Contents

Abstract

We modify the very well known theory of normed spaces $(E, \|\cdot\|)$ within functional analysis by considering a sequence $(\|\cdot\|_n : n \in \mathbb{N})$ of norms, where $\|\cdot\|_n$ is defined on the product space E^n for each $n \in \mathbb{N}$.

Our theory is analogous to, but distinct from, an existing theory of 'operator spaces'; it is designed to relate to general spaces L^p for $p \in [1, \infty]$, and in particular to L^1-spaces, rather than to L^2-spaces.

After recalling in Chapter 1 some results in functional analysis, especially in Banach space, Hilbert space, Banach algebra, and Banach lattice theory, that we shall use, we shall present in Chapter 2 our axiomatic definition of a 'multi-normed space' $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$, where $(E, \|\cdot\|)$ is a normed space. Several different, equivalent, characterizations of multi-normed spaces are given, some involving the theory of tensor products; key examples of multi-norms are the minimum, maximum, and (p, q)-multi-norms based on a given space. Multi-norms measure 'geometrical features' of normed spaces, in particular by considering their 'rate of growth'. There is a strong connection between multi-normed spaces and the theory of absolutely summing operators.

A substantial number of examples of multi-norms will be presented.

Following the pattern of standard presentations of the foundations of functional analysis, we consider generalizations to 'multi-topological linear spaces' through 'multi-null sequences', and to 'multi-bounded' linear operators, which are exactly the 'multi-continuous' operators. We define a new Banach space $\mathcal{M}(E, F)$ of multi-bounded operators, and show that it generalizes well-known spaces, especially in the theory of Banach lattices.

We conclude with a theory of 'orthogonal decompositions' of a normed space with respect to a multi-norm, and apply this to construct a 'multi-dual' space.

Applications of this theory will be presented elsewhere.

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1. Introduction

In this introductory chapter, we shall recall some background that we shall require, and establish our notation; many of the results are well known. We shall conclude the chapter with a summary, with some history of our project, and with some acknowledgements.

1.1. Basic notation. We begin by recalling some standard notation that will be fixed throughout this memoir.

1.1.1. Sets and sequences. We write $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Z}^+$ for the three sets $\{1, 2, \dots\}$ of natural numbers, $\{0, \pm 1, \pm 2, \dots\}$ of integers, and $\{0, 1, 2, \dots\}$ of non-negative integers, respectively. For each $n \in \mathbb{N}$, we denote by $\mathbb{N}_n$ and $\mathbb{Z}_n^+$ the sets $\{1, \dots, n\}$ and $\{0, 1, \dots, n\}$, respectively. Also, we denote by $\mathfrak{S}_n$ the group of permutations on n symbols; we write $\mathfrak{S}_{\mathbb{N}}$ for the group of all permutations of $\mathbb{N}$.

The real field is $\mathbb{R}$, and $\mathbb{R}^+ = [0, \infty)$; the *unit interval* $[0, 1]$ in $\mathbb{R}$ is denoted by $\mathbb{I}$. The complex field is $\mathbb{C}$; the *open unit disc* in $\mathbb{C}$ is always denoted by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and its closure is $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$, the *closed unit disc.* We write $[x]$ for the integer part of $x \in \mathbb{R}^+$.

For $i \in \mathbb{N}_n$, the i^{th} *coordinate functional* on $\mathbb{C}^n$ or $\mathbb{R}^n$ is denoted by Z_i, so that

$$Z_i : (z_1, \dots, z_n) \mapsto z_i, \qquad \mathbb{C}^n \to \mathbb{C}.$$

The *cardinality* of a set S is denoted by $|S|$, and the *symmetric difference* of two sets S and T is $S \bigtriangleup T$.

The space of all complex-valued sequences on $\mathbb{N}$ is $\mathbb{C}^{\mathbb{N}}$, and we often write (α_i) for $\alpha = (\alpha_i : i \in \mathbb{N}) \in \mathbb{C}^{\mathbb{N}}$. Let $\alpha, \beta \in \mathbb{C}^{\mathbb{N}}$. Then:

- $\alpha = O(\beta)$ if there is a constant K with $|\alpha_i| \leq K|\beta_i|$ $(i \in \mathbb{N})$;
- $\alpha = o(\beta)$ if, for each $\varepsilon > 0$, there exists $i_0 \in \mathbb{N}$ with $|\alpha_i| \leq \varepsilon|\beta_i|$ $(i \geq i_0)$;
- $\alpha \sim \beta$ if $\alpha = O(\beta)$ and $\beta = O(\alpha)$, in which case α and β are said to be *similar* sequences.

1.1.2. Inequalities. We shall use various inequalities; for an attractive discussion of many inequalities in related areas, see [31].

Take p with $1 < p < \infty$. Then the *conjugate index* to p is q, where

$$\frac{1}{p} + \frac{1}{q} = 1;$$

we also regard 1 and ∞ as being conjugates of each other; later we shall sometimes denote the conjugate of p by p'. We shall interpret $(\alpha_1^q + \cdots + \alpha_n^q)^{1/q}$, where $\alpha_1, \dots, \alpha_n \in \mathbb{R}^+$, as $\max\{\alpha_1, \dots, \alpha_n\}$ when $q = \infty$.

First, an easy form of *Hölder's inequality* gives the following. Let $p, q \in [1,\infty]$ be conjugate indices. Then, for each $n \in \mathbb{N}$ and each $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{C}$, we have

$$\sum_{j=1}^{n} |x_j y_j| \le \Big(\sum_{j=1}^{n} |x_j|^p\Big)^{1/p} \Big(\sum_{j=1}^{n} |y_j|^q\Big)^{1/q}. \tag{1.1}$$

Now take $a_1, \dots, a_n \in \mathbb{R}^+$ and r, s with $1 \le r \le s$. Then (in the case where $r < s$) we apply (1.1) with $x_j = a_j^r$ and $y_j = 1$ for $j \in \mathbb{N}_n$ and with $p = s/r$ and $q = s/(s-r)$ to see that

$$\frac{1}{n^{1/r}}(a_1^r + \cdots + a_n^r)^{1/r} \le \frac{1}{n^{1/s}}(a_1^s + \cdots + a_n^s)^{1/s}. \tag{1.2}$$

For $k \in \mathbb{N}$ with $k \ge 2$, set $\zeta = \exp(2\pi \mathrm{i}/k)$, so that $1 + \zeta^t + \cdots + \zeta^{t(k-1)} = 0$ for $\pm t \in \mathbb{N}_{k-1}$, and then take $\zeta_1, \dots, \zeta_k \in \mathbb{C}$ and set

$$z_i = \sum_{j=1}^{k} \zeta_j \zeta^{ij} \qquad (i \in \mathbb{N}_k).$$

LEMMA 1.1. *Let $k \in \mathbb{N}$, and let $q \in [1,2]$.*

(i) *Take $\zeta_1, \dots, \zeta_k \in \mathbb{C}$ with $\sum_{i=1}^k |\zeta_i|^2 = 1$. Then $\sum_{i=1}^k |z_i|^2 = k$ and*

$$\Big(\sum_{i=1}^{k} |z_i|^q\Big)^{1/q} \le k^{1/q}.$$

(ii) *Take $\zeta_1, \dots, \zeta_k \in \mathbb{T}$. Then $\sum_{i=1}^k |z_i|^2 = k^2$ and*

$$\Big(\sum_{i=1}^{k} |z_i|^q\Big)^{1/q} \le k^{1/2+1/q}.$$

Proof. For $r, s \in \mathbb{N}_k$ with $r \ne s$, the coefficient of $\zeta_r \overline{\zeta}_s$ in the expansion of $\sum_{i=1}^k z_i \overline{z}_i$ is $\sum_{i=1}^k \zeta^{it}$, where $t = r - s$, so that $|t| \in \mathbb{N}_{k-1}$. Hence this coefficient is 0. For $r \in \mathbb{N}_k$, the coefficient of $\zeta_r \overline{\zeta}_r$ in the expansion is k, so that $\sum_{i=1}^k |z_i|^2 = k \sum_{i=1}^k |\zeta_i|^2$, and this is k in case (i) and k^2 in case (ii), giving the equalities in the two results. The subsequent inequalities follow from (1.2). ∎

1.1.3. Linear spaces. Let E be a linear space over the real or complex field. In fact, we shall usually implicitly assume that E is taken over the complex field $\mathbb{C}$; small modifications usually give the same result for spaces over the real field $\mathbb{R}$, but at a few points it will be important to specify the underlying field. Note that a linear space E over $\mathbb{C}$ can be regarded as a linear space over $\mathbb{R}$ by restricting the scalars to $\mathbb{R}$; we obtain the *underlying real-linear space*.

A real-linear space V has a standard *complexification* of the form $E = V \oplus \mathrm{i}V$, where $(\alpha + \mathrm{i}\beta)(x + \mathrm{i}y) = \alpha x - \beta y + \mathrm{i}(\beta x + \alpha y)$ for $\alpha, \beta \in \mathbb{R}$ and $x, y \in V$, so that E is a complex linear space; we set $E_{\mathbb{R}} = V$.

The *dimension* of E over the underlying field and the linear subspace spanned by a subset S of E are denoted by

$$\dim E \quad \text{and} \quad \operatorname{lin} S,$$

respectively.

Let F and G be linear subspaces of a linear space E. Then we set

$$F + G = \{x + y : x \in F,\, y \in G\},$$

so that $F + G$ is a linear subspace of E; further, we write $E = F \oplus G$ if $F \cap G = \{0\}$ and $F + G = E$. More generally, let $E_1, \dots, E_n$ be linear subspaces of E such that $E_1 + \cdots + E_n = E$ and $E_i \cap E_j = \{0\}$ whenever $i, j \in \mathbb{N}_n$ with $i \neq j$. Then we write

$$E = E_1 \oplus \cdots \oplus E_n;$$

this is a *direct sum decomposition* of E. In this case, each $x \in E$ has a unique expression as $x = x_1 + \cdots + x_n$, where $x_i \in E_i$ $(i \in \mathbb{N}_n)$. Two direct sum decompositions $E_1 \oplus \cdots \oplus E_m$ and $F_1 \oplus \cdots \oplus F_n$ of E are *equal* if $n = m$ and $F_i = E_i$ $(i \in \mathbb{N}_m)$.

Let E be a linear space. For $x, y \in E$, define

$$[x, y] = \{tx + (1 - t)y : t \in \mathbb{I}\}.$$

A non-empty subset K of a linear space E is *convex* if $[x, y] \subset K$ whenever $x, y \in K$. The *convex hull* of a non-empty subset S of E is the intersection of the convex subsets of E that contain S; it is denoted by $\mathrm{co}(S)$, so that

$$\mathrm{co}(S) = \left\{t_1 x_1 + \cdots + t_n x_n : t_1, \dots, t_n \in \mathbb{I},\, \sum_{i=1}^{n} t_i = 1,\, x_1, \dots, x_n \in S\right\}.$$

The set of *extreme points* of a convex subset K of E is denoted by $\mathrm{ex}\, K$, so that, for $x \in K$, we have $x \in \mathrm{ex}\, K$ if and only if $K \setminus \{x\}$ is convex.

