}} &= (x_{0,0,3m,n,g})_{g=1}^{G_{0,0,3m,0}} \cup (x_{3m,n,g})_{g=1}^{G_{3m,0}},\\ (x_{0,0,3m,n,g})_{g=1}^{G_{0,0,3m,0}} &= \omega_{3m,n,0} \cup (\omega_{3m,n,k,0})_{k=1}^{2^{M_{3m}}}\\ &\quad \cup (x_{3m,n,0,k})_{k=0}^{2^{M_{3m}}} \cup ((x_{3m,n,k,0,l})_{l=0}^{2^{M_{3m}}})_{k=1}^{2^{M_{3m}}} \ \text{for } 1 \le n \le P_{3m}.\end{aligned}$$

In the next lemma we will use the notations $n'(3m)$, $t(n'(3m))$, $n(3m)$, ε'_{3m} and ε_{3m} of the proof of LCL in the operating chain condition where now we can suppose that $|u^*_{3m,n'(3m),t}(\overline{x})| < \varepsilon_{3m}$ for $t(n'(3m)) + 1 \le t \le 2^{2B_{3m}}$.

LEMMA 16 (Completeness Lemma, CL). *Let $\overline{x} \in X$ with $\|\overline{x}\| = 1$. Then there exists a sequence (η_m) of positive numbers, with $\eta_m \to 0$, such that, for each m,*

(i) *there exists a subsequence $(u'_{3m,n_k})_{k=1}^{K+1}$ of $(u'_{3m,s})_{s=1}^{A''_{3m}}$ and $0 < a < 1$ with*

$$\Big\|\overline{x} - \Big(\sum_{n=1}^{q(3m)} x^*_n(\overline{x})x_n + \sum_{k=1}^{K} u'^*_{3m,n_k}(\overline{x})u'_{3m,n_k} + a u'^*_{3m,n_{K+1}}(\overline{x})u'_{3m,n_{K+1}}\Big)\Big\| < \eta_m$$

(*in particular* $(n_k)_{k=1}^{M_{3m-1,0}+Q''_{3m}} = (k)_{k=1}^{M_{3m-1,0}+Q''_{3m}}$);

(ii)

$$\max\Big(\Big\|\sum_{n=q(3m)+1}^{q(3m+1)} x_n^*(\overline{x})x_{\mathrm{brd},p,n} + E_{3m,0}\Big\|, \Big\|\sum_{n=q(3m+1)+1}^{q(3m+2)} x_n^*(\overline{x})x_{\mathrm{brd},p,n} + E_{3m+1,0}\Big\|,$$
$$\Big\|\sum_{g=1}^{G_{0,3m}} x_{0,3m,g}^*(\overline{x})(x'_{0,3m,g} + x''_{0,3m,g}) + E_{3m,0}\Big\|,$$
$$\Big\|\sum_{g=1}^{G_{0,3m+1}} x_{0,3m+1,g}^*(\overline{x})(x'_{0,3m+1,g} + x''_{0,3m+1,g}) + E_{3m+1,0}\Big\|,$$
$$\Big\|\sum_{g=1}^{G_{3m+2}} x_{3m+2,g}^*(\overline{x})(x'_{3m+2,g} + x''_{3m+2,g}) + E_{3m+2,0}\Big\|\Big) < \eta_m;$$

(iii) $t(n'(3m)) > 2^{B_{3m}/2}$ *and there always is* $\widetilde{u}'_m \in \mathrm{span}(u_{3m,n'(3m),t})_{t=1}^{t(n'(3m))-1}$ *with* $\|\widetilde{u}'_m - u'^*_{3m,n'(3m)}(\overline{x})u'_{3m,n'(3m)}\| < \eta_m$.

Proof. First let us point out some facts.

FACT 1. $\mathrm{dist}(\overline{x} - \sum_{n=1}^{q(3m+2)} x_n^*(\overline{x})x_n, \mathrm{span}(x_n)_{n=q(3m+2)+1}^{q(3m+3)}) \to 0$.

Indeed, by Step 4 of SC III.2, and by (41.1), (42.1) and (43.1) of Construction II, we know that

$$\mathrm{dist}\Big(\overline{x} - \sum_{n=1}^{q(3m+2)} x_n^*(\overline{x})x_n, \mathrm{span}((u'_{3m+2,s})_{s=1}^{A''_{3m+2}} \cup (u'_{3m+2,s})_{s=A_{3m+2}-A'_{3m+2}+1}^{A_{3m+2}})\Big) \to 0,$$

since $(u'_{3m+2,s})_{s=M_{3m+1,0}+Q''_{3m+2}+1}^{M_{3m+1,0}+P''_{3m+2}} = (u''_{3m+2,s})_{s=Q''_{3m+2}+1}^{P''_{3m+2}}$, $\mathrm{span}(u''_{3m+2,s})_{s=Q''_{3m+2}+1}^{P''_{3m+2}} = \mathrm{span}((v'_{3m+2,n})_{n=Q''_{3m+2}+1}^{Q'_{3m+2}} \cup ((v'_{3m+2,n,k})_{k=1}^{Q'_{3m+2,n}} \cup (v''_{3m+2,n,k})_{k=1}^{Q''_{3m+2,n}})_{n=Q''_{3m+2}+1}^{Q'_{3m+2}})$, while

$$(x_n)_{n=1}^{q(3m+2)} \cup (u'_{3m+2,s})_{s=1}^{M_{3m+1,0}} \cup (u'_{3m+2,s})_{s=M_{3m+1,0}+P''_{3m+2}+1}^{A''_{3m+2}}$$
$$\cup\, (u'_{3m+2,s})_{s=A_{3m+2}-A'_{3m+2}+1}^{A_{3m+2}}$$

corresponds to $(y_n)_{n=1}^{Q(m)}$ of Construction II; on the other hand, from Step 4 of SC III.2 it follows that

$$\mathrm{span}(x_n)_{n=q(3m+2)+1}^{q(3m+3)} \supset \mathrm{span}(u_{3m+2,s})_{s=1}^{A_{3m+2}} \supset \mathrm{span}(u'_{3m+2,s})_{s=1}^{A_{3m+2}}$$
$$\supset \mathrm{span}((u'_{3m+2,s})_{s=1}^{A''_{3m+2}} \cup (u'_{3m+2,s})_{s=A_{3m+2}-A'_{3m+2}+1}^{A_{3m+2}}).$$

FACT 2.

$$\max((\max(|v'^*_{3m+i,n}(\overline{x})| : 1 \le n \le Q'_{3m+i}) : 0 \le i \le 2) \to 0,$$
$$\max((\max(|u^*_{3m+i,n}(\overline{x})| : 1 \le n \le P_{3m+i}) : 0 \le i \le 2) \to 0.$$

It is sufficient to prove the second relation for $i = 0$ and we recall, by Step 5 of SC III.1, Step 3 of SC III.2 and Step 2 of SC III.3, that $(u^*_{3m+i})_{n=1}^{P_{3m+i}} \in \mathrm{span}(x^*_{3m+i,n})_{n=1}^{G_{3m+i}}$ for $0 \le i \le 2$ (indeed, for $1 \le n \le P_{3m+i}$, $\sum_{k=1}^{2^{M_{3m}}} \sum_{l=1}^{2^{M_{3m}}} x^*_{3m,n,k,l} = 2^{M_{3m}} w^*_{3m,n}$,

$\sum_{k=1}^{2^{M_{3m+1}}}\sum_{k=1}^{2^{4Q_{3m+1}}} x^*_{3m+1,n,k,l} = 2^{P_{3m+1}Q_{0,3m+1}} v^*_{3m+1,n}$ and $\sum_{k=1}^{2^{M_{3m+2}}} x^*_{3m+2,n,k} = w^*_{3m+1,n}$). Therefore $\operatorname{span}(x^*_n)_{n=q(3m)+1}^{q(3m+1)} \supset \operatorname{span}(u^*_{3m,n})_{n=1}^{P_{3m}}$. The space

$$X'^*_m = \operatorname{span}(x^*_n)_{n=q(3m)+1}^{q(3m+1)}$$

is the dual of $X'_m = \operatorname{span}(x_n)_{n=q(3m)+1}^{q(3m+1)}$ hence also $X'_m = (X'^*_m)^*$. On the other hand, by Step 4 of SC III.1, $(u^*_{3m,n}, u_{3m,n})_{n=1}^{P_{3m}}$ is biorthogonal with $1 < \|u^*_{3m,n}\| < 5$ and $1 \le \|u_{3m,n}\| < 3$ for $1 \le n \le P_{3m}$, that is,

$$\operatorname{dist}(u^*_{3m,n}/\|u^*_{3m,n}\|, \operatorname{span}(u^*_{3m,k})_{k(\neq n)=1}^{P_{3m}}) > 1/15 \quad \text{for } 1 \le n \le P_{3m}$$

Therefore there exists $(\widetilde{u}_{3m,n})_{n=1}^{P_{3m}}$ in X'_m with $(u^*_{3m,n}, \widetilde{u}_{3m,n})_{n=1}^{P_{3m}}$ biorthogonal and $1 \le \|\widetilde{u}_{3m,n}\| < 3$ for $1 \le n \le P_{3m}$. It is now easy (for instance by means of a biorthogonalization procedure) to find $(\widetilde{x}_n)_{n=\widetilde{q}(3m)+1}^{\widetilde{q}(3m+1)}$ in X'_m and $(\widetilde{x}^*_n)_{n=\widetilde{q}(3m)+1}^{\widetilde{q}(3m+1)}$ in $X^{*\prime}_m$ such that $(\widetilde{x}_n, \widetilde{x}^*_n)_{n=\widetilde{q}(3m)+1}^{\widetilde{q}(3m+1)} \cup (\widetilde{u}_{3m,n}, u^*_{3m,n})_{n=1}^{P_{3m}}$ is biorthogonal, with $X'_m = \operatorname{span}((\widetilde{x}_n)_{n=\widetilde{q}(3m)+1}^{\widetilde{q}(3m+1)} \cup (\widetilde{u}_{3m,n})_{n=1}^{P_{3m}})$ and $X'^*_m = \operatorname{span}((\widetilde{x}^*_n)_{n=\widetilde{q}(3m)+1}^{\widetilde{q}(3m+1)} \cup (u^*_{3m,n})_{n=1}^{P_{3m}})$. On the other hand, by Fact 1, for each m there exists a sequence $(a_n)_{n=q(3m+2)+1}^{q(3m+3)}$ of numbers such that

$$\Big\|\overline{x} - \Big(\sum_{n=1}^{q(3m+2)} x^*_n(\overline{x})x_n + \sum_{n=q(3m+2)+1}^{q(3m+3)} a_n x_n\Big)\Big\| < \eta_m \to 0,$$

hence

$$\begin{aligned}
\eta_{m-1} + \eta_m &> \Big\|\overline{x} - \Big(\sum_{n=1}^{q(3m-1)} x^*_n(\overline{x})x_n + \sum_{n=q(3m-1)+1}^{q(3m)} a_n x_n\Big)\Big\| \\
&\quad + \Big\|\overline{x} - \Big(\sum_{n=1}^{q(3m+2)} x^*_n(\overline{x})x_n + \sum_{n=q(3m+2)+1}^{q(3m+3)} a_n x_n\Big)\Big\| \\
&\ge \Big\|\Big(\overline{x} - \Big(\sum_{n=1}^{q(3m-1)} x^*_n(\overline{x})x_n + \sum_{n=q(3m-1)+1}^{q(3m)} a_n x_n\Big)\Big) \\
&\quad - \Big(\overline{x} - \Big(\sum_{n=1}^{q(3m+2)} x^*_n(\overline{x})x_n + \sum_{n=q(3m+2)+1}^{q(3m+3)} a_n x_n\Big)\Big)\Big\| \\
&= \Big\|\sum_{n=q(3m-1)+1}^{q(3m)} (x^*_n(\overline{x})x_n - a_n)x_n + \sum_{n=q(3m)+1}^{q(3m+2)} x^*_n(\overline{x})x_n + \sum_{n=q(3m+2)+1}^{q(3m+3)} a_n x_n\Big\| \\
&= \Big\|\sum_{n=q(3m-1)+1}^{q(3m)} (x^*_n(\overline{x})x_n - a_n)x_n + \sum_{n=\widetilde{q}(3m)+1}^{\widetilde{q}(3m+1)} \widetilde{x}^*_n(\overline{x})\widetilde{x}_n \\
&\quad + \sum_{n=1}^{P_{3m}} u^*_{3m,n}(\overline{x})\widetilde{u}_{3m,n} + \sum_{n=q(3m+1)+1}^{q(3m+2)} x^*_n(\overline{x})x_n + \sum_{n=q(3m+2)+1}^{q(3m+3)} a_n x_n\Big\| \\
&\ge \frac{1}{5}\max(|u^*_{3m,n}(\overline{x})| : 1 \le n \le P_{3m}).
\end{aligned}$$

This completes the proof of Fact 2.

Proof of Lemma 16(i). By the procedure of the proof of Fact 1 we can state Fact 1 with $3m+2$ replaced by $3m$ (hence now we use the end of SC III.3 and the beginning of SC III.1) in the following more precise form:

$$\operatorname{dist}\Big(\overline{x}-\Big(\sum_{n=1}^{q(3m)} x_n^*(\overline{x})x_n+\sum_{s=1}^{M_{3m-1,0}+Q''_{3m}} u'^*_{3m,s}(\overline{x})u'_{3m,s}\Big),\\ \operatorname{span}(u'_{3m,s})_{s=M_{3m-1,0}+Q''_{3m}+1}^{A''_{3m}}\Big)\to 0$$

(since $(u'_{3m,s})_{s=M_{3m-1,0}+Q''_{3m}+1}^{A''_{3m}}=(u''_{3m,s})_{s=Q''_{3m}+1}^{P''_{3m}}$ with

$$\operatorname{span}(u''_{3m,s})_{s=Q''_{3m}+1}^{P''_{3m}}=\operatorname{span}((v'_{3m,n})_{n=Q''_{3m}+1}^{Q'_{3m}}\cup((v'_{3m,n,k})_{k=1}^{Q'_{3m,n}}\cup(v''_{3m,n,k})_{k=1}^{Q''_{3m,n}})_{n=Q''_{3m}+1}^{Q'_{3m}})$$

and $(x_n)_{n=1}^{q(3m)}\cup(u'_{3m,s})_{s=1}^{M_{3m-1,0}}$ corresponds to $(y_n)_{n=1}^{Q(m)}$ of CII); hence now we can set $Q(3m)=q(3m)+M_{3m-1,0}$ and $\eta_{3m}=1/2^{2Q(3m)+1}$.

Suppose that $|v'^*_{3m,n'}(\overline{x})|>\eta_{3m}/2Q'_{3m}$ for some n' with $Q''_{3m}+1\le n'\le Q'_{3m}$, while, for $n'+1\le n\le Q'_{3m}$, $|v'^*_{3m,n}(\overline{x})|\le\eta_{3m}/2Q'_{3m}$. We set $x'_{3m}=\overline{x}-\sum_{n=1}^{q(3m)}x_n^*(\overline{x})x_n$, moreover (see (43.1) of Lemma 12)

$$(u'_{3m,n_k})_{k=1}^{K}=(u'_{3m,s})_{s=1}^{M_{3m-1,0}+Q''_{3m}}\cup((u''_{3m,n,k})_{k=0}^{Q''_{3m,n}}\cup(v'_{3m,n,k})_{k=1}^{Q'_{3m,n}})_{n=Q''_{3m}+1}^{n'-1}.$$

By Fact 1 there exists $x_{0,3m}\in\operatorname{span}(x_n)_{n=1}^{q(3m)}$ with $\|x'-x_{0,3m}\|\le\eta''_{3m}\to 0$. We set

$$x'_{0,3m}=x_{0,3m}-\sum_{n=1}^{q(3m)}x_n^*(\overline{x})x_n,\ x''_{0,3m}=x'_{0,3m}-\sum_{k=1}^{K}u'^*_{3m,n_k}(\overline{x})u'_{3m,n_k},$$

$$x''_{3m}=x'_{3m}-\sum_{n=1}^{K}u'^*_{3m,n_k}(\overline{x})u'_{3m,n_k},$$

$$x'''_{0,3m}=x''_{0,3m}-\sum_{n=n'}^{Q'_{3m}}v'^*_{3m,n}(\overline{x})v'_{3m,n},\qquad x'''_{3m}=x''_{3m}-\sum_{n=n'}^{Q'_{3m}}v'^*_{3m,n}(\overline{x})v'_{3m,n}.$$

By (43.1) of Lemma 12,

$$\operatorname{span}((v'_{3m,n})_{n=n'}^{Q'_{3m}}\cup((u''_{3m,n,k})_{k=0}^{Q''_{3m,n}}\cup(v'_{3m,n,k})_{k=1}^{Q'_{3m,n}})_{n=1}^{n'-1})\\ =\operatorname{span}((v'_{3m,n})_{n=Q''_{3m}+1}^{Q'_{3m}}\cup((v'_{3m,n,k})_{k=1}^{Q'_{3m,n}}\cup(v''_{3m,n,k})_{k=1}^{Q''_{3m,n}})_{n=Q''_{3m}+1}^{n'-1})$$

and by (43.2) there exist a with $0<|a|<1$ and k' with $1\le k'\le Q''_{3m,n'}$ so that, if $u'_{3m,n_{K+1}}=u''_{3m,n',k'}$ and $u'^*_{3m,n_{K+1}}=u''^*_{3m,n',k'}$, then

$$\|x'''_{0,3m}-au'^*_{3m,n_{K+1}}(\overline{x})u'_{3m,n_{K+1}}\|=\|x'''_{0,3m}-au''^*_{3m,n',k'}(\overline{x})u''_{3m,n',k'}\|\\ <\operatorname{dist}(x'''_{0,3m},\operatorname{span}((v'_{3m,n,k})_{k=1}^{Q'_{3m,n}}\cup(v''_{3m,n,k})_{k=1}^{Q''_{3m,n}})_{n=n'}^{Q'_{3m}}+U''_{3m})+\eta_{3m}/2+\eta_{3m}$$

(where we used (42.3) of CII and (43.3) of Lemma 12), with

$$U''_{3m}=X\cap\bigcap_{n=1}^{q(3m)}x^*_{n\perp}\cap\bigcap_{s=1}^{A''_{3m}}u'^*_{(3m,s)\perp}.$$

On the other hand, by the construction of x'''_{3m} and of U''_{3m},

$$\begin{aligned}&\operatorname{dist}(x'''_{0,3m}, \operatorname{span}((v'_{3m,n,k})_{k=1}^{Q'_{3m,n}} \cup (v''_{3m,n,k})_{k=1}^{Q''_{3m,n}})_{n=n'}^{Q'_{3m}} + U''_{3m})\\ &\quad\le \operatorname{dist}(x'''_{3m}, \operatorname{span}((v'_{3m,n,k})_{k=1}^{Q'_{3m,n}} \cup (v''_{3m,n,k})_{k=1}^{Q''_{3m,n}})_{n=n'}^{Q'_{3m}} + U''_{3m}) + \eta''_{3m} = 0 + \eta''_{3m} = \eta''_{3m}.\end{aligned}$$

Hence by the above

$$\|x'''_{0,3m} - a.u'^{*}_{3m,n_{K+1}}(\overline{x})u'_{3m,n_{K+1}}\| < \eta''_{3m} + \eta_{3m}/2 + \eta_{3m}.$$

Therefore we conclude that

$$\begin{aligned}&\Big\|\overline{x} - \Big(\sum_{n=1}^{q(3m)} x_n^*(\overline{x})x_n + \sum_{k=1}^{K} u'^{*}_{3m,n_k}(\overline{x})u'_{3m,n_k} + au'^{*}_{3m,n_{K+1}}(\overline{x})u'_{3m,n_{K+1}}\Big)\Big\|\\
&\qquad = \|x''_{3m} - au'^{*}_{m,n_{K+1}}(\overline{x})u'_{m,n_{K+1}}\|\\
&\qquad = \Big\|\Big(x'''_{3m} + \sum_{n=n'}^{Q'_{3m}} v'^{*}_{3m,n}(\overline{x})v'_{3m,n}\Big) - au'^{*}_{m,n_{K+1}}(\overline{x})u'_{m,n_{K+1}}\Big\|\\
&\qquad \le \|x'''_{3m} - au'^{*}_{3m,n_{K+1}}(\overline{x})u'_{3m,n_{K+1}}\| + \Big\|\sum_{n=n'}^{Q'_{3m}} v'^{*}_{3m,n}(\overline{x})v'_{3m,n}\Big\|\\
&\qquad \le \|x'''_{0,3m} - au'^{*}_{3m,n_{K+1}}(\overline{x})u'_{3m,n_{K+1}}\| + \|x'''_{3m} - x'''_{0,3m}\| + \Big\|\sum_{n=n'}^{Q'_{3m}} v'^{*}_{3m,n}(\overline{x})v'_{3m,n}\Big\|\\
&\qquad < \eta''_{3m} + (\eta_{3m}/2 + \eta_{3m}) \text{ (by above)} + \eta''_{3m} + \sum_{n=n'}^{Q'_{3m}} |v'^{*}_{3m,n}(\overline{x})|\\
&\qquad = 2\eta''_{3m} + \frac{3}{2}\eta_{3m} + |v'^{*}_{3m,n'}(\overline{x})| + \sum_{n=n'+1}^{Q'_{3m}} |v'^{*}_{3m,n}(\overline{x})|\\
&\qquad \le 2\eta''_{3m} + \frac{3}{2}\eta_{3m} + |v'^{*}_{3m,n'}(\overline{x})| + \sum_{n=n'+1}^{Q'_{3m}} \frac{\eta_{3m}}{2Q'_{3m}}\\
&\qquad < 2\eta''_{3m} + 2\eta_{3m} + |v'^{*}_{3m,n'}(\overline{x})|.\end{aligned}$$

Now, since by the first part of Fact 2 (for $i = 0$) $|v'^{*}_{3m,n}(\overline{x})| < \widehat{\eta}_{3m} \to 0$ for $1 \le n \le Q'_{3m}$ for each m, the assertion is proved for $\eta'_m = 2\eta''_{3m} + 2\eta_{3m} + \widehat{\eta}_{3m}$.

Suppose now that $|v'^{*}_{3m,n}(\overline{x})| \le \eta_{3m}/2Q'_{3m}$ for $Q''_{3m} + 1 \le n \le Q'_{3m}$.

Let $x_{0,3m}$, η''_{3m}, x'_{3m} and $x'_{0,3m}$ be as above with

$$(u'_{3m,n_k})_{k=1}^{K} = (u'_{3m,s})_{s=1}^{M_{3m-1,0}+Q''_{3m}}, \qquad x''_{3m} = x'_{3m} - \sum_{n=1}^{M_{3m-1,0}+Q''_{3m}} u'^{*}_{3m,n}(\overline{x})u'_{3m,n},$$

$$\begin{aligned}x''_{0,3m} = x'_{0,3m} - \sum_{n=1}^{M_{3m-1,0}+Q''_{3m}} u'^{*}_{3m,n}(\overline{x})u'_{3m,n} = x_{0,3m} - \Big(\sum_{n=1}^{q(3m)} x_n^*(\overline{x})x_n\\ + \sum_{n=1}^{M_{3m-1,0}+Q''_{3m}} u'^{*}_{3m,n}(\overline{x})u'_{3m,n}\Big) \in \operatorname{span}(x_n)_{n=1}^{q(3m)} + \operatorname{span}(u'_{3m,s})_{s=1}^{M_{3m-1,0}+Q''_{3m}}.\end{aligned}$$

Moreover, by the definition of U''_{3m},

$$\begin{aligned}0 &= \operatorname{dist}(x''_{3m}, \operatorname{span}(v'_{3m,n} \cup (v'_{3m,n,k})_{k=1}^{Q'_{3m,n}} \cup (v''_{3m,n,k})_{k=1}^{Q''_{3m,n}})_{n=Q''_{3m}+1}^{Q'_{3m}} + U''_{3m}) \\ &= \Big\| x''_{3m} - \Big(\widetilde{x}_{3m} + \sum_{n=Q''_{3m}+1}^{Q'_{3m}} v'^{*}_{3m,n}(\overline{x}) v'_{3m,n}\Big)\Big\|,\end{aligned}$$

hence

$$\begin{aligned}\widetilde{x}_{3m} &= x''_{3m} - \sum_{n=Q''_{3m}+1}^{Q'_{3m}} v'^{*}_{3m,n}(\overline{x}) v'_{3m,n} \\ &= \overline{x} - \Big(\sum_{n=1}^{q(3m)} x^{*}_{n}(\overline{x}) x_n + \sum_{n=1}^{M_{3m-1,0}+Q''_{3m}} u'^{*}_{3m,n}(\overline{x}) u'_{3m,n} + \sum_{n=Q''_{3m}+1}^{Q'_{3m}} v'^{*}_{3m,n}(\overline{x}) v'_{3m,n}\Big) \\ &\in U''_{3m} + \operatorname{span}((v'_{3m,n,k})_{k=1}^{Q'_{3m,n}} \cup (v''_{3m,n,k})_{k=1}^{Q''_{3m,n}})_{n=Q''_{3m}+1}^{Q'_{3m}}.\end{aligned}$$

Hence $\|x''_{0,3m} - \widetilde{x}_{3m}\| > \|x''_{0,3m}\| - \eta_{3m}/2$ by what we said above about $x''_{0,3m}$, hence by the above

$$\begin{aligned}&\Big\| x''_{0,3m} - \Big(\widetilde{x}_{3m} + \sum_{n=Q''_{3m}+1}^{Q'_{3m}} v'^{*}_{3m,n}(\overline{x}) v'_{3m,n}\Big)\Big\| \\ &\qquad \geq \|x''_{0,3m} - \widetilde{x}_{3m}\| - \Big\| \sum_{n=Q''_{3m}+1}^{Q'_{3m}} v'^{*}_{3m,n}(\overline{x}) v'_{3m,n}\Big\| \\ &\qquad \geq \|x''_{0,3m} - \widetilde{x}_{3m}\| - \sum_{n=Q''_{3m}+1}^{Q'_{3m}} |v'^{*}_{3m,n}(\overline{x})| > \|x''_{0,3m} - \widetilde{x}_{3m}\| - \eta_{3m}/2 \\ &\qquad > \|x''_{0,3m}\| - \eta_{3m} \geq \|x''_{3m}\| - (\eta''_{3m} + \eta_{3m});\end{aligned}$$

that is, by the above,

$$\begin{aligned}0 &= \Big\| x''_{3m} - \Big(\widetilde{x}_{3m} + \sum_{n=Q''_{3m}+1}^{Q'_{3m}} v'^{*}_{3m,n}(\overline{x}) v'_{3m,n}\Big)\Big\| \\ &\geq \Big\| x''_{0,3m} - \Big(\widetilde{x}_{3m} + \sum_{n=Q''_{3m}+1}^{Q'_{3m}} v'^{*}_{3m,n}(\overline{x}) v'_{3m,n}\Big)\Big\| - \eta''_{3m} \geq \|x''_{3m}\| - (\eta''_{3m} + \eta_{3m}) - \eta''_{3m}.\end{aligned}$$

Thus $\|x''_{3m}\| \leq \eta'''_{3m} = 2\eta''_{3m} + \eta_{3m}$ and the assertion holds for $\eta'_m = \eta'''_{3m}$. This completes the proof of (i).

Proof of Lemma 16(ii). By Steps 1 and 2 of SC III.1, $(e_{3m,\mathrm{brd},p,s} + E_{3m,0})_{s=1}^{N_{3m,\mathrm{brd},p}}$ is 1-equivalent to the natural basis of $l_\infty^{N_{3m,\mathrm{brd},p}}$. Therefore, by the procedure of the proof of Fact 2 (see also Steps 5 and 6 of SC III.1) and setting

$$A = \sum_{n=q(3m-1)+1}^{q(3m)} (x^{*}_{n}(\overline{x}) x_n - a_n) x_n + \sum_{n=q(3m+2)+1}^{q(3m+3)} a_n x_n,$$

we have

$$\begin{aligned}
\eta_{m-1}+\eta_m &> \Big\| \sum_{n=q(3m)+1}^{q(3m+2)} x_n^*(\overline{x})x_n + A\Big\| \\
&= \Big\| \sum_{n=q(3m)+1}^{q(3m+2)} x_n^*(\overline{x})x_n \\
&\quad + \Big(A - \sum_{s=n'(3m+1)+1}^{A_{3m+1}} u'^*_{3m+1,s}(\overline{x})u'_{3m+1,s}\Big) + \sum_{s=n'(3m+1)+1}^{A_{3m+1}} u'^*_{3m+1,s}(\overline{x})u'_{3m+1,s}\Big\| \\
&\ge \Big\| \sum_{n=q(3m)+1}^{q(3m+2)} x_n^*(\overline{x})x_n + \Big(A - \sum_{s=n'(3m+1)+1}^{A_{3m+1}} u'^*_{3m+1,s}(\overline{x})u'_{3m+1,s}\Big)\Big\| \\
&\quad - \sum_{s=n'(3m+1)+1}^{A_{3m+1}} |u'^*_{3m+1,s}(\overline{x})| \\
&\ge \Big\| \sum_{n=q(3m)+1}^{q(3m+2)} x_n^*(\overline{x})x_n + \Big(A - \sum_{s=n'(3m+1)+1}^{A_{3m+1}} u'^*_{3m+1,s}(\overline{x})u'_{3m+1,s}\Big)\Big\| \\
&\quad - \frac{A_{3m+1}-n'(3m+1)}{A_{3m+1}2^{2q(3m+1)A_{3m+1}}} \\
&\ge \frac{1}{2}\Big\| \sum_{n=q(3m)+1}^{q(3m+1)} x_n^*(\overline{x})x_{\mathrm{brd},p,n} + E_{3m,0}\Big\| - \frac{A_{3m+1}-n'(3m+1)}{A_{3m+1}2^{2q(3m+1)A_{3m+1}}};
\end{aligned}$$

that is,

$$\begin{aligned}
&\Big\| \sum_{n=q(3m)+1}^{q(3m+1)} x_n^*(\overline{x})x_{\mathrm{brd},p,n} + E_{3m,0}\Big\| \\
&\quad < 2\Big(\eta_{m-1}+\eta_m + \frac{A_{3m+1}-n'(3m+1)}{A_{3m+1}2^{2q(3m+1)A_{3m+1}}}\Big) < 2\Big(\eta_{m-1}+\eta_m+\frac{1}{2^{2q(3m+1)A_{3m+1}}}\Big).
\end{aligned}$$

Hence the assertion is proved if we replace $2(\eta_{m-1}+\eta_m+1/2^{2q(3m+1)A_{3m+1}})$ by η_m. The same procedure works also for the other cases (we have to use also (39.6) of Theorem 11). This completes the proof of (ii).

Proof of Lemma 16(iii). By Steps 2 and 4 of SC III.1, $\|\sum_{t=1}^{t'} e_{3m,\mathrm{aux},n'(3m),t}\| = 1$ for $1 \le t' \le 2^{2B_{3m}}$ and $2^{B_{3m}}u'^*_{3m,n'(3m)}(\overline{x}) = \sum_{t=1}^{2^{2B_{3m}}} u^*_{3m,n'(3m),t}(\overline{x})$. Hence, if $\max(|u^*_{3m,n}(\overline{x})| : 1 \le n \le P_{3m}) = \eta'_m \to 0$ (by the second part of Fact 2), it follows that

$$\begin{aligned}
t(n'(3m))\eta'_m &> \Big| \sum_{t=1}^{t(n'(3m))} u^*_{3m,n'(3m),t}(\overline{x})\Big| \\
&\ge 2^{B_{3m}}|u'^*_{3m,n'(3m)}(\overline{x})| - \sum_{t=t(n'(3m))+1}^{2^{2B_{3m}}} |u^*_{3m,n'(3m),t}(\overline{x})| \\
&> 2^{B_{3m}}\frac{1}{A_{3m}2^{2q(3m)A_{3m}}} - \frac{2^{2B_{3m}}-t(n'(3m))}{P_{3m}2^{2q(3m)P_{3m}}} > 2^{B_{3m}/2},
\end{aligned}$$

that is, $t(n'(3m)) > 2^{B_{3m}/2}/\eta'_m > 2^{B_{3m}/2}$; therefore, setting

$$\widetilde{u}'_m = \sum_{t=1}^{t(n'(3m))-1} \widetilde{a} u_{3m,n'(3m),t}, \qquad \widetilde{a} = \frac{1}{t(n'(3m))-1} \sum_{j=1}^{t(n'(3m))-1} u^*_{3m,n'(3m),j}(\overline{x}),$$

we have $|\widetilde{a}| < \eta'_m$ and

$$\widetilde{u}'_m = \Big(\sum_{j=1}^{t(n'(3m))-1} u^*_{3m,n'(3m),j}(\overline{x}) \Big) \frac{u'_{3m,n'(3m)}}{2^{B_{3m}}} + \widetilde{a} \sum_{t=1}^{t(n'(3m))-1} e_{3m,n'(3m),t},$$

$$\begin{aligned}
&\|\widetilde{u}'_m - u'^*_{3m,n'(3m)}(\overline{x}) u'_{3m,n'(3m)}\| \\
&\quad = \bigg\| \widetilde{u}'_m - \bigg(\frac{1}{2^{B_{3m}}} \sum_{1}^{2^{2B_{3m}}} u^*_{3m,n'(3m),t}(\overline{x}) \bigg) u'_{3m,n'(3m)} \bigg\| \\
&\quad = \bigg\| \widetilde{a} \sum_{t=1}^{t(n'(3m))-1} e_{3m,\mathrm{aux},n'(3m),t} - \bigg(\frac{1}{2^{B_{3m}}} \sum_{t=t(n'(3m))}^{2^{2B_{3m}}} u^*_{3m,n'(3m),t}(\overline{x}) \bigg) u'_{3m,n'(3m)} \bigg\| \\
&\quad \le |\widetilde{a}| \cdot \bigg\| \sum_{t=1}^{t(n'(3m))-1} e_{3m,\mathrm{aux},n'(3m),t} \bigg\| \\
&\qquad + \frac{\|u'_{3m,n'(3m)}\|}{2^{B_{3m}}} \bigg(|u^*_{3m,n'(3m),t(n'(3m))}(\overline{x})| + \sum_{t=t(n'(3m))+1}^{2^{2B_{3m}}} |u^*_{3m,n'(3m),t}(\overline{x})| \bigg) \\
&\quad < \eta'_m \cdot 1 + 1 \cdot \frac{1}{2^{B_{3m}}} \bigg(\eta'_m + \sum_{t=t(n'(3m))+1}^{2^{2B_{3m}}} \frac{1}{P_{3m} 2^{2q(3m)P_{3m}}} \bigg) \\
&\quad < \eta'_m \bigg(1 + \frac{1}{2^{B_{3m}}} \bigg) + \frac{1}{2^{2q(3m)P_{3m}}}.
\end{aligned}$$

Hence the assertion is proved for $\eta_m = \eta'_m(1 + 1/2^{B_{3m}}) + 1/2^{2q(3m)P_{3m}}$. This completes the proof of Lemma 16. ∎

Before the regularization lemma, let us state an easy lemma on numerical permutations.

LEMMA 17 (on numerical permutations, NPL). *Suppose we have two sequences of numbers* $(n_k)_{k=1}^M$ *and* $(m_k)_{k=1}^M$ *so that*

$$(*) \qquad |n_k| \le 1 \quad \textit{and} \quad |m_k| \le 1 \quad \textit{for } 1 \le k \le M \quad \textit{and} \quad \sum_{k=1}^M n_k = \sum_{k=1}^M m_k = 0.$$

Then there exists a permutation $(\pi(k))_{k=1}^M$ *of* $(k)_{k=1}^M$ *such that* $(|\sum_{g=1}^G n_{\pi(k)}|)_{G=1}^M$ *and* $(|\sum_{g=1}^G m_{\pi(k)}|)_{G=1}^M$ *are both* $(2,0)$*-monotone* (*which in this case is tantamount to* $(0,2)$*-monotone*). *Analogously, if* $(p_k)_{k=1}^M$ *is another sequence of numbers with* $|p_k| \le 1$ *for* $1 \le k \le M$ *and* $\sum_{k=1}^M p_k = 0$, *then there exists a permutation* $(\pi(k))_{k=1}^M$ *of* $(k)_{k=1}^M$ *such that* $(|\sum_{g=1}^G n_{\pi(k)}|)_{G=1}^M$, $(|\sum_{g=1}^G m_{\pi(k)}|)_{G=1}^M$ *and* $(|\sum_{g=1}^G p_{\pi(k)}|)_{G=1}^M$ *are all* $(3,0)$*-*

monotone. Of course the $(0,3)$*-monotonicity continues to hold also if we pass to* $(n_k+a)_{k=1}^M$, $(m_k+b)_{k=1}^M$ *and* $(p_k+c)_{k=1}^M$ *for three fixed numbers* a, b *and* c.

