1. Introduction

In this paper we are concerned with the semilinear Neumann problem

$$
\begin{cases} -\Delta u + \lambda u = Q(x)|u|^{2^*-2}u + h(x)|u|^{q-2}u & \text{in } \Omega, \\ \partial u/\partial \nu(x) = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}
$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$, the coefficients Q and h are continuous on $\overline{\Omega}$, Q is positive on $\overline{\Omega}$ and $\lambda > 0$ is a parameter. We take $N \geq 3$ and denote by $2^* = 2N/(N-2)$ the critical Sobolev exponent. The exponent q satisfies the inequality $2 \leq q < 2^*$. In the second part of this work we consider a modified problem (1.1) with Q replaced by $-Q$. In this case the exponent 2^* can be replaced by any $p > q$ and we no longer require the coefficients to be smooth.

Throughout this work by a solution of problem (1.1) we mean a nontrivial solution.

Solutions of (1.1) are sought in the Sobolev space $H^1(\Omega)$. We recall that by $H^1(\Omega)$ we denote the usual Sobolev space equipped with the norm

$$
\|u\|^2 = \int_\Omega (|\nabla u|^2 + u^2)\,dx.
$$

Semilinear Neumann problems arise in the study of mathematical models in biological formation theory governed by diffusion and cross-diffusion systems [42]. Such problems also have a number of applications in various branches of differential geometry [32], [46]. The pioneering paper by Brézis and Nirenberg [21] has inspired research on elliptic equations with critical Sobolev exponents.

If $Q \equiv 1$ and $h \equiv 0$, then problem (1.1) has an extensive literature. We refer to the papers [2]–[7], [34], [51]–[57], [43], [44], where the existence of least energy solutions and their properties have been investigated. In these papers, solutions of (1.1) were obtained as minimizers of a functional

$$
I_\lambda(u) = \frac{\int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx}{(\int_\Omega |u|^{2^*}\,dx)^{2/2^*}}
$$

on $H^1(\Omega) \setminus \{0\}$. A minimizer u of I_λ over $H^1(\Omega) \setminus \{0\}$ is called a *least energy solution*, that is,

$$
m_\lambda = \inf_{v \in H^1(\Omega)\setminus\{0\}} I_\lambda(v) = I_\lambda(u).
$$

The main idea in the proof of the existence of a least energy solution is to show that $m_\lambda < S/2^{2/N}$, where S is the best Sobolev constant. The inequality $m_\lambda < S/2^{2/N}$ allows us to show that every minimizing sequence is relatively compact in $H^1(\Omega)$.

It is easily verified that problem (1.1) always has a constant solution $\lambda^{1/(2^*-2)}$. However, comparing the energy levels of least energy solutions and constant solutions, one can show that least energy solutions are nonconstant for large λ. Moreover, it has been proved in [8] that there exists $\lambda_\circ > 0$ such that least energy solutions for $\lambda < \lambda_\circ$ are constant. The least energy solutions u_λ can be chosen to be positive and have the following concentration property: they are single-peaked in the sense that every u_λ, for λ large, attains its unique maximum at a point $P_\lambda \in \partial\Omega$ and $P_\lambda \to P_\circ$, as $\lambda \to \infty$, with $H(P_\circ) = \max_{P\in\partial\Omega} H(P)$, where H is the mean curvature of $\partial\Omega$ with respect to the inner normal. These results have been extended to the case $Q \not\equiv \text{const}$ and $h \equiv 0$ in the papers [23], [24] and [27].

The purpose of this work is twofold. Firstly we investigate the combined effect of both coefficients Q and h and the mean curvature of $\partial\Omega$ on the existence and nonexistence of solutions of problem (1.1). The existence results depend on the relationship between the global maximum $Q_{\mathrm{M}} = \max_{x\in\overline{\Omega}} Q(x)$ and $Q_{\mathrm{m}} = \max_{x\in\partial\Omega} Q(x)$. The first part of this work focuses on seeking the so-called *low energy* solutions, generated as the limits of Palais–Smale sequences. According to [25] a higher energy Palais–Smale sequence of (1.1), with a nonconstant coefficient Q, displays a very complicated behaviour and can concentrate at any point of Ω. The only Palais–Smale sequences that are relatively easy to control are those corresponding to a low energy level of a variational functional of (1.1).

In the second part of this work we consider problem (1.1) with Q replaced by $-Q$. The existence results will be described in terms of some integrability conditions imposed on Q and h. In this case the influence of the relationship of Q_{M} and Q_{m} as well as of the mean curvature of $\partial\Omega$ completely disappears. Moreover the term $-Q(x)|u|^{2^*-2}u$ can be replaced by $-Q(x)|u|^{p-2}u$ with $q < p$. The underlying Sobolev space $H^1(\Omega)$ is now replaced by a weighted Sobolev space. However, in order to get the existence of solutions we must restrict the parameter λ to an interval $(-\infty, \lambda_\circ]$ with $0 < \lambda_\circ \le \infty$. We present conditions guaranteeing $\lambda_\circ < \infty$ and $\lambda_\circ = \infty$.

Throughout this work we use standard notations. The norms in the Lebesgue spaces $L^p(\Omega)$, $1 \le p < \infty$, are denoted by $\|\cdot\|_p$. If h is a measurable and positive a.e. function on Ω, then by $L^p(\Omega, h)$ we denote the weighted Lebesgue space equipped with the norm

$$\|u\|_{p,h}^p = \int_\Omega |u(x)|^p h(x)\,dx.$$

The symbol $|A|$ stands for the Lebesgue measure of $A \subset \mathbb{R}$. In a Banach space X we denote by "$\to$" the strong convergence and a weak convergence is denoted by "$\rightharpoonup$". We always denote by $\langle\cdot,\cdot\rangle$ the duality pairing between the Banach space X and its dual X^*.

The paper is organized as follows. The first part of this paper consisting of Sections 1–8 is devoted to problem (1.1). In the remaining sections we examine equation (1.1) with Q replaced by $-Q$. In Sections 3–4 solutions to problem (1.1) are found through the mountain-pass principle [12]. These solutions are low energy solutions. To apply the mountain-pass principle we need the Palais–Smale condition. Energy levels of the variational functional for problem (1.1) below which the Palais–Smale condition holds are investigated in Section 3. The existence and nonexistence results are given in Sections 4 and 6. In Section 5, we study the existence of multiple solutions in terms of the Lusternik–Schnirelmann category of level sets of Q on the boundary or interior of the domain. In Section 7 we consider the problem (1.1) at resonance, that is, for $\lambda = 0$. This problem has already been studied in the paper [23] with $h \equiv 0$. However, in this case solutions exist if Q changes sign in Ω and $\int_\Omega Q(x)\,dx < 0$. Here the presence of a lower order nonlinearity with a coefficient h changing sign allows us to establish the existence of a solution when Q is positive. In Section 8 we implement the fountain theorem [16] to generate infinitely many solutions when $1 < q < 2$.

Sections 9, 10 and 11 are devoted to problem (1.1) with Q replaced by $-Q$. In this case the variational functional no longer has the mountain-pass geometry and instead we seek solutions through a local minimization. This approach does not require smoothness of the coefficients Q and h. We obtain some existence results under rather general integrability conditions on Q and h. This situation is discussed in Section 9. Some results without the integrability conditions on Q and h are given in Section 10. The main ingredient here is the use of the Hardy inequality. In Section 11 we consider the case where Q and h vanish on some subsets of Ω. In this situation it is not clear whether the Palais–Smale condition holds. The existence of solutions is obtained through constrained minimization. The set of constraints consists of functions between a sub and supersolution of (11.1). It is relatively easy to construct a subsolution. However the construction of a supersolution is more involved and it is obtained through the bifurcation theorem [30]. These solutions are of negative energy. In the second part of Section 12 we adopt a different approach to problem (11.1). We apply the mountain-pass theorem to a truncated variational functional to obtain solutions with positive energy. Section 12 concentrates on problem (1.1) when 2^* is replaced by a supercritical exponent, and the coefficient Q is replaced with μQ, some small $\mu > 0$. Assuming that the coefficients Q and h are positive and in $L^\infty(Q)$ we establish the existence of a solution in $H^1(\Omega) \cap L^\infty(\Omega)$. The final Section 13 is devoted to the study of semilinear parabolic equations involving the critical Sobolev exponent. The optimal Sobolev inequalities from Section 3 are used to derive criteria for blow-up and no blow-up of solutions.

2. Preliminaries

In this work we frequently use an equivalent norm in $H^1(\Omega)$: $\|\cdot\|_\lambda$ defined by

$$\|u\|_\lambda = \int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx.$$

By S we denote the best Sobolev constant defined by

$$S = \inf\Big\{ \int_{\mathbb{R}^N} |\nabla u|^2\,dx : u \in D^{1,2}(\mathbb{R}^N),\ \int_{\mathbb{R}^N} |u|^{2^*}\,dx = 1 \Big\},$$

where $D^{1,2}(\mathbb{R}^N)$ is the Sobolev space obtained as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|^2_{D^{1,2}} = \int_{\mathbb{R}^N} |\nabla u|^2\, dx.$$

The best Sobolev constant S is achieved by

$$U(x) = \frac{c_N}{(1+|x|^2)^{(N-2)/2}},$$

where $c_N > 0$ is a constant depending on N. The function U, called an *instanton*, satisfies the equation

$$-\Delta U = U^{2^*-1} \quad \text{in } \mathbb{R}^N.$$

We use the notation

$$U_{\varepsilon,y}(x) = \varepsilon^{-(N-2)/2} U\left(\frac{x-y}{\varepsilon}\right), \qquad \varepsilon > 0,\ y \in \mathbb{R}^N.$$

We frequently refer to the Sobolev inequality

$$\left(\int_\Omega |u|^{2^*}\, dx\right)^{2/2^*} \le C_s \int_\Omega (|\nabla u|^2 + u^2)\, dx \tag{2.1}$$

for all $u \in H^1(\Omega)$, where $C_s > 0$ is a constant. Letting $C_s(\lambda) = C_s$ for $\lambda \ge 1$ and $C_s(\lambda) = C_s/\lambda$ for $0 < \lambda < 1$ we can write inequality (2.1) in the following form:

$$\left(\int_\Omega |u|^{2^*}\, dx\right)^{2/2^*} \le C_s(\lambda) \int_\Omega (|\nabla u|^2 + \lambda u^2)\, dx \tag{2_λ}$$

for all $u \in H^1(\Omega)$. Throughout this work we shall often use P. L. Lions's concentration-compactness principle [39]:

If $u_m \rightharpoonup u$ in $H^1(\Omega)$, then there exist Borel measures μ and ν such that

$$|\nabla u_m|^2 \overset{*}{\rightharpoonup} \mu \quad \text{and} \quad |u_m|^{2^*} \overset{*}{\rightharpoonup} \nu$$

weakly in the sense of measures, where

$$\mu \ge |\nabla u|^2 + \sum_{j\in J} \mu_j \delta_{x_j}, \qquad \nu = |u|^{2^*} + \sum_{j\in J} \nu_j \delta_{x_j}.$$

Here the set of indices J is at most countable and the constants $\nu_j > 0$ and $\mu_j > 0$ satisfy:

$$\text{if } x_j \in \Omega, \quad \text{then} \quad S\nu_j^{2/2^*} \le \mu_j; \tag{2.2}$$

$$\text{if } x_j \in \partial\Omega, \quad \text{then} \quad \frac{S\nu_j^{2/2^*}}{2^{2/N}} \le \mu_j. \tag{2.3}$$

We associate with problem (1.1) a variational functional J_λ given by

$$J_\lambda(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + \lambda u^2)\, dx - \frac{1}{2^*}\int_\Omega Q(x)|u|^{2^*}\, dx - \frac{1}{q}\int_\Omega h(x)|u|^q\, dx.$$

Critical points of J_λ are solutions of problem (1.1). Critical points of J_λ can be taken to be positive on Ω. Indeed, we can modify the nonlinearity in (1.1) by setting $f(t) = Q(x)t^{2^*-1} + h(x)t^{q-1}$ for $t \ge 0$ and $f(x,t) = 0$ for $t < 0$. If $u \not\equiv 0$ is a critical point of J_λ, then

$$0 = \langle J'_\lambda(u), u^-\rangle = \int_\Omega (|\nabla u^-|^2 + \lambda (u^-)^2)\, dx = 0.$$

Thus $u^- \equiv 0$ and by Hopf's boundary point lemma $u > 0$ on Ω. To find critical points of J_λ we use the mountain-pass theorem [12]. First we check that J_λ has a mountain-pass geometry. If $u \in H^1(\Omega)$, then

$$\begin{aligned} J_\lambda(u) &\ge \frac{\min(1,\lambda)}{2}\|u\|^2 - \|Q\|_\infty C_s^{2^*/2}\|u\|^{2^*} - |\Omega|^{1-q/2^*}\|h\|_\infty C_s^{q/2}\|u\|^q \\ &= \|u\|^2\biggl(\frac{\min(1,\lambda)}{2} - \|Q\|_\infty C_s^{2^*/2}\|u\|^{2^*-2} - \|h\|_\infty C_s^{q/2}\|u\|^{q-2}\biggr). \end{aligned}$$

Hence there exists a constant $\varrho = \varrho(\lambda, C_s, \|Q\|_\infty, \|h\|_\infty) > 0$ such that

$$J_\lambda(u) \ge \frac{\min(1,\lambda)}{4}\varrho^2 \quad \text{for } \|u\| = \varrho. \tag{2.4}$$

It is easy to see that for each $\varepsilon \in (0, \varepsilon_\circ]$, with $\varepsilon_\circ$ small and fixed, and $y \in \mathbb{R}^N$ we have

$$J_\lambda(tU_{\varepsilon,y}) < 0 \quad \text{and} \quad \|tU_{\varepsilon,y}\| > \varrho \tag{2.5}$$

for $t > 0$ sufficiently large. The mountain-pass level is defined by

$$c_\lambda = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_\lambda(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], H^1(\Omega)) : \gamma(0) = 0,\ \gamma(1) = tU_{\varepsilon,y}\}$. In the next section we shall examine Palais–Smale sequences for J_λ.

Solutions of problem (1.1) are in $C^{1,\alpha}(\overline{\Omega})$. This can be deduced from the following two lemmas whose proofs can be found in the paper [51].

Lemma 2.1. *Suppose that $\partial\Omega \in C^1$ and $u \in H^1(\Omega)$ is a solution of the problem*

$$\begin{cases} -\Delta u = a(x)u & \text{in } \Omega, \\ \partial u/\partial\nu = 0 & \text{on } \partial\Omega, \end{cases}$$

with $a \in L^{N/2}(\Omega)$, then $u \in L^t(\Omega)$ for every $t \ge 1$.

Lemma 2.2. *Suppose that $\partial\Omega \in C^2$ and $f \in L^p(\Omega)$ with $1 < p < \infty$. If u is a solution of the problem*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \partial u/\partial\nu = 0 & \text{on } \partial\Omega, \end{cases}$$

then $\|u\|_{H^{2,p}(\Omega)} \le C\|f\|_p$ for some constant $C > 0$.

First we apply Lemma 2.1 with $a(x) = Q(x)|u|^{2^*-2} + h(x)|u|^{q-2}$ and then Lemma 2.2 with $f(x) = Q(x)|u|^{2^*-2}u + h(x)|u|^{q-2}u$.

3. The Palais–Smale condition

We recall that $\{u_m\} \subset H^1(\Omega)$ is said to be a *Palais–Smale sequence* at a level c ($(\mathrm{PS})_c$ sequence for short) if $J_\lambda(u_m) \to c$ and $J'_\lambda(u_m) \to 0$ in $H^{-1}(\Omega)$.

We say that J_λ satisfies the *Palais–Smale condition* at a level c ($(\mathrm{PS})_c$ condition for short) if every $(\mathrm{PS})_c$ sequence is relatively compact in $H^1(\Omega)$.

Let $Q_{\mathrm{m}} = \max_{x\in\partial\Omega} Q(x)$ and $Q_{\mathrm{M}} = \max_{x\in\overline{\Omega}} Q(x)$. We set

$$S_\infty = \min\left(\frac{S^{N/2}}{2NQ_{\mathrm{m}}^{(N-2)/2}}, \frac{S^{N/2}}{NQ_{\mathrm{M}}^{(N-2)/2}}\right).$$

Proposition 3.1. *For every $\lambda > 0$ the functional J_λ satisfies the* $(\mathrm{PS})_c$ *condition with* $c < S_\infty$.

Proof. The concentration-compactness principle is invoked to establish that any $(\mathrm{PS})_c$ sequence is relatively compact. We establish this result only in the case $Q_{\mathrm{M}} \le 2^{2/(N-2)} Q_{\mathrm{m}}$. In this case $S_\infty = S^{N/2}/(2NQ_{\mathrm{m}}^{(N-2)/2})$ but the case $Q_{\mathrm{M}} > 2^{2/(N-2)}Q_{\mathrm{m}}$ with $S_\infty = S^{N/2}/(NQ_{\mathrm{M}}^{(N-2)/2})$ can be treated in the same way. Let $\{u_m\} \subset H^1(\Omega)$ be such that

$$J_\lambda(u_m) \to c < \frac{S^{N/2}}{2NQ_{\mathrm{m}}^{(N-2)/2}} \quad \text{and} \quad J'_\lambda(u_m) \to 0 \ \text{in} \ H^{-1}(\Omega). \tag{3.1}$$

Indeed, for large m, say $m \ge m_\circ$, we have

$$\begin{aligned} c + 1 + o(\|u_m\|) &\ge J_\lambda(u_m) - \frac{1}{q}\langle J'_\lambda(u_m), u_m\rangle \\ &= \left(\frac{1}{2} - \frac{1}{q}\right)\int_\Omega (|\nabla u_m|^2 + \lambda u_m^2)\,dx + \left(\frac{1}{q} - \frac{1}{2^*}\right)\int_\Omega Q|u_m|^{2^*}\,dx. \end{aligned}$$

From this we deduce that $\{u_m\}$ is bounded in $H^1(\Omega)$. Therefore we may assume that $u_m \rightharpoonup u$ in $H^1(\Omega)$, $u_m \to u$ in $L^q(\Omega)$ and a.e. on Ω. By the concentration-compactness principle we have

$$|\nabla u_m|^2 \overset{*}{\rightharpoonup} |\nabla u|^2 + \sum_{J\in J} \mu_j \delta_{x_j}, \qquad |u_m|^{2^*} \overset{*}{\rightharpoonup} |u|^{2^*} + \sum_{j\in J} \nu_j \delta_{x_j},$$

with μ_j and ν_j satisfying (2.2) and (2.3), respectively. Applying a family of test functions concentrating at x_j and using the second condition of (3.1) we get $Q(x_j)\nu_j = \mu_j$ for each $j \in J$. It then follows from (2.2) and (2.3) that if $\nu_j > 0$ for some $j \in J$, then

$$\nu_j \ge \frac{S^{N/2}}{Q(x_j)^{N/2}} \qquad \text{if } x_j \in \Omega, \tag{3.2}$$

$$\nu_j \ge \frac{S^{N/2}}{2Q(x_j)^{N/2}} \qquad \text{if } x_j \in \partial\Omega. \tag{3.3}$$

We now show that $\nu_j = 0$ for each $j \in J$. We write

$$\begin{aligned} J_\lambda(u_m) - \frac{1}{q}\langle J'_\lambda(u_m), u_m\rangle &= \left(\frac{1}{2} - \frac{1}{q}\right)\int_\Omega (|\nabla u_m|^2 + \lambda u_m^2)\,dx \\ &\quad + \left(\frac{1}{q} - \frac{1}{2^*}\right)\int_\Omega Q(x)|u_m|^{2^*}\,dx. \end{aligned} \tag{3.4}$$

Letting $m \to \infty$ we get

$$\begin{aligned} c &\ge \left(\frac{1}{2} - \frac{1}{q}\right)\sum_{j\in J}\mu_j + \left(\frac{1}{q} - \frac{1}{2^*}\right)\sum_{j\in J} Q(x_j)\nu_j \\ &= \frac{1}{N}\sum_{j\in J} Q(x_j)\nu_j = \frac{1}{N}\sum_{x_j\in\Omega}\nu_j Q(x_j) + \frac{1}{N}\sum_{x_j\in\partial\Omega}\nu_j Q(x_j). \end{aligned}$$

If $\nu_j > 0$ for some $x_j \in \Omega$, then by (3.2)

$$c \geq \frac{S^{N/2}}{NQ(x_j)^{(N-2)/2}} \geq \frac{S^{N/2}}{NQ_{\mathrm{M}}^{(N-2)/2}} \geq \frac{S^{N/2}}{2NQ_{\mathrm{m}}^{(N-2)/2}},$$

which is impossible. Similarly, if $x_j \in \partial\Omega$, then by (3.3)

$$c \geq \frac{S^{N/2}}{2NQ(x_j)^{(N-2)/2}} \geq \frac{S^{N/2}}{2NQ_{\mathrm{m}}^{(N-2)/2}}$$

and again we have arrived at a contradiction. This means that $\nu_j = 0$ for each $j \in J$ and $u_m \to u$ in $L^{2^*}(\Omega)$. This yields, using $J'_\lambda(u_m) \to 0$ in $H^{-1}(\Omega)$, $u_m \to u$ in $H^1(\Omega)$. ∎

We need some weighted Sobolev inequalities whose proof can be found in the papers [27] and [24] (see also [59] for some related results):

(I) Let $N \geq 5$ and $Q_{\mathrm{M}} > 2^{2/(N-2)} Q_{\mathrm{m}}$. Then there exists a constant $\Lambda_1 > 0$ such that

$$\Big(\int_\Omega Q(x)|u|^{2^*}\,dx\Big)^{2/2^*} \leq \frac{Q_{\mathrm{M}}^{(N-2)/N}}{S}\Big(\int_\Omega |\nabla u|^2\,dx + \Lambda_1 \int_\Omega u^2\,dx\Big)$$

for all $u \in H^1(\Omega)$.

To formulate other weighted Sobolev inequalities we need the following assumptions:

(S_1) $\{x \in \partial\Omega : H(x) < 0\} \neq \emptyset$ and $\{x \in \partial\Omega : Q(x) = Q_{\mathrm{m}}\} \subset \{x \in \partial\Omega : H(x) < 0\}$ and

$$|Q(x) - Q(x_\circ)| = o(|x - x_\circ|) \quad \text{as } |x - x_\circ| \to 0$$

for every $x_\circ \in \partial\Omega$ such that $Q_{\mathrm{m}} = Q(x_\circ)$.

(S_2) $D(0,a) \subset \partial\Omega$ for some $a > 0$, where $D(0,a) = B(0,a) \cap \{x_N = 0\}$ and $\{x \in \partial\Omega : Q(x) = Q_{\mathrm{m}}\} \subset D(0,a/2)$ and moreover for every $x_\circ \in D(0,a/2)$,

$$|Q(x) - Q(x_\circ)| = o(|x - x_\circ|^2) \quad \text{as } x \to x_\circ.$$

We are now in a position to state two weighted Sobolev inequalities corresponding to assumptions (S_1) and (S_2):

(II) Let $N \geq 5$ and $Q_{\mathrm{M}} \leq 2^{2/(N-2)} Q_{\mathrm{m}}$ and suppose that (S_1) holds. Then there exists a constant $\Lambda_2 > 0$ such that

$$\Big(\int_\Omega Q(x)|u|^{2^*}\,dx\Big)^{2/2^*} \leq \frac{2^{2/N} Q_{\mathrm{m}}^{(N-2)/N}}{S}\Big(\int_\Omega |\nabla u|^2\,dx + \Lambda_2 \int_\Omega u^2\,dx\Big)$$

for all $u \in H^1(\Omega)$.

(III) Let $N \geq 5$ and $Q_{\mathrm{M}} \leq 2^{2/(N-2)} Q_{\mathrm{m}}$ and suppose that (S_2) holds. Then there exists a constant $\Lambda_3 > 0$ such that

$$\Big(\int_\Omega Q(x)|u|^{2^*}\,dx\Big)^{2/2^*} \leq \frac{2^{2/N} Q_{\mathrm{m}}^{(N-2)/N}}{S}\Big(\int_\Omega |\nabla u|^2\,dx + \Lambda_3 \int_\Omega u^2\,dx\Big)$$

for every $u \in H^1(\Omega)$.