Now suppose that E is a complex linear space. For $\alpha \in \mathbb{C}$ and a subset S of E, we write $\alpha S = \{\alpha x : x \in S\}$; S is *absorbing* if

$$\bigcup \{\alpha S : \alpha > 0\} = E,$$

balanced if $\alpha S \subset S$ $(\alpha \in \overline{\mathbb{D}})$, and *absolutely convex* if S is convex and balanced. Equivalently, S is absolutely convex if $\alpha x + \beta y \in S$ whenever $x, y \in S$ and $\alpha, \beta \in \mathbb{C}$ with $|\alpha| + |\beta| \leq 1$. The *absolutely convex hull* of a non-empty subset S of E is the intersection of the absolutely convex subsets of E that contain S; it is denoted by $\mathrm{aco}(S)$, so that

$$\mathrm{aco}(S) = \left\{\alpha_1 x_1 + \cdots + \alpha_n x_n : \sum_{i=1}^{n} |\alpha_i| \leq 1,\, x_1, \dots, x_n \in S\right\},$$

where $\alpha_1, \dots, \alpha_n \in \mathbb{C}$. In the case where S is balanced, $\mathrm{aco}(S) = \mathrm{co}(S)$.

Let K be an absolutely convex, absorbing subset of the space E. Then the *Minkowski functional* p_K of K, defined by

$$p_K(x) = \inf\{\alpha > 0 : x \in \alpha K\} \quad (x \in E),$$

is a seminorm on E; p_K is a norm if and only if

$$\bigcap \{(1/n)K : n \in \mathbb{N}\} = \{0\}.$$

Of course, we have

$$\{x \in E : p_K(x) < 1\} \subset K \subset \{x \in E : p_K(x) \leq 1\}.$$

Let S be a non-empty set. The linear spaces of all functions from S to $\mathbb{C}$ and $\mathbb{R}$ are denoted by $\mathbb{C}^S$ and $\mathbb{R}^S$, respectively; $\mathbb{C}^S$ and $\mathbb{R}^S$ are complex and real algebras, respectively, for the pointwise operations. There is an obvious ordering on the space $\mathbb{R}^S$: for each $f, g \in \mathbb{R}^S$, we set $f \leq g$ if $f(s) \leq g(s)\ (s \in S)$, so that $(\mathbb{R}^S, \leq)$ is a partially ordered linear space. Indeed, $fg \geq 0$ whenever $f, g \geq 0$ in $\mathbb{R}^S$, and so $(R^S, \leq)$ is a partially ordered algebra. For a subset F of $\mathbb{R}^S$, we set

$$F^+ = \{f \in F : f \geq 0\}.$$

The functions $|f|$ and $\exp f$, etc., for functions $f, g \in \mathbb{C}^S$, and $f \vee g$ and $f \wedge g$ for functions $f, g \in \mathbb{R}^S$, are defined pointwise. For example,

$$(f \vee g)(s) = \max\{f(s), g(s)\}, \quad (f \wedge g)(s) = \min\{f(s), g(s)\} \quad (s \in S).$$

We then define the functions $f^+ = f \vee 0$, $f^- = (-f) \vee 0$, and

$$|f| = f^+ + f^- = f \vee (-f),$$

so that $f = f^+ - f^-$ and $f^+ f^- = 0$.

Let E be a linear space, and take $n \in \mathbb{N}$. Then we denote by E^n the linear space

$$\overbrace{E \times \cdots \times E}^{n},$$

where there are n copies of the space E. Thus E^n consists of n-tuples $(x_1, \dots, x_n)$, where $x_1, \dots, x_n \in E$. As a matter of notational convenience, we regard the generic element $(x_1, \dots, x_{k-1}, y_1, \dots, y_m)$ for $k, m \in \mathbb{N}$ as $(y_1, \dots, y_m)$ in the special case where $k = 1$, and we write x, rather than (x), in the case where $n = 1$. The linear operations on E^n are defined coordinatewise. The zero element of either E or E^n is denoted by 0. When we write

$$(0, \dots, 0, x_i, 0, \dots, 0)$$

for an element in E^n, we understand that x_i appears in the i^{th} coordinate, unless we say otherwise. An element x of E^n is often written as either $(x_1, \dots, x_n)$ or (x_i). For each $x \in E$, the *constant sequence with value* x is the sequence $(x) = (x, \dots, x) \in E^n$.

Definition 1.2. Let E be a linear space.

Take $n \in \mathbb{N}$ and $k \in \mathbb{N}_n$, and let $(x_1, \dots, x_n) \in E^n$. Then an element $(y_1, \dots, y_k) \in E^k$ is a *coagulation* of $(x_1, \dots, x_n)$ if there is a partition $\{S_j : j \in \mathbb{N}_k\}$ of $\mathbb{N}_n$ such that $y_j = \sum\{x_i : i \in S_j\}$ for each $j \in \mathbb{N}_k$.

Let $n, k \in \mathbb{N}$, and take $x = (x_1, \dots, x_k) \in E^k$. Then

$$x^{[n]} = (x_1, \dots, x_k, x_1, \dots, x_k, \ \dots, x_1, \dots, x_k) \in E^{nk},$$

where there are n copies of each block $(x_1, \dots, x_k)$; $x^{[n]}$ is the n^{th} *amplification* of x.

Let E be a linear space, and consider the space $E^{\mathbb{N}}$, which is also a linear space. A generic element of $E^{\mathbb{N}}$ is often written as

$$x = (x_i) = (x_i : i \in \mathbb{N});$$

the *zero* element of $E^{\mathbb{N}}$ is $0 = (0, 0, 0, \dots)$, and, for $x \in E$, the 'constant sequence with value x' is again (x). Define $\iota : x \mapsto (x)$, $E \to E^{\mathbb{N}}$, so that $\iota(E)$ is a linear subspace of $E^{\mathbb{N}}$.

1.1.4. Linear operators and matrices. Let E and F be linear spaces. Then the linear space of all linear operators from E to F is denoted by $\mathcal{L}(E,F)$; we set $\mathcal{L}(E) = \mathcal{L}(E,E)$. The identity operator on E is denoted by I_E. Thus $\mathcal{L}(E)$ is a unital algebra with respect to the composition of operators.

Now let V and W be real-linear spaces, and let T be a real-linear map from V to W. Set $E = V \oplus \mathrm{i}V$ and $F = W \oplus \mathrm{i}W$. The *complexification* $T_{\mathbb{C}}$ of T is defined by

$$T_{\mathbb{C}}(x + \mathrm{i}y) = Tx + \mathrm{i}Ty \quad (x, y \in V),$$

so that $T_{\mathbb{C}}$ is a complex-linear map from E to F.

Let E be a linear space, and take $m, n \in \mathbb{N}$. Then we denote by $\mathbb{M}_{m,n}(E)$ the linear space of all $m \times n$ matrices with coefficients in E; also, we write $\mathbb{M}_n(E)$ for $\mathbb{M}_{n,n}(E)$. We write $\mathbb{M}_{m,n}$ and $\mathbb{M}_n$ for $\mathbb{M}_{m,n}(\mathbb{C})$ and $\mathbb{M}_n(\mathbb{C})$, respectively. Let $v \in \mathbb{M}_m(E)$ and $w \in \mathbb{M}_n(E)$. Then $v \oplus w$ is the matrix in $\mathbb{M}_{m+n}(E)$ of the form

$$\begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix}.$$

Let $x = (x_{ij}) \in \mathbb{M}_{m,n}(E)$. Then the *transpose* of x is the matrix

$$x^t = (x_{ji}) \in \mathbb{M}_{n,m}(E).$$

Let E be a linear space, and take $m, n \in \mathbb{N}$. Then each element $a \in \mathbb{M}_{m,n}$ defines an element of $\mathcal{L}(E^n, E^m)$ by matrix multiplication.

Let $E_1, \dots, E_n$ and F be linear spaces. Then the linear space of n-linear maps from $E_1 \times \cdots \times E_n$ to F is denoted by $\mathcal{L}^n(E_1, \dots, E_n; F)$.

Let E be a linear space, take $n \in \mathbb{N}$, and let S be a subset of $\mathbb{N}_n$. For $x = (x_i) \in E^n$, we set

$$\begin{aligned} P_S(x) &= (y_i), \quad \text{where} \quad y_i = x_i \ (i \in S) \text{ and } y_i = 0 \ (i \notin S), \\ Q_S(x) &= (y_i), \quad \text{where} \quad y_i = x_i \ (i \notin S) \text{ and } y_i = 0 \ (i \in S). \end{aligned}$$

Thus P_S is the *projection onto* S and Q_S is the *projection onto the complement of* S. Clearly P_S and Q_S are idempotents in the algebra $\mathcal{L}(E^n)$, and $P_S + Q_S = I_{E^n}$. Also, for $i \in \mathbb{N}_n$, we set

$$\left.\begin{aligned} P_i(x) &= (0, \dots, 0, x_i, 0, \dots, 0), \\ Q_i(x) &= (x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) \end{aligned}\right\} \quad (x = (x_1, \dots, x_n) \in E^n),$$

so that $P_i = P_{\{i\}}$ and $Q_i = Q_{\{i\}}$.

We conclude this section by defining more formally some operators that will be important for us.

Definition 1.3. Let E be a linear space, and take $n \in \mathbb{N}$. For $\sigma \in \mathfrak{S}_n$, define

$$A_\sigma(x) = (x_{\sigma(1)}, \dots, x_{\sigma(n)}) \quad (x = (x_1, \dots, x_n) \in E^n).$$

For $\alpha = (\alpha_i) \in \mathbb{C}^n$, define

$$M_\alpha(x) = (\alpha_i x_i) \quad (x = (x_1, \dots, x_n) \in E^n).$$

Let E and F be linear spaces, and let $T \in \mathcal{L}(E,F)$. For $n \in \mathbb{N}$, define

$$T^{(n)} : (x_1, \dots, x_n) \mapsto (Tx_1, \dots, Tx_n), \quad E^n \to F^n; \tag{1.3}$$

$T^{(n)}$ is the n^{th} *amplification* of T.

Thus we see that $A_\sigma \in \mathcal{L}(E^n)$ for each $\sigma \in \mathfrak{S}_n$, that $M_\alpha \in \mathcal{L}(E^n)$ for each $\alpha \in \mathbb{C}^n$, and that $T^{(n)} \in \mathcal{L}(E^n, F^n)$.

1.2. Banach spaces and Banach algebras. We recall some basic facts about Banach spaces and algebras that we shall use.

1.2.1. Banach spaces and operators. For attractive introductions to Banach space theory, see [6, 8, 54, 74], for example; standard and beautiful classical texts on functional analysis are [27] and [65]. Most of the results on these topics that we shall use are summarized in [16, Appendix A.3].

Suppose that $(E, \|\cdot\|)$ is a normed space (over a scalar field $\mathbb{K}$, always taken to be $\mathbb{R}$ or $\mathbb{C}$). We denote by $E_{[r]}$ the closed ball in E with centre 0 and radius $r \geq 0$. We recall that each $E_{[r]}$ is an absolutely convex, absorbing, and closed neighbourhood of 0. We also denote by S_E the *unit sphere* of E, so that

$$S_E = \{x \in E : \|x\| = 1\}.$$

We shall later consider *direct sum decompositions* of a Banach space E, say

$$E = E_1 \oplus \cdots \oplus E_n.$$

In this situation, we shall always suppose that each of the linear subspaces $E_1, \dots, E_n$ is closed in E.

A sequence $(x_n : n \in \mathbb{N})$ in a normed space E is a *null sequence* if

$$\lim_{n\to\infty} x_n = 0;$$

the subspace of $E^{\mathbb{N}}$ consisting of all null sequences in E is denoted by $c_0(E)$.