Proof. Let us consider in $\mathbb{R}^2$, for $1\le k\le M$, the vector $v_k=\sqrt{{m_k}^2+n_k^2}e^{i\theta_k}$ corresponding to the point $P_k=(m_k,n_k)$. We start with $\pi(1)=1$, hence $v_{\pi(1)}=v_1$, and suppose for instance that $v_{\pi(1)}=\sqrt{2}e^{i\frac{1}{4}\pi}$ (that is, $P_{\pi(1)}=(1,1)$). Suppose that there exists $\pi(2)\in(k)_{k=2}^M$ with $\pi\le\theta_{\pi(2)}\le\frac{3}{2}\pi$. Then

$$\max(|m_{\pi(1)}+m_{\pi(2)}|,|n_{\pi(1)}+n_{\pi(2)}|)\le 1$$

and we are done for what concerns the choice of $v_{\pi(2)}$; then we can repeat the previous procedure but starting from $\sum_{k=1}^2 v_{\pi(k)}$ instead of from $v_{\pi(1)}$. While, if $\pi(2)$ does not exist, since the last part of $(*)$ says that $\sum_{k=2}^M v_k=-v_{\pi(1)}$, there exists $(v_{k'})_{k=1}^{M'}\cup(v_{k''})_{k=1}^{M''}\subseteq(v_k)_{k=2}^M$ such that $\frac{3}{4}\pi\le\theta_{k'}<\pi$ for $1\le k\le M'$ and $\frac{3}{2}\pi<\theta_{k''}\le\frac{7}{4}\pi$ for $1\le k\le M''$. Then we set $\pi(2)=1'$ and $\pi(3)=1''$; next, if $\pi\le\theta_{\pi(2)}+\theta_{\pi(3)}\le\frac{3}{2}\pi$, we repeat the whole previous procedure starting from $\sum_{k=1}^3 v_{\pi(k)}$; while, if $\frac{3}{4}\pi\le\theta_{\pi(2)}+\theta_{\pi(3)}<\frac{3}{2}\pi$, we set $\pi(4)=2''$ (analogously if $\frac{3}{2}\pi<\theta_{\pi(2)}+\theta_{\pi(3)}\le\frac{7}{8}\pi$, we set $\pi(4)=2'$). Proceeding in this way we get $(k''')_{k=1}^{M'''}\subseteq(k')_{k=1}^{M'}\cup(k'')_{k=1}^{M''}$ such that $\pi\le\sum_{k=1}^{M'''}\theta_{\pi(k''')}\le\frac{3}{2}\pi$; then we repeat the whole procedure starting from $\sum_{k=1}^3 v_{\pi(k)}+\sum_{k=1}^{M'''}v_{\pi(k''')}$ and so on. Now the procedure is clear. An analogous 3-dimensional geometrical proof works if we consider also the third sequence $(p_k)_{k=1}^M$. This completes the proof of Lemma 17. ∎

LEMMA 18 (Regularization lemma, RL). *Let* $\overline{x}\in X$ *with* $\|\overline{x}\|=1$ *and, for a fixed* m, *let* $\overline{a}_m(\overline{x})$ *be the number appearing at the end of* SC III.1, III.2 *and* III.3. *Then*

(A) *There exists a permutation* $(x_{\pi(0,3m+1,g)})_{g=1}^{G_{0,3m+1}}$ *of*

$$(x_{3m+1,g})_{g=1}^{G_{3m+1}}\cup(\omega'_{3m+1,n,0}\cup x_{3m+1,n.0}\cup(x_{3m+1,n,k,0})_{k=1}^{2^{M_{3m+1}}})_{n=1}^{P_{3m+1}}$$

such that, if $(x_{\pi(3m+1,g)})_{g=1}^{G_{3m+1}}$ *is the permutation induced on* $(x_{3m+1,g})_{g=1}^{G_{3m+1}}$, *then the permutation*

$$\Big(\Big\|\sum_{g=1}^{G}x^*_{\pi(0,3m+1,g)}(\overline{x})(x'_{\pi(0,3m+1,g)}+x_{\pi(0,3m+1,\mathrm{brd},g)})\\ +\operatorname{span}(u'_{3m+2,s})_{s=A_{3m+2}-A'_{3m+2}+1}^{n'(3m+2)}+E_{3m+1,0}\Big\|\Big)_{G=1}^{G_{0,3m+1}}$$

is $(6,0)$*-monotone; the permutations*

$$\Big(\Big\|\sum_{g=1}^{G}x^*_{\pi(3m+1,g)}(\overline{x})\widetilde{x}_{\pi(3m+1,g)}+E_{3m+1,0}\Big\|\Big)_{Q=1}^{G_{3m+1}}$$

and

$$\Big(\Big\|\sum_{g=1}^{G}x^*_{\pi(3m+1,g)}(\overline{x})\widetilde{x}_{\pi(3m+1,g)}\Big\|\Big)_{Q=1}^{G_{3m+1}}$$

are $(0,\overline{a}_{3m+1}/2^{Q_{0,3m+1}})$*-monotone; and*

$$\Big(\Big\|\sum_{g=1}^{G}x^*_{\pi(0,3m+1,g)}(\overline{x})x_{\pi(0,3m+1,g)}+\operatorname{span}(u'_{3m+2,s})_{s=A_{3m+2}-A'_{3m+2}+1}^{n'(3m+2)}+E_{3m+1,0}\Big\|\Big)_{Q=1}^{G_{0,3m+1}}$$

is $(2,6,\overline{a}_{3m+1}/2^{Q_{0,3m+1}})$*-monotone.*

(B) *There exists a permutation* $(x_{\pi(0,3m,g)})_{g=1}^{G_{0,3m}}$ *of* $(x_{0,3m,g})_{g=1}^{G_{0,3m}}$ *such that, if* $(x_{\pi(3m,g)})_{g=1}^{G_{3m}}$ *is the permutation induced on* $(x_{3m,g})_{g=1}^{G_{3m}}$, *then*

$$\Big(\Big\|\sum_{g=1}^{G} x^*_{\pi(0,3m,g)}(\overline{x})(x_{\pi(0,3m,g)}-\widetilde{x}_{\pi(0,3m,g)})+E_{3m,0} + \operatorname{span}(u'_{3m+1,s})_{s=G_{0,0,0,3m}-G_{3m}+1}^{G_{0,0,0,3m}} + \operatorname{span}(u'_{3m+1,s})_{s=A_{3m+1}-A'_{3m+1}+1}^{n'(3m+1)}\Big\|\Big)_{G=1}^{G_{0,3m}}$$

is $(3,0)$-*monotone; moreover*

$$\Big(\Big\|\sum_{g=1}^{G} x^*_{\pi(3m,g)}(\overline{x})\widetilde{x}_{\pi(3m,g)}+E_{3m,0}\Big\|\Big)_{G=1}^{G_{3m}} \quad \text{and} \quad \Big(\Big\|\sum_{g=1}^{G} x^*_{\pi(3m,g)}(\overline{x})\widetilde{x}_{\pi(3m,g)}\Big\|\Big)_{G=1}^{G_{3m}}$$

are $(2,\overline{a}_{3m}/2^{M_{3m}/2})$-*monotone; and*

$$\Big(\Big\|\sum_{g=1}^{G} x^*_{\pi(0,3m,g)}(\overline{x})x_{\pi(0,3m,g)}+E_{3m,0} + \operatorname{span}(u'_{3m+1,s})_{s=G_{0,0,0,3m}-G_{3m}+1}^{G_{0,0,0,3m}} + \operatorname{span}(u'_{3m+1,s})_{s=A_{3m+1}-A'_{3m+1}+1}^{n'(3m+1)}\Big\|\Big)_{G=1}^{G_{0,3m}}$$

is $(2,3,\overline{a}_{3m}/2^{M_{3m}/2})$-*monotone.*

(C) *There exists a permutation* $(\pi(3m+2,g))_{g=1}^{G_{3m+2}}$ *of* $((3m+2,g))_{g=1}^{G_{3m+2}}$ *such that*

$$\Big(\Big\|\sum_{g=1}^{G} x^*_{\pi(3m+2,g)}(\overline{x})\widetilde{x}_{\pi(3m+2,g)}+E_{3m+2,0}\Big\|\Big)_{G=1}^{G_{3m+2}}$$

and

$$\Big(\Big\|\sum_{g=1}^{G} x^*_{\pi(3m+2,g)}(\overline{x})\widetilde{x}_{\pi(3m+2,g)}\Big\|\Big)_{G=1}^{G_{3m+2}}$$

are $(0,\overline{a}_{3m+2}/2^{Q_{0,3m+2}})$-*monotone, and*

$$\Big(\Big\|\sum_{g=1}^{G} x^*_{\pi(3m+2,g)}(\overline{x})x_{\pi(3m+2,g)}+E_{3m+2,0}\Big\|\Big)_{G=1}^{G_{3m+2}}$$

is $(2,1,\overline{a}_{3m+2}/2^{Q_{0,3m+2}})$-*monotone.*

Proof. First part. Let us consider the block $(x^*_n(\overline{x})x_n)_{n=q(3m+1)+1}^{q(3m+2)}$. We point out that, in what follows, we could also use the alternative construction of Step 4′ of SC III.2 and the procedure of the proof would be the same, only with more formalism. We set

$$(x_n)_{n=q(3m+1)+1}^{q(3m+2)} = (x_{3m+1,g})_{g=1}^{G_{3m+1}} = ((x_{3m+1,n,g})_{=1}^{G_{3m+1,0}})_{n=1}^{P_{3m+1}},$$
$$G_{3m+1,0} = 2^{M_{3m+1}+4Q_{3m+1}},$$
$$(x_{3m+1,n,g})_{=1}^{G_{3m+1,0}} = ((x_{3m+1,n,k,l})_{l=1}^{2^{4Q_{3m+1}}})_{k=1}^{2^{M_{3m+1}}}$$

for $1\le n\le P_{3m+1}$. Then by Steps 1 and 4 of SC III.2 we find that, for each sequence

$(a_n)_{n=q(3m+1)+1}^{q(3m+2)}$ of numbers,

$$\Big\|\sum_{n=q(3m+1)+1}^{q(3m+2)} a_n x_n + E_{3m+1,0}\Big\|$$
$$\geq \max\Big(\Big\|\sum_{n=q(3m+1)+1}^{q(3m+2)} a_n(x'_n + x''_n + x_{\mathrm{brd},n}) + E_{3m+1,0}\Big\|\Big/2, \quad \Big\|\sum_{n=q(3m+1)+1}^{q(3m+2)} a_n x'''_n + E_{3m+1,0}\Big\|\Big);$$

$$\Big\|\sum_{n=q(3m+1)+1}^{q(3m+2)} a_n x'''_n + E_{3m+1,0}\Big\| = \Big\|\sum_{n=q(3m+1)+1}^{q(3m+2)} a_n x'''_n\Big\|;$$

$$\Big\|\sum_{n=q(3m+1)+1}^{q(3m+2)} a_n(x'_n + x''_n + x_{\mathrm{brd},n}) + E_{3m+1,0}\Big\|$$
$$= \max\Big(\Big\|\sum_{g=1}^{G_{3m+1,0}} a_{3m+1,n,g}(x'_{3m+1,n,g} + x''_{3m+1,n,g} + x_{3m+1,\mathrm{brd},n,g}) + E_{3m+1,0}\Big\| : 1 \leq n \leq P_{3m+1}\Big);$$

$$\Big\|\sum_{g=1}^{G_{3m+1,0}} a_{3m+1,n,g}(x'_{3m+1,n,g} + x''_{3m+1,n,g} + x_{3m+1,\mathrm{brd},n,g}) + E_{3m+1,0}\Big\|$$
$$= \max\Big(\Big\|\sum_{g=1}^{G_{3m+1,0}} a_{3m+1,n,g}x'_{3m+1,n,g} + E_{3m+1,0}\Big\|, \Big\|\sum_{g=1}^{G_{3m+1,0}} a_{3m+1,n,g}x''_{3m+1,n,g} + E_{3m+1,0}\Big\|, \Big\|\sum_{g=1}^{G_{3m+1,0}} a_{3m+1,n,g}x_{3m+1,n,\mathrm{brd},g} + E_{3m+1,0}\Big\|\Big)$$

for $1 \leq n \leq P_{3m+1}$. If $(\pi(n))_{n=q(3m+1)+1}^{q(3m+2)} = (\pi(3m+1,g))_{g=1}^{G_{3m+1}}$ is a permutation of $(n)_{n=q(3m+1)+1}^{q(3m+2)}$ and if, for a fixed n with $1 \leq n \leq P_{3m+1}$, $(\pi(3m+1,n,g))_{g=1}^{G_{3m+1,0}}$ is the permutation induced on $((3m+1,n,g))_{g=1}^{G_{3m+1,0}}$, we denote, for each k and l with $1 \leq k \leq 2^{M_{3m+1}}$ and $1 \leq l \leq 2^{4Q_{3m+1}}$, by $(\pi(3m+1,n,k,g))_{g=1}^{2^{M_{3m+1}}}$ and $(\pi(3m+1, n,g,l))_{g=1}^{2^{4Q_{3m+1}-1}}$ the permutations induced on $((3m+1,n,k,g))_{g=1}^{2^{M_{3m+1}}}$ and $((3m+1, n,g,l))_{g=1}^{2^{4Q_{3m+1}-1}}$ respectively. Then (recall that, by Steps 1, 2 and 3 of SC III.2, $\|\omega'_{3m+1,n,0}\| = 1 = \|x_{3m+1,n,0}\| = \|x_{3m+1,n,k,0}\|$ for $1 \leq k \leq 2^{M_{3m+1}}$) we have

$$\Big\|\omega'^*_{3m+1,n,0}(\overline{x})\omega'_{3m+1,n,0} + x^*_{3m+1,n,0}(\overline{x})x_{3m+1,n,0}$$
$$+ \sum_{k=1}^{2^{M_{3m+1}}}\Big(x^*_{3m+1,n,k,0}(\overline{x})x_{3m+1,n,k,0} + \sum_{l=1}^{2^{4Q_{3m+1}}} x^*_{3m+1,n,k,l}(\overline{x})x'_{3m+1,n,k,l}\Big) + E_{3m+1,0}\Big\|$$

$$
\begin{aligned}
&= \Big\| \omega'^{*}_{3m+1,n,0}(\overline{x}) \Big(\sum_{f=1}^{2^{M_{3m+1}}} e_{3m+1,n,f,2^{4Q_{3m+1}}} \Big) + x^{*}_{3m+1,n,0}(\overline{x}) \Big(\sum_{f=1}^{2^{M_{3m+1}}} e_{3m+1,n,f} \Big) \\
&\quad + \sum_{k=1}^{2^{M_{3m+1}}} \bigg(x^{*}_{3m+1,n,k,0}(\overline{x}) \Big(\sum_{g=1}^{2^{4Q_{3m+1}}} e_{3m+1,n,k,g} + \omega'_{3m+1,n} \Big) \\
&\quad + \sum_{l=1}^{2^{4Q_{3m+1}}-1} x^{*}_{3m+1,n,k,l}(\overline{x}) \bigg(e_{3m+1,n,k,l} + \frac{e_{3m+1,n,k}}{2^{P_{3m+1}Q_{0,3m+1}} 2^{2Q_{3m+1}}} \bigg) \\
&\quad + x^{*}_{3m+1,n,k,2^{4Q_{3m+1}}}(\overline{x}) \bigg(e_{3m+1,n,k,2^{4Q_{3m+1}}} + \omega'_{3m+1,n} + \frac{e_{3m+1,n,k}}{2^{P_{3m+1}Q_{0,3m+1}} 2^{2Q_{3m+1}}} \bigg) \bigg) \\
&\quad + E_{3m+1,0} \Big\| \\
&= \Big\| \sum_{k=1}^{2^{M_{3m+1}}} \bigg(x^{*}_{3m+1,n,0}(\overline{x}) + \sum_{l=1}^{2^{4Q_{3m+1}}} \frac{x^{*}_{3m+1,n,k,l}(\overline{x})}{2^{P_{3m+1}Q_{0,3m+1}} 2^{2Q_{3m+1}}} \bigg) e_{3m+1,n,k} \\
&\quad + \sum_{k=1}^{2^{M_{3m+1}}} \Big(\sum_{l=1}^{2^{4Q_{3m+1}}-1} (x^{*}_{3m+1,n,k,0}(\overline{x}) + x^{*}_{3m+1,n,k,l}(\overline{x})) e_{3m+1,n,k,l} \\
&\quad + (\omega'^{*}_{3m+1,n,0}(\overline{x}) + x^{*}_{3m+1,n,k,0}(\overline{x}) + x^{*}_{3m+1,n,k,2^{4Q_{3m+1}}}(\overline{x}) \Big) e_{3m+1,n,k,2^{4Q_{3m+1}}} \\
&\quad + \Big(\sum_{k=1}^{2^{M_{3m+1}}} (x^{*}_{3m+1,n,k,0}(\overline{x}) + x^{*}_{3m+1,n,k,2^{4Q_{3m+1}}}(\overline{x})) \Big) \omega'_{3m+1,n} + E_{3m+1,0} \Big\|.
\end{aligned}
$$

By Steps 2 and 3 of SC III.2, setting

$$
A = \frac{1}{2^{4Q_{3m+1}}} \Big(\sum_{g=1}^{2^{4Q_{3m+1}}-1} e^{*}_{3m+1,n,k,g}(\overline{x}) + e''^{*}_{3m+1,n,k,2^{4Q_{3m+1}}}(\overline{x}) \Big),
$$

we have

$$
x^{*}_{3m+1,n,k,0}(\overline{x}) = A - B, \quad B = \frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{2Q_{3m+1}}} x^{*}_{3m+1,n,k}(\overline{x}),
$$

$$
\begin{aligned}
x^{*}_{3m+1,n,0}(\overline{x}) &+ \sum_{l=1}^{2^{4Q_{3m+1}}} \frac{x^{*}_{3m+1,n,k,l}(\overline{x})}{2^{P_{3m+1}Q_{0,3m+1}} 2^{2Q_{3m+1}}} \\
&= x^{*}_{3m+1,n,0}(\overline{x}) \\
&\quad + \frac{\sum_{l=1}^{2^{4Q_{3m+1}}-1} (e^{*}_{3m+1,n,k,l}(\overline{x}) - A + B) + (e''^{*}_{3m+1,n,k,2^{4Q_{3m+1}}}(\overline{x}) - A + B)}{2^{P_{3m+1}Q_{0,3m+1}} 2^{2Q_{3m+1}}} \\
&= x^{*}_{3m+1,n,0}(\overline{x}) + \frac{2^{2Q_{3m+1}}}{2^{P_{3m+1}Q_{0,3m+1}}} B = x^{*}_{3m+1,n,0}(\overline{x}) + x^{*}_{3m+1,n,k}(\overline{x}) \\
&= e^{*}_{3m+1,n,k}(\overline{x}),
\end{aligned}
$$

$$
x^{*}_{3m+1,n,k,0}(\overline{x}) + x^{*}_{3m+1,n,k,l}(\overline{x}) = e^{*}_{3m+1,n,k,l}(\overline{x}) \quad \text{for } 1 \le l \le 2^{4Q_{3m+1}} - 1,
$$

$$x^*_{3m+1,n,k,0}(\overline{x})+x^*_{3m+1,n,k,2^{4Q_{3m+1}}}(\overline{x})=e''^*_{3m+1,n,k,2^{4Q_{3m+1}}}(\overline{x}),$$

$$\begin{aligned}\omega'^*_{3m+1,n,0}(\overline{x})&+x^*_{3m+1,n,k,0}(\overline{x})+x^*_{3m+1,n,k,2^{4Q_{3m+1}}}(\overline{x})\\&=\omega'^*_{3m+1,n,0}(\overline{x})+e''^*_{3m+1,n,k,2^{4Q_{3m+1}}}(\overline{x})=e^*_{3m+1,n,k,2^{4Q_{3m+1}}}(\overline{x}),\end{aligned}$$

$$\begin{aligned}\sum_{k=1}^{2^{M_{3m+1}}}&(x^*_{3m+1,n,k,0}(\overline{x})+x^*_{3m+1,n,k,2^{4Q_{3m+1}}}(\overline{x}))\\&=\sum_{k=1}^{2^{M_{3m+1}}}e''^*_{3m+1,n,k,2^{4Q_{3m+1}}}(\overline{x})=\omega'^*_{3m+1,n}(\overline{x}).\end{aligned}$$

Therefore

$$\begin{aligned}(45)\quad&\Big\|\omega'^*_{3m+1,n,0}(\overline{x})\omega'_{3m+1,n,0}+x^*_{3m+1,n,0}(\overline{x})x_{3m+1,n,0}\\&+\sum_{k=1}^{2^{M_{3m+1}}}\Big(x^*_{3m+1,n,k,0}(\overline{x})x_{3m+1,n,k,0}+\sum_{l=1}^{2^{4Q_{3m+1}}}x^*_{3m+1,n,k,l}(\overline{x})x'_{3m+1,n,k,l}\Big)+E_{3m+1,0}\Big\|\\&=\Big\|\sum_{k=1}^{2^{M_{3m+1}}}e^*_{3m+1,n,k}(\overline{x})e_{3m+1,n,k}\\&+\sum_{k=1}^{2^{M_{3m+1}}}\sum_{l=1}^{2^{4Q_{3m+1}}}e^*_{3m+1,n,k,l}(\overline{x})e_{3m+1,n,k,l}+\omega'^*_{3m+1,n}(\overline{x})\omega'_{3m+1,n}+E_{3m+1,0}\Big\|\\&=\max(D_{1,3m+1,n},D_{2,3m+1,n},D_{3,3m+1,n},D_{4,3m+1,n});\end{aligned}$$

$$D_{1,3m+1,n}=\max\Big(\Big|e^*_{3m+1,n,k}(\overline{x})=x^*_{3m+1,n,0}(\overline{x})+\sum_{l=1}^{2^{4Q_{3m+1}}}\frac{x^*_{3m+1,n,k,l}(\overline{x})}{2^{P_{3m+1}Q_{0,3m+1}}2^{2Q_{3m+1}}}\Big|:1\le k\le 2^{M_{3m+1}}\Big);$$

$$D_{2,3m+1,n}=\max\Big(\Big\|\sum_{k=1}^{2^{M_{3m+1}}}(e^*_{3m+1,n,k,l}(\overline{x})=x^*_{3m+1,n,k,0}(\overline{x})+x^*_{3m+1,n,k,l}(\overline{x}))e_{3m+1,n,k,l}+E_{3m+1,0}\Big\|:1\le l\le 2^{4Q_{3m+1}}-1\Big);$$

$$D_{3,3m+1,n}=\max(|e^*_{3m+1,n,k,2^{4Q_{3m+1}}}(\overline{x})=\omega'^*_{3m+1,n,0}(\overline{x})+x^*_{3m+1,n,k,0}(\overline{x})+x^*_{3m+1,n,k,2^{4Q_{3m+1}}}(\overline{x})|:1\le k\le 2^{M_{3m+1}});$$

$$D_{4,3m+1,n}=\Big|\omega'^*_{3m+1,n}(\overline{x})=\sum_{k=1}^{2^{M_{3m+1}}}(x^*_{3m+1,n,k,0}(\overline{x})+x^*_{3m+1,n,k,2^{4Q_{3m+1}}}(\overline{x}))\Big|.$$

On the other hand,

$$\Big\|\sum_{n=q(3m+1)+1}^{q(3m+2)}x^*_n(\overline{x})x_{\mathrm{brd},n}+E_{3m+1,0}+\operatorname{span}(u'_{3m+2,s})_{s=A_{3m+2}-A'_{3m+2}+1}^{n'(3m+2)}\Big\|$$

$$\begin{aligned}
&= \Big\| \sum_{n=1}^{P_{3m+1}} \sum_{k=1}^{2^{M_{3m+1}}} x^*_{3m+1,n,k,2^{4Q_{3m+1}}}(\overline{x})(e_{3m+1,\mathrm{brd},n,k} + \omega_{3m+1,n}) \\
&\quad + E_{3m+1,0} + \mathrm{span}(u'_{3m+2,s})_{s=A_{3m+2}-A'_{3m+2}+1}^{n'(3m+2)} \Big\| \\
&= \Big\| \sum_{k=k(3m+1,\mathrm{brd})+1}^{2^{M_{3m+i}}} x^*_{3m+1,n''(3m+1,\mathrm{brd}),k,2^{4Q_{3m+1}}}(\overline{x})(e_{3m+1,\mathrm{brd},n''(3m+1,\mathrm{brd}),k} \\
&\quad + \omega_{3m+1,n''(3m+1,\mathrm{brd})}) + \sum_{n=n''(3m+1,\mathrm{brd})+1}^{P_{3m+1}} \sum_{k=1}^{2^{M_{3m+1}}} x^*_{3m+1,n,k,2^{4Q_{3m+1}}}(\overline{x}) \\
&\quad \cdot (e_{3m+1,\mathrm{brd},n,k} + \omega_{3m+1,n})) + E_{3m+1,0} \Big\| \\
&= \Big\| \sum_{k=k(3m+1,\mathrm{brd})+1}^{2^{M_{3m+i}}} x^*_{3m+1,n''(3m+1,\mathrm{brd}),k,2^{4Q_{3m+1}}}(\overline{x}) e_{3m+1,\mathrm{brd},n''(3m+1,\mathrm{brd}),k} \\
&\quad + \sum_{n=n''(3m+1,\mathrm{brd})+1}^{P_{3m+1}} \sum_{k=1}^{2^{M_{3m+1}}} x^*_{3m+1,n,k,2^{4Q_{3m+1}}}(\overline{x}) e_{3m+1,\mathrm{brd},n,k} \\
&\quad + \Big(\sum_{k=k(3m+1,\mathrm{brd})+1}^{2^{M_{3m+i}}} x^*_{3m+1,n,k,2^{4Q_{3m+1}}}(\overline{x}) \Big) \omega_{3m+1,n''(3m+1,\mathrm{brd})} \\
&\quad + \sum_{n=n''(3m+1,\mathrm{brd})+1}^{P_{3m+1}} \Big(\sum_{k=1}^{2^{M_{3m+1}}} x^*_{3m+1,n,k,2^{4Q_{3m+1}}}(\overline{x}) \Big) \omega_{3m+1,n} + E_{3m+1,0} \Big\| \\
&= \max(D_{5,3m+1}, D_{6,3m+1}),
\end{aligned}$$

$$\begin{aligned}
D_{5,3m+1} &= \max(\max(|x^*_{3m+1,n''(3m+1,\mathrm{brd}),k,2^{4Q_{3m+1}}}(\overline{x})| : \\
&\qquad\qquad k(3m+1,\mathrm{brd}) + 1 \le k \le 2^{M_{3m+i}}), \\
&\quad \max(|x^*_{3m+1,n,k,2^{4Q_{3m+1}}}(\overline{x})| : 1 \le k \le 2^{M_{3m+i}}, n''(3m+1,\mathrm{brd}) + 1 \le n \le P_{3m+1})), \\
D_{6,3m+1} &= \max\Big(\Big| \sum_{k=k(3m+1,\mathrm{brd})+1}^{2^{M_{3m+i}}} x^*_{3m+1,n''(3m+1,\mathrm{brd}),k,2^{4Q_{3m+1}}}(\overline{x}) \Big|, \\
&\qquad \max\Big(\Big| \sum_{k=1}^{2^{M_{3m+1}}} x^*_{3m+1,n,k,2^{4Q_{3m+1}}}(\overline{x}) \Big| : n''(3m+1,\mathrm{brd}) + 1 \le n \le P_{3m+1} \Big)\Big).
\end{aligned}$$

Second part. We have to consider (also for each permutation $(x_{\pi(3m+1,g)})_{g=1}^{G_{3m+1,0}}$ of $(x_{3m+1,g})_{g=1}^{G_{3m+1,0}}$)

$$\begin{aligned}
\overline{D}_{3m+1} = \max(\max((&\overline{D}_{1,3m+1,n}, \overline{D}_{2,3m+1,n}, \overline{D}_{3,3m+1,n}, \overline{D}_{4,3m+1,n} : \\
&1 \le n \le P_{3m+1}), D_{5,3m+1}, D_{6,3m+1})
\end{aligned}$$

where

$$\begin{aligned}
\overline{D}_{1,3m+1,n} = \max\Big(\Big| x^*_{3m+1,n,0}(\overline{x}) + \sum_{l=1}^{L} \frac{x^*_{3m+1,n,k,l}(\overline{x})}{2^{P_{3m+1}Q_{0,3m+1}} 2^{2Q_{3m+1}}} \Big| : \\
1 \le L \le 2^{4Q_{3m+1}}, 1 \le k \le 2^{M_{3m+1}} \Big);
\end{aligned}$$

$$\overline{D}_{2,3m+1,n} = \max\Big(\max\Big(\Big\|\sum_{k=1}^{K-1}(x^*_{3m+1,n,k,0}(\overline{x}) + x^*_{3m+1,n,k,l}(\overline{x}))e_{3m+1,n,k,l} + E_{3m+1,0}\Big\| : 1 \le l \le 2^{4Q_{3m+1}} - 1\Big),$$

$$\max\Big(\Big\|\sum_{k=1}^{K}(x^*_{3m+1,n,k,0}(\overline{x}) + x^*_{3m+1,n,k,l}(\overline{x}))e_{3m+1,n,k,l} + E_{3m+1,0}\Big\| : 1 \le l \le L\Big) : 1 \le L \le 2^{4Q_{3m+1}} - 1, 1 \le K \le 2^{M_{3m+1}}\Big);$$

$$\overline{D}_{3,3m+1,n} = \max(|e^*_{3m+1,n,k,2^{4Q_{3m+1}}}(\overline{x}) = \omega'^*_{3m+1,n,0}(\overline{x}) + x^*_{3m+1,n,k,0}(\overline{x}) + x^*_{3m+1,n,k,2^{4Q_{3m+1}}}(\overline{x})| : 1 \le k \le 2^{M_{3m+1}});$$

$$\overline{D}_{4,3m+1,n} = \max\Big(\Big|\sum_{k=1}^{K}(x^*_{3m+1,n,k,0}(\overline{x}) + x^*_{3m+1,n,k,2^{4Q_{3m+1}}}(\overline{x}))\Big| : 1 \le K \le 2^{M_{3m+1}}\Big).$$

Fix n with $1 \le n \le P_{3m+1}$. For $\overline{D}_{1,3m+1,n}$ it is sufficient that, for $1 \le k \le 2^{M_{3m+1}}$, $(|\sum_{l=1}^{L} x^*_{\pi(3m+1,n,k,l)}(\overline{x})|)_{L=1}^{2^{4Q_{3m+1}}-1}$ is $(1,0)$-monotone (hence $(|\sum_{l=1}^{L} \frac{x^*_{\pi(3m+1,n,k,l)}(\overline{x})}{2^{P_{3m+1}Q_{0,3m+1}}2^{2Q_{3m+1}}}|)_{L=1}^{2^{4Q_{3m+1}}}$ becomes $(1,0)$-monotone, precisely $(0,\overline{a}_{3m+1}/2^{P_{3m+1}Q_{0,3m+1}}2^{2Q_{3m+1}})$-monotone); $\overline{D}_{3,3m+1,n} \le 3\overline{a}_{3m+1}$ and $D_{5,3m+1} \le \overline{a}_{3m+1}$; and $\overline{D}_{4,3m+1,n} \le 3\overline{a}_{3m+1}$ if

$$\Big(\Big|\sum_{k=1}^{K} e''^*_{\pi(3m+1,n,k,2^{4Q_{3m+1}})}(\overline{x}) \ (= x^*_{\pi(3m+1,n,k,2^{4Q_{3m+1}})}(\overline{x}) + x^*_{\pi(3m+1,n,k,0)}(\overline{x}))\Big|\Big)_{K=1}^{2^{M_{3m+1}}}$$

is $(3,0)$-monotone, $D_{6,3m+1} \le 3\overline{a}_{3m+1}$ if $(|\sum_{k=1}^{K} x^*_{\pi(3m+1,n,k,2^{4Q_{3m+1}})}(\overline{x})|)_{K=1}^{2^{M_{3m+1}}}$ is $(3,0)$-monotone and we recall the second part of (ii) of CL. Turning to $\overline{D}_{2,3m+1,n}$ it is necessary to have together $x^*_{3m+1,n,k,0}(\overline{x})x_{3m+1,n,k,0}$ and $x^*_{3m+1,n,k,l}(\overline{x})\,x_{3m+1,n,k,l}$, in order to avoid the following drawback: Suppose that, for $n = 1$ and $1 \le k \le 2^{M_{3m+1}}$, $x^*_{3m+1,1,k,l}(\overline{x}) = (-1)^k$ for $1 \le l \le 2^{4Q_{3m+1}}$. Then (by Step 1 of SC III.2 and by Theorem 9 and in particular by the same tools of the proof of (39.5), in particular by (23.5) of Remark 6)

$$\Big\|\sum_{k=1}^{2^{M_{3m+1}}}\sum_{l=1}^{2^{4Q_{3m+1}}-1} x^*_{3m+1,1,k,l}(\overline{x})e_{3m+1,1,k,l} + E_{3m+1,0}\Big\| = \Big\|\sum_{k=1}^{2^{M_{3m+1}}}(-1)^k e_{3m+1,1,k,1} + E_{3m+1,0}\Big\| \ge 2^{M_{3m+1}/2};$$

$$\begin{aligned}\Big\|\sum_{k=1}^{2^{M_{3m+1}}}\sum_{l=1}^{2^{4Q_{3m+1}}} x^*_{3m+1,1,k,l}(\overline{x})e_{3m+1,1,k,l} - \sum_{k=1}^{2^{M_{3m+1}}}(-1)^k x_{3m+1,1,k,0} + E_{3m+1,0}\Big\| \\ = \Big\|\sum_{k=1}^{2^{M_{3m+1}}}\sum_{l=1}^{2^{4Q_{3m+1}}}(-1)^k e_{3m+1,1,k,l} - \sum_{k=1}^{2^{M_{3m+1}}}(-1)^k\Big(\sum_{l=1}^{2^{4Q_{3m+1}}} e_{3m+1,1,k,l} + \omega'_{3m+1,1}\Big) \\ + E_{3m+1,0}\Big\| \\ = \Big\|\sum_{k=1}^{2^{M_{3m+1}}}(-1)^k\omega'_{3m+1,1} + E_{3m+1,0}\Big\| = 0.\end{aligned}$$

On the other hand, in the third part of the proof we will use the fact that

$$\Big(\Big|\sum_{g=1}^{G} x^*_{\pi(3m+1,n,g)}(\overline{x})\Big|\Big)_{G=1}^{G_{3m+1,0}}$$

is $(6,0)$-monotone for $1 \le n \le P_{3m+1}$; therefore we can settle this fact and avoid the possibility of the previous drawback, by means of the following property:

Suppose we have a sequence of numbers $(b_k)_{k=1}^{2^{M_{3m+1}}} \cup ((a_{k,l})_{l=1}^{2^{4Q_{3m+1}}-1})_{k=1}^{2^{M_{3m+1}}}$ with $a_{k,l} = b_k$ for $1 \le l \le 2^{4Q_{3m+1}} - 1$ and $1 \le k \le 2^{M_{3m+1}}$, moreover set $a = \max(|b_k| : 1 \le k \le 2^{M_{3m+1}})$ and suppose that $\sum_{k=1}^{2^{M_{3m+1}}} b_k = 0$. Setting $(b_k)_{k=1}^{2^{M_{3m+1}}} = (b_{k'})_{k=1}^{M'} \cup (b_{k''})_{k=1}^{M''}$ with $b_{k'} > 0$ for $1 \le k \le M'$ and $b_{k''} < 0$ for $1 \le k \le M''$ (we can suppose $|b_k| > 0$ for $1 \le k \le 2^{M_{3m+1}}$), we start with $\pi(1) = 1'$. There are two possibilities: if $b_{1'} \le |b_{1''}|$, then there exists a permutation $(a_{\pi(g)})_{g=1}^{G_1}$ of $(a_{1',l})_{l=1}^{2^{4Q_{3m+1}}-1} \cup (a_{1'',l})_{l=1}^{L_1}$ $(L_1 \le 2^{4Q_{3m+1}} - 1)$ such that $(|\sum_{g=1}^{G} a_{\pi(g)}|)_{G=1}^{G_1}$ is $(1,0)$-monotone; otherwise $b_{1'} > |b_{1''}|$ and then there exist two positive integers $S(1) > 1$ and $L_1 \le 2^{4Q_{3m+1}} - 1$ such that $b_{1'} > |\sum_{k=1}^{S(1)-1} b_{k''}|$ while $b_{1'} \le |\sum_{k=1}^{S(1)} b_{k''}|$, hence again there exists a permutation $(a_{\pi(g)})_{g=1}^{G_1}$ of $(a_{1',l})_{l=1}^{2^{4Q_{3m+1}}-1} \cup ((a_{k'',l})_{l=1}^{2^{4Q_{3m+1}}-1})_{k=1}^{S(1)-1} \cup (a_{S(1)'',l})_{l=1}^{L_1}$ such that $(|\sum_{g=1}^{G} a_{\pi(g)}|)_{G=1}^{G_1}$ is $(1,0)$-monotone. Since the second case includes also the first one, setting $S(1) = 1$, we can start again from this general situation and there are three possibilities:

(i) It is not possible to get $L_1 = 2^{4Q_{3m+1}} - 1$ (hence $b_{1'} < |\sum_{k=1}^{S(1)} b_{k''}|$). In this case there exist again two positive integers $T(2) > 2$ and $L_2 \le 2^{4Q_{3m+1}} - 1$ such that $\sum_{k=1}^{T(2)-1} b_{k'} < |\sum_{k=1}^{S(1)} b_{k''}|$ while $\sum_{k=1}^{T(2)} b_{k'} \ge |\sum_{k=1}^{S(1)} b_{k''}|$, hence there exists a permutation $(a_{\pi(g)})_{g=G_1+1}^{G_2}$ of

$$(a_{S(1)'',l})_{l=L_1+1}^{2^{4Q_{3m+1}}-1} \cup ((a_{k',l})_{l=1}^{2^{4Q_{3m+1}}-1})_{k=2}^{T(2)-1} \cup (a_{T(2)',l})_{l=1}^{L_2}$$

such that $(|\sum_{g=1}^{G} a_{\pi(g)}|)_{G=1}^{G_2}$ is $(1,0)$-monotone.

(ii) $L_1 = 2^{4Q_{3m+1}} - 1$ and $b_{1'} < |\sum_{k=1}^{S(1)} b_{k''}|$. Then we proceed as for (i); while, if $b_{1'} = |\sum_{k=1}^{S(1)} b_{k''}|$, we can repeat the whole procedure starting from $(b_{k'})_{k=2}^{M'} \cup (b_{k''})_{k=S(1)+1}^{M''}$ instead of from $(b_{k'})_{k=1}^{M'} \cup (b_{k''})_{k=1}^{M''}$.

Now the procedure is clear and we conclude that there exist two permutations

$$(\pi(k))_{k=1}^{2^{M_{3m+1}}} \text{ of } (k)_{k=1}^{2^{M_{3m+1}}}, \quad (a_{\pi(g)})_{g=1}^{2^{M_{3m+1}}(2^{4Q_{3m+1}}-1)} \text{ of } ((a_{k,l})_{l=1}^{2^{4Q_{3m+1}}-1})_{k=1}^{2^{M_{3m+1}}}$$

with both

$(46.1)'$ $(|\sum_{g=1}^{G} a_{\pi(g)}|)_{G=1}^{2^{M_{3m+1}}(2^{4Q_{3m+1}}-1)}$, $(|\sum_{k=1}^{K} b_{\pi(k)}|)_{K=1}^{2^{M_{3m+1}}}$ $(1,0)$-monotone;

$(46.2)'$ for each G, $1 \le G \le 2^{M_{3m+1}}(2^{4Q_{3m+1}} - 1)$, there are four integers S, T, $L(s)$ and $L(T)$ with $1 \le S \le T \le 2^{M_{3m+1}}$ and $1 \le L(S),\ L(T) \le 2^{4Q_{3m+1}} - 1$ so that $(a_{\pi(g)})_{g=1}^{G} = ((a_{\pi(k),l})_{l=1}^{2^{4Q_{3m+1}}-1})_{k(\ne S)=1}^{T-1} \cup (a_{\pi(S),l})_{l=1}^{L(S)} \cup (a_{\pi(T),l})_{l=1}^{L(T)}$

(that is, apart from two rows at the most, all the rows present in $(\pi(g))_{g=1}^{G}$ are totally included). Indeed, this property is sufficient to settle $\overline{D}_{2,3m+1,n}$ since, taking into account also (39.4) of Theorem 11 and (30.4) of Theorem 8, we see that, for $1 \le n \le P_{3m+1}$,

$((e_{3m+1,n,k,l}+E_{3m+1,0})_{l=1}^{2^{4Q_{3m+1}}-1})_{k=1}^{2^{M_{3m+1}}}$ is 2-unconditional and in particular, for $1 \leq 1 \leq 2^{4Q_{3m+1}}-1$, $(e_{3m+1,n,k,l}+E_{3m+1,0})_{k=1}^{2^{M_{3m+1}}}$ is 2-indiscernible.

Now, for a fixed n with $1 \leq n \leq P_{3m+1}$, by means of NPL (Lemma 17), setting

$$E_k^*(\overline{x}) = e^*_{3m+1,n,k}(\overline{x}) - \frac{1}{2^{M_{3m+1}}}\sum_{f=1}^{2^{M_{3m+1}}} e^*_{3m+1,n,f}(\overline{x}) \quad \text{for } 1 \leq k \leq 2^{M_{3m+1}},$$

let $(\pi(k))_{k=1}^{2^{M_{3m+1}}}$ be a permutation of $(k)_{k=1}^{2^{M_{3m+1}}}$ such that $(|\sum_{k=1}^{K} E^*_{\pi(k)}(\overline{x})|)_{K=1}^{2^{M_{3m+1}}}$, $(|\sum_{k=1}^{K} x^*_{3m+1,n,\pi(k),2^{4Q_{3m+1}}}(\overline{x})|)_{K=1}^{2^{M_{3m+1}}}$, $(|\sum_{k=1}^{K} e''^*_{3m+1,n,\pi(k),2^{4Q_{3m+1}}}(\overline{x})|)_{K=1}^{2^{M_{3m+1}}}$ are all $(3,0)$-monotone; moreover, for each k with $1 \leq k \leq 2^{M_{3m+1}}$, let $(\pi(k,l))_{l=1}^{2^{4Q_{3m+1}}-1}$ be a permutation of $(l)_{l=1}^{2^{4Q_{3m+1}}-1}$ such that $(|\sum_{l=1}^{L} x^*_{3m+1,n,\pi(k),\pi(k,l)}(\overline{x})|)_{L=1}^{2^{4Q_{3m+1}}-1}$ is $(1,0)$-monotone. We recall by Step 3 of SC III.2 that

$$\begin{aligned} x^*_{3m+1,n,\pi(k),\pi(k,l)}(\overline{x}) = x'^*_{3m+1,n,\pi(k),\pi(k,l)}(\overline{x}) &+ \frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{2Q_{3m+1}}}E_k^*(\overline{x}) \\ &+ \frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{M_{3m+1}+4Q_{3m+1}}}v^*_{3m+1,n} \end{aligned}$$

for $1 \leq l \leq 2^{4Q_{3m+1}}$ with $\sum_{l=1}^{2^{4Q_{3m+1}}} x'^*_{3m+1,n,\pi(k),\pi(k,l)}(\overline{x}) = 0$. So there exists a permutation

$$(x_{\pi(0,3m+1,n,g)})_{g=1}^{2^{M_{3m+1}+4Q_{3m+1}}} \text{ of } (x_{3m+1,n,\pi(k),0} \cup (x_{3m+1,n,\pi(k),\pi(k,l)})_{l=1}^{2^{4Q_{3m+1}}-1})_{k=1}^{2^{M_{3m+1}}},$$

which includes a permutation

$$(x_{\pi'(3m+1,n,g)})_{g=1}^{2^{M_{3m+1}}(2^{4Q_{3m+1}}-1)} \text{ of } ((x_{3m+1,n,\pi(k),\pi(k,l)})_{l=1}^{2^{4Q_{3m+1}}-1})_{k=1}^{2^{M_{3m+1}}},$$

such that (according to $(46.1)'$ and $(46.2)'$ where by the above we can replace b_k by $(2^{P_{3m+1}Q_{0,3m+1}}/2^{2Q_{3m+1}})E_k^*(\overline{x})$)

(46.1) $(|\sum_{g=1}^{G} x^*_{\pi'(3m+1,n,g)}(\overline{x})|)_{G=1}^{2^{M_{3m+1}}(2^{4Q_{3m+1}}-1)}$ is $(3,0)$-monotone;

(46.2) for each G with $1 \leq G \leq 2^{M_{3m+1}+4Q_{3m+1}} = G_{3m+1,0}$ there are $(S_i)_{i=1}^3$, T, $(L(S_i))_{i=1}^3$ and $L(T)$ with $1 \leq S_1 \leq S_2 \leq S_3 \leq T \leq 2^{M_{3m+1}}$, $1 \leq L(S_i), L(T) \leq 2^{4Q_{3m+1}}-1$ for $1 \leq i \leq 3$, with

$$\begin{aligned} (x^*_{\pi'(3m+1,n,g)})_{g=1}^{G} = {}& (x_{3m+1,n,\pi(k),0} \cup (x_{3m+1,n,\pi(k),\pi(k,l)})_{l=1}^{2^{4Q_{3m+1}}-1})_{k(\notin(S_i)_{i=1}^3)=1}^{T-1} \\ & \cup (x_{3m+1,n,\pi(S_i),0} \cup (x_{3m+1,n,\pi(S_i),\pi(S_i,l)})_{l=1}^{L(S_i)})_{i=1}^3 \\ & \cup (x_{3m+1,n,\pi(T),0} \cup (x_{3m+1,n,\pi(T),\pi(T,l)})_{l=1}^{L(T)}) \end{aligned}$$

(it is possible to check this fact directly by combining the proof of $(46.1)'$ and $(46.2)'$ and the proof of NPL, better in the 2-dimensional case since the procedure is the same but simpler, in this case we find $(S_i)_{i=1}^2$ instead of $(S_i)_{i=1}^3$). Therefore setting

$$\begin{aligned} &(x'_{\pi(0,3m+1,n,g)} + x_{\pi(0,3m+1,\mathrm{brd},n,g)})_{g=1}^{G_{0,3m+1,0}} \\ &\quad = \omega'_{3m+1,n,0} \cup x_{3m+1,n,0} \\ &\qquad \cup (x_{3m+1,n,\pi(k),0} \cup (x'_{3m+1,n,\pi(k),\pi(k,l)} + x_{3m+1,\mathrm{brd},n,\pi(k),\pi(k,l)})_{l=1}^{2^{4Q_{3m+1}}})_{k=1}^{2^{M_{3m+1}}}, \end{aligned}$$

$$((x'_{\pi(3m+1,n,g)} + x_{\pi(3m+1,\mathrm{brd},n,g)})_{g=1}^{G_{3m+1,0}} = ((x'_{3m+1,n,\pi(k),\pi(k,l)} + x_{3m+1,\mathrm{brd},n,\pi(k),\pi(k,l)})_{l=1}^{2^{4Q_{3m+1}}})_{k=1}^{2^{M_{3m+1}}},$$

$$(x'_{\pi(0,3m+1,g)} + x_{\pi(0,3m+1,\mathrm{brd},g)})_{g=1}^{G_{0,3m+1}} = ((x'_{\pi(0,3m+1,n,g)} + x_{\pi(0,3m+1,\mathrm{brd},n,g)})_{g=1}^{G_{0,3m+1,0}})_{n=1}^{P_{3m+1}},$$

$$(x'_{\pi(3m+1,g)} + x_{\pi(3m+1,\mathrm{brd},g)})_{g=1}^{G_{3m+1}} = ((x'_{\pi(3m+1,n,g)} + x_{\pi(3m+1,\mathrm{brd},n,g)})_{g=1}^{G_{3m+1,0}})_{n=1}^{P_{3m+1}},$$

since $\overline{D} \le 6$, we can state the following

FACT 1.