Solutions u of (1.1) satisfying $J_\lambda(u) < S_\infty$ will be referred to as *low energy solutions.*

LEMMA 3.2. *Let $N \geq 5$.*

(i) *Suppose that $Q_{\mathrm{M}} > 2^{2/(N-2)} Q_{\mathrm{m}}$. If $h \leq 0$ on Ω, then problem* (1.1) *does not have a low energy solution for $\lambda \geq \Lambda_1$.*

(ii) *Suppose that $Q_{\mathrm{M}} \leq 2^{2/(N-2)} Q_{\mathrm{m}}$. If* ($S_1$) *holds and $h \leq 0$ on Ω, then problem* (1.1) *does not have a low energy solution for $\lambda \geq \Lambda_2$.*

(iii) *Suppose that $Q_{\mathrm{M}} \leq 2^{2/(N-2)} Q_{\mathrm{m}}$. If* ($S_2$) *holds and $h \leq 0$ on Ω, then problem* (1.1) *does not have a low energy solution for $\lambda \geq \Lambda_3$.*

Proof. We only prove (i). Let u be a solution of (1.1), with $\lambda \geq \Lambda_1$, satisfying $J_\lambda(u) < S^{N/2}/(N Q_{\mathrm{M}}^{(N-2)/2})$. Then it follows from inequality (I) that

$$\int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx = \int_\Omega Q|u|^{2^*}\,dx + \int_\Omega h|u|^q\,dx \leq \int_\Omega Q|u|^{2^*}\,dx \leq \left(\frac{Q_{\mathrm{M}}^{(N-2)/N}}{S}\right)^{2^*/2} \Big(\int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx\Big)^{2^*/2}.$$

Hence

$$\frac{S^{N/(N-2)}}{Q_{\mathrm{M}}} < \Big(\int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx\Big)^{2/(N-2)}$$

and

$$J_\lambda(u) - \frac{1}{2^*}\langle J'_\lambda(u), u\rangle \geq \frac{1}{N}\int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx \geq \frac{S^{N/2}}{N Q_{\mathrm{M}}^{(N-2)/2}},$$

which is impossible for $\lambda > 0$ large. ∎

It is easy to see that one can always obtain a solution through the mountain-pass theorem for $\lambda > 0$ small and for h with small norm $\|h\|_\infty$.

4. Existence of solutions of problem (1.1) for every $\lambda > 0$

In this section we present some existence results for each $\lambda > 0$ in both cases $Q_{\mathrm{M}} > 2^{2/(N-2)} Q_{\mathrm{m}}$ and $Q_{\mathrm{M}} \leq 2^{2/(N-2)} Q_{\mathrm{m}}$. If $Q_{\mathrm{M}} > 2^{2/(N-2)} Q_{\mathrm{m}}$ and $h \equiv 0$ on Ω, then problem (1.1) does not have low energy solutions for $\lambda > \Lambda_1$, where Λ_1 is a constant from inequality (I) (see [27]). However, the presence of a coefficient h with $h(y) > 0$ for some $y \in \{x \in \Omega : Q(x) = Q_{\mathrm{M}}\}$ produces low energy solutions for all $\lambda > 0$.

THEOREM 4.1. *Suppose that $Q_{\mathrm{M}} \geq 2^{2/(N-2)} Q_{\mathrm{m}}$, $h(x) \geq 0$ on Ω and $h(x) > 0$ for each $x \in \{x : Q(x) = Q_{\mathrm{M}}\}$.*

(i) *If $N \geq 4$, $2 < q < 2^*$ and Q is C^2 on $B(y, \delta) \subset \Omega$ and $D_{ij} Q(y) = 0$, $i, j = 1, \ldots, N$, for some $y \in \{x; Q(x) = Q_{\mathrm{M}}\}$, then problem* (1.1) *has a solution for every $\lambda > 0$.*

(ii) *If $N \geq 3$, $2(N-1)/(N-2) < q < 2^*$ and Q is differentiable at some point $y \in \{x : Q(x) = Q_{\mathrm{M}}\}$, then problem* (1.1) *has a solution for every $\lambda > 0$.*

Proof. We set

$$c^*_\lambda = \inf_{u \in H^1(\Omega)} \max_{t \geq 0} J_\lambda(tu).$$

It is well known that $c_\lambda \le c_\lambda^*$. We only consider the case $2(N-1)/(N-2) < q < 2^*$. For simplicity we assume that $0 \in \{x : Q(x) = Q_{\mathrm{M}}\}$ and $D_iQ(0) = 0$, $i = 1, \dots, N$, and set $u_\varepsilon = U_{\varepsilon,0}$. For each $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$J_\lambda(t_\varepsilon u_\varepsilon) = \max_{0\le t<\infty} J_\lambda(tu_\varepsilon) = \frac{t_\varepsilon^2}{2}\int_\Omega (|\nabla u_\varepsilon|^2 + \lambda u_\varepsilon^2)\,dx - \frac{t_\varepsilon^{2^*}}{2^*}\int_\Omega Q(x)u_\varepsilon^{2^*}\,dx - \frac{t_\varepsilon^q}{q}\int_\Omega h(x)u_\varepsilon^q\,dx,$$

where t_ε satisfies $0 < t_\circ \le t_\varepsilon \le M < \infty$, with $t_\circ$ and M independent of ε for ε small. We may assume that $\int_\Omega h(x)u_\varepsilon^q\,dx > 0$ for small $\varepsilon > 0$. We now choose $0 < t_* \le t_\circ$ so that

$$J_\lambda(tu_\varepsilon) < \frac{S^{N/2}}{NQ_{\mathrm{M}}^{(N-2)/2}}$$

for $0 \le t \le t_*$. Then for $t_* \le t$ we have

$$\begin{aligned} J_\lambda(tu_\varepsilon) &\le \frac{t^2}{2}\int_\Omega (|\nabla u_\varepsilon|^2 + \lambda u_\varepsilon^2)\,dx - \frac{t^{2^*}}{2^*}\int_\Omega Q(x)u_\varepsilon^{2^*}\,dx - \frac{t_*^q}{q}\int_\Omega h(x)u_\varepsilon^q\,dx \\ &\le \max_{0\le t<\infty}\left[\frac{t^2}{2}\int_\Omega (|\nabla u_\varepsilon|^2 + \lambda u_\varepsilon^2)\,dx - \frac{t^{2^*}}{2^*}\int_\Omega Q(x)u_\varepsilon^{2^*}\,dx\right] - \frac{t_*^q}{q}\int_\Omega h(x)u_\varepsilon^q\,dx \\ &:= M_\varepsilon - \frac{t_*^q}{q}\int_\Omega h(x)u_\varepsilon^q\,dx. \end{aligned} \tag{4.1}$$

We now observe that there exists $\bar t_\varepsilon > 0$ such that

$$M_\varepsilon = \frac{\bar t_\varepsilon^2}{2}\int_\Omega (|\nabla u_\varepsilon|^2 + \lambda u_\varepsilon^2)\,dx - \frac{\bar t_\varepsilon^{2^*}}{2^*}\int_\Omega Q(x)u_\varepsilon^{2^*}\,dx = \frac{1}{N}\left(\frac{\int_\Omega (|\nabla u_\varepsilon|^2 + \lambda u_\varepsilon^2)\,dx}{(\int_\Omega Q(x)u_\varepsilon^{2^*}\,dx)^{(N-2)/N}}\right)^{N/2}.$$

To proceed further we need the following asymptotic formulae:

$$\int_\Omega |\nabla u_\varepsilon|^2\,dx = K_1 + O(\varepsilon^{N-2}), \qquad \int_\Omega Q(x)u_\varepsilon^{2^*}\,dx = K_2Q_{\mathrm{M}} + O(\varepsilon),$$

where $K_1 = \int_{\mathbb{R}^N} |\nabla U|^2\,dx$, $K_2 = \int_{\mathbb{R}^N} U^{2^*}\,dx$ and $S = K_1/K_2^{(N-2)/N}$. Thus

$$\begin{aligned} M_\varepsilon &= \frac{1}{N}\left(\frac{K_1 + O(\varepsilon^{N-2})}{(Q_{\mathrm{M}}K_2 + O(\varepsilon))^{(N-2)/N}}\right)^{N/2} + \lambda O(\varepsilon^2) \\ &= \frac{1}{N}[K_1^{N/2} + O(\varepsilon^{N-2})][(Q_{\mathrm{M}}K_2)^{-(N-2)/2} + O(\varepsilon)] + O(\lambda\varepsilon^2) \\ &= \frac{1}{N}\frac{K_1^{N/2}}{Q_{\mathrm{M}}^{(N-2)/2}K_2^{(N-2)/2}} + O(\varepsilon) + O(\lambda\varepsilon^2). \end{aligned}$$

From this we deduce the estimate

$$J_\lambda(tu_\varepsilon) \le \frac{S^{N/2}}{NQ_{\mathrm{M}}^{(N-2)/2}} + O(\varepsilon) + O(\lambda\varepsilon^2) - \frac{t_*^q}{q}\int_\Omega h(x)u_\varepsilon^q\,dx \tag{4.2}$$

for each $t \ge 0$. We now use the following estimate:

$$\int_\Omega u_\varepsilon^q\,dx \ge C\varepsilon^{N(1-q/2^*)}$$

for some $C > 0$ independent of ε, which is valid for $N/(N-2) < q$. Since $N/(N-2) < 2(N-1)/(N-2) < q$ we derive from this estimate that

$$\int_\Omega h(x)u_\varepsilon^q\,dx \geq ah(0)\varepsilon^{N-(N-2)q/2} \tag{4.3}$$

for some constant $a > 0$ independent of ε. Since $0 < N-(N-2)q/2 < 1$, we deduce from (4.2) and (4.3) that

$$J_\lambda(tu_\varepsilon) \leq \frac{S^{N/2}}{NQ_{\mathrm{M}}^{(N-2)/2}} + \lambda O(\varepsilon^2) + O(\varepsilon) - t_*^q ah(0)\varepsilon^{N-(N-2)q/2} < \frac{S^{N/2}}{NQ_{\mathrm{M}}^{(N-2)/2}}$$

for small $\varepsilon > 0$. The proof in case (ii) when $2 < q < 2^*$ is similar. In this case the asymptotic formula for $\int_\Omega Q(x)u_\varepsilon^{2^*}\,dx$ is replaced by

$$\int_\Omega Q(x)u_\varepsilon^{2^*}\,dx = K_2Q_{\mathrm{M}} + O(\varepsilon^2),$$

which yields

$$\frac{1}{N}M_\varepsilon = \begin{cases} S^{N/2}/Q_{\mathrm{M}}^{(N-2)/2} + \lambda c\varepsilon^2\log(\varepsilon) + O(\varepsilon^2) & \text{if } N = 4,\\ S^{N/2}/Q_{\mathrm{M}}^{(N-2)/2} + \lambda c\varepsilon^2 + O(\varepsilon^2) & \text{if } N \geq 5. \ \blacksquare\end{cases}$$

We now turn our attention to the case $Q_{\mathrm{M}} \leq 2^{2/(N-2)}Q_{\mathrm{m}}$.

THEOREM 4.2. (i) *Let* $Q_{\mathrm{M}} \leq 2^{2/(N-2)}Q_{\mathrm{m}}$, $h \geq 0$ *on* Ω *and* $\{x \in \partial\Omega : Q(x) = Q_{\mathrm{m}}\} \subset \{x \in \partial\Omega : H(x) > 0\}$. *Moreover, assume that* $|Q(x)-Q(y)| = o(|x-y|)$ *for some* $y \in \partial\Omega$ *with* $Q_{\mathrm{m}} = Q(y)$. *Then problem* (1.1) *has a low energy solution for every* $\lambda > 0$.

(ii) *If* $2 < q < 2(N-1)/(N-2)$ *for* $N \geq 4$ *and* $3 < q < 4$ *for* $N = 3$, *then the assumption* $h \geq 0$ *on* Ω *can be dropped and a low energy solution of problem* (1.1) *exists for every* $\lambda > 0$.

Proof. Since part (i) is well established (see [27], [51]), we only prove part (ii). For simplicity we assume that $0 \in \partial\Omega$ and $Q(0) = Q_{\mathrm{m}}$. Let

$$f_\lambda(t_\varepsilon) = \max_{0\leq t<\infty}\left[\frac{t^2}{2}\int_\Omega(|\nabla u_\varepsilon|^2 + \lambda u_\varepsilon^2)\,dx - \frac{t^{2^*}}{2^*}\int_\Omega Q(x)u_\varepsilon^{2^*}\,dx - \frac{t^q}{q}\int_\Omega h(x)u_\varepsilon^q\,dx\right],$$

where $u_\varepsilon = U_{\varepsilon,0}$. If $0 < t_\varepsilon \leq 1$, then

$$f_\lambda(t_\varepsilon) \leq \frac{1}{N}\frac{(\int_\Omega(|\nabla u_\varepsilon|^2 + \lambda u_\varepsilon^2)\,dx)^{N/2}}{(\int_\Omega Q(x)u_\varepsilon^{2^*}\,dx)^{(N-2)/2}} + \frac{1}{q}\int_\Omega |h|u_\varepsilon^q\,dx. \tag{4.4}$$

If $t_\varepsilon \geq 1$, then setting $A = \int_\Omega(|\nabla u_\varepsilon|^2 + \lambda u_\varepsilon^2)\,dx$ and $B = \int_\Omega Qu_\varepsilon^{2^*}\,dx$, and observing that $B + \int_\Omega hu_\varepsilon^q\,dx > 0$ for small $\varepsilon > 0$ we have

$$At_\varepsilon = Bt_\varepsilon^{2^*-1} + t_\varepsilon^{q-1}\int_\Omega hu_\varepsilon^q\,dx \geq Bt_\varepsilon^{q-1} + t_\varepsilon^{q-1}\int_\Omega hu_\varepsilon^q\,dx.$$

Hence

$$t_\varepsilon \leq \left(\frac{A}{B + \int_\Omega hu_\varepsilon^q\,dx}\right)^{1/(q-2)}.$$

This implies that

$$
\begin{aligned}
(4.5)\qquad f_\lambda(t_\varepsilon) &\le \max_{0\le t<\infty}\left[\frac{t^2}{2}\int_\Omega(|\nabla u_\varepsilon|^2+\lambda u_\varepsilon^2)\,dx-\frac{t^{2^*}}{2^*}\int_\Omega Q(x)u_\varepsilon^{2^*}\,dx\right]\\
&\quad+\frac{1}{q}\int_\Omega|h|u_\varepsilon^q\,dx\left(\frac{A}{B+\int_\Omega hu_\varepsilon^q\,dx}\right)^{q/(q-2)}\\
&\le\frac{1}{N}\frac{(\int_\Omega(|\nabla u_\varepsilon|^2+\lambda u_\varepsilon^2)\,dx)^{N/2}}{(\int_\Omega Qu_\varepsilon^{2^*}\,dx)^{(N-2)/2}}\\
&\quad+\frac{1}{q}\int_\Omega|h|u_\varepsilon^q\,dx\,\frac{(\int_\Omega(|\nabla u_\varepsilon|^2+\lambda u_\varepsilon^2)\,dx)^{q/(q-2)}}{(\int_\Omega Qu_\varepsilon^{2^*}\,dx+\int_\Omega hu_\varepsilon^q\,dx)^{q/(q-2)}}.
\end{aligned}
$$

We now need the following estimate (see [2]):

$$
\begin{aligned}
(4.6)\qquad &\frac{\int_\Omega(|\nabla u_\varepsilon|^2+\lambda u_\varepsilon^2)\,dx}{(\int_\Omega u_\varepsilon^{2^*}\,dx)^{2/2^*}}\\
&\le\begin{cases}
S/2^{2/N}-A_NH(0)\varepsilon\log\frac{1}{\varepsilon}+a_N\lambda\varepsilon+O(\varepsilon)+o(\lambda\varepsilon) & \text{if } N=3,\\
S/2^{2/N}-A_NH(0)\varepsilon+a_N\lambda\varepsilon^2\log\frac{1}{\varepsilon}+O\big(\varepsilon^2\log\frac{1}{\varepsilon}\big)+o\big(\lambda\varepsilon^2\log\frac{1}{\varepsilon}\big) & \text{if } N=4,\\
S/2^{2/N}-A_NH(0)\varepsilon+a_N\lambda\varepsilon^2+O(\varepsilon^2)+o(\lambda\varepsilon^2) & \text{if } N\ge 5,
\end{cases}
\end{aligned}
$$

where A_N and a_N are positive constants depending on N. The integral $\int_\Omega hu_\varepsilon^q\,dx$ in (4.4) and (4.5) satisfies the estimate

$$\int_\Omega|h|u_\varepsilon^q\,dx=O(\varepsilon^{-q(N-2)/2+N})\qquad\text{provided } q>N/(N-2)$$

with $-q(N-2)/2+N>1$. Combining this with (4.4)–(4.6) we deduce that the mountain-pass level satisfies the inequality

$$c_\lambda<\frac{S^{N/2}}{2NQ_{\mathrm m}^{(N-2)/2}}$$

and this completes the proof. ∎

We now consider the case $\{x\in\partial\Omega: Q(x)=Q_{\mathrm m}\}\subset\{x\in\partial\Omega: H(x)<0\}$.

THEOREM 4.3. *Suppose that* $Q_{\mathrm M}\le 2^{2/(N-2)}Q_{\mathrm m}$ *and that the assumption* (S_1) *holds. Further, assume that* $h(x)\ge 0$ *on* Ω *and* $h(x)>0$ *for all* $x\in\{x\in\partial\Omega: H(x)<0\}$. *If* $2(N-1)/(N-2)<q<2^*$, *then there exists a low energy solution of problem* (1.1) *for every* $\lambda>0$.

Proof. As in the proof of Theorem 4.2 we have

$$f_\lambda(t_\varepsilon)=\max_{0\le t<\infty}\left[\frac{t^2}{2}\int_\Omega(|\nabla u_\varepsilon|^2+\lambda u_\varepsilon^2)\,dx-\frac{t^{2^*}}{2^*}\int_\Omega Qu_\varepsilon^{2^*}\,dx-\frac{t^q}{q}\int_\Omega hu_\varepsilon^q\,dx\right],$$

with $0<t_*\le t_\varepsilon\le M<\infty$ for some constants t_* and M independent of ε. Here we have assumed that $0\in\{x\in\partial\Omega: H(x)<0\}$, $Q(0)=Q_{\mathrm m}$ and $h(0)>0$. By straightforward estimates we obtain

$$J_\lambda(tu_\varepsilon)\le\frac{S^{N/2}}{2NQ_{\mathrm m}^{(N-2)/2}}+O(\varepsilon)+\lambda O(\varepsilon^2)-\frac{bt_*}{q}h(0)\varepsilon^{-q(N-2)/2+N}$$

for some constant $b>0$ independent of ε. Since $-(N-2)q/2+N<1$, we conclude that

$$c_\lambda < \frac{S^{N/2}}{2NQ_{\mathrm{m}}^{(N-2)/2}}$$

for sufficiently small $\varepsilon > 0$ and the result follows. ∎

Finally, we establish the existence result in the flat case.

THEOREM 4.4. *Let $N \geq 5$ and suppose that* (S_2) *holds. If $2 < q < 2^*$, $h(x) \geq 0$ on Ω and $h(x) > 0$ for $x \in \{x \in \partial\Omega : Q(x) = Q_{\mathrm{m}}\}$, then problem* (1.1) *has a low energy solution for every $\lambda > 0$.*

Proof. For simplicity we assume that 0 belongs to the flat part of the boundary $\partial\Omega$, $Q(0) = Q_{\mathrm{m}}$ and $h(0) > 0$. The proof is parallel to the arguments from Theorems 4.1–4.3. The only change is in the estimation of

$$\frac{\int_\Omega (|\nabla u_\varepsilon|^2 + \lambda u_\varepsilon^2)\,dx}{(\int_\Omega Q u_\varepsilon^{2^*}\,dx)^{(N-2)/2}} = \frac{(K_1/2 + O(\varepsilon^{N-2}) + \lambda O(\varepsilon^2))^{N/2}}{(K_2 Q_{\mathrm{m}}/2 + O(\varepsilon^N) + o(\varepsilon^2))^{(N-2)/2}}.$$

Since $\int_\Omega h u_\varepsilon^q\,dx \geq b\varepsilon^{-q(N-2)/2+N}$ for some constant $b > 0$, with $-q(N-2)/2 + N < 2$ we derive the estimate for the mountain-pass level $c_\lambda < S^{N/2}/(2NQ_{\mathrm{M}}^{(N-2)/2})$. ∎

5. Multiple solutions in terms of Lusternik–Schnirelmann category

In this section, we relate the number of solutions of (1.1) to the category of a maximal level set of the coefficient Q. Willem [58] details the Lusternik–Schnirelmann category and its applications. Let $M^0 = \{x \in \partial\Omega : Q(x) = Q_{\mathrm{m}}\}$, and $M^I = \{x \in \Omega : Q(x) = Q_{\mathrm{M}}\}$ and for small $\varrho > 0$, let $M_\varrho^j = \{x \in \overline{\Omega} : \mathrm{dist}(x, M^j) < \varrho\}$, $j = 0, I$. Let $\mathcal{N}_\lambda$ be a Nehari manifold for J_λ:

$$\mathcal{N}_\lambda = \{u \in H^1(\Omega) \setminus \{0\} : \langle J'_\lambda(u), u\rangle = 0\}.$$

For $k \in \mathbb{R}$, denote $E_\lambda^k = \{u \in \mathcal{N}_\lambda : J_\lambda(u) \leq k\}$.

By the implicit function theorem, $\mathcal{N}_\lambda$ is a smooth manifold of codimension 1. In fact, for each $u \in H^1$, there is a unique $s(u) > 0$ such that $s(u)u \in \mathcal{N}_\lambda$, where $s(u)$ maximizes

$$f(s) = \frac{s^2}{2}\int_\Omega \left(|\nabla u|^2 + \frac{s^2}{2}\lambda u^2\right) dx - \frac{s^{2^*}}{2^*}\int_\Omega Q u^{2^*}\,dx - \frac{s^q}{q}\int_\Omega h u^q\,dx.$$

In a standard way [1], $\mathcal{N}_\lambda$ is a natural constraint and any critical point $u \neq 0$ of J_λ in $H^1(\Omega)$ corresponds to a critical point of the restriction of J_λ to $\mathcal{N}_\lambda$.

We introduce a barycentre for $u \in H^1(\Omega)$, defined by

$$\beta(u) = \frac{\int_\Omega x|u|^{2^*}\,dx}{\int_\Omega |u|^{2^*}\,dx}.$$

LEMMA 5.1. *For every (sufficiently small) $\varrho > 0$, there exists $\overline{\lambda} > 1$ such that*

(i) *if* $Q_{\mathrm{M}} < 2^{2/(N-2)}Q_{\mathrm{m}}$, $\beta(u) \in M_\varrho^0$,
(ii) *if* $Q_{\mathrm{M}} > 2^{2/(N-2)}Q_{\mathrm{m}}$, $\beta(u) \in M_\varrho^I$,
(iii) *if* $Q_{\mathrm{M}} = 2^{2/(N-2)}Q_{\mathrm{m}}$, $\beta(u) \in M_\varrho^0 \cup M_\varrho^I$ *for* $u \in E_\lambda^{S_\infty}$, $\lambda \geq \overline{\lambda}$.

Proof. Suppose to the contrary that there is a sequence $\lambda_n \to \infty$ and $u_n \in E_{\lambda_n}^{S_\infty}$ with (i) $\lim_{n\to\infty} \beta(u_n) \notin M_\varrho^I$, (ii) $\lim_{n\to\infty} \beta(u_n) \notin M_\varrho^0$, (iii) $\lim_{n\to\infty} \beta(u_n) \notin M_\varrho^I \cup M_\varrho^0$.

By considering $J_\lambda(u) - \frac{1}{q}\langle J_\lambda'(u), u\rangle$, we note that for any $K > 0$, $\bigcup_{\lambda>1} E_\lambda^K$ is bounded. It also follows that $\lambda_n \int u_n^2\, dx$ is bounded, so $u_n \to 0$ in $L^2(\Omega)$ and $L^q(\Omega)$ and $u_n \rightharpoonup 0$ in $H^1(\Omega)$ and $L^{2^*}(\Omega)$.

Let $v_n = u_n(\int Q u_n^{2^*}\, dx)^{-1/2^*}$. Then $v_n \rightharpoonup 0$ in $H^1(\Omega)$ and the concentration-compactness principle states that

$$|\nabla v_n|^2 \overset{*}{\rightharpoonup} \sum_{j\in J} \mu_j \delta_{x_j}, \qquad |v_n|^{2^*} \overset{*}{\rightharpoonup} \sum_{j\in J} \nu_j \delta_{x_j}$$

weakly in the sense of measures. Hence

$$\begin{aligned}
S_\infty \geq \lim_{n\to\infty} J_\lambda(u_n) &= \lim_{n\to\infty} [\max_{t>0} J_\lambda(tv_n)] \\
&= \lim_{n\to\infty} \max_{t>0} \left\{ \frac{t^2}{2} \int_\Omega |\nabla v_n|^2\, dx + \frac{t^2}{2}\lambda \int_\Omega v_n^2\, dx - \frac{t^{2^*}}{2^*} \int_\Omega Q|v_n|^{2^*}\, dx \right\} + o(1) \\
&= \lim_{n\to\infty} \frac{1}{N} \Big(\int_\Omega |\nabla v_n|^2\, dx + \lambda_n v_n^2 \Big)^{N/2} \\
&\geq \lim_{n\to\infty} \frac{1}{N} \Big(\int_\Omega |\nabla v_n|^2\, dx \Big)^{N/2} = \frac{1}{N} \Big(\sum_{x_j\in\overline{\Omega}} \mu_j \Big)^{N/2} \\
&\geq \frac{1}{N} \Big(\sum_{x_j\in\Omega} S\nu_j^{2/2^*} + \sum_{x_j\in\partial\Omega} \frac{S\nu_j^{2/2^*}}{2^{2/N}} \Big)^{N/2} \\
&\geq \frac{S^{N/2}}{N} \Big(\sum_{x_j\in\Omega} \frac{(Q(x_j)\nu_j)^{2/2^*}}{Q_{\mathrm{M}}^{2/2^*}} + \sum_{x_j\in\partial\Omega} \frac{(Q(x_j)\nu_j)^{2/2^*}}{2^{2/N} Q_{\mathrm{m}}^{2/2^*}} \Big)^{N/2}.
\end{aligned}$$

Now, $1 = \int_\Omega Q v_n^{2^*}\, dx = \sum_J Q(x_j)\nu_j$. Suppose that for some i (and hence all i), $Q(x_i)\nu_i < 1$. Then $(Q(x_i)\nu_i)^{2/2^*} > Q(x_i)\nu_i$. In the case that $Q_{\mathrm{M}} \leq 2^{2/(N-2)} Q_{\mathrm{m}}$,

$$S_\infty \geq \frac{S^{N/2}}{N} \frac{(\sum_j Q(x_j)\nu_j)^{N/2}}{Q_{\mathrm{M}}^{(N-2)/2}} > \frac{S^{N/2}}{2NQ_{\mathrm{m}}^{(N-2)/2}} = S_\infty$$

while if $Q_{\mathrm{M}} \geq 2^{2/(N-2)} Q_{\mathrm{m}}$, we similarly obtain a contradiction. Thus there can be only one point of concentration and $Q(x_i)\nu_i = 1$ for some $x_i \in \overline{\Omega}$.