The dual space of a normed space $(E, \|\cdot\|)$ is denoted by E'; the action of $\lambda \in E'$ on $x \in E$ gives the number $\langle x, \lambda\rangle$. We shall sometimes denote the *dual norm* on E' by $\|\cdot\|'$. The second dual space of E is denoted by E'', and the action of $\Phi \in E''$ on $\lambda \in E'$ gives $\langle \Phi, \lambda\rangle$ in our notation; we shall sometimes denote the dual norm on E'' by $\|\cdot\|''$. The *canonical embedding* $\iota : E \to E''$ is defined by the equation

$$\langle \iota(x), \lambda\rangle = \langle x, \lambda\rangle \quad (x \in E,\ \lambda \in E'),$$

so that ι is an isometry; the space E is *reflexive* if ι is a surjection. In fact, we shall usually identify x with $\iota(x)$ and sometimes write $\|\cdot\|$ for the second dual norm on E''.

The weak topology on E is denoted by $\sigma(E, E')$, the weak-$*$ topology on E' is $\sigma(E', E)$, and the weak-$*$ topology on E'' is $\sigma(E'', E')$, so that $(E', \sigma(E', E))$ is a locally convex space whose dual space is E. Of course, by *Goldstein's theorem*, $E_{[1]}$ is $\sigma(E'', E')$-dense in $E''_{[1]}$, and, by the *Banach–Alaoglu theorem*, $E'_{[1]}$ is $\sigma(E', E)$-compact.

For a subset $X \subset E$, we define its *annihilator* X° to be

$$X^\circ = \{\lambda \in E : \langle x, \lambda\rangle = 0 \ \ (x \in X)\}.$$

Evidently X° is a $\sigma(E', E)$-closed linear subspace of E'.

A form of the *Hahn–Banach separation theorem* [65, Theorem 3.7] is the following. Let (E, τ) be a locally convex space. Suppose that S is a closed, absolutely convex subset of E and that $x_0 \in E \setminus S$. Then there exists $\lambda \in (E, \tau)'$ such that $\langle x_0, \lambda\rangle > 1$ and $|\langle x, \lambda\rangle| \leq 1$ $(x \in S)$.

Let E and F be normed spaces. We denote by $\mathcal{B}(E,F)$ the normed space (with respect to the operator norm) of bounded linear operators from E to F; $\mathcal{B}(E,F)$ is a Banach space whenever F is a Banach space. Let $T \in \mathcal{B}(E,F)$. Then we denote the operator norm by $\|T\|$ or, occasionally, by

$$\|T : E \to F\|.$$

We set $\mathcal{B}(E) = \mathcal{B}(E,E)$, so that $\mathcal{B}(E)$ is a unital normed algebra. A map $T \in \mathcal{B}(E,F)$ is an *isometry* if $\|Tx\| = \|x\|$ $(x \in E)$; T is a *contraction* if $\|Tx\| \le \|x\|$ $(x \in E)$; T is an *isometric isomorphism* if T is a bijection and T and T^{-1} are isometries.

Let E and F be two Banach spaces. The space E is *linearly homeomorphic*, or *isomorphic, to* F if there exists a bijection $T \in \mathcal{B}(E,F)$ (so that we have $T^{-1} \in \mathcal{B}(F,E)$); such a map T is a *linear homeomorphism* or an *isomorphism*. In this case, we write

$$E \sim F;$$

the *Banach–Mazur distance* from E to F is

$$d(E,F) = \inf\{\|T\|\,\|T^{-1}\| : T \in \mathcal{B}(E,F) \text{ is an isomorphism}\};$$

see [6, Definition 7.4.5], for example. The space E is *isometrically isomorphic* to F if there is an isometric isomorphism $T \in \mathcal{B}(E,F)$, so that $d(E,F) = 1$; in this case, we shall write

$$E \cong F.$$

For $\lambda_0 \in E'$ and $y_0 \in F$, set

$$y_0 \otimes \lambda_0 : x \mapsto \langle x,\, \lambda_0\rangle y_0, \qquad E \to F.$$

Then $y_0 \otimes \lambda_0$ is a *rank-one operator* in $\mathcal{B}(E,F)$ with $\|y_0 \otimes \lambda_0\| = \|y_0\|\,\|\lambda_0\|$, and each *finite-rank operator* in $\mathcal{B}(E,F)$ is a finite sum of such operators. The linear subspace of $\mathcal{B}(E,F)$ consisting of the finite-rank operators is denoted by $\mathcal{F}(E,F)$. An operator $T \in \mathcal{B}(E,F)$ is *nuclear* if it can be expressed in the form $T = \sum_{i=1}^{\infty} y_i \otimes \lambda_i$, where (λ_i) is a sequence in E', (y_i) is a sequence in F, and

$$\sum_{i=1}^{\infty} \|y_i\|\,\|\lambda_i\| < \infty;$$

the *nuclear norm* $\nu(T)$ of the operator T is defined to be the infimum of the specified sums $\sum_{i=1}^{\infty} \|y_i\|\,\|\lambda_i\|$. In particular,

$$\nu(y_0 \otimes \lambda_0) = \|y_0 \otimes \lambda_0\| = \|y_0\|\,\|\lambda_0\| \quad (\lambda_0 \in E',\, y_0 \in F).$$

The space of nuclear operators is denoted by $\mathcal{N}(E,F)$; $(\mathcal{N}(E,F),\nu)$ is a Banach space when E and F are Banach spaces, and $\mathcal{F}(E,F)$ is dense in $(\mathcal{N}(E,F),\nu)$.

The closure of the space $\mathcal{F}(E,F)$ in $(\mathcal{B}(E,F),\|\cdot\|)$ forms the closed subspace of *approximable operators*. The spaces of approximable and compact operators from E to F are denoted by

$$\mathcal{A}(E,F) \quad \text{and} \quad \mathcal{K}(E,F),$$

respectively. In the case where $F = E$, we write $\mathcal{F}(E)$, $\mathcal{N}(E)$, $\mathcal{A}(E)$, and $\mathcal{K}(E)$ for $\mathcal{F}(E,E)$, $\mathcal{N}(E,E)$, $\mathcal{A}(E,E)$, and $\mathcal{K}(E,E)$ respectively; each of these is an ideal in the normed algebra $\mathcal{B}(E)$.

For $T \in \mathcal{B}(E, F)$, the *dual operator* T' of T is defined by the equation

$$\langle x, T'\lambda \rangle = \langle Tx, \lambda \rangle \quad (x \in E, \lambda \in F');$$

we have $T' \in \mathcal{B}(F', E')$ and $\|T'\| = \|T\|$. The dual of an isometry is also an isometry.

A closed subspace F of a Banach space E is *complemented* if there is a projection $P \in \mathcal{B}(E, F)$ with $P(E) = F$, and λ-*complemented* (for $\lambda \geq 1$) if there is a projection P of E onto F with $\|P\| \leq \lambda$.

We shall sometimes use the following *Principle of Local Reflexivity*, proved in [6, Theorem 11.2.4] and [66, Theorem 5.54], for example.

THEOREM 1.4. *Let E be a Banach space, let X and Y be finite-dimensional subspaces of E'' and E', respectively, and take $\varepsilon > 0$. Then there is an injective, bounded linear map $S : X \to E$ with the following properties:*

(i) $Sx = x$ $(x \in X \cap E)$;
(ii) $\langle S(\Lambda), \lambda \rangle = \langle \Lambda, \lambda \rangle$ $(\lambda \in Y, \Lambda \in X)$;
(iii) $(1 - \varepsilon)\|\Lambda\| \leq \|S(\Lambda)\| \leq (1 + \varepsilon)\|\Lambda\|$ $(\Lambda \in X)$. ∎

Let $E_1, \dots, E_n$ and F be normed spaces. Then the space of bounded n-linear maps from $E_1 \times \cdots \times E_n$ to F is denoted by $\mathcal{B}^n(E_1, \dots, E_n; F)$. This is a normed space for the norm $\|\cdot\|$ defined by

$$\|T\| = \sup\{\|T(x_1, \dots, x_n)\| : x_j \in (E_j)_{[1]}, \ j \in \mathbb{N}_n\}$$

for $T \in \mathcal{B}^n(E_1, \dots, E_n; F)$, and it is a Banach space whenever F is complete.

1.2.2. Tensor products. Let E and F be linear spaces. Each element of the (algebraic) tensor product $E \otimes F$ has the form $\sum_{i=1}^m x_i \otimes y_i$ for some $m \in \mathbb{N}$, $x_1, \dots, x_m \in E$, and $y_1, \dots, y_m \in F$; such a representation is not unique.

Let G be a third linear space. For each bilinear map $T : E \times F \to G$, there is a unique linear map $\widetilde{T} : E \otimes F \to G$ such that

$$\widetilde{T}(x \otimes y) = T(x, y) \quad (x \in E, y \in F).$$

Let $S \in \mathcal{L}(E)$ and $T \in \mathcal{L}(F)$. Then there exists a map $S \otimes T \in \mathcal{L}(E \otimes F)$ such that

$$(S \otimes T)(x \otimes y) = Sx \otimes Ty \quad (x \in E, y \in F).$$

Now suppose that E and F are normed spaces, and that $\|\cdot\|$ is a norm on the linear space $E \otimes F$. Then $\|\cdot\|$ is a *sub-cross-norm* if

$$\|x \otimes y\| \leq \|x\| \, \|y\| \quad (x \in E, y \in F)$$

and a *cross-norm* if

$$\|x \otimes y\| = \|x\| \, \|y\| \quad (x \in E, y \in F).$$

Further, a sub-cross-norm $\|\cdot\|$ on $E \otimes F$ is a *reasonable cross-norm* if the linear functional $\lambda \otimes \mu$ is bounded and $\|\lambda \otimes \mu\| \leq \|\lambda\| \, \|\mu\|$ for each $\lambda \in E'$ and $\mu \in F'$. For these definitions and the properties stated below, see [25, §VIII,1] and [66, §6.1].

PROPOSITION 1.5. *Let E and F be normed spaces. Then each reasonable cross-norm on $E \otimes F$ is a cross-norm, and*

$$\|\lambda \otimes \mu\| = \|\lambda\| \, \|\mu\| \quad (\lambda \in E', \mu \in F'). \ ∎$$

The *projective norm* $\|\cdot\|_\pi$ on $E \otimes F$ is defined by

$$\|z\|_\pi = \inf\Big\{\sum_{i=1}^m \|x_i\|\,\|y_i\| : z = \sum_{i=1}^m x_i \otimes y_i \in E \otimes F\Big\},$$

where the infimum is taken over all representations $z = \sum_{i=1}^m x_i \otimes y_i$ of $z \in E \otimes F$; $(E \otimes F, \|\cdot\|_\pi)$ is then a normed space, and its completion

$$(E \mathbin{\widehat{\otimes}} F, \|\cdot\|_\pi)$$

is the *projective tensor product* of E and F. We note that

$$\|x \otimes y\|_\pi = \|x\|\,\|y\| \quad (x \in E,\ y \in F), \tag{1.4}$$

so that $\|\cdot\|_\pi$ is a cross-norm on $E \otimes F$. In fact, $\|\cdot\|_\pi$ is a reasonable cross-norm, and $\|z\| \leq \|z\|_\pi$ $(z \in E \otimes F)$ for each reasonable cross-norm $\|\cdot\|$ on $E \otimes F$. The key property of this tensor product is the following.