$$\Big(\Big\|\sum_{g=1}^{G} x^*_{\pi(0,3m+1,g)}(\overline{x})(x'_{\pi(0,3m+1,g)} + x_{\pi(0,3m+1,\mathrm{brd},g)}) + E_{3m+1,0} + \operatorname{span}(u'_{3m+2,s})_{s=A_{3m+2}-A'_{3m+2}+1}^{n'(3m+2)}\Big\|\Big)_{G=1}^{G_{0,3m+1,0}}$$

and $(|\sum_{g=1}^{G} x^*_{\pi(3m+1,n,g)}(\overline{x})|)_{G=1}^{G_{3m+1,0}}$ *for* $1 \le n \le P_{3m+1}$ *are* $(6,0)$*-monotone.* ∎

THIRD PART. We turn to considering

$$\Big\|\sum_{n=q(3m+1)+1}^{q(3m+2)} x^*_{\pi'(n)}(\overline{x})(x''_n + x'''_n) + E_{3m+1,0}\Big\| = \Big\|\sum_{n=1}^{P_{3m+1}}\Big(\sum_{k=1}^{2^{M_{3m+1}}}\sum_{l=1}^{2^{4Q_{3m+1}}} x^*_{3m+1,n,k,l}(\overline{x})\Big)\frac{v_{3m+1,n}}{2^{P_{3m+1}Q_{0,m}}} + E_{3m+1,0}\Big\|.$$

If we fix n with $1 \le n \le P_{3m+1}$, by the end of Step 3 of SC III.2 we know that, for $1 \le l \le 2^{4Q_{3m+1}}$ and $1 \le k \le 2^{M_{3m+1}}$,

$$x^*_{3m+1,n,k,l} = x^*_{0,3m+1,n,k,l} + \frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{M_{3m+1}+4Q_{3m+1}}} v^*_{3m+1,n},$$

$$\sum_{k=1}^{2^{M_{3m+1}}}\sum_{l=1}^{2^{4Q_{3m+1}}} x^*_{0,3m+1,n,k,l} = 0,$$

$$\sum_{k=1}^{2^{M_{3m+1}}}\sum_{l=1}^{2^{4Q_{3m+1}}} x^*_{3m+1,n,k,l} = 2^{P_{3m+1}Q_{0,3m+1}} v^*_{3m+1,n}.$$

We set $T_{3m+1} = 2^{M_{3m+1}+2Q_{3m+1}}$. Let $(\pi'(3m+1,g))_{g=1}^{G_{3m+1}}$ be any permutation of $(n)_{n=q(3m+1)+1}^{q(3m+2)}$ which satisfies Fact 1, so $(|\sum_{g=1}^{G} x^*_{\pi'(3m+1,g)}(\overline{x})|)_{G=1}^{G_{3m+1}}$ is $(6,0)$-monotone. For $1 \le n \le P_{3m+1}$, if $(\pi'(3m+1,n,g))_{g=1}^{G_{3m+1,0}}$ is the permutation induced on $(((3m+1,n,g))_{g=1}^{G_{3m+1,0}}$, we use $(t(3m+1,n,p))_{p=1}^{T_{3m+1}}$ for a partition of $(\pi'(3m+1,n,g))_{g=1}^{G_{3m+1,0}}$ (whence $t(3m+1,n,0) = 0$ and $t(3m+1,n,T_{3m+1}) = G_{3m+1,0}$) so that, for $1 \le p \le T_{3m+1}$,

$$\Big|\sum_{g=t(3m+1,n,p-1)+1}^{t(3m+1,n,p)} x^*_{\pi'(3m+1,n,g)}(\overline{x}) - \frac{2^{P_{3m+1}Q_{0,3m+1}} v^*_{3m+1,n}(\overline{x})}{T_{3m+1}}\Big| < \overline{a}_{3m+1},$$

$$\Big|\sum_{g=1}^{t(3m+1,n,p)} x^*_{\pi'(3m+1,n,g)}(\overline{x}) - p\frac{2^{P_{3m+1}Q_{0,3m+1}}v^*_{3m+1,n}(\overline{x})}{T_{3m+1}}\Big| < \overline{a}_{3m+1}.$$

Since by the above, for $1 \le g, G \le G_{3m+1,0}$,

$$x^*_{\pi'(3m+1,n,g)}(\overline{x}) = x^*_{0,\pi'(3m+1,n,g)}(\overline{x}) + \frac{2^{P_{3m+1}Q_{0,3m+1}}}{2^{M_{3m+1}+4Q_{3m+1}}}v^*_{3m+1,n}(\overline{x}),$$

$(|\sum_{g=1}^{G} x^*_{0,\pi'(3m+1,n,g)}(\overline{x})|)_{G=1}^{G_{3m+1}}$ is $(6,0)$-monotone and hence $|\sum_{g=1}^{G} x^*_{0,\pi'(3m+1,n,g)}(\overline{x})| \le 6\overline{a}_{3m+1}$, we can directly write, for $1 \le p \le T_{3m+1}$, $t(3m+1,n,p) = t(3m+1,p) = p2^{2Q_{3m+1}}$ and hence

$$\Big|\sum_{g=t(3m+1,p-1)+1}^{t(3m+1,p)} x^*_{\pi'(3m+1,n,g)}(\overline{x}) - \frac{2^{P_{3m+1}Q_{0,3m+1}}v^*_{3m+1,n}(\overline{x})}{T_{3m+1}}\Big| < 7\overline{a}_{3m+1},$$

$$\begin{aligned}\Big|\sum_{g=t(3m+1,p-1)+1}^{t(3m+1,p)} x^*_{\pi'(3m+1,n,g)}(\overline{x})\Big| &\le \frac{2^{P_{3m+1}Q_{0,3m+1}}|v^*_{3m+1,n}(\overline{x})|}{T_{3m+1}} + 7\overline{a}_{3m+1}\\ &< \frac{2^{P_{3m+1}Q_{0,3m+1}}Q_{0,3m+1}2^{M_{3m+1}+Q_{3m+1}}}{2^{M_{3m+1}+2Q_{3m+1}}} + 7\overline{a}_{3m+1}\\ &= \frac{2^{P_{3m+1}Q_{0,3m+1}}Q_{0,3m+1}}{2^{Q_{3m+1}}} + 7\overline{a}_{3m+1},\end{aligned}$$

$$\Big|\sum_{g=t(3m+1,p-1)+1}^{t(3m+1,p)} \frac{x^*_{\pi'(3m+1,n,g)}(\overline{x})}{2^{P_{3m+1}Q_{0,3m+1}}}\Big| < \frac{Q_{0,3m+1}}{2^{Q_{3m+1}}} + \frac{7\overline{a}_{3m+1}}{2^{P_{3m+1}Q_{0,3m+1}}},$$

$$\Big|\sum_{g=t(3m+1,p-1)+1}^{t(3m+1,p)} \frac{x^*_{\pi'(3m+1,n,g)}(\overline{x})}{2^{P_{3m+1}Q_{0,3m+1}}} - \frac{v^*_{3m+1,n}(\overline{x})}{T_{3m+1}}\Big| < \frac{7\overline{a}_{3m+1}}{2^{P_{3m+1}Q_{0,3m+1}}},$$

$$\Big|\sum_{g=1}^{t(3m+1,p)} \frac{x^*_{\pi'(3m+1,n,g)}(\overline{x})}{2^{P_{3m+1}Q_{0,3m+1}}} - \frac{p}{T_{3m+1}}v^*_{3m+1,n}(\overline{x})\Big| < \frac{7\overline{a}_{3m+1}}{2^{P_{3m+1}Q_{0,3m+1}}}.$$

For $1 \le n \le P_{3m+1}$, $1 \le p \le T_{3m+1}$, we set $\widetilde{W}_{3m+1,p} = \sum_{g=1}^{p} W_{3m+1,g}$, where

$$W_{3m+1,p} = \sum_{n=1}^{P_{3m+1}} W_{3m+1,n,p},$$

$$W_{3m+1,n,p} = \sum_{g=t(3m+1,p-1)+1}^{t(3m+1,p)} x^*_{\pi'(3m+1,n,g)}(\overline{x})\frac{v_{3m+1,n}}{2^{P_{3m+1}Q_{0,3m+1}}}.$$

Then, by Step 2 of SC III.2,

$$\begin{aligned}W_{3m+1} = \widetilde{W}_{3m+1,T_{3m+1}} &= \sum_{n=1}^{P_{3m+1}} v^*_{3m+1,n}(\overline{x})v_{3m+1,n}\\ &= \sum_{n=1}^{P_{3m+1}} u^*_{3m+1,n}(\overline{x})u_{3m+1,n} = W''_{3m+1} + W'''_{3m+1},\end{aligned}$$

$$
\begin{aligned}
W''_{3m+1} = &\sum_{s=A''_{3m+1}+1}^{A_{3m+1}/2} e^*_{3m+1,\mathrm{aux},s}(\overline{x})e_{3m+1,\mathrm{aux},s}\\
&+\sum_{s=A_{3m+1}/2+M_{3m+1-1,\mathrm{arm},0}+1}^{A_{3m+1}-A'_{3m+1}} e^*_{3m+1,\mathrm{aux},s}(\overline{x})e_{3m+1,\mathrm{aux},s}\\
&+\sum_{s=1}^{A_{3m+1}}\sum_{t=1}^{2^{2B_{3m+1}}} e^*_{3m+1,\mathrm{aux},s,t}(\overline{x})e_{3m+1,\mathrm{aux},s,t},\\
W'''_{3m+1} = &\sum_{s=1}^{A''_{3m+1}} u'^*_{3m+1,s}(\overline{x})u'_{3m+1,s}\\
&+\sum_{s=A_{3m+1}/2+1}^{A_{3m+1}/2+M_{3m+1-1,\mathrm{arm},0}} u'^*_{3m+1,s}(\overline{x})u'_{3m+1,s}\\
&+\sum_{s=A_{3m+1}-A'_{3m+1}+1}^{A_{3m+1}} u'^*_{3m+1,s}(\overline{x})u'_{3m+1,s}.
\end{aligned}
$$

For $1 \le p \le T_{3m+1}$, we have

$$
\begin{aligned}
W_{3m+1,p}-\frac{W_{3m+1}}{T_{3m+1}} &= \sum_{n=1}^{P_{3m+1}} W_{3m+1,n,p} - \frac{1}{T_{3m+1}}\sum_{n=1}^{P_{3m+1}} v^*_{3m+1,n}(\overline{x})v_{3m+1,n}\\
&= \sum_{n=1}^{P_{3m+1}}\left(W_{3m+1,n,p}-\frac{v^*_{3m+1,n}(\overline{x})v_{3m+1,n}}{T_{3m+1}}\right)\\
&= \sum_{n=1}^{P_{3m+1}}\left(\sum_{g=t(3m+1,p-1)+1}^{t(3m+1,p)} x^*_{\pi'(3m+1,n,g)}(\overline{x})\frac{1}{2^{P_{3m+1}Q_{0,3m+1}}}-\frac{v^*_{3m+1,n}(\overline{x})}{T_{3m+1}}\right)v_{3m+1,n},
\end{aligned}
$$

$$
\begin{aligned}
&\left\|W_{3m+1,p}-\frac{W_{3m+1}}{T_{3m+1}}\right\|\\
&\quad < 2\sum_{n=1}^{P_{3m+1}}\left\|\sum_{g=t(3m+1,p-1)+1}^{t(3m+1,p)} x^*_{\pi'(3m+1,n,g)}(\overline{x})\frac{1}{2^{P_{3m+1}Q_{0,3m+1}}}-\frac{v^*_{3m+1,n}(\overline{x})}{T_{3m+1}}\right\|\\
&\qquad < 2P_{3m+1}\frac{7\overline{a}_{3m+1}}{2^{P_{3m+1}Q_{0,3m+1}}} < \frac{\overline{a}_{3m+1}}{4P_{3m+1}2^{Q_{0,3m+1}}},\\
&\qquad \left\|\widetilde{W}_{3m+1,p}-\frac{p}{T_{3m+1}}W_{3m+1}\right\| < \frac{\overline{a}_{3m+1}}{4P_{3m+1}2^{Q_{0,3m+1}}}
\end{aligned}
$$

by the same proof. Moreover by the above also

$$
\begin{aligned}
\|W_{3m+1,n,p}\| &= \Big|\sum_{g=t(3m+1,p-1)+1}^{t(3m+1,p)} x^*_{\pi'(3m+1,n,g)}(\overline{x})\Big|\frac{\|v_{3m+1,n}\|}{2^{P_{3m+1}Q_{0,3m+1}}}\\
&< 2\left(\frac{Q_{0,3m+1}}{2^{Q_{3m+1}}}+\frac{7\overline{a}_{3m+1}}{2^{P_{3m+1}Q_{0,3m+1}}}\right) < \frac{15\overline{a}_{3m+1}}{2^{P_{3m+1}Q_{0,3m+1}}} < \frac{\overline{a}_{3m+1}}{4P_{3m+1}2^{Q_{0,3m+1}}}.
\end{aligned}
$$

At this point, setting

$$(\pi(3m+1,g))_{g=1}^{G_{3m+1,0}} = (((\pi'(3m+1,n,g))_{g=t(3m+1,p-1)+1}^{t(3m+1,p)})_{n=1}^{P_{3m+1}})_{p=1}^{T_{3m+1}},$$

it follows that $(\|\sum_{g=1}^{G} x^*_{\pi(3m+1,g)}(\overline{x})(x''_{\pi(3m+1,g)} + x'''_{\pi(3m+1,g)}) + E_{3m+1,0}\|)_{G=1}^{G_{3m+1}}$ and $(\|\sum_{g=1}^{G} x^*_{\pi(3m+1,g)}(\overline{x})x'''_{\pi(3m+1,g)}\|)_{G=1}^{G_{3m+1}}$ are $(0,\overline{a}_{3m+1}/2^{Q_{0,3m+1}})$-monotone (since, for $1\le n\le P_{3m+1}$ and $1\le p\le T_{3m+1}$,

$$\Big(\Big\|\sum_{g=t(3m+1,p-1)+1}^{G} \frac{x^*_{\pi'(3m+1,n,g)}(\overline{x})}{2^{P_{3m+1}Q_{0,3m+1}}}\Big\|\Big)_{G=t(3m+1,p-1)+1}^{t(3m+1,p)} \quad \text{is } (6,0)\text{-monotone},$$

hence also $(0,6\overline{a}_{3m+1}/2^{P_{3m+1}Q_{0,3m+1}})$-monotone. Moreover

$$\Big\|\sum_{k=1}^{n} W_{3m+1,k,p}\Big\| \le \sum_{k=1}^{n}\|W_{3m+1,k,p}\| < n\frac{\overline{a}_{3m+1}}{4P_{3m+1}2^{Q_{0,3m+1}}} < \frac{\overline{a}_{3m+1}}{4\cdot 2^{Q_{0,3m+1}}},$$

therefore $(\|\sum_{k=1}^{N} W_{3m+1,k,p}\|)_{N=1}^{n}$ is $(0,\frac{\overline{a}_{3m+1}}{4.2^{Q_{0,3m+1}}})$-monotone, while

$$\Big(\Big\|\sum_{p=1}^{P} W_{3m+1,p}\Big\|\Big)_{P=1}^{T_{3m+1}} = (\|\widetilde{W}_{3m+1,p}\|)_{P=1}^{T_{3m+1}}$$

is $(0,\overline{a}_{3m+1}/4\cdot P_{3m+1}2^{Q_{0,3m+1}})$-monotone since by the above

$$\Big\|\widetilde{W}_{3m+1,p} - \frac{p}{T_{3m+1}}W_{3m+1}\Big\| < \frac{\overline{a}_{3m+1}}{4\cdot P_{3m+1}2^{Q_{0,3m+1}}}$$

for $1\le p\le T_{3m+1}$ and $(\|\frac{p}{T_{3m+1}}W_{3m+1}\|)_{P=1}^{D_{3m+1}}$ is $(0,0)$-monotone; finally $(2,0,0)$-monotonicity comes from the inequality

$$\|y+e\| \ge \max(\|y\|,\|e\|/2)$$

for each $y\in\operatorname{span}(y_{3m+1,n})_{n=1}^{Q(3m+1)}$ and $e\in\operatorname{span}(\widehat{e}_{3m+1,n})_{n=1}^{S'_{3m+1}+M'_{3m+1,0}}$ of Step 1 of SC III.2).

At this point we can state the following

Fact 2. *Let $(x_{\pi(0,3m+1,g)})_{g=1}^{G_{0,3m+1}}$ be the permutation of*

$$(x_{3m+1,g})_{g=1}^{G_{3m+1}} \cup (\omega'_{3m+1,n,0} \cup x_{3m+1,n,0} \cup (x_{3m+1,n,k,0})_{k=1}^{2^{M_{3m+1}}})_{n=1}^{P_{3m+1}}$$

of Fact 1 such that the permutation induced on $(x_{3m+1,g})_{g=1}^{G_{3m+1}}$ is just $(x_{\pi(3m+1,g)})_{g=1}^{G_{3m+1}}$. Then

$$\Big(\Big\|\sum_{g=1}^{G} x^*_{\pi(0,3m+1,g)}(\overline{x})(x'_{\pi(0,3m+1,g)} + x_{\pi(0,3m+1,\mathrm{brd},g)}) + \operatorname{span}(u'_{3m+2,s})_{s=A_{3m+2}-A'_{3m+2}+1}^{n'(3m+2)} + E_{3m+1,0}\Big\|\Big)_{G=1}^{G_{3m+1}}$$

is $(6,0)$-monotone and

$$\Big(\Big\|\sum_{g=1}^{G} x^*_{\pi(0,3m+1,g)}(\overline{x})x_{\pi(0,3m+1,g)} + \operatorname{span}(u'_{3m+2,s})_{s=A_{3m+2}-A'_{3m+2}+1}^{n'(3m+2)} + E_{3m+1,0}\Big\|\Big)_{Q=1}^{G_{3m+1}}$$

is $(2,6,\overline{a}_{3m+1}/2^{Q_{0,3m+1}})$-monotone.

This follows since all the properties of Fact 1 continue to hold. Now (A) follows from Fact 1 and from what we stated just above. ∎

FOURTH PART. We now turn to considering the block $(x_n^*(\overline{x})x_n)_{n=q(3m)+1}^{q(3m+1)}$. We already know, by Step 1 of SC III.1, that $(e'''_{3m,n}+E_{3m,0})_{n=1}^{S_{3m}}$ is 1-equivalent to the natural basis of $l_\infty^{S_{3m}}$ (2-equivalent if $E_{3m,0}$ is replaced by $E_{3m,0}+\mathrm{span}(y_{3m,n})_{n=1}^{Q(3m)}+U_{3m}$ for $U_{3m}=X\cap\bigcap_{n=1}^{Q(3m)}y^*_{(3m,n)\perp}\cap\bigcap_{n=1}^{S'_{3m}+M'_{3m,0}}\widehat{e}^*_{(3m,n)\perp}$). Analogously, if X' and $U_{3m,\mathrm{arm}}$ are the subspaces of Step 3 of SC III.1, then $((e_{3m,\mathrm{arm},n,j}+E_{3m,\mathrm{arm},0})_{i=1}^{J_{3m,\mathrm{arm}}})_{n=1}^{P_{3m}}$ is 1-equivalent to the natural basis of $l_\infty^{P_{3m}J_{3m,\mathrm{arm}}}$ (2-equivalent if $E_{3m,\mathrm{arm},0}$ is replaced by $E_{3m,\mathrm{arm},0}+X'+U_{3m,\mathrm{arm}}$). Moreover, if we set $E=E_{3m,0}+E_{3m,\mathrm{arm},0}$, then if $u\in\mathrm{span}(e'''_{3m,n})_{n=1}^{S'_{3m}}+E$ and $v\in\mathrm{span}((e_{3m,\mathrm{arm},n,j})_{i=1}^{J_{3m,\mathrm{arm}}})_{n=1}^{P_{3m}}+E$, we have

$$\|u+v\|\ge\max(\|u\|,\|v\|/2)).$$

Owing to the first part of (ii) of CL and in order to decrease the formalism, in what follows we can first suppose (see the notations before CL) that

$$(x_{\mathrm{brd},n})_{n=q(3m)+1}^{q(3m+1)}=((e_{3m,\mathrm{brd},n,k}+\omega'_{3m,n})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}}=(x_{3m,\mathrm{brd},a,g})_{g=1}^{N_{3m,\mathrm{brd},a}},$$

that is, $n'(3m+1)=A_{3m+1}$; then we will turn to the general case. Now, for $1\le n\le P_{3m}$, by the procedure of the second part it is possible to check (see Steps 4, 5 and 6 of SC III.1, in particular the definition of $(e'_{3m,\mathrm{arm},g})_{g=1}^{G_{3m}}=(x_{0,0,0,3m,g})_{g=G_{0,0,0,3m}-G_{3m}+1}^{G_{0,0,0,3m}}=(u'_{3m+1,s})_{s=G_{0,0,0,3m}-G_{3m}+1}^{G_{0,0,0,3m}}$) that

$$\begin{aligned}
&\Big\|\sum_{g=1}^{G_{0,3m,0}}x^*_{0,3m,n,g}(\overline{x})(x_{0,3m,n,g}-\widetilde{x}_{0,3m,n,g})+E_{3m,0}\\
&\qquad\qquad+\mathrm{span}(u'_{3m+1,s})_{s=G_{0,0,0,3m}-G_{3m}+1}^{G_{0,0,0,3m}}+\mathrm{span}(u'_{3m+1,s})_{s=A_{3m+1}-A'_{3m+1}+1}^{A_{3m+1}}\Big\|\\
&\qquad=\Big\|\omega^*_{3m,n,0}(\overline{x})\omega_{3m,n,0}+\sum_{k=1}^{2^{M_{3m}}}\omega^*_{3m,n,k,0}(\overline{x})\omega_{3m,n,k,0}\\
&\qquad\quad+x^*_{3m,n,0}(\overline{x})x_{3m,n,0}+\sum_{k=1}^{2^{M_{3m}}}\sum_{l=0}^{2^{M_{3m}}}x^*_{3m,n,k,0,l}(\overline{x})x_{3m,n,k,0,l}\\
&\qquad\quad+\sum_{g=1}^{G_{3m,0}}x^*_{3m,n,g}(\overline{x})x'_{3m,n,g}+E_{3m,0}\Big\|\\
&\qquad=\Big\|\omega^*_{3m,n,0}(\overline{x})\sum_{f=1}^{2^{M_{3m}}}e_{3m,n,f,2^{M_{3m}}}+\sum_{k=1}^{2^{M_{3m}}}\omega^*_{3m,n,k,0}(\overline{x})\sum_{g=1}^{2^{M_{3m}}-1}e_{3m,n,k,g}\\
&\qquad\quad+x^*_{3m,n,0}(\overline{x})\Big(\sum_{f=1}^{2^{M_{3m}}}e_{3m,n,f}\Big)\\
&\qquad\quad+\sum_{k=1}^{2^{M_{3m}}}\Big(x^*_{3m,n,k,0,0}(\overline{x})\sum_{g=1}^{2^{M_{3m}}}e_{3m,n,k,0,g}+\sum_{l=1}^{2^{M_{3m}}}x^*_{3m,n,k,0,l}(\overline{x})
\end{aligned}$$

$$\cdot\Big(e_{3m,n,k,0,l}+\frac{1}{2^{M_{3m}}}\sum_{g=1}^{2^{M_{3m}}}e_{3m,n,k,g}+(1-1/2^{M_{3m}})\omega_{3m,n,k}+\omega_{3m,n}/2^{M_{3m}}\Big)\Big)$$

$$+\sum_{k=1}^{2^{M_{3m}}}\Big(\sum_{l=1}^{2^{M_{3m}}-1}x^*_{3m,n,k,l}(\overline{x})(e_{3m,n,k,l}+\omega_{3m,n,k}+e_{3m,n,k}/2^{M_{3m}})$$

$$+x^*_{3m,n,k,2^{M_{3m}}}(\overline{x})(e_{3m,n,k,2^{M_{3m}}}+\omega_{3m,n}+e_{3m,n,k}/2^{M_{3m}})\Big)+E_{3m,0}\Big\|$$

$$=\Big\|\sum_{k=1}^{2^{M_{3m}}}\Big(x^*_{3m,n,0}(\overline{x})+\frac{1}{2^{M_{3m}}}\sum_{l=1}^{2^{M_{3m}}}x^*_{3m,n,k,l}(\overline{x})\Big)e_{3m,n,k}$$

$$+\sum_{k=1}^{2^{M_{3m}}}\sum_{l=1}^{2^{M_{3m}}-1}\Big(\omega^*_{3m,n,k,0}(\overline{x})+\frac{1}{2^{M_{3m}}}\sum_{g=1}^{2^{M_{3m}}}x^*_{3m,n,k,0,g}(\overline{x})+x^*_{3m,n,k,l}(\overline{x})\Big)e_{3m,n,k,l}$$

$$+\sum_{k=1}^{2^{M_{3m}}}\Big(\omega^*_{3m,n,0}(\overline{x})+\frac{1}{2^{M_{3m}}}\sum_{g=1}^{2^{M_{3m}}}x^*_{3m,n,k,0,g}(\overline{x})+x^*_{3m,n,k,2^{M_{3m}}}(\overline{x})\Big)e_{3m,n,k,2^{M_{3m}}}$$

$$+\sum_{k=1}^{2^{M_{3m}}}\sum_{l=1}^{2^{M_{3m}}}(x^*_{3m,n,k,0,0}(\overline{x})+x^*_{3m,n,k,0,l}(\overline{x}))e_{3m,n,k,0,l}$$

$$+\sum_{k=1}^{2^{M_{3m}}}\Big(\sum_{l=1}^{2^{M_{3m}}}(1-1/2^{M_{3m}})x^*_{3m,n,k,0,l}(\overline{x})+\sum_{l=1}^{2^{M_{3m}}-1}x^*_{3m,n,k,l}(\overline{x})\Big)\omega_{3m,n,k}$$

$$+\sum_{k=1}^{2^{M_{3m}}}\Big(\sum_{l=1}^{2^{M_{3m}}}x^*_{3m,n,k,0,l}(\overline{x})/2^{M_{3m}}+x^*_{3m,n,k,2^{M_{3m}}}(\overline{x})\Big)\Big)\omega_{3m,n}+E_{3m,0}\Big\|$$

$$=\Big\|\sum_{k=1}^{2^{M_{3m}}}e^*_{3m,n,k}(\overline{x})e_{3m,n,k}+\sum_{k=1}^{2^{M_{3m}}}\sum_{l=1}^{2^{M_{3m}}}e^*_{3m,n,k,l}(\overline{x})e_{3m,n,k,l}$$

$$+\sum_{k=1}^{2^{M_{3m}}}\sum_{l=1}^{2^{M_{3m}}}e^*_{3m,n,k,0,l}(\overline{x})e_{3m,n,k,0,l}$$

$$+\sum_{k=1}^{2^{M_{3m}}}\omega^*_{3m,n,k}(\overline{x})\omega_{3m,n,k}+\omega^*_{3m,n}(\overline{x})\omega_{3m,n}+E_{3m,0}\Big\|.$$

Therefore we have

$$\Big\|\sum_{g=1}^{G_{0,3m,0}}x^*_{0,3m,n,g}(\overline{x})(x_{0,3m,n,g}-\widetilde{x}_{0,3m,n,g})+E_{3m,0}$$

$$+\operatorname{span}(u'_{3m+1,s})_{s=G_{0,0,0,3m}-G_{3m}+1}^{G_{0,0,0,3m}}+\operatorname{span}(u'_{3m+1,s})_{s=A_{3m+1}-A'_{3m+1}+1}^{A_{3m+1}}\Big\|$$

$$=\max(D_{i,3m,n},1\le i\le 7),$$

where

$$D_{1,3m,n} = \max\Big(\Big|x^*_{3m,n,0}(\overline{x}) + \frac{1}{2^{M_{3m}}}\sum_{l=1}^{2^{M_{3m}}} x^*_{3m,n,k,l}(\overline{x}) = e^*_{3m,n,k}(\overline{x})\Big| : 1 \le k \le 2^{M_{3m}}\Big);$$

$$D_{2,3m,n} = \max\Big(\Big|\omega^*_{3m,n,k,0}(\overline{x}) + \frac{1}{2^{M_{3m}}}\sum_{g=1}^{2^{M_{3m}}} x^*_{3m,n,k,0,l}(\overline{x}) + x^*_{3m,n,k,l}(\overline{x}) = e^*_{3m,n,k,l}(\overline{x})\Big| : 1 \le k \le 2^{M_{3m}}, 1 \le l \le 2^{M_{3m}} - 1\Big);$$

$$D_{3,3m,n} = \max\Big(\Big|\omega^*_{3m,n,0}(\overline{x}) + \frac{1}{2^{M_{3m}}}\sum_{g=1}^{2^{M_{3m}}} x^*_{3m,n,k,0,l}(\overline{x}) + x^*_{3m,n,k,2^{M_{3m}}}(\overline{x}) = e^*_{3m,n,k,2^{M_{3m}}}(\overline{x})\Big| : 1 \le k \le 2^{M_{3m}}\Big);$$

$$D_{4,3m,n} = \max(|x^*_{3m,n,k,0,0}(\overline{x}) + x^*_{3m,n,k,0,l}(\overline{x}) = e^*_{3m,n,k,0,l}(\overline{x})| : 1 \le l \le 2^{M_{3m}}, 1 \le k \le 2^{M_{3m}});$$

$$D_{5,3m,n} = \max\Big(\Big|\sum_{l=1}^{2^{M_{3m}}}(1 - 1/2^{M_{3m}})x^*_{3m,n,k,0,l}(\overline{x}) + \sum_{l=1}^{2^{M_{3m}}-1} x^*_{3m,n,k,l}(\overline{x}) = \omega^*_{3m,n,k}(\overline{x})\Big| : 1 \le k \le 2^{M_{3m}}\Big);$$

$$D_{6,3m,n} = \Big|\sum_{k=1}^{2^{M_{3m}}}\Big(\sum_{l=1}^{2^{M_{3m}}} x^*_{3m,n,k,0,l}(\overline{x})/2^{M_{3m}} + x^*_{3m,n,k,2^{M_{3m}}}(\overline{x})\Big) = \omega^*_{3m,n}(\overline{x})\Big|.$$

At this point we turn to the general case and we need also consider the terms analogous to $D_{5,3m+1}$ and $D_{6,3m+1}$ of the first part; a consequence will be that also

$$\Big(\Big|\sum_{k=1}^{G} x^*_{3m,n,k,2^{M_{3m}}}(\overline{x})\Big|\Big)_{G=1}^{2^{M_{3m}}}$$

will have to be monotone for some permutation. In order to construct the permutation $(\pi(0,3m,n,g))_{g=1}^{G_{0,3m,0}}$ of the assertion we have to pay attention only to $D_{i,3m,n}$ for $i \in (5,6)$. For $D_{5,3m,n}$, for each l with $1 \le l \le 2^{M_{3m}}$, we have

$$\sum_{l=1}^{2^{M_{3m}}}(1 - 1/2^{M_{3m}})x^*_{3m,n,k,0,l} = (2^{M_{3m}} - 1)x^*_{3m,n,k,0},$$

$$\sum_{l=1}^{2^{M_{3m}}-1} x^*_{3m,n,k,l} = \omega^*_{3m,n,k} - (2^{M_{3m}} - 1)x^*_{3m,n,k,0};$$

analogously for $D_{6,3m,n}$ we have

$$\sum_{l=1}^{2^{M_{3m}}} x^*_{3m,n,k,0,l}/2^{M_{3m}} = x^*_{3m,n,k,0}, \qquad x^*_{3m,n,k,2^{M_{3m}}} = e''^*_{3m,n,k,2^{M_{3m}}} - x^*_{3m,n,k,0}$$

where

$$\sum_{k=1}^{2^{M_{3m}}} e''^*_{3m,n,k,2^{M_{3m}}} = \omega^*_{3m,n}.$$

In the next part we will need that $(|\sum_{g=1}^{G} x^*_{3m,n,g}(\overline{x})|)_{G=1}^{G_{3m,0}}$ be $(2,0)$-monotone; therefore, if again by NPL, $(\pi(k))_{k=1}^{2^{M_{3m}}}$ is a permutation of $(k)_{k=1}^{2^{M_{3m}}}$ such that

$$\Big(\Big|\sum_{k=1}^{K}\Big(\sum_{l=1}^{2^{M_{3m}}-1} x^*_{3m,n,\pi(k),l}(\overline{x})\Big)\Big|\Big)_{K=1}^{2^{M_{3m}}},\quad \Big(\Big|\sum_{k=1}^{K} e''^*_{3m,n,\pi(k),2^{M_{3m}}}(\overline{x})\Big|\Big)_{K=1}^{2^{M_{3m}}}$$

and $(|\sum_{k=1}^{K} x^*_{3m,n,\pi(k),2^{M_{3m}}}(\overline{x})|)_{K=1}^{2^{M_{3m}}}$ are $(3,0)$-monotone and if, for $1\le k\le 2^{M_{3m}}$, $(\pi(k,l))_{l=1}^{2^{M_{3m}}}$ is a permutation of $(l)_{l=1}^{2^{M_{3m}}}$ such that $(|\sum_{l=1}^{G} x^*_{3m,n,\pi(k),\pi(k,l)}(\overline{x})|)_{G=1}^{2^{M_{3m}}-1}$ is $(1,0)$-monotone, we can state the following

Fact 3. *Set $\pi(k,2^{M_{3m}})=2^{M_{3m}}$ for $1\le k\le 2^{M_{3m}}$ and*

$$\begin{aligned}(x_{\pi(0,3m,n,g)})_{g=1}^{G_{0,3m,0}} &= (\omega_{3m,n,0},x_{3m,n,0})\\ &\cup((\omega_{3m,n,\pi(k),0},x_{3m,n,\pi(k),0,0})\cup(x_{3m,n,\pi(k),0,l},x_{3m,n,\pi(k),\pi(k,l)})_{l=1}^{2^{M_{3m}}})_{k=1}^{2^{M_{3m}}}.\end{aligned}$$

Then, for each permutation $(x_{\pi(0,3m,g)})_{g=1}^{G_{0,3m}}$ of $(x_{0,3m,g})_{g=1}^{G_{0,3m}}$ such that, for $1\le n\le P_{3m}$, the permutation induced on $(x_{0,3m,n,g})_{g=1}^{G_{0,3m,0}}$ is just $(x_{\pi(0,3m,n,g)})_{g=1}^{G_{0,3m,0}}$ above,

$$\begin{aligned}\Big(\Big\|\sum_{g=1}^{G} x^*_{\pi(0,3m,g)}(\overline{x})(x_{\pi(0,3m,g)}-\widetilde{x}_{\pi(0,3m,g)})+E_{3m,0}\\ +\operatorname{span}(u'_{3m+1,s})_{s=G_{0,0,0,3m}-G_{3m}+1}^{G_{0,0,0,3m}}+\operatorname{span}(u'_{3m+1,s})_{s=A_{3m+1}-A'_{3m+1}+1}^{A_{3m+1}}\Big\|\Big)_{G=1}^{G_{3m,0}}\end{aligned}$$

is $(3,0)$-monotone (where 3 comes also from $D_{2,3m,n}$ and $D_{3,3m,n}$), hence also $(0,3\overline{a}_{3m})$-monotone.

Fifth part. By the above it follows that, for $1\le n\le P_{3m}$, since

$$\Big(\Big|\sum_{g=1}^{G} x^*_{\pi(3m,n,g))}(\overline{x})\Big|\Big)_{G=1}^{G_{3m,0}}$$

is $(2,0)$-monotone, it follows that $(\|\sum_{g=1}^{Q} x^*_{\pi(3m,n,g)}(\overline{x})\widetilde{x}_{\pi(3m,n,g)}\|)_{Q=1}^{G_{3m,0}}$ is also $(2,0)$-monotone, hence $(0,2\overline{a}_{3m}/2^{Q_{3m}})$-monotone. Moreover

$$\begin{aligned}\Big(\Big\|\sum_{g=1}^{G} x^*_{\pi(0,3m,n,g)}(\overline{x})x_{\pi(0,3m,n,g)}+E_{3m,0}+\operatorname{span}(u'_{3m+1,s})_{s=G_{0,0,0,3m}-G_{3m}+1}^{G_{0,0,0,3m}}\\ +\operatorname{span}(u'_{3m+1,s})_{s=A_{3m+1}-A'_{3m+1}+1}^{n'(3m+1)}\Big\|\Big)_{G=1}^{G_{3m,0}}\end{aligned}$$

is $(2,3,2\overline{a}_{3m}/2^{Q_{3m}})$-monotone (by $\|y+e\|\ge\max(\|y\|,\|e\|/2)$ of Step 1 of SC III.1). In order to pass, from the single permutation $(\pi(0,3m,n,g))_{g=1}^{G_{0,3m,0}}$ for each n with $1\le n\le P_{3m}$, to the global permutation $(\pi(0,3m,g))_{g=1}^{G_{0,3m}}$, we will use the procedure of the third part of the proof. Fix n with $1\le n\le P_{3m}$. Recall from Step 5 of SC III.1 that $\sum_{k=1}^{2^{M_{3m}}}\sum_{l=1}^{2^{M_{3m}}} x^*_{3m,n,k,l}=2^{M_{3m}}w^*_{3m,n}$; moreover we set $T_{3m}=2^{M_{3m}}$. Then there exists a partition $((\pi(3m,n,g))_{g=t(3m,p-1)+1}^{t(3m,p)})_{p=1}^{T_{3m}}$ of $(\pi(3m,n,g))_{g=1}^{G_{3m,0}}$ (with $t(3m,0)=0$ and

$t(3m,p) = p2^{M_{3m}}$ for $1 \le p \le T_{3m}$) such that, for $1 \le p \le T_{3m}$,

$$\Big|\sum_{g=t(3m,p-1)+1}^{t(3m,p)} x^*_{\pi(3m,n,g)}(\overline{x}) - \frac{2^{M_{3m}} w^*_{3m,n}(\overline{x})}{T_{3m}}\Big| = \Big|\sum_{g=t(3m,p-1)+1}^{t(3m,p)} x^*_{\pi(3m,n,g)}(\overline{x}) - w^*_{3m,n}(\overline{x})\Big| < \overline{a}_{3m},$$

$$\Big|\sum_{g=t(3m,p-1)+1}^{t(3m,p)} x^*_{\pi(3m,n,g)}(\overline{x})\Big| \le |w^*_{3m,n}(\overline{x})| + \overline{a}_{3m} < Q_{0,3m} + \overline{a}_{3m},$$

$$\Big|\sum_{g=t(3m,p-1)+1}^{t(3m,p)} \frac{x^*_{\pi(3m,n,g)}(\overline{x})}{2^{M_{3m}}} - \frac{w^*_{3m,n}(\overline{x})}{2^{M_{3m}}}\Big| < \frac{\overline{a}_{3m}}{2^{M_{3m}}},$$

$$\Big|\sum_{g=1}^{t(3m,p)} \frac{x^*_{\pi(3m,n,g)}(\overline{x})}{2^{M_{3m}}} - p\frac{w^*_{3m,n}(\overline{x})}{2^{M_{3m}}}\Big| < \frac{\overline{a}_{3m}}{2^{M_{3m}}},$$

$$W_{3m,p} = \sum_{n=1}^{P_{3m}} W_{3m,n,p}, \qquad \widetilde{W}_{3m,p} = \sum_{g=1}^{p} W_{3m,g},$$

$$W_{3m,n,p} = \sum_{g=t(3m,p-1)+1}^{t(3m,p)} x^*_{\pi(3m,n,g)}(\overline{x}) \frac{w_{3m,n}}{2^{M_{3m}}}$$

for $1 \le n \le P_{3m}$. Hence, by Step 4 of SC III.1,

$$W_{3m} = \widetilde{W}_{3m,D_{3m}} = \sum_{n=1}^{P_{3m}} w^*_{3m,n}(\overline{x}) w_{3m,n} = \sum_{n=1}^{P_{3m}} u^*_{3m,n}(\overline{x}) u_{3m,n}.$$

Then, for $1 \le p \le T_{3m}$, we have

$$\begin{aligned}\Big\|W_{3m,p} - \frac{W_{3m}}{T_{3m}}\Big\| &= \Big\|\sum_{n=1}^{P_{3m}}\Big(\sum_{g=t(3m,p-1)+1}^{t(3m,p)} x^*_{\pi(3m,n,g)}(\overline{x}) - w^*_{3m,n}(\overline{x})\Big)\frac{w_{3m,n}}{2^{M_{3m}}}\Big\| \\ &< \sum_{n=1}^{P_{3m}}\Big\|\sum_{g=t(3m,p-1)+1}^{t(3m,p)} x^*_{\pi(3m,n,g)}(\overline{x}) - w^*_{3m,n}(\overline{x})\Big\|\frac{2}{2^{M_{3m}}} \\ &< \frac{P_{3m}2\overline{a}_{3m}}{2^{M_{3m}}} < \frac{\overline{a}_{3m}}{2^{M_{3m}/2}},\end{aligned}$$

$$\Big\|\widetilde{W}_{3m,p} - \frac{p}{T_{3m}} W_{3m}\Big\| < P_{3m}\frac{2\overline{a}_{3m}}{2^{M_{3m}}} < \frac{\overline{a}_{3m}}{2^{M_{3m}/2}};$$

$$\begin{aligned}\|W_{3m,n,p}\| &= \Big|\sum_{g=t(3m,p-1)+1}^{t(3m,p)} x^*_{\pi(3m,n,g)}(\overline{x})\Big|\frac{\|w_{3m,n}\|}{2^{M_{3m}}} < \frac{2}{2^{M_{3m}}}\Big|\sum_{g=t(3m,p-1)+1}^{t(3m,p)} x^*_{\pi(3m,n,g)}(\overline{x})\Big| \\ &< \frac{2}{2^{M_{3m}}}(Q_{0,3m} + \overline{a}_{3m}) < \frac{4Q_{0,3m}}{2^{M_{3m}}} < \frac{\overline{a}_{3m}}{2^{M_{3m}/2}}.\end{aligned}$$

By Fact 3 and by the procedure of the proof of Fact 2, if we set

$$(\pi(3m,g))_{g=1}^{G_{3m,0}} = (((\pi(3m,n,g))_{g=t(3m,p-1)+1}^{t(3m,p)})_{n=1}^{P_{3m}})_{p=1}^{T_{3m}},$$

then $(\|\sum_{g=1}^{G} x^*_{\pi(3m,g)}(\overline{x})\widetilde{x}_{\pi(3m,g)}+E_{3m,0}\|)_{G=1}^{G_{3m}}$ and $(\|\sum_{g=1}^{G} x^*_{\pi(3m,g)}(\overline{x})\widetilde{x}_{\pi(3m,g)}\|)_{G=1}^{G_{3m}}$ are $(2,\overline{a}_{3m}/2^{M_{3m}/2})$-monotone. At this point we can state the following

FACT 4. *If $(x_{\pi(0,3m,g)})_{g=1}^{G_{0,3m}}$ is any permutation of $(x_{0,3m,g})_{g=1}^{G_{0,3m}}$ of Fact 3 such that the permutation induced on $(x_{3m,n,g})_{g=1}^{G_{3m}}$ is $(x_{\pi(3m,g)})_{g=1}^{G_{3m,0}}$ above, then*

$$\Big(\Big\|\sum_{g=1}^{G} x^*_{\pi(0,3m,g)}(\overline{x})x_{\pi(0,3m,g)} + E_{3m,0} + \operatorname{span}(u'_{3m+1,s})_{s=G_{0,0,0,3m}-G_{3m}+1}^{G_{0,0,0,3m}} + \operatorname{span}(u'_{3m+1,s})_{s=A_{3m+1}-A'_{3m+1}+1}^{n'(3m+1)}\Big\|\Big)_{G=1}^{G_{0,3m}}$$

is $(2,3,\overline{a}_{3m}/2^{M_{3m}/2})$-monotone.