Consider case (i) when $Q_{\mathrm{M}} < 2^{2/(N-2)} Q_{\mathrm{m}}$. If $x_i \in \Omega$ then a contradiction arises as the sequence of inequalities implies that

$$S_\infty \geq \frac{S^{N/2}}{NQ(x_i)^{(N-2)/2}} > \frac{S^{N/2}}{2NQ_{\mathrm{m}}^{(N-2)/2}}$$

while if $x_i \in \partial\Omega \setminus M^0$ then

$$S_\infty \geq \frac{S^{N/2}}{2NQ(x_i)^{(N-2)/2}} > \frac{S^{N/2}}{2NQ_{\mathrm{m}}^{(N-2)/2}}$$

and consequently $x_i \in M^0$.

Consider case (ii) when $Q_{\mathrm{M}} > 2^{2/(N-2)}Q_{\mathrm{m}}$. If $x_i \in \partial\Omega$, then again we have a contradiction as

$$S_\infty \geq \frac{S^{N/2}}{2NQ(x_i)^{(N-2)/2}} > \frac{S^{N/2}}{Q_{\mathrm{M}}^{(N-2)/2}}$$

while if $x_i \in \Omega \setminus M^I$, then

$$S_\infty \geq \frac{S^{N/2}}{NQ(x_i)^{(N-2)/2}} > \frac{S^{N/2}}{NQ_{\mathrm{M}}^{(N-2)/2}}.$$

Similarly, case (iii) when $Q_{\mathrm{M}} = 2^{2/(N-2)}Q_{\mathrm{m}}$ rejects the possibility that $x_i \in \overline{\Omega} \setminus (M^0 \cup M^I)$.

In case (i), we have $|v_n|^{2^*} \stackrel{*}{\rightharpoonup} Q_{\mathrm{m}}^{-1}\delta_{x_i}$, while in case (ii) we have $|v_n|^{2^*} \stackrel{*}{\rightharpoonup} Q_{\mathrm{M}}^{-1}\delta_{x_i}$, and in case (iii) either may occur. In all cases $\beta(v_n) = \beta(u_n) \to x_i$. ∎

Theorem 5.2. *Suppose that*

(i) *the conditions of Theorem* 4.2 *or* 4.3 *or* 4.4 *hold, along with* $Q_{\mathrm{M}} < 2^{2/(N-2)}Q_{\mathrm{m}}$;
(ii) *the conditions of Theorem* 4.1 *hold with* $Q_{\mathrm{M}} > 2^{2/(N-2)}Q_{\mathrm{m}}$;
(iii) *the conditions of Theorem* 4.1 *hold with* $Q_{\mathrm{M}} = 2^{2/(N-2)}Q_{\mathrm{m}}$.

There exists $\lambda' > 1$ *sufficiently large that for all* $\lambda > \lambda'$ *problem* (1.1) *possesses at least* (i) $\mathrm{cat}(M^0)$ *solutions*, (ii) $\mathrm{cat}(M^i)$ *solutions or* (iii) $\mathrm{cat}(M^0) + \mathrm{cat}(M^I)$ *solutions.*

Proof. (i) Let $\varrho > 0$ be sufficiently small that $\mathrm{cat}_{M_\varrho^0} M^0 = \mathrm{cat}(M^0)$. By the estimates in Section 4, we know that for each $\lambda > 1$ and $x \in M^0$ there is $\varepsilon > 0$ such that

$$J_\lambda(s(U_{\varepsilon,x})U_{\varepsilon,x}) < S_\infty.$$

We choose $\overline{\varepsilon}(\lambda)$ sufficienly small to construct $c(\lambda)$ satisfying

$$\max_{x \in M^0}\{J_\lambda(s(U_{\overline{\varepsilon},x})U_{\overline{\varepsilon},x})\} < c(\lambda) < S_\infty.$$

Define a map $\Phi_\lambda : M^0 \mapsto E_\lambda^{c_\lambda}$ by $\Phi_\lambda(x) = s(U_{\overline{\varepsilon},x})U_{\overline{\varepsilon},x}$. By the previous lemma, for all $\lambda > \lambda'$, $\beta(\Phi_\lambda(x)) \in M_\varrho^0$.

It is easy to see that $\beta \circ \Phi_\lambda(x)$ is homotopic to the inclusion $M^0 \to M_\varrho^0$. Let

$$\mathcal{H}(t,x) = x + t\beta(\Phi_\lambda(x) - x).$$

Then $\mathrm{dist}(\mathcal{H}(t,x), M^0) \leq |\beta(\Phi_\lambda(x)) - x| < \varrho$ for every $x \in M$ and $t \in [0,1]$, so $\mathcal{H} : [0,1] \times M^0 \to M_\varrho^0$.

Recall that for each λ, J_λ satisfies the $(\mathrm{PS})_{c(\lambda)}$ condition. By Lusternik–Schnirelmann theory, in order to show the theorem, it suffices to confirm that

$$\mathrm{cat}(E_\lambda^{c(\lambda)}) \geq \mathrm{cat}_{M_\varrho^0}(M^0).$$

This follows in an identical way to [17] (see also [19], [18]). Suppose that $\mathrm{cat}(E_\lambda^{c(\lambda)}) = n$. Then

$$E_\lambda^{c(\lambda)} \subset A_1 \cup \ldots \cup A_n,$$

where each A_i is closed in $E_\lambda^{c(\lambda)}$ and is contractible in $E_\lambda^{c(\lambda)}$:

$$h_i(0,u) = u, \quad h_i(1,u) = w_i \in E_\lambda^{c(\lambda)}.$$

Set $C_i = \Phi_\lambda^{-1}(A_i)$. Then $\mathrm{cat}_{M_\varrho^0}(M^0) \le \sum_{i=1}^n \mathrm{cat}_{M_\varrho^0}(C_i)$. The map $h \circ \beta \circ \mathcal{H}_i(1,\cdot) \circ \Phi_\lambda : K_i \mapsto M$ is homotopic to the identity, yielding

$$\mathrm{cat}_{M_\varrho}(K_i) \le \mathrm{cat}_{M_\varrho}(h \circ \beta \circ \mathcal{H}_i(1,\cdot) \circ \Phi_\lambda(K_i)) \le \mathrm{cat}_{M_\varrho}(h \circ \beta \circ \mathcal{H}_i(1, A_i)) = 1$$

so $\mathrm{cat}(M^0) = \mathrm{cat}_{M_\varrho^0}(M^0) \le n$.

(ii) We remark that M^I lies entirely within Ω, so for small enough $\varrho > 0$, $M_\varrho^I \in \Omega$ and $\mathrm{cat}_{M_\varrho^I}(M^I) = \mathrm{cat}(M^I)$. Again, Theorem 4.1 shows that for each $\lambda > \lambda'$, there exist $c(\lambda)$ and $\overline{\varepsilon}(\lambda) > 0$ such that for all $x \in M^I$,

$$J_\lambda(s(U_{\overline{\varepsilon},x})U_{\overline{\varepsilon},x}) < c(\lambda) < S_\infty.$$

Let $\Phi_\lambda(x) = s(U_{\overline{\varepsilon},x})U_{\overline{\varepsilon},x}$. Again $\beta(\Phi_\lambda(x))$ is homotopic to $M^I \to M_\varrho^I$ as $|\beta \circ \Phi_\lambda(x) - x| < \varrho$ for all $x \in M^I$. The remainder of the proof follows part (i).

(iii) We note that $\varrho > 0$ can be taken sufficiently small that M_ϱ^i and M_ϱ^0 are disjoint and $\mathrm{cat}_{M_\varrho^I}(M^I) = \mathrm{cat}(M^I)$ and $\mathrm{cat}_{M_\varrho^0}(M^0) = \mathrm{cat}(M^0)$. For given $\lambda > 0$, $\Phi_\lambda(x) = s(U_{\overline{\varepsilon}})U_{\overline{\varepsilon}}$, where $\overline{\varepsilon}$ is chosen so that

$$\max_{x \in M^0 \cup M^I} J_\lambda(\Phi_\lambda(x)) < c(\lambda) < S_\lambda.$$

It follows that for any $x \in M^0 \cup M^I$, $\beta(\Phi_\lambda(x)) \in M_\varrho^I \cup M_\varrho^0$. The remainder of the proof follows as before, and there are at least $\mathrm{cat}(M^0 \cup M^I)$ solutions. Since M_ϱ^0 and M_ϱ^I are disjoint sets, $\mathrm{cat}(M^0 \cup M^I) = \mathrm{cat}(M^0) + \mathrm{cat}(M^I)$. ■

6. Nonexistence results

We commence by considering the case $Q_{\mathrm{M}} > 2^{2/(N-2)} Q_{\mathrm{m}}$.

PROPOSITION 6.1. *Let $Q_{\mathrm{M}} > 2^{2/(N-2)} Q_{\mathrm{m}}$ and suppose that for every $\lambda > 0$ there exists a low energy solution u_λ. Then*

$$\lim_{\lambda\to\infty} J_\lambda(u_\lambda) = \frac{S^{N/2}}{N Q_{\mathrm{M}}^{(N-2)/2}}, \quad \lim_{\lambda\to\infty} \int_\Omega (|\nabla u_\lambda|^2 + \lambda u_\lambda^2)\, dx = \lim_{\lambda\to\infty} \int_\Omega Q(x) u_\lambda^{2^*}\, dx = \frac{S^{N/2}}{Q_{\mathrm{M}}^{(N-2)/2}}$$

and

$$\lim_{\lambda\to\infty} \int_\Omega h(x) u_\lambda^q\, dx = 0.$$

Proof. First we observe that if $\lambda \ge \Lambda_1$, then $\int_\Omega h u_\lambda^q\, dx \ge 0$. This follows from the proof of Lemma 3.2. Hence

$$\frac{S^{N/2}}{N Q_{\mathrm{M}}^{(N-2)/2}} > J_\lambda(u_\lambda) - \frac{1}{2}\langle J'_\lambda(u_\lambda), u_\lambda\rangle = \left(\frac{1}{2} - \frac{1}{2^*}\right) \int_\Omega Q u_\lambda^{2^*}\, dx + \left(\frac{1}{2} - \frac{1}{q}\right) \int_\Omega h u_\lambda^q\, dx. \tag{6.1}$$

This implies that $\{u_\lambda\}$ is bounded in $H^1(\Omega)$. Therefore we may assume that $u_\lambda \rightharpoonup u$ in $H^1(\Omega)$. Since $u_\lambda \to 0$ in $L^2(\Omega)$, $u \equiv 0$. It then follows from (6.1) that

$$\limsup_{\lambda\to\infty} \int_\Omega Q u_\lambda^{2^*}\, dx \le \frac{S^{N/2}}{Q_{\mathrm{M}}^{(N-2)/2}}.$$

Suppose that

$$\lim_{\lambda_k\to\infty} \int_\Omega Q u_{\lambda_k}^{2^*}\, dx < \frac{S^{N/2}}{Q_{\mathrm{M}}^{(N-2)/2}} \tag{6.2}$$

for some sequence $\lambda_k \to \infty$. For $\lambda_k \ge \Lambda_1$ we have

$$\int_\Omega (|\nabla u_{\lambda_k}|^2 + \lambda_k u_{\lambda_k}^2)\, dx = \int_\Omega Q u_{\lambda_k}^{2^*}\, dx + \int_\Omega h u_{\lambda_k}^q\, dx \le C(\|u_{\lambda_k}\|^{2^*} + \|u_{\lambda_k}\|^q)$$

for some constant $C > 0$ independent of λ_k. Hence

$$\|u_{\lambda_k}\| \ge \mathrm{const} > 0 \tag{6.3}$$

for $\lambda_k \ge \Lambda_1$ and also

$$\lim_{k\to\infty} \int_\Omega Q u_{\lambda_k}^{2^*}\, dx > 0.$$

Using the concentration-compactness principle we show that (6.2) is impossible. Indeed, we have

$$\lim_{k\to\infty} \int_\Omega Q u_{\lambda_k}^{2^*}\, dx = \sum_{j\in J} Q(x_j)\nu_j.$$

Using a family of functions concentrating at x_j we check that $\mu_j \le \nu_j Q(x_j)$. Combining this with (3.2) and (3.3) we get

$$\lim_{k\to\infty} \int_\Omega Q u_{\lambda_k}^{2^*}\, dx \ge \frac{S^{N/2}}{Q_{\mathrm{M}}^{(N-2)/2}},$$

which contradicts (6.2). ■

THEOREM 6.2. *Let* $Q_{\mathrm{M}} > 2^{2/(N-2)} Q_{\mathrm{m}}$, $h \ge 0$ *on* Ω *and* $h(x) = 0$ *for* $x \in \{x \in \Omega : Q(x) = Q_{\mathrm{M}}\}$ *and* $h \in C^2(\overline{\Omega})$.

(i) *If* $2 < q < 2(N-1)/(N-2)$, $D_i h(x) = 0$, $i = 1, \ldots, N$, *for* $x \in \{x : Q(x) = Q_{\mathrm{M}}\}$, *then there exists* $\widetilde{\Lambda} > 0$ *such that problem* (1.1) *for* $\lambda \ge \widetilde{\Lambda}$ *has no low energy solution.*

(ii) *If* $D_i h(x) = 0$, $D_{ij} h(x) = 0$, $i, j = 1, \ldots, N$, *for* $x \in \{x : Q(x) = Q_{\mathrm{M}}\}$ *and* $2 < q < 2^*$, *then there exists* $\Lambda^* > 0$ *such that problem* (1.1) *with* $\lambda \ge \Lambda^*$ *has no low energy solution.*

Proof. Suppose that problem (1.1) has a solution u_λ for every $\lambda > 0$. Let

$$M_\lambda = \max_{x\in\overline{\Omega}} u_\lambda(x) = u_\lambda(x_\lambda)$$

for some $x_\lambda \in \overline{\Omega}$. It is easy to check that $M_\lambda \to \infty$. We now use a blow-up technique. We follow the ideas from the paper [5]. Define $\varepsilon_\lambda = M_\lambda^{2/(2-N)}$, $\Omega_\lambda = (\Omega - x_\lambda)/\varepsilon_\lambda$ and set

$$v_\lambda(x) = \varepsilon_\lambda^{(N-2)/2} u_\lambda(\varepsilon_\lambda x + x_\lambda).$$

By a simple rescaling argument we can assume that $Q_{\mathrm{M}} = 1$. Then we have

$$-\Delta v_\lambda + \lambda\varepsilon_\lambda^2 v_\lambda = Q(\varepsilon_\lambda x + x_\lambda)v_\lambda^{2^*-1} + h(\varepsilon_\lambda x + x_\lambda)\varepsilon_\lambda^{(N+2)/2-(q-1)(N-2)/2}v_\lambda^{q-1} \quad \text{in } \Omega_\lambda,$$
$$0 < v_\lambda(x) \le v_\lambda(0) = 1 \quad \text{in } \Omega_\lambda, \qquad \frac{\partial v_\lambda}{\partial\nu} = 0 \quad \text{on } \partial\Omega_\lambda.$$

The term $\varepsilon_\lambda^2\lambda$ is bounded as $\lambda \to \infty$. Indeed, we have

$$0 \le \int_\Omega (Q|u_\lambda|^{2^*} + h|u_\lambda|^q - \lambda u_\lambda^2)\,dx = \int_\Omega u_\lambda^2(Q|u_\lambda|^{2^*-2} + h|u_\lambda|^{q-2} - \lambda)\,dx.$$

From this we deduce that

$$Q_{\mathrm{M}}M_\lambda^{2^*-2} + \|h\|_\infty M_\lambda^{q-2} - \lambda \ge 0.$$

By the Young inequality for every $\delta > 0$ we can find $C(\delta) > 0$ such that

$$\|h\|_\infty M_\lambda^{q-2} \le \delta M_\lambda^{2^*-2} + C(\delta)\|h\|_\infty^{\frac{2^*-2}{2^*-q}}$$

and consequently

$$(Q_{\mathrm{M}} + \delta)M_\lambda^{2^*-2} \ge \lambda - C(\delta)\|h\|_\infty^{\frac{2^*-2}{2^*-q}}$$

and our claim follows. We can assume that for a sequence $\lambda_k \to \infty$, $x_{\lambda_k} \to y$, $\lambda_k\varepsilon_k^2 \to a$ and

$$\frac{\operatorname{dist}(x_{\lambda_k}, \partial\Omega)}{\varepsilon_{\lambda_k}} \to \alpha.$$

Let $\lim_{k\to\infty} \Omega_{\lambda_k} = \Omega_\infty$. By standard elliptic estimates ([33]), we obtain $v_{\lambda_k} \to w$ in $C^2_{\mathrm{loc}}(\Omega)$, where w satisfies

$$-\Delta w + aw = Q(y)w^{2^*-1} \quad \text{in } \Omega_\infty,$$
$$0 \le w(x) \le w(0) = 1 \quad \text{in } \Omega_\infty, \qquad \frac{\partial w}{\partial\nu} = 0 \quad \text{on } \partial\Omega_\infty.$$

It is easy to check that $\int_{\Omega_\infty} |\nabla w|^2\,dx < \infty$ and $\int_{\Omega_\infty} w^{2^*}\,dx < \infty$. By Pokhozhaev's identity [45], $a = 0$ and

$$w(x) = Q(y)^{-1/(2^*-2)}U_{\varepsilon,z}(x).$$

Since $\max_{\overline{\Omega}_\infty} w = w(0) = 1$, we see that $z = 0$, $\varepsilon = Q(y)^{-1/2}$ and $\Omega_\infty = \mathbb{R}^N_+$ ($\alpha = 0$) or $\Omega_\infty = \mathbb{R}^N$ ($\alpha = \infty$). If $\Omega_\infty = \mathbb{R}^N_+$, then $y \in \partial\Omega$ and by Proposition 6.1,

$$Q(y)^{-2/(2^*-2)}\frac{S^{N/2}}{2} = Q(y)^{-2/(2^*-2)}\int_{\mathbb{R}^N_+} |\nabla U|^2\,dx \le \lim_{k\to\infty}\int_{\Omega_{\lambda_k}} |\nabla v_{\lambda_k}|^2\,dx \le \frac{S^{N/2}}{Q_{\mathrm{M}}^{(N-2)/2}}.$$

This implies that $Q_{\mathrm{M}} \le 2^{2/(N-2)}Q_{\mathrm{m}}$, which is impossible. Thus the case $\Omega_\infty = \mathbb{R}^N$ prevails and by Proposition 6.1 we have

$$\frac{S^{N/2}}{Q_{\mathrm{M}}^{(N-2)/2}} \le Q(y)^{-2/(2^*-2)}S^{N/2} = Q(y)^{-2/(2^*-2)}\int_{\mathbb{R}^N} |\nabla U|^2\,dx$$
$$\le \lim_{k\to\infty}\int_{\Omega_{\lambda_k}} |\nabla v_{\lambda_k}|^2\,dx \le \frac{S^{N/2}}{Q_{\mathrm{M}}^{(N-2)/2}}.$$

It then follows that $Q(y) = Q_{\mathrm{M}}$. Then we have

$$\lim_{k\to\infty} \int_{\Omega} |\nabla(u_{\lambda_k} - U_{\varepsilon_k,x_k})|^2\,dx = 0. \tag{6.4}$$

We now follow the argument from the paper of Z. Q. Wang [54]. First we observe that (6.4) implies the representation

$$u_{\lambda_k} = C_k U_{\varepsilon_k,y_k} + w_k$$

with $w_k \to 0$ in $H^1(\Omega)$, $C_k \to 1$ and $\int_\Omega \nabla w_k \nabla U_{\varepsilon_k,y_k}\,dx = 0$. It then follows from Lemma 4.6 in Z. Q. Wang [54] that

$$J_{\lambda_k}(U_{\varepsilon_k,y_k}) = \frac{S^{N/2}}{N} + b_N\lambda_k\varepsilon_k^2 + O(\varepsilon_k^{N-2}) + O(\varepsilon_k^{-q(N-2)/2+N+i})$$

with $i = 1$ if $D_k h(0) = 0$, $k = 1,\ldots,N$, and $i = 2$ if $D_{rs}h(0) = 0$, $r,s = 1,\ldots,N$. This can be used to show as in Lemma 4.7 in [54] that $J_{\lambda_k}(u_{\lambda_k}) > S^{N/2}/(NQ_{\mathrm{M}}^{(N-2)/2})$, which is impossible. ■

Related nonexistence results in the case $Q(x) = \mathrm{const}$ can be found in [29]. In a similar manner we can establish the nonexistence results under assumptions (S_1) and (S_2).

THEOREM 6.3. *Let $N \ge 5$ and let (S_1) hold. Suppose that $Q_{\mathrm{M}} \le 2^{2/(N-2)}Q_{\mathrm{m}}$, $h(x) \ge 0$ on Ω and $h(x) = 0$ for $x \in \{x \in \partial\Omega : Q(x) = Q_{\mathrm{m}}\}$. If $2 < q < 2(N-1)/(N-2)$, then there exists $\Lambda_2 > 0$ such that problem (1.1) with $\lambda \ge \Lambda_2$ does not have a low energy solution. If in addition $D_i h(y) = 0$, $i = 1,\ldots,N$, for some $y \in \{x \in \partial\Omega : Q(x) = Q_{\mathrm{m}}\}$, then there exists a constant $\Lambda_2 > 0$ such that problem (1.1) with $\lambda \ge \Lambda_2$ does not have a low energy solution.*

THEOREM 6.4. *Let $N \ge 5$ and (S_2) hold. Suppose that $Q_{\mathrm{M}} \le 2^{2/(N-2)}Q_{\mathrm{m}}$, $h(x) \ge 0$ on Ω and $h(x) = 0$ for $x \in \{\partial\Omega : Q(x) = Q_{\mathrm{m}}\}$. Then there exists $\Lambda_3 > 0$ such that problem (1.1) with $\lambda \ge \Lambda_3$ does not have a low energy solution.*

7. Problem at resonance

The value $\lambda = 0$ is the first eigenvalue of the Laplace operator $-\Delta$ with zero Neumann boundary conditions. The corresponding eigenfunctions are constant. This section is devoted to the discussion of the solvability of the problem

$$\begin{cases} -\Delta u = Q(x)|u|^{2^*-2}u + h(x)|u|^{q-2}u & \text{in } \Omega, \\ \partial u/\partial\nu = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{7.1}$$

If $h \equiv 0$ and $Q(x) > 0$ on Ω, then problem (7.1) does not have a positive solution. This follows from the direct integration of (7.1):

$$0 = -\int_\Omega \Delta u\,dx = \int_\Omega Q(x)u^{2^*-1}\,dx.$$

In this case a positive solution exists if Q changes sign and $\int_\Omega Q(x)\,dx < 0$. This result can be found in the paper [24] (for a related result see [40]). Here we continue the investigation of the solvability of the Neumann problem assuming that $Q(x) > 0$ on

$\overline{\Omega}$ and that h changes sign on Ω. As in the previous sections we also assume that the coefficients Q and h are smooth on Ω.

We decompose $H^1(\Omega)$ as

$$H^1(\Omega) = \operatorname{span} 1 \oplus V,$$

where $V = \{v \in H^1(\Omega) : \int_\Omega v\,dx = 0\}$. Having this decomposition we define an equivalent norm in $H^1(\Omega)$ by

$$\|u\|_V^2 = t^2 + \|\nabla v\|_2^2.$$

To check the mountain-pass geometry for the variational functional

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{2^*}\int_\Omega Q(x)|u|^{2^*}\,dx - \frac{1}{q}\int_\Omega h(x)|u|^q\,dx$$

we need the following quantitative statement:

LEMMA 7.1. *Suppose that $h(x)$ changes sign on Ω and $\int_\Omega h(x)\,dx < 0$. Then there exists a constant $\eta > 0$ such that for each $t \in \mathbb{R}$ and $v \in V$ the inequality*

$$\Big(\int_\Omega |\nabla v(x)|^2\,dx\Big)^{1/2} \le \eta|t|$$

implies

$$\int_\Omega h(x)|t+v(x)|^q\,dx \le \frac{|t|^q}{2}\int_\Omega h(x)\,dx.$$

For the proof we refer to the paper [20].