PROPOSITION 1.6. *Let E, F, and G be three Banach spaces. Then, for each bilinear operator $T \in \mathcal{B}(E, F; G)$, there exists a unique linear operator $\widetilde{T} \in \mathcal{B}(E \mathbin{\widehat{\otimes}} F, G)$ such that*

$$\widetilde{T}(x \otimes y) = T(x, y) \quad (x \in E,\ y \in F),$$

and the map $T \mapsto \widetilde{T}$, $\mathcal{B}(E, F; G) \to \mathcal{B}(E \mathbin{\widehat{\otimes}} F, G)$, is an isometric isomorphism. ■

Let E and F be two Banach spaces. For $\mu \in (E \mathbin{\widehat{\otimes}} F)'$, define T_μ by

$$\langle y,\, T_\mu x\rangle = \langle x \otimes y,\, \mu\rangle \quad (x \in E,\ y \in F).$$

Then $T_\mu x \in F'$ $(x \in E)$, $T_\mu \in \mathcal{B}(E, F')$, and the map

$$\mu \mapsto T_\mu, \quad (E \mathbin{\widehat{\otimes}} F)' \to \mathcal{B}(E, F'),$$

is an isometric isomorphism, and so

$$(E \mathbin{\widehat{\otimes}} F)' \cong \mathcal{B}(E, F'). \tag{1.5}$$

Let E and F be normed spaces over a field $\mathbb{K}$. For $x \in E$ and $y \in F$, set

$$T_{x,y}(\lambda, \mu) = \langle x,\, \lambda\rangle\langle y,\, \mu\rangle \quad (\lambda \in E',\ \mu \in F'),$$

so that $T_{x,y} \in \mathcal{B}(E', F'; \mathbb{K})$; the map

$$(x, y) \mapsto T_{x,y}, \quad E \times F \to \mathcal{B}(E', F'; \mathbb{K}),$$

is bilinear. There is an injective linear map $\iota : E \otimes F \to \mathcal{B}(E', F'; \mathbb{K})$ such that

$$\iota(x \otimes y) = T_{x,y} \quad (x \in E,\ y \in F),$$

and so we may regard $E \otimes F$ as a linear subspace of $\mathcal{B}(E', F'; \mathbb{K})$. The *injective norm* $\|\cdot\|_\varepsilon$ on $E \otimes F$ is the norm inherited from $\mathcal{B}(E', F'; \mathbb{K})$, and so

$$\|z\|_\varepsilon = \sup\Big\{\Big|\sum_{i=1}^m \langle x_i,\, \lambda\rangle\langle y_i,\, \mu\rangle\Big| : \lambda \in E'_{[1]},\ \mu \in F'_{[1]}\Big\},$$

for any representation $z = \sum_{i=1}^m x_i \otimes y_i$ of $z \in E \otimes F$. The closure of $E \otimes F$ in $\mathcal{B}(E', F'; \mathbb{K})$, denoted by

$$(E \mathbin{\check{\otimes}} F, \|\cdot\|_\varepsilon),$$

is the *injective tensor product* of E and F. We note that

$$\|x \otimes y\|_\varepsilon = \|x\|\,\|y\| \quad (x \in E,\ y \in F), \tag{1.6}$$

so that $\|\cdot\|_\varepsilon$ is also cross-norm on $E \otimes F$. In fact, $\|\cdot\|_\varepsilon$ is a reasonable cross-norm, and $\|z\|_\varepsilon \leq \|z\|$ $(z \in E \otimes F)$ for each reasonable cross-norm $\|\cdot\|$ on $E \otimes F$.

It is shown in [66, Proposition 6.1] that a norm $\|\cdot\|$ on $E \otimes F$ is a reasonable cross-norm if and only if

$$\|z\|_\varepsilon \leq \|z\| \leq \|z\|_\pi \qquad (z \in E \otimes F). \tag{1.7}$$

1.2.3. Direct sum decompositions. Let $(E, \|\cdot\|)$ be a normed space, and suppose that $E = E_1 \oplus \cdots \oplus E_k$ is a direct sum decomposition of E, where $E_1, \ldots, E_k$ are closed subspaces of E; we allow the possibility that $E_j = \{0\}$ for some $j \in \mathbb{N}_k$. We say that the decomposition has *length* k in this case. Thus each element $x \in E$ has a unique expression as $x = x_1 + \cdots + x_k$, where $x_j \in E_j$ $(j \in \mathbb{N}_k)$. The decomposition is *trivial* if $E = E_j$ for some $j \in \mathbb{N}_k$. We write $P_j : E \to E_j$ $(j \in \mathbb{N}_k)$ for the natural projections.

Now suppose that $E = E_1 \oplus \cdots \oplus E_k$ is a Banach space. Then, for each $j \in \mathbb{N}_k$, the map P_j is continuous, and is regarded as a member of the Banach space $\mathcal{B}(E, E_j)$. It is not necessarily true that $\|P_j\| \leq 1$.

Definition 1.7. Let $(E, \|\cdot\|)$ be a normed space, and consider a family $\mathcal{K}$ of direct sum decompositions of E. The family $\mathcal{K}$ is *closed* provided that the following conditions are satisfied for each $k \in \mathbb{N}$:

(C1) $E_{\sigma(1)} \oplus \cdots \oplus E_{\sigma(k)} \in \mathcal{K}$ whenever $E_1 \oplus \cdots \oplus E_k \in \mathcal{K}$, $\sigma \in \mathfrak{S}_k$, and $k \in \mathbb{N}$;
(C2) $F \oplus E_3 \oplus \cdots \oplus E_k \in \mathcal{K}$ whenever $E_1 \oplus \cdots \oplus E_k \in \mathcal{K}$, $F = E_1 \oplus E_2$, and $k \geq 3$;
(C3) $\mathcal{K}$ contains all trivial direct sum decompositions.

It follows from (C3) that, for each $k \in \mathbb{N}$, there exists an element of $\mathcal{K}$ with length k.

For example, the families of all direct sum decompositions and of all trivial direct sum decompositions of E are closed families.

We see immediately that the intersection of a collection of closed families of direct sum decompositions of a normed space is also a closed family of direct sum decompositions. Thus the following notion is well-defined.

Definition 1.8. Let $(E, \|\cdot\|)$ be a normed space, and consider a family $\mathcal{K}$ of direct sum decompositions of E. Then the smallest closed family $\mathcal{L}$ of direct sum decompositions of E such that $\mathcal{L}$ contains $\mathcal{K}$ is the *closed family generated by* $\mathcal{K}$.

Let $E = E_1 \oplus \cdots \oplus E_k$ be a direct sum decomposition. For $j \in \mathbb{N}_k$, the dual map

$$P_j' : \lambda \mapsto \lambda \circ P_j, \qquad E_j' \to E',$$

is a continuous linear embedding, and the image $P_j'(E_j')$ is a closed subspace of E'; we shall usually regard E_j' as a subspace of E' by identifying $\lambda \in E_j'$ with $\lambda \circ P_j \in E'$, and then $E' = E_1' \oplus \cdots \oplus E_k'$.

Definition 1.9. Let $(E, \|\cdot\|)$ be a normed space, and let $\mathcal{K}$ be a closed family of direct sum decompositions of E. The *dual* to the family $\mathcal{K}$ is

$$\mathcal{K}' = \{E_1' \oplus \cdots \oplus E_k' : E_1 \oplus \cdots \oplus E_k \in \mathcal{K}\}.$$

Thus $\mathcal{K}'$ is a closed family of direct sum decompositions of E'.

1.2.4. Duals of products of Banach spaces. Let $(E, \|\cdot\|)$ be a normed space, and take $k \in \mathbb{N}$. Let $\|\!|\cdot|\!\|$ be any norm on the linear space E^k such that

$$\|\!|x|\!\| \geq \max\{\|x_i\| : i \in \mathbb{N}_k\} \quad (x = (x_i) \in E^k) \tag{1.8}$$

and

$$\|\!|(0, \dots, 0, x_i, 0, \dots, 0)|\!\| = \|x_i\| \quad (x_i \in E,\, i \in \mathbb{N}_k). \tag{1.9}$$

For $\lambda_1, \dots, \lambda_k \in E'$, define λ on E^k by

$$\langle x,\, \lambda\rangle = \sum_{i=1}^{k} \langle x_i,\, \lambda_i\rangle \quad (x = (x_1, \dots, x_k) \in E^k). \tag{1.10}$$

Then λ is a linear functional on E^k, and

$$|\langle x,\, \lambda\rangle| \leq \Bigl(\sum_{i=1}^{k} \|\lambda_i\|\Bigr) \max\{\|x_i\| : i \in \mathbb{N}_k\} \leq \Bigl(\sum_{i=1}^{k} \|\lambda_i\|\Bigr) \|\!|x|\!\|$$

for each $x = (x_1, \dots, x_k) \in E^k$. Thus $\lambda \in (E^k, \|\!|\cdot|\!\|)'$ with

$$\max\{\|\lambda_i\| : i \in \mathbb{N}_k\} \leq \|\!|\lambda|\!\|' \leq \sum_{i=1}^{k} \|\lambda_i\|, \tag{1.11}$$

where $\|\!|\cdot|\!\|'$ is the dual norm to $\|\!|\cdot|\!\|$. Further, each element in $(E^k, \|\!|\cdot|\!\|)'$ arises in this way. Thus we may regard $(E')^k$ as a Banach space for the norm $\|\!|\cdot|\!\|'$, identifying $\lambda \in (E^k, \|\!|\cdot|\!\|)'$ with $(\lambda_1, \dots, \lambda_k) \in (E')^k$.

In this case, it is easily seen that $\|\!|\cdot|\!\|'$ is a norm on $(E')^k$ that also satisfies (1.8) and (1.9), and so we may also regard $(E'')^k$ as a Banach space for the norm $\|\!|\cdot|\!\|''$. The weak-$*$ topology on $(E^k, \|\!|\cdot|\!\|)'$ as the dual of $(E^k, \|\!|\cdot|\!\|)$ is equal to the product topology on $(E', \sigma(E', E))^k$.

Let E be a normed space, and suppose that, for each $k \in \mathbb{N}$, $\|\cdot\|_k$ is a norm on E^k satisfying (1.8) and (1.9), so that $\|\cdot\|_k'$ is a norm on $(E')^k$. Then $(\|\cdot\|_k' : k \in \mathbb{N})$ is the *dual sequence* to $(\|\cdot\|_k : k \in \mathbb{N})$.

1.2.5. Families of Banach spaces. Let $\{(E_\alpha, \|\cdot\|_\alpha) : \alpha \in A\}$ be a family of normed spaces, defined for each α in a non-empty index set A (perhaps finite). Then we shall consider the following spaces.

First set

$$\ell^\infty(E_\alpha) = \Bigl\{(x_\alpha : \alpha \in A) : \|(x_\alpha)\| = \sup_\alpha \|x_\alpha\|_\alpha < \infty\Bigr\}.$$

Similarly, for p with $1 \leq p < \infty$, we define

$$\ell^p(E_\alpha) = \Bigl\{(x_\alpha : \alpha \in A) : \|(x_\alpha)\| = \Bigl(\sum_\alpha \|x_\alpha\|_\alpha^p\Bigr)^{1/p} < \infty\Bigr\}.$$

Clearly, $\ell^\infty(E_\alpha)$ and $\ell^p(E_\alpha)$ are normed spaces; they are Banach spaces if each of the spaces E_α is a Banach space. We write

$$F \oplus_\infty G \quad \text{and} \quad F \oplus_p G$$

for the sum of two normed spaces F and G with the appropriate norms, etc., and we write $\ell^p_n(E)$ for E^n with the norm given by

$$\|(x_1,\dots,x_n)\| = \Big(\sum_{i=1}^n \|x_i\|^p\Big)^{1/p} \qquad (x_1,\dots,x_n \in E).$$

1.2.6. Hilbert spaces and C^*-algebras. We recall some basic facts about Hilbert spaces; for further background, see [8, 43], for example.