SIXTH PART. We now turn to considering the block $(x^*_n(\overline{x})x_n)_{n=q(3m+2)+1}^{q(3m+3)}$. We already know, by Step 1 of SC III.3, that $(e'''_{3m+2,n}+E_{3m+2,0})_{n=1}^{S_{3m+2}}$ is 1-equivalent to the natural basis of $l_\infty^{S_{3m+2}}$ (2-equivalent if $E_{3m+2,0}$ is replaced by

$$E_{3m+2,0} + \operatorname{span}(y_{3m+2,n})_{n=1}^{Q(3m+2)} + U_{3m+2}$$

for $U_{3m+2} = X \cap \bigcap_{n=1}^{Q(3m+2)} y^*_{(3m+2,n)\perp} \cap \bigcap_{n=1}^{S'_{3m+2}+M'_{3m+2,0}} \widehat{e}^*_{(3m+2,n)\perp}$. Then, for $1 \le n \le P_{3m+2}$, we have (recall that, by Step 2 of SB III.3, $\|x_{3m+2,n,0}\| = 1$)

$$\begin{aligned}
&\Big\|x^*_{3m+2,n,0}(\overline{x})x_{3m+2,n,0} + \sum_{k=1}^{2^{M_{3m+2}}} x^*_{3m+2,n,k}(\overline{x})x'_{3m+2,n,k} + E_{3m+2,0}\Big\|\\
&\quad= \Big\|x^*_{3m+2,n,0}(\overline{x})\Big(\sum_{f=1}^{2^{M_{3m+2}}} e_{3m+2,n,f}\Big) + \sum_{k=1}^{2^{M_{3m+2}}} x^*_{3m+2,n,k}(\overline{x})e_{3m+2,n,k} + E_{3m+2,0}\Big\|\\
&\quad= \Big\|\sum_{k=1}^{2^{M_{3m+2}}} (x^*_{3m+2,n,0}(\overline{x}) + x^*_{3m+2,n,k}(\overline{x}))e_{3m+2,n,k} + E_{3m+2,0}\Big\|\\
&\quad= \Big\|\sum_{k=1}^{2^{M_{3m+2}}} e^*_{3m+2,n,k}(\overline{x})e_{3m+2,n,k} + E_{3m+2,0}\Big\|\\
&\quad= \max(|x^*_{3m+2,n,0}(\overline{x}) + x^*_{3m+2,n,k}(\overline{x}) = e^*_{3m+2,n,k}(\overline{x})| : 1 \le k \le 2^{M_{3m+2}}).
\end{aligned}$$

That is, for any permutation

$$(\pi(3m+2,g))_{g=1}^{G_{3m+2}} \text{ of } ((3m+2,g))_{g=1}^{G_{3m+2}},$$

$(\|\sum_{g=1}^{G} x^*_{\pi(3m+2,g)}(\overline{x})x'_{\pi(3m+2,g)} + E_{3m+2,0}\|)_{G=1}^{G_{3m+2}}$ is $(1,0)$-monotone. On the other hand, by the procedure of the proof of Fact 4, there exists a permutation $(\pi(3m+2,g))_{g=1}^{G_{3m+2}}$ of $((3m+2,g))_{g=1}^{G_{3m+2}}$ (we can use for instance $T_{3m+2} = 2^{M_{3m+2}/2}$) such that

$$\Big(\Big\|\sum_{g=1}^{G} x^*_{\pi(3m+2,g)}(\overline{x})(x''_{\pi(3m+2,g)} + x'''_{\pi(3m+2,g)}) + E_{3m+2,0}\Big\|\Big)_{G=1}^{G_{3m+2}}$$

and

$$\Big(\Big\|\sum_{g=1}^{G} x^*_{\pi(3m+2,g)}(\overline{x})(x''_{\pi(3m+2,g)} + x'''_{\pi(3m+2,g)})\Big\|\Big)_{G=1}^{G_{3m}}$$

are $(2,1,\overline{a}_{3m+2}/2^{Q_{0,3m+2}})$-monotone; therefore

$$\Big(\Big\|\sum_{g=1}^{G} x^*_{\pi(3m+2,g)}(\overline{x})x_{\pi(3m+2,g)} + E_{3m+2,0}\Big\|\Big)_{G=1}^{G_{3m+2}}$$

is $(2,1,\overline{a}_{3m+2}/2^{Q_{0,3m+2}})$-monotone.

Seventh part (proof of (A) for C IV). By the beginning and Step 1 of SC IV.2, and by Step 3 of SC III.2, it follows, for $1 \le d \le L_{3m}$, that the definition of

$$(x_{d,0,3m+1,g}, x^*_{d,0,3m+1,g})_{g=1}^{G_{d,0,3m+1}}$$

is simpler than the definition of $(x_{0,3m+1,g}, x^*_{0,3m+1,g})_{g=1}^{G_{0,3m+1}}$ in SC III.2, hence also a more simplified procedure of regularization works, in particular

(a) for what concerns the first part of the proof of RL we have to consider, passing to SC IV.2, only $D_{1,3m+1}$ and $D_{2,3m+1}$, since the bridge sequence does not appear;
(b) for the same reason of (a) it is not necessary to use NPL;
(c) (46.1) and (46.2) hold for $(S_i)_{i=1}^3$ replaced by (S);
(d) Fact 1 continues to hold, only with $(3,0)$-monotone instead of $(6,0)$-monotone.

Now let us fix $1 \le d \le L_{3m}$. We will proceed through two steps.

Step 1 (regularization of the support sequence). By the above it follows that there exists a permutation $(\pi(d,0,3m+1,n,g))_{g=1}^{G_{d,0,3m+1,0}}$ of $((d,0,3m+1,n,g))_{g=1}^{G_{d,0,3m+1,0}}$ for $1 \le n \le P_{d,3m+1}$ such that if $(\pi(d,0,3m+1,g))_{g=1}^{G_{d,0,3m+1}}$ is any permutation of $((d,0,3m+1,g))_{g=1}^{G_{d,0,3m+1}}$ with the property that, for each n with $1 \le n \le P_{d,3m+1}$, the permutation induced on $((d,0,3m+1,n,g))_{g=1}^{G_{d,0,3m+1,0}}$ is just $(\pi(d,0,3m+1,n,g))_{g=1}^{G_{d,0,3m+1,0}}$ above, then

$$\Big(\Big\|\sum_{g=1}^{G} x^*_{\pi(d,0,3m+1,g)}(\overline{x})(x_{\pi(d,0,3m+1,g)} - \widetilde{x}_{\pi(d,0,3m+1,g)}) + E_{d,3m+1,0}\Big\|\Big)_{G=1}^{G_{d,0,3m+1}}$$

is $(3,0)$-monotone (we recall that $x_{\pi(d,0,3m+1,g)} - \widetilde{x}_{\pi(d,0,3m+1,g)} = x'_{\pi(d,0,3m+1,g)}$ for $1 \le g \le G_{d,0,3m+1}$).

Step 2 (regularization of the connection sequences). We turn to considering

$$(x_{d,0,3m+1,g})_{G=1}^{G_{d,0,3m+1}}$$

and we will follow the whole procedure of the third part till Fact 2, but for the whole sequence $(x_{0,3m+1,g})_{G=1}^{G_{0,3m+1}}$ and not only for each sequence $(x_{d,0,3m+1,g})_{G=1}^{G_{d,0,3m+1}}$ one by one.

For $1 \le n \le P_{d,3m+1}$ we start from the permutation $(\pi(d,0,3m+1,n,g))_{g=1}^{G_{d,0,3m+1,0}}$ of Step 1 and let $(\pi(d,3m+1,n,g))_{g=1}^{G_{d,3m+1,0}}$ be the permutation induced on $((d,3m+1,n,g))_{g=1}^{G_{d,3m+1,0}}$. We also set (we follow the notations of the third part, see also the last part of Step 3 of SC IV.3) $T_{d,3m+1} = 2^{M_{d,3m+1}+2Q_{d,3m+1}}$ and, for $1 \le p \le T_{d,3m+1}$,

$$t(d,3m+1,p) = p2^{2Q_{d,3m+1}},$$

$$W_{d,3m+1,n,p} = \sum_{g=t(d,3m+1,p-1)+1}^{t(d,3m+1,p)} x^*_{\pi(d,3m+1,n,g)}(\overline{x})\widetilde{x}_{\pi(d,3m+1,n,g)},$$

$$W_{d,3m+1,p} = \sum_{n=1}^{P_{d,3m+1}} W_{d,3m+1,n,p},$$

$$W_{d,3m+1} = \sum_{p=1}^{T_{d,3m+1}} W_{d,3m+1,p};$$

now we have to put together in a suitable way all these $W_{d,3m+1,p}$ for $1 \le p \le T_{d,3m+1}$ and $1 \le d \le L_{3m}$.

Then we set, for $1 \le p \le T_{L_{3m},3m+1}$,

$$U_{L_{3m},3m+1,p} = W_{L_{3m},3m+1,p},$$

moreover we set, for $1 \le p \le T_{L_{3m}-1,3m+1}$,

$$U_{L_{3m}-1,3m+1,p} = W_{L_{3m}-1,3m+1,p} + \sum_{p'=(p-1)T_{L_{3m},3m+1}/T_{L_{3m}-1,3m+1}+1}^{pT_{L_{3m},3m+1}/T_{L_{3m}-1,3m+1}} U_{L_{3m},3m+1,p'}$$

and in general, for each d with $1 \le d \le L_{3m} - 1$ and for $1 \le p \le T_{d,3m+1}$,

$$U_{d,3m+1,p} = W_{d,3m+1,p} + \sum_{p'=(p-1)T_{d+1,3m+1}/T_{d,3m+1}+1}^{pT_{d+1,3m+1}/T_{d,3m+1}} U_{d+1,3m+1,p'}.$$

In particular we get, for $1 \le p \le T_{1,3m+1}$,

$$U_{1,3m+1,p} = W_{1,3m+1,p} + \sum_{p'=(p-1)T_{2,3m+1}/T_{1,3m+1}+1}^{pT_{2,3m+1}/T_{1,3m+1}} U_{2,3m+1,p'}.$$

Finally, we can define the permutation $(\pi(3m+1,g))_{g=1}^{G_{3m+1}}$ of $((3m+1,g))_{g=1}^{G_{3m+1}}$ which we can recognize by the order of the summands in the following sum:

$$U_{3m+1} = \sum_{p=1}^{T_{1,3m+1}} U_{1,3m+1,p} = \sum_{g=1}^{G_{3m+1}} x^*_{\pi(3m+1,g)}(\overline{x})\widetilde{x}_{\pi(3m+1,g)}.$$

Indeed, we already know, by the last part of the procedure of Fact 2, that, for $1 \le p \le T_{L_{3m},3m+1}$,

$$\|U_{L_{3m},3m+1,p} - W_{L_{3m},3m+1}/T_{L_{3m},3m+1}\| < \overline{a}_{3m+1}(1/2^{Q_{L_{3m},0,3m+1}}).$$

Then by the above it follows that also, for $1 \le p \le T_{L_{3m}-1,3m+1}$,

$$\left\| U_{L_{3m}-1,3m+1,p} - \frac{W_{L_{3m}-1,3m+1} + W_{L_{3m},3m+1}}{T_{L_{3m}-1,3m+1}} \right\| < \overline{a}_{3m+1} \sum_{c=L_{3m}-1}^{L_{3m}} \frac{1}{2^{Q_{c,0,3m+1}}}.$$

In general, for $1 \le d \le L_{3m}$ and $1 \le p \le T_{d,3m+1}$, we have

$$\Big\|U_{d,3m+1,p} - \Big(\sum_{c=d}^{L_{3m}} W_{c,3m+1}\Big)\frac{1}{T_{d,3m+1}}\Big\| < \overline{a}_{3m+1}\sum_{c=d}^{L_{3m}} \frac{1}{2^{Q_{c,0,3m+1}}}.$$

Therefore, since by the beginning of Step 2 of SC IV.2 and by the definitions of GBST we can see that $Q_{d,0,3m+1} > dQ_{1,0,3m+1}$ for $1 \le d \le L_{3m}$, hence

$$\sum_{d=1}^{L_{3m}} \frac{1}{2^{Q_{d,0,3m+1}}} < \sum_{d=1}^{L_{3m}} \Big(\frac{1}{2^{Q_{1,0,3m+1}}}\Big)^d < \frac{2}{2^{Q_{1,0,3m+1}}},$$

we have defined a permutation $(\pi(3m+1,g))_{g=1}^{G_{3m+1}}$ of $((3m+1,g))_{g=1}^{G_{3m+1}}$ such that, for each permutation $(\pi(0,3m+1,g))_{g=1}^{G_{0,3m+1}}$ of $((0,3m+1,g))_{g=1}^{G_{0,3m+1}}$ with the property of Step 1, which induces $(\pi(3m+1,g))_{g=1}^{G_{3m+1}}$ on $((3m+1,g))_{g=1}^{G_{3m+1}}$, the sequence

$$\Big(\Big\|\sum_{g=1}^{G} x^*_{\pi(0,3m+1,g)}(\overline{x})x_{\pi(0,3m+1,g)} + E_{3m+1,0}\Big\|\Big)_{G=1}^{G_{0,3m+1}}$$

is $(2,3,2\overline{a}_{3m+1}/2^{Q_{1,0,3m+1}})$-monotone; in particular, since by Steps 2 and 4 of SC IV.2 we find that $\operatorname{span}(\widetilde{e}_{3m+1,n} + E_{3m+1,0})_{n=1}^{\widetilde{L}_{3m+1}}$ is 2-complemented in $X/E_{3m+1,0}$, by the same proof of (ii) of CL it follows that

$$\Big\|\sum_{g=1}^{G} x^*_{\pi(0,3m+1,g)}(\overline{x})(x_{\pi(0,3m+1,g)} - x'''_{\pi(0,3m+1,g)}) + E_{3m+1,0}\Big\| < \eta_m \to 0;$$

this completes the proof of (A) for C IV.

This completes the proof of Lemma 18. ■

In Step 1 of the proof of (i) of the next lemma we will give the reason for the relation

$$A_{3m+1} = 2^{q(3m+1)(A'_{3m+1}+M_{3m,\mathrm{arm},0}+A''_{3m+1})}$$

from the beginning of SC III.2; moreover $(v'_{3m,n}, v'^*_{3m,n})_{n=1}^{Q''_{3m}}$ is the sequence

$$(v'_{m+2,n}, v'^*_{m+2,n})_{n=1}^{Q''_{m+2}}$$

of (42.4) of C II, for $m+2$ replaced by $3m$.

LEMMA 19 (First Completeness Regularization Lemma, FCRL). *Let $\overline{x} \in X$ with $\|\overline{x}\| = 1$. Then there exists a sequence (η_m) of positive numbers, with $\eta_m \to 0$, such that, for each m,*

(i) *for each $e_{0,m}$ in $\operatorname{span}(((x_{3m,n,k,l} - \widetilde{x}_{3m,n,k,l})_{l=1}^{2^{M_{3m}}-1})_{k=1}^{2^{M_{3m}}})_{k=1}^{P_{3m}}$ with*

$$\|e_{0,m}\| \le 10.7q(3m+1),$$

under the "operating chain condition" there exists $u_{0,m} = \sum_{i=1}^{R_m} x^*_{r(m)_i}(\overline{x})x_{r(m)_i}$, with $(r(m)_i)_{i=1}^{R_m}$ a subsequence of $(n)_{n=q(3m+1)+1}^{q(3m+2)}$, such that $\|e_{0,m} + u_{0,m}\| < \|e_{0,m} + E_{3m,0}\| + 1/(q(3m+1)2^{2P_{3m+1}})$;

(ii) *under the "operating chain condition" there exists a subsequence* $(\overline{q}(m)_i)_{i=1}^{\overline{Q}_m}$ *of* $(n)_{n=q(3m)+1}^{q(3m+2)}$ *such that*

$$\Big\|\overline{x}-\Big(\sum_{n=1}^{q(3m)}x_n^*(\overline{x})x_n+\overline{u}_m\Big)\Big\|<\eta_m,\quad \textit{with}\quad \overline{u}_m=\sum_{i=1}^{\overline{Q}_m}x_{\overline{q}(m)_i}^*(\overline{x})x_{\overline{q}(m)_i},$$

while under the "disconnected chain condition" we have directly

$$\Big\|\overline{x}-\sum_{n=1}^{q(3m)}x_n^*(\overline{x})x_n\Big\|<\eta_m=\eta'_m+\widetilde{\eta}_m+\frac{2}{2^{2q(3m)A_{3m}}}+4\overline{a}_{3m},$$

where η'_m *is the number* η_m *of* (i) *of* CL *and* $\widetilde{\eta}_m\to 0$ *with* m;

(iii) *under the "operating chain condition", if* $(\widetilde{u}_s)_{s=1}^{q(3m)}\subset \operatorname{span}(u_{3m,k})_{k=1}^{n(3m)-1}$ *with* $\|\widetilde{u}_s\|<30q(3m)$ *for* $1\le s\le q(3m)$, *then there exists* $(\overline{u}_s)_{s=1}^{q(3m)}$ *with*

$$\Big\|\sum_{s=1}^{S}(\overline{u}_s-\widetilde{u}_s)\Big\|<\eta_m-\eta'_m,\qquad \overline{u}_s=\sum_{g=G(3m,s-1)+1}^{G(3m,s)}(x_{\pi(3m,g)}^*(\overline{x})x_{\pi(3m,g)}+\overline{u}_{s,g}),$$

$$\overline{u}_{s,g}=\sum_{g'=G(3m+1,g-1)+1}^{G(3m+1,g)}x_{\pi(3m+1,g')}^*(\overline{x})x_{\pi(3m+1,g')},$$

for $1\le s,S\le q(3m)$ *and* $G(3m,s-1)+1\le g\le G(3m,s)$, *where* $(\pi(3m,g))_{g=G(3m,s-1)+1}^{G(3m,s)}$ *and* $(\pi(3m+1,g'))_{g'=G(3m+1,g-1)+1}^{G(3m+1,g)}$ *are permutations of the kinds of* (B) *and* (A) *of* RL *respectively*; *moreover also the whole permutations*

$$(x_{\pi(3m,g)}-\widetilde{x}_{\pi(3m,g)})_{g=1}^{G(3m,q(3m))},\qquad (x_{\pi(3m+1,g')}-\widetilde{x}_{\pi(3m+1,g')})_{g'=1}^{G(3m+1,G(3m,q(3m)))}$$

have the same properties of the analogous permutations of (B) *and* (A) *of* RL *respectively.*

Proof of (i). Since the elements of $(e_{3m,\mathrm{brd},n})_{n=1}^{P_{3m}}$ do not appear in the expression of $e_{0,m}$, we also have $e_{0,m}=e'_{0,m}+e_{\mathrm{arm},0,m}$ with $e'_{0,m}\in\operatorname{span}(x'_n+x''_n)_{n=q(3m)+1}^{q(3m+1)}$ and $e_{\mathrm{arm},0,m}\in\operatorname{span}(x_{\mathrm{arm},n})_{n=q(3m)+1}^{q(3m+1)}$. Since the operating chain condition holds, by LCL there exists $n'(3m+1)$ with $A_{3m+1}-A'_{3m+1}+1\le n'(3m+1)\le A_{3m+1}$ such that $|u'^*_{3m+1,n'(3m+1)}(\overline{x})|\ge\varepsilon'_{3m+1}$ while $|u'^*_{3m+1,n}(\overline{x})|<\varepsilon'_{3m+1}$ for $n'(3m+1)+1\le n\le A_{3m+1}$. Hence (see the proof of (ii) of LCL) there exists $\overline{n}(3m+1)$ with $P_{3m+1}-A'_{3m+1}(1+2^{2B_{3m+1}})+1\le\overline{n}(3m+1)+1\le P_{3m+1}$ so that $|u^*_{3m+1\overline{n}(3m+1)+1}(\overline{x})|\ge\varepsilon_{3m+1}$ while $|u^*_{3m+1,n}(\overline{x})|<\varepsilon_{3m+1}$ for $\overline{n}(3m+1)+2\le n\le A_{3m+1}$. So, if $(g(3m+1,n))_{n=1}^{\overline{n}(3m+1)+1}$ is the subsequence of $(n)_{n=1}^{P_{3m+1}}$ corresponding to $(g(n))_{n=1}^{\overline{n}+1}$ of GBST, we will proceed through the following six steps:

STEP 1. By Step 6 of SC III.1 and by the beginning of SC III.2 (in particular by the definition of A_{3m+1}), moreover by the definition of $\overline{n}(3m+1)+1$ above, $(g(3m+1,n))_{n=1}^{\overline{n}(3m+1)+1}$ has a subsequence of the kind $(2n'-1,2n')_{n=1}^{A''_{3m+1}(1+2^{2B_{3m+1}})+1}$.

STEP 2. Always in the spirit of the approximations of Subsection 1.5, we can suppose that for each $e\in\operatorname{span}(x'_n+x''_n)_{n=q(3m)+1}^{q(3m+1)}$ there exists $u\in E_{0,3m}\subset\operatorname{span}(u'_{3m+1,s})_{s=1}^{A''_{3m+1}}$ such that $\|e+u\|=\|e+E_{0,3m}\|$. Moreover we know that $((e''_{3m,\mathrm{arm},n,j})_{j=1}^{J_{3m,\mathrm{arm}}})_{n=1}^{P_{3m}}=$

$(u'_{3m+1,s})_{s=G_{0,0,0,3m}-G_{3m}+1}^{G_{0,0,0,3m}}$. Hence there is $\widetilde{u}_{0,m} \in \operatorname{span}(u'_{3m+1,s})_{s=1}^{A''_{3m+1}}$ so that $\|e_{0,m} + \widetilde{u}_{0,m}\| = \|e'_{0,m} + E_{3m,0}\|$, where $e_{0,m} = e'_{0,m} + e_{0,m,\mathrm{arm}}$ with $e'_{0,m} \in \operatorname{span}(x'_n + x''_n)_{n=q(3m)+1}^{q(3m+1)}$ and $e_{0,m,\mathrm{arm}} \in \operatorname{span}(x_{\mathrm{arm},n})_{n=q(3m)+1}^{q(3m+1)}$.

STEP 3. We recall that, by GBST where we can suppose that $M > q(3m+1)$ for the $(3m+1)$th block, since $\|\widetilde{u}_{0,m}\| < 2\|\overline{e}_{0,m}\| < 140q(3m+1)$, there are sequences $(f(n))_{n=1}^{A''_{3m+1}(1+2^{2B_{3m+1}})} \subset (g(3m+1,n))_{n=1}^{\overline{n}(3m+1)+1}$ and $(\widehat{a}(3m+1,n))_{n=1}^{A''_{3m+1}(1+2^{2B_{3m+1}})}$ of numbers such that if we set

$$\widehat{u}_{0,m} = \sum_{n=1}^{A''_{3m+1}(1+2^{2B_{3m+1}})} \widehat{a}(3m+1,n) w_{3m+1,f(n)},$$

then

$$\begin{gathered}\|\widehat{u}_{0,m} - \widetilde{u}_{0,m}\| < \|\widetilde{u}_{0,m}\|/2^{MP_{3m+1}} < 140q(3m+1)/2^{MP_{3m+1}} < 1/(2q(3m+1)2^{2P_{3m+1}}),\\ 0 < \widehat{a}(3m+1,n)/w^*_{3m+1,f(n)}(\overline{x}) < \|\widetilde{u}_{0,m}\|/2^{MP_{3m+1}} < 1/(2q(3m+1)2^{2P_{3m+1}})\end{gathered}$$

for $1 \le n \le A''_{3m+1}(1+2^{2B_{3m+1}})$; recall that $(w^*_{3m+1,f(n)}(\overline{x}))_{n=1}^{A''_{3m+1}(1+2^{2B_{3m+1}})}$ has alternate signs. On the other hand, by the end of the statement of GBST, we can have the same fact also if we use another subsequence $(h'(n))_{n=1}^{A''_{3m+1}(1+2^{2B_{3m+1}})}$ of $(g(3m+1,n))_{n=1}^{\overline{n}(3m+1)+1}$ such that the two sequences

$$\left(\frac{w^*_{3m+1,h'(n)}(\overline{x})}{|w^*_{3m+1,h'(n)}(\overline{x})|}\right)_{n=1}^{A''_{3m+1}(1+2^{2B_{3m+1}})}$$

and

$$\left(\frac{w^*_{3m+1,f(n)}(\overline{x})}{|w^*_{3m+1,f(n)}(\overline{x})|}\right)_{n=1}^{A''_{3m+1}(1+2^{2B_{3m+1}})}$$

are equal. The only difference is that we will use another sequence $(\widehat{a}'(n))_{n=1}^{A''_{3m+1}(1+2^{2B_{3m+1}})}$ of numbers and if we set $\widehat{u}'_{0,m} = \sum_{n=1}^{A''_{3m+1}(1+2^{2B_{3m+1}})} \widehat{a}'(n) w_{3m+1,h'(n)}$, then again

$$\|\widehat{u}'_{0,m} - \widetilde{u}_{0,m}\| < \frac{1}{2q(3m+1)2^{2P_{3m+1}}}, \qquad 0 < \frac{\widehat{a}'(n)}{w^*_{3m+1,f(n)}(\overline{x})} < \frac{1}{2q(3m+1)2^{2P_{3m+1}}}$$

for $1 \le n \le A''_{3m+1}(1+2^{2B_{3m+1}})$.

STEP 4. Therefore by Step 1 we now have at our disposal

$$(v^*_{3m+1,2n'-1}(\overline{x}) v_{3m+1,2n'-1}, v^*_{3m+1,2n'}(\overline{x}) v_{3m+1,2n'})_{n=1}^{A''_{3m+1}(1+2^{2B_{3m+1}})+1},$$

where, for each n with $1 \le n \le A''_{3m+1}(1+2^{2B_{3m+1}})+1$, since

$$(2n'-1, 2n')_{n=1}^{A''_{3m+1}(1+2^{2B_{3m+1}})+1}$$

is a subsequence of $(g(3m+1,n))_{n=1}^{\overline{n}(3m+1)+1}$, by (ii) of (13) and (14) of GBST we have

$$\begin{aligned} v_{3m+1,2n'-1} &= w_{3m+1,2n'} + \frac{w_{3m+1,2n'-1}}{2^{M_{3m+1}+Q_{3m+1}}},\\ v_{3m+1,2n'} &= w_{3m+1,2n'} - \frac{w_{3m+1,2n'-1}}{2^{M_{3m+1}+Q_{3m+1}}},\end{aligned}$$

$$\begin{gathered}
v^*_{3m+1,2n'-1} = (w^*_{3m+1,2n'} + 2^{M_{3m+1}+Q_{3m+1}} w^*_{3m+1,2n'-1})/2, \\
v^*_{3m+1,2n'} = (w^*_{3m+1,2n'} - 2^{M_{3m+1}+Q_{3m+1}} w^*_{3m+1,2n'-1})/2, \\
\|v_{3m+1,2n'-1} - w_{3m+1,2n'}\| = \|v_{3m+1,2n'} - w_{3m+1,2n'}\| < 2/2^{M_{3m+1}+Q_{3m+1}}, \\
|w^*_{3m+1,2n'-1}(\overline{x})| > |w^*_{3m+1,2n'}(\overline{x})|,
\end{gathered}$$

hence $v^*_{3m+1,2n'-1}(\overline{x})$ has the sign of $w^*_{3m+1,2n'-1}(\overline{x})$ while $v^*_{3m+1,2n'}(\overline{x})$ has the sign of $-w^*_{3m+1,2n'-1}(\overline{x})$. Therefore we can choose $h(n') \in (2n'-1, 2n')$ such that the whole sequence $(v^*_{3m+1,h(n')}(\overline{x}))_{n=1}^{A''_{3m+1}(1+2^{2B_{3m+1}})+1}$ has alternate signs.

STEP 5. By the previous steps we conclude that, for the approximation of $\widetilde{u}_{0,m}$ of Step 2, $(w^*_{3m+1,f(n)}(\overline{x}) w_{3m+1,f(n)})_{n=1}^{A''_{3m+1}(1+2^{2B_{3m+1}})}$ of Step 3 has the same properties as either

$$\left(\frac{v^*_{3m+1,h(n')}(\overline{x})}{|v^*_{3m+1,h(n')}(\overline{x})|} |w^*_{3m+1,2n'}(\overline{x})| w_{3m+1,2n'}\right)_{n=1}^{A''_{3m+1}(1+2^{2B_{3m+1}})}$$

or the same sequence but for $2 \le n \le A''_{3m+1}(1+2^{2B_{3m+1}})+1$; we can suppose the former to be the case. Hence now, by the procedure of the proof of GBST, we can get a sequence $(b(3m+1,n))_{n=1}^{A''_{3m+1}(1+2^{2B_{3m+1}})}$ of numbers so that if we set

$$u'_{0,m} = \sum_{n=1}^{A''_{3m+1}(1+2^{2B_{3m+1}})} b(3m+1,n) w_{3m+1,2n'},$$

then

$$\|u'_{0,m} - \widetilde{u}_{0,m}\| < \frac{1}{2q(3m+1)2^{2P_{3m+1}}}, \quad 0 < \frac{b(3m+1,n)}{w^*_{3m+1,2n'}(\overline{x})} < \frac{1}{2q(3m+1)2^{2P_{3m+1}}}$$

for $1 \le n \le A''_{3m+1}(1+2^{2B_{3m+1}})$. But we do not have directly at our disposal

$$\left(\frac{v^*_{3m+1,h(n')}(\overline{x})}{|v^*_{3m+1,h(n')}(\overline{x})|} |w^*_{3m+1,2n'}(\overline{x})| w_{3m+1,2n'}\right)_{n=1}^{A''_{3m+1}(1+2^{2B_{3m+1}})+1}.$$

However, let us point out that, by Step 3 of SC III.2, for $1 \le n \le A''_{3m+1}(1+2^{2B_{3m+1}})$ and $1 \le k \le 2^{M_{3m+1}}$ we could have at our disposal, by means of

$$\sum_{l=1}^{2^{4Q_{3m+1}}} (x^*_{3m+1,h(n'),k,l}(\overline{x}) - x^*_{0,3m+1,h(n'),k,l}(\overline{x})) x'''_{3m+1,h(n'),k,l},$$

$(v^*_{3m+1,h(n')}(\overline{x})/2^{M_{3m+1}}) v_{3m+1,h(n')}$ distributed in the $2^{4Q_{3m+1}}$ summands

$$(x^*_{3m+1,h(n'),k,l}(\overline{x}) - x^*_{0,3m+1,h(n'),k,l}(\overline{x})) v_{3m+1,h(n')}/2^{P_{3m+1}Q_{0,3m+1}} \qquad \text{for } 1 \le l \le 2^{4Q_{3m+1}},$$

with

$$\sum_{l=1}^{2^{4Q_{3m+1}}} (x^*_{3m+1,h(n'),k,l}(\overline{x}) - x^*_{0,3m+1,h(n'),k,l}(\overline{x})) = 2^{P_{3m+1}Q_{0,3m+1}} \frac{v^*_{3m+1,h(n')}(\overline{x})}{2^{M_{3m+1}}}$$

and where, setting $v_{3m+1,h(n')} - w_{3m+1,2n'} = a$ and

$$v^*_{3m+1,h(n')}(\overline{x})/2^{M_{3m+1}} - 2^{Q_{3m+1}} w^*_{3m+1,2n'-1}(\overline{x})/2 = b$$

(if $h(n')=2n'-1$, while if $h(n')=2n'$ we replace $-$ by $+$), we have

$$\|a\|<\frac{2}{2^{M_{3m+1}+Q_{3m+1}}}\quad \text{and}\quad |b|=\frac{|w^*_{3m+1,2n'}(\overline{x})|}{2\cdot 2^{M_{3m+1}}}<\frac{Q_{0,3m+1}}{2\cdot 2^{M_{3m+1}}}.$$

Hence also

$$\begin{aligned}
&\left\|\frac{v^*_{3m+1,h(n')}(\overline{x})}{2^{M_{3m+1}}}v_{3m+1,h(n')}-\frac{1}{2}2^{Q_{3m+1}}w^*_{3m+1,2n'-1}(\overline{x})w_{3m+1,2n'}\right\|\\
&\qquad=\left\|\left(\frac{1}{2}\cdot 2^{Q_{3m+1}}w^*_{3m+1,2n'-1}(\overline{x})+b\right)(w_{3m+1,2n'}+a)\right.\\
&\qquad\qquad\left.-\frac{1}{2}\cdot 2^{Q_{3m+1}}w^*_{3m+1,2n'-1}(\overline{x})w_{3m+1,2n'}\right\|\\
&\qquad\le\left|\frac{1}{2}\cdot 2^{Q_{3m+1}}w^*_{3m+1,2n'-1}(\overline{x})\right|\cdot\|a\|+|b|\cdot\|w_{3m+1,2n'}\|+|b|\cdot\|a\|\\
&\qquad<\frac{1}{2^{M_{3m+1}+Q_{3m+1}}}|2^{Q_{3m+1}}w^*_{3m+1,2n'-1}(\overline{x})|+\frac{Q_{0,3m+1}}{2\cdot 2^{M_{3m+1}}}\|w_{3m+1,2n'}\|\\
&\qquad\qquad+\frac{2}{2^{M_{3m+1}+Q_{3m+1}}}\frac{Q_{0,3m+1}}{2\cdot 2^{M_{3m+1}}}\\
&\qquad<\frac{2^{Q_{3m+1}}Q_{0,3m+1}}{2^{M_{3m+1}+Q_{3m+1}}}+\frac{Q_{0,3m+1}}{2^{M_{3m+1}}}+\frac{Q_{0,3m+1}}{2^{2M_{3m+1}+Q_{3m+1}}}<3\frac{Q_{0,3m+1}}{2^{M_{3m+1}}}.
\end{aligned}$$

So

$$\begin{aligned}
&\left\|\sum_{l=1}^{2^{4Q_{3m+1}}}(x^*_{3m+1,h(n'),k,l}(\overline{x})-x^*_{0,3m+1,h(n'),k,l}(\overline{x}))x'''_{3m+1,h(n'),k,l}\right.\\
&\qquad\qquad\left.-\frac{1}{2}\cdot 2^{Q_{3m+1}}w^*_{3m+1,2n'-1}(\overline{x})w_{3m+1,2n'}\right\|<3\frac{Q_{0,3m+1}}{2^{M_{3m+1}}};
\end{aligned}$$

where by the above we need only $b(3m+1,n)w_{3m+1,2n'}$ with

$$|b(3m+1,n)|<\frac{|w^*_{3m+1,2n'}(\overline{x})|}{2q(3m+1)2^{2P_{3m+1}}}<\frac{|w^*_{3m+1,2n'-1}(\overline{x})|}{2q(3m+1)2^{2P_{3m+1}}}.$$

But again we have at our disposal only $\sum_{l=1}^{2^{4Q_{3m+1}}}x^*_{3m+1,h(n'),k,l}(\overline{x})x_{3m+1,h(n'),k,l}$ and not

$$\sum_{l=1}^{2^{4Q_{3m+1}}}(x^*_{3m+1,h(n'),k,l}(\overline{x})-x^*_{0,3m+1,h(n'),k,l}(\overline{x}))x'''_{3m+1,h(n'),k,l};$$

so let us settle this in the next step.

Step 6. From (v) of RBL and from Step 5 there exist $k(n)\in(k)_{k=1}^{2^{M_{3m+1}}}$ and $L(n)$ with $1\le L(n)<Q_{0,3m+1}2^{3Q_{3m+1}}$, for $1\le n\le A''_{3m+1}(1+2^{2B_{3m+1}})$, so that

$$\left\|\sum_{l=1}^{L(n)}x^*_{3m+1,h(n'),k(n),l}(\overline{x})x_{3m+1,h(n'),k(n),l}-b(3m+1,n)w_{3m+1,2n'}\right\|<1/2^{Q_{0m}2P_{3m+1}}.$$

Hence, setting

$$u_{0,m} = \sum_{n=1}^{A''_{3m+1}(1+2^{2B_{3m+1}})} \sum_{l=1}^{L(n)} x^*_{3m+1,h(n'),k(n),l}(\overline{x})x_{3m+1,h(n'),k(n),l} = \sum_{i=1}^{R_m} x^*_{r(m)_i}(\overline{x})x_{r(m)_i},$$

we have

$$\begin{aligned}\|e_{0,m}+u_{0,m}\| &\le \|u_{0,m}-u'_{0,m}\| + \|u'_{0,m}-\widetilde{u}_{0,m}\| + \|e_{0,m}+\widetilde{u}_{0,m}\| \\ &< A''_{3m+1}(1+2^{2B_{3m+1}})/2^{Q_{0m}2P_{3m+1}}\end{aligned}$$

(by the above in this step, and by Steps 5 and 2)

$$\begin{aligned}+1/(2q(3m+1)2^{2P_{3m+1}}) + \|e_{0,m}+\widetilde{u}_{0,m}\| &< 1/(q(3m+1)2^{2P_{3m+1}}) + \|\overline{e}_{0,m}+\widetilde{u}_{0,m}\| \\ &= 1/(q(3m+1)2^{2P_{3m+1}}) + \|e'_{0,m}+E_{0,3m}\|.\end{aligned}$$

This completes the proof of (i).