PROPOSITION 7.2. *Suppose that $h(x)$ changes sign on Ω and $\int_\Omega h(x)\,dx < 0$. Then there exist constants $\varrho > 0$ and $\beta > 0$ such that*

$$J(u) \ge \beta \quad \textit{for all } u \textit{ satisfying } \|u\|_V = \varrho.$$

Proof. Let $\eta > 0$ be the constant from Lemma 7.1. We distinguish two cases: (i) $\|\nabla v\|_2 \le \eta|t|$ and (ii) $\|\nabla v\|_2 > \eta|t|$. If $\|\nabla v\|_2 \le \eta|t|$ and $\|\nabla v\|_2^2 + t^2 = \varrho^2$, then $t^2 \ge \varrho^2/(1+\eta^2)$. By Lemma 7.1 we get

$$\int_\Omega h(x)|t+v(x)|^q\,dx \le \frac{|t|^q}{2}\int_\Omega h(x)\,dx = -|t|^q\alpha,$$

with $\alpha = -\frac{1}{2}\int_\Omega h(x)\,dx > 0$. Using this and the Sobolev inequality in V we obtain the estimate of J from below:

$$\begin{aligned} J(u) &\ge -\frac{C}{2^*}\|\nabla v\|_2^{2^*} - \frac{C}{2^*}|t|^{2^*} + \frac{|t|^q}{q}\alpha \ge -\frac{2C}{2^*}\varrho^{2^*} + \frac{\alpha\varrho^q}{q(1+\eta^2)^{q/2}} \\ &= \varrho^q\bigg(\frac{\alpha}{q(1+\eta^2)^{q/2}} - \frac{2C\varrho^{2^*-q}}{2^*}\bigg) \ge \frac{\varrho^q\alpha}{2q(1+\eta^2)^{q/2}} \end{aligned}$$

for $\varrho > 0$ small enough, say $\varrho \le \varrho_\circ$, and some constant $C > 0$. In case (ii) we have $\|u\|_V \le \|\nabla v\|_2(1+1/\eta^2)^{1/2}$. Thus applying the Sobolev inequality we get

$$\int_\Omega Q(x)|u|^{2^*}\,dx \le C_1\|u\|_V^{2^*} \le C_1\Big(1+\frac{1}{\eta^2}\Big)^{2^*/2}\|\nabla v\|_2^{2^*}$$

and

$$\Big|\int_\Omega h(x)|u|^q\,dx\Big| \le C_2\|u\|_V^q \le C_2\Big(1+\frac{1}{\eta^2}\Big)^{q/2}\|\nabla v\|_2^q$$

for some constants $C_1>0$ and $C_2>0$. Hence

$$J(u) \ge \frac{1}{2}\|\nabla v\|_2^2 - C_1\Big(1+\frac{1}{\eta^2}\Big)^{2^*/2}\|\nabla v\|_2^{2^*} - C_2\Big(1+\frac{1}{\eta^2}\Big)^{q/2}\|\nabla v\|_2^q.$$

Taking $\|\nabla v\|_2 \le \varrho$ small enough we derive from the above inequality the estimate

$$J(u) \ge \frac{1}{4}\,\|\nabla v\|_2^2.$$

On the other hand if $\|u\|_V=\varrho$, then $\varrho \le \|\nabla v\|_2(1+\eta^2)^{1/2}/\eta$. Consequently,

$$J(u) \ge \frac{\eta^2\varrho^2}{4(1+\eta^2)}.$$

If we take $\beta=\min\big(\frac{\eta^2\varrho^2}{4(1+\eta^2)}, \frac{\varrho^q\alpha}{2q(1+\eta^2)^{q/2}}\big)$ the result follows. ■

Proposition 7.3. *Suppose that h changes sign in Ω and $\int_\Omega h(x)\,dx<0$. Then J satisfies the* $(\mathrm{PS})_c$ *condition with $c<S_\infty$.*

Proof. We commence by showing that $\{u_m\}$ is bounded in $H^1(\Omega)$. We have

$$\begin{aligned} J(u_m) - \frac{1}{q}\,\langle J'(u_m),u_m\rangle &= \Big(\frac{1}{2}-\frac{1}{q}\Big)\int_\Omega |\nabla u_m|^2\,dx + \Big(\frac{1}{q}-\frac{1}{2^*}\Big)\int_\Omega Q(x)|u_m|^{2^*}\,dx \\ &= \varepsilon_m\|u_m\| + o(1) + c. \end{aligned}$$

By the Hölder inequality we have

$$\begin{aligned} \int_\Omega (|\nabla u_m|^2+u_m^2)\,dx &\le \int_\Omega |\nabla u_m|^2\,dx + |\Omega|^{1-2/2^*}\Big(\int_\Omega |u_m|^{2^*}\,dx\Big)^{2/2^*} \\ &\le \int_\Omega |\nabla u_m|^2\,dx + Q_*^{-2/2^*}\Big(\int_\Omega Q|u_m|^{2^*}\,dx\Big)^{2/2^*}|\Omega|^{1-2/2^*}, \end{aligned}$$

where $Q_*=\min_{x\in\overline{\Omega}}Q(x)$. These two relations show that the sequence $\{u_m\}$ is bounded in $H^1(\Omega)$. The remaining part of the proof is the same as in the proof of Proposition 3.1. ■

We are now in a position to formulate the following existence results:

Theorem 7.4. *Suppose that $Q_{\mathrm{M}} \le 2^{2/(N-2)}Q_{\mathrm{m}}$, $h(x)$ changes sign on Ω and $\int_\Omega h(x)\,dx$* <0.

(i) *If $2<q<2(N-1)/(N-2)$ for $N\ge 4$ and $3<q<4$ for $N=3$, then problem* (7.1) *has a solution.*

(ii) *If*

$$\Big(\frac{1}{q}-\frac{1}{2^*}\Big)\frac{(-\int_\Omega h(x)\,dx)^{2^*/(2^*-q)}}{(\int_\Omega Q(x)\,dx)^{q/(2^*-q)}} < S_\infty,$$

then problem (7.1) *has a solution.*

Proof. The proof of assertion (i) is identical to that of Theorem 4.2(ii). Part (ii) follows by observing that

$$\max_{t\geq 0} J(t) = \left(\frac{1}{q} - \frac{1}{2^*}\right) \frac{(-\int_\Omega h(x)\,dx)^{2^*/(2^*-q)}}{(\int_\Omega Q(x)\,dx)^{q/(2^*-q)}}$$

and an application of the mountain-pass theorem. ■

Further results in the case when λ interferes with the eigenvalues of higher order can be found in [26]. In this paper solutions are obtained via a topological linking.

8. Existence of infinitely many solutions

To establish the existence of infinitely many solutions we assume that $1 < q < 2$ and that $h(x) > 0$ on Ω. Moreover we replace the coefficient $h(x)$ by $\mu h(x)$, where $\mu > 0$ is a parameter. Explicitly, we consider the problem

$$\begin{cases} -\Delta u + \lambda u = Q(x)|u|^{2^*-2}u + \mu h(x)|u|^{q-2}u & \text{in } \Omega, \\ \partial u/\partial\nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.1}$$

By $J_{\lambda,\mu}$ we denote the corresponding variational functional

$$J_{\lambda,\mu}(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx - \frac{1}{2^*}\int_\Omega Q(x)|u|^{2^*} - \frac{\mu}{q}\int_\Omega h(x)|u|^q\,dx.$$

Our approach is based on the fountain theorem due to Bartsch–Willem [16, 58]. Let $\{e_k\}$ be an orthonormal base of $H^1(\Omega)$. We set

$$X(j) = \operatorname{span}(e_1, \ldots, e_j), \qquad X_k = \bigoplus_{j\geq k} X(j), \qquad X^k = \bigoplus_{j\leq k} X(j).$$

THEOREM 8.1 (Bartsch–Willem). *Let $\phi \in C^1(H^1(\Omega), \mathbb{R})$ be an even functional. Suppose that*

- (A_1) *there exists $k_\circ$ such that for every $k \geq k_\circ$ we can find $R_k > 0$ such that $\phi(u) \geq 0$ for all $u \in X_k$ with $\|u\| = R_k$,*
- (A_2) *$b_k = \inf_{B(0,R_k)} \phi(u) \to 0$ as $k \to \infty$, where $B(0, R_k)$ is a ball of radius R_k in X_k,*
- (A_3) *for every $k \geq 1$ there exist $r_k \in (0, R_k)$ and $d_k < 0$ such that $\phi(u) \leq d_k$ for every $u \in X^k$ with $\|u\| = r_k$,*
- (A_4) *every sequence $\{u_m\}$ such that $u_m \in X^m$, $\phi(u_m) < 0$ and $\phi'|_{X^m}(u_m) \to 0$ as $m \to \infty$ has a subsequence convergent to a critical point of ϕ.*

Then for each $k \geq k_\circ$, ϕ has a critical value $c_k \in [b_k, d_k]$, with $c_k \to 0$ as $k \to \infty$.

To formulate the $(\mathrm{PS})_c$ condition we need the following estimate:

$$\begin{aligned} J_{\lambda,\mu}(u) &- \frac{1}{2}\langle J'_{\lambda,\mu}(u), u\rangle \\ &= \frac{1}{N}\int_\Omega Q|u|^{2^*}\,dx - \left(\frac{1}{q} - \frac{1}{2}\right)\mu\int_\Omega h|u|^q\,dx \\ &\geq \frac{1}{N}\int_\Omega Q|u|^{2^*}\,dx - \left(\frac{1}{q} - \frac{1}{2}\right)\mu\|h\|_\infty Q_*^{q/2^*}|\Omega|^{(2^*-q)/2^*}\left(\int_\Omega Q|u|^{2^*}\,dx\right)^{q/2^*} \\ &\geq -\mu^r C_* \end{aligned}$$

for all $u \in H^1(\Omega)$, where $r = 2^*/(2^* - q)$ and $C_* = C_*(N, q, Q_*, \|h\|_\infty) > 0$ is a constant and $Q_* = \min_{x \in \bar{Q}} Q(x)$.

With the aid of this estimate we can establish the $(\mathrm{PS})_c$ condition for $J_{\lambda,\mu}$.

LEMMA 8.2. *The functional $J_{\lambda,\mu}$ satisfies the modified $(\mathrm{PS})_c$ condition (A_4) with $c < S_\infty - \mu^r C_*$ for each $\lambda > 0$ and $\mu > 0$.*

The proof parallels Lemma 3.21 in [58] and is omitted.

We now choose $\mu_\circ > 0$ so that $S_\infty - \mu^r C_* \geq 0$ for all $0 < \mu \leq \mu_\circ$.

THEOREM 8.3. *Let $0 < \mu \leq \mu_\circ$. Suppose that $h(x) > 0$ on Ω and $1 < q < 2$. Then problem* (1.1) *has infinitely many solutions for $\lambda > 0$.*

Proof. We apply Theorem 8.1 to the functional $J_{\lambda,\mu}$. Obviously this functional is even. We define

$$\mu_k = \sup_{u \in X_k \setminus \{0\}} \frac{(\int_\Omega h(x)|u|^q \, dx)^{1/q}}{\|u\|_\lambda}.$$

Since the space $L^q(\Omega, h)$ is compactly embedded in $H^1(\Omega)$ we see that $\mu_k \to 0$ as $k \to \infty$. For every $u \in X_k$ we have

$$J_{\lambda,\mu}(u) \geq \frac{1}{2}\, \|u\|_\lambda^2 - \frac{\mu\mu_k^q}{q}\, \|u\|_\lambda^q - C_1\|u\|_\lambda^{2^*}$$

for some constant $C_1 > 0$ independent of k. For $R > 0$ sufficiently small we have

$$C_1\|u\|_\lambda^{2^*} \leq \frac{1}{4}\, \|u\|_\lambda^2 \quad \text{for } \|u\| \leq R.$$

Thus

$$J_{\lambda,\mu}(u) \geq \frac{1}{4}\, \|u\|_\lambda^2 - \frac{\mu\mu_k}{q}\, \|u\|_\lambda^q.$$

If $R_k = \left(\frac{q}{4\mu\mu_k}\right)^{1/(q-2)}$, then $R_k \to 0$ and $J_\lambda(u) \geq 0$ for $\|u\|_\lambda = R_k$. Then (A_1) of Theorem 8.1 is satisfied. Assumption (A_2) follows from the fact that $R_k \to 0$. The Palais–Smale condition appearing in (A_4) follows from Lemma 8.2. Since all norms in X^k are equivalent it is easy to check that (A_3) also holds by choosing $r_k > 0$ sufficiently small. ■

Alternatively, the existence of infinitely many solutions can be established using Clark's critical point theorem [28], [48]. This approach has been exploited in the paper [38] in the case of equation (8.1) with the Dirichlet boundary conditions (see also [47]).

We remark that if $Q \leq 0$ in Ω then the geometry of $J_{\lambda,\mu}$ is maintained. Condition (A_4) holds for all $c \in \mathbb{R}$ and all $\mu > 0$, so $\mu_\circ = \infty$ in Theorem 8.3.

9. Existence results under integrability conditions on Q and h

In this section we consider problem (1.1) with Q replaced by $-Q$. We rewrite this problem as

$$\text{(9.1)} \qquad \begin{cases} -\Delta u + \lambda u = h(x)|u|^{q-2}u - Q(x)|u|^{p-2}u & \text{in } \Omega, \\ \partial u/\partial \nu = 0 & \text{on } \partial\Omega, \end{cases}$$

where h and Q are measurable and positive a.e. functions on Ω. We no longer assume the continuity of Q and h. It is assumed that $2 < q < p < \infty$ and we do not require $p = 2^*$, that is, p can be a supercritical exponent.

We assume that h and Q are in $L^1(\Omega)$ and

$$(\mathrm{A}_{Q,h}) \qquad \int_\Omega \left(\frac{h(x)^p}{Q(x)^q}\right)^{1/(p-q)} dx < \infty.$$

In what follows we also need either

$$(\mathrm{A}_h) \qquad \int_\Omega \frac{1}{h(x)^{2/(q-2)}}\, dx < \infty$$

or

$$(\mathrm{A}_Q) \qquad \int_\Omega \frac{1}{Q(x)^{2/(p-2)}}\, dx < \infty.$$

We notice that if $h(x) = Q(x)$ on Ω then $(\mathrm{A}_{Q,h})$ is obviously satisfied. We follow some ideas from the paper of Alama–Tarantello [10], where the Dirichlet problem for equation (9.1) under assumption $(\mathrm{A}_{Q,h})$ was investigated (see also [9], [11]). Solutions to problem (9.1) will be sought in the weighted Sobolev space E_Q defined by

$$E_Q = \Big\{u : \nabla u \in L^2(\Omega) \text{ and } \int_\Omega Q(x)|u|^p\, dx < \infty\Big\}.$$

The norm in E_Q is given by

$$\|u\|^2_{E_Q} = \int_\Omega |\nabla u|^2\, dx + \Big(\int_\Omega Q(x)|u|^p\, dx\Big)^{2/p}.$$

If $u \in E_Q$ and $(\mathrm{A}_{Q,h})$ holds, then by the Hölder inequality we have

$$\int_\Omega h|u|^q\, dx \le \Big(\int_\Omega Q|u|^p\, dx\Big)^{q/p}\Big(\int_\Omega \frac{h^{p/(p-q)}}{Q^{q/(p-q)}}\, dx\Big)^{(p-q)/p}.$$

If $u \in E_Q$, then $u \in L^2_{\mathrm{loc}}(\Omega)$ (see [41], p. 7). To ensure that $u \in L^2(\Omega)$ we need either (A_Q) or (A_h). Indeed, using (A_Q), we check that if $u \in E_Q$, then

$$\int_\Omega u^2\, dx \le \Big(\int_\Omega Q|u|^p\, dx\Big)^{2/p}\Big(\int_\Omega \frac{1}{Q^{2/(p-2)}}\, dx\Big)^{(p-2)/p}$$

while if $u \in E_Q$ and (A_h) hold, then

$$\int_\Omega u^2\, dx \le \Big(\int_\Omega h|u|^q\, dx\Big)^{2/q}\Big(\int_\Omega \frac{1}{h^{2/(q-2)}}\, dx\Big)^{(q-2)/q}.$$

These estimates show that under assumptions $(\mathrm{A}_{Q,h})$ and (A_Q) (or (A_h)) E_Q is continuously embedded into $L^2(\Omega)$ and $L^q(\Omega, h)$. We also note that E_Q is continuously embedded into $H^1(\Omega)$.

We now make the following remarks about assumptions $(\mathrm{A}_{Q,h})$, (A_Q) and (A_h). If $0 < m \le Q(x) \le M$ on Ω for some constants m and M, then assumption (A_Q) is automatically satisfied. Assumption $(\mathrm{A}_{Q,h})$ takes the form

$$\int_\Omega h(x)^{p/(p-q)}\,dx < \infty.$$

In this case we can take as a norm in the space E_Q:

$$\|u\|_E^2 = \int_\Omega |\nabla u|^2\,dx + \Big(\int_\Omega |u|^p\,dx\Big)^{2/p}.$$

E_Q is continuously embedded in $L^2(\Omega)$ under only the assumption $(\mathrm{A}_{Q,h})$.

If $0 < m \le h(x) \le M$ on Ω, where m and M are some constants, then assumption $(\mathrm{A}_{Q,h})$ can be written as

$$\int_\Omega \frac{1}{Q(x)^{q/(p-q)}}\,dx < \infty.$$

This inequality implies that E_Q is continuously embedded in $L^2(\Omega)$ and $L^q(\Omega)$. On the other hand by the Hölder inequality we have

$$\int_\Omega \frac{1}{Q^{2/(p-2)}}\,dx \le |\Omega|^{\frac{p(q-2)}{q(p-2)}}\Big(\int_\Omega \frac{1}{Q^{q/(p-q)}}\,dx\Big)^{\frac{2(p-q)}{q(p-2)}},$$

so (A_Q) holds.

Finally, assumptions $(\mathrm{A}_{Q,h})$ and (A_h) imply (A_Q). Indeed, by the Hölder inequality we have

$$\int_\Omega \frac{1}{Q^{2/(p-2)}}\,dx \le \Big(\int_\Omega \frac{h^{p/(p-q)}}{Q^{q/(p-q)}}\,dx\Big)^{\frac{2(p-q)}{q(p-2)}}\Big(\int_\Omega \frac{1}{h^{2/(q-2)}}\,dx\Big)^{\frac{p(q-2)}{q(p-2)}}.$$

We associate with (9.1) the variational functional

$$I_\lambda(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx + \frac{1}{p}\int_\Omega Q(x)|u|^p\,dx - \frac{1}{q}\int_\Omega h(x)|u|^q\,dx.$$

It is easy to check that I_λ is of class C^1 on E_Q.

Throughout this and next sections we shall frequently refer to the following inequality: for all $a > 0$ and $b > 0$ and $s < r$ we have

$$a|u|^s - b|u|^r \le C_{rs}a\Big(\frac{a}{b}\Big)^{s/(r-s)}, \tag{9.2}$$

for every $u \in \mathbb{R}$, where $C_{rs} > 0$ is a constant depending on r and s.

Proposition 9.1. *Suppose that* $(\mathrm{A}_{Q,h})$ *and either* (A_Q) *or* (A_h) *hold. Then for each* $\lambda \in \mathbb{R}$ *the functional* I_λ *is bounded from below on* E_Q.

Proof. Suppose that $(\mathrm{A}_{Q,h})$ and (A_Q) hold. We use the Young inequality: for every $\delta > 0$ there exist constants $C_1(\delta) > 0$ and $C_2(\delta) > 0$ such that

$$\int_\Omega u^2\,dx \le \delta\int_\Omega Q|u|^p\,dx + C_1(\delta)\int_\Omega \frac{1}{Q^{2/(p-2)}}\,dx, \tag{9.3}$$

$$\int_\Omega h|u|^q\,dx \le \delta\int_\Omega Q|u|^p\,dx + C_2(\delta)\int_\Omega \frac{h^{p/(p-q)}}{Q^{q/(p-q)}}\,dx. \tag{9.4}$$

If $\lambda < 0$ we insert (9.3) and (9.4) into I_λ to obtain

$$I_\lambda(u) \geq \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \left(\frac{1}{p} + \delta\lambda - \frac{\delta}{q}\right)\int_\Omega Q|u|^p\,dx + \lambda C_1(\delta)\int_\Omega \frac{1}{Q^{2/(p-2)}}\,dx - \frac{C_2(\delta)}{q}\int_\Omega \frac{h^{p/(p-q)}}{Q^{q/(p-q)}}\,dx.$$

We now select δ so that $1/p + \lambda\delta - \delta/q > 0$ and the assertion follows. If $\lambda \geq 0$ we need only inequality (9.4). We argue in a similar manner if $(A_{Q,h})$ and (A_h) hold. ∎

PROPOSITION 9.2. *Suppose that* $(\mathrm{A}_{Q,h})$ *and either* (A_Q) *or* (A_h) *hold. Then* I_λ *satisfies the* $(\mathrm{PS})_c$ *condition for every* c.

Proof. Let $\{u_m\}$ be a $(\mathrm{PS})_c$ sequence for I_λ. First we show that it is bounded in E_Q. This is obvious if $\lambda \geq 0$. So we consider the case $\lambda < 0$. We have $I_\lambda(u_m) \leq c+1$ for large m, say $m \geq m_\circ$. Using the Young inequality, in conjunction with $(\mathrm{A}_{Q,h})$ and (A_Q) for every $\delta > 0$ we can find a constant $C(\delta) > 0$ such that

$$\frac{1}{2}\int_\Omega |\nabla u_m|^2\,dx + \frac{1}{p}\int_\Omega Q|u_m|^p\,dx \leq \frac{|\lambda|}{2}\int_\Omega u_m^2\,dx + \frac{1}{q}\int_\Omega h|u_m|^q\,dx + c + 1$$
$$\leq \delta(1+|\lambda|^{p/2})\int_\Omega Q|u_m|^p\,dx + C(\delta)\left(\int_\Omega \frac{1}{Q^{2/(p-2)}}\,dx + \int_\Omega \frac{h^{p/(p-q)}}{Q^{q/(p-q)}}\,dx\right) + c + 1. \tag{9.5}$$

If (A_h) holds then the integral $\int_\Omega \frac{1}{Q^{2/(p-2)}}\,dx$ in the above inequality is replaced by $\int_\Omega \frac{1}{h^{2/(q-2)}}\,dx$. Taking $0 < \delta(1+|\lambda|^{p/2}) < 1/p$ we derive from (9.5) that $\{\|u_m\|_{E_Q}\}$ is bounded. The sequence is also bounded in $H^1(\Omega)$. Therefore we can assume that $u_m \rightharpoonup u$ in E_Q, $H^1(\Omega)$, $L^q(\Omega, Q)$ and $L^q(\Omega, h)$ and $u_m \to u$ in $L^2(\Omega)$. We now show that $u_m \to u$ in $H^1(\Omega)$. We set

$$F(x,u) = h(x)\,\frac{|u|^q}{q} - Q(x)\,\frac{|u|^p}{p} \quad \text{and} \quad f(x,u) = F_u(x,u).$$

We then have

$$f_u(x,u) = (q-1)h(x)|u|^{q-2} - (p-1)Q(x)|u|^{p-2} \leq C_{pq}h\left(\frac{h}{Q}\right)^{\frac{q-2}{p-q}}$$

for all $u \in \mathbb{R}$ and some constant $C_{pq} > 0$. Hence

$$\begin{aligned}\int_\Omega |\nabla(u_m - u_n)|^2\,dx &= -\lambda\int_\Omega (u_m - u_n)^2\,dx \\ &\quad + \int_\Omega (f(x,u_m) - f(x,u_n))(u_m - u_n)\,dx + o(1) \\ &= \int_\Omega\int_0^1 f_u(x, u_n + t(u_m - u_n))\,dt(u_m - u_n)^2\,dx + o(1) \\ &\leq C_{pq}\int_\Omega h(x)\left(\frac{h(x)}{Q(x)}\right)^{\frac{q-2}{p-q}}(u_m - u_n)^2\,dx + o(1).\end{aligned} \tag{9.6}$$

According to $(\mathrm{A}_{Q,h})$, $(h/Q)^{(q-2)/(p-q)} \in L^{q/(q-2)}(\Omega, h)$ and moreover $(u_m - u_n)^2 \rightharpoonup 0$ as $m, n \to \infty$ in $L^{q/2}(\Omega, h)$. Therefore the right hand side of (9.6) tends to 0 as $m, n \to \infty$. This shows that $u_m \to u$ in $H^1(\Omega)$. In the final step of the proof we show that $u_m \to u$ in E_Q. To show this we use $\langle I'_\lambda(u), u\rangle = 0$ to write

$$o(1) = \langle I'_\lambda(u), u\rangle - \langle I'_\lambda(u_m), u_m\rangle = \int_\Omega h(x)(|u_m|^q - |u|^q)\, dx - \int_\Omega Q(|u_m|^p - |u|^p)\, dx + o(1)$$

and

$$o(1) = I_\lambda(u) - I_\lambda(u_m) = \frac{1}{q}\int_\Omega h(x)(|u_m|^q - |u|^q)\, dx - \frac{1}{p}\int_\Omega Q(|u_m|^p - |u|^p)\, dx + o(1).$$

These two relations show that $u_m \to u$ in $L^p(\Omega, Q)$ and $L^q(\Omega, h)$ and this completes the proof. ∎

THEOREM 9.3. *Suppose that* $(\mathrm{A}_{Q,h})$ *and* (A_Q) *(or* (A_h)*) hold. Then there exists* $0 < \lambda_* \le \infty$ *such that for every* $\lambda \le \lambda_*$ *problem* (9.1) *has a solution which is a global minimizer of* I_λ *on* E_Q.