Let H be a Hilbert space, with inner product denoted by $[\,\cdot\,,\cdot\,]$. For example, let $H=\ell^2$, where the inner product is specified by

$$[(z_j),\,(w_j)] = \sum_{j=1}^\infty z_j\overline{w}_j \qquad ((z_j),\,(w_j)\in \ell^2).$$

We recall that $\|x\|^2=[x,\,x]$ $(x\in H)$ and that

$$\|x+y\|^2 = \|x\|^2 + 2\Re\,[x,y] + \|y\|^2 \qquad (x,y\in H). \tag{1.12}$$

The *Cauchy–Schwarz inequality* asserts that

$$|\,[x,y]\,| \le \|x\|\,\|y\| \qquad (x,y\in H).$$

Two vectors $x,y\in H$ are *orthogonal*, written $x\perp y$, if $[x,\,y]=0$; a subset S of S_H is *orthonormal* if $x\perp y$ whenever $x,y\in S$ with $x\neq y$, and an n-tuple $(e_1,\dots,e_n)$ of elements in S_H is *orthonormal* if $e_i\perp e_j$ whenever $i,j\in\mathbb{N}_n$ with $i\neq j$.

Let S be an orthonormal set in H. Then

$$\sum_{e\in S}|[x,e]|^2 \le \|x\|^2 \qquad (x\in H),$$

with equality if and only if $x\in\overline{\mathrm{lin}}\,S$. A maximal orthonormal set is an *orthonormal basis* for H; an orthonormal set is an orthonormal basis if and only if its closed linear span is H. The *Hilbert dimension* of H is the cardinality of such a basis; it is independent of the choice of the basis. Every Hilbert space is isomorphic to one of the form $\ell^2(I)$, where I is an index set with $|I|$ equal to the Hilbert dimension of H.

Two linear subspaces F and G of H are *orthogonal* if

$$[x,\,y]=0 \qquad (x\in F,\,y\in G),$$

and we write $F\perp G$ in this case. Suppose that $H=F\oplus G$, where $F\perp G$. Then $H=F\oplus G$ is an *orthogonal decomposition*, and we write

$$H = F\oplus_\perp G.$$

Let H be a Hilbert space. There is a standard involution $*$ on $\mathcal{B}(H)$, defined by the condition that

$$[T^*x,y]=[x,Ty] \qquad (x,y\in H,\,T\in\mathcal{B}(H)),$$

and then

$$\|T^*T\| = \|T\|^2 \qquad (T\in\mathcal{B}(H)),$$

showing that $\mathcal{B}(H)$ is a C^*-algebra; see [43, Chapter 4]. Subalgebras of $\mathcal{B}(H)$ that are $*$-closed and norm-closed are also C^*-algebras, and the *Gel'fand–Naimark representation theorem* asserts that every abstractly defined C^*-algebra has this form.

Let A be a unital C^*-algebra, with identity e_A. An element $u \in A$ is *unitary* if $u^*u = uu^* = e_A$; the set of unitary elements is the *unitary group*, $\mathcal{U}(A)$, of A. We shall use the *Russo–Dye theorem* [16, Theorem 3.2.18], which asserts that

$$A_{[1]} = \overline{\mathrm{co}}(\mathcal{U}(A)). \tag{1.13}$$

Suppose that $(e_1, \dots, e_n)$ is an orthonormal n-tuple in H^n and that U is a unitary operator on H. Then $(Ue_1, \dots, Ue_n)$ is also an orthonormal n-tuple in H^n.

Let H be a Hilbert space. A *projection* in $\mathcal{B}(H)$ is an element P in $\mathcal{B}(H)$ such that $P = P^* = P^2$. In the case where $H = F \oplus_\perp G$, set $y = P_F x$ and $z = P_G x$, so that $x = y + z$. Then P_F and P_G are projections in $\mathcal{B}(H)$ such that $P_F + P_G = I_H$ and $P_F P_G = P_G P_F = 0$, so that P_F and P_G are *orthogonal projections*. Conversely, each pair of orthogonal projections gives an orthogonal decomposition of H. Take $x \in H = F \oplus_\perp G$, and set $e = P_F x / \|P_F x\|$, with $e = 0$ when $P_F x = 0$. Then

$$e \in F \quad \text{and} \quad \|P_F x\| = [e, x]. \tag{1.14}$$

We set

$$H = H_1 \oplus_\perp \cdots \oplus_\perp H_n$$

when $H_1, \dots, H_n$ are closed subspaces of H with $H = H_1 \oplus \cdots \oplus H_n$ and $H_i \perp H_j$ whenever $i, j \in \mathbb{N}_n$ with $i \neq j$; this is an *orthogonal decomposition*. It corresponds to an orthogonal family $\{P_1, \dots, P_n\}$ of projections, where $P_i P_j = 0$ whenever $i, j \in \mathbb{N}_n$ with $i \neq j$.

1.2.7. Standard Banach spaces. Throughout we have certain fixed notations for some standard elements and Banach spaces.

Consider the space $\mathbb{C}^{\mathbb{N}}$, which consists of all complex-valued sequences, regarded as functions from $\mathbb{N}$ to $\mathbb{C}$. For $n \in \mathbb{N}$, set

$$\delta_n = (\delta_{m,n} : m \in \mathbb{N}) \in \mathbb{C}^{\mathbb{N}},$$

where $\delta_{m,n} = 1$ $(m = n)$ and $\delta_{m,n} = 0$ $(m \neq n)$. Define

$$c_{00} = \mathrm{lin}\{\delta_n : n \in \mathbb{N}\} \subset \mathbb{C}^{\mathbb{N}},$$

and, for p with $1 \leq p < \infty$, set

$$\ell^p = \Big\{(\alpha_i) \in \mathbb{C}^{\mathbb{N}} : \sum_{i=1}^{\infty} |\alpha_i|^p < \infty\Big\},$$

so that ℓ^p is a Banach space for the norm given by

$$\|(\alpha_i)\|_{\ell^p} = \|(\alpha_i)\| = \Big(\sum_{i=1}^{\infty} |\alpha_i|^p\Big)^{1/p} \qquad ((\alpha_i) \in \ell^p).$$

(We shall usually suppress the dependence of the norm $\|\cdot\|$ on the index p in the notation, but we shall occasionally write $\|\cdot\|_{\ell^p}$ when there is a possibility of confusion.)

Further, we set

$$\ell^\infty = \Big\{(\alpha_i) \in \mathbb{C}^{\mathbb{N}} : |(\alpha_i)|_{\mathbb{N}} = \sup_{i \in \mathbb{N}} |\alpha_i| < \infty\Big\},$$

so that $(\ell^\infty, |\cdot|_{\mathbb{N}})$ is a Banach space; the spaces

$$c_0 = \left\{(\alpha_i) \in \mathbb{C}^{\mathbb{N}} : \lim_{i\to\infty} \alpha_i = 0\right\} \quad \text{and} \quad c = \left\{(\alpha_i) \in \mathbb{C}^{\mathbb{N}} : \lim_{i\to\infty} \alpha_i \text{ exists}\right\}$$

of *null sequences* and *convergent sequences*, respectively, are each closed subspaces of $(\ell^\infty, |\cdot|_{\mathbb{N}})$. Of course, $c = c_0 \oplus \mathbb{C}1$, where 1 is the sequence identically equal to 1, c_{00} is a dense linear subspace of each ℓ^p for $p \geq 1$ and of c_0, and $\{\delta_n : n \in \mathbb{N}\}$ is a Schauder basis for each of these spaces; we call it the *standard basis*. Note that $\|\delta_n\| = 1$ $(n \in \mathbb{N})$, where $\|\cdot\|$ is calculated in any of the spaces ℓ^p (for $p \geq 1$) or c_0.

Similarly, we regard $\{\delta_1, \dots, \delta_n\}$ as the standard basis of $\mathbb{C}^n$ for $n \in \mathbb{N}$.

The real-valued versions of these spaces are $\ell^p_{\mathbb{R}}$, $\ell^\infty_{\mathbb{R}}$, $c_{0,\mathbb{R}}$ and $c_{\mathbb{R}}$, regarded as subspaces of $\mathbb{R}^{\mathbb{N}}$.

We note that the spaces ℓ^p for $1 < p < \infty$ are reflexive, that the spaces ℓ^p for $1 \leq p < \infty$ and c_0 are separable, that ℓ^∞ is not separable, and that

$$\ell^1 \subset \ell^p \subset \ell^q \subset c_0 \subset \ell^\infty \quad \text{whenever} \quad 1 \leq p \leq q < \infty.$$

Of course, $c_0' \cong \ell^1$, $(\ell^1)' \cong \ell^\infty$, and $(\ell^p)' \cong \ell^q$ for $1 < p < \infty$, with the standard duality, where q is the conjugate index to p.

Let $n \in \mathbb{N}$. The n-dimensional versions of the above spaces are denoted by ℓ^p_n (for $p \geq 1$) and by ℓ^∞_n. Now $(\ell^\infty_n)' = \ell^1_n$.

Let $m, n \in \mathbb{N}$. Then we can identify $\mathbb{M}_{m,n}$ with the Banach space $\mathcal{B}(\ell^\infty_n, \ell^\infty_m)$, so that $(\mathbb{M}_{m,n}, \|\cdot\|)$ is a Banach space. Indeed, the formula for the norm in $\mathbb{M}_{m,n}$ of an element $a = (a_{ij})$ is then

$$\|a : \ell^\infty_n \to \ell^\infty_m\| = \max\left\{\sum_{j=1}^n |a_{ij}| : i \in \mathbb{N}_m\right\}. \tag{1.15}$$

In the case where $m = n$, we obtain a unital Banach algebra $(\mathbb{M}_n, \|\cdot\|)$. More generally, let $p, q \in [1, \infty]$. Then we can also identify $\mathbb{M}_{m,n}$ with $\mathcal{B}(\ell^p_n, \ell^q_m)$, and in this case we may denote the norm of $a \in \mathbb{M}_{m,n}$ by

$$\|a : \ell^p_n \to \ell^q_m\|.$$

For example,

$$\|a : \ell^1_n \to \ell^1_m\| = \max\left\{\sum_{i=1}^m |a_{ij}| : j \in \mathbb{N}_n\right\}. \tag{1.16}$$

Let $p_1, p_2 \in [1, \infty]$, and take q_1, q_2 to be the two conjugate indices to p_1 and p_2, respectively. For each $a \in \mathbb{M}_{m,n}$, we have $a^t = a'$ and

$$\|a : \ell^{p_1}_n \to \ell^{p_2}_m\| = \|a^t : \ell^{q_2}_m \to \ell^{q_1}_n\|. \tag{1.17}$$

Let μ be a positive measure on a measure space Ω. (We use the terminology concerning measures of [15, 64, 65].) An *ordered partition* of Ω is an n-tuple $(S_1, \dots, S_n)$ of measurable subsets of Ω such that $S_1 \cup \cdots \cup S_n = \Omega$ and $S_i \cap S_j = \emptyset$ whenever $i, j \in \mathbb{N}_n$ with $i \neq j$. (We allow some of the sets S_j to be empty.)

We shall consider measurable functions $f : \Omega \to \mathbb{C}$. For $p \geq 1$, we set

$$L^p(\Omega, \mu) = \left\{f : \int_\Omega |f|^p \, \mathrm{d}\mu < \infty\right\},$$

so that $L^p(\Omega, \mu)$ is a Banach space for the norm $\|\cdot\|$, where

$$\|f\|_{L^p} = \|f\| = \left(\int_\Omega |f|^p \,\mathrm{d}\mu\right)^{1/p} = \left(\int_\Omega |f(x)|^p \,\mathrm{d}\mu(x)\right)^{1/p} \qquad (f \in L^p(\Omega, \mu)).$$

Here we equate functions that are equal almost everywhere with respect to μ in the usual way. We shall often write $L^p(\Omega)$ for $L^p(\Omega, \mu)$ and $\int_\Omega f$ or $\int f$ for $\int_\Omega f \,\mathrm{d}\mu$.