Proof of (ii). By (i) of CL, we know that there exists a subsequence $(u'_{3m,n_k})_{k=1}^{K+1}$ of $(u'_{3m,s})_{s=1}^{A''_{3m}}$ and $0<a<1$, with

$$\Big\|\overline{x} - \Big(\sum_{n=1}^{q(3m)} x^*_n(\overline{x})x_n + u'_m\Big)\Big\| < \eta'_m, \qquad u'_m = \sum_{k=1}^{K} u'^*_{3m,n_k}(\overline{x})u'_{3m,n_k} + a u'^*_{3m,n_{K+1}}(\overline{x})u'_{3m,n_{K+1}}.$$

So, under the "disconnected chain condition", by (i) of LCL, we know that

$$|u'^*_{3m,n}(\overline{x})| < \varepsilon'_{3m} = \frac{1}{A_{3m}2^{2q(3m)A_{3m}}} \quad \text{for } 1 \le n \le A_{3m}.$$

Therefore from the above it also follows that

$$\begin{aligned}&\Big\|\overline{x} - \sum_{n=1}^{q(3m)} x^*_n(\overline{x})x_n\Big\| \\ &\quad = \Big\|\overline{x} - \Big(\sum_{n=1}^{q(3m)} x^*_n(\overline{x})x_n + u'_m\Big) + u'_m\Big\| \le \Big\|\overline{x} - \Big(\sum_{n=1}^{q(3m)} x^*_n(\overline{x})x_n + u'_m\Big)\Big\| + \|u'_m\| \\ &\quad < \eta'_m + \|u'_m\| = \eta'_m + \Big\|\sum_{k=1}^{K} u'^*_{3m,n_k}(\overline{x})u'_{3m,n_k} + a u'^*_{3m,n_{K+1}}(\overline{x})u'_{3m,n_{K+1}}\Big\| \\ &\quad \le \eta'_m + \sum_{k=1}^{K+1} |u'^*_{3m,n_k}(\overline{x})| < \eta'_m + \frac{K+1}{A_{3m}2^{2q(3m)A_{3m}}} \le \eta'_m + \frac{1}{2^{2q(3m)A_{3m}}}.\end{aligned}$$

Hence, for $\eta_m = \eta'_m + 1/2^{2q(3m)A_{3m}}$, (ii) is directly proved. Therefore suppose that the "operating chain condition" holds. Then, by (i) and (iii) of CL and by the definitions of $n'(3m)$ and $n(3m)$ in LCL, setting $(n_k)_{k=1}^{K+1} = (n_{k'})_{k=1}^{K'} \cup (n_{k''})_{k=1}^{K''}$ with $(n_{k'})_{k=1}^{K'} \subseteq (n)_{n=1}^{n'(3m)}$ and $(n_{k''})_{k=1}^{K''} \subseteq (n)_{n=n'(3m)+1}^{A_{3m}}$, we find that there exist $(\widetilde{u}_{n_{k'}})_{k=1}^{K'} \subset \operatorname{span}(u_{3m,k})_{k=1}^{n(3m)-1}$ with $\widetilde{u}_{n_{k'}} = u'^*_{3m,n_{k'}}(\overline{x})u'_{3m,n_{k'}}$ if $(n_{k'})_{k=1}^{K'} \subseteq (n)_{n=1}^{n'(3m)-1}$, while if there is $\overline{k}$ with $1 \le \overline{k} \le K'$ such that $n_{\overline{k}'} = n_{K+1} = n'(3m)$, by (iii) of CL we only have

$\|\widetilde{u}_{n_{\overline{k}'}} - au'^{*}_{3m,n_{K+1}}(\overline{x})u'_{3m,n_{K+1}}\| < \eta''_m$ for some $\eta''_m \to 0$. Finally,

$$\sum_{k=1}^{K''} \|u'_{3m,n_{k''}}\| < \frac{K''}{A_{3m}2^{2q(3m)A_{3m}}} < \frac{1}{2^{2q(3m)A_{3m}}},$$

therefore, setting $\widetilde{u}'_m = \sum_{k=1}^{K'} u'_{3m,n_{k'}}$, we have $\|\widetilde{u}'_m - u'_m\| < \eta''_m + 1/2^{2q(3m)A_{3m}}$. At this point, since $\widetilde{u}'_m \in \operatorname{span}(u_{3m,k})_{k=1}^{n(3m)-1}$ and since obviously $\|u'_m\| < 2.3.5q(3m)$ and hence $\|\widetilde{u}'_m\| < 30q(3m) + \eta''_m + 1/2^{2q(3m)A_{3m}} < 31q(3m)$, since moreover we can suppose $M > q(3m)$ for the integer M of GBST for the $(3m)$th block, if $(g(3m,n))_{n=1}^{\overline{n}(3m)+1}$ is the subsequence of $(n)_{n=1}^{P_{3m}}$ corresponding to $(g(n))_{n=1}^{\overline{n}+1}$ of GBST hence in our case $\overline{n}(3m) = n(3m)-1$, then by the statement of GBST there exist sequences $(f(n))_{n=1}^{n(3m)-1} \subset (g(3m,n))_{n=1}^{n(3m)}$ and $(a(3m,n))_{n=1}^{n(3m)-1}$ of numbers such that, setting

$$w_{0,m} = \sum_{n=1}^{n(3m)-1} a(3m,n)w_{3m,f(n)},$$

we have

$$\|w_{0,m} - \widetilde{u}'_m\| < \frac{1}{2q(3m)2^{2P_{3m}}},$$

$$a(3m,n)/w^*_{3m,f(n)}(\overline{x}) < 1/(2q(3m)2^{2P_{3m}}) \quad \text{for } 1 \le n \le n(3m)-1.$$

At this point, if we use the permutation $(x_{\pi(0,3m,g)})_{g=1}^{G_{0,3m}}$ of $(x_{0,3m,g})_{g=1}^{G_{0,3m}}$ of Fact 4 of the fifth part of the proof of RL, where, for each n with $1 \le n \le P_{3m}$, $(x_{\pi(3m,n,g)})_{g=1}^{G_{3m,0}}$ is the permutation induced on $(x_{3m,n,g})_{g=1}^{G_{3m,0}}$, then the sequence $(|\sum_{g=1}^{G} x^*_{\pi(3m,n,g)}(\overline{x})|)_{G=1}^{G_{3m,0}}$ is $(1,0)$-monotone and hence $(0,\overline{a}_{3m})$-monotone. So we set, for $1 \le n \le n(3m)-1$,

$$\widetilde{w}_{u,m,f(n)} = \sum_{g=1}^{G(3m,f(n))} x^*_{\pi(3m,f(n),g)}(\overline{x})\widetilde{x}_{\pi(3m,f(n),g)} = \sum_{g=1}^{G(3m,f(n))} \frac{x^*_{\pi(3m,f(n),g)}(\overline{x})}{2^{M_{3m}}} w_{3m,f(n)}$$

with $G(3m,f(n)) \le 2^{M_{3m}}a(3m,n) < G(3m,f(n))+1$ and $\widetilde{w}_{u,m} = \sum_{n=1}^{n(3m)-1} \widetilde{w}_{u,m,f(n)}$, hence

$$\begin{aligned}\|\widetilde{w}_{u,m} - w_{0,m}\| &\le \sum_{n=1}^{n(3m)-1} \Bigg\|\Bigg(\sum_{g=1}^{G(3m,f(n))} \frac{x^*_{\pi(3m,f(n),g)}(\overline{x})}{2^{M_{3m}}} - a(3m,n)\Bigg) w_{3m,f(n)}\Bigg\| \\ &< \sum_{n=1}^{n(3m)-1} \frac{\overline{a}_{3m}}{2^{M_{3m}}}\|w_{3m,f(n)}\| < \frac{2(n(3m)-1)}{2^{M_{3m}}}\overline{a}_{3m},\end{aligned}$$

$$\|\widetilde{w}_{u,m} - \widetilde{u}'_m\| \le \|\widetilde{w}_{u,m} - w_{0,m}\| + \|w_{0,m} - \widetilde{u}'_m\| < \frac{P_{3m}\overline{a}_{3m}}{2^{M_{3m}}} + \frac{1}{2q(3m)2^{2P_{3m}}}.$$

Here we recall that $G_{3m,0} = 2^{2M_{3m}}$ and by the above

$$|a(3m,n)| < \frac{|w^*_{3m,f(n)}(\overline{x})|}{2q(3m)2^{2P_{3m}}} < \frac{Q_{0,3m}}{2q(3m)2^{2P_{3m}}} < \frac{2^{M_{3m}}}{2q(3m)2^{2P_{3m}}},$$

therefore, for $1 \le n \le n(3m)-1$, we will not use all the elements of $(x_{3m,n,g})_{g=1}^{G_{3m,0}}$. We can also write

$$\widetilde{w}_{0,m} = \widetilde{w}_{u,m} + \widetilde{w}_{e,m} = \sum_{n=1}^{n(3m)-1} \sum_{g=1}^{G(3m,f(n))} x^*_{\pi(3m,f(n),g)}(\overline{x}) x_{\pi(3m,f(n),g)},$$

hence

$$\widetilde{w}_{e,m} = \sum_{n=1}^{n(3m)-1} \sum_{g=1}^{G(3m,f(n))} x^*_{\pi(3m,f(n),g)}(\overline{x})(x_{\pi(3m,f(n),g)} - \widetilde{x}_{\pi(3m,f(n),g)}).$$

We recall, by Step 6 of SC III 1 and by the notations before CL, that

$$(u'_{3m+1,s})_{s=M_{3m,0}+1}^{M_{3m,0}+G_{0,0,0,3m}} = (x_{0,0,3m,g})_{g=1}^{G_{0,0,3m}} \cup (e'_{3m,\mathrm{arm},g})_{g=1}^{G_{3m}},$$

$$(x_{0,0,3m,g})_{g=1}^{G_{0,0,3m}} = ((x_{0,0,3m,n,g})_{g=1}^{G_{0,0,3m,0}})_{n=1}^{P_{3m}},$$

$$(x_{0,0,3m,n,g})_{g=1}^{G_{0,0,3m,0}} = (\omega_{3m,n,0} \cup x_{3m,n,0} \cup (\omega_{3m,n,k,0} \cup (x_{3m,n,k,0,l})_{l=0}^{2^{M_{3m}}})_{k=1}^{2^{M_{3m}}})_{n=1}^{P_{3m}}.$$

Moreover, setting $\widetilde{E}_{3m,0} = E_{3m,0} + \operatorname{span}(u'_{3m+1,s})_{s=M_{3m,0}+1}^{M_{3m,0}+G_{0,0,0,3m}}$, if we use any permutation $(x_{\pi(0,3m,g)})_{g=1}^{G_{0,3m}}$ of $(x_{0,3m,g})_{g=1}^{G_{0,3m}}$ of Fact 3 of the fourth part of the proof of RL, we find that

$$\Big(\Big\|\sum_{g=1}^{G} x^*_{\pi(0,3m,g)}(\overline{x})(x_{\pi(0,3m,g)} - \widetilde{x}_{\pi(0,3m,g)}) + E_{3m,0} + \operatorname{span}(u'_{3m+1,s})_{s=G_{0,0,0,3m}-G_{3m}+1}^{G_{0,0,0,3m}} + \operatorname{span}(u'_{3m+1,s})_{s=A_{3m+1}-A'_{3m+1}+1}^{n'(3m+1)}\Big\|\Big)_{G=1}^{G_{0,3m}}$$

$$= \Big(\Big\|\sum_{g=1}^{G} x^*_{\pi(0,3m,g)}(\overline{x})(x_{\pi(0,3m,g)} - \widetilde{x}_{\pi(0,3m,g)}) + \widetilde{E}_{3m,0} + \operatorname{span}(u'_{3m+1,s})_{s=A_{3m+1}-A'_{3m+1}+1}^{n'(3m+1)}\Big\|\Big)_{G=1}^{G_{0,3m}}$$

is $(3,0)$-monotone, hence also $(0, 3\overline{a}_{3m})$-monotone. On the other hand, if $(x_{\pi(3m,g)})_{g=1}^{G_{3m}}$ is the corresponding permutation induced on $(x_{3m,g})_{g=1}^{G_{3m}}$, we also know that, for each G_0 with $1 \le G_0 \le G_{0,3m}$, if $(x_{\pi(3m,g)})_{g=1}^{G}$ is the permutation induced by $(x_{\pi(0,3m,g)})_{g=1}^{G_0}$ on $(x_{\pi(3m,g)})_{g=1}^{G_{3m}}$, then

$$\begin{aligned}\Big\|\sum_{g=1}^{G} x^*_{\pi(3m,g)}(\overline{x}) x'_{\pi(3m,g)} + \widetilde{E}_{3m,0}\Big\| &\le \Big\|\sum_{g=1}^{G_0} x^*_{\pi(0,3m,g)}(\overline{x})(x_{\pi(0,3m,g)} - \widetilde{x}_{\pi(0,3m,g)}) + E_{3m,0} \\ &\qquad + \operatorname{span}(u'_{3m+1,s})_{s=G_{0,0,0,3m}-G_{3m}+1}^{G_{0,0,0,3m}} \\ &\qquad + \operatorname{span}(u'_{3m+1,s})_{s=A_{3m+1}-A'_{3m+1}+1}^{n'(3m+1)}\Big\| \\ &= \Big\|\sum_{g=1}^{G_0} x^*_{\pi(0,3m,g)}(\overline{x}) x'_{\pi(0,3m,g)} + E_{3m,0}\Big\|.\end{aligned}$$

Therefore $(\|\sum_{g=1}^{G} x^*_{\pi(3m,g)}(\overline{x}) x'_{\pi(3m,g)} + \widetilde{E}_{3m,0}\|)_{G=1}^{G_{3m}}$ is also $(0, 3\overline{a}_{3m})$-monotone, hence, by (ii) of CL, also

$$\Big\|\sum_{g=1}^{G} x^*_{\pi(3m,g)}(\overline{x}) x'_{\pi(3m,g)} + \widetilde{E}_{3m,0}\Big\| < \eta'''_m + 3\overline{a}_{3m} \quad \text{for } 1 \le G \le G_{3m}$$

for some $\eta'''_m \to 0$ as $m \to \infty$. Therefore let us state the following hypothesis, which we will prove later in C IV, before the proof of (iii).

HYPOTHESIS (*). For each subsum u of $\sum_{g=1}^{G_{3m}} x^*_{3m,g}(\overline{x})x_{3m,brd,g}$ there is a subsum u_0 of $\sum_{g=1}^{G_{3m+1}} x^*_{3m+1,g}(\overline{x})x_{3m+1,g}$ such that $\|u - u_0\| < \eta_m \to 0$.

If we include η_m in η'''_m and u_0 in $\widetilde{E}_{3m,0}$, it follows that

$$\begin{aligned}\|\widetilde{w}_{e,m} + \widetilde{E}_{3m,0}\| &= \Big\| \sum_{n=1}^{n(3m)-1} \sum_{g=1}^{G(3m,f(n))} x^*_{\pi(3m,n,g)}(\overline{x})(x_{\pi(3m,n,g)} - \widetilde{x}_{\pi(3m,n,g)}) + \widetilde{E}_{3m,0}\Big\| \\ &< \eta'''_m + 3\overline{a}_{3m}.\end{aligned}$$

On the other hand, we actually have at our disposal all the elements of $\widetilde{E}_{3m,0}$ that we need; therefore by the procedure of the proof of (i) there exists $u_{0,m}$ as in (i) such that

$$\begin{aligned}\|\widetilde{w}_{e,m} + u_{0,m}\| &< \Big\| \sum_{n=1}^{n(3m)-1} \sum_{g=1}^{G(3m,f(n))} x^*_{\pi(3m,n,g)}(\overline{x})x'_{\pi(3m,n,g)} + \widetilde{E}_{3m,0}\Big\| \\ &\quad + 1/(q(3m+1)2^{2P_{3m+1}}) \\ &< \eta'''_m + 3\overline{a}_{3m} + 1/(q(3m+1)2^{2P_{3m+1}}).\end{aligned}$$

It is now sufficient to set

$$\overline{u}_m = \widetilde{w}_{0,m} + u_{0,m}, \quad \eta_m = \eta'_m + \widetilde{\eta}_m + \frac{3}{2^{2q(3m)A_{3m}}} + 4\overline{a}_{3m},$$

where $\widetilde{\eta}_m = \eta''_m + \eta'''_m$, since

$$\frac{2}{2^{2q(3m)A_{3m}}} + \frac{P_{3m}\overline{a}_{3m}}{2^{M_{3m}}} + \frac{1}{2q(3m)2^{2P_{3m}}} + 3\overline{a}_{3m} + \frac{1}{2q(3m+1)2^{2P_{3m+1}}} < \frac{3}{2^{2q(3m)A_{3m}}} + 4\overline{a}_{3m}.$$

Proof of Hypothesis (*) *for C IV.* Fix $1 \le n \le P_{3m}$; our aim is to get $aw^*_{3m,n}(\overline{x})w_{3m,n}$ for some number a with $|a| \le 1$. We will proceed through six steps.

STEP 1 (the problem). We recall that, by Steps 5 and 6 of SC III.1,

$$\sum_{g=1}^{G_{3m,0}} x^*_{3m,n,g}(\overline{x})\widetilde{x}_{3m,n,g} = \sum_{k=1}^{2^{M_{3m}}} \sum_{l=1}^{2^{M_{3m}}} x^*_{3m,n,k,l}(\overline{x})\widetilde{x}_{3m,n,k,l} = w^*_{3m,n}(\overline{x})w_{3m,n};$$

moreover, since $E_{3m,0} \supset (x_{3m,\mathrm{arm},n,g})_{g=1}^{G_{3m,0}}$, also

$$\begin{aligned}&\Big\| \sum_{g=1}^{G_{3m,0}} x^*_{3m,n,g}(\overline{x})(x_{3m,n,g} - \widetilde{x}_{3m,n,g} - x_{3m,\mathrm{brd},n,g}) + E_{3m,0}\Big\| \\ &\quad = \Big\| \sum_{g=1}^{G_{3m,0}} x^*_{3m,n,g}(\overline{x})x'_{3m,n,g} + E_{3m,0}\Big\| \le 8\max(|x^*_{3m,n,g}(\overline{x})| : 1 \le g \le G_{3m,0}) \le 8\overline{a}_{3m};\end{aligned}$$

but we have to point out that, setting

$$\begin{aligned}A_n &= \sum_{g=1}^{G_{3m,0}} x^*_{3m,n,g}(\overline{x})x_{3m,\mathrm{brd},n,g} = \sum_{k=1}^{2^{M_{3m}}} x^*_{3m,n,k,2^{M_{3m}}}(\overline{x})x_{3m,\mathrm{brd},n,k} \\ &= \sum_{k=1}^{2^{M_{3m}}} x^*_{3m,n,k,2^{M_{3m}}}(\overline{x})(e_{3m,\mathrm{brd},n,k} + \omega'_{3m,n}),\end{aligned}$$

it follows that

$$\begin{aligned}\|A_n+E_{3m,0}\| &= \Big\|\sum_{k=1}^{2^{M_{3m}}} x^*_{3m,n,k,2^{M_{3m}}}(\overline{x})(e_{3m,\mathrm{brd},n,k}+\omega'_{3m,n})+E_{3m,0}\Big\|\\ &= \max\Big(3|x^*_{3m,n,k,2^{M_{3m}}}(\overline{x})| : 1\le k\le 2^{M_{3m}}, \Big|\sum_{k=1}^{2^{M_{3m}}} x^*_{3m,n,k,2^{M_{3m}}}(\overline{x})\Big|\Big)\end{aligned}$$

where $|\sum_{k=1}^{2^{M_{3m}}} x^*_{3m,n,k,2^{M_{3m}}}(\overline{x})|$ can be large and this happens for each n with $1\le n\le P_{3m}$. However, in order to remedy this, it is not necessary to remove the whole $A=\sum_{n=1}^{P_{3m}} A_n$ by means of subsums of $\sum_{g=1}^{G_{0,3m+1,0}} x^*_{0,3m+1,g}(\overline{x})x_{0,3m+1,n,g}$, since it is sufficient by means of these subsums to get only

$$\begin{aligned}B=\sum_{i=1}^{L'_{3m}} b'^*_{3m,i}(\overline{x})b'_{3m,i} = \sum_{n=1}^{P_{3m}}\Big(&\omega'^*_{3m,n,0}(\overline{x})\sum_{f=1}^{2^{M_{3m}}} e_{3m,\mathrm{brd},n,f}\\ &+\sum_{k=1}^{2^{M_{3m}}}(e''^*_{3m,\mathrm{brd},n,k}(\overline{x})-x^*_{3m,n,k,2^{M_{3m}}}(\overline{x}))(e_{3m,\mathrm{brd},n,k}+\omega'_{3m,n})\Big);\end{aligned}$$

indeed, it will follow that

$$A+B=\sum_{n=1}^{P_{3m}}\Big(\omega'^*_{3m,n}(\overline{x})\omega'_{3m,n}+\sum_{k=1}^{2^{M_{3m}}} e^*_{3m,\mathrm{brd},n,k}(\overline{x})e_{3m,\mathrm{brd},n,k}\Big)$$

and we are done since $\|A+B+E_{3m,0}\|\le 2\overline{a}_{3m}$.

Step 2 (first approximation). There exists $i(m)$ such that

$$|b'^*_{3m,i(m)}(\overline{x})|\ge 1/(L'_{3m}2^{2L'_{3m}}),\quad |b'^*_{3m,i}(\overline{x})|<1/(L'_{3m}2^{2L'_{3m}})\quad \text{for } i(m)+1\le i\le L'_{3m};$$

since $\|\sum_{i=i(m)+1}^{L'_{3m}} b'^*_{3m,i}(\overline{x})b'_{3m,i}\|<1/2^{2L'_{3m}}$ we can disregard these last elements; moreover, in order to decrease the formalism, we can suppose also

$$|b'^*_{3m,i}(\overline{x})|\ge 1/(L'_{3m}2^{2L'_{3m}})\quad \text{for } 1\le i\le i(m)-1.$$

Now we fix i with $1\le i\le i(m)$ and we pass to the corresponding system

$$(b_{3m,i,j},b^*_{3m,i,j})_{j=0}^{2^{2B_{3m+1}}} = (b_{3m,d},b^*_{3m,d})_{d=d(0,i)+1}^{d(0,i)+2^{2B_{3m+1}}+1}$$

where $d(0,i)=(2^{2B_{3m+1}}+1)(i-1)$; since we know that

$$|b'^*_{3m,i}(\overline{x})|\ge 1/(L'_{3m}2^{2L'_{3m}}),$$

$$\sum_{d=d(0,i)+2}^{d(0,i)+2^{2B_{3m+1}}+1} b^*_{3m,d}(\overline{x}) = \sum_{j=1}^{2^{2B_{3m+1}}} b^*_{3m,i,j}(\overline{x}) = 2^{B_{3m+1}}b'^*_{3m,i}(\overline{x}),$$

there exists (see also (iii) of CL) $d(m,i)$ with $d(0,i)+2<d(m,i)\le d(0,i)+2^{2B_{3m+1}}+1$ such that

$$|b^*_{3m,d(m,i)}(\overline{x})|\ge 1/(L_{3m}2^{2L_{3m}}),\quad |b^*_{3m,d}(\overline{x})|<1/(L_{3m}2^{2L_{3m}})$$

for $d(m,i)+1 \le d \le d(0,i)+2^{2B_{3m+1}}+1$. Let us prove that

$$\Big\|\sum_{d=d(0,i)+2}^{d(m,i)-1} b^*_{3m,d}(\overline{x})b_{3m,d} - b'^*_{3m,i}(\overline{x})b'_{3m,i} + E_{1,3m+1,0}\Big\| < 3\overline{a}_{3m+1} + \frac{1}{L_{3m}2^{B_{3m+1}+L_{3m}}}.$$

Indeed, from the definitions of (i) of Step 3 of SC IV.2 it follows that

$$\begin{aligned}\sum_{d=d(0,i)+2}^{d(m,i)-1} b^*_{3m,d}(\overline{x})b_{3m,d} - b'^*_{3m,i}(\overline{x})b'_{3m,i} &= \sum_{j=1}^{d(m,i)-d(0,i)} b^*_{3m,i,j}(\overline{x})b_{3m,i,j} - b'^*_{3m,i}(\overline{x})b'_{3m,i}\\ &= \sum_{j=1}^{d(m,i)-d(0,i)} b^*_{3m,i,j}(\overline{x})\Big(e_{3m+1,\mathrm{aux},i,j} + \frac{b'_{3m,i}}{2^{B_{3m+1}}}\Big) - \Big(\sum_{j=1}^{2^{2B_{3m+1}}} \frac{b^*_{3m,i,j}(\overline{x})}{2^{B_{3m+1}}}\Big)b'_{3m,i}\\ &= \sum_{j=1}^{d(m,i)-d(0,i)} b^*_{3m,i,j}(\overline{x})e_{3m+1,\mathrm{aux},i,j} - \Big(\sum_{j=d(m,i)-d(0,i)+1}^{2^{2B_{3m+1}}} \frac{b^*_{3m,i,j}(\overline{x})}{2^{B_{3m+1}}}\Big)b'_{3m,i}\end{aligned}$$

where, by the definition of $\overline{a}_{3m+1}$,

$$\begin{aligned}&\Big\|\sum_{j=1}^{d(m,i)-d(0,i)} b^*_{3m,i,j}(\overline{x})e_{3m+1,\mathrm{aux},i,j} + E_{1,3m+1,0}\Big\|\\ &\qquad\le 2\max(|b^*_{3m,i,j}(\overline{x})| : 1 \le j \le d(m,i)-d(0,i)) \le 2\overline{a}_{3m+1}),\\ &\Big\|\Big(\sum_{j=d(m,i)-d(0,i)+1}^{2^{2B_{3m+1}}} \frac{b^*_{3m,i,j}(\overline{x})}{2^{B_{3m+1}}}\Big)b'_{3m,i}\Big\| \le 3\sum_{j=d(m,i)-d(0,i)+1}^{2^{2B_{3m+1}}} \frac{|b^*_{3m,i,j}(\overline{x})|}{2^{B_{3m+1}}}\\ &\qquad< 3\frac{|b^*_{3m,i,d(m,i)-d(0,i)+1}(\overline{x})|}{2^{B_{3m+1}}} + \sum_{j=d(m,i)-d(0,i)+2}^{2^{2B_{3m+1}}} \frac{1}{2^{B_{3m+1}}L_{3m}2^{2L_{3m}}}\\ &\qquad< 3\overline{a}_{3m+1} + \frac{1}{2^{B_{3m+1}}L_{3m}2^{L_{3m}}};\end{aligned}$$

therefore we can approximate $b'^*_{3m,i}(\overline{x})b'_{3m,i}$ by $\sum_{d=d(0,i)+2}^{d(m,i)-1} b^*_{3m,d}(\overline{x})b_{3m,d}$.

Step 3 (second approximation). Our next aim is to approximate sufficiently these $b^*_{3m,d}(\overline{x})b_{3m,d}$ by subsums of $\sum_{g=1}^{G_{d,0,3m+1,0}} x^*_{d,0,3m+1,g}(\overline{x})\widetilde{x}_{d,0,3m+1,g}$, for each d with $d(0,i) + 1 \le d \le d(m,i)-1$. For these d we recall, by the end of Step 3 of SC IV.2, that

$$(\widetilde{x}_{d,0,3m+1,g})_{g=1}^{G(d,0,3m+1)} = (v_{d,0,3m+1,n}/2^{P_{d,3m+1}Q_{0,d,3m+1}})_{n=1}^{P_{d,3m+1}}$$

where $(v_{d,0,3m+1,n})_{n=1}^{P_{d,3m+1}}$ comes by means of the procedure of MGBS from

$$(u_{d,0,3m+1,n})_{n=1}^{P_{d,3m+1}}$$

where $(u_{d,0,3m+1,n})_{n=P_{d,3m+1}-2^{B_{0,3m+1}}}^{P_{d,3m+1}}$ comes from

$$(e_{d,3m+1,\mathrm{aux},A_{d,3m+1},t}, e^*_{d,3m+1,\mathrm{aux},A_{d,3m+1},t})_{t=1}^{2^{2B_{0,3m+1}}} \cup (b_{3m,d}, b^*_{3m,d});$$

in order to decrease the formalism we will suppose also

$$|b^*_{3m,d}(\overline{x})| \ge 1/(L_{3m}2^{2L_{3m}}) \quad \text{for all } 1 \le d \le d(m,i)-1$$

(since it would be unnecessary to get $b^*_{3m,d}(\overline{x})b_{3m,d}$ if $|b^*_{3m,d}(\overline{x})| < 1/(L_{3m}2^{2L_{3m}})$). Hence there exists $n(d)$ with $P_{d,3m+1} - 2^{B_{0,3m+1}} < n(d) \le P_{d,3m+1}$, so that (we have only to replace L_{3m} by $2^{2B_{0,3m+1}}$)

$$|u^*_{d,0,3m+1,n(d)}(\overline{x})| \ge \frac{1}{2^{2B_{0,3m+1}}2^{2\cdot 2^{2B_{0,3m+1}}}} > \frac{1}{2^{2P_{d,3m+1}}},$$
$$|u^*_{d,0,3m+1,n}(\overline{x})| < \frac{1}{2^{2B_{0,3m+1}}2^{2\cdot 2^{2B_{0,3m+1}}}}$$

for $n(d)+1 \le n \le P_{d,3m+1}$; therefore, by the same proof above for the approximation of $b'^*_{3m,i}(\overline{x})b'_{3m,i}$, we can get

$$\Big\| \sum_{n=P_{d,3m+1}-2^{B_{0,3m+1}}+1}^{n(d)-1} u^*_{d,0,3m+1,n}(\overline{x})u_{d,0,3m+1,n} - b^*_{3m,d}(\overline{x})b_{3m,d} + E_{d,3m+1,0}\Big\|$$
$$< 3\overline{a}_{3m+1} + 1/(2^{2B_{0,3m+1}}2^{B_{0,3m+1}+2^{2B_{0,3m+1}}}).$$

STEP 4 (third approximation). At this point, since $1/(L_{3m}2^{2L_{3m}}) > 1/2^{2P_{d,3m+1}}$, from the procedures of MGBS and GBST (where now $\overline{n}+1$ of (7) of GBST is replaced by $n(d)$) it follows that we can approximate $b^*_{3m,d}(\overline{x})b_{3m,d}$ by a subsum of

$$\sum_{g=1}^{G_{d,0,3m+1}} x^*_{d,0,3m+1,g}(\overline{x})\widetilde{x}_{d,0,3m+1,g}.$$

Therefore, for $1 \le i \le i(m)$ and $d(0,i)+1 \le d \le d(m,i)-1$, there exists a subsequence $(s(d,g))_{g=1}^{G(d,0,3m+1)}$ of $(g)_{g=1}^{G_{d,0,3m+1}}$ such that, setting

$$u_{0,d,3m+1} = \sum_{g=1}^{G(d,0,3m+1)} x^*_{d,0,3m+1,s(d,g)}(\overline{x})\widetilde{x}_{d,0,3m+1,s(d,g)},$$

we have

$$\|u_{0,d,3m+1} - b^*_{3m,d}(\overline{x})b_{3m,d}\| < 2/2^{2P_{d,3m+1}}.$$

STEP 5 (panorama of the first three approximations). Hence by the above, setting $D(m) = d(m,i(m))$, it follows that (we also use the last but one property of Step 2 of SC IV.2)

$$\Big\|\sum_{i=1}^{i(m)}\Big(\sum_{d=d(0,i)+2}^{d(m,i)-1}\sum_{g=1}^{G(d,0,3m+1)} x^*_{d,0,3m+1,s(d,g)}(\overline{x})\widetilde{x}_{d,0,3m+1,s(d,g)} - b'^*_{3m,i}(\overline{x})b'_{3m,i}\Big)$$
$$+ E_{D(m),3m+1,0}\Big\| < 3\overline{a}_{3m+1} + \frac{1}{2^{L_{3m}}}$$

since (we point out that $B_{d,3m+1}$—hence $P_{d,3m+1}$—is much larger than $B_{0,3m+1}$)

$$\sum_{i=1}^{i(m)}\Big(\frac{1}{L_{3m}2^{B_{3m+1}+L_{3m}}} + \sum_{d=d(0,i)+2}^{d(m,i)-1}\Big(\frac{2}{2^{2P_{d,3m+1}}} + \frac{1}{2^{B_{0,3m+1}+2^{2B_{0,3m+1}}}}\Big)\Big)$$
$$< \frac{L'_{3m}}{L_{3m}2^{B_{3m+1}+L_{3m}}} + \frac{L'_{3m}2^{2B_{0,3m+1}}}{2^{2P_{d,3m+1}}} + \frac{L'_{3m}2^{2B_{0,3m+1}}}{2^{B_{0,3m+1}+2^{2B_{0,3m+1}}}} < \frac{1}{2^{L_{3m}}}.$$

Step 6 (fourth approximation). On the other hand by the above it also follows that, for $i=i(m)$, since $D(m)=d(m,i(m))$, $|b^*_{3m,D(m)}(\overline{x})|\geq 1/(L_{3m}2^{2L_{3m}})$, hence also

$$|u^*_{D(m),0,3m+1,n(D(m))}(\overline{x})|\geq 1/(L_{3m}2^{2.2^{2B_{0,3m+1}}})>1/2^{2P_{D(m),3m+1}}.$$

Therefore for $(x_{D(m),0,3m+1,g})_{g=1}^{G_{D(m),0,3m+1}}$ we are in the situation of (i) of FRCL (it is possible by the definitions of $B_{0,3m+1}$ and $B_{d,3m+1}$ for $1\leq d\leq L_{3m}$); moreover by the above there exists

$$((u_{e,0,D(m),d,3m+1,s(d,g)})_{g=1}^{G(d,0,3m+1)})_{d=d(0,i)+2}^{d(m,i)-1})_{i=1}^{L'_{3m}}\subset E_{D(m),3m+1,0}$$

such that

$$\begin{aligned}&\Big\|\sum_{i=1}^{i(m)}\sum_{d=d(0,i)+2}^{d(m,i)-1}\sum_{g=1}^{G(d,0,3m+1)}(x^*_{d,0,3m+1,s(d,g)}(\overline{x})x'_{d,0,3m+1,s(d,g)}+u_{e,0,D(m),d,3m+1,s(d,g)})\Big\|\\&\quad=\Big\|\sum_{i=1}^{i(m)}\sum_{d=d(0,i)+2}^{d(m,i)-1}\sum_{g=1}^{G(d,0,3m+1)}x^*_{d,0,3m+1,s(d,g)}(\overline{x})x'_{d,0,3m+1,s(d,g)}+E_{D(m),3m+1,0}\Big\|;\end{aligned}$$

therefore by (i) of FRCL and by the above it follows that

$$\begin{aligned}\Big\|\sum_{i=1}^{i(m)}\Big(&\sum_{d=d(0,i)+2}^{d(m,i)-1}\sum_{g=1}^{G(d,0,3m+1)}(x^*_{d,0,3m+1,s(d,g)}(\overline{x})x_{d,0,3m+1,s(d,g)}+u_{u,0,D(m),d,3m+1,s(d,g)})\\&-b'^*_{3m,i}(\overline{x})b'_{3m,i}\Big)+E_{D(m),3m+1,0}\Big\|<3\overline{a}_{3m+1}+\frac{1}{2^{L_{3m}}}+\frac{1}{2^{2P_{D(m),3m+1}}},\end{aligned}$$

$$u_{u,0,D(m),d,3m+1,s(d,g)}=\sum_{g'=G(D(m),0,3m+1,g-1)+1}^{G(D(m),0,3m+1,g)}x^*_{D(m),0,3m+1,s'(d,g')}(\overline{x})x_{D(m),0,3m+1,s'(d,g')}$$

for $1\leq g\leq G(d,0,3m+1)$, $d(0,i)+2\leq d\leq d(m,i)-1$ and $1\leq i\leq i(m)$, where

$$((((x_{D(m),0,3m+1,s'(d,g')})_{g'=G(D(m),0,3m+1,g-1)+1}^{G(D(m),0,3m+1,g)})_{g=1}^{G(d,0,3m+1)})_{d=d(0,i)+2}^{d(m,i)-1})_{i=1}^{i(m)}$$

is a suitable subsequence of $(x_{D(m),0,3m+1,g})_{g=1}^{G_{D(m),0,3m+1}}$; this completes the proof of $(*)$ for C IV.

This completes the proof of (ii).

Proof of (iii). It is sufficient to follow the procedure of (i) and (ii). This completes the proof of Lemma 19. ∎

5. Properties of the whole construction

In the proof of Step 2 of the next lemma there is an explanation of the reason of the construction of the sequence $(v'_{3m+3,n})_{n=1}^{Q''_{3m+3}}$ in Step 2 of SC III.3; the same reason holds also for $(v'_{3m+1,n})_{n=1}^{Q''_{3m+1}}$ in Step 6 of SC III.1 and for $(v'_{3m+2,n})_{n=1}^{Q''_{3m+2}}$ in Step 4 of SC III.2.

Lemma 20 (Second Completeness-Regularization Lemma, SCRL). *Concerning the third sub-block, we have the following properties*:

(A) *There is always a permutation* $(\pi(3m+2,g))_{g=1}^{G_{3m+2}}$ *of* $((3m+2,g))_{g=1}^{G_{3m+2}}$ *and a sequence* $(\overline{u}_{3m+3,g})_{g=1}^{G_{3m+2}}$ *of the kind of* $(\overline{u}_s)_{s=1}^{q(3m)}$ *of* (iii) *of* FCRL *with* m *replaced by* $m+1$, *such that*

$$\Big(\Big\|\sum_{g=1}^{G}(x^*_{\pi(3m+2,g)}(\overline{x})x_{\pi(3m+2,g)}+\overline{u}_{3m+3,g})\Big\|\Big)_{G=1}^{G_{3m+2}}$$

is $(2,1,\overline{a}_{3m+2}/2^{Q_{0,3m+2}})$-*monotone, and*

$$\Big\|\sum_{g=1}^{G}(x^*_{\pi(3m+2,g)}(\overline{x})x_{\pi(3m+2,g)}+\overline{u}_{3m+3,g})\Big\|<\eta_{3m+2}$$

for $1\le G\le G_{3m+2}$ *where* $\eta_{3m+2}\to 0$ *as* $m\to\infty$.

In particular this fact continues to hold also if, setting

$$\overline{u}_{3m+3,g}=\sum_{g'=G(3m+3,g-1)+1}^{G(3m+3,g)}(x^*_{\pi(3m+3,g')}(\overline{x})x_{\pi(3m+3,g')}+\overline{u}_{3m+4,g'}),$$

$$\overline{u}_{3m+4,g'}=\sum_{g''=G(3m+4,g'-1)+1}^{G(3m+4,g')}x^*_{\pi(3m+4,g'')}(\overline{x})x_{\pi(3m+4,g'')}$$

for $1\le g\le G_{3m+2}$ *and* $G(3m+3,g-1)+1\le g'\le G(3m+3,g)$, *we consider the whole sequence*

$$\begin{aligned}
&\Big(\sum_{g=1}^{\widetilde{G}}x^*_{\widetilde{\pi}(g)}(\overline{x})x_{\widetilde{\pi}(g)}\Big)_{\widetilde{G}=1}^{\widetilde{G}_{3m+2}}\\
&\quad=\bigg(\sum_{g=1}^{G-1}\Big(x^*_{\pi(3m+2,g)}(\overline{x})x_{\pi(3m+2,g)}+\sum_{g'=G(3m+3,g-1)+1}^{G(3m+3,g)}\Big(x^*_{\pi(3m+3,g')}(\overline{x})x_{\pi(3m+3,g')}\\
&\qquad+\sum_{g''=G(3m+4,g'-1)+1}^{G(3m+4,g')}x^*_{\pi(3m+4,g'')}(\overline{x})x_{\pi(3m+4,g'')}\Big)\Big)\\
&\qquad+\sum_{g'=G(3m+3,G-1)+1}^{G'-1}\Big(x^*_{\pi(3m+3,g')}(\overline{x})x_{\pi(3m+3,g')}\\
&\qquad+\sum_{g''=G(3m+4,g'-1)+1}^{G(3m+4,g')}x^*_{\pi(3m+4,g'')}(\overline{x})x_{\pi(3m+4,g'')}\Big)\\
&\qquad+\sum_{g''=G(3m+4,G'-1)+1}^{G''}\\
&\qquad x^*_{\pi(3m+4,g'')}(\overline{x})x_{\pi(3m+4,g'')}\Big)\Big)_{G''=G(3m+4,G'-1)+1}^{G(3m+4,G')}\bigg)_{G'=G(3m+3,G-1)+1}^{G(3m+3,G)}\bigg)_{G=1}^{G_{3m+2}},\\
&\qquad\widetilde{G}_{3m+2}=G_{3m+2}+G(3m+3,G_{3m+2})+G(3m+4,G(3m+3,G_{3m+2}));
\end{aligned}$$

that is, again $(\|\sum_{g=1}^{\widetilde{G}} x^*_{\widetilde{\pi}(g)}(\overline{x})x_{\widetilde{\pi}(g)}\|)_{\widetilde{G}=1}^{\widetilde{G}_{3m+2}}$ *is* $(0,\widetilde{\eta}_{3m+2})$*-monotone and*

$$\Big\|\sum_{g=1}^{\widetilde{G}} x^*_{\widetilde{\pi}(g)}(\overline{x})x_{\widetilde{\pi}(g)}\Big\| < \widetilde{\eta}_{3m+2}$$

for $1 \le \widetilde{G} \le \widetilde{G}_{3m+2}$ *where* $\widetilde{\eta}_{3m+2} \to 0$ *as* $m \to \infty$.

(B) *For each* $(\widetilde{u}_{3m+2,s})_{s=1}^{q(3m+2)} \subset \operatorname{span}(u'_{3m+2,s})_{s=1}^{n'(3m+2)-1}$ *with*

$$\|\widetilde{u}_{3m+2,s}\| < 2.3.5q(3m+2)$$

for $1 \le s \le q(3m+2)$ *and for each sequence* $(a_s)_{s=1}^{q(3m+2)}$ *of numbers with* $|a_s| \le 1$ *for* $1 \le s \le q(3m+2)$, *there exists* $(\overline{u}_{3m+2,s})_{s=1}^{q(3m+2)}$ *with, for* $1 \le s, S \le q(3m+2)$ *and for some* $\eta_{3m+2} \to 0$,

$$\Big\|\sum_{s=1}^{S}(\overline{u}_{u,3m+2,s} - \widetilde{u}_{3m+2,s}) + a_S u'^*_{3m+2,n'(3m+2)}(\overline{x})u'_{3m+2,n'(3m+2)}\Big\| < \eta_{3m+2},$$

$$\begin{aligned}
\overline{u}_{3m+2,s} &= \sum_{g=G(3m+2,s-1)+1}^{G(3m+2,s)} (x^*_{\pi(3m+2,g)}(\overline{x})x_{\pi(3m+2,g)} + \overline{u}_{3m+3,3m+4,g}) \\
&= \overline{u}_{u,3m+2,s} + \overline{u}_{e,3m+2,s}, \\
\overline{u}_{u,3m+2,s} &= \sum_{g=G(3m+2,s-1)+1}^{G(3m+2,s)} x^*_{\pi(3m+2,g)}(\overline{x})\widetilde{x}_{\pi(3m+2,g)}, \\
\overline{u}_{3m+3,3m+4,g} &= \sum_{n=G(3m+3,3m+4,g-1)+1}^{G(3m+3,3m+4,g)} x^*_{\pi(3m+3,3m+4,n)}(\overline{x})x_{\pi(3m+3,3m+4,n)},
\end{aligned}$$

where $(x_{\pi(3m+3,3m+4,n)})_{n=1}^{G(3m+3,3m+4,q(3m+2))}$ *is a permutation of a subsequence of* $(x_n)_{n=q(3m+3)+1}^{q(3m+5)}$ *such that, for the elements* $x_{\pi(3m+3,3m+4,n)} - \widetilde{x}_{\pi(3m+3,3m+4,n)}$, *the permutations induced on* $(x_n)_{n=q(3m+3)+1}^{q(3m+4)}$ *and on* $(x_n)_{n=q(3m+4)+1}^{q(3m+5)}$ *are of the same kind of the permutations of* (B) *and* (A) *of* RL *respectively, and such that, for* $1 \le G \le G(3m+2,q(3m+2))$ *and* $1 \le G' \le G(3m+3,3m+4,G(3m+2,q(3m+2)))$,

$$\Big\|\sum_{g=1}^{G}\Big(x^*_{\pi(3m+2,g)}(\overline{x})(x_{\pi(3m+2,g)} - \widetilde{x}_{\pi(3m+2,g)}) + \overline{u}_{3m+3,3m+4,g}\Big)\Big\| < \eta_{3m+2},$$

$$\Big\|\sum_{g=1}^{G'} x^*_{\pi(3m+3,3m+4,n)}(\overline{x})x_{\pi(3m+3,3m+4,n)}\Big\| < \eta_{3m+2}.$$

(C) *For each sequence* $(\widetilde{u}_{3m+1,s})_{s=1}^{q(3m+1)} \subset \operatorname{span}(u'_{3m+1,s})_{s=1}^{n'(3m+1)-1}$ *with* $\|\widetilde{u}_{3m+1,s}\| < 2.10.7q(3m+1)$ *for* $1 \le s \le q(3m+1)$, *and for each sequence* $(a_s)_{s=1}^{q(3m+1)}$ *of numbers with* $|a_s| \le 1$ *for* $1 \le s \le q(3m+1)$, *there exists* $(\overline{u}_{3m+1,s})_{s=1}^{q(3m+1)}$ *with, for* $1 \le s, S \le q(3m+1)$ *and for some* $\eta_{3m+1} \to 0$,

$$\Big\|\sum_{s=1}^{S}(\overline{u}_{u,3m+1,s} - \widetilde{u}_{3m+1,s}) + a_S u'^*_{3m+1,n'(3m+1)}(\overline{x})u'_{3m+1,n'(3m+1)}\Big\| < \eta_{3m+1},$$

$$\begin{aligned}\overline{u}_{3m+1,s} &= \sum_{g=G(3m+1,s-1)+1}^{G(3m+1,s)} (x^*_{\pi(3m+1,g)}(\overline{x})x_{\pi(3m+1,g)} + \overline{u}_{3m+2,3m+3,3m+4,g}) \\ &= \overline{u}_{u,3m+1,s} + \overline{u}_{e,3m+1,s}, \\ \overline{u}_{u,3m+1,s} &= \sum_{g=G(3m+1,s-1)+1}^{G(3m+1,s)} x^*_{\pi(3m+1,g)}(\overline{x})\widetilde{x}_{\pi(3m+1,g)},\end{aligned}$$

$$\overline{u}_{3m+2,3m+3,3m+4,g} = \sum_{n=G(3m+2,3m+3,3m+4,g-1)+1}^{G(3m+2,3m+3,3m+4,g)} x^*_{\pi(3m+2,3m+3,3m+4,n)}(\overline{x})x_{\pi(3m+2,3m+3,3m+4,n)},$$

where $(x_{\pi(3m+2,3m+3,3m+4,n)})_{n=1}^{G(3m+2,3m+3,3m+4,G(3m+1,q(3m+1)))}$ *is a permutation of a subsequence of* $(x_n)_{n=q(3m+2)+1}^{q(3m+5)}$ *such that, for what concerns the elements*

$$x_{\pi(3m+2,3m+3,3m+4,n)} - \widetilde{x}_{\pi(3m+2,3m+3,3m+4,n)},$$

the permutations induced on $(x_n)_{n=q(3m+2)+1}^{q(3m+3)}$ *and on* $(x_n)_{n=q(3m+3)+1}^{q(3m+4)}$ *and on* $(x_n)_{n=q(3m+4)+1}^{q(3m+5)}$ *are of the same kind of the permutations of* (C), (B) *and* (A) *of* RL *respectively, and such that, for* $1 \le G \le G(3m+1, q(3m+1))$ *and* $1 \le G' \le G(3m+2, 3m+3, 3m+4, G(3m+1, q(3m+1)))$,

$$\Big\|\sum_{g=1}^{G}\Big(x^*_{\pi(3m+1,g)}(\overline{x})(x_{\pi(3m+1,g)} - \widetilde{x}_{\pi(3m+1,g)}) + \overline{u}_{3m+2,3m+3,3m+4,g}\Big)\Big\| < \eta_{3m+1},$$

$$\Big\|\sum_{g=1}^{G'} x^*_{\pi(3m+2,3m+3,3m+4,n)}(\overline{x})x_{\pi(3m+2,3m+3,3m+4,n)}\Big\| < \eta_{3m+1}.$$

Proof. FIRST PART. *Proof of (A).* Suppose that the operating chain condition holds for $\sum_{n=q(3m+3)+1}^{q(3m+6)} x^*_n(\overline{x})x_n$. We will proceed through 3 steps.