Proof. If $\lambda < 0$, then for $t > 0$ we have

$$I_\lambda(t) = \frac{\lambda t^2|\Omega|}{2} + \frac{t^p}{p}\int_\Omega Q\, dx - \frac{t^q}{q}\int_\Omega h\, dx \le \frac{t^p}{p}\int_\Omega Q\, dx - \frac{t^q}{q}\int_\Omega h\, dx < 0$$

taking t sufficiently small. If $\lambda \ge 0$, we first choose $t > 0$ so that

$$\frac{t^p}{p}\int_\Omega Q\, dx - \frac{t^q}{q}\int_\Omega h\, dx < 0.$$

We then select $\lambda_\circ > 0$ such that $I_\lambda(t) < 0$ for all $0 \le \lambda \le \lambda_\circ$. Thus for each $\lambda \le \lambda_\circ$,

$$\inf_{u\in E_Q} I_\lambda(u) < 0.$$

Applying the Ekeland variational principle [31] for each $\lambda \le \lambda_\circ$ we can find a sequence $\{u_m\} \subset E_Q$ such that $I_\lambda(u_m) \to \inf_{u\in E_Q} I_\lambda(u)$ and $I'_\lambda(u_m) \to 0$ in E_Q^*. By Proposition 9.2 up to a subsequence $u_m \to u$ in E_Q and u is a minimizer of I_λ. We set

$$\lambda_* = \sup\{\lambda : \text{problem (9.1) has a solution}\}.$$

It is clear that $\lambda_* > 0$. To complete the proof we show that for each $\lambda_1 < \lambda_*$ problem (9.1) has a solution. It is sufficient to consider $\lambda_1 > 0$. There exists $\lambda_1 < \mu < \lambda_*$ such that problem (9.1) with $\lambda = \mu$ has a solution u_μ. We now consider the minimization problem

$$\inf\{I_{\lambda_1}(u) : u \in E_Q,\ u(x) \ge u_\mu\}.$$

It follows from the proof of Proposition 9.1 that I_{λ_1} is lower semicontinuous. Thus I_{λ_1} attains its minimum at some $u \ge u_\mu$. Since u_μ is a supersolution, u must be a solution of problem (9.1). ∎

Under a stronger assumption on h we can show that $\lambda_* < \infty$. We impose the following condition on Q and h:

$$(\mathrm{B}_{Q,h}) \qquad \int_\Omega \left(\frac{h(x)^{\frac{p-1}{p-q}}}{Q(x)^{\frac{q-1}{p-q}}}\right)^2 dx < \infty.$$

To compare assumptions $(\mathrm{A}_{Q,h})$ and $(\mathrm{B}_{Q,h})$ we use the Hölder inequality to obtain

$$\begin{aligned}\int_\Omega h\Big(\frac{h}{Q}\Big)^{\frac{q}{p-q}}\,dx &= \int_\Omega h^{\frac{q}{q-1}}\Big(\frac{h}{Q}\Big)^{\frac{q}{p-q}} h^{-\frac{1}{q-1}}\,dx\\ &\le \Big(\int_\Omega h^2\Big(\frac{h}{Q}\Big)^{\frac{2(q-1)}{p-q}}\,dx\Big)^{\frac{q}{2(q-1)}}\Big(\int_\Omega \frac{1}{h^{\frac{2}{q-2}}}\,dx\Big)^{\frac{q-2}{2(q-1)}}\end{aligned}$$

and

$$\begin{aligned}\int_\Omega h\Big(\frac{h}{Q}\Big)^{\frac{q}{p-q}}\,dx &= \int_\Omega \frac{h^{\frac{p}{p-q}}}{Q^{\frac{p(q-1)}{(p-q)(p-1)}}}Q^{-\frac{1}{p-1}}\,dx\\ &\le \Big(\int_\Omega \frac{h^{\frac{2(p-1)}{p-q}}}{Q^{\frac{2(q-1)}{p-q}}}\,dx\Big)^{\frac{p}{2(p-1)}}\Big(\int_\Omega \frac{1}{Q^{\frac{2}{p-2}}}\,dx\Big)^{\frac{p-2}{2(p-1)}}.\end{aligned}$$

Thus if $(\mathrm{B}_{Q,h})$ and either (A_Q) or (A_h) hold, then $(\mathrm{A}_{Q,h})$ is satisfied.

PROPOSITION 9.4. *Let $2 < q < 2^*$. Suppose that $(\mathrm{B}_{Q,h})$ and either (A_Q) or (A_h) hold. If $h \in L^{2^*/(2^*-q)}(\Omega)$, then $\lambda_* < \infty$.*

Proof. Suppose that $\lambda_* = \infty$. Then for each $\lambda > 0$ problem (9.1) has a solution u_λ. We then have

$$\begin{aligned}\int_\Omega (|\nabla u_\lambda|^2 + \lambda u_\lambda^2)\,dx &= \int_\Omega h|u_\lambda|^q\,dx - \int_\Omega Q|u_\lambda|^p\,dx \le C_{pq}\int_\Omega \frac{h^{\frac{p-1}{p-q}}}{Q^{\frac{q-1}{p-q}}}|u_\lambda|\,dx\\ &\le \frac{\lambda}{2}\int_\Omega u_\lambda^2\,dx + \frac{C_{pq}^2}{2\lambda}\int_\Omega \Big(\frac{h^{\frac{p-1}{p-q}}}{Q^{\frac{q-1}{p-q}}}\Big)^2\,dx.\end{aligned}$$

Assuming that $\lambda > 2$ we deduce from this that

$$\int_\Omega (|\nabla u_\lambda|^2 + u_\lambda^2)\,dx \le \frac{C_{pq}^2}{2\lambda}\int_\Omega \Big(\frac{h^{\frac{p-1}{p-q}}}{Q^{\frac{q-1}{p-q}}}\Big)^2\,dx.$$

On the other hand by (2.1) we have for $\lambda \ge 1$ that

$$\begin{aligned}\int_\Omega (|\nabla u_\lambda|^2 + u_\lambda^2)\,dx &\le \int_\Omega (|\nabla u_\lambda|^2 + \lambda u_\lambda^2)\,dx\\ &\le \Big(\int_\Omega |h|^{2^*/(2^*-q)}\,dx\Big)^{(2^*-q)/2^*}\Big(\int_\Omega |u_\lambda|^{2^*}\,dx\Big)^{q/2^*}\\ &\le C_s\|h\|_{2^*/(2^*-q)}\Big(\int_\Omega (|\nabla u_\lambda|^2 + u_\lambda^2)\,dx\Big)^{q/2}.\end{aligned}$$

Hence

$$(C_s\|h\|_{2^*/(2^*-q)})^{-2/(q-2)} \le \int_\Omega (|\nabla u_\lambda|^2 + u_\lambda^2)\,dx,$$

which is impossible for large $\lambda > 0$. ■

The proof of Proposition 9.4 shows that

$$\lambda_* \le \max\left(2, \frac{1}{2}C_{pq}^2(C_s\|h\|_{2^*/(2^*-q)})^{2/(q-2)} \int_\Omega \left(\frac{h^{\frac{p-1}{p-q}}}{Q^{\frac{q-1}{p-q}}}\right)^2 dx\right).$$

Proposition 9.5. *Let $\lambda = 0$, $Q(x) = h(x)$ a.e. on Ω and suppose that either* (A_Q) *or* (A_h) *holds. Then $u \equiv 1$ is a global minimizer of $I_\circ$.*

Proof. Obviously $u \equiv 1$ is a solution of problem (9.1) with $\lambda = 0$. By Theorem 9.3 problem (9.1) has a global minimizer $u_\circ \in E_Q$. Thus $I_\circ(u_\circ) \le I_\circ(1)$. Applying the Young inequality we deduce from this that

$$\frac{1}{2}\int_\Omega |\nabla u_\circ|^2\,dx + \frac{1}{p}\int_\Omega h|u_\circ|^p\,dx + \left(\frac{1}{q} - \frac{1}{p}\right)\int_\Omega h\,dx$$
$$\le \frac{1}{q}\int_\Omega h|u_\circ|^q\,dx \le \frac{1}{q}\left(\frac{q}{p}\int_\Omega h|u_\circ|^p\,dx + \frac{p-q}{p}\int_\Omega h\,dx\right).$$

Hence $\int_\Omega |\nabla u_\circ|^2\,dx = 0$, that is, $u_\circ \equiv 1$ on Ω. ∎

We now state the existence results for (9.1) with $(\mathrm{A}_{Q,h})$ replaced by the integrability condition:

$$(\mathrm{A}_{Q,h,N}) \qquad \int_\Omega \left(\frac{h(x)^{\frac{p-2}{p-q}}}{Q(x)^{\frac{q-2}{p-q}}}\right)^{N/2} dx < \infty.$$

Lemma 9.6. *If* $(\mathrm{A}_{Q,h,N})$ *and either* (A_Q) *or* (A_h) *hold, then for every $\lambda \in \mathbb{R}$, the functional I_λ is bounded from below on E_Q.*

Proof. We only consider the case $\lambda \le 0$ assuming (A_Q). First by the Young inequality and (A_Q) for every $\delta > 0$ there exists a constant $C(\delta) > 0$ such that

$$\int_\Omega u^2\,dx \le \delta\int_\Omega Q|u|^p\,dx + C(\delta)\int_\Omega \frac{1}{Q^{2/(p-2)}}\,dx. \tag{9.7}$$

As in the paper [10] we define for $\eta > 0$ and $M > 0$ the sets

$$X = \{x \in \Omega : h(x) < M \text{ and } Q(x) > \eta\},$$
$$Y = \{x \in \Omega : h(x) < M \text{ and } Q(x) \le \eta\},$$
$$Z = \{x \in \Omega : h(x) \ge M\}.$$

We now use inequality (9.2) and the Sobolev inequality to get

$$\int_X \left(\frac{h}{q}|u|^q - \frac{Q}{2p}|u|^p\right) dx \le C_{pq}\int_X \frac{h(x)^{p/(p-q)}}{Q(x)^{q/(p-q)}}\,dx \tag{9.8}$$

and

$$\int_{Y\cup Z} \left(\frac{h}{q}|u|^q - \frac{Q}{2p}|u|^p\right) dx \le C_{pq}\int_{Y\cup Z} \left(\frac{h(x)^{\frac{p-2}{p-q}}}{Q(x)^{\frac{q-2}{p-q}}}\right) u^2\,dx \tag{9.9}$$
$$\le C_1\left(\int_{Y\cup Z}\left(\frac{h^{\frac{p-2}{p-q}}}{Q^{\frac{q-2}{p-q}}}\right)^{N/2} dx\right)^{2/N}\|u\|_{2^*}^2 \le C_2\left(\int_{Y\cup Z}\left(\frac{h^{\frac{p-2}{p-q}}}{Q^{\frac{q-2}{p-q}}}\right)^{N/2} dx\right)^{2/N}\int_\Omega (|\nabla u|^2 + u^2)\,dx.$$

We now observe that $|Z| \to 0$ as $M \to \infty$ and for every M, $|Y| \to 0$ as $\eta \to 0$. Given $\varepsilon > 0$ we first select $M > 0$ large and then $\eta > 0$ small enough so that

$$C_2\bigg(\int_{Y\cup Z}\bigg(\frac{h^{\frac{p-2}{p-q}}}{Q^{\frac{q-2}{p-q}}}\bigg)^{N/2}dx\bigg)^{2/N} < \varepsilon. \tag{9.10}$$

It then follows from (9.7)–(9.9) and (9.10)

$$\begin{aligned} I_\lambda(u) \geq{}& \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2p}\int_\Omega Q(x)|u|^p\,dx - C_{pq}\int_X \frac{h(x)^{p/(p-q)}}{Q(x)^{q/(p-q)}}\,dx \\ &- \varepsilon\int_\Omega(|\nabla u|^2+u^2)\,dx - \frac{\lambda\delta}{2}\int_\Omega Q|u|^p\,dx - \lambda C(\delta)\int_\Omega \frac{1}{Q^{2/(p-q)}}\,dx. \end{aligned} \tag{9.11}$$

If we apply again inequality (9.7) and choose $\delta > 0$ and $\varepsilon > 0$ so that

$$\frac{1}{2p} - \varepsilon\delta - \frac{|\lambda|\delta}{2} > 0 \quad \text{and} \quad \frac{1}{2} - \varepsilon > 0$$

the result readily follows. ∎

PROPOSITION 9.7. *If* $(\mathrm{A}_{Q,h,N})$ *and either* (A_Q) *or* (A_h) *hold, then for every* $\lambda \in \mathbb{R}$, I_λ *satisfies the* $(\mathrm{PS})_c$ *condition for every* $c \in \mathbb{R}$.

Proof. We only consider the case $\lambda < 0$ assuming that (A_Q) holds. As in the proof of Lemma 9.6 we obtain estimate (9.11) for the $(\mathrm{PS})_c$ sequence $\{u_m\}$. From this we deduce that $\{u_m\}$ is bounded in E_Q. So we can assume that $u_m \rightharpoonup u$ in $H^1(\Omega)$, E_Q, $L^p(\Omega, Q)$, $L^q(\Omega, h)$ and $u_m \to u$ in $L^2(\Omega)$. Repeating the final part of the proof of Proposition 9.2, with obvious modifications, we show that up to a subsequence $u_m \to u$ in E_Q. ∎

It is worth mentioning that under the assumptions of Proposition 9.7 one can show that a solution of (9.1) in E_Q belongs to $C^{1,\alpha}(\overline{\Omega})$ for each $0 < \alpha < 1$. This can be proved using the iteration technique from the paper [10] (pp. 170–171).

THEOREM 9.8. *If* $(\mathrm{A}_{Q,h,N})$ *and either* (A_Q) *or* (A_h) *hold, then there exists* $0 < \lambda_\circ < \infty$ *such that for each* $\lambda < \lambda_\circ$ *problem* (9.1) *has a solution.*

Proof. It is evident that there exists $\overline{\lambda} > 0$ such that for every $\lambda < \overline{\lambda}$, $\inf_{E_Q} I_\lambda(u) < 0$. It then follows from Proposition 9.7 that for each $\lambda < \overline{\lambda}$ there exists a global minimizer of I_λ which is a solution of problem (9.1). We now set

$$\lambda_\circ = \sup\{\lambda :\ \text{problem (9.1) has a solution}\}.$$

It is clear that $\lambda_\circ > 0$ and as in the proof of Theorem 9.3 we check that for each $\lambda < \lambda_\circ$ problem (9.1) has a solution. In the final step of the proof we show that $\lambda_\circ < \infty$. We follow the argument from the paper [10] (pp. 176–177). We define

$$a(x) = \frac{p-q}{p-2}\,h(x)\bigg[\frac{2(q-2)h(x)}{(p-2)Q(x)}\bigg]^{\frac{q-2}{p-q}}.$$

Let μ_1 be the first eigenvalue of the eigenvalue problem

$$\begin{cases} -\Delta u - a(x)u = \mu u & \text{in } \Omega, \\ \partial u/\partial\nu = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $a(x) \not\equiv 0$ and $a(x) \geq 0$ on Ω, we see that $\mu_1 < 0$. Let e_1 be the corresponding eigenfunction. If u_λ is a solution of (9.1), then

$$\begin{aligned} 0 &= \int_\Omega \nabla u_\lambda \nabla e_1\,dx + \lambda \int_\Omega u_\lambda e_1\,dx + \int_\Omega Q u_\lambda^{p-1} e_1\,dx - \int_\Omega h u_\lambda^{q-1} e_1\,dx \\ &\geq \int_\Omega \nabla u_\lambda \nabla e_1\,dx + \lambda \int_\Omega u_\lambda e_1\,dx - \int_\Omega a u_\lambda e_1\,dx + \frac{1}{2}\int_\Omega Q u_\lambda^{p-1} e_1\,dx \\ &\geq \mu_1 \int_\Omega u_\lambda e_1\,dx + \lambda \int_\Omega u_\lambda e_1\,dx. \end{aligned}$$

This obviously implies that $\lambda_\circ \leq -\mu_1$. ■

The existence results in Theorems 9.3 and 9.8 were obtained for $\lambda \leq \overline{\lambda}$ and in general $\overline{\lambda} < \infty$. We now consider a situation where a solution of problem (9.1) exists for every λ. To achieve this we introduce a new parameter $\gamma > 0$ with the coefficient Q. The problem we shall consider is

$$\begin{cases} -\Delta u + \lambda u = h(x)|u|^{q-2}u - \gamma Q(x)|u|^{p-2}u & \text{in } \Omega, \\ \partial u/\partial \nu = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{9.12}$$

Let

$$I_{\lambda,\gamma}(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx - \frac{1}{q}\int_\Omega h(x)|u|^q\,dx + \frac{\gamma}{p}\int_\Omega Q(x)|u|^p\,dx$$

be a variational functional for problem (9.12).

THEOREM 9.9. *Suppose that* $(\mathrm{A}_{Q,h,N})$ *and either* (A_Q) *or* (A_h) *hold. Then for every* λ *there exists* $\gamma^* = \gamma^*(\lambda)$ *such that problem* (9.12) *with* $0 < \gamma < \gamma^*$ *admits a solution.*

Proof. Let us assume that $(\mathrm{A}_{Q,h,N})$ and (A_Q) hold. Let $t \in \mathbb{R}$. Then

$$\begin{aligned} I_{\lambda,\gamma}(t) &= \frac{\lambda t^2}{2}|\Omega| - \frac{|t|^q}{q}\int_\Omega h(x)\,dx + \frac{\gamma|t|^p}{p}\int_\Omega Q(x)\,dx \\ &= \frac{|t|^q}{2}\left(\frac{\lambda|\Omega|}{t^{q-2}} - \int_\Omega h(x)\,dx + \frac{\gamma|t|^{p-q}}{2}\int_\Omega Q(x)\,dx\right). \end{aligned}$$

First we choose t large to satisfy

$$\frac{\lambda|\Omega|}{t^{q-2}} - \int_\Omega h(x)\,dx < 0.$$

Then there exists $\overline{\gamma} = \overline{\gamma}(\lambda)$ such that $I_{\lambda,\gamma}(t) < 0$ for every $0 < \gamma \leq \overline{\gamma}$. By Lemma 9.6, $I_{\lambda,\gamma}$ is bounded from below. Therefore if we fix $\lambda \in \mathbb{R}$ and take $0 < \gamma < \overline{\gamma}(\lambda)$, then $\inf_{u\in H^1(\Omega)} I_{\lambda,\gamma}(u) < 0$. By Proposition 9.7 with the aid of the Ekeland variational principle we can show that problem (9.12), with $0 < \gamma < \overline{\gamma}(\lambda)$, has a solution. To complete the proof we define for a fixed $\lambda \in \mathbb{R}$,

$$\gamma^*(\lambda) = \sup\{\gamma : \text{problem (9.12) has a solution in } E_Q\}.$$

We now show that $\gamma^*(\lambda) < \infty$ for every λ. If $u \in E_Q$ is a solution of problem (9.12) then

$$\int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx = \int_\Omega h(x)|u|^q\,dx - \gamma\int_\Omega Q(x)|u|^p\,dx$$

$$\leq C_{pq} \int_\Omega \frac{h^{\frac{p-2}{p-q}}}{(\gamma Q)^{\frac{q-2}{p-q}}} u^2\,dx$$

$$\leq C_{pq} \bigg(\int_\Omega \bigg(\frac{h^{\frac{p-2}{p-q}}}{(\gamma Q)^{\frac{q-2}{p-q}}} \bigg)^{N/2} dx \bigg)^{2/N} \bigg(\int_\Omega |u|^{2^*}\,dx \bigg)^{(N-2)/N}$$

$$\leq C_{pq} C_s(\lambda) \bigg(\int_\Omega \bigg(\frac{h^{\frac{p-2}{p-q}}}{(\gamma Q)^{\frac{q-2}{p-q}}} \bigg)^{N/2} dx \bigg)^{2/N} \int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx,$$

where $C_s(\lambda)$ is the best Sobolev constant from inequality (2_λ). From this estimate we derive that

$$\gamma^*(\lambda) \leq \bigg(C_{pq} C_s(\lambda) \int_\Omega \bigg(\frac{h^{\frac{p-2}{p-q}}}{Q^{\frac{q-2}{p-q}}} \bigg)^{N/2} dx \bigg)^{\frac{2(p-q)}{N(q-2)}}. \ \blacksquare$$

The above estimates show that $\gamma^*(\lambda) < \infty$ for $\lambda > 0$ and $\gamma^*(\lambda)$ is a bounded function for large $\lambda > 0$. On the other hand by Theorem 9.8, $\gamma^*(\lambda) = \infty$ for $\lambda \leq 0$.

A similar result can be established under assumptions $(\mathrm{B}_{Q,h})$ and (A_Q) (or (A_h)).

10. Problem (9.1) without the integrability condition $(\mathrm{A}_{Q,h})$

In this section we briefly discuss the case when the assumption $(\mathrm{A}_{Q,h})$ is not satisfied. For simplicity we assume that $h \in L^\infty(\Omega)$, $h(x) \geq a$ on $B(x_\circ, \delta)$ and $\limsup_{x \to x_\circ} Q(x)/|x - x_\circ|^s = \beta$ for some $x_\circ \in \Omega$ and constants $a > 0$, $\beta > 0$ and $s > 0$. It is easy to check that if $s \geq N(p-q)/q$, then $(\mathrm{A}_{Q,h})$ does not hold. On the other hand if $s \geq 2(p-q)/(q-2)$, then $(\mathrm{A}_{Q,h,N})$ does not hold. We now observe that $2(p-q)/(q-2) \geq N(p-q)/q$ if and only if $q \leq 2^*$. This means that if $(\mathrm{A}_{Q,N,h})$ is not satisfied then also $(\mathrm{A}_{Q,h})$ does not hold.

Let $Q(x) = \gamma |x - x_\circ|^{2(p-q)/(q-2)}$, $\gamma > 0$. It can be verified that if q and p satisfy $2 < q < p < 4/(4-q)$ with $2 < q \leq 2^*$, then $1/(q-2) - 1/(p-2) \leq 1/2$. This in turn implies that $4/(q-2) - 4/(p-2) < 2 < N$. Hence $N - 4(p-q)/((q-2)(p-2)) \geq 0$, that is, condition (A_Q) holds. As in the paper [10] in the case $q \leq 2^*$ one can construct a sequence $\{u_n\} \subset E_Q$ such that $I_\lambda(u_n) \to -\infty$ and $\|u_n\|_{E_Q} \to \infty$, which shows that I_λ is not coercive. In this section we establish the existence result for $Q(x) = \gamma |x - x_\circ|^{2(p-q)/(q-2)}$, $\gamma > 0$.

We need a version of the Hardy inequality in $H^1(\Omega)$. We recall that if $0 \in \Omega$, then

$$\int_\Omega \frac{u(x)^2}{|x|^2}\,dx \leq \bigg(\frac{N-2}{2} \bigg)^2 \int_\Omega |\nabla u(x)|^2\,dx$$

for every $u \in H^1_\circ(\Omega)$ (see [22]). Let $\phi \in C^1_\circ(\Omega)$ and $u \in H^1(\Omega)$. Then by the Hardy inequality we have

$$\int_\Omega \frac{u^2}{|x|^2}\,dx \leq \bigg(\frac{N-2}{2} \bigg)^2 \int_\Omega |\nabla (u\phi)|^2\,dx + \int_\Omega \frac{u^2(1-\phi^2)}{|x|^2}\,dx.$$

Assuming that $\phi = 1$ in a ball $B(0,\delta) \subset \Omega$ we deduce from this inequality that there is a constant $C_1 > 0$ such that

$$\int_\Omega \frac{u^2}{|x|^2}\,dx \le C_1 \int_\Omega (|\nabla u|^2 + u^2)\,dx \tag{10.1}$$

for every $u \in H^1(\Omega)$.

PROPOSITION 10.1. *Let $Q(x) = \gamma|x - x_\circ|^{2(p-q)/(q-2)}$, $\gamma > 0$ and let $q \le 2N/(N-2)$ and $2 < q < p < 4/(4-q)$. Then for every λ there exists $\gamma_\circ = \gamma_\circ(\lambda) > 0$ such that for $\gamma \ge \gamma_\circ$, I_λ is bounded from below on E_Q.*

Proof. For simplicity assume that $x_\circ = 0$ and $0 \in \Omega$. We shall only consider the case $\lambda \le 0$. Applying (9.2) and (10.1) we have

$$\begin{aligned} I_\lambda(u) &\ge \frac{1}{2}\int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx + \frac{\gamma}{2p}\int_\Omega |x|^{\frac{2(p-q)}{q-2}}|u|^p\,dx \\ &\quad + \frac{\gamma}{2p}\int_\Omega |x|^{\frac{2(p-q)}{q-2}}|u|^p\,dx - \frac{1}{q}\int_\Omega h|u|^q\,dx \\ &\ge \frac{1}{2}\int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx + \frac{\gamma}{2p}\int_\Omega |x|^{\frac{2(p-q)}{q-2}}|u|^p\,dx \\ &\quad - C_{pq}\int_\Omega h\left(\frac{h}{\gamma}\right)^{\frac{q-2}{p-q}}\frac{u^2}{|x|^2}\,dx \\ &\ge \frac{1}{2}\int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx + \frac{\gamma}{2p}\int_\Omega |x|^{\frac{2(p-q)}{q-2}}|u|^p\,dx \\ &\quad - C_{pq}C_1\frac{\|h\|_\infty^{\frac{p-2}{p-q}}}{\gamma^{\frac{q-2}{p-q}}}\int_\Omega (|\nabla u|^2 + u^2)\,dx. \end{aligned} \tag{10.2}$$

On the other hand by (A_Q) we have

$$\int_\Omega u^2\,dx \le \delta\int_\Omega Q|u|^p\,dx + C(\delta)\int_\Omega \frac{1}{Q^{2/(p-2)}}\,dx \tag{10.3}$$

for every $\delta > 0$. Inserting this into (10.2) and taking $\delta > 0$ sufficiently small and $\gamma > 0$ large the boundedness from below of I_λ follows. ■

An inspection of the proof of Proposition 10.1 shows that the choice of $\gamma_\circ$ can be made independent of λ for $\lambda \ge 0$.