The space $L^\infty(\Omega, \mu)$ consists of the essentially bounded functions on Ω, with the essential supremum norm.

The real-linear subspaces of $L^p(\Omega, \mu)$ and $L^\infty(\Omega, \mu)$ consisting of the real-valued functions are denoted by $L^p_{\mathbb{R}}(\Omega, \mu)$ and $L^\infty_{\mathbb{R}}(\Omega, \mu)$, respectively.

We shall use *Hölder's inequality* in the following form. Take $p \geq 1$, with conjugate index q. Then, for $f \in L^p(\Omega, \mu)$ and $g \in L^q(\Omega, \mu)$, we have $fg \in L^1(\Omega, \mu)$ and

$$\int_\Omega |fg| \leq \left(\int_\Omega |f|^p\right)^{1/p} \left(\int_\Omega |g|^q\right)^{1/q}. \tag{1.18}$$

We shall identify the dual space $L^p(\Omega, \mu)'$ with $L^q(\Omega, \mu)$ in the cases where $p > 1$, where $p = 1$ and μ is σ-finite, and where μ is counting measure on a non-empty set S, so that $\ell^1(S)' = \ell^\infty(S)$; the duality is specifed by

$$\langle f,\, g\rangle = \int_\Omega fg \,\mathrm{d}\mu \qquad (f \in L^p(\Omega, \mu),\, g \in L^q(\Omega, \mu)).$$

See [15, Theorem 4.5.1] or [64, Theorem 6.16]. Again the spaces $L^p(\Omega, \mu)$ for $1 < p < \infty$ are reflexive.

When we consider the spaces $L^p(\mathbb{I})$, we always suppose that the measure on $\mathbb{I}$ is the Lebesgue measure.

Throughout, a locally compact topological space is supposed to be Hausdorff.

Let K be a non-empty, locally compact space. Then $C_0(K)$ is the space of all complex-valued, continuous functions on K that vanish at infinity, and $C_{0,\mathbb{R}}(K)$ is the real-linear subspace of real-valued functions in $C_0(K)$. We write $C(K)$ for $C_0(K)$ in the case where K is compact. Thus $C_0(K)$ is a Banach space with respect to the uniform norm $|\cdot|_K$ on Ω, defined by

$$|f|_K = \sup\{|f(x)| : x \in K\} \qquad (f \in C_0(K)). \tag{1.19}$$

Let $f, g \in C_{0,\mathbb{R}}(K)$. Then $|f|$, f^+, f^-, $|f| \vee |g|$, $|f| \wedge |g|$ belong to $C_{0,\mathbb{R}}(K)$.

Let K be a non-empty, locally compact space. We denote by $M(K)$ the space of all complex-valued, regular Borel measures on K, taken with the total variation norm

$$\|\mu\| = |\mu|(K) \qquad (\mu \in M(K));$$

the subspace of real-valued measures is $M_{\mathbb{R}}(K)$. We shall write δ_x for the measure which is the point mass at x for $x \in K$. A subset S of K is said to be *measurable* if it is measurable with respect to the σ-algebra of Borel subsets of K. For a (Borel) measurable subset X of K, we define the restriction measure $\mu | X$ for $\mu \in M(K)$ by $(\mu | X)(B) = \mu(X \cap B)$ for each Borel subset B of K. We identify the dual space $C_0(K)'$ with $M(K)$; the duality is

$$\langle f,\, \mu\rangle = \int_K f \,\mathrm{d}\mu \qquad (f \in C_0(K),\, \mu \in M(K)).$$

See [15, §4.1] and [64, Chapter 6].

A measure $\mu \in M(K)$ is *discrete* if there is a countable subset S of K such that $|\mu|(K \setminus S) = 0$; the closed subspace of $M(K)$ consisting of the discrete measures is denoted by $M_d(K)$, and identified with $\ell^1(K)$. A measure $\mu \in M(K)$ is *continuous* if $\mu(\{x\}) = 0$ $(x \in K)$; the closed subspace of $M(K)$ consisting of the continuous measures is denoted by $M_c(K)$. We have $M(K) = M_d(K) \oplus M_c(K)$, and $\|\mu + \nu\| = \|\mu\| + \|\nu\|$ for each $\mu \in M_d(K)$ and $\nu \in M_c(K)$, so that

$$M(K) = \ell^1(K) \oplus_1 M_c(K).$$

We shall use *Hahn's decomposition theorem* in the following form. Let $\mu \in M_{\mathbb{R}}(K)$. Then there exist measurable subsets P and N of K such that $\mu(S) \geq 0$ for each measurable subset S of P and $\mu(S) \leq 0$ for each measurable subset S of N.

1.2.8. Banach algebras. We shall sometimes refer to Banach algebras. As a standard reference for this topic, we shall cite [16], and we shall use the terminology of that book. For an introduction to the theory of Banach algebras that is sufficient for our purposes, see [8, Part II].

Thus a *Banach algebra* is a linear, associative algebra A over $\mathbb{C}$ such that A is also a Banach space and

$$\|ab\| \leq \|a\| \, \|b\| \quad (a, b \in A).$$

Let G be a locally compact group. Then the *group algebra* $L^1(G)$ and the *measure algebra* $M(G)$ are Banach algebras with respect to convolution multiplication. For details of these algebras, see [16, 18, 22, 36].

The *spectrum* of an element a in a Banach algebra A is denoted by $\sigma_A(a)$ or $\sigma(a)$ [16, Definition 1.5.27]; $\sigma_A(a)$ is always a non-empty, compact subset of $\mathbb{C}$. The corresponding *spectral radius* is denoted by $\nu(a)$; by definition, $\nu(a) = \sup\{|z| : z \in \sigma_A(a)\}$, and the *spectral radius formula* [16, Theorem 2.3.8(iii)] states that

$$\nu(a) = \lim_{n \to \infty} \|a^n\|^{1/n}.$$

For example, the spaces ℓ^p (for $p \geq 1$), ℓ^∞, and c_0 are Banach algebras with respect to the product defined by coordinatewise multiplication; indeed, they are *Banach sequence algebras* in the sense of [16, §4.1].

Let E be a Banach space. A *Banach operator algebra* is a subalgebra $\mathfrak{A}$ of $\mathcal{B}(E)$ containing $\mathcal{F}(E)$ such that $\mathfrak{A}$ is a Banach algebra with respect to a norm, say $\|\!|\cdot|\!\|$; necessarily $\|\!|T|\!\| \geq \|T\|$ $(T \in \mathfrak{A})$. For example, $(\mathcal{K}(E), \|\cdot\|)$, $(\mathcal{B}(E), \|\cdot\|)$, and $(\mathcal{N}(E), \nu)$ are Banach operator algebras. The spectrum $\sigma(T)$ of $T \in \mathcal{K}(E)$ is always either finite or a sequence converging to 0, together with $\{0\}$. See [16, §2.5], for example.

For each compact space K (always assumed to be Hausdorff), the algebra $C(K)$ is a commutative, unital C^*-algebra, and each commutative C^*-algebra A has the form $C_0(\Phi_A)$, where Φ_A is the (locally compact) character space of A.

We quote the following form of the *Banach–Stone theorem*, as stated in [27, V.8.8], for example. For an account of related results, see [22, Chapter 2] and [30].

Theorem 1.10. *Let K and L be two compact spaces, and suppose that $T : C(K) \to C(L)$ is an isometric isomorphism. Then there is a homeomorphism $\eta : L \to K$ and a function*

$h \in C(L)$ with $h(L) \subset \mathbb{T}$ such that

$$(Tg)(x) = h(x)(g \circ \eta)(x) \quad (g \in C(K),\ x \in L). \ \blacksquare$$

A related result is described in [30, §3.2], which is largely an exposition of results of Lamperti [51]; see [30, Theorem 3.2.5]. We first recall some background.

Let (Ω_1, μ_1) and (Ω_2, μ_2) be measure spaces. A map σ from the measurable subsets of Ω_1 to the measurable subsets of Ω_2 is defined to be a *regular set isomorphism* if $\sigma(\Omega_1 \setminus X) = \sigma(\Omega_1) \setminus \sigma(X)$ for each measurable subset of Ω_1, if $\sigma(\bigcup X_n) = \bigcup \sigma(X_n)$ for pairwise-disjoint families $\{X_n : n \in \mathbb{N}\}$ of measurable subsets of Ω_1, and, for a measurable subset X of Ω_1, $\sigma(X)$ is a μ_2-null set if and only if X is a μ_1-null set. In the case where Ω is discrete, such a map $\sigma : \Omega \to \Omega$ is just a permutation of Ω. A regular set isomorphism σ induces a unique linear map T_σ on the space of measurable functions on Ω such that $T_\sigma(\chi_X) = \chi_{\sigma(X)}$ for all measurable subsets X of Ω and $T_\sigma(fg) = T_\sigma(f) \cdot T_\sigma(g)$ for all measurable functions f and g on Ω.

THEOREM 1.11 (Lamperti). *Let (Ω_1, μ_1) and (Ω_2, μ_2) be measure spaces. Suppose that $p \in [1, \infty)$ with $p \neq 2$. Then an isometric isomorphism U from $L^p(\Omega_1, \mu_1)$ to $L^p(\Omega_2, \mu_2)$ has the form*

$$U : f \mapsto h \cdot T_\sigma f, \qquad L^p(\Omega_1, \mu_1) \to L^p(\Omega_2, \mu_2), \tag{1.20}$$

where $h : \Omega \to \mathbb{C}$ is such that

$$\int_{\sigma(X)} |h|^p \, \mathrm{d}\mu_2 = \mu_1(X)$$

for each measurable subset X of Ω, and $T_\sigma \in \mathcal{B}(L^p(\Omega_1), L^p(\Omega_2))$ is induced by a regular set isomorphism σ. $\blacksquare$

In the case where $L^p(\Omega_1, \mu_1) = L^p(\Omega_2, \mu_2) = \ell^p$, the function $h : \mathbb{N} \to \mathbb{C}$ is such that $|h(i)| = 1$ $(i \in \mathbb{N})$.

A Hausdorff topological space X is *extremely disconnected* if the closure of every open set is itself open; this is equivalent to requiring that, for every pair $\{U, V\}$ of open sets in X with $U \cap V = \emptyset$, we have $\overline{U} \cap \overline{V} = \emptyset$. A compact, extremely disconnected space is called a *Stonean space*. A Stonean space has a basis for its topology consisting of clopen sets. By definition, a compact space K such that $C(K)$ is isometrically isomorphic to the dual of a Banach space is a *hyper-Stonean space* (in which case the predual is unique up to isometric isomorphism). A hyper-Stonean space is Stonean. For a discussion and further characterizations of Stonean and hyper-Stonean spaces, see [22, Theorems 2.5 and 2.9].

Let Ω be a measure space, and consider the Banach space $L^\infty(\Omega)$. This is a commutative C^*-algebra for the pointwise product (defined almost everywhere), and so this space is isometrically isomorphic by the Gel'fand transform to $C(\Phi)$ for a certain compact space Φ; the identification is an isomorphism of commutative C^*-algebras. Thus, when Ω is σ-finite, the dual space and the second dual space of $L^1(\Omega)$ are isometrically isomorphic to $C(\Phi)$ and $M(\Phi)$, respectively. In the case where $\Omega = S$ is discrete, we have $\Phi = \beta S$, the Stone–Čech compactification of the set S; we shall sometimes identify ℓ^∞ with the space $C(\beta\mathbb{N})$.