STEP 1. Let $(\pi(3m+2,g))_{g=1}^{G_{3m+2}}$ be the permutation of $((3m+2,g))_{g=1}^{G_{3m+2}}$ of Fact 4 of the sixth part of the proof of RL and consider

$$\sum_{g=1}^{G_{3m+2}} x^*_{\pi(3m+2,g)}(\overline{x})(x_{\pi(3m+2,g)} - x'''_{\pi(3m+2,g)}) = \sum_{g=1}^{G_{3m+2}} x^*_{\pi(3m+2,g)}(\overline{x})(x'_{\pi(3m+2,g)} + x''_{\pi(3m+2,g)}).$$

If $E_{3m+2,0} \subset \operatorname{span}(u_{3m+3,k})_{k=1}^{n(3m+3)-1}$ (that is, by Step 2 of SC III.3, if $M_{3m+2,0} \le n'(3m+3)-1$) we have, by (iii) of FCRL with m replaced by $m+1$, the existence of $(\overline{u}_{e,3m+3,g})_{g=1}^{G_{3m+2}}$ and $(\overline{u}_{e,3m+4,g})_{g=1}^{G(e,3m+4,G(e,3m+3,G_{3m+2}))}$ with (we set $G(e,3m+3,0) = 0 = G(e,3m+4,0)$)

$$\overline{u}_{e,3m+3,g} = \sum_{g'=G(e,3m+3,g-1)+1}^{G(e,3m+3,g)} (x^*_{\pi(3m+3,g')}(\overline{x})x_{\pi(3m+3,g')} + \overline{u}_{e,3m+4,g'}),$$

$$\overline{u}_{e,3m+4,g'} = \sum_{g''=G(e,3m+4,g'-1)+1}^{G(e,3m+4,g')} x^*_{\pi(3m+4,g'')}(\overline{x})x_{\pi(3m+4,g'')}$$

for $1 \le g \le G_{3m+2}$ and $G(e,3m+3,g-1)+1 \le g' \le G(e,3m+3,g)$, such that, for $1 \le G \le G_{3m+2}$,

$$\Big\| \sum_{g=1}^{G} (x^*_{\pi(3m+2,g)}(\overline{x})(x'_{\pi(3m+2,g)} + x''_{\pi(3m+2,g)}) + \overline{u}_{e,3m+3,g}) \Big\|$$
$$< \Big\| \sum_{g=1}^{G} x^*_{\pi(3m+2,g)}(\overline{x})(x'_{\pi(3m+2,g)} + x''_{\pi(3m+2,g)}) + E_{3m+2,0} \Big\| + \eta_{3m+2}$$

for some $\eta_{3m+2} \to 0$. Therefore, by the procedure of (iii) of FCRL with m replaced by $m+1$, we deduce that

$$\Big(\Big\| \sum_{g=1}^{G} (x^*_{\pi(3m+2,g)}(\overline{x})(x'_{\pi(3m+2,g)} + x''_{\pi(3m+2,g)}) + \overline{u}_{e,3m+3,g}) \Big\| \Big)_{G=1}^{G_{3m+2}}$$

is $(1, \overline{a}_{3m+2}/2^{Q_{0,3m+2}})$-monotone, hence $(0, \overline{a}_{3m+2}(1+1/2^{Q_{0,3m+2}}))$-monotone, with

$$\Big\| \sum_{g=1}^{G} (x^*_{\pi(3m+2,g)}(\overline{x})(x'_{\pi(3m+2,g)} + x''_{\pi(3m+2,g)}) + \overline{u}_{e,3m+3,g}) \Big\| < \eta_{3m+2}$$

for $1 \le G \le G_{3m+2}$ for some $\eta_{3m+2} \to 0$.

While, if $n'(3m+3) \le M_{3m+2,0}$, by Step 1 of SC III.3 and by the equivalent, for SC III.3, of the fact at the end of Step 1 of SC III.1, there exists $\widetilde{S}$ with $1 \le \widetilde{S} \le S_{3m+2}$ such that, for some $\eta_m \to 0$,

$$\Big| \Big\| \sum_{g=1}^{\widetilde{S}-1} e'''^*_{3m+2,g}(\overline{x})e'''_{3m+2,g} + \operatorname{span}(u'_{3m+3.s})_{s=1}^{A_{3m+3}} \Big\|$$
$$- \max(|e'''^*_{3m+2,g}(\overline{x})| : 1 \le g \le \widetilde{S}-1) \Big| < \eta_m,$$

$$\Big| \Big\| \sum_{g=\widetilde{S}+1}^{S_{3m+2}} e'''^*_{3m+2,g}(\overline{x})e'''_{3m+2,g} + \operatorname{span}(u'_{3m+3.s})_{s=1}^{A_{3m+3}} \Big\|$$
$$- \sum_{g=\widetilde{S}+1}^{S_{3m+2}} |e'''^*_{3m+2,g}(\overline{x})| \Big| < \eta_m + \sum_{s=n'(3m+3)+1}^{A_{3m+3}} |u'^*_{3m+3,s}(\overline{x})|,$$

in particular $|e'''^*_{3m+2,\widetilde{S}}(\overline{x})| \le \overline{a}_{3m+2}$ and, also by the same proof of (ii) of CL and by (42.3) of C II, $\sum_{g=\widetilde{S}+1}^{S_{3m+2}} |e'''^*_{3m+2,g}(\overline{x})| < \eta'_m$ for some $\eta'_m \to 0$; therefore we have the same conclusion as above for $M_{3m+2,0} \le n'(3m+3)-1$. Moreover the construction of $(\overline{u}_{e,3m+3,g})_{g=1}^{G_{3m+2}}$ and $(\overline{u}_{e,3m+4,g})_{g=1}^{G(e,3m+3,G_{3m+2})}$ has to be such that the whole sequences

$$(x^*_{\pi(3m+3,g')}(\overline{x})(x_{\pi(3m+3,g')} - \widetilde{x}_{\pi(3m+3,g')}))_{g=1}^{G(e,3m+3,G_{3m+2})} \quad \text{and}$$
$$(x^*_{\pi(3m+4,g'')}(\overline{x})(x_{\pi(3m+4,g'')} - \widetilde{x}_{\pi(3m+4,g'')}))_{g=1}^{G(e,3m+4,G(e,3m+3,G_{3m+2}))}$$

have the properties of the permutations of (B) and (A) of RL respectively; analogously, for each g and g' with $1 \le g \le G_{3m+2}$ and $G(e,3m+3,g-1)+1 \le g' \le G(e,3m+3,g)$, the permutations of each

$$(x^*_{\pi(3m+3,g')}(\overline{x})\widetilde{x}_{\pi(3m+3,g')})_{g'=G(e,3m+3,g-1)+1}^{G(e,3m+3,g)} \quad \text{and}$$
$$(x^*_{\pi(3m+4,g'')}(\overline{x})\widetilde{x}_{\pi(3m+4,g'')})_{g''=G(e,3m+4,g'-1)+1}^{G(e,3m+4,g')}$$

have the same constructions and properties of the corresponding permutations of the last parts of (B) and (A) of RL respectively; that is (recall for (A) of RL, from the end of the proof of Fact 2 in the third part of the proof of RL,

$$\left\|\widetilde{W}_{3m+1,p} - \frac{p}{T_{3m+1}} W_{3m+1}\right\| < \frac{\overline{a}_{3m+1}}{4 \cdot P_{3m+1} 2^{Q_{0,3m+1}}}, \quad T_{3m+1} = 2^{M_{3m+1}+2Q_{3m+1}}$$

and analogously for (B) of RL) practically

$$\Big(\sum_{g'=G(e,3m+3,g-1)+1}^{G} x^*_{\pi(3m+3,g')}(\overline{x})\widetilde{x}_{\pi(3m+3,g')}\Big)_{G=G(e,3m+3,g-1)+1}^{G(e,3m+3,g)}$$

is a progressive enlargement of a $1/2^{M_{3m+3}}$-miniature of the whole

$$\sum_{g'=G(e,3m+3,g-1)+1}^{G(e,3m+3,g)} x^*_{\pi(3m+3,g')}(\overline{x})\widetilde{x}_{\pi(3m+3,g')}$$

and analogously

$$\Big(\sum_{g''=G(e,3m+4,g'-1)+1}^{G} x^*_{\pi(3m+4,g'')}(\overline{x})\widetilde{x}_{\pi(3m+4,g'')}\Big)_{G=G(e,3m+4,g'-1)+1}^{G(e,3m+4,g')}$$

is a progressive enlargement of a $1/2^{M_{3m+4}+2Q_{3m+4}}$-miniature of the whole

$$\sum_{g''=G(e,3m+4,g'-1)+1}^{G(e,3m+4,g')} x^*_{\pi(3m+4,g'')}(\overline{x})\widetilde{x}_{\pi(3m+4,g'')}.$$

Step 2. Let us now turn to $\sum_{g=1}^{G_{3m+2}} x^*_{\pi(3m+2,g)}(\overline{x})x'''_{\pi(3m+2,g)}$. We recall, by Step 2 of SC III.3 and by the beginning of SC III.1, that

$$(u'_{3m+3,s})_{s=1}^{A_{3m+3}} = (u'_{3m+3,s})_{s=1}^{A''_{3m+3}} = (e_{3m+2,0,n})_{n=1}^{M_{3m+2,0}} \cup (u''_{3m+3,s})_{s=1}^{P''_{3m+3}},$$

where $(u''_{3m+3,s})_{s=1}^{P''_{3m+3}} \cup (v'_{3m+4,n})_{n=1}^{Q''_{3m+4}}$ comes from the procedure of C II and of Lemma 12 and in particular $(u''_{3m+3,s})_{s=1}^{Q''_{3m+3}} = (v'_{3m+3,n})_{n=1}^{Q''_{3m+3}}$ was already defined in Step 6 of SC III.2.

Therefore suppose that $M_{3m+2,0} + Q''_{3m+3} \le n'(3m+3) - 1$. By (42.4) of C II with $m+2$ replaced by $3m+4$, in particular in the definition of $(v'_{3m+3,n})_{n=1}^{Q''_{3m+3}}$, we see that $(y_n, y^*_n)_{n=1}^{Q(m)}$ is replaced by $B'_4(3m+1)$ of Step 4 of SC III.2, hence for $(y_n)_{n=1}^{Q(m)}$ replaced by $(x_n)_{n=1}^{q(3m+1)} \cup (x_n - x_{\mathrm{brd},n})_{n=q(3m+1)+1}^{q(3m+2)} \cup (u'_{3m+2,s})_{s=1}^{A''_{3m+2}}$, there exists

$$(\widetilde{u}_{u,3m+3,g})_{g=1}^{G_{3m+2}} \subset \operatorname{span}(v'_{3m+3,n})_{n=1}^{Q''_{3m+3}} \subset \operatorname{span}(u'_{3m+3,s})_{s=1}^{n'(3m+3)-1}$$

such that, for each G with $1 \le G \le G_{3m+2}$,

$$(\circ)\qquad \Big\| \sum_{g=1}^{G} (x^*_{\pi(3m+2,g)}(\overline{x}) x'''_{\pi(3m+2,g)} + \widetilde{u}_{u,3m+3,g}) \Big\| < \operatorname{dist}\Big(\sum_{g=1}^{G} x^*_{\pi(3m+2,g)}(\overline{x}) x'''_{\pi(3m+2,g)}, X \cap \bigcap_{n=1}^{q(3m+2)} x^*_{n\perp} \cap \bigcap_{s=1}^{A''_{3m+2}} u'^*_{(3m+2,s)\perp} \Big) + 1/2^{2(q(3m+1)+A''_{3m+2})+2}$$

(this relation is possible because, by the beginning of Step 1 of SC III.3,

$$\operatorname{span}(x'''_{\pi(3m+2,g)})_{g=1}^{G_{3m+2}} = \operatorname{span}(u'_{3m+2,s})_{s=1}^{A''_{3m+2}},$$

$$(u'_{3m+2,s})_{s=A''_{3m+2}+1}^{A_{3m+2}} \subset \operatorname{span}(x'_{3m+2,g}, x''_{3m+2,g})_{g=1}^{G_{3m+2}} \subset \operatorname{span}(e'''_{3m+2,n})_{n=1}^{S_{3m+2}} \subset X \cap \bigcap_{n=1}^{q(3m+2)} x^*_{n\perp} \cap \bigcap_{s=1}^{A''_{3m+2}} u'^*_{(3m+2,s)\perp} \cap \bigcap_{s=1}^{Q''_{3m+3}} v'^*_{(3m+3,s)\perp}$$

where we recall that, since $(x'''_{\pi(3m+2,g)})_{g=1}^{G_{3m+2}}$ is the permutation of

$$(x''_{\pi(3m+2,g)} + x'''_{\pi(3m+2,g)})_{g=1}^{G_{3m+2}}$$

of the sixth part of the proof of RL,

$$(*)\qquad \Big(\sum_{g=1}^{G} x^*_{\pi(3m+2,g)}(\overline{x}) x'''_{\pi(3m+2,g)} \Big)_{G=1}^{G_{3m+2}}$$

is a progressive enlargement of a $1/2^{M_{3m+2}/2}$-miniature of the whole

$$\sum_{g=1}^{G_{3m+2}} x^*_{\pi(3m+2,g)}(\overline{x}) x'''_{\pi(3m+2,g)} = \sum_{s=1}^{A''_{3m+2}} u'^*_{3m+2,s}(\overline{x}) u'_{3m+2,s}.$$

Therefore, by the procedure of Step 1, we have the existence of $(\overline{u}_{u,3m+3,g})_{g=1}^{G_{3m+2}}$ and $(\overline{u}_{u,3m+4,g})_{g=1}^{G(u,3m+3,G_{3m+2})}$ with

$$\overline{u}_{u,3m+3,g} = \sum_{g'=G(u,3m+3,g-1)+1}^{G(ue,3m+3,g)} (x^*_{\pi(3m+3,g')}(\overline{x}) x_{\pi(3m+3,g')} + \overline{u}_{u,3m+4,g'}),$$

$$\overline{u}_{u,3m+4,g'} = \sum_{g''=G(u,3m+4,g'-1)+1}^{G(u,3m+4,g')} x^*_{\pi(3m+4,g'')}(\overline{x}) x_{\pi(3m+4,g'')}$$

for $1 \le g \le G_{3m+2}$ and $G(u,3m+3,g-1)+1 \le g' \le G(u,3m+3,g)$ (where now $G(u,3m+3,0) = G(e,3m+3,G_{3m+2})$ and $G(u,3m+4,0) = G(e,3m+4,G(e,3m+3,$

$G_{3m+2})))$, such that, for $1 \le G \le G_{3m+2}$,

$$\Big\| \sum_{g=1}^{G} \Big(x^*_{\pi(3m+2,g)}(\overline{x}) x'''_{\pi(3m+2,g)} + \overline{u}_{u,3m+3,g} \Big) \Big\| < \Big\| \sum_{g=1}^{G} x^*_{\pi(3m+2,g)}(\overline{x}) x'''_{\pi(3m+2,g)} + \widetilde{u}_{u,3m+3,g} \Big\| + \eta_{3m+2}$$

for some $\eta_{3m+2} \to 0$. On the other hand, by ($\circ$) above and by the proof of (ii) of CL, we have

$$\Big\| \sum_{g=1}^{G_{3m+2}} (x^*_{\pi(3m+2,g)}(\overline{x}) x'''_{\pi(3m+2,g)} + \widetilde{u}_{u,3m+3,g}) \Big\| < \eta'_{3m+2}$$

for some $\eta'_{3m+2} \to 0$. Therefore, by the procedure of the proof of (iii) of FCRL with m replaced by $m+1$, again

$$\Big(\Big\| \sum_{g=1}^{G} (x^*_{\pi(3m+2,g)}(\overline{x}) x'''_{\pi(3m+2,g)} + \overline{u}_{u,3m+2,g}) \Big\| \Big)_{G=1}^{G_{3m+2}}$$

is $(0, \overline{a}_{3m+2}/2^{Q_{0,3m+2}})$-monotone, with

$$\Big\| \sum_{g=1}^{G} (x^*_{\pi(3m+2,g)}(\overline{x}) x'''_{\pi(3m+2,g)} + \overline{u}_{u,3m+2,g}) \Big\| < \eta_{3m+2}$$

for $1 \le G \le G_{3m+2}$ for some $\eta_{3m+2} \to 0$.

Suppose now that

$$M_{3m+2,0} + 1 \le n'(3m+3) \le M_{3m+2,0} + Q''_{3m+3}.$$

Then, since

$$\sum_{s=n'(3m+3)+1}^{M_{3m+2,0}+Q''_{3m+3}} |u'^*_{3m+3,s}(\overline{x})| < 1/2^{2q(3m+3)A_{3m+3}},$$

for the effect of $\operatorname{span}(v'_{3m+3,n})_{n=1}^{n'(3m+3)-1-M_{3m+2,0}}$, where

$$(v'_{3m+3,n})_{n=1}^{n'(3m+3)-1-M_{3m+2,0}} = (u'_{3m+3,s})_{s=M_{3m+2,0}+1}^{n'(3m+3)-1},$$

we can proceed as above, but now we have to consider also the effect of

$$u'^*_{3m+3,n'(3m+3)}(\overline{x}) u'_{3m+3,n'(3m+3)}.$$

Therefore by a variation of ($\circ$) we have also

$$\inf \Big(\operatorname{dist} \Big(\sum_{g=1}^{G_{3m+2}} x^*_{\pi(3m+2,g)}(\overline{x}) x'''_{\pi(3m+2,g)} + a u'^*_{3m+3,n'(3m+3)}(\overline{x}) u'_{3m+3,n'(3m+3)}, \operatorname{span}(u'_{3m+3,s})_{s=M_{3m+2,0}+1}^{n'(3m+3)-1} \Big) : 0 \le |a| \le 1 \Big)$$

$$< \operatorname{dist}\Big(\sum_{g=1}^{G_{3m+2}} x^*_{\pi(3m+2,g)}(\overline{x})x'''_{\pi(3m+2,g)}, X\cap \bigcap_{n=1}^{q(3m+2)} x^*_{n\perp}\cap \bigcap_{s=1}^{A''_{3m+2}} u'^*_{(3m+2,s)\perp}\Big)$$
$$+\frac{1}{2^{2(q(3m+2)+A''_{3m+2})+2}}+\frac{1}{2^{2q(3m+3)A_{3m+3}}}.$$

Indeed, by the definition of $(x'''_{3m+3,g})_{g=1}^{G_{3m+3}}$ in Step 6 of SC III.1 and by the fifth part of the proof of RL (where we replace $(\widetilde{x}_{\pi(3m+3,g)})_{g=1}^{G_{3m+3}}$ by $(x'''_{\pi(3m+3,g)})_{g=1}^{G_{3m+3}}$),

$$(**)\qquad \Big(\sum_{g=1}^{G} x^*_{\pi(3m+3,g)}(\overline{x})x'''_{\pi(3m+3,g)}\Big)_{G=1}^{G_{3m+3}}$$

is a progressive enlargement of a $1/2^{M_{3m+2}}$-miniature of the whole

$$\sum_{g=1}^{G_{3m+3}} x^*_{\pi(3m+3,g)}(\overline{x})x'''_{\pi(3m+3,g)}=\sum_{s=1}^{A_{3m+3}} u'^*_{3m+3,s}(\overline{x})u'_{3m+3,s},$$

that is, $u'_{3m+3,n'(3m+3)}$ can actually act on $\sum_{g=1}^{G_{3m+2}} x^*_{\pi(3m+2,g)}(\overline{x})x'''_{\pi(3m+2,g)}$ only by means of a portion of $u'^*_{3m+3,n'(3m+3)}(\overline{x})u'_{3m+3,n'(3m+3)}$. Therefore again there exist $(\widetilde{u}_{u,3m+3,g})_{g=1}^{G_{3m+2}}\subset \operatorname{span}(u'_{3m+3,s})_{s=M_{3m+2,0}+1}^{n'(3m+3)-1}$ and a sequence $(\overline{a}_g)_{g=1}^{G_{3m+2}}$ of numbers with $|\overline{a}_g|\le 1$ for $1\le g\le G_{3m+2}$ such that, for $1\le G\le G_{3m+2}$,

$$\Big\|\sum_{g=1}^{G}(x^*_{\pi(3m+2,g)}(\overline{x})x'''_{\pi(3m+2,g)}+\widetilde{u}_{u,3m+3,g})+\overline{a}_G u'^*_{3m+3,n'(3m+3)}(\overline{x})u'_{3m+3,n'(3m+3)}\Big\|$$
$$< \operatorname{dist}\Big(\sum_{g=1}^{G} x^*_{\pi(3m+2,g)}(\overline{x})x'''_{\pi(3m+2,g)}, X\cap \bigcap_{n=1}^{q(3m+2)} x^*_{n\perp}\cap \bigcap_{s=1}^{A''_{3m+2}} u'^*_{(3m+2,s)\perp}\Big)$$
$$+1/2^{2(q(3m+2)+A''_{3m+2})+2}+1/2^{2q(3m+3)A_{3m+3}}$$

(since by the above $\sum_{s=n'(3m+3)+1}^{A_{3m+3}}|u'^*_{3m+3,s}(\overline{x})|<1/2^{2q(3m+3)A_{3m+3}}$).

At this point, we can use the property (iii) of CL and hence, by the elements of $\operatorname{span}(u_{3m+3,n})_{n=1}^{n(3m+3)-1}$, we can really approximate

$$(\widetilde{u}_{u,3m+3,g})_{g=1}^{G_{3m+2}}\cup(\overline{a}_g u'^*_{3m+3,n'(3m+3)}(\overline{x})u'_{3m+3,n'(3m+3)})_{g=1}^{G_{3m+2}}$$

(precisely, for each g with $1\le g\le G_{3m+2}$, we have to approximate the element

$$\widetilde{u}_{u,3m+3,g}+(\overline{a}_g-\overline{a}_{g-1})u'^*_{3m+3,n'(3m+3)}(\overline{x})u'_{3m+3,n'(3m+3)}$$

and on the other hand $(*)$ holds, hence actually each number of $(\overline{a}_g)_{g=1}^{G_{3m+2}}$ concerns a progressive enlargement of a $1/2^{M_{3m+2}/2}$-miniature of the whole

$$\sum_{g=1}^{G_{3m+2}} x^*_{\pi(3m+2,g)}(\overline{x})x'''_{\pi(3m+2,g)},$$

that is, these numbers have the same sign and $(|\overline{a}_g|)_{g=1}^{G_{3m+2}}$ is not decreasing with

$$\Big|\sum_{g=1}^{G_{3m+2}}(\overline{a}_g-\overline{a}_{g-1})\Big|=\sum_{g=1}^{G_{3m+2}}|\overline{a}_g-\overline{a}_{g-1}|=|\overline{a}_{G_{3m+2}}|\le 1),$$

therefore the same procedure for $M_{3m+2,0}+Q''_{3m+3}\le n'(3m+3)-1$ works, with the same conclusions.

STEP 3. In order to have the assertion, it is now sufficient to set $\overline{u}_{3m+3,g}=\overline{u}_{e,3m+3,g}+\overline{u}_{u,3m+3,g}$ for $1\le g\le G_{3m+2}$. ■

SECOND PART. Suppose now that, for the block $\sum_{n=q(3m+3)+1}^{q(3m+6)}x_n^*(\overline{x})x_n$, the disconnected chain condition holds; hence $\sum_{s=1}^{A_{3m+3}}|u'^{*}_{3m+3,s}(\overline{x})|<1/2^{2q(3m+3)A_{3m+3}}$. By SC III.1 with m replaced by $m+1$ and in particular by

$$(\circ\circ)\qquad \max(\|y\|,\|e\|/2),\quad y\in\operatorname{span}(y_{3m,n})_{n=1}^{Q(3m)},\quad e\in\operatorname{span}(\widehat{e}_{3m,n})_{n=1}^{S'_{3m}+M'_{3m,0}},$$

of Step 1 and by Steps 5 and 6; by the definition of $(u'_{3m+3,s},u'^{*}_{3m+3,s})_{s=1}^{A''_{3m+3}}$ in Step 2 of SC III.3; by Steps 3 and 4 of SC III.2 and by (42.2) and (42.3) of C II, we know that

$$\operatorname{dist}\Big(\sum_{n=1}^{q(3m+3)}x_n^*(\overline{x})x_n,\operatorname{span}(x_n-x'''_n)_{n=q(3m+3)+1}^{q(3m+4)}+\operatorname{span}(x_n)_{n\ge q(3m+4)+1}\Big)$$
$$>\Big\|\sum_{n=1}^{q(3m+3)}x_n^*(\overline{x})x_n\Big\|-\frac{1}{2^{2q(3m+3)+2}},$$

hence

$$\operatorname{dist}\Big(\sum_{n=q(3m+2)+1}^{q(3m+3)}x_n^*(\overline{x})x_n,S+\operatorname{span}((x_n-x'''_n)_{n=q(3m+3)+1}^{q(3m+4)}\cup\operatorname{span}(x_n)_{n\ge q(3m+4)+1})\Big)$$
$$>\Big\|\sum_{n=1}^{q(3m+3)}x_n^*(\overline{x})x_n\Big\|-(1/2^{2q(3m+3)+2}+1/2^{2q(3m+3)A_{3m+3}}),$$

where

$$S=\text{partial sums of }\sum_{n=q(3m+3)+1}^{q(3m+4)}x^*_{\pi(n)}(\overline{x})x'''_{\pi(n)}$$

(always by $(**)$ above in Step 2 of the first part of the proof). Therefore it also follows, by $(\circ\circ)$ above and by the proof of (ii) of CL, that

$$\Big\|\sum_{n=q(3m+2)+1}^{q(3m+3)}x_n^*(\overline{x})(x_n-x'''_n)\Big\|+\Big\|\sum_{n=q(3m+2)+1}^{q(3m+3)}x_n^*(\overline{x})x'''_n\Big\|\to 0\qquad\text{as } m\to\infty.$$

On the other hand, by the sixth part of the proof of RL, since by $(*)$ above in Step 2 of the first part of the proof, $(\sum_{g=1}^{G}x^*_{\pi(3m+2,g)}(\overline{x})x'''_{\pi(3m+2,g)})_{G=1}^{G_{3m+2}}$ and

$$\Big(\sum_{g=1}^{G}x^*_{\pi(3m+2,g)}(\overline{x})x''_{\pi(3m+2,g)}\Big)_{G=1}^{G_{3m+2}}$$

are progressive enlargements of the $1/2^{M_{3m+2}/2}$-miniatures of the whole

$$\sum_{g=1}^{G_{3m+2}} x^*_{3m+2,g}(\overline{x})x'''_{3m+2,g}$$

and $\sum_{g=1}^{G_{3m+2}} x^*_{3m+2,g}(\overline{x})x''_{3m+2,g}$ respectively. Since by Step 1 of SC III.3,

$$\Big\|\sum_{n=q(3m+2)+1}^{q(3m+3)} x^*_n(\overline{x})(x_n - x'''_n)\Big\| = \Big\|\sum_{n=q(3m+2)+1}^{q(3m+3)} x^*_n(\overline{x})x'_n\Big\| + \Big\|\sum_{n=q(3m+2)+1}^{q(3m+3)} x^*_n(\overline{x})x''_n\Big\|,$$

since finally by the sixth part of the proof of RL and by Step 1 of SC III.3,

$$\begin{aligned}\Big\|\sum_{n=q(3m+2)+1}^{q(3m+3)} x^*_n(\overline{x})x'_n\Big\| &= \Big\|x^*_{3m+2,n,0}(\overline{x})x_{3m+2,n,0} + \sum_{k=1}^{2^{M_{3m+2}}} x^*_{3m+2,n,k}(\overline{x})x'_{3m+2,n,k}\Big\| \\ &= \sum_{k=1}^{2^{M_{3m+2}}} |x^*_{3m+2,n,0}(\overline{x}) + x^*_{3m+2,n,k}(\overline{x})| = \sum_{k=1}^{2^{M_{3m+2}}} |e^*_{3m+2,n,k}(\overline{x})|\end{aligned}$$

and hence also $(\|\sum_{g=1}^{G} x^*_{\pi(3m+2,g)}(\overline{x})x'_{\pi(3m+2,g)}\|)_{G=1}^{G_{3m+2}}$ is $(1,0)$-monotone, we conclude that the assertion directly follows by setting $\overline{u}_{3m+2,g} = 0$ for $1 \le g \le G_{3m+2}$.

Proof of (B) and (C). The procedures and ideas of the proof of (A) work for (B), where now we point out that the construction of the sequence $(\overline{u}_{3m+3,3m+4,g})_{g=1}^{G(3m+2,q(3m+2))}$ is of the same kind of the construction of the sequence $(\overline{u}_{3m+3,g})_{g=1}^{G_{3m+2}}$ in the proof of (A); analogously for (C) the construction of the sequence

$$(\overline{u}_{3m+2,3m+3,3m+4,g})_{g=1}^{G(3m+2,3m+2,q(3m+2))}$$

is of the same kind of the construction of the sequence $(\overline{u}_{3m+2,s})_{g=1}^{G_{3m+2}}$ in the proof of (B). Moreover we point out that, in contrast to the proof of (i) of FCRL, we now have to use MGBS. This completes the proof of Lemma 20. ■

THEOREM 21. *For every separable Banach space X where l_1 is finitely represented, there exists a biorthogonal system $(x_n, x^*_n)_{n=1}^\infty = ((x_n, x^*_n)_{n=q(3m)+1}^{q(3m+3)})_{m=1}^\infty$ $(q(0) = 0)$ with $\|x_n\| < 10$ and $\|x^*_n\| < 7$ for each n, such that, for each $\overline{x} \in X$ with $\|\overline{x}\| = 1$, there exist sequences $(\overline{q}(m))_{m=1}^\infty$ of increasing positive integers (with $\overline{q}(0) = 0$) and $(\overline{\eta}_m)_{m=1}^\infty$ non-increasing with $\overline{\eta}_m \to 0$, and for each m a partition $(x_n)_{n=q(3m)+1}^{q(3m+2)} = (x_{n'})_{n=q'(3m)+1}^{q'(3m+2)} \cup (x_{n''})_{n=q''(3m)+1}^{q''(3m+2)}$ and a permutation $(x_{\overline{\pi}(n)})_{n=\overline{q}(m-1)+1}^{\overline{q}(m)}$ of*

$$(x_{n''})_{n=q''(3m-3)+1}^{q''(3m-1)} \cup (x_n)_{n=q(3m-1)+1}^{q(3m)} \cup (x_{n'})_{n=q'(3m)+1}^{q'(3m+2)},$$

such that

$$\Big\|\overline{x} - \sum_{n=1}^{\overline{q}(m)} x^*_{\overline{\pi}(n)}(\overline{x})x_{\overline{\pi}(n)}\Big\| < \overline{\eta}_m, \qquad \Big\|\sum_{n=\overline{q}(m-1)+1}^{q} x^*_{\overline{\pi}(n)}(\overline{x})x_{\overline{\pi}(n)}\Big\| < \overline{\eta}_m$$

for $\overline{q}(m-1) + 1 \le q \le \overline{q}(m)$; hence $\overline{x} = \sum_{n=1}^\infty x^*_{\overline{\pi}(n)}(\overline{x})x_{\overline{\pi}(n)}$.

Indeed, for $\overline{q}(m-1)+1 \le q \le \overline{q}(m)$,

$$\Big\|\overline{x} - \sum_{n=1}^{q} x^*_{\overline{\pi}(n)}(\overline{x})x_{\overline{\pi}(n)}\Big\| \le \Big\|\overline{x} - \sum_{n=1}^{\overline{q}(m-1)} x^*_{\overline{\pi}(n)}(\overline{x})x_{\overline{\pi}(n)}\Big\| + \Big\|\sum_{n=\overline{q}(m-1)+1}^{q} x^*_{\overline{\pi}(n)}(\overline{x})x_{\overline{\pi}(n)}\Big\| < \overline{\eta}_{m-1} + \overline{\eta}_m.$$

We point out that, in the regularization procedure, only a little portion of the elements of each sub-block can be consumed in the regularization of the previous sub-blocks, by the first relation of (9) of GBST.

Proof. We will proceed by induction, that is, we suppose to have constructed $(x_{\overline{\pi}(n)})_{n=1}^{\overline{q}(m)}$ and we are going to construct $(x_{\overline{\pi}(n)})_{n=\overline{q}(m)+1}^{\overline{q}(m+1)}$; moreover in this construction we can also suppose to be in the particular case of $\overline{q}(m) = q(3m)$, that is, $(x_{n''})_{n=q''(3m)+1}^{q''(3m+2)} = (x_n)_{n=q(3m)+1}^{q(3m+2)}$, since the procedure of the construction in the general case is the same. We will proceed through four steps.

First step. Let us consider the first sub-block $\sum_{n=q(3m)+1}^{q(3m+1)} x_n^*(\overline{x})x_n$. Suppose that (see Step 6 of SC III.1 and LCL)

$$n'(3m+1) \le G_{0,0,0,3m} - G_{3m} = M_{3m,0} + G_{0,0,3m}.$$

Then, by the definition of $(u'_{3m+1,s})_{s=A_{3m+1}/2+1}^{A_{3m+1}/2+M_{3m,\mathrm{arm},0}}$ and by Step 3 of SC III.1 and in particular by the relation $\|x+e\| \ge \|e\|/2$ for each $x \in X' + U_{3m,\mathrm{arm}}$ and

$$e \in \operatorname{span}\big((e_{3m,\mathrm{arm},n,j})_{j=1}^{J_{3m,\mathrm{arm}}}\big)_{n=1}^{P_{3m}} \cup (e_{3m,\mathrm{arm},0,n})_{n=1}^{M'_{3m,\mathrm{arm},0}}\big),$$

we see that, for each positive integer p,

$$\Big\|\sum_{n=1}^{q(3m+p)} x_n^*(\overline{x})x_n\Big\| > \Big\|\sum_{n=q(3m)+1}^{q(3m+1)} x_n^*(\overline{x})x_{\mathrm{arm},n}\Big\| - \frac{1}{2^{2q(3m+1)A_{3m+1}}}$$

with

$$\Big\|\sum_{n=q(3m)+1}^{q(3m+1)} x_n^*(\overline{x})x_{\mathrm{arm},n}\Big\| = \sum_{n=q(3m)+1}^{q(3m+1)} |x_n^*(\overline{x})| > \frac{1}{10}\Big\|\sum_{n=q(3m)+1}^{q(3m+1)} x_n^*(\overline{x})x_n\Big\|.$$

Hence the first sub-block is automatically regularized with, by the proof of (ii) of CL, $\|\sum_{n=q(3m)+1}^{q} x_n^*(\overline{x})x_n\| < \eta_m \to 0$ for $q(3m)+1 \le q \le q(3m+1)$ and for each permutation $(\pi(3m,g))_{g=1}^{G_{3m}}$ of $((3m,g))_{g=1}^{G_{3m}}$.

Suppose now that $G_{0,0,0,3m} < n'(3m+1)$. We turn to $\sum_{n=q(3m)+1}^{q(3m+1)} x_n^*(\overline{x})(x_n - x'''_n)$ and we recall that

$$\Big\|\sum_{n=q(3m)+1}^{q(3m+1)} x_n^*(\overline{x})(x_n - x'''_n) + \operatorname{span}(u'_{3m+1,s})_{s=1}^{G_{0,0,0,3m}}\Big\|$$
$$= \Big\|\sum_{n=q(3m)+1}^{q(3m+1)} x_n^*(\overline{x})(x'_n + x''_n + x_{\mathrm{brd},n}) + \operatorname{span}(u'_{3m+1,s})_{s=1}^{G_{0,0,0,3m}-G_{3m}}\Big\|$$

(where $G_{0,0,0,3m} - G_{3m} = M_{3m,0} + G_{0,0,3m}$). Moreover, by the notations before CL, we also have

$$\Big\| \sum_{n=q(3m)+1}^{q(3m+1)} x_n^*(\overline{x})(x'_n + x''_n + x_{\mathrm{brd},n}) + \mathrm{span}(u'_{3m+1,s})_{s=1}^{M_{3m,0}+G_{0,0,3m}} + \mathrm{span}(u'_{3m+1,s})_{s=A_{3m+1}-A'_{3m+1}+1}^{n'(3m+1)} \Big\| = \Big\| \sum_{n=q(3m)+1}^{q(3m+1)} x_n^*(\overline{x})(x'_n + x''_n + x_{\mathrm{brd},p,n}) + \mathrm{span}(u'_{3m+1,s})_{s=1}^{M_{3m,0}+G_{0,0,3m}} \Big\|$$

(if $A_{3m+1} - A'_{3m+1} + 1 \leq n'(3m+1) \leq A_{3m+1}$; otherwise, if $n'(3m+1) \leq A_{3m+1} - A'_{3m+1}$, we directly have $(x_{\mathrm{brd},p,n})_{s=q(3m)+1}^{q(3m+1)} = (x_{\mathrm{brd},n})_{s=q(3m)+1}^{q(3m+1)}$). Therefore there exists $(\widetilde{u}_{3m+1,g})_{g=1}^{G_{3m}}$ with, for $1 \leq g \leq G_{3m}$,

$$\begin{aligned}\widetilde{u}_{3m+1,g} = \widetilde{u}_{e,3m+1,g} &\ (\in \mathrm{span}(u'_{3m+1,s})_{s=1}^{M_{3m,0}+G_{0,0,3m}}) \\ + \widetilde{u}_{\mathrm{arm},3m+1,g} &\ (\in \mathrm{span}(u'_{3m+1,s})_{s=M_{3m,0}+G_{0,0,3m}+1}^{G_{0,0,0,3m}}) \\ + \widetilde{u}_{\mathrm{brd},3m+1,g} &\ (\in \mathrm{span}(u'_{3m+1,s})_{s=A_{3m+1}-A'_{3m+1}+1}^{n'(3m+1)}),\end{aligned}$$

such that, for $q(3m)+1 \leq G \leq q(3m+1)$ and for some $\eta'_m \to 0$,

$$\Big\| \sum_{n=q(3m)+1}^{q(3m+1)} (x_n^*(\overline{x})(x_n - x'''_n) + \widetilde{u}_{\mathrm{arm},3m+1,g} + \widetilde{u}_{\mathrm{brd},3m+1,g}) + \mathrm{span}(u'_{3m+1,s})_{s=1}^{G_{0,0,0,3m}} + \mathrm{span}(u'_{3m+1,s})_{s=A_{3m+1}-A'_{3m+1}+1}^{n'(3m+1)} \Big\| = \Big\| \sum_{n=q(3m)+1}^{q(3m+1)} (x_n^*(\overline{x})(x'_n + x''_n + x_{\mathrm{brd},p,n}) + \mathrm{span}(u'_{3m+1,s})_{s=1}^{M_{3m,0}+G_{0,0,3m}} \Big\|,$$

$$\Big\| \sum_{n=q(3m)+1}^{G} (x_n^*(\overline{x})(x'_n + x''_n + x_{\mathrm{brd},p,n}) + \widetilde{u}_{e,3m+1,n}) \Big\| < \Big\| \sum_{n=q(3m)+1}^{G} x_n^*(\overline{x})(x'_n + x''_n + x_{\mathrm{brd},p,n}) + \mathrm{span}(u'_{3m+1,s})_{s=1}^{M_{3m,0}+G_{0,0,3m}} \Big\| + \eta'_m.$$

On the other hand (see Step 1 of SC III.1),

$$\Big\| \sum_{n=q(3m)+1}^{q(3m+1)} x_n^*(\overline{x})(x'_n + x''_n + x_{\mathrm{brd},p,n}) + \mathrm{span}(u'_{3m+1,s})_{s=1}^{M_{3m,0}+G_{0,0,3m}} \Big\| \leq \Big\| \sum_{g=1}^{G_{0,3m}} x_{0,3m,g}^*(\overline{x})(x'_{0,3m,g} + x''_{0,3m,g} + x_{0,3m,\mathrm{brd},p,g}) + E_{3m,0} \Big\|.$$

Therefore, always by (ii) of CL, if we use the permutation $(\pi(3m,g))_{g=1}^{G_{3m}}$ of $((3m,g))_{g=1}^{G_{3m}}$ of Fact 3 of the fifth part of the proof of RL, there exists another sequence of the kind of $(\widetilde{u}_{3m+1,g})_{g=1}^{G_{3m}}$ of above, which we call again $(\widetilde{u}_{3m+1,g})_{g=1}^{G_{3m}}$, such that

$$\Big\|\sum_{g=1}^{G}(x^*_{\pi(3m,g)}(\overline{x})(x'_{\pi(3m,g)}+x''_{\pi(3m,g)}+x_{\pi(3m,\mathrm{brd},p,g)})+\widetilde{u}_{3m+1,g})\Big\|<\eta'_m$$

for $1\le G\le G_{3m}$ (hence in particular

$$\Big(\Big\|\sum_{g=1}^{G}(x^*_{\pi(3m,g)}(\overline{x})(x'_{\pi(3m,g)}+x''_{\pi(3m,g)}+x_{\pi(3m,\mathrm{brd},p,g)})+\widetilde{u}_{3m+1,g})\Big\|\Big)_{G=1}^{G_{3m}}$$

is $(0,2\eta'_m)$-monotone).