PROPOSITION 10.2. *Under the assumptions of Proposition* 10.1 *there exists $\lambda_\circ > 0$ such that for every $\lambda \le \lambda_\circ$ there exists $\gamma_\circ = \gamma_\circ(\lambda)$ such that problem* (9.1) *admits a global minimizer for I_λ for every $\lambda \le \lambda_\circ$ and $\gamma > \gamma_\circ$.*

Proof. By Proposition 10.1 for $\lambda \le \lambda_\circ$ we have

$$-\infty < \inf_{u \in E_Q} I_\lambda(u) < 0,$$

where $\lambda_\circ > 0$ is determined as in the proof of Theorem 9.3. By the Ekeland variational principle there exists a minimizing sequence $\{u_m\}$ satisfying $I'_\lambda(u_m) \to 0$ in E_Q^*. For large m, say $m \geq m_\circ$, we have

$$\frac{1}{2}\int_\Omega (|\nabla u_m|^2 + \lambda u_m^2)\,dx + \frac{1}{2p}\int_\Omega Q|u_m|^p\,dx$$
$$\leq \inf_{u\in H^1(\Omega)} I_\lambda(u) + 1 + \frac{1}{q}\int_\Omega h|u_m|^q\,dx - \frac{1}{2p}\int_\Omega Q|u_m|^p\,dx$$
$$\leq \inf_{u\in H^1(\Omega)} I_\lambda(u) + 1 + C_{pq}\int_\Omega h\left(\frac{h}{\gamma}\right)^{\frac{q-2}{p-q}}\frac{u_m^2}{|x|^2}\,dx$$
$$\leq \inf_{u\in H^1(\Omega)} I_\lambda(u) + 1 + \frac{C_1}{\gamma^{\frac{q-2}{p-q}}}\int_\Omega (|\nabla u_m|^2 + u_m^2)\,dx.$$

This combined with estimate (10.3) applied to u_m implies the boundedness of $\{u_m\}$ in E_Q. It is now routine to show that $u_m \to u$ in E_Q. ■

11. Case where the (PS) condition fails

In this section we investigate problem (9.1) assuming that

(A) $Q(x) \geq 0$, $\not\equiv 0$ on Ω and $Q(x) = 0$ on a nonempty subdomain $\Omega_\circ \subset \Omega$, $h(x) \geq 0$ and $h \not\equiv 0$ on Ω.

It is assumed that the coefficients Q and h are smooth on $\overline{\Omega}$ with a smooth boundary $\partial\Omega$. We also insert a new parameter $\gamma > 0$ in problem (9.1), that is, we now consider the problem

$$\begin{cases} -\Delta u + \lambda u = h(x)|u|^{q-2}u - \gamma Q(x)|u|^{p-2}u & \text{in } \Omega,\\ \partial u/\partial\nu = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{11.1}$$

We shall prove the existence of solutions for large $\gamma > 0$. By $J_{\lambda,\gamma}$ we denote the variational functional for (11.1)

$$J_{\lambda,\gamma}(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx + \frac{\gamma}{p}\int_\Omega Q(x)|u|^p\,dx - \frac{1}{q}\int_\Omega h(x)|u|^q\,dx.$$

Under assumption (A), $J_{\lambda,\gamma}$ is not well defined on E_Q nor on $H^1(\Omega)$. Therefore we adopt a more regular approach. We shall work with $J_{\lambda,\gamma}$ on a subset of H^1 where its regularity can be controlled. This particular region is bounded by a sub- and supersolution. First we construct a sub- and supersolution for problem (11.1). These functions will be used to define a closed and convex subset of $H^1(\Omega)$ and a solution will be found by the minimization of $J_{\lambda,\gamma}$ restricted to this set. The construction of a subsolution is based on the bifurcation theorem.

Let X and Y be Banach spaces. Let $F : X \times \mathbb{R} \to Y$ be a continuously differentiable mapping. We assume that $F(0,\lambda) = 0$ for every $\lambda \in \Lambda$, where $\Lambda \subset \mathbb{R}$ is an open interval containing $\lambda_\circ$ and every neighbourhood of $(0,\lambda_\circ)$ contains a zero of $F(x,\lambda)$ which does not belong to the curve $\Gamma = \{(0,\lambda) : \lambda \in \Lambda\}$. Then $(0,\lambda_\circ)$ is said to be a *bifurcation point* of $F(x,\lambda)$ with respect to Γ.

Bifurcation theorem [30]. *Let X and Y be Banach spaces and let V be a neighbourhood of* 0 *in* X. *Suppose that* $F:(-1,1)\times U\to Y$ *satisfies*:

(i) $F(0,\lambda)=0$ *for* $|\lambda|\le 1$,
(ii) *the partial derivatives* F_t, F_x *and* F_{tx} *exist and are continuous*,
(iii) $\mathcal{N}(F_x(0,0))$ *and* $Y/\mathcal{R}(F_x(0,0))$ *are one-dimensional*,
(iv) $F_{tx}(0,0)x_\circ\notin\mathcal{R}(F_x(0,0))$,

where $\mathcal{N}(F_x(0,0))=\operatorname{span}\{x_\circ\}$. *If* Z *is any complement of* $\mathcal{N}(F_x(0,0))$ *in* X, *then there exist a neighbourhood* U *of* $(0,0)$ *in* $X\times\mathbb{R}$, *an interval* $(-a,a)$, *continuous functions* $\phi:(-a,a)\to\mathbb{R}$ *and* $\psi:(-a,a)\to Z$ *such that* $\phi(0)=0$, $\psi(0)=0$ *and*

$$F^{-1}(0)\cap U=\{(\phi(\alpha),\alpha x_\circ+\alpha\psi(\alpha)):|\alpha|<a\}\cup\{(0,t):(0,t)\in U\}.$$

Theorem 11.1. *Suppose that* (A) *holds. Then for every* $-\lambda_1<\overline{\lambda}\le 0$, *there exist* $\gamma_1=\gamma_1(\bar{\lambda})$ *and* $\varepsilon_\circ>0$ *such that problem* (11.1) *has a solution for* $\overline{\lambda}\le\lambda\le\varepsilon_\circ$ *and* $\gamma\ge\gamma_1$. (*Here* λ_1 *is the first eigenvalue of* $-\Delta$ *on* $\Omega_\circ$ *with the Dirichlet boundary condition.*)

Proof. We begin by constructing a supersolution for (11.1). We consider the problem

$$\begin{cases}-\Delta u+\mu u=-\gamma Q(x)|u|^{p-2}u & \text{in } \Omega,\\ \partial u/\partial\nu=0 \quad \text{on } \Omega,\end{cases}\tag{11.2}$$

where $\mu\in\mathbb{R}$ is a parameter. It is easy to check that this problem has no nontrivial solution for $\mu\ge 0$. By the result of Ouyang [46] problem (11.2) has a unique smooth positive solution w_γ if $-\lambda_1<\mu<0$. We rescale the solution w_γ as $w_\gamma=\gamma^{-1/(p-2)}w_1$. Let $-\lambda_1<\overline{\lambda}\le 0$ and choose $-\lambda_1<\widetilde{\lambda}<\overline{\lambda}<0$. We now select $\gamma_1>0$ so that

$$h(x)w_{\gamma_1}^{q-2}\le\overline{\lambda}-\widetilde{\lambda}\quad \text{on } \Omega.$$

We use a notation w_{γ_1} for a solution of (11.2) with $\mu=\widetilde{\lambda}$. Then for $\overline{\lambda}\le\lambda$ and $\gamma_1\le\gamma$ we have

$$\begin{aligned}&-\Delta w_{\gamma_1}+\lambda w_{\gamma_1}+\gamma Q w_{\gamma_1}^{p-1}-hw_{\gamma_1}^{q-1}\\ &\qquad\ge-\Delta w_{\gamma_1}+\overline{\lambda}w_{\gamma_1}+\gamma_1 Q w_{\gamma_1}^{p-1}-hw_{\gamma_1}^{q-1}\ge-\Delta w_{\gamma_1}+\widetilde{\lambda}w_{\gamma_1}+\gamma_1 Q w_{\gamma_1}^{p-1}=0\end{aligned}$$

in Ω. Thus w_{γ_1} is a supersolution of problem (11.1) with $\gamma\ge\gamma_1$ and $\lambda\ge\overline{\lambda}$. To construct a subsolution we employ a bifurcation argument from the trivial solution at $\lambda=0$ (see [30]). We set $X=C^{2,\beta}(\Omega)\cap C^1(\overline{\Omega})$ and $Y=C^{0,\beta}(\Omega)$, $0<\beta<1$. We define a map $\mathcal{F}:X\times\mathbb{R}\to Y$ by

$$\mathcal{F}(u,\lambda)=-\Delta u+\lambda u-h(x)u^{q-1}+\gamma Q(x)u^{p-1}.$$

We have $\mathcal{F}(0,\lambda)=0$ for every $\lambda\in\mathbb{R}$, $\mathcal{F}_u(0,0)v=-\Delta v$, $\mathcal{N}(\mathcal{F}_u(0,0))=\operatorname{span}\{1\}$. Since $\mathcal{R}(\mathcal{F}_u(0,0))=\{f\in Y:\int_\Omega f\,dx=0\}$, we see that $\mathcal{F}_{\lambda u}(0,0)1\notin\mathcal{R}(\mathcal{F}_u(0,0))$. Obviously

$$\dim\mathcal{N}(\mathcal{F}_u(0,0))=\dim Y/\mathcal{R}(\mathcal{F}_u(0,0))=1.$$

By the Crandall–Rabinowitz bifurcation theorem [30], $(0,0)$ is a bifurcation point for $\mathcal{F}$. Therefore we obtain a decomposition $X=\operatorname{span}\{1\}\oplus Z$, a neighbourhood U of $(0,0)$ in $X\times\mathbb{R}$, and continuous functions $\phi:(-a,a)\to\mathbb{R}$, $\psi:(-a,a)\to Z$, with $\phi(0)=0$, $\psi(0)=0$, such that

$$\mathcal{F}^{-1}(0,0)\cap U=\{(\alpha\cdot 1+\alpha\psi(\alpha),\phi(\alpha)):\alpha\in(-a,a)\}\cup\{(0,\mu):(0,\mu)\in U\}.$$

The curve $u_\alpha = \alpha(1+\psi(\alpha))$ represents solutions of (11.1) with $\lambda = \phi(\alpha)$. Since $\psi(\alpha) \to 0$ as $\alpha \to 0$ uniformly on Ω, we may assume that $u_\alpha > 0$ on Ω for $\alpha > 0$ small enough. Testing equation (11.1) with the constant function 1 we obtain

$$\phi(\alpha) \int_\Omega u_\alpha \, dx = \int_\Omega h u_\alpha^{q-1} \, dx - \gamma \int_\Omega Q u_\alpha^{p-1} \, dx = \alpha^{q-1} \int_\Omega h \, dx + o(\alpha^{q-1}).$$

This in turn implies that

$$\frac{\phi(\alpha)|\Omega|}{\alpha^{q-2}} = \int_\Omega h(x) \, dx + o(1) > 0$$

for $\alpha > 0$ small. Hence

$$\lim_{\alpha \to 0} \frac{\phi(\alpha)}{\alpha^{q-2}} = \frac{1}{|\Omega|} \int_\Omega h(x) \, dx. \tag{11.3}$$

This relation implies that $\phi(\alpha) > 0$ for $\alpha > 0$ small. We now observe that for $\gamma \geq \gamma_1$ and $\lambda \in (-\lambda_1, 0)$ we have

$$\begin{aligned}
\frac{J_{\lambda,\gamma}(u_\alpha)}{\alpha^q} &\leq \frac{J_{0,\gamma}(u_\alpha)}{\alpha^q} \\
&= \frac{1}{\alpha^q}\left(-\frac{1}{2} \int_\Omega \phi(\alpha) u_\alpha^2 \, dx + \left(\frac{1}{2} - \frac{1}{q}\right) \int_\Omega h u_\alpha^q \, dx + \left(-\frac{1}{2} + \frac{1}{p}\right)\gamma \int_\Omega Q u_\alpha^p \, dx \right) \\
&= -\frac{\phi(\alpha)|\Omega|}{2\alpha^{q-2}} + \left(\frac{1}{2} - \frac{1}{q}\right) \int_\Omega h \, dx + o(1).
\end{aligned}$$

This combined with (11.3) gives

$$\lim_{\alpha \to 0} \frac{J_{\mu,\gamma}(u_\alpha)}{\alpha^q} \leq -\frac{1}{q} \int_\Omega h \, dx < 0.$$

We fix $\gamma > \gamma_1$ and taking $\alpha_\circ > 0$ sufficiently small we get $u_{\alpha_\circ} \leq w_{\gamma_1}$ on Ω and $J_{\lambda,\gamma}(u_{\alpha_\circ}) \leq J_{0,\gamma}(u_{\alpha_\circ}) < 0$ for every $\lambda \in (-\lambda_1, 0)$. We now choose $0 < \varepsilon_\circ < \phi(\alpha_\circ)$ so small that

$$J_{\lambda,\gamma}(u_{\alpha_\circ}) \leq J_{0,\gamma}(u_{\alpha_\circ}) + \frac{\varepsilon_\circ}{2} \|u_{\alpha_\circ}\|_2^2 = J_{\varepsilon_\circ,\gamma}(u_{\alpha_\circ}) < 0$$

for every $\overline{\lambda} < \lambda \leq \varepsilon_\circ$. Since for $\lambda \leq \varepsilon_\circ$ we have

$$-\Delta u_{\alpha_\circ} + \lambda u_{\alpha_\circ} - h u_{\alpha_\circ}^{q-1} + \gamma Q u_{\alpha_\circ}^{p-1} \leq -\Delta u_{\alpha_\circ} + \phi(\alpha_\circ) u_{\alpha_\circ} - h u_{\alpha_\circ}^{q-1} + \gamma Q u_{\alpha_\circ}^{p-1} = 0$$

in Ω, we see that $u_{\alpha_\circ}$ is a subsolution of (11.1) for $\lambda \leq \varepsilon_\circ$ and $\gamma \geq \gamma_1$. A solution u_λ of (11.1) for every $\lambda \in [\overline{\lambda}, \varepsilon_\circ]$ is obtained through the minimization

$$J_{\lambda,\gamma}(u_\lambda) = \inf\{J_{\lambda,\gamma}(w) : w \in H^1(\Omega),\ w_{\gamma_1} \leq w \leq u_{\alpha_\circ}\} \leq J_{\lambda,\gamma}(u_{\alpha_\circ}) < 0. \ \blacksquare$$

Inspection of the proof of Theorem 11.1 shows that we may relax the hypothesis $h > 0$ on Ω assuming that $\int_\Omega h \, dx > 0$. Also, assuming that $\int_\Omega h \, dx < 0$ we can obtain a solution bifurcating to the left at 0.

The following definition is suggested by Theorem 11.1:

$$\overline{\gamma}(\lambda) = \inf\{\gamma > 0 : \text{problem (11.1) has a solution } u \in E_Q \text{ satisfying } I_{\lambda,\gamma}(u) < 0\}.$$

If h and Q satisfy the assumptions of Theorem 9.3 then for every $\lambda \leq \lambda_*$, we have $\overline{\gamma}(\lambda) = 0$. Here λ_* is the constant determined by Theorem 9.3. This is no longer true if h

and Q vanish on some subsets of Ω. In Proposition 11.2 we show that if $\operatorname{supp} h \cap \operatorname{supp} Q = \emptyset$, then $\overline{\gamma} > 0$.

PROPOSITION 11.2. *Let $q < p = 2^*$. Suppose that h and Q satisfy the assumptions of Theorem* 11.1 *and that* $\operatorname{supp} h \cap \operatorname{supp} Q = \emptyset$. *Moreover, we assume that $h(x) > 0$ on some neighbourhood of $\partial\Omega$. Then for every $-\lambda_1 < \lambda \leq \varepsilon_\circ$ we have $\overline{\gamma}(\lambda) > 0$. (Here λ_1 and $\varepsilon_\circ$ are constants from Theorem* 11.1.)

Proof. Let $-\lambda_1 < \lambda \leq \varepsilon_\circ$. Arguing by contradiction we assume $\overline{\gamma}(\lambda) = 0$. Let $\gamma_n \to 0$ and $\{u_n\}$ be a corresponding sequence of solutions of (11.1) with $I_{\lambda,\gamma_n}(u_n) < 0$. Then

$$\left(\frac{1}{2} - \frac{1}{q}\right) \int_\Omega h|u_n|^q \, dx \leq \left(\frac{1}{2} - \frac{1}{p}\right) \gamma_n \int_\Omega Q|u_n|^p \, dx. \tag{11.4}$$

Let $\eta(x)$ be a smooth function such that $\eta(x) = 0$ on $\operatorname{supp} h$ and $\eta(x) = 1$ on $\operatorname{supp} Q$. Testing equation (11.1) with $u_n\eta^2$ we obtain

$$\int_\Omega \eta^2 |\nabla u_n|^2 \, dx + \lambda \int_\Omega u_n^2 \eta^2 \, dx + \gamma_n \int_\Omega Q|u_n|^p \, dx = -2 \int_\Omega u_n \nabla u_n \eta \nabla \eta \, dx.$$

By the Young inequality we get

$$\frac{1}{2} \int_\Omega |\nabla u_n|^2 \eta^2 \, dx + \gamma_n \int_\Omega Q|u_n|^p \, dx \leq C \int_\Omega u_n^2 \, dx$$

for some constant $C > 0$ independent of n. This implies that

$$\gamma_n \int_\Omega Q|u_n|^p \, dx \leq C \int_\Omega u_n^2 \, dx \tag{11.5}$$

and by (11.4)

$$\int_\Omega h|u_n|^q \, dx \leq C \int_\Omega u_n^2 \, dx. \tag{11.6}$$

Since $I_{\lambda,\gamma_n}(u_n) < 0$ we see that

$$\int_\Omega |\nabla u_n|^2 \, dx \leq C \int_\Omega u_n^2 \, dx. \tag{11.7}$$

We claim that $\{u_n\}$ is bounded in $L^2(\Omega)$. In the contrary case we may assume that $\int_\Omega u_n^2 \, dx \to \infty$ and set $v_n = u_n/\|u_n\|_2$. Then by (11.7), the sequence $\{v_n\}$ is bounded in $H^1(\Omega)$. So we may assume that $v_n \rightharpoonup v_\circ$ in $H^1(\Omega)$ and $v_n \to v_\circ$ in $L^2(\Omega)$ and $L^q(\Omega)$. It then follows from (11.6) that $\int_\Omega h|v_\circ|^q \, dx = 0$ and $v_\circ(x) = 0$ on $\operatorname{supp} h$. Testing (11.1) with $v_\circ$ we obtain

$$\int_\Omega \nabla v_n \nabla v_\circ \, dx + \lambda \int_\Omega v_n v_\circ \, dx + \gamma_n \|u_n\|_2^{p-2} \int_\Omega Q|v_n|^{p-2} v_n v_\circ \, dx = 0.$$

Letting $n \to \infty$ we get

$$\int_\Omega |\nabla v_\circ|^2 \, dx + \lambda \int_\Omega v_\circ^2 \, dx \leq 0.$$

Since $\lambda > -\lambda_1$ we get a contradiction. Since $\{u_n\}$ is bounded in $L^2(\Omega)$, we see that according to (11.7) $\{u_n\}$ is also bounded in $H^1(\Omega)$. Hence we may assume that $u_n \rightharpoonup u$ in

$H^1(\Omega)$, $u_n \to u$ in $L^r(\Omega)$ for every $2 \le r < 2^*$. Then by (11.4), $\lim_{n\to\infty} \int_\Omega h|u_n|^q\,dx = 0$. Thus

$$\int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx \le \lim_{n\to\infty} \int_\Omega |\nabla u_n|^2\,dx + \lambda \int_\Omega u^2\,dx + \lim_{n\to\infty} \gamma_n \int_\Omega Q|u_n|^p\,dx$$
$$= \lim_{n\to\infty} \int_\Omega h|u_n|^q\,dx = 0.$$

Consequently $u_n \to 0$ in $H^1(\Omega)$. On the other hand since u_n is a solution of (11.1) we have

$$\left(\frac{1}{2} - \frac{1}{q}\right) \int_\Omega (|\nabla u_n|^2 + \lambda u_n^2)\,dx \le \left(\frac{1}{2} - \frac{1}{p}\right)\gamma_n \int_\Omega Q|u_n|^2\,dx$$
$$\le \gamma_n C_s(\lambda)\left(\frac{1}{2} - \frac{1}{p}\right)\Big(\int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx\Big)^{p/2}.$$

Therefore there exists a constant $C > 0$, independent of n, such that

$$\int_\Omega (|\nabla u_n|^2 + \lambda u_n^2)\,dx \ge C\gamma_n^{-2/(p-2)},$$

which contradicts the fact that $u_n \to 0$ in $H^1(\Omega)$. ∎

Solutions of problem (11.1) from Theorem 11.1 have negative energy. In Section 12 we establish the existence of solutions with positive energy for small $\gamma > 0$. This will be accomplished through the mountain-pass theorem applied to the truncated variational functional.

12. Supercritical problem for (1.1)

In Sections 9, 10 and 11 we have considered problem (9.1) which has been obtained from (1.1) by replacing Q by $-Q$. This allowed us to replace 2^* by any $q < p < \infty$. A question arises whether in problem (1.1) we can directly replace 2^* by any $q < p < \infty$ and obtain some existence results. In this section we show that this is possible provided Q is replaced by μQ with μ being a positive parameter whose range will depend on λ.

Therefore we are led to consider the following problem:

$$\begin{cases} -\Delta u + \lambda u = \mu Q(x)|u|^{p-2}u + h(x)|u|^{q-2}u & \text{in } \Omega, \\ \partial u/\partial\nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{12.1}$$

where $\lambda > 0$ and $\mu > 0$ are parameters. We assume that the coefficients Q and h are positive, measurable and bounded on Ω. Moreover we assume that $2 < q < 2^* < p < \infty$. To obtain a solution of (12.1) we first consider a truncated problem. Let $K > 0$ be a constant and define

$$g(u) = \begin{cases} 0 & \text{for } u < 0, \\ u^{p-1} & \text{for } 0 \le u < K, \\ K^{p-q}u^{q-1} & \text{for } u \ge K, \end{cases}$$

and set $G(u) = \int_0^u g(s)\,ds$. It is easy to verify that

$$g(u) \le K^{p-q}u^{q-1} \quad \text{for every } u \ge 0,$$

$$G(u) \le \frac{1}{q}\, g(u)u \quad \text{and} \quad G(u) \le \frac{K^{p-q}}{q}\, u^q \quad \text{for } u \ge 0.$$

We commence by solving the truncated problem

$$\begin{cases} -\Delta u + \lambda u = \mu Q(x) g(u) + h(x)|u|^{q-2}u & \text{in } \Omega, \\ \partial u/\partial\nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{12.2}$$

The variational functional for (12.2) given by

$$J_{\lambda,K}(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx - \mu\int_\Omega Q(x)G(u)\,dx - \frac{1}{q}\int_\Omega h(x)|u|^q\,dx$$

is well defined on $H^1(\Omega)$. It is easy to verify that $J_{\lambda,K}$ has a mountain-pass structure. Since $2 < q < 2^*$, $J_{\lambda,K}$ satisfies the $(\mathrm{PS})_c$ condition for every $c \in \mathbb{R}$. Let $t_\circ > 0$ be a constant sufficiently large so that $J_{\lambda,K}(t_\circ) < 0$ and set

$$\Gamma_{\lambda,K} = \{\gamma \in C([0,1], H^1(\Omega)) : \gamma(0) = 0,\ \gamma(1) = t_\circ\}$$

and let

$$c_{\lambda,K} = \inf_{\gamma\in\Gamma_{\lambda,K}} \max_{t\in[0,1]} J_{\lambda,K}(\gamma(t)).$$

PROPOSITION 12.1. *For each $\lambda > 0$, $\mu > 0$ and $K > 0$ problem* (12.2) *admits a mountain-pass solution $u_{\lambda,\mu,K} > 0$.*

For brevity we set $u = u_{\lambda,\mu,K}$.