Let K be a non-empty, locally compact space. We shall also identify the space $M(K)' = C_0(K)''$ as $C(\widetilde{K})$, where $\widetilde{K}$ is a certain hyper-Stonean space, called the *hyper-Stonean envelope* of K in [22]; in particular, $\widetilde{K}$ is compact and extremely disconnected. Thus we are identifying $M(K)''$ with $M(\widetilde{K})$. For further details of $\widetilde{K}$ and these identifications, see [22].

1.2.9. Hermitian elements. We shall require some notions concerned with numerical ranges and hermitian elements of a Banach algebra.

Let A be a unital Banach algebra, with identity e_A. Then the *state space* of A is

$$S(A) = \{\lambda \in A' : \|\lambda\| = \langle e_A,\, \lambda\rangle = 1\}.$$

Clearly $S(A)$ contains the character space Φ_A, and so is non-empty; it is convex and closed in the weak-$*$ topology. The *numerical range* of $a \in A$ is

$$V(A, a) = \{\langle a,\, \lambda\rangle : \lambda \in S(A)\}.$$

See [13, 14]. We see that $V(A, a) \supset \sigma(a)$, the spectrum of a.

Let E be a Banach space. Then

$$\Pi(E) = \{(x,\, \lambda) \in E \times E' : \|x\| = \|\lambda\| = \langle x,\, \lambda\rangle = 1\}.$$

Take $T \in \mathcal{B}(E)$. Then the *spatial numerical range* of T is

$$V(T) = \{\langle Tx,\, \lambda\rangle : (x, \lambda) \in \Pi(E)\}.$$

Clearly, $V(T) \subset V(\mathcal{B}(E), T)$, and, in fact, $V(\mathcal{B}(E), T) = \overline{\mathrm{co}}\, V(T)$ [13, §9, Theorem 4(i)].

Definition 1.12. Let $(A, \|\cdot\|)$ be a unital Banach algebra. Then an element $a \in A$ is *hermitian* if $\|\exp(\mathrm{i}ta)\| = 1$ for all $t \in \mathbb{R}$.

The following result is basic; see [13, §5] and [30, Theorem 5.2.6] for other equivalences.

Proposition 1.13. (i) *Let A be a unital Banach algebra, and take $a \in A$. Then a is hermitian if and only if $V(A, a) \subset \mathbb{R}$.*

(ii) *Let E be a Banach space, and take $T \in \mathcal{B}(E)$. Then T is hermitian if and only if $V(T) \subset \mathbb{R}$.* ∎

The following result is close to [14, §29, Theorem 3]. Let K be a compact space. Point evaluation at $x \in K$ is denoted by ε_x.

Theorem 1.14. *Let K be a compact space, and let T be a hermitian operator on $C(K)$. Then there exists an element $h \in C_{\mathbb{R}}(K)$ such that $Tf = hf$ $(f \in C(K))$.*

Proof. We define $h = T(1) \in C(K)$.

Let $g \in C(K)$ with $|g|_K = 1$, and take $x \in K$ with $|g(x)| = 1$. Then clearly we have $\langle g,\, \varepsilon_x\rangle \in \Pi(C(K))$, and so $\langle Tg,\, \varepsilon_x\rangle \in V(T) \subset \mathbb{R}$. In particular, $h \in C_{\mathbb{R}}(K)$.

Now take $f \in C_{\mathbb{R}}(K)$ with $|f|_K = 1$, and write $f = f^+ - f^-$, where $f^+, f^- \in C_{\mathbb{R}}(K)^+$. Suppose that $x \in K$ with $f(x) = 0$, so that $f^+(x) = 0$ and $f(K) \subset \mathbb{I}$. Then

$$|1 - f^+|_K = |(1 - f^+)(x)| = 1,$$

and so $T(1 - f^+)(x) \in \mathbb{R}$, whence $(Tf^+)(x) \in \mathbb{R}$. Similarly, $(Tf^-)(x) \in \mathbb{R}$, and so $(Tf)(x) \in \mathbb{R}$. Next set

$$v = (1 - f^2)^{1/2} \quad \text{and} \quad g = v + \mathrm{i}f,$$

so that $v, g \in C(K)$. Further, $v(x) = g(x) = 1$ and $|v|_K = |g|_K = 1$, and so we have both $(Tv)(x) \in \mathbb{R}$ and $(Tg)(x) \in \mathbb{R}$. Since $Tg = Tv + \mathrm{i}Tf$, it follows that $(Tf)(x) = 0$. Thus, by scaling, we see that $(Tf)(x) = 0$ whenever $f \in C_{\mathbb{R}}(K)$ with $f(x) = 0$.

Next, take an arbitrary $f \in C_{\mathbb{R}}(K)$ and $x \in K$. Then $(f - f(x)1)(x) = 0$, and so $T(f - f(x)1)(x) = 0$. This says that $Tf(x) = h(x)f(x)$. Hence $Tf = hf \in C_{\mathbb{R}}(K)$.

Finally, take $f \in C(K)$, say $f = f_1 + \mathrm{i}f_2$, where $f_1, f_2 \in C_{\mathbb{R}}(K)$. Then

$$Tf = T(f_1 + \mathrm{i}f_2) = h(f_1 + \mathrm{i}f_2) = hf. \ \blacksquare$$

An elementary argument, given in [14, §29] gives the following, related result; the result is due to Tam [70].

THEOREM 1.15. *Suppose that* $p \in [1, \infty]$ *with* $p \neq 2$. *Then each hermitian operator on* ℓ^p *has the form* $\alpha \mapsto \beta\alpha$ *for some* $\beta \in \ell^\infty_{\mathbb{R}}$. $\blacksquare$

1.3. Banach lattices. There is a strong connection between the old theory of Banach lattices and our new theory of multi-Banach spaces. This will be explained in Chapter 4, §4.3. Here we recall briefly some basic notions of the theory of Banach lattices; for details, see [1], [7, Chapter 4], [52, Volume II], [55], and [67]. In fact, we choose forms of the standard definitions and notations that are most convenient for us.

1.3.1. Definitions. Let $(S, \leq)$ be a partially ordered set. For $x, y \in S$, the *order-interval* $[x, y]$ is the set $\{z \in S : x \leq z \leq y\}$, and a subset T of S is *order-bounded* if there exist $x, y \in S$ such that $T \subset [x, y]$. A net $(x_\alpha : \alpha \in A)$ in S is *order-bounded* if $\{x_\alpha : \alpha \in A\}$ is order-bounded. Further, $(x_\alpha : \alpha \in A)$ is *increasing* (respectively, *decreasing*) if $x_\alpha \leq x_\beta$ (respectively, $x_\alpha \geq x_\beta$) whenever $\alpha \leq \beta$ in A. We write

$$x_\alpha \downarrow x \quad \text{and} \quad x_\alpha \uparrow x$$

if (x_α) is a decreasing net in S and $x = \inf\{x_\alpha : \alpha \in A\}$, or if (x_α) is an increasing net in S and $x = \sup\{x_\alpha : \alpha \in A\}$, respectively.

DEFINITION 1.16. A partially ordered set $(S, \leq)$ is a *lattice* if, for each pair $\{s, t\}$ of elements of S, there is a supremum, denoted by $s \vee t$, and an infimum, denoted by $s \wedge t$.

A lattice is *Dedekind complete* (respectively, *σ-Dedekind complete*) if every non-empty (respectively, every countable, non-empty) subset which is bounded above has a supremum and every non-empty (respectively, every countable, non-empty) subset which is bounded below has an infimum.

The supremum and infimum of a non-empty subset S of a lattice E are denoted by $\bigvee S$ and $\bigwedge S$, respectively (if they exist). Suppose that $x_0 = \bigvee S$. Then the family $\mathcal{F}$ of finite subsets of S forms a directed set when ordered by inclusion; in this case, set $x_F = \bigvee\{y : y \in F\}$ for $F \in \mathcal{F}$. Then $\{x_F : F \in \mathcal{F}\}$ is a net and $x_F \uparrow x_0$.

Let E be a linear space over the real field $\mathbb{R}$ such that $(E, \leq)$ is also a partially ordered set for an order $\leq$. Then E is an *ordered linear space* if the linear space and order structures are compatible, in the sense that:

(i) $x + z \leq y + z$ whenever $x, y, z \in E$ and $x \leq y$;
(ii) $\alpha x \leq \alpha y$ whenever $\alpha \in \mathbb{R}^+$ and $x, y \in E$ with $x \leq y$.

DEFINITION 1.17. An ordered linear space E is a *Riesz space* if $(E, \leq)$ is a lattice.

Let $(E, \leq)$ be a Riesz space. Then the operations $(x, y) \mapsto x \vee y$ and $(x, y) \mapsto x \wedge y$ are the *lattice operations*. A linear subspace F of E is a *sublattice* if $x \vee y, x \wedge y \in F$ whenever $x, y \in F$. The *positive cone* of E is

$$E^+ = \{x \in E : x \geq 0\}.$$

The ordering on a Riesz space is determined by E^+. For $x \in E$, set

$$x^+ = x \vee 0, \qquad x^- = (-x) \vee 0, \qquad |x| = x \vee (-x);$$

thus x^+, x^-, and $|x|$ are the *positive part*, the *negative part*, and the *modulus* of x, respectively. Elements x and y of E are *disjoint*, written $x \perp y$, if $|x| \wedge |y| = 0$. Two subsets S and T of E are *disjoint*, written $S \perp T$, if $x \perp y$ whenever $x \in S$ and $y \in T$.

For each non-empty set S, the space $\mathbb{R}^S$ is a Riesz space with the pointwise lattice operations, and the definitions of $|f|$, etc., coincide with the ones given on page 10.

Let E be a Riesz space. Here are some elementary consequences of the above definitions; they hold for all $x, y, z \in E$ and $\alpha \in \mathbb{R}$:

$$x = x^+ - x^-; \qquad |x| = x^+ + x^-; \qquad |\alpha x| = |\alpha|\,|x|; \qquad |x + y| \leq |x| + |y|.$$

PROPOSITION 1.18. *Let E be a Riesz space, and take $x, y, z \in E$. Then:*

(i) $x + y = x \vee y + x \wedge y$;
(ii) $(x \vee y) + z = (x + z) \vee (y + z)$;
(iii) $x \perp y$ *if and only if* $|x| \vee |y| = |x| + |y|$, *and then* $|x + y| = |x| + |y|$;
(iv) $|x| \vee |y| = (|x + y| + |x - y|)/2$;
(v) $\alpha x + \beta y \leq x \vee y$ *whenever* $\alpha, \beta \in \mathbb{I}$ *with* $\alpha + \beta = 1$. ■

An element e in a Riesz space E is an *order-unit* if, for each $x \in E$, there exists $\alpha > 0$ such that $|x| \leq \alpha e$.

A net $(x_\alpha : \alpha \in A)$ in a Riesz space E is *order-convergent* to $x \in E$ if there exists a net $(y_\alpha : \alpha \in A)$ and $\alpha_0 \in A$ such that $|x_\alpha - x| \leq y_\alpha$ $(\alpha \geq \alpha_0)$ and $y_\alpha \downarrow 0$; in this case, the element x is the *order-limit* of $(x_\alpha : \alpha \in A)$, and we write

$$x = \operatorname*{o-lim}_\alpha x_\alpha.$$

An order-limit is unique. A net $(x_\alpha : \alpha \in A)$ is *order-null* if

$$\operatorname*{o-lim}_\alpha x_\alpha = 0,$$

and a subset T of E is *order-closed* if $x \in T$ whenever $(x_\alpha : \alpha \in A)$ is a net in T with $x = \text{o-lim}_\alpha x_\alpha$. For a discussion of the notion of order-convergence of nets in a Riesz space, see [3, 45].