By the notations before CL we know that $(\widetilde{u}_{3m+1,\mathrm{brd},g})_{g=1}^{G_{3m}}$ is in

$$\mathrm{span}(u'_{3m+1,s})_{s=A_{3m+1}-A'_{3m+1}+1}^{n'(3m+1)},$$

and $x_{\pi(3m,\mathrm{brd},g)}=x_{\pi(3m,\mathrm{brd},a,g)}+x_{\pi(3m,\mathrm{brd},p,g)}$ with $x_{\pi(3m,\mathrm{brd},a,g)}$ in

$$(u'_{3m+1,s})_{s=A_{3m+1}-A'_{3m+1}+1}^{n'(3m+1)}$$

and $x_{\pi(3m,\mathrm{brd},p,g)}$ in $(u'_{3m+1,s})_{s=n'(3m+1)+1}^{A_{3m+1}}$ for $1\le g\le G_{3m}$, hence (see Steps 5 and 6 of SC III.1) there is $\overline{g}$ with $1\le\overline{g}\le G_{3m}$ such that $x_{\pi(3m,\mathrm{brd},a,\overline{g})}=u'_{3m+1,n'(3m+1)}$. On the other hand, by the same procedure for the case $n'(3m+3)\le M_{3m+2,0}$ in Step 1 of the first part of the proof SCRL, the contribution of this term is $x^*_{\pi(3m,\overline{g})}(\overline{x})x_{\pi(3m,\mathrm{brd},a,\overline{g})}$ where $|x^*_{\pi(3m,\overline{g})}(\overline{x})|\le\overline{a}_{3m}\to 0$. Therefore, in view of our aim, we can disregard this term and hence we can suppose in what follows that $(\widetilde{u}_{3m+1,\mathrm{brd},g})_{g=1}^{G_{3m}}$ is in

$$\mathrm{span}(u'_{3m+1,s})_{s=A_{3m+1}-A'_{3m+1}+1}^{n'(3m+1)-1},$$

therefore

$$(\widetilde{u}_{3m+1,g})_{g=1}^{G_{3m}}\subset\mathrm{span}(u'_{3m+1,s})_{s=A_{3m+1}-A'_{3m+1}+1}^{n'(3m+1)-1}.$$

At this point we can use (C) of SCRL, in the particular case of absence of the sequence $(a_s)_{s=1}^{q(3m+1)}$ of numbers. So we get $(\overline{u}'_{3m+1,g})_{g=1}^{G_{3m}}$, with the same properties of construction of the sequence $(\overline{u}_{3m+1,s})_{s=1}^{q(3m+1)}$ of (C) of SCRL, such that, for $1\le G\le G_{3m}$ and for some $\eta''_m\to 0$,

$$\Big\|\sum_{g=1}^{G}(\overline{u}'_{3m+1,g}-\widetilde{u}_{3m+1,g})\Big\|<\eta''_m.$$

If $G_{0,0,0,3m}-G_{3m}+1\le n'(3m+1)\le G_{0,0,0,3m}$, there is again some $\overline{g}$ with $1\le\overline{g}\le G_{3m}$ such that for $\sum_{g=1}^{\overline{g}-1}x^*_{\pi(3m,g)}(\overline{x})(x_{\pi(3m,g)}-x'''_{\pi(3m,g)})$ we are in the case of $G_{0,0,0,3m}<n'(3m+1)$ and for $\sum_{g=\overline{g}+1}^{G_{3m}}x^*_{\pi(3m,g)}(\overline{x})(x_{\pi(3m,g)}-x'''_{\pi(3m,g)})$ in the case of $n'(3m+1)\le M_{3m,0}+G_{0,0,3m}$, while, analogously to the above for $x^*_{\pi(3m,\overline{g})}(\overline{x})x_{\pi(3m,\mathrm{brd},a,\overline{g})}$, we can disregard the term $x^*_{\pi(3m,\overline{g})}(\overline{x})(x_{\pi(3m,\overline{g})}-x'''_{\pi(3m,\overline{g})})$ since $|x^*_{\pi(3m,\overline{g})}(\overline{x})|\le\overline{a}_{3m}\to 0$.

Second step. Now we turn to considering

$$\sum_{n=q(3m)+1}^{q(3m+1)}x^*_n(\overline{x})x'''_n=\sum_{g=1}^{G_{3m}}x^*_{3m,g}(\overline{x})x'''_{3m,g}.$$

Then the procedure of Step 2 of the first part of the proof of SCRL works, hence also the assertion of (C) of SCRL works, hence there exists $(\overline{u}''_{3m+1,g})_{g=1}^{G_{3m}}$, again with the same properties of construction of the sequence $(\overline{u}_{3m+1,s})_{s=1}^{q(3m+1)}$ of (C) of SCRL, such that, setting $\overline{u}_{3m+1,g} = \overline{u}'_{3m+1,g} + \overline{u}''_{3m+1,g}$ for $1 \leq g \leq G_{3m}$, there exists a subsequence $(x_{n'})_{n=q'(3m+3)+1}^{q'(3m+5,1)}$ of $(x_n)_{n=q(3m+3)+1}^{q(3m+5)}$, a subsequence $(\widetilde{x}'_n)_{n=\widetilde{q}(3m+1)+1}^{\widetilde{q}(3m+2)}$ of $(x_n)_{n=q(3m+1)+1}^{q(3m+2)}$ and a subsequence $(\widetilde{x}'_n)_{n=\widetilde{q}(3m+2)+1}^{\widetilde{q}(3m+3)}$ of $(x_n)_{n=q(3m+2)+1}^{q(3m+3)}$ such that there exist $\overline{\eta}_{m,1} \to 0$ and a permutation $(x_{\overline{\pi}(n)})_{n=\overline{q}(m)+1}^{\overline{q}(m+1,1)}$ of

$$(x_n)_{n=q(3m)+1}^{q(3m+1)} \cup (\widetilde{x}'_n)_{n=\widetilde{q}(3m+1)+1}^{\widetilde{q}(3m+2)} \cup (\widetilde{x}'_n)_{n=\widetilde{q}(3m+2)+1}^{\widetilde{q}(3m+3)} \cup (x_{n'})_{n=q'(3m+3)+1}^{q'(3m+5,1)},$$

with

$$\sum_{g=1}^{G_{3m}} (x^*_{\pi(3m,g)}(\overline{x}) x_{\pi(3m,g)} + \overline{u}_{3m+1,g}) = \sum_{n=\overline{q}(m)+1}^{\overline{q}(m+1,1)} x^*_{\overline{\pi}(n)}(\overline{x}) x_{\overline{\pi}(n)},$$

$$\Big\| \sum_{n=\overline{q}(m)+1}^{q} x^*_{\overline{\pi}(n)}(\overline{x}) x_{\overline{\pi}(n)} \Big\| < \overline{\eta}_{m,1} \quad \text{for } \overline{q}(m)+1 \leq q \leq \overline{q}(m+1,1).$$

Third step. Now we turn to considering the subsum of $\sum_{n=q(3m+1)+1}^{q(3m+2)} x^*_n(\overline{x}) x_n$ complementary to $\sum_{n=\widetilde{q}(3m+1)+1}^{\widetilde{q}(3m+2)} \widetilde{x}'^*_n(\overline{x}) \widetilde{x}'_n$. Then, since the procedure in the general case is the same, we consider only the particular case when $(\widetilde{x}'_n)_{n=\widetilde{q}(3m+1)+1}^{\widetilde{q}(3m+2)}$ and $(\widetilde{x}'_n)_{n=\widetilde{q}(3m+2)+1}^{\widetilde{q}(3m+3)}$ do not appear; that is, we consider the whole sub-block $\sum_{n=q(3m+1)+1}^{q(3m+2)} x^*_n(\overline{x}) x_n$ and we suppose that also the sub-block $\sum_{n=q(3m+2)+1}^{q(3m+3)} x^*_n(\overline{x}) x_n$ is still intact. Then the whole procedure of the proof of (A) of SCRL works. We only point out the following fact, which is important for the regularization when we do not have at our disposal all the elements of $(e_{3m+1,0,n})_{n=1}^{M_{3m+1,0}}$, that is (see Step 4 of SC III.2), when $n'(3m+2) < M_{3m+1,0}$. Precisely, in order to decrease the formalism, suppose that $n'(3m+2) = 0$ and consider $\sum_{n=q(3m+1)+1}^{q(3m+2)} x^*_n(\overline{x})(x_n - \widetilde{x}_n)$ (we did not consider this fact in the first step because it was not necessary, just owing to the presence of the armouring sequence). Then, by the definitions of Steps 1 and 2 of SC III.2, for $1 \leq n \leq P_{3m+1}$ we have

$$\begin{aligned}
&\Big\| \sum_{k=1}^{2^{M_{3m+1}}} \sum_{l=1}^{2^{4Q_{3m+1}}} x^*_{3m+1,n,k,l}(\overline{x})(x_{3m+1,n,k,l} - \widetilde{x}_{3m+1,n,k,l}) \Big\| \\
&\quad = \Big\| \sum_{k=1}^{2^{M_{3m+1}}} \sum_{l=1}^{2^{4Q_{3m+1}}} x^*_{3m+1,n,k,l}(\overline{x})(x'_{3m+1,n,k,l} + x_{3m+1,\mathrm{brd},n,k,l}) \Big\| \\
&\quad = \Big\| \sum_{k=1}^{2^{M_{3m+1}}} \Big(\sum_{l=1}^{2^{4Q_{3m+1}}-1} x^*_{3m+1,n,k,l}(\overline{x}) \Big(e_{3m+1,n,k,l} + \frac{e_{3m+1,n,k}}{2^{P_{3m+1}Q_{0,3m+1}} 2^{2Q_{3m+1}}} \Big) \\
&\qquad + x^*_{3m+1,n,k,2^{4Q_{3m+1}}}(\overline{x}) \Big(e_{3m+1,n,k,2^{4Q_{3m+1}}} + \omega'_{3m+1,n} \\
&\qquad + \frac{e_{3m+1,n,k}}{2^{P_{3m+1}Q_{0,3m+1}} 2^{2Q_{3m+1}}} + e_{3m+1,\mathrm{brd},n,k} + \omega_{3m+1,n} \Big) \Big) \Big\| \\
&\quad = \Big\| \sum_{k=1}^{2^{M_{3m+1}}} \Big(\sum_{l=1}^{2^{4Q_{3m+1}}} \frac{x^*_{3m+1,n,k,l}(\overline{x})}{2^{P_{3m+1}Q_{0,3m+1}} 2^{2Q_{3m+1}}} \Big) e_{3m+1,n,k}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{2^{M_{3m+1}}}\sum_{l=1}^{2^{4Q_{3m+1}}-1} x^*_{3m+1,n,k,l}(\overline{x})e_{3m+1,n,k,l}\\
&+\sum_{k=1}^{2^{M_{3m+1}}} x^*_{3m+1,n,k,2^{4Q_{3m+1}}}(\overline{x})e_{3m+1,n,k,2^{4Q_{3m+1}}}\\
&+\sum_{k=1}^{2^{M_{3m+1}}} x^*_{3m+1,n,k,2^{4Q_{3m+1}}}(\overline{x})e_{3m+1,\mathrm{brd},n,k}\\
&+\Big(\sum_{k=1}^{2^{M_{3m+1}}} x^*_{3m+1,n,k,2^{4Q_{3m+1}}}(\overline{x})\Big)\omega'_{3m+1,n}\\
&+\Big(\sum_{k=1}^{2^{M_{3m+1}}} x^*_{3m+1,n,k,2^{4Q_{3m+1}}}(\overline{x})\Big)\omega_{3m+1,n}\Big\|.
\end{aligned}
$$

Now, in order to define a good permutation, we could use a combination of Lemma 9 and of NPL; but, in order to avoid further pages of proof, it is convenient to study the definition of this permutation in the case of the alternative construction of Step 4′ of SC III.2 (since the same proofs work also for this second alternative, with some more formalism for the proof of (A) of RL). Indeed, in this case, since

$$
\begin{aligned}
\sum_{l=1}^{2^{4Q_{3m+1}}-1} x^*_{3m+1,n,k,l}(\overline{x})e_{3m+1,n,k,l} &= x^*_{3m+1,n,k,1}(\overline{x})e'_{3m+1,n,k,1}\\
&\quad+\sum_{l=2}^{2^{4Q_{3m+1}}-1} x^*_{3m+1,n,k,l}(\overline{x})(e'_{3m+1,n,k,l}-e'_{3m+1,n,k,l-1})\\
&=\sum_{l=1}^{2^{4Q_{3m+1}}-2}(x^*_{3m+1,n,k,l}(\overline{x})-x^*_{3m+1,n,k,l+1}(\overline{x}))e'_{3m+1,n,k,l}\\
&\quad+x^*_{3m+1,n,k,2^{4Q_{3m+1}}-1}(\overline{x})e'_{3m+1,n,k,2^{4Q_{3m+1}}-1}
\end{aligned}
$$

for $1\le k\le 2^{M_{3m+1}}$, we have

$$
\begin{aligned}
&\Big\|\sum_{k=1}^{2^{M_{3m+1}}}\sum_{l=1}^{2^{4Q_{3m+1}}} x^*_{3m+1,n,k,l}(\overline{x})(x_{3m+1,n,k,l}-\widetilde{x}_{3m+1,n,k,l})\Big\|\\
&=\sum_{k=1}^{2^{M_{3m+1}}}\Big|\sum_{l=1}^{2^{4Q_{3m+1}}}\frac{x^*_{3m+1,n,k,l}(\overline{x})}{2^{P_{3m+1}Q_{0,3m+1}}2^{2Q_{3m+1}}}\Big|\\
&\quad+\sum_{k=1}^{2^{M_{3m+1}}}\Big(\sum_{l=1}^{2^{4Q_{3m+1}}-2}|x^*_{3m+1,n,k,l}(\overline{x})-x^*_{3m+1,n,k,l+1}(\overline{x})|+|x^*_{3m+1,n,k,2^{4Q_{3m+1}}-1}(\overline{x})|\Big)\\
&\quad+2\sum_{k=1}^{2^{M_{3m+1}}}|x^*_{3m+1,n,k,2^{4Q_{3m+1}}}(\overline{x})|+2\Big|\sum_{k=1}^{2^{M_{3m+1}}}x^*_{3m+1,n,k,2^{4Q_{3m+1}}}(\overline{x})\Big|.
\end{aligned}
$$

Therefore, by NPL, let $(\pi(k))_{k=1}^{2^{M_{3m+1}}}$ be a permutation of $(k)_{k=1}^{2^{M_{3m+1}}}$ such that the three sequences

$$\Big(\sum_{k=1}^{K}\Big(\sum_{l=1}^{2^{4Q_{3m+1}}} x^*_{3m+1,n,\pi(k),l}(\overline{x})\Big)\Big)_{K=1}^{2^{M_{3m+1}}}, \quad \Big(\sum_{k=1}^{K}\Big(\sum_{l=1}^{2^{4Q_{3m+1}-1}} x^*_{3m+1,n,\pi(k),l}(\overline{x})\Big)\Big)_{K=1}^{2^{M_{3m+1}}}$$

and $(\sum_{k=1}^{K} x^*_{3m+1,n,\pi(k)k,2^{4Q_{3m+1}}}(\overline{x}))_{K=1}^{2^{M_{3m+1}}}$ are all $(3,0)$-monotone. Moreover, by Lemma 9, for $1 \le k \le 2^{M_{3m+1}}$ let $(\pi(k,l))_{l=1}^{2^{4Q_{3m+1}-1}}$ be a permutation of $(l)_{l=1}^{2^{4Q_{3m+1}-1}}$ such that (32.1) and (32.2) of Lemma 9 are satisfied for the sequence

$$\Big(\sum_{l=1}^{L} x^*_{3m+1,n,\pi(k),\pi(k,l)}(\overline{x}) e_{3m+1,n,\pi(k),\pi(k,l)}\Big)_{L=1}^{2^{4Q_{3m+1}-1}}.$$

Finally, set

$$\begin{aligned}(\pi(3m+1,n,g))_{g=1}^{G_{3m+1,0}} = (((3m+1,n,\pi(k),\pi(k,l)))_{l=1}^{2^{4Q_{3m+1}-1}} \\ \cup (3m+1,n,\pi(k),2^{4Q_{3m+1}}))_{k=1}^{2^{M_{3m+1}}}.\end{aligned}$$

It follows that $(|\sum_{g=1}^{G} x^*_{\pi(3m+1,n,g)}(\overline{x})|)_{G=1}^{G_{3m+1,0}}$ is $(3,0)$-monotone and, for any permutation $(\pi(3m+1,g))_{g=1}^{G_{3m+1}}$ of $((3m+1,g))_{g=1}^{G_{3m+1}}$ such that the permutation induced on $((3m+1,n,g))_{g=1}^{G_{3m+1,0}}$ is just $(\pi(3m+1,n,g))_{g=1}^{G_{3m+1,0}}$ for $1 \le n \le P_{3m+1}$,

$$\Big(\Big\|\sum_{g=1}^{G} x^*_{\pi(3m+1,g)}(\overline{x})(x_{\pi(3m+1,g)} - \widetilde{x}_{\pi(3m+1,g)})\Big\|\Big)_{G=1}^{G_{3m+1}}$$

is $(3,0)$-monotone. Therefore again there exists $(\overline{u}_{3m+2,g})_{g=1}^{G_{3m+1}}$, again with the same properties of construction of the sequence $(\overline{u}_{3m+1,s})_{s=1}^{q(3m+1)}$ of (C) of SCRL, such that there exists a subsequence $(x_{n'})_{n=q'(3m+5,1)+1}^{q'(3m+5,2)}$ of the subsequence of $(x_n)_{n=q(3m+3)+1}^{q(3m+5)}$ which is complementary to $(x_n)_{n=q'(3m+3)+1}^{q'(3m+5,1)}$, and again a subsequence $(\widetilde{x}'_n)_{n=\widetilde{q}(3m+2)+1}^{\widetilde{q}(3m+3)}$ of $(x_n)_{n=q(3m+2)+1}^{q(3m+3)}$, such that there exist $\overline{\eta}_{m,2} \to 0$ and a permutation $(x_{\overline{\pi}(n)})_{n=\overline{q}(m+1,1)+1}^{\overline{q}(m+1,2)}$ of

$$(x_n)_{n=q(3m+1)+1}^{q(3m+2)} \cup (\widetilde{x}'_n)_{n=\widetilde{q}(3m+2)+1}^{\widetilde{q}(3m+3)} \cup (x_{n'})_{n=q'(3m+5,1)+1}^{q'(3m+5,2)},$$

with

$$\sum_{g=1}^{G_{3m+1}} x^*_{\pi(3m+1,g)}(\overline{x})(x_{\pi(3m+1,g)} + \overline{u}_{3m+2,g}) = \sum_{n=\overline{q}(m+1,1)+1}^{\overline{q}(m+1,2)} x^*_{\overline{\pi}(n)}(\overline{x}) x_{\overline{\pi}(n)},$$

$$\Big\|\sum_{n=\overline{q}(m+1,1)+1}^{q} x^*_{\overline{\pi}(n)}(\overline{x}) x_{\overline{\pi}(n)}\Big\| < \overline{\eta}_{m,2} \quad \text{for } \overline{q}(m+1,1)+1 \le q \le \overline{q}(m+1,2).$$

FOURTH STEP. Now we turn to the subsum of $\sum_{n=q(3m+2)+1}^{q(3m+3)} x^*_n(\overline{x}) x_n$ complementary to $\sum_{n=\widetilde{q}(3m+2)+1}^{\widetilde{q}(3m+3)} \widetilde{x}'^*_n(\overline{x})\widetilde{x}'_n$. Then again, since the procedure in the general case is the same, we can consider only the particular case when $(\widetilde{x}'_n)_{n=\widetilde{q}(3m+2)+1}^{\widetilde{q}(3m+3)}$ do not appear; that is, $\sum_{n=q(3m+2)+1}^{q(3m+3)} x^*_n(\overline{x}) x_n$, which on the other hand has already been considered in (A) of SCRL. Therefore again there exists $(\overline{u}_{3m+3,g})_{g=1}^{G_{3m+2}}$, again with the same properties of construction of the sequence $(\overline{u}_{3m+1,s})_{s=1}^{q(3m+1)}$ of (C) of SCRL, such that there exists a subsequence $(x_{n'})_{n=q'(3m+5,2)+1}^{q'(3m+5,3)}$ of the subsequence of $(x_n)_{n=q(3m+3)+1}^{q(3m+5)}$ which is complementary to $(x_n)_{n=q'(3m+3)+1}^{q'(3m+5,2)}$, such that there exist $\overline{\eta}_{m,3} \to 0$ and a permutation $(x_{\overline{\pi}(n)})_{n=\overline{q}(m+1,2)+1}^{\overline{q}(m+1,3)}$ of $(x_n)_{n=q(3m+2)+1}^{q(3m+3)} \cup (x_{n'})_{n=q'(3m+5,2)+1}^{q'(3m+5,3)}$, with

$$\sum_{g=1}^{G_{3m+2}} x^*_{\pi(3m+2,g)}(\overline{x})(x_{\pi(3m+2,g)} + \overline{u}_{3m+3,g}) = \sum_{n=\overline{q}(m+1,2)+1}^{\overline{q}(m+1,3)} x^*_{\overline{\pi}(n)}(\overline{x})x_{\overline{\pi}(n)},$$

$$\Big\| \sum_{n=\overline{q}(m+2,1)+1}^{q} x^*_{\overline{\pi}(n)}(\overline{x})x_{\overline{\pi}(n)} \Big\| < \overline{\eta}_{m,3} \quad \text{for } \overline{q}(m+1,2)+1 \le q \le \overline{q}(m+1,3).$$

Finally, by (ii) of FRCL and by the previous procedures, there exists a subsequence $(x_{n'})_{n=q'(3m+5,3)+1}^{q'(3m+5)}$ of the subsequence of $(x_n)_{n=q(3m+3)+1}^{q(3m+5)}$ which is complementary to $(x_n)_{n=q'(3m+3)+1}^{q'(3m+5,3)}$, such that there exist $\overline{\eta}_{m,4} \to 0$ and a permutation $(x_{\overline{\pi}(n)})_{n=\overline{q}(m+1,3)+1}^{\overline{q}(m+1)}$ of $(x_{n'})_{n=q'(3m+5,3)+1}^{q'(3m+5)}$ with

$$\Big\| \overline{x} - \sum_{n=1}^{\overline{q}(m+1} x^*_{\overline{\pi}(n)}(\overline{x})x_{\overline{\pi}(n)} \Big\| < \overline{\eta}_{m,4}, \qquad \Big\| \sum_{n=\overline{q}(m+3,1)+1}^{q} x^*_{\overline{\pi}(n)}(\overline{x})x_{\overline{\pi}(n)} \Big\| < \overline{\eta}_{m,4}$$

for $\overline{q}(m+1,2)+1 \le q \le \overline{q}(m+1,3)$. It is now sufficient to set $\overline{\eta}_m = \sum_{i=1}^{4} \overline{\eta}_{m,i}$ to complete the proof of Theorem 21. ∎

6. The special case of spaces of type > 1

For the proof of the existence of a basis with permutations in these spaces we need only (i) of CL and a simplification of CSL (Lemma 12 of Subsection 3.1).

THEOREM 22. *Let X be a separable Banach space of type $p > 1$, that is, l_1 is not finitely represented in X. Then there exists a biorthogonal system (x_n, x_n^*), with $\|x_n\| < 2$ and $\|x_n^*\| < 2K+1$ for each n, and with a fixed increasing sequence $(q(m))$ of positive integers, such that for each $\overline{x} \in X$ with $\|\overline{x}\| = 1$ there exist*

(a) *a permutation $(\overline{\pi}(n))$ of $\{n\}$,*
(b) *a nonincreasing sequence $(\overline{\eta}_m)$ of positive numbers with $\overline{\eta}_m \to 0$,*
(c) *a sequence $(\overline{q}(m))$ of positive integers, so that, for each m:*

(48.1) $(\overline{\pi}(n))_{n=1}^{\overline{q}(m)}$ *is a permutation of* $(n)_{n=1}^{q(m)} \cup (m'_n)_{n=1}^{h'(m)}$ *where* $(n)_{n=q(m)+1}^{q(m+1)} = (m'_n)_{n=1}^{h'(m)} \cup (m''_n)_{n=1}^{h''(m)}$, *hence* $(n)_{n=1}^{q(m)} \subseteq (\overline{\pi}(n))_{n=1}^{\overline{q}(m)} \subseteq (n)_{n=1}^{q(m+1)}$;

(48.2) $\|\overline{x} - \sum_{n=1}^{q} x^*_{\overline{\pi}(n)}(\overline{x})x_{\overline{\pi}(n)}\| < \overline{\eta}_m$ *for each* $q \ge \overline{q}(m)$;

(48.3) $(\|\sum_{n=1}^{q} x^*_{\overline{\pi}(n)}(\overline{x})x_{\overline{\pi}(n)}\|)_{q=\overline{q}(m)+1}^{\overline{q}(m+1)}$ *is* $(K, 0, 1/2^{P_m})$*-monotone.*

Proof (*Construction of a basis with permutations*). We proceed by induction and we are now going to construct, starting from the mth step, the $(m+1)$th step through the following six substeps.

SUBSTEP 1. Our starting point is the biorthogonal system

$$(x_n, x_n^*)_{n=1}^{q(m)} \cup (u'_{m,n}, u'^*_{m,n})_{n=1}^{A_m}, \qquad \|u'_{m,n}\| = 1, \|u'^*_{m,n}\| < K'_m$$

for $1 \le n \le A_m$. We set $X = X_m + V_m$ with

$$X_m = \operatorname{span}((x_n)_{n=1}^{q(m)} \cup (u'_{m,n})_{n=1}^{A_m}), \qquad V_m = X \cap \bigcap_{n=1}^{q(m)} x^*_{n\perp} \cap \bigcap_{n=1}^{A_m} u'^*_{(m,n)\perp}.$$

SUBSTEP 2. We are going to prove that there exists $K > 0$ independent of m such that we can pass to the biorthogonal system

$$\begin{gathered}(x_n, x^*_n)_{n=1}^{q(m)} \cup (u'_{m,n}, u'^*_{m,n})_{n=1}^{A_m} \cup (e_{m,n}, e^*_{m,n})_{n=1}^{Q_m},\\ (e_{m,n})_{n=1}^{Q_m} = ((e_{m,n,k})_{k=1}^{16^{B_m}} \cup ((e_{m,n,k,j})_{j=1}^{16^{M_m}})_{k=0}^{16^{B_m}})_{n=1}^{A_m},\\ Q_m = A_m(16^{B_m} + (1+16^{B_m})16^{M_m}).\end{gathered}$$

Then if $E_m = \operatorname{span}(e_{m,n})_{n=1}^{Q_m}$, $U_m = V_m \cap \bigcap_{n=1}^{Q_m} e^*_{(m,n)\perp}$ and $P_m = (1+16^{B_m})A_m$, we have $X = X_m + E_m + U_m$ and

(49.1) $(e_{m,n})_{n=1}^{Q_m}$ and $(e_{m,n} + X_m + U_m)_{n=1}^{Q_m} = (e_{m,n} + X \cap \bigcap_{k=1}^{Q_m} e^*_{(m,k)\perp})_{n=1}^{Q_m}$ (basis of $X/(X_m+U_m)$) are respectively 1-equivalent and K-equivalent to the natural basis of $l_2^{Q_m}$;

(49.2) $(e^*_{m,n})_{n=1}^{Q_m}$ is K-equivalent to the natural basis of $l_2^{Q_m}$.

By Theorem I* of the Introduction for $C = K/\sqrt{2}$ there exists $(e_{m,n})_{n=1}^{D_m}$ in V_m which is 1-equivalent to the natural basis of $l_2^{D_m}$, and there exists a projection $R_m : V_m \to \operatorname{span}(e_{m,n})_{n=1}^{D_m}$ with $\|R_m\| < K/\sqrt{2}$. Then $(e_{m,n} + R_{m\perp} \cap V_m)_{n=1}^{D_m}$ is $K/\sqrt{2}$-equivalent to the natural basis of $l_2^{D_m}$. But D_m is the dimension of $\operatorname{span}((e_{m,n} + R_{m\perp} \cap V_m) + X_m)_{n=1}^{D_m}$ while by Substep 1 the dimension of X_m is $q(m) + A_m$; hence if D_m is sufficiently larger than $q(m) + A_m$ there exists a subsequence of $2Q_m$ elements of $(e_{m,n})_{n=1}^{D_m}$, which we can call $(e_{m,n})_{n=1}^{2Q_m}$, such that (always following Subsection 1.5 of the Introduction) we can suppose that, for each given sequence $(a_n)_{n=1}^{Q_m}$ of numbers with $\sum_{n=1}^{Q_m} a_n^2 = 1$, there is $x \in X_m$ so that

$$\begin{aligned}\Big\|\sum_{n=1}^{Q_m} a_n e_{m,n} + R_{m\perp} \cap V_m + X_m\Big\| &= \Big\|\sum_{n=1}^{Q_m} a_n e_{m,n} + R_{m\perp} \cap V_m + x\Big\|\\ &= \Big\|\sum_{n=Q_m+1}^{2Q_m} a_{n-Q_m} e_{m,n} + R_{m\perp} \cap V_m + x\Big\|\\ &= \Big\|\sum_{n=Q_m+1}^{2Q_m} a_{n-Q_m} e_{m,n} + R_{m\perp} \cap V_m + X_m\Big\|.\end{aligned}$$

Hence it follows that

$$\begin{aligned}\frac{\sqrt{2}}{K} &= \frac{\sqrt{2}}{K}\sqrt{\sum_{n=1}^{Q_m} a_n^2} \le \Big\|\sum_{n=1}^{Q_m} a_n \frac{e_{m,n} - e_{m,n+Q_m}}{\sqrt{2}} + R_{m\perp} \cap V_m\Big\|\\ &= \frac{1}{\sqrt{2}}\Big\|\Big(\sum_{n=1}^{Q_m} a_n e_{m,n} + R_{m\perp} \cap V_m + x\Big) - \Big(\sum_{n=Q_m+1}^{2Q_m} a_{n-Q_m} e_{m,n} + R_{m\perp} \cap V_m + x\Big)\Big\|\end{aligned}$$

$$\leq \frac{1}{\sqrt{2}}\Big\|\sum_{n=1}^{Q_m} a_n e_{m,n} + R_{m\perp}\cap V_m + x\Big\| + \frac{1}{\sqrt{2}}\Big\|\sum_{n=Q_m+1}^{2Q_m} a_{n-Q_m} e_{m,n} + R_{m\perp}\cap V_m + x\Big\|$$

$$= \sqrt{2}\Big\|\sum_{n=1}^{Q_m} a_n e_{m,n} + R_{m\perp}\cap V_m + X_m\Big\| \leq \sqrt{2}\Big\|\sum_{n=1}^{Q_m} a_n e_{m,n}\Big\| = \sqrt{2}\sqrt{\sum_{n=1}^{Q_m} a_n^2} = \sqrt{2};$$

that is,

$$\frac{1}{K} \leq \Big\|\sum_{n=1}^{Q_m} a_n e_{m,n} + R_{m\perp}\cap V_m + X_m\Big\| \leq 1.$$

Therefore, for each sequence $\{a_n\}_{n=1}^{Q_m}$ of numbers,

$$\frac{1}{K}\sqrt{\sum_{n=1}^{Q_m} a_n^2} \leq \sqrt{\sum_{n=1}^{Q_m} a_n^2}\Big\|\sum_{n=1}^{Q_m}\Big(a_n\Big/\sqrt{\sum_{k=1}^{Q_m} a_k^2}\Big)e_{m,n} + R_{m\perp}\cap V_m + X_m\Big\|$$

$$\Big(= \Big\|\sum_{n=1}^{Q_m} a_n e_{m,n} + R_{m\perp}\cap V_m + X_m\Big\|\Big) \leq \sqrt{\sum_{n=1}^{Q_m} a_n^2};$$

that is, $\{e_{m,n} + R_{m\perp}\cap V_m + X_m\}_{n=1}^{Q_m}$ (basis of $X/(R_{m\perp}\cap V_m + X_m)$) is K-equivalent to the natural basis of $l_2^{Q_m}$. Hence, setting $U_m = R_{m\perp}\cap V_m$, there is $((e^*_{m,n,k})_{k=1}^{16^{B_m}} \cup ((e^*_{m,n,k,j})_{j=1}^{16^{M_m}})_{k=0}^{16^{B_m}})_{n=1}^{A_m}$ in X^* so that $U_m = V_m \cap \bigcap_{k=1}^{Q_m} e^*_{(m,k)\perp}$ and (49.1) and (49.2) are satisfied.

Substep 3. We pass from $(u'_{m,n},u'^*_{m,n})_{n=1}^{A_m} \cup ((e_{m,n,k},e^*_{m,n,k})_{k=1}^{16^{B_m}})_{n=1}^{A_m}$ to $(u_{m,n},u^*_{m,n})_{n=1}^{P_m} = ((u_{m,n,k},u^*_{m,n,k})_{k=0}^{16^{B_m}})_{n=1}^{A_m}$ biorthogonal by means of the procedure of Subconstructions III.1 (Step 4) and III.2 (Step 2) with some slight difference, with $\|u_{m,n}\| < 2$ and $\|u^*_{m,n}\| < 2K+1$ for $1 \leq n \leq P_m$; precisely we set, for $1 \leq n \leq A_m$, $1 \leq k \leq 16^{B_m}$,

$$u_{m,n,0} = \frac{1}{4^{B_m}}\sum_{f=1}^{16^{B_m}} e_{m,n,f}, \qquad u^*_{m,n,0} = \frac{1}{4^{B_m}}\sum_{f=1}^{16^{B_m}} e^*_{m,n,f} - \frac{u'^*_{m,n}}{2^{B_m}},$$

$$u_{m,n,k} = e_{m,n,k} + u'_{m,n}/2^{B_m},$$

$$u^*_{m,n,k} = e^*_{m,n,k} - u^*_{m,n,0}/4^{B_m} = e^*_{m,n,k} - \frac{1}{16^{B_m}}\sum_{f=1}^{16^{B_m}} e^*_{m,n,f} + \frac{u'^*_{m,n}}{2^{3B_m}}.$$

Substep 4. We go from $(u_{m,n},\, u^*_{m,n})_{n=1}^{P_m}$ to its generating form

$$(w_{m,n}, w^*_{m,n})_{n=1}^{P_m} = ((w_{m,n,k}, w^*_{m,n,k})_{k=0}^{16^{B_m}})_{n=1}^{A_m}$$

by means of the GBST.

Substep 5. If B_m, M_m and T_m are integers with

$$B_m \geq 2^{A_m+q(m)+K+K_m}, \qquad T_m \geq 4^{P_m}\sum_{n=1}^{P_m}\|w^*_{m,n}\|, \qquad M_m \geq 2^{B_m+T_m},$$

we pass from

$$(w_{m,n}, w^*_{m,n})_{n=1}^{P_m} \cup (((e_{m,n,k,j}, e^*_{m,n,k,j})_{j=1}^{16^{M_m}})_{k=0}^{16^{B_m}})_{n=1}^{A_m}$$
$$= (((w_{m,n,k}, w^*_{m,n,k}) \cup (e_{m,n,k,j}, e^*_{m,n,k,j})_{j=1}^{16^{M_m}})_{k=0}^{16^{B_m}})_{n=1}^{A_m}$$

to $(((x_{m,n,k,j}, x^*_{m,n,k,j})_{j=0}^{16^{M_m}})_{k=0}^{16^{B_m}})_{n=1}^{A_m}$ biorthogonal with $\|x_{m,n,k,j}\| < 2$ and $\|x^*_{m,n,k,j}\| < 2K+1$ for $0 \le j \le 16^{M_m}$, $0 \le k \le 16^{B_m}$ and $1 \le n \le A_m$, by means of the following procedure: we set, for $1 \le n \le A_m$, $0 \le k \le 16^{B_m}$, $1 \le j \le 16^{M_m}$,

$$x_{m,n,k,0} = \frac{1}{4^{M_m}} \sum_{f=1}^{16^{M_m}} e_{m,n,k,f}, \qquad x^*_{m,n,k,0} = \frac{1}{4^{M_m}} \sum_{f=1}^{16^{M_m}} e^*_{m,n,k,f} - w^*_{m,n,k}/2^{M_m},$$

$$x'_{m,n,k,j} = e_{m,n,k,j}, \quad x''_{m,n,k,j} = x''_{m,n,k} = w_{m,n,k}/2^{M_m}, \quad x_{m,n,k,j} = x'_{m,n,k,j} + x''_{m,n,k,j},$$

$$x^*_{m,n,k,j} = e^*_{m,n,k,j} - \frac{x^*_{m,n,k,0}}{4^{M_m}} = e^*_{m,n,k,j} - \frac{1}{16^{M_m}} \sum_{f=1}^{16^{M_m}} e^*_{m,n,k,f} + \frac{w^*_{m,n,k}}{8^{M_m}}.$$

SUBSTEP 6. We set

$$(x_n, x^*_n)_{n=q(m)+1}^{q(m+1)} = (((x_{m,n,k,j}, x^*_{m,n,k,j})_{j=1}^{16^{M_m}})_{k=0}^{16^{B_m}})_{n=1}^{A_m},$$
$$(u'_{m+1,n}, u'^*_{m+1,n})_{n=1}^{P'_{m+1}} = ((x_{m,n,k,0}, x^*_{m,n,k,0})_{k=0}^{16^{B_m}})_{n=1}^{A_m}.$$

Now, by a simplified version of Construction II (precisely without the construction of $(v'_{m+1,n}, v'^*_{m+1,n})_{n=1}^{Q'_{m+1}}$ in two distinct stages by means of $(v'_{m+1,n}, v'^*_{m+1,n})_{n=1}^{Q''_{m+1}}$ and $(v'_{m+1,n}, v'^*_{m+1,n})_{n=Q''_{m+1}+1}^{Q'_{m+1}}$; we now use only the second stage, that is, we define directly $(v'_{m+1,n}, v'^*_{m+1,n})_{n=1}^{Q'_{m+1}}$) and by Lemma 12, we define $(u'_{m+1,n}, u'^*_{m+1,n})_{n=P'_{m+1}+1}^{A_{m+1}}$ (which corresponds to $(u''_{m+1,n}, u''^*_{m+1,n})_{n=1}^{P''_{m+1}}$ of (43.1) of Lemma 12), so we are again in the situation of Substep 1 but with $m+1$ instead of m. Hence by the same procedure we can define $(x_n, x^*_n)_{n=q(m+1)+1}^{q(m+2)} \cup (u'_{m+2,n}, u'^*_{m+2,n})_{n=1}^{A_{m+2}}$ and so on. Proceeding in this way we define $\{x_n, x^*_n\}$ biorthogonal such that, by (i) of CL (Lemma 16; the proof of Lemma 16 holds also for spaces of type > 1, actually in the situation of this section the proof is simpler), we have the following property:

(50) for each $x' \in X$ with $\|x'\| = 1$ there exists a nonincreasing sequence $\{\eta'_m\}$ of positive numbers with $\eta'_m \to 0$ such that, for each m, $\operatorname{dist}(x' - (\sum_{n=1}^{q(m)} x^*_n(x')x_n$, set of all $\sum_{k=1}^{K} u'^*_{m,n_k}(x')u'_{m,n_k} + a u'^*_{m,n_{K+1}}(x')u'_{m,n_{K+1}}$ for $0 < |a| < 1$ and for a subsequence $(n_k)_{k=1}^{K+1}$ of $(n)_{n=1}^{A_m}) < \eta'_m$.

We point out that, by (49.1) and (49.2) and by our definition of "(H, M, ε)-monotone", it follows that

(51) $(\|\sum_{i=1}^{P} a_i x'_{m,1,i} + \operatorname{span}(x'_{m,2,j})_{j=1}^{P(2)} + \operatorname{span}(x''_n)_{n=q(m)+1}^{q(m+1)} + \operatorname{span}((x_n)_{n=1}^{q(m)} \cup (x_n)_{n \ge q(m+1)+1})\|)_{P=1}^{P(1)}$ is $(K, 0, 0)$-monotone for each m, for each partition $(n)_{n=q(m)+1}^{q(m+1)} = ((m,1,i))_{i=1}^{P(1)} \cup ((m,2,i))_{i=1}^{P(2)}$ and for each sequence $(a_i)_{i=1}^{P(1)}$ of numbers. ■

Proof (*Properties of a basis with permutations*). Suppose that $\overline{x} \in X$ with $\|\overline{x}\| = 1$.We proceed by induction.

We fix a positive integer m and we suppose to have found $(\overline{q}(k))_{k=1}^{m}$ and $(\overline{\pi}(n))_{n=1}^{\overline{q}(m)}$ such that, for each k with $1 \le k \le m$, (48.1) is satisfied when m is replaced by k, and such that (48.2) is satisfied in the following form: there exist four nonincreasing sequences $(\widehat{\eta}_k)_{k=1}^{m}$, $(\overline{\eta}_k)_{k=1}^{m}$, $(\varepsilon_k)_{k=1}^{m}$ and $(\eta'_k)_{k=1}^{m}$ of numbers, with $\widehat{\eta}_m \to 0$, $\overline{\eta}_m \to 0$, $\varepsilon_m \to 0$ and $\eta'_m \to 0$, such that:

(52.1) $\|\overline{x} - \sum_{n=1}^{\overline{q}(m)} x^*_{\overline{\pi}(n)}(\overline{x})x_{\overline{\pi}(n)}\| < \widehat{\eta}_m$, where $\widehat{\eta}_m = \eta'_m + 1/2^{2q(m)} + \varepsilon_{m-1} + P_m 2(2K + 1)/2^{M_m} + K(\eta'_{m-1} + \eta'_{m+1})$;

(52.2) $\|\overline{x} - \sum_{n=1}^{q} x^*_{\overline{\pi}(n)}(\overline{x})x_{\overline{\pi}(n)}\| < \overline{\eta}_k$ for $\overline{q}(k) \le q \le \overline{q}(m)$ and for $1 \le k \le m$, with $\overline{\eta}_m = (1+K)\widehat{\eta}_{m-1} + K\widehat{\eta}_m + K/2^{P_{m-1}}$;

(52.3) $(\|\sum_{n=1}^{q} x^*_{\overline{\pi}(n)}(\overline{x})x_{\overline{\pi}(n)}\|)_{q=\overline{q}(m)+1}^{\overline{q}(m+1)}$ is $(K, 0, 1/2^{P_m})$-monotone.