PROPOSITION 12.2. *For every $\lambda > 0$ there exist $\mu_\circ > 0$ and $K_\circ > 0$ such that for every $0 < \mu \le \mu_\circ$ and $K > K_\circ$ the truncated problem* (12.2) *has a solution $u > 0$ satisfying*

$$\|u\|_{L^\infty} \le K.$$

Proof. We follow a standard bootstrap argument (see for example [33, Section 8.6]). For every $L \ge K$ we define a function u_L by

$$u_L = \begin{cases} u & \text{for } u < L, \\ L & \text{for } u > L. \end{cases}$$

For a constant $\beta > 1$, to be determined later, we set $\phi = u_L^{2(\beta-1)}u$. Taking ϕ as a test function for (12.2) we obtain

$$\int_\Omega (u_L^{2(\beta-1)}|\nabla u|^2 + 2(\beta-1)u_L^{2\beta-3}u\nabla u\nabla u_L + \lambda u^2 u_L^{2(\beta-1)})\,dx = \mu\int_\Omega Q(x)g(u)uu_L^{2(\beta-1)}\,dx + \int_\Omega h(x)|u|^q u_L^{2(\beta-1)}\,dx. \tag{12.3}$$

We now note that

$$\int_\Omega u_L^{2\beta-3}u\nabla u\nabla u_L\,dx = \int_{\Omega\cap(|u|<L)} u^{2(\beta-1)}|\nabla u|^2\,dx \ge 0 \tag{12.4}$$

and

$$\int_\Omega Q(x)g(u)uu^{2(\beta-1)}\,dx \le K^{p-q}\int_\Omega Q(x)|u|^q u_L^{2(\beta-1)}\,dx. \tag{12.5}$$

Combining (12.3), (12.4) and (12.5) we obtain

$$\int_\Omega (u_L^{2(\beta-1)}|\nabla u|^2 + \lambda u^2 u_L^{2(\beta-1)})\,dx \le \mu K^{p-q}\int_\Omega Q(x)|u|^q u_L^{2(\beta-1)}\,dx + \int_\Omega h(x)|u|^q u_L^{2(\beta-1)}\,dx.$$

Let $M_1 = \sup_{x\in\Omega} Q(x) + \sup_{x\in\Omega} h(x)$ and $C_{\mu,K} = M_1(\mu K^{p-q}+1)$. We rewrite the previous estimate

$$\begin{aligned}(12.6)\qquad \int_\Omega (u_L^{2(\beta-1)}|\nabla u|^2 + \lambda u^2 u_L^{2(\beta-1)})\,dx &\le M_1(\mu K^{p-q}+1)\int_\Omega |u|^q u_L^{2(\beta-1)}\,dx\\ &= C_{\mu,K}\int_\Omega |u|^q u_L^{2(\beta-1)}\,dx.\end{aligned}$$

It is now convenient to introduce a function w_L defined by $w_L = uu_L^{\beta-1}$ and note that

$$\nabla w_L = \nabla u\cdot u_L^{\beta-1} + (\beta-1)uu^{\beta-2}\nabla u_L.$$

From this we deduce the following estimate:

$$\begin{aligned}(12.7)\qquad \int_\Omega |\nabla w_L|^2\,dx &\le 2\int_\Omega |\nabla u|^2 u_L^{2(\beta-1)}\,dx + 2(\beta-1)^2\int_\Omega u_L^{2(\beta-1)}|\nabla u_L|^2\,dx\\ &= 2(1+(\beta-1)^2)\int_\Omega |\nabla u|^2 u_L^{2(\beta-1)}\,dx \le 4\beta^2\int_\Omega |\nabla u|^2 u_L^{2(\beta-1)}\,dx.\end{aligned}$$

It then follows from (12.6) and (12.7) that

$$\int_\Omega |\nabla w_L|^2\,dx + \lambda\int_\Omega w_L^2\,dx \le 4\beta^2 C_{\mu,K}\int_\Omega |u|^q u_L^{2(\beta-1)}\,dx.$$

Using inequality (2.1) we deduce from this that

$$C_s^{-1}\min(1,\lambda)\Big(\int_\Omega |w_L|^{2^*}\,dx\Big)^{2/2^*} \le 4\beta^2 C_{\mu,K}\int_\Omega |u|^q u_L^{2(\beta-1)}\,dx.$$

Using the Hölder inequality we obtain

$$(12.8)\quad C_s^{-1}\min(1,\lambda)\Big(\int_\Omega |w_L|^{2^*}\,dx\Big)^{2/2^*} \le 4\beta^2 C_{\mu,K}\|u\|_{2^*}^{q-2}\Big(\int_\Omega w_L^{\frac{2\cdot 2^*}{2^*-q+2}}\,dx\Big)^{(2^*-q+2)/2^*}.$$

It remains to estimate $\|u\|_{2^*}$. We accomplish this by estimating the mountain-pass level

$$\begin{aligned}(12.9)\qquad c_{\lambda,K} &\le \max_{t\ge 0} J_{\lambda,K}(t) \le \max_{t\ge 0}\Big(\frac{\lambda t^2|\Omega|}{2} - \frac{t^q}{q}\int_\Omega h\,dx\Big)\\ &= \frac{(q-2)\lambda|\Omega|}{2q}\Big(\frac{\lambda|\Omega|}{\int_\Omega h\,dx}\Big)^{2/(q-2)}.\end{aligned}$$

On the other hand since u is a critical point of $J_{\lambda,K}$ at level $c_{\lambda,K}$ we get

$$\begin{aligned}c_{\lambda,K} = J_{\lambda,K}(u) &\ge \frac{1}{2}\int_\Omega (|\nabla u|^2+\lambda u^2)\,dx - \frac{\mu}{q}\int_\Omega Q(x)g(u)u\,dx - \frac{1}{q}\int_\Omega h(x)|u|^q\,dx\\ &= \Big(\frac{1}{2}-\frac{1}{q}\Big)\int_\Omega (|\nabla u|^2+\lambda u^2)\,dx\end{aligned}$$

$$\geq \frac{(q-2)\min(1,\lambda)}{2q}\int_\Omega(|\nabla u|^2+u^2)\,dx \geq \frac{(q-2)\min(1,\lambda)}{2qC_s}\Big(\int_\Omega |u|^{2^*}\,dx\Big)^{2/2^*}.$$

Combining the above estimate with (12.9), we therefore have

$$\Big(\int_\Omega |u|^{2^*}\,dx\Big)^{2/2^*} \leq \frac{C_s}{\min(1,\lambda)}\frac{(\lambda|\Omega|)^{q/(q-2)}}{(\int_\Omega h(x)\,dx)^{2/(q-2)}}. \tag{12.10}$$

Using (12.10) and (12.8) we can conclude that

$$\Big(\int_\Omega |w_L|^{2^*}\,dx\Big)^{2/2^*} \leq \frac{4\beta^2 C_{\mu,K}C_s^{q/2}(\lambda|\Omega|)^{q/2}}{\min(1,\lambda)^{q/2}\int_\Omega h\,dx}\Big(\int_\Omega |w_L|^{2\cdot 2^*/(2^*-q+2)}\,dx\Big)^{(2^*-q+2)/2^*}. \tag{12.11}$$

Let $\alpha = 2\cdot 2^*/(2^*-q+2)$. One easily verifies that $\alpha < 2^*$. We now set

$$M_2 = \frac{2C_{\mu,K}^{1/2}C_s^{q/4}(\lambda|\Omega|)^{q/4}}{\min(1,\lambda)^{q/4}(\int_\Omega h\,dx)^{1/2}}.$$

If $\int_\Omega |u|^{\beta\alpha}\,dx < \infty$, letting $L \to \infty$, we conclude from (12.11) that

$$\Big(\int_\Omega |u|^{\beta 2^*}\,dx\Big)^{1/(\beta 2^*)} \leq M_2^{1/\beta}\beta^{1/\beta}\Big(\int_\Omega |u|^{\beta\alpha}\,dx\Big)^{1/(\beta\alpha)}. \tag{12.12}$$

This inequality can now be iterated to yield the boundedness of u. First we choose $\beta = \beta_1$ in (12.12) so that $\beta_1\alpha = 2^*$. For this choice of β we have

$$\Big(\int_\Omega |u|^{2^*}\,dx\Big)^{1/(\beta_1 2^*)} \leq M_2^{1/\beta_1}\beta_1^{1/\beta_1}\Big(\int_\Omega |u|^{2^*}\,dx\Big)^{1/2^*}. \tag{12.13}$$

In the next step we choose β_2 so that $\beta_2\alpha = \beta_1 2^*$, that is, $\beta_2 = (2^*/\alpha)^2$. It then follows from (12.12) and (12.13) that

$$\begin{aligned}\Big(\int_\Omega |u|^{\beta_2 2^*}\,dx\Big)^{1/(\beta_2 2^*)} &\leq M_2^{1/\beta_2}\beta_2^{1/\beta_2}\Big(\int_\Omega |u|^{\beta_1 2^*}\,dx\Big)^{1/(\beta_1 2^*)}\\ &\leq M_2^{1/\beta_1+1/\beta_2}\beta_1^{1/\beta_1}\beta_2^{1/\beta_2}\Big(\int_\Omega |u|^{2^*}\,dx\Big)^{1/2^*}.\end{aligned}$$

In the kth step we obtain the following estimate

$$\Big(\int_\Omega |u|^{\beta_k 2^*}\,dx\Big)^{1/(\beta_k 2^*)} \leq M_2^{1/\beta_1+1/\beta_2+\ldots+1/\beta_k}\beta_1^{1/\beta_1}\beta_2^{1/\beta_2}\ldots\beta_k^{1/\beta_k}\Big(\int_\Omega |u|^{2^*}\,dx\Big)^{1/2^*} \tag{12.14}$$

with $\beta_k = (2^*/\alpha)^k$. Since

$$\sum_{k=1}^\infty \frac{1}{\beta_k} = \sum_{k=1}^\infty\Big(\frac{\alpha}{2^*}\Big)^k = \sigma_1 > 0$$

and

$$\lim_{k\to\infty}\beta_1^{1/\beta_1}\ldots\beta_k^{1/\beta_k} = \lim_{k\to\infty}\Big(\frac{2^*}{\alpha}\Big)^{\alpha/2^*+2(\alpha/2^*)^2+\ldots+k(\alpha/2^*)^k} = \beta_1^{\sigma_2} > 0,$$

letting $k \to \infty$ in (12.14) and using (12.10) we get

$$\|u\|_{L^\infty(\Omega)} \le \frac{M_2^{\sigma_1}\beta_1^{\sigma_2}C_s^{1/2}}{\min(1,\lambda)^{1/2}} \frac{(\lambda|\Omega|)^{q/(2(q-2))}}{(\int_\Omega h\,dx)^{1/(q-2)}}. \tag{12.15}$$ ■

Estimate (12.15) will be used to show that problem (12.1) has a solution.

THEOREM 12.3. *For every $\lambda > 0$ there exist $\mu_\circ = \mu_\circ(\lambda) > 0$ and $K_\circ = K_\circ(\lambda) > 0$ such that for every $0 < \mu \le \mu_\circ$ and $K \ge K_\circ$ problem* (12.1) *has a solution $u \in H^1(\Omega)\cap L^\infty(\Omega)$ such that*

$$\|u\|_{L^\infty(\Omega)} \le K.$$

Proof. We use (12.15). It is sufficient to choose K so that

$$\frac{M_2^{\sigma_1}\beta_1^{\sigma_2}C_s^{1/2}(\lambda|\Omega|)^{q/(2(q-2))}}{\min(1,\lambda)^{1/2}(\int_\Omega h\,dx)^{1/(q-2)}} \le K. \tag{12.16}$$

This is equivalent to the inequality

$$(\mu K^{p-2}+1)^{\sigma_1/2} \le KM_3$$

with

$$M_3 = \frac{\min(1,\lambda)^{1/2+(q/\sigma_1)/4}(\int_\Omega h\,dx)^{\sigma_1/2+1/(q-2)}}{2^{\sigma_1}C_s^{\sigma_1 q/4+1/2}(\lambda|\Omega|)^{q/(2(q-2))+q\sigma_1/4}}.$$

Given $\lambda > 0$ we choose $K_\circ > M_3$. Hence for $K \ge K_\circ$ we have $KM_3 > 1$. Consequently, we can choose $\mu_\circ > 0$ so that for $0 < \mu \le \mu_\circ$ inequality (12.16) is satisfied. This completes the proof of Theorem 12.3. ■

We now turn our attention to problem (11.1). First we consider the truncated problem

$$\begin{cases} \Delta u + \lambda u = h(x)|u|^{q-2}u - \gamma Q(x)g(u) & \text{on } \Omega,\\ \partial u/\partial\nu = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{12.17}$$

where g is the truncation of the homogeneous term $|u|^{p-2}u$ defined at the beginning of this section. In the sequel we assume that assumption (A) holds. However, no smoothness of the coefficients Q and h is required here. It is sufficient to assume that Q and h are in $L^\infty(Q)$, For every $K > 0$ appearing in the definition of g, we define a truncated functional

$$J_{\lambda,\gamma,K}(u) = \frac12\int_\Omega(|\nabla u|^2+\lambda u^2)\,dx + \gamma\int_\Omega Q(x)G(u)\,dx - \frac1q\int_\Omega h(x)|u|^q\,dx.$$

PROPOSITION 12.4. *For every $K > 0$ and $\gamma > 0$ problem* (12.17) *has a positive solution.*

Proof. We first show that the functional $J_{\lambda,\gamma,K}$ has a mountain-pass structure. With the aid of the Sobolev inequality we obtain

$$\begin{aligned} J_{\lambda,\gamma,K}(u) &\ge \frac12\min(1,\lambda)\|u\|^2 - \frac{K^{p-q}\|Q\|_\infty}{q}\int_\Omega |u|^q\,dx - \frac{\|h\|_\infty}{q}\int_\Omega |u|^q\,dx\\ &\ge \|u\|^2\biggl(\frac{\min(1,\lambda)}{2} - C_1\|u\|^{q-2}\biggr) \end{aligned}$$

for some constant $C_1 > 0$. From this we deduce that there exist constants $\varrho > 0$ and $\alpha > 0$ such that

$$J_{\lambda,\gamma,K}(u) \ge \alpha \ \text{ for } \|u\| = \varrho.$$

We now fix a function $\phi \in H^1(\Omega)$ with $\operatorname{supp}\phi \subset \Omega_\circ$ and $\phi \not\equiv 0$. (We recall that $Q(x) = 0$ on $\Omega_\circ$.) Therefore,

$$J_{\lambda,\gamma,K}(t\phi) = \frac{t^2}{2}\int_\Omega (|\nabla\phi|^2 + \lambda\phi^2)\,dx - \frac{|t|^q}{q}\int_\Omega h(x)|\phi|^q\,dx.$$

We choose a constant $t_\circ > 0$ such that $J_{\lambda,\gamma,K}(t_\circ\phi) < 0$ and $\|t_\circ\phi\| > \varrho$. Let

$$\Gamma_{\lambda,\gamma,K} = \{\xi \in C^1([0,1], H^1(\Omega)) : \xi(0) = 0,\, \xi(1) = t_\circ\phi\}$$

and set

$$c_{\lambda,\gamma,K} = \inf_{\xi\in\Gamma_{\lambda,\gamma,K}} \max_{t\in[0,1]} J_{\lambda,\gamma,K}(\xi(t)).$$

We now show that $J_{\lambda,\gamma,K}$ satisfies the $(\mathrm{PS})_c$ condition for every $c \in \mathbb{R}$. First we observe that $G(s) = s^p/p$ for $0 < s \le K$ and $G(s) = K^p/p + K^{p-q}s^q/q - K^p/q$ for $s > K$. Let $\{u_m\} \subset H^1(\Omega)$ be a $(\mathrm{PS})_c$-sequence for the functional $J_{\lambda,\gamma,K}$. Then

$$\begin{aligned}
&J_{\lambda,\gamma,K}(u_m) - \frac{1}{q}\langle J'_{\lambda,\gamma,K}(u_m), u_m\rangle \\
&\quad= \left(\frac{1}{2} - \frac{1}{q}\right)\|u_m\|_\lambda^2 + \gamma\int_\Omega Q(x)G(u_m)\,dx - \frac{\gamma}{q}\int_\Omega Q(x)g(u_m)u_m\,dx \\
&\quad= \left(\frac{1}{2} - \frac{1}{q}\right)\|u_m\|_\lambda^2 + \gamma\int_{\Omega\cap(0\le u_m\le K)} Q(x)G(u_m)\,dx - \frac{\gamma}{q}\int_{\Omega\cap(0\le u_m\le K)} Q(x)g(u_m)u_m\,dx \\
&\qquad+ \gamma\int_{\Omega\cap(u_m\ge K)} Q(x)G(u_m)\,dx - \frac{\gamma}{q}\int_{\Omega\cap(u_m\ge K)} Q(x)g(u_m)u_m\,dx \\
&\quad= \left(\frac{1}{2} - \frac{1}{q}\right)\|u_m\|_\lambda^2 + \frac{\gamma}{p}\int_{\Omega\cap(0\le u_m\le K)} Q(x)u_m^p\,dx - \frac{\gamma}{q}\int_{\Omega\cap(0\le u_m\le K)} \gamma u_m^p\,dx \\
&\qquad+ \gamma\int_{\Omega\cap(u_m\ge K)} Q(x)\left(\frac{K^p}{p} + \frac{K^{p-q}}{q}u_m^q - \frac{K^p}{q}\right)dx - \frac{\gamma}{q}\int_{\Omega\cap(u_m\ge K)} Q(x)K^{p-q}u_m\,dx \\
&\quad= \left(\frac{1}{2} - \frac{1}{q}\right)\|u_m\|_\lambda^2 + \frac{\gamma}{p}\int_{\Omega\cap(0\le u_m\le K)} Q(x)u_m^p\,dx - \frac{\gamma}{q}\int_{\Omega\cap(0\le u_m\le K)} Q(x)u_m^p\,dx \\
&\qquad+ \gamma K^p\left(\frac{1}{p} - \frac{1}{q}\right)\int_{\Omega\cap(u_m\ge K)} Q(x)\,dx.
\end{aligned}$$

It is now routine to deduce from this identity the boundedness of $\{u_m\}$ in $H^1(\Omega)$. Since problem (12.17) is subcritical the $(\mathrm{PS})_c$ condition readily follows. This obviously yields the existence of a solution u of (12.7). ∎

For future use we now estimate $\|u\|$. Since

$$c_{\lambda,\gamma,K} \le \max_{t\ge 0}\left(\frac{t^2\|\phi\|_\lambda^2}{2} - \frac{1}{t^q}\int_\Omega h(x)|\phi|^q\,dx\right)$$

we have

$$c_{\lambda,\gamma,K} \leq \|\phi\|_\lambda^2 \left(\frac{\|\phi\|_\lambda^2}{\int_\Omega h(x)|\phi|^q\,dx} \right)^{2/(q-2)}. \tag{12.18}$$

On the other hand we have

$$\begin{aligned}
c_{\lambda,\gamma,K} &= J_{\lambda,\gamma,K}(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx + \gamma \int_\Omega Q(x)G(u)\,dx - \frac{1}{q}\int_\Omega h(x)|u|^q\,dx \\
&= \left(\frac{1}{2} - \frac{1}{q}\right) \int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx + \gamma \int_\Omega Q(x)G(u)\,dx - \frac{\gamma}{q}\int_\Omega Q(x)g(u)u\,dx \\
&= \left(\frac{1}{2} - \frac{1}{q}\right) \int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx + \frac{\gamma}{p}\int_{\Omega\cap(u\leq K)} Q(x)u^p\,dx \\
&\quad - \frac{\gamma}{q}\int_{\Omega\cap(u\leq K)} Q(x)u^p\,dx + \gamma\left(\frac{K^p}{p} - \frac{K^p}{q}\right)\int_{\Omega\cap(u\geq K)} Q(x)\,dx \\
&\geq \frac{q-2}{2q}\|u\|_\lambda^2 - \frac{\gamma K^p}{q}\int_{\Omega\cap(u\geq K)} Q(x)\,dx - \frac{\gamma}{q}\int_{\Omega\cap(u\leq K)} Q(x)u^p\,dx.
\end{aligned}$$

This estimate combined with (12.18) gives

$$\|u\|_\lambda \leq \frac{2q}{q-2}\left[\frac{q-2}{2q}\,\|\phi\|_\lambda^2 \left(\frac{\|\phi\|_\lambda^2}{\int_\Omega h(x)|\phi|^q\,dx} \right)^{2/(q-2)} + \frac{2K^p\gamma}{q}\int_\Omega Q(x)\,dx\right].$$

We set

$$M_1 = \|\phi\|_\lambda^2 \left(\frac{\|\phi\|_\lambda^2}{\int_\Omega h(x)|\phi|^q\,dx} \right)^{2/(q-2)} + \frac{4K^p}{q-2}\int_\Omega Q(x)\,dx$$

and using (2.1) we deduce from the above estimate that

$$\left(\int_\Omega |u|^{2^*}\,dx\right)^{2/2^*} \leq \frac{M_1 C_s}{\min(1,\lambda)}. \tag{12.19}$$

Proposition 12.5. *A mountain-pass solution u of problem* (12.7) *is bounded.*

Proof. We follow the proof of Proposition 12.1. Using the same notations as in Proposition 12.1 we obtain the estimate of the form (12.8) with $C_{\mu,K} = \|h\|_\infty$, that is,

$$C_s^{-1}\min(1,\lambda)\left(\int_\Omega |w_L|^{2^*}\,dx\right)^{2/2^*} \leq 4\beta^2\|h\|_\infty\|u\|_{2^*}^{q-2}\left(\int_\Omega |w_L|^{2\cdot 2^*/(2^*-q+2)}\,dx\right)^{(2^*-q+2)/2^*}.$$

This estimate combined with (12.19) leads to the following estimate:

$$\left(\int_\Omega |w_L|^{2^*}\,dx\right)^{2/2^*} \leq \frac{4\beta^2\|h\|_\infty M_1^{(q-2)/2}C_s^{q/2}}{\min(1,\lambda)^{q/2}}\left(\int_\Omega |w_L|^{2\cdot 2^*/(2^*-q+2)}\,dx\right)^{(2^*-q+2)/2^*}.$$

Letting $L \to \infty$, we find as in Proposition 12.1 that

$$\Big(\int_\Omega u^{\beta 2^*}\,dx\Big)^{1/(\beta 2^*)} \le \beta^{1/\beta} M_2^{1/\beta}\Big(\int_\Omega u^{\beta\alpha}\,dx\Big)^{1/(\beta\alpha)},$$

where

$$M_2 = \frac{2M_1^{(q-4)/4} C_s^{q/4}\|h\|_\infty^{1/2}}{\min(1,\lambda)^{q/4}}.$$

By the iterating procedure with the aid of (12.19) we conclude that

$$\|u\|_\infty \le \frac{M_2^{\sigma_1}\beta_1^{\sigma_2} M_1^{1/2} C_s^{1/2}}{\min(1,\lambda)^{1/2}} \tag{12.20}$$

for some constants $\sigma_1 > 0$ and $\sigma_2 > 0$. ■

We now define a constant A by

$$A = 2^{\sigma_1} C_s^{(q\sigma_1+2)/4}\beta_1^{\sigma_2}\|h\|_\infty^{\sigma_1/2}.$$

It is convenient to write (12.20) in an explicit form

$$\|u\|_\infty \le \frac{A}{\min(1,\lambda)^{(q\sigma_1+2)/4}}\bigg[\|\phi\|_\lambda^2\bigg(\frac{\|\phi\|_\lambda^2}{\int_\Omega h(x)|\phi|^q\,dx}\bigg)^{2/(q-2)} + \frac{4K^p\gamma}{q-2}\int_\Omega Q(x)\,dx\bigg]^{(q\sigma_1+2)/4}. \tag{12.21}$$

THEOREM 12.6. *Suppose that* (A) *holds. Then for every $\lambda > 0$ there exists $\gamma_\circ > 0$ such that problem* (11.1) *for $0 < \gamma \le \gamma_\circ$ has a solution.*

Proof. It is sufficient to choose a constant $K > 0$ so that

$$\frac{A}{\min(1,\lambda)^{(q\sigma_1+2)/4}}\bigg[\|\phi\|_\lambda^2\bigg(\frac{\|\phi\|_\lambda^2}{\int_\Omega h(x)|\phi|^q\,dx}\bigg)^{2/(q-2)} + \frac{4K^p\gamma}{q-2}\int_\Omega Q(x)\,dx\bigg]^{(q\sigma_1+2)/4} \le K. \tag{12.22}$$

To accomplish this we first choose $K > 0$ so that

$$\frac{A}{\min(1,\lambda)^{(q\sigma_1+2)/4}}\bigg[\|\phi\|_\lambda^2\bigg(\frac{\|\phi\|_\lambda^2}{\int_\Omega h(x)|\phi|^q\,dx}\bigg)^{2/(q-2)}\bigg]^{(q\sigma_1+2)/4} < K.$$

Then we select $\gamma > 0$ small enough so that (12.22) holds. ■

13. Blow-up for semilinear parabolic equations

As an application of the optimal Sobolev inequalities we investigate the blow-up for the Neumann problem for the semilinear parabolic equation

$$
(13.1)\qquad \begin{cases} \partial u/\partial t - \Delta u + \lambda u = Q(x)|u|^{2^*-2}u & \text{for } (x,t) \in \Omega \times (0,T], \\ u(x,t) = 0 & \text{for } (x,t) \in \partial\Omega \times (0,T], \\ u(x,0) = u_\circ(x) & \text{for } x \in \Omega,\ u_\circ \geq 0 \text{ and } u_\circ \not\equiv 0, \end{cases}
$$

where $\lambda > 0$ is a parameter and $u_\circ \in H^1(\Omega)$. As in the previous sections we assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$.

By $u(x,t) = u(x,t,u_\circ)$ we denote a solution of problem (13.1) defined for $(x,t) \in \Omega\times(0,T_m)$, where $(0,T_m)$ is the maximal interval of existence of the solution u. A solution is understood in the weak sense, it belongs to $H^1(\Omega \times (0,T_m))$ and is continuous in t with respect to the norm in $L^2(\Omega)$ on $[0,T_m)$ ([13], [14] and [36]).

Let

$$
J_\lambda(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx - \frac{1}{2^*}\int_\Omega Q(x)|u|^{2^*}\,dx.
$$

Proposition 13.1. *If $J_\lambda(u_\circ) \leq 0$, then $u(x,t)$ blows up at a finite time, that is, $T_m < \infty$.*

Proof. We follow some ideas from the paper [49]. We set

$$
f(t) = \frac{1}{2}\int_0^t \|u(\cdot,s)\|_2^2\,ds.
$$

It is easy to check that

$$
(13.2)\qquad \int_0^t\int_\Omega u_t^2\,dxds + \frac{1}{2}\int_\Omega (|\nabla_x u(x,t)|^2 + \lambda u(x,t)^2)\,dx - \frac{1}{2^*}\int_\Omega Q(x)|u(\cdot,t)|^{2^*}\,dx = J_\lambda(u_\circ),
$$

$$
(13.3)\qquad f'(t) = \frac{1}{2}\|u_\circ\|_2^2 + \int_0^t\int_\Omega (-|\nabla_x u(x,s)|^2\,dx - \lambda u(x,s)^2 + Q(x)|u(x,s)|^{2^*})\,dx\,ds,
$$

$$
(13.4)\qquad f''(t) = -\int_\Omega (|\nabla_x u(x,t)|^2 + \lambda u(x,t)^2)\,dx + \int_\Omega Q(x)|u(x,t)|^{2^*}\,dx.
$$

We deduce from (13.2) and (13.4) that

$$
(13.5)\qquad f''(t) \geq \left(\frac{2^*}{2} - 1\right)\int_\Omega (|\nabla_x u(x,t)|^2 + \lambda u(x,t)^2)\,dx - 2^* J_\lambda(u_\circ) + 2^*\int_0^t\int_\Omega u_t(x,s)^2\,dxds.
$$

Suppose that $T_m = \infty$. Since $J_\lambda(u_\circ) \leq 0$, we see that

$$
(13.6)\qquad f''(t) \geq \left(\frac{2^*}{2} - 1\right)\int_\Omega (|\nabla_x u(x,t)|^2 + \lambda u(x,t)^2)\,dx - 2^* J_\lambda(u_\circ) > 0.
$$

Selecting $t_1 > 0$ we derive from (13.5) that

$$
f''(t) > 2^*\int_0^{t_1}\int_\Omega u_t^2\,dx\,ds
$$

for $t > t_1$. This inequality implies that

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} f'(t) = \infty.$$

On the other hand, from (13.5) and (13.6) we have

$$f''(t) \geq 2^* \int_0^t \int_\Omega u_t(x,s)^2 \, dxds.$$

Applying the Hölder inequality we get

$$\begin{aligned} f(t)f''(t) &\geq \frac{2^*}{2}\Big(\int_0^t \|u(\cdot,s)\|_2^2 \, ds\Big)\Big(\int_0^t \|u_s(\cdot,s)\|_2^2 \, ds\Big) \\ &\geq \frac{2^*}{2}\Big(\int_0^t \int_\Omega uu_s \, dxds\Big)^2 = \frac{2^*}{2}\bigg(\frac{1}{2}\int_\Omega u(x,t)^2 \, dx - \frac{1}{2}\int_\Omega u_\circ(x) \, dx\bigg)^2 \\ &= \frac{2^*}{2}(f'(t) - f(0))^2. \end{aligned}$$

Since $\lim_{t\to\infty} f'(t) = \infty$, there exist $t_2 > 0$ and $\alpha > 0$ such that

$$f(t)f''(t) \geq (1+\alpha)f'(t)^2$$

for $t \geq t_2$. Thus $f^{-\alpha}$ is concave on (t_2, ∞) and this contradicts the fact that $\lim_{t\to\infty} f^{-\alpha}(t) = 0$. ■

We now define

$$S_\lambda = \inf\Big\{\int_\Omega (|\nabla u|^2 + \lambda u^2) \, dx : \int_\Omega Q(x)u^{2^*} \, dx = 1, \, u \in H^1(\Omega)\Big\}.$$

It follows from the definition of S_λ that

$$S_\lambda\Big(\int_\Omega Q(x)|u|^{2^*} \, dx\Big)^{2/2^*} \leq \int_\Omega (|\nabla_x u|^2 + \lambda u^2) \, dx \tag{13.7}$$

for every $u \in H^1(\Omega)$.

PROPOSITION 13.2. *Suppose that*

(a) $$J_\lambda(u_\circ) < \frac{S_\lambda^{N/2}}{N} \quad \textit{and} \quad \int_\Omega Q(x)|u_\circ|^{2^*} \, dx < S_\lambda^{N/2}.$$

Then the solution u of (13.1) *exists for every $t > 0$.*

Proof. We write (13.2) as

$$\int_0^t \|u_s(\cdot,s)\|_2^2 \, ds + J_\lambda(u(\cdot,t)) = J_\lambda(u_\circ) < \frac{S_\lambda^{N/2}}{N}. \tag{13.8}$$

Suppose that

$$\int_\Omega Q(x)|u(x,t^*)|^{2^*} \, dx = S_\lambda^{N/2} \tag{13.9}$$

for some $0 < t^* < T_m$. We derive from (13.8) that

$$J_\lambda(u(\cdot, t^*)) < \frac{1}{N} \int_\Omega Q(x)|u(x,t^*)|^{2^*}\, dx$$

and consequently

$$\int_\Omega (|\nabla_x u(x,t^*)|^2 + \lambda u(x,t^*)^2)\, dx < \int_\Omega Q(x)|u(x,t^*)|^{2^*}\, dx.$$

Combining this with (13.7) we obtain

$$S_\lambda^{N/2} < \int_\Omega Q(x)|u(x,t^*)|^{2^*}\, dx,$$

which contradicts (13.9). Therefore for each $0 < t < T_m$ we have

$$\int_\Omega Q(x)|u(x,t)|^{2^*}\, dx < S_\lambda^{N/2}.$$

This in turn implies that

$$\int_\Omega (|\nabla_x u(x,t)|^2 + \lambda u(x,t)^2)\, dx > \int_\Omega Q(x)|u(x,t)|^{2^*}\, dx$$

for each $0 \le t < T_m$. Combining this with (13.8) we obtain

$$\int_0^t \|u_s(\cdot,s)\|_2^2\, ds + \frac{1}{N}\int_\Omega (|\nabla_x u(x,t)|^2 + \lambda u(x,t)^2)\, dx < J_\lambda(u_\circ) < \frac{1}{N} S_\lambda^{N/2}.$$

This obviously implies that $T_m = \infty$. ■

In general, it is difficult to estimate S_λ. However, under conditions guaranteeing the validity of optimal Sobolev inequalities, S_λ is constant for large λ. Therefore using Proposition 13.1 we can formulate the following theorem giving conditions for no blow-up.

Theorem 13.3. (i) *Let $N \ge 5$ and $Q_{\mathrm{M}} > 2^{2/(N-2)} Q_{\mathrm{m}}$. Suppose that $\int_\Omega Q(x)|u_\circ|^{2^*}\, dx < S^{N/2}/Q_{\mathrm{M}}^{(N-2)/2}$ and $J_\lambda(u_\circ) < S^{N/2}/(N Q_{\mathrm{M}}^{(N-2)/2})$. Then there exists a $\Lambda_1 > 0$ such that problem* (13.1) *for $\lambda \ge \Lambda_1$ has a solution for all $t \ge 0$.*

(ii) *Let $N \ge 5$ and suppose that* (S_1) *holds. If $J_\lambda(u_\circ) < S^{N/2}/(2N Q_{\mathrm{m}}^{(N-2)/2})$ and $\int_\Omega Q(x)|u_\circ|^{2^*}\, dx < S^{N/2}/(2N Q_{\mathrm{m}}^{(N-2)/2})$, then there exists a $\Lambda_2 > 0$ such that problem* (13.1) *for $\lambda \ge \Lambda_2$ has a solution for all $t \ge 0$.*

(iii) *Let $N \ge 5$ and suppose that* (S_2) *holds. If $J_\lambda(u_\circ) < S^{N/2}/(2N Q_{\mathrm{m}}^{(N-2)/2})$ and $\int_\Omega Q(x)|u_\circ|^{2^*} < S^{N/2}/(2N Q_{\mathrm{m}}^{(N-2)/2})$, then there exists a $\Lambda_3 > 0$ such that problem* (13.1) *for $\lambda \ge \Lambda_3$ has a solution for all $t \ge 0$.*

In Proposition 13.4 we examine the behaviour of the norm $\|u(\cdot,t)\|_\lambda$ of a solution of (13.1).

Proposition 13.4. *Let $N \ge 5$. Suppose that*

$$Q_{\mathrm{M}} > 2^{2/(N-2)} Q_{\mathrm{m}}, \qquad J_\lambda(u_\circ) < \frac{S^{N/2}}{N Q_{\mathrm{M}}^{(N-2)/2}}, \qquad \int_\Omega Q(x)|u_\circ|^{2^*}\, dx < \frac{S^{N/2}}{Q_{\mathrm{M}}^{(N-2)/2}}.$$

Then the global solution u of (13.1) *satisfies*

$$\int_\Omega (|\nabla_x u(x,t)|^2 + \lambda u(x,t)^2)\,dx = O(e^{-\alpha t})$$

for large λ, for every $t \geq 0$ and some constant $\alpha > 0$.

Proof. We set

$$H(u(\cdot,t)) = \int_\Omega (|\nabla_x u(x,t)|^2 + \lambda u(x,t)^2)\,dx - \int_\Omega Q(x)|u(x,t)|^{2^*}\,dx.$$

We assume that $\lambda \geq \Lambda_1$. It follows from the proof of Proposition 13.2 that $H(u(\cdot,t)) > 0$ for every $t \geq 0$. The Sobolev inequality (I) of Section 3 and the inequality

$$J_\lambda(u_\circ) > \frac{1}{N}\int_\Omega (|\nabla_x u(x,t)|^2 + \lambda u(x,t)^2)\,dx$$

yield the estimate

$$\int_\Omega Q(x)|u(x,t)|^{2^*}\,dx \leq \left(\frac{Q_{\mathrm{M}}^{(N-2)/N}}{S}\right)^{2^*/2} (NJ_\lambda(u_\circ))^{2^*/2-1} \int_\Omega (|\nabla_x u(x,t)|^2 + \lambda u(x,t)^2)\,dx.$$

According to our assumption

$$\delta = \left(\frac{Q_{\mathrm{M}}^{(N-2)/N}}{S}\right)^{2^*/2} (NJ_\lambda(u_\circ))^{2^*/2-1} < 1,$$

so setting $\gamma = 1 - \delta$, we can write the last estimate in the form

$$\int_\Omega Q(x)|u(x,t)|^{2^*}\,dx \leq (1-\gamma)\int_\Omega (|\nabla_x u(x,t)|^2 + \lambda u(x,t)^2)\,dx. \tag{13.10}$$

For a fixed $T > 0$ the integration over (t,T) of

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u(x,t)^2\,dx = -H(u(\cdot,t))$$

gives

$$\begin{aligned}\int_t^T H(u(\cdot,s))\,ds &= \frac{1}{2}\int_\Omega u(x,t)^2\,dx - \frac{1}{2}\int_\Omega u(x,T)^2\,dx \\ &\leq \frac{1}{2\lambda}\int_\Omega (|\nabla_x u(x,t)|^2 + \lambda u(x,t)^2)\,dx.\end{aligned}$$

On the other hand we have

$$\begin{aligned} J_\lambda(u(\cdot,t)) &= \frac{1}{2}\int_\Omega (|\nabla_x u(x,t)|^2 + \lambda u(x,t)^2)\,dx - \frac{1}{2^*}\int_\Omega Q(x)|u(x,t)|^{2^*}\,dx \\ &= \frac{1}{2}\int_\Omega (|\nabla_x u(x,t)|^2 + \lambda u(x,t)^2)\,dx \\ &\quad + \frac{1}{2^*}\left[H(u(\cdot,t)) - \int_\Omega (|\nabla_x u(x,t)|^2 + \lambda u(x,t)^2)\,dx\right] \end{aligned} \tag{13.11}$$

$$= \frac{1}{N} \int_\Omega (|\nabla_x u(x,t)|^2 + \lambda u(x,t)^2)\,dx + \frac{1}{2^*} H(u(\cdot,t))$$
$$\geq \frac{1}{N} \int_\Omega (|\nabla_x u(x,t)|^2 + \lambda u(x,t)^2)\,dx.$$

Combining the last two inequalities we get

$$\int_t^T H(u(\cdot,s))\,ds \leq \frac{N}{2\lambda} J_\lambda(u(\cdot,t)). \tag{13.12}$$

We now rewrite inequality (13.10) as

$$\gamma \int_\Omega (|\nabla_x u(x,t)|^2 + \lambda u(x,t)^2)\,dx \leq H(u(\cdot,t)). \tag{13.13}$$

Inequality (13.13) and the equality part of (13.11) imply that

$$J_\lambda(u(\cdot,t)) \leq \left(\frac{1}{N\gamma} + \frac{1}{2^*}\right) H(u(\cdot,t)).$$

Combining the last estimate with (13.12) we get

$$\int_t^T J_\lambda(u(\cdot,s))\,ds \leq \left(\frac{1}{2\lambda\gamma} + \frac{N}{2\cdot 2^*\lambda}\right) J_\lambda(u(\cdot,t)).$$

We choose a constant $T_\circ > 1/(2\lambda\gamma) + N/(2\cdot 2^*\lambda)$ and write the last inequality in the form

$$\int_t^\infty J_\lambda(u(\cdot,s))\,ds \leq T_\circ J_\lambda(u(\cdot,t)) \tag{13.14}$$

for every $t \geq T_\circ$. By standard calculations we deduce from (13.14) the inequality

$$\int_t^\infty J_\lambda(u(\cdot,s))\,ds \leq T_\circ J_\lambda(u(\cdot,T_\circ))e^{1-t/T_\circ} \tag{13.15}$$

for every $t \geq T_\circ$. Since

$$\int_t^\infty J_\lambda(u(\cdot,s))\,ds \geq \int_t^{T_\circ+t} J_\lambda(u(\cdot,s))\,ds \geq T_\circ J_\lambda(u(\cdot,T_\circ+t))$$

we deduce from (13.15) that

$$J_\lambda(u(\cdot,T_\circ+t)) \leq J_\lambda(u(\cdot,T_\circ))e^{1-t/T_\circ}.$$

The assertion of Proposition follows from (13.11). ■

A similar asymptotic estimate of u can be obtained with the aid of the optimal Sobolev inequalities (II) and (III).

PROPOSITION 13.5. *Let $N \geq 5$ and $Q_{\mathrm{M}} > 2^{2/(N-2)}$. Suppose that*

$$0 < J_\lambda(u_\circ) < \frac{S^{N/2}}{NQ_{\mathrm{M}}^{(N-2)/2}} \quad \text{and} \quad \int_\Omega Q(x)|u_\circ|^{2^*}\,dx \geq \frac{S^{N/2}}{Q_{\mathrm{M}}^{(N-2)/2}}$$

for $\lambda \geq \Lambda_1$. Then a solution of (13.1) *blows up at a finite time.*

Proof. We commence by showing that there is no function $u_\circ$ satisfying

$$J_\lambda(u_\circ) < \frac{S^{N/2}}{NQ_{\mathrm{M}}^{(N-2)/2}} \quad \text{and} \quad \int_\Omega Q(x)|u_\circ|^{2^*}\,dx = \frac{S^{N/2}}{Q_{\mathrm{M}}^{(N-2)/2}}. \tag{13.16}$$

Indeed, assuming that there is a function $u_\circ$ satisfying (13.16) we get

$$\frac{1}{2}\int_\Omega (|\nabla u_\circ|^2 + \lambda u_\circ^2)\,dx - \frac{1}{2^*}\int_\Omega Q(x)|u_\circ|^{2^*}\,dx < \frac{1}{N}\int_\Omega Q(x)|u_\circ|^{2^*}\,dx.$$

Hence

$$\int_\Omega (|\nabla u_\circ|^2 + \lambda u_\circ^2)\,dx < \int_\Omega Q(x)|u_\circ|^{2^*}\,dx.$$

By (I) we have

$$\Big(\int_\Omega Q(x)|u_\circ|^{2^*}\,dx\Big)^{2/2^*} \frac{S}{Q_{\mathrm{M}}^{(N-2)/N}} < \int_\Omega Q(x)|u_\circ|^{2^*}\,dx$$

and consequently

$$\frac{S^{N/2}}{Q_{\mathrm{M}}^{(N-2)/2}} < \int_\Omega Q(x)|u_\circ|^{2^*}\,dx,$$

which is impossible. Therefore we only consider the case

$$\int_\Omega Q(x)|u_\circ|^{2^*}\,dx > \frac{S^{N/2}}{Q_{\mathrm{M}}^{(N-2)/2}}.$$

We then have

$$\frac{S^{N/2}}{Q_{\mathrm{M}}^{(N-2)/2}} < \int_\Omega (|\nabla_x u_\circ|^2 + \lambda u_\circ^2)\,dx < \int_\Omega Q(x)|u_\circ|^{2^*}\,dx. \tag{13.17}$$

If the second part of this inequality is not true, then

$$J_\lambda(u_\circ) \ge \frac{1}{N}\int_\Omega Q(x)|u_\circ|^{2^*}\,dx \ge \frac{S^{N/2}}{NQ_{\mathrm{M}}^{(N-2)/2}},$$

which is impossible. Similarly, if the first part of (13.17) does not hold, then we easily arrive at a contradiction with the aid of inequality (I). Obviously by continuity we have

$$\frac{S^{N/2}}{Q_{\mathrm{M}}^{(N-2)/2}} < \int_\Omega (|\nabla_x u(x,t)|^2 + \lambda u(x,t)^2)\,dx < \int_\Omega Q(x)|u(x,t)|^{2^*}\,dx \tag{13.18}$$

for small $t \ge 0$. We now show that (13.18) remains valid for $t \in [0, T_m)$. If for some $\bar t \in [0,T)$,

$$\int_\Omega (|\nabla_x u(x,\bar t)|^2 + \lambda u(x,\bar t)^2)\,dx = \int_\Omega Q(x)|u(x,\bar t)|^{2^*}\,dx,$$

then

$$\int_\Omega (|\nabla_x u(x,\bar t)|^2 + \lambda u(x,\bar t)^2)\,dx \ge \frac{S^{N/2}}{Q_{\mathrm{M}}^{(N-2)/2}}.$$

Since $J_\lambda(u\cdot,t)$ is decreasing in t we must have

$$J_\lambda(u(\cdot,\bar{t})) = \frac{1}{N}\int_\Omega (|\nabla_x u(x,\bar{t})|^2 + \lambda u(x,\bar{t})^2)\,dx < \frac{S^{N/2}}{NQ_{\mathrm{M}}^{(N-2)/2}},$$

which is impossible. Therefore the second part of inequality (13.18) is valid for $t \in [0, T_m)$. To proceed further we employ the method due to Ishii [37] (see also [49]). We define

$$X(t) = \frac{1}{2}\int_\Omega (|\nabla_x u(x,t)|^2 + \lambda u(x,t)^2)\,dx, \qquad Y(t) = \frac{1}{2^*}\int_\Omega Q(x)|u(x,t)|^{2^*}\,dx.$$

By inequality (I) we have

$$Y \le \frac{1}{2^*}\left(\frac{2Q_{\mathrm{M}}^{(N-2)/N}}{S}\right)^{2^*/2} X^{2^*/2}.$$

We now set

$$c = \frac{1}{2^*}\left(\frac{2Q_{\mathrm{M}}^{(N-2)/N}}{S}\right)^{2^*/2} \quad \text{and} \quad \alpha = \frac{2^*}{2} > 1.$$

Let $(X_\circ, Y_\circ)$ be a point where the curve $Y = cX^\alpha$ crosses the line $Y = \alpha^{-1}X$ in the half plane $X \ge 0$. It is easy to check that

$$\frac{d}{dX}(cX^\alpha) = 1 \quad \text{at } X = X_\circ$$

Thus the tangent line to the curve $Y = cX^\alpha$ at $(X_\circ, Y_\circ)$ is given by

$$Y = X - d \quad \text{with} \quad d = X_\circ - Y_\circ.$$

Easy calculations show that

$$d = \frac{\alpha-1}{\alpha}(c\alpha)^{-1/(\alpha-1)} = \max(X - Y; Y = cX^\alpha) = \frac{S^{N/2}}{NQ_{\mathrm{M}}^{(N-2)/2}}.$$

We now consider the set U defined by

$$U = \{(X,Y) \in \mathbb{R}^2 : X \ge 0,\ Y \le cX^\alpha,\ Y > X - d\},$$

which is the union of two connected components

$$W = \{(X,Y) \in U : Y \le \alpha^{-1}X\}, \qquad V = \{(X,Y) \in U :\ Y > \alpha^{-1}X\}.$$

The component W is a bounded subset of $\mathbb{R}^2$ and $X - Y \ge 0$ for $(X,Y) \in W$. We also have

$$X > (c\alpha)^{-1/(\alpha-1)} = \frac{S^{N/2}}{Q_{\mathrm{M}}^{(N-2)/2}}$$

for $(X,Y) \in V$. This shows that the first part of inequality (13.18) is valid for every $t \in (0, T_m)$. We now consider the line $Y = X - r$ with $0 \le r < d$. Let us denote by (X_-, Y_-) and (X_+, Y_+) the intersection points of $Y = X - r$ with the curve $Y = cX^\alpha$ such that $X_- < X_\circ < X_+$. Let $Y = \beta_- X$ and $Y = \beta_+ X$ be the lines passing through the points (X_-, Y_-) and (X_+, Y_+), respectively. Since $X_-(r)$ is strictly increasing and $X_+(r)$ is strictly decreasing in $r \in (0,d)$, we see that $\beta_-(r) = cX_-^{\alpha-1}(r)$ is strictly increasing and $\beta_+(r) = cX_+(r)^{\alpha-1}$ is strictly decreasing in $r \in (0,d)$. Thus we have

$$Y \le \beta_-(r)X \quad \text{for } (X,Y) \in W \text{ with } X - Y = r,$$
$$Y \ge \beta_+(r)X \quad \text{for } (X,Y) \in V \text{ with } X - Y = r$$

and moreover

$$\beta_-(r) < \alpha^{-1} < \beta_+(r) \quad \text{for } 0 \le r < d.$$

Since $d = S^{N/2}/(NQ_{\mathrm{M}}^{(N-2)/2})$, we have

$$\beta_-\left(\frac{S^{N/2}}{NQ_{\mathrm{M}}^{(N-2)/2}}\right) = \beta_+\left(\frac{S^{N/2}}{NQ_{\mathrm{M}}^{(N-2)/2}}\right) = \frac{1}{\alpha} = \frac{2}{2^*}.$$

Taking $r = J_\lambda(u_\circ)$, we get

$$\frac{S^{N/2}}{NQ_{\mathrm{M}}^{(N-2)/2}} = d > r = J_\lambda(u_\circ) \ge J_\lambda(u(\cdot,t)) = X(t) - Y(t).$$

Hence

$$Y(t) \ge \beta_+(J_\lambda(u(\cdot,t)))X(t) \ge \beta_+(J_\lambda(u_\circ))X(t)$$

and

$$\beta_+(J_\lambda(u_\circ)) > \beta_+\left(\frac{S^{N/2}}{NQ_{\mathrm{M}}^{(N-2)/2}}\right) = \frac{2}{2^*}.$$

Thus we can find an $\eta > 0$ such that

$$Y(t) \ge \frac{2}{2^*}(1+\eta)X(t)$$

for all $t \in [0, T_m)$ or equivalently

$$(1+\eta)\int_\Omega (|\nabla u(x,t)|^2 + \lambda u(x,t)^2)\,dx \le \int_\Omega Q(x)|u(x,t)|^{2^*}\,dx \tag{13.19}$$

for all $t \in [0, T_m)$. Inequality (13.19) is crucial to prove that the blow-up occurs at a finite time. Assume that $T_m = \infty$ and define

$$f(t) = \frac{1}{2}\int_o^t \|u(\cdot,s)\|_2^2\,ds.$$

We follow the argument from the proof of Proposition 13.1. From (13.4) we derive that

$$f''(t) \ge \eta \int_\Omega (|\nabla u(x,t)|^2 + \lambda u(x,t)^2)\,dx.$$

From this we deduce that $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} f'(t) = \infty$. We now observe that

$$\left(\frac{2^*}{2} - 1\right)\int_\Omega (|\nabla u(x,t)|^2 + \lambda u(x,t)^2)\,dx - 2^* J_\lambda(u_\circ) \ge 0.$$

It then follows from (13.5) that

$$f''(t) \ge 2^* \int_0^t \int_\Omega u_t^2\,dxds.$$

The remaining part of the proof is similar to that of Proposition 13.1. ■

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