At this point suppose that, for a fixed m, (48.1) and (52.1)–(52.3) are satisfied; we are going to verify them also for $m+1$. We have two possibilities. The first one is that

$$|u'^*_{m+1,n}(\overline{x})| \le \frac{1}{A_{m+1}2^{2q(m+1)+1}} \quad \text{for } 1 \le n \le A_{m+1}. \tag{53}$$

Again, by (52.1)–(52.3) with x' and m replaced by $\overline{x}$ and $m+1$ respectively and by (53),

$$\Big\|\overline{x} - \Big(\sum_{n=1}^{\overline{q}(m)} x^*_{\overline{\pi}(n)}(\overline{x})x_{\overline{\pi}(n)} + \sum_{n=1}^{h''(m)} x^*_{m''_n}(\overline{x})x_{m''_n}\Big)\Big\| = \Big\|\overline{x} - \sum_{n=1}^{q(m+1)} x^*_n(\overline{x})x_n\Big\|$$
$$< \widehat{\eta}_{m+1} = \eta'_{m+1} + \frac{1}{2^{2q(m+1)+1}},$$
$$((m+1)''_n)_{n=1}^{h''(m+1)} = (n)_{n=q(m+1)+1}^{q(m+2)}.$$

We claim that there exists a permutation $(\overline{\pi}(n))_{n=\overline{q}(m)+1}^{\overline{q}(m+1)}$ of $((m)''_n)_{n=1}^{h''(m)}$ $(\overline{q}(m+1) = q(m+1))$ so that

$$\Big(\Big\|\sum_{n=\overline{q}(m)+1}^{q} x^*_{\overline{\pi}(n)}(\overline{x})x''_{\overline{\pi}(n)}\Big\|\Big)_{q=\overline{q}(m)+1}^{\overline{q}(m+1)} \text{ is } (1, 1/2^{P_m})\text{-monotone.} \tag{54}$$

Indeed, suppose, in order to decrease the formality (in the general case the procedure is the same), that $((m)''_n)_{n=1}^{h''(m)} = (n)_{n=q(m)+1}^{q(m+1)}$. By Substeps 5 and 6,

$$\sum_{n=q(m)+1}^{q(m+1)} x^*_n(\overline{x})x''_n = \sum_{n=1}^{A_m}\sum_{k=0}^{2^{4B_m}}\Big(\sum_{j=1}^{2^{4M_m}} \frac{x^*_{m,n,k,j}(\overline{x})}{2^{M_m}}\Big)w_{m,n,k},$$

hence, for $1 \le n \le A_m$ and $0 \le k \le 2^{4B_m}$, there is a permutation $(\pi(m,n,k,j))_{j=1}^{2^{4M_m}}$ of $(j)_{j=1}^{2^{4M_m}}$ such that $(|\sum_{j=1}^{J} x^*_{\pi(m,n,k,j)}(\overline{x})|)_{J=1}^{2^{4M_m}}$ is $(1,0)$-monotone. Then there is a partition $((\pi(m,n,k,j))_{j=t(m,n,k,p-1)+1}^{t(m,n,k,p)})_{p=1}^{T_m}$ of $(\pi(m,n,k,j))_{j=1}^{2^{4M_m}}$ (hence with $t(m,n,k,0)=0$ and $t(m,n,k,T_m) = 2^{4M_m}$) such that, for $1 \le p \le T_m$,

$$\Big|\sum_{j=t(m,n,k,p-1)+1}^{t(m,n,k,p)} \frac{x^*_{\pi(m,n,k,j)}(\overline{x})}{2^{M_m}} - \frac{1}{T_m}\sum_{j=1}^{2^{4M_m}} \frac{x^*_{\pi(m,n,k,j)}(\overline{x})}{2^{M_m}}\Big| < \frac{2K+1}{2^{M_m}},$$
$$\Big|\sum_{j=1}^{t(m,n,k,p)} \frac{x^*_{\pi(m,n,k,j)}(\overline{x})}{2^{M_m}} - \frac{p}{T_m}\sum_{j=1}^{2^{4M_m}} \frac{x^*_{\pi(m,n,k,j)}(\overline{x})}{2^{M_m}}\Big| < \frac{2K+1}{2^{M_m}},$$

because

$$\left|\frac{1}{T_m}\sum_{j=1}^{2^{4M_m}}\frac{x^*_{\pi(m,n,k,j)}(\overline{x})}{2^{M_m}}\right| = \left|\frac{w^*_{m,n,k}(\overline{x})}{T_m}\right| \le \frac{\|w^*_{m,n,k}\|}{T_m} \le \frac{\|w^*_{m,n,k}\|}{4^{P_m}\sum_{n=1}^{P_m}\|w^*_{m,n}\|} < \frac{1}{4^{P_m}}.$$

Hence, for $1 \le p \le T_m$, since

$$\begin{aligned}\sum_{n=q(m)+1}^{q(m+1)} x^*_n(\overline{x})x''_n &= \sum_{n=1}^{A_m}\sum_{k=0}^{2^{4B_m}}\sum_{j=1}^{16^{M_m}} x^*_{m,n,k,j}(\overline{x})\frac{w_{m,n,k}}{2^{M_m}}\\ &= \sum_{n=1}^{A_m}\sum_{k=0}^{2^{4B_m}} w^*_{m,n,k}(\overline{x})w_{m,n,k} = \sum_{n=1}^{P_m} w^*_{m,n}(\overline{x})w_{m,n} = \sum_{n=1}^{P_m} u^*_{m,n}(\overline{x})u_{m,n},\end{aligned}$$

we have

$$\begin{aligned}&\left\|\sum_{n=1}^{A_m}\sum_{k=0}^{2^{4B_m}}\sum_{j=t(m,n,k,p-1)+1}^{t(m,n,k,p)} x^*_{\pi(m,n,k,j)}(\overline{x})\frac{w_{m,n,k}}{2^{M_m}} - \frac{\sum_{n=1}^{P_m}u^*_{m,n}(\overline{x})u_{m,n}}{T_m}\right\|\\ &\quad= \left\|\sum_{n=1}^{A_m}\sum_{k=0}^{2^{4B_m}}\sum_{j=t(m,n,k,p-1)+1}^{t(m,n,k,p)} x^*_{\pi(m,n,k,j)}(\overline{x})\frac{w_{m,n,k}}{2^{M_m}} - \frac{\sum_{n=q(m)+1}^{q(m+1)}x^*_n(\overline{x})x''_n}{T_m}\right\|\\ &\quad= \left\|\sum_{n=1}^{A_m}\sum_{k=0}^{2^{4B_m}}\frac{w_{m,n,k}}{2^{M_m}}\left(\sum_{j=t(m,n,k,p-1)+1}^{t(m,n,k,p)} x^*_{\pi(m,n,k,j)}(\overline{x}) - \frac{\sum_{j=1}^{2^{4M_m}}x^*_{\pi(m,n,k,j)}(\overline{x})}{T_m}\right)\right\|\\ &\quad< \sum_{n=1}^{A_m}\sum_{k=0}^{2^{4B_m}}\frac{2K+1}{2^{M_m}}\|w_{m,n,k}\| < \frac{(4K+2)P_m}{2^{M_m}}\end{aligned}$$

and

$$\left\|\sum_{n=1}^{A_m}\sum_{k=0}^{2^{4B_m}}\sum_{j=1}^{t(m,n,k,p)} x^*_{\pi(m,n,k,j)}(\overline{x})\frac{w_{m,n,k}}{2^{M_m}} - \frac{p}{T_m}\sum_{n=1}^{P_m}u^*_{m,n}(\overline{x})u_{m,n}\right\| < \frac{(4K+2)P_m}{2^{M_m}}.$$

Therefore, since $(\frac{p}{T_m}\sum_{n=1}^{P_m}u^*_{m,n}(\overline{x})u_{m,n})_{p=1}^{T_m}$ is $(0,0)$-monotone,

$$\begin{aligned}&\left(\left\|\sum_{p=1}^{P}\sum_{n=1}^{A_m}\sum_{k=0}^{2^{4B_m}}\sum_{j=t(m,n,k,p-1)+1}^{t(m,n,k,p)} x^*_{\pi(m,n,k,j)}(\overline{x})x''_{\pi(m,n,k,j)}\right\|\right)_{P=1}^{T_m}\\ &\qquad\qquad= \left(\left\|\sum_{n=1}^{A_m}\sum_{k=0}^{2^{4B_m}}\sum_{j=1}^{t(m,n,k,P)} x^*_{\pi(m,n,k,j)}(\overline{x})x''_{\pi(m,n,k,j)}\right\|\right)_{P=1}^{T_m}\end{aligned}$$

is $(0,(4K+2)P_m/2^{M_m})$-monotone. On the other hand, for $1 \le p \le T_m$, $1 \le n \le A_m$ and $0 \le k \le 2^{4B_m}$, by construction and by the above we have

$$\left\|\sum_{j=t(m,n,k,p-1)+1}^{t(m,n,k,p)} x^*_{\pi(m,n,k,j)}(\overline{x})x''_{\pi(m,n,k,j)}\right\| \le \left|\sum_{j=t(m,n,k,p-1)+1}^{t(m,n,k,p)} x^*_{\pi(m,n,k,j)}(\overline{x})\right|\frac{2}{2^{M_m}}$$

$$\leq 2\Big|\sum_{j=t(m,n,k,p-1)+1}^{t(m,n,k,p)} \frac{x^*_{\pi(m,n,k,j)}(\overline{x})}{2^{M_m}} - \frac{1}{T_m}\sum_{j=1}^{2^{4M_m}} \frac{x^*_{\pi(m,n,k,j)}(\overline{x})}{2^{M_m}}\Big|$$

$$+ 2\Big|\sum_{j=1}^{2^{4M_m}} \frac{x^*_{\pi(m,n,k,j)}(\overline{x})}{T_m 2^{M_m}}\Big| < \frac{4K+2}{2^{M_m}} + \frac{2}{4^{P_m}}$$

and

$$\sum_{n=1}^{A_m}\sum_{k=0}^{2^{4B_m}}\Big\|\sum_{j=t(m,n,k,p-1)+1}^{t(m,n,k,p)} x^*_{\pi(m,n,k,j)}(\overline{x})x''_{\pi(m,n,k,j)}\Big\| \leq P_m\frac{4K+2}{2^{M_m}} + \frac{2P_m}{4^{P_m}}.$$

It follows that

$$\Big(\Big(\Big(\Big(\Big\|\sum_{f=1}^{A_m}\sum_{g=0}^{2^{4B_m}}\sum_{j=1}^{t(m,f,g,P-1)} x^*_{\pi(m,f,g,j)}(\overline{x})x''_{\pi(m,f,g,j)}$$

$$+\sum_{f=1}^{N-1}\sum_{g=0}^{2^{4B_m}}\sum_{j=1}^{t(m,f,g,P)} x^*_{\pi(m,f,g,j)}(\overline{x})x''_{\pi(m,f,g,j)}$$

$$+\sum_{g=0}^{K-1}\sum_{j=1}^{t(m,N,g,P)} x^*_{\pi(m,N,g,j)}(\overline{x})x''_{\pi(m,N,g,j)}$$

$$+\sum_{j=1}^{T} x^*_{\pi(m,N,K,j)}(\overline{x})x''_{\pi(m,N,K,j)}\Big\|\Big)_{T=t(m,N,K,P-1)+1}^{t(m,N,K,P)}\Big)_{K=0}^{2^{4B_m}}\Big)_{N=1}^{A_m}\Big)_{P=1}^{T_m}$$

(the second sum appears only if $N \geq 2$ and the third sum only if $K \geq 1$) is $(0, 1/2^{P_m})$-monotone since

$$\Big(\Big\|\sum_{j=t(m,N,K,p-1)+1}^{T} x^*_{\pi(m,N,K,j)}(\overline{x})x''_{\pi(m,N,K,j)}\Big\|\Big)_{T=t(m,N,K,p-1)+1}^{t(m,N,K,p)}$$

is $(1,0)$-monotone and hence $(0, 4K+2/2^{M_m})$-monotone for each N and K with $1 \leq N \leq A_m$ and $0 \leq K \leq 2^{4B_m}$, and since, by Substep 5,

$$\frac{(4K+2)P_m}{2^{M_m}} + P_m\frac{4K+2}{2^{M_m}} + \frac{2P_m}{4^{P_m}} + \frac{4K+2}{2^{M_m}} < \frac{1}{2^{P_m}}.$$

This completes the proof of (54) if we set

$$(\overline{\pi}(n))_{n=q(m)+1}^{q(m+1)} = ((((\pi(m,n,k,j))_{j=t(m,n,k,p-1)+1}^{t(m,n,k,p)})_{k=0}^{2^{4B_m}})_{n=1}^{A_m})_{p=1}^{T_m}.$$

Hence we conclude that, for $\overline{q}(m)+1 \leq q \leq \overline{q}(m+1)$,

$$\Big\|\overline{x} - \sum_{n=1}^{\overline{q}(m+1)} x^*_{\overline{\pi}(n)}(\overline{x})x_{\overline{\pi}(n)}\Big\| < \widehat{\eta}_{m+1} = \eta'_{m+1} + 1/2^{2q(m+1)+1}; \tag{55.1}$$

$$\Big(\Big\|\sum_{n=1}^{q} x^*_{\overline{\pi}(n)}x_{\overline{\pi}(n)}\Big\|\Big)_{q=\overline{q}(m)+1}^{\overline{q}(m+1)} \quad \text{is } (K, 0, 1/2^{P_m})\text{-monotone}; \tag{55.2}$$

$$\Big\|\sum_{n=\overline{q}(m)+1}^{q} x^*_{\overline{\pi}(n)}(\overline{x})x'_{\overline{\pi}(n)}\Big\| < K^2(\widehat{\eta}_m + \widehat{\eta}_{m+1}) + K/2^{P_m}; \tag{55.3}$$

$$\Big\|\overline{x} - \sum_{n=1}^{q} x^*_{\overline{\pi}(n)}(\overline{x}) x_{\overline{\pi}(n)}\Big\| < \overline{\eta}_{m+1} = (1+K)\widehat{\eta}_m + K\widehat{\eta}_{m+1} + K/2^{P_m}. \tag{55.4}$$

Indeed, (55.1) follows from (50) and (53); (55.2) follows from (51) and (54); for (55.3), by (51) and (55.2), it follows that

$$\begin{aligned}
&\Big\| \sum_{n=\overline{q}(m)+1}^{q} x^*_{\overline{\pi}(n)}(\overline{x}) x'_{\overline{\pi}(n)}\Big\| \\
&\qquad \le K\Big\| \sum_{n=\overline{q}(m)+1}^{q} x^*_{\overline{\pi}(n)}(\overline{x}) x_{\overline{\pi}(n)}\Big\| < K^2\Big\| \sum_{n=\overline{q}(m)+1}^{\overline{q}(m+1)} x^*_{\overline{\pi}(n)}(\overline{x}) x_{\overline{\pi}(n)}\| + K/2^{P_m} \\
&\qquad = K^2\Big\|\Big(\overline{x} - \sum_{n=1}^{\overline{q}(m)} x^*_{\overline{\pi}(n)}(\overline{x}) x_{\overline{\pi}(n)}\Big) - \Big(\overline{x} - \sum_{n=1}^{\overline{q}(m+1)} x^*_{\overline{\pi}(n)}(\overline{x}) x_{\overline{\pi}(n)}\Big)\Big\| + K/2^{P_m} \\
&\qquad \le K^2\Big\|\overline{x} - \sum_{n=1}^{\overline{q}(m)} x^*_{\overline{\pi}(n)}(\overline{x}) x_{\overline{\pi}(n)}\Big\| + K^2\Big\|\overline{x} - \sum_{n=1}^{\overline{q}(m+1)} x^*_{\overline{\pi}(n)}(\overline{x}) x_{\overline{\pi}(n)}\Big\| + K/2^{P_m} \\
&\qquad < K^2(\widehat{\eta}_m + \widehat{\eta}_{m+1}) + K/2^{P_m}.
\end{aligned}$$

For (55.4), by (55.2) we have

$$\begin{aligned}
&\Big\|\overline{x} - \sum_{n=1}^{q} x^*_{\overline{\pi}(n)}(\overline{x}) x_{\overline{\pi}(n)}\Big\| \le \Big\|\overline{x} - \sum_{n=1}^{\overline{q}(m)} x^*_{\overline{\pi}(n)}(\overline{x}) x_{\overline{\pi}(n)}\Big\| + \Big\| \sum_{n=\overline{q}(m)+1}^{q} x^*_{\overline{\pi}(n)}(\overline{x}) x_{\overline{\pi}(n)}\Big\| \\
&\quad < \Big\|\overline{x} - \sum_{n=1}^{\overline{q}(m)} x^*_{\overline{\pi}(n)}(\overline{x}) x_{\overline{\pi}(n)}\Big\| + K\Big\| \sum_{n=\overline{q}(m)+1}^{\overline{q}(m+1)} x^*_{\overline{\pi}(n)}(\overline{x}) x_{\overline{\pi}(n)}\Big\| + K/2^{P_m} \\
&\quad = \Big\|\overline{x} - \sum_{n=1}^{\overline{q}(m)} x^*_{\overline{\pi}(n)}(\overline{x}) x_{\overline{\pi}(n)}\Big\| + K\Big\|\Big(\overline{x} - \sum_{n=1}^{\overline{q}(m)} x^*_{\overline{\pi}(n)}(\overline{x}) x_{\overline{\pi}(n)}\Big) \\
&\qquad - \Big(\overline{x} - \sum_{n=1}^{\overline{q}(m+1)} x^*_{\overline{\pi}(n)}(\overline{x}) x_{\overline{\pi}(n)}\Big)\Big\| + K/2^{P_m} \\
&\quad \le (1+K)\Big\|\overline{x} - \sum_{n=1}^{\overline{q}(m)} x^*_{\overline{\pi}(n)}(\overline{x}) x_{\overline{\pi}(n)}\Big\| + K\Big\|\overline{x} - \sum_{n=1}^{\overline{q}(m+1)} x^*_{\overline{\pi}(n)}(\overline{x}) x_{\overline{\pi}(n)}\Big\| + K/2^{P_m} \\
&\quad = (1+K)\widehat{\eta}_m + K\widehat{\eta}_{m+1} + K/2^{P_m}.
\end{aligned}$$

The second possibility is that (53) is not true, hence there exists $\widetilde{n}$ so that

(56) $1 \le \widetilde{n} \le A_{m+1}$ and $|u'^*_{m+1,\widetilde{n}}(\overline{x})| > 1/A_{m+1}2^{2q(m+1)+1}$ while

$$|u'^*_{m+1,n}(\overline{x})| \le 1/A_{m+1}2^{2q(m+1)+1}$$

for $\widetilde{n}+1 \le n \le A_{m+1}$; we set $\overline{n} = (\widetilde{n}-1)(1+2^{4B_{m+1}}) + \widetilde{k}$ with $\widetilde{k} > 2^{B_{m+1}/2}$ so that $|u^*_{m+1,\widetilde{n},\widetilde{k}+1}(\overline{x})| > 1/2^{2P_{m+1}}$ (hence $|u^*_{m+1,\overline{n}+1}(\overline{x})| > 1/2^{2P_{m+1}}$) while $|u^*_{m+1,\widetilde{n},k}(\overline{x})| \le 1/2^{2P_{m+1}}$ for $\widetilde{k}+2 \le k \le 2^{4B_{m+1}}$.

Indeed, we have

$$\frac{1}{A_{m+1}2^{2q(m+1)+1}} < |u'^*_{m+1,\widetilde{n}}(\overline{x})| = \frac{|\sum_{k=1}^{2^{4B_{m+1}}} u^*_{m+1,\widetilde{n},k}(\overline{x})|}{2^{B_{m+1}}}$$
$$\le \frac{|\sum_{k=1}^{\widetilde{k}+1} u^*_{m+1,\widetilde{n},k}(\overline{x})|}{2^{B_{m+1}}} + \frac{\sum_{k=\widetilde{k}+2}^{2^{4B_{m+1}}} |u^*_{m+1,\widetilde{n},k}(\overline{x})|}{2^{B_{m+1}}} < \frac{(\widetilde{k}+1)(2K+1)}{2^{B_{m+1}}} + \frac{2^{4B_{m+1}} - \widetilde{k} - 1}{2^{B_{m+1}}2^{2P_{m+1}}}$$
$$< \frac{(\widetilde{k}+1)(2K+1)}{2^{B_{m+1}}} + \frac{2^{4B_{m+1}}}{2^{B_{m+1}}2^{2P_{m+1}}} = \frac{(\widetilde{k}+1)(2K+1)}{2^{B_{m+1}}} + \frac{2^{3B_{m+1}}}{2^{2(1+16^{B_{m+1}})A_{m+1}}}$$
$$< \frac{(\widetilde{k}+1)(2K+1)}{2^{B_{m+1}}} + \frac{1}{4^{2^{4B_{m+1}}}},$$

that is,

$$\widetilde{k} > \left(\frac{1}{A_{m+1}2^{2q(m+1)+1}} - \frac{1}{4^{2^{4B_{m+1}}}}\right)\frac{2^{B_{m+1}}}{2K+1} - 1 > 2^{B_{m+1}/2}.$$

So, according to (50) with x', m replaced respectively by $\overline{x}$, $m+1$, we have

$$u'_{m+1} = \sum_{n=1}^{A_{m+1}} a_n u'^*_{m+1,n}(\overline{x})u'_{m+1,n} = \sum_{n=1}^{A_{m+1}} u'^*_{m+1,n}(u'_{m+1})u'_{m+1,n}$$
$$= \sum_{n=1}^{A_{m+1}} \sum_{k=0}^{16^{B_{m+1}}} u^*_{m+1,n,k}(u'_{m+1})u_{m+1,n,k}, \qquad 0 \le |a_n| \le 1,$$
$$u'^*_{m+1,n}(u'_{m+1}) = a_n u'^*_{m+1,n}(\overline{x}), \qquad u^*_{m+1,n,0}(u'_{m+1}) = -a_n u'^*_{m+1,n}(\overline{x})/2^{B_{m+1}},$$
$$u^*_{m+1,n,k}(u'_{m+1}) = a_n u'^*_{m+1,n}(\overline{x})/2^{B_{m+1}}$$

for $1 \le n \le A_{m+1}$, $1 \le k \le 16^{B_{m+1}}$ (since by Substep 3,

$$\operatorname{span}(u_{m+1,n})_{n=1}^{P_{m+1}} = \operatorname{span}(u'_{m+1,n})_{n=1}^{A_{m+1}} + \operatorname{span}((e_{m+1,n,k})_{k=1}^{16^{B_{m+1}}})_{n=1}^{A_{m+1}},$$

hence $e^*_{m+1,n,k}(u'_{m+1}) = 0$ for $0 \le k \le 16^{B_{m+1}}$ and $1 \le n \le A_{m+1}$), so that

$$\Big\|\overline{x} - \Big(\sum_{n=1}^{\overline{q}(m)} x^*_{\overline{\pi}(n)}(\overline{x})x_{\overline{\pi}(n)} + \sum_{n=1}^{h''(m)} x^*_{m''_n}(\overline{x})x_{m''_n} + u'_{m+1}\Big)\Big\| \tag{57}$$
$$= \Big\|\overline{x} - \Big(\sum_{n=1}^{q(m+1)} x^*_n(\overline{x})x_n + u'_{m+1}\Big)\Big\| < \eta'_{m+1}.$$

Hence, setting

$$\widetilde{u}_{m+1} = \sum_{n=1}^{\widetilde{n}-1} \sum_{k=0}^{2^{4B_{m+1}}} u^*_{m+1,n,k}(u'_{m+1})u_{m+1,n,k} + \sum_{k=0}^{\widetilde{k}} u^*_{m+1,\widetilde{n},k}(u'_{m+1})u_{m+1,\widetilde{n},k}$$

(then $\widetilde{u}_{m+1} \in \operatorname{span}(u_{m+1,n})_{n=1}^{\overline{n}}$) we have

$$\|u'_{m+1} - \widetilde{u}_{m+1}\| < \frac{1}{2^{2q(m+1)}}. \tag{58}$$

Indeed, by the above and by Substep 3 it follows that

$$\begin{aligned}
u'_{m+1}-\widetilde{u}_{m+1} &= \sum_{k=\widetilde{k}+1}^{2^{4B_{m+1}}} u^*_{m+1,\widetilde{n},k}(u'_{m+1})u_{m+1,\widetilde{n},k} + \sum_{n=\widetilde{n}+1}^{A_{m+1}} \sum_{k=0}^{2^{4B_{m+1}}} u^*_{m+1,n,k}(u'_{m+1})u_{m+1,n,k} \\
&= a_{\widetilde{n}}(u'^*_{m+1,\widetilde{n}}(\overline{x})/2^{3B_{m+1}})u_{m+1,\widetilde{n},\widetilde{k}+1} + \sum_{k=\widetilde{k}+2}^{2^{4B_{m+1}}} a_{\widetilde{n}}\frac{u'^*_{m+1,\widetilde{n}}(\overline{x})}{2^{3B_{m+1}}}u_{m+1,\widetilde{n},k} \\
&\quad + \sum_{n=\widetilde{n}+1}^{A_{m+1}} a_n u'^*_{m+1,n}(\overline{x})u'_{m+1,n}, \|u'_{m+1}-\widetilde{u}_{m+1}\| \\
&< \frac{K_{m+1}}{2^{3B_{m+1}}}2 + \sum_{k=\widetilde{k}+2}^{2^{4B_{m+1}}} \frac{2}{2^{2P_{m+1}}2^{3B_{m+1}}} + \sum_{n=\widetilde{n}+1}^{A_{m+1}} \frac{1}{A_{m+1}2^{2q(m+1)+1}} \\
&< \frac{2K_{m+1}}{2^{3B_{m+1}}} + \frac{2^{B_{m+1}}}{2^{2P_{m+1}}} + \frac{1}{2^{2q(m+1)+1}} < \frac{1}{2^{2q(m+1)}}.
\end{aligned}$$

Hence, by the above and by the Generating Biorthogonal System Theorem, there exists a sequence $(\widehat{f}(n))_{n=1}^{\overline{n}}$ of increasing integers with $\overline{n} \le \widehat{f}(\overline{n}) \le P_{m+1}$ and $n \le \widehat{f}(n) < \widehat{f}(n+1)$ for $1 \le n \le \overline{n}-1$, and a sequence $(\widehat{\varepsilon}_n)_{n=1}^{\overline{n}}$ of numbers, so that, for a suitable $\varepsilon_m > 0$, setting $\overline{f}(\widetilde{n},k) = \widehat{f}((\widetilde{n}-1)(1+2^{4B_{m+1}})+k)$ and $\overline{\varepsilon}_{\widetilde{n},k} = \widehat{\varepsilon}_{(\widetilde{n}-1)(1+2^{4B_{m+1}})+k}$ for $0 \le k \le \widetilde{k}$, and $\overline{f}(n,k) = \widehat{f}((n-1)(1+2^{4B_{m+1}})+k)$ and $\overline{\varepsilon}_{n,k} = \widehat{\varepsilon}_{(n-1)(1+2^{4B_{m+1}})+k}$ for $1 \le n \le \widetilde{n}-1$ and $0 \le k \le 2^{4B_{m+1}}$, we have

$$0 < \frac{\widehat{\varepsilon}_n}{w^*_{m+1,\widehat{f}(n)}(\overline{x})} < \varepsilon_m \quad \text{for } 1 \le n \le \overline{n}, \qquad \|\widehat{u}_{m+1}-\widetilde{u}_{m+1}\| < \varepsilon_m \tag{59}$$

for

$$\begin{aligned}
\widehat{u}_{m+1} &= \sum_{n=1}^{\overline{n}} \widehat{\varepsilon}_n w^*_{m+1,\widehat{f}(n)}(\overline{x})w_{m+1,\widehat{f}(n)} \\
&= \sum_{n=1}^{\widetilde{n}-1}\sum_{k=0}^{2^{4B_{m+1}}} \overline{\varepsilon}_{n,k} w^*_{m+1,\overline{f}(n,k)}(\overline{x})w_{m+1,\overline{f}(n,k)} + \sum_{k=0}^{\widetilde{k}} \overline{\varepsilon}_{\widetilde{n},k} w^*_{m+1,\overline{f}(\widetilde{n},k)}(\overline{x})w_{m+1,\overline{f}(\widetilde{n},k)}.
\end{aligned}$$

At this point, proceeding as for the proof of (54), for $1 \le n \le \widetilde{n}-1$ and $0 \le k \le 2^{4B_{m+1}}$, and for $0 \le k \le \widetilde{k}$ if $n = \widetilde{n}$, there exists a permutation $(\pi(m+1,\overline{f}(n,k),j))_{j=1}^{2^{4M_{m+1}}}$ of $(j)_{j=1}^{2^{4M_{m+1}}}$ such that $(|\sum_{j=1}^{J} x^*_{\pi(m+1,\overline{f}(n,k),j)}(\overline{x})|)_{J=1}^{2^{4M_{m+1}}}$ is $(1,0)$-monotone, and a partition

$$((\pi(m+1,\overline{f}(n,k),j))_{j=t(m+1,\overline{f}(n,k),p-1)+1}^{t(m+1,\overline{f}(n,k),p)})_{p=1}^{T_{m+1}}$$

of $(\pi(m+1,\overline{f}(n,k),j))_{j=1}^{M_{m+1,\overline{f}(n,k)}}$, $M_{m+1,\overline{f}(n,k)} < 2^{4M_{m+1}}$, $t(m+1,\overline{f}(n,k),0) = 0$ and $t(m+1,\overline{f}(n,k),T_{m+1}) = M_{m+1,\overline{f}(n,k)}$, so that, for $1 \le p \le T_{m+1}$,

$$\Big|\sum_{j=t(m+1,\overline{f}(n,k),p-1)+1}^{t(m+1,\overline{f}(n,k),p)} \frac{x^*_{\pi(m+1,\overline{f}(n,k),j)}(\overline{x})}{2^{M_{m+1}}} - \frac{\overline{\varepsilon}_{n,k}w^*_{m+1,\overline{f}(n,k)}(\overline{x})}{T_{m+1}}\Big| < \frac{2K+1}{2^{M_{m+1}}},$$

$$\Big|\sum_{j=1}^{t(m+1,\overline{f}(n,k),p)} \frac{x^*_{\pi(m+1,\overline{f}(n,k),j)}(\overline{x})}{2^{M_{m+1}}} - \frac{p\overline{\varepsilon}_{n,k}w^*_{m+1,\overline{f}(n,k)}(\overline{x})}{T_{m+1}}\Big| < \frac{2K+1}{2^{M_{m+1}}}.$$

By Substep 5 and by (59), for $1 \le P \le T_{m+1}$, setting

$$\overline{w}_{m+1,p} = \sum_{n=1}^{\widetilde{n}-1} \sum_{k=0}^{2^{4B_{m+1}}} \sum_{j=t(m+1,\overline{f}(n,k),p-1)+1}^{t(m+1,\overline{f}(n,k),p)} x^*_{\pi(m+1,\overline{f}(n,k),j)}(\overline{x}) x''_{\pi(m+1,\overline{f}(n,k),j)},$$

we have

$$\Big\| \overline{w}_{m+1,p} - \widehat{u}_{m+1}/T_{m+1} \Big\| < \frac{P_{m+1} 2(2K+1)}{2^{M_{m+1}}},$$

$$\Big\| \sum_{p=1}^{P} \overline{w}_{m+1,p} - \frac{P}{T_{m+1}} \widehat{u}_{m+1} \Big\| < \frac{P_{m+1} 2(2K+1)}{2^{M_{m+1}}}.$$

Now, setting $\overline{w}_{m+1} = \sum_{p=1}^{T_{m+1}} \overline{w}_{m+1,p}$, it follows that

$$\|\overline{w}_{m+1} - \widehat{u}_{m+1}\| < \frac{P_{m+1} 2(2K+1)}{2^{M_{m+1}}}. \tag{60}$$

Moreover, analogously to the proof of (55.2), there exists a permutation $(\pi(m''_n))_{n=1}^{h''(m)}$ of $(m''_n)_{n=1}^{h''(m)}$ such that

$$\Big(\Big\| \sum_{n=1}^{\overline{q}(m)} x^*_{\overline{\pi}(n)}(\overline{x}) x_{\overline{\pi}(n)} + \sum_{n=1}^{q} x^*_{\pi(m''_n)}(\overline{x}) x_{\pi(m''_n)} \Big\| \Big)_{q=1}^{h''(m)} \text{ is } (K, 0, 1/2^{P_m})\text{-monotone.} \tag{61}$$

Therefore by the above, setting

$$\begin{aligned}
(\overline{\pi}(n))_{n=\overline{q}(m)+1}^{\overline{q}(m+1)} &= (\pi(m''_n))_{n=1}^{h''(m)} \\
&\quad \cup ((((\pi(m+1, \overline{f}(n,k), j))_{j=t(m+1,\overline{f}(n,k),p-1)+1}^{t(m+1,\overline{f}(n,k),p)})_{k=0}^{2^{4B_{m+1}}})_{n=1}^{\widetilde{n}-1} \\
&\quad \cup (((\pi(m+1, \overline{f}(\widetilde{n},k), j))_{j=t(m+1,\overline{f}(\widetilde{n},k),p-1)+1}^{t(m+1,\overline{f}(\widetilde{n},k),p)})_{k=0}^{\widetilde{k}})_{p=1}^{T_{m+1}}, \\
(n)_{n=q(m+1)+1}^{q(m+2)} &= ((m+1)'_n)_{n=1}^{h'(m+1)} \cup ((m+1)''_n)_{n=1}^{h''(m+1)}, \\
(\pi(m+1)'_n)_{n=1}^{h'(m+1)} &= ((((\pi(m+1, \overline{f}(n,k), j))_{j=t(m+1,\overline{f}(n,k),p-1)+1}^{t(m+1,\overline{f}(n,k),p)})_{k=0}^{2^{4B_{m+1}}})_{n=1}^{\widetilde{n}-1} \\
&\quad \cup (((\pi(m+1, \overline{f}(\widetilde{n},k), j))_{j=t(m+1,\overline{f}(\widetilde{n},k),p-1)+1}^{t(m+1,\overline{f}(\widetilde{n},k),p)})_{k=0}^{\widetilde{k}})_{p=1}^{T_{m+1}}
\end{aligned}$$

(hence $\overline{q}(m+1) = \overline{q}(m) + h''(m) + h'(m+1)$), we have

$$\begin{aligned}
\Big\| \overline{x} - \Big(\sum_{n=1}^{\overline{q}(m)} x^*_{\overline{\pi}(n)}(\overline{x}) x_{\overline{\pi}(n)} + \sum_{n=1}^{h''(m)} x^*_{\pi((m)''_n)}(\overline{x}) x_{\pi((m)''_n)} \\
+ \sum_{n=1}^{h'(m+1)} x^*_{\pi(m+1)'_n}(\overline{x}) x''_{\pi(m+1)'_n} \Big) \Big\| \\
< \eta'_{m+1} + \frac{1}{2^{2q(m+1)}} + \varepsilon_m + \frac{P_{m+1} 2(2K+1)}{2^{M_{m+1}}},
\end{aligned} \tag{62}$$

since

$$\Big\| \overline{x} - \Big(\sum_{n=1}^{\overline{q}(m)} x^*_{\overline{\pi}(n)}(\overline{x}) x_{\overline{\pi}(n)} + \sum_{n=1}^{h''(m)} x^*_{(m)''_n}(\overline{x}) x_{(m)''_n} + \sum_{n=1}^{h'(m+1)} x^*_{(m+1)'_n}(\overline{x}) x''_{(m+1)'_n} \Big) \Big\|$$

$$
\begin{aligned}
&\le \Big\|\overline{x} - \Big(\sum_{n=1}^{\overline{q}(m)} x^*_{\overline{\pi}(n)}(\overline{x})x_{\overline{\pi}(n)} + \sum_{n=1}^{h''(m)} x^*_{m''_n}(\overline{x})x_{m''_n} + u'_{m+1}\Big)\Big\| \\
&\quad + \|u'_{m+1} - \widetilde{u}_{m+1}\| + \|\widehat{u}_{m+1} - \widetilde{u}_{m+1}\| + \|\overline{w}_{m+1} - \widehat{u}_{m+1}\| \\
&\quad + \Big\|\overline{w}_{m+1} - \sum_{n=1}^{h'(m+1)} x^*_{\pi(m+1)'_n}(\overline{x})x''_{\pi(m+1)'_n}\Big\| \\
&< \eta'_{m+1} \text{ (by (57))} + \frac{1}{2^{2q(m+1)}} \text{ (by (58))} + \varepsilon_m \text{ (by (59))} \\
&\quad + \frac{P_{m+1}2(2K+1)}{2^{M_{m+1}}} \text{ (by (60))} \\
&\quad + 0 \text{ (by the definition of } \overline{w}_{m+1} \text{ and by the above).}
\end{aligned}
$$

Now we claim that

$$
\Big\|\sum_{n=q(m)+1}^{q(m+1)} x^*_n(\overline{x})x'_n\Big\| < K(\eta'_{m-1} + \eta'_{m+1}) \quad \text{for each } m. \tag{63}
$$

Indeed, by (57), setting $E_m = \sum_{n=1}^{q(m)} x^*_n(\overline{x})x_n + u'_m$ for each m, we have $\|\overline{x} - E_{m-1}\| < \eta'_{m-1}$ and $\|\overline{x} - E_{m+1}\| < \eta'_{m+1}$ with

$$
\begin{aligned}
\eta'_{m-1} + \eta'_{m+1} &\ge \|\overline{x} - E_{m-1}\| + \|\overline{x} - E_{m+1}\| \ge \|(\overline{x} - E_{m-1}) - (\overline{x} - E_{m+1})\| \\
&= \|E_{m+1} - E_{m-1}\| \\
&= \Big\|\Big(\sum_{n=1}^{q(m+1)} x^*_n(\overline{x})x_n + u'_{m+1}\Big) - \Big(\sum_{n=1}^{q(m-1)} x^*_n(\overline{x})x_n + u'_{m-1}\Big)\Big\| \\
&> \frac{1}{K}\Big\|\sum_{n=q(m)+1}^{q(m+1)} x^*_n(\overline{x})x'_n\Big\|,
\end{aligned}
$$

by (49.1) and (49.2) and by Substeps 5 and 6 since $u'_{m-1} \in \operatorname{span}(x_n)_{n=q(m-1)+1}^{q(m)}$, $u'_{m+1} \in \operatorname{span}(x_n)_{n=q(m+1)+1}^{q(m+2)}$.

Since $((m+1)'_n)_{n=1}^{h'(m+1)} \subset (n)_{n=q(m+1)+1}^{q(m+2)}$, by (49.1) and by Substep 5 we have

$$
\begin{aligned}
\Big\|\sum_{n=1}^{h'(m+1)} x^*_{(m+1)'_n}(\overline{x})x'_{(m+1)'_n}\Big\| &= \sqrt{\sum_{n=1}^{h'(m+1)} (x^*_{(m+1)'_n}(\overline{x}))^2} \\
&\le \sqrt{\sum_{n=q(m+1)+1}^{q(m+2)} (x^*_n(\overline{x}))^2} = \Big\|\sum_{n=q(m+1)+1}^{q(m+2)} x^*_n(\overline{x})x'_n\Big\|.
\end{aligned}
$$

Hence we conclude that, by (62) and (63),

$$
\begin{aligned}
&\Big\|\overline{x} - \sum_{n=1}^{\overline{q}(m+1)} x^*_{\overline{\pi}(n)}(\overline{x})x_{\overline{\pi}(n)}\Big\| < \widehat{\eta}_{m+1} \\
&\qquad = \eta'_{m+1} + \frac{1}{2^{2q(m+1)}} + \varepsilon_m + P_{m+1}2(2K+1)/2^{M_{m+1}} + K(\eta'_m + \eta'_{m+2}).
\end{aligned}
$$

Now $(\|\sum_{n=1}^{q} x^*_{\overline{\pi}(n)} x_{\overline{\pi}(n)}\|)_{q=\overline{q}(m)+1}^{\overline{q}(m+1)}$ is $(K,0,1/2^{P_m})$-monotone since

$$\Big(\Big\|\sum_{n=1}^{\overline{q}(m)} x^*_{\overline{\pi}(n)}(\overline{x})x_{\overline{\pi}(n)} + \sum_{n=1}^{q} x^*_{\pi(m''_n)}(\overline{x})x_{\pi(m''_n)}\Big\|\Big)_{q=1}^{h''(m)}$$

is $(K,0,1/2^{P_m})$-monotone by (61), while

$$\Big(\Big\|\sum_{n=1}^{\overline{q}(m)} x^*_{\overline{\pi}(n)}(\overline{x})x_{\overline{\pi}(n)} + \sum_{n=1}^{h''(m)} x^*_{(m)''_n}(\overline{x})x_{(m)''_n} + \sum_{n=1}^{q} x^*_{(m+1)'_n}(\overline{x})x''_{(m+1)'_n}\Big\|\Big)_{q=1}^{h'(m+1)}$$

is $(1,0,1/2^{P_{m+1}})$-monotone (hence also $(K,0,1/2^{P_m})$-monotone) by the procedure before (60) and by a proof analogous to the proof of (54). Moreover, by (51),

$$\Big(\Big\|\sum_{n=1}^{\overline{q}(m)} x^*_{\overline{\pi}(n)}(\overline{x})x_{\overline{\pi}(n)} + \sum_{n=1}^{h''(m)} x^*_{(m)''_n}(\overline{x})x_{(m)''_n} + \sum_{n=1}^{q} x^*_{(m+1)'_n}(\overline{x})x'_{(m+1)'_n}\Big\|\Big)_{q=1}^{h'(m+1)}$$

is $(K,0,0)$-monotone; and

$$\|\overline{x} - \sum_{n=1}^{q} x^*_{\overline{\pi}(n)}(\overline{x})x_{\overline{\pi}(n)}\| < \overline{\eta}_{m+1} = (1+K)\widehat{\eta}_m + K\widehat{\eta}_{m+1} + 1/2^{P_m}$$

for $\overline{q}(m)+1 \le q \le \overline{q}(m+1)$ (see the proof of (55.4)). Hence, by (55.1), (55.2) and (55.4), we see that both (48.1) and (52) are satisfied, hence also (48.2) and (48.3). This completes the proof of Theorem 22. ■

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