Let $(E, \leq)$ be a Riesz space. A subset S of E is *solid* if $x \in S$ whenever $x \in E$ and $|x| \leq |y|$ for some $y \in S$; a solid linear subspace of E is an *order-ideal* in E. Clearly each order-ideal in E is a sublattice of E. Let F be an order-ideal in E, and let $\pi : E \to E/F$ be the quotient map. Then the space E/F, with positive cone $\pi(E^+)$, is a Riesz space. An order-closed order-ideal in E is a *band*. Suppose that $E = E_1 \oplus \cdots \oplus E_n$ is a direct-sum decomposition, where each of $E_1, \ldots, E_n$ is an order-ideal. Then each of $E_1, \ldots, E_n$ is a band, and the decomposition is a *band decomposition*.

It is clear that a Riesz space $(E, \leq)$ is Dedekind complete if every non-empty subset which is bounded above has a supremum.

Let $(E, \leq)$ and $(F, \leq)$ be two Riesz spaces. An operator $T \in \mathcal{L}(E, F)$ is an *order-homomorphism* if

$$T(x \vee y) = Tx \vee Ty \quad (x, y \in E);$$

a bijective order-homomorphism is an *order-isomorphism*, and then $(E, \leq)$ and $(F, \leq)$ are *order-isomorphic*. We see easily that the operator T is an order-homomorphism if and only if $T(|x|) = |Tx|$ $(x \in E)$.

DEFINITION 1.19. Let $(E, \leq)$ be a Riesz space. A norm $\|\cdot\|$ on E is a *lattice norm* if $\|x\| \leq \|y\|$ whenever $x, y \in E$ with $|x| \leq |y|$. A *normed Riesz space* is a Riesz space equipped with a lattice norm. A *real Banach lattice* is a normed Riesz space which is a real Banach space with respect to the norm.

For example, the spaces $L^p_{\mathbb{R}}(\Omega, \mu)$ for $p \geq 1$ and $L^\infty_{\mathbb{R}}(\Omega, \mu)$ for a measure space (Ω, μ) and the spaces $C_{0,\mathbb{R}}(K)$ for a non-empty, locally compact space K are real Banach lattices with respect to the pointwise lattice operations. In the case where K is compact, the constant function 1 is an order-unit of $C_{\mathbb{R}}(K)$.

In a normed Riesz space $(E, \|\cdot\|, \leq)$, we have

$$\|x\| = \|\,|x|\,\| \quad (x \in E);$$

further, the lattice operations are uniformly continuous, and so the positive cone E^+ and each order-interval $[x, y]$ in E are closed in $(E, \|\cdot\|)$.

Let E and F be normed Riesz spaces, and take $T \in \mathcal{L}(E, F)$. Then T is an *order-isometry* if it is an order-homomorphism and an isometry; if there is such a map which is a bijection, E and F are *order-isometric*.

The *functional calculus* or *Krivine calculus* for a real Banach lattice E is described in [52, II, §1.d], for example. Indeed, a function $f : \mathbb{R}^n \to \mathbb{R}$ is *homogeneous of degree* 1 if

$$f(\alpha t_1, \dots, \alpha t_n) = \alpha f(t_1, \dots, t_n) \quad (\alpha \in \mathbb{R}^+,\ t_1, \dots, t_n \in \mathbb{R}).$$

The lattice of all such continuous functions is denoted by $\mathcal{H}_n$. Then, by [26, Chapter 16] or [52, II, Theorem 1.d.1], for each $x_1, \dots, x_n \in E$, there is a unique order-homomorphism $\tau : \mathcal{H}_n \to E$ such that $\tau(Z_i) = x_i$ $(i \in \mathbb{N}_n)$. In particular, for $x_1, \dots, x_n \in E$, we can define

$$\left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \in E$$

for each $p \geq 1$.

1.3.2. Complexifications. Suppose that $(E_{\mathbb{R}}, \|\cdot\|)$ is a real Banach lattice. Then we make the following definitions. Take $z \in E$, say $z = x + \mathrm{i}y$, where $x, y \in E_{\mathbb{R}}$, and first define the *modulus* $|z| \in E^+$ of z by

$$|z| = \bigvee \{x \cos\theta + y \sin\theta : 0 \leq \theta \leq 2\pi\} = (|x|^2 + |y|^2)^{1/2}. \tag{1.21}$$

We see that, for $\alpha \in \mathbb{C}$ and $z, w \in E$, we have: $|z| = 0$ if and only if $z = 0$; $|\alpha z| = |\alpha|\,|z|$; $|z + w| \le |z| + |w|$. Next, define

$$\|z\| = \|\,|z|\,\| \qquad (z \in E).$$

Then $\|\cdot\|$ is a norm on E, and $(E, \|\cdot\|)$ is a Banach space. In fact, we have

$$\frac{1}{2}(\|x\| + \|y\|) \le \|z\| \le \|x\| + \|y\| \qquad (z = x + \mathrm{i}y \in E).$$

For details of these remarks, see [1, §3.2], [67, Chapter II, §11], and [75].

The above complexification of a real Banach lattice is defined to be a (complex) *Banach lattice* [1, §3.2]. We denote such a Banach lattice by E or $(E, \|\cdot\|, \le)$, although, strictly, the order $\le$ is only defined on the real part, $E_{\mathbb{R}}$, of E. For example, the spaces $L^p(\Omega, \mu)$ for $p \ge 1$ and $L^\infty(\Omega, \mu)$ for a measure space (Ω, μ) and the spaces $C_0(K)$ for a non-empty, locally compact space K are Banach lattices which are the complexifications of the analogous real Banach lattices.

We write E^+ for $E_{\mathbb{R}}^+$, and set $E_{[1]}^+ = \{x \in E^+ : \|x\| \le 1\}$. For $v \in E^+$, we set

$$\Delta_v = \{z \in E : |z| \le v\}.$$

Let E be a Banach lattice. Again, elements z and w of E are *disjoint*, written $z \perp w$, if $|z| \wedge |w| = 0$, and so $z \perp w$ if and only if $|z| \vee |w| = |z| + |w|$. In this case, take $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$ in E. Then we see that

$$\begin{aligned} |z + w| &= \bigvee\{(x + u)\cos\theta + (y + v)\sin\theta\} \\ &= \bigvee\{|x\cos\theta + y\sin\theta| \vee |u\cos\theta + v\sin\theta|\} \\ &= \bigvee\{|x\cos\theta + y\sin\theta|\} \vee \bigvee\{|u\cos\theta + v\sin\theta|\}, \end{aligned}$$

where we are taking suprema over $\theta \in [0, 2\pi]$, and so

$$|z + w| = |z| \vee |w| = |z| + |w|. \tag{1.22}$$

A sequence (z_i) in E is *pairwise-disjoint* if $z_i \perp z_j$ for $i, j \in \mathbb{N}$ with $i \ne j$.

Two subsets S and T of E are *disjoint*, written $S \perp T$, if $z \perp w$ whenever $z \in S$ and $w \in T$. The *disjoint complement* $S^\perp$ of a non-empty subset S of E is defined by

$$S^\perp = \{w \in E : w \perp z \ (z \in S)\}.$$

Note that $S \cap S^\perp \subset \{0\}$.

Let $(E, \|\cdot\|, \le)$ be a Banach lattice. A subset S is *order-bounded* if $\{|z| : z \in S\}$ is order-bounded in $E_{\mathbb{R}}$; this holds if and only if there exists $x \in E^+$ with $|z| \le x$ $(z \in S)$. Similarly, we define *solid subsets*, *order-closed subsets*, *order-ideals*, and *bands*. It is easy to see that a subset of E is a subspace (respectively, an order-ideal) if and only if it has the form $V \oplus \mathrm{i}V$, where V is a real subspace (respectively, order-ideal) in $E_{\mathbb{R}}$.

The smallest band containing a subset A of E is denoted by $B(A)$, and we also set $B_x = B(\{x\})$ for $x \in E$; the latter set is a *principal band*. An element $x \in E^+$ is a *weak order unit* if $B_x = E$. A band B in E is a *projection band* if there exists a projection $P \in \mathcal{B}(E)$ with $P(E) = B$ and $0 \le Px \le x$ $(x \in E^+)$, and then $E = B \oplus_\perp B^\perp$.

It follows from (c) and (d) of [7, pp. 259–260] that every separable Banach lattice contains a weak order unit. In particular, the Banach lattices $L^p(\Omega,\mu)$ contain a weak order unit whenever $p \in [1,\infty)$ and the measure space is σ-finite.

Let $E = E_1 \oplus \cdots \oplus E_n$ be a direct sum decomposition of a Banach lattice E. This decomposition is a *band decomposition* if each of $E_1,\dots,E_n$ is a band, or equivalently, if $E_i \perp E_j$ whenever $i,j \in \mathbb{N}_n$ and $i \neq j$. We then write

$$E = E_1 \oplus_\perp \cdots \oplus_\perp E_n.$$

In this case, each of $E_1,\dots,E_n$ is a projection band, and, using [1, Theorem 1.34], each $P_i : E \to E_i$ is a contraction with

$$|P_i x| = P_i(|x|) \le |x| \quad (x \in E,\ i \in \mathbb{N}_n). \tag{1.23}$$

Further,

$$\sum_{i=1}^n |x_i| = \left|\sum_{i=1}^n x_i\right| \quad (x_i \in E_i,\ i \in \mathbb{N}_n). \tag{1.24}$$

Indeed, set $x = \sum_{i=1}^n x_i$. Then $\sum_{i=1}^n |x_i| = \sum_{i=1}^n |P_i x| = \sum_{i=1}^n P_i(|x|)$.

Suppose that $E = E_1 \oplus_\perp \cdots \oplus_\perp E_n$, and take $z_i \in E_i$ for $i \in \mathbb{N}_n$. Then we have

$$|z_1 + \cdots + z_n| = |z_1| \vee \cdots \vee |z_n| = |z_1| + \cdots + |z_n|, \tag{1.25}$$

and so

$$\|z_1 + \cdots + z_n\| = \|\,|z_1| \vee \cdots \vee |z_n|\,\| = \|\,|z_1| + \cdots + |z_n|\,\|. \tag{1.26}$$

Definitions are carried over from real Banach lattices to Banach lattices in the obvious way; for example, a Banach lattice E is *Dedekind complete* if $(E_{\mathbb{R}},\le)$ is a Dedekind complete real Banach lattice.

In general, a Banach lattice is not necessarily Dedekind complete. Indeed, the Banach lattices $L^p(\Omega)$ are always Dedekind complete [55, Example (v), p. 9], but the Banach lattice $C(K)$ is Dedekind complete if and only if the compact space K is Stonean [16, Proposition 4.2.29(i)], [52, II, Proposition 1.a.4(ii)], [55, Proposition 2.1.4]; $C(K)$ is σ-Dedekind complete if and only if K is basically disconnected [52, II, Proposition 1.a.4(i)]. A simple example of a σ-Dedekind complete space of the form $C(K)$ which is not Dedekind complete is the subspace of $\ell^\infty(S)$, for S an uncountable set, spanned by the constant functions and the functions with countable support.

We shall use the following theorem of F. Riesz; see [7, Theorems 3.8 and 3.13], [55, Theorem 1.2.9 and Proposiiton 1.2.11], and [67, Chapter II, §2].

PROPOSITION 1.20.

(i) *Every band in a Dedekind complete Riesz space is a projection band.*
(ii) *Every principal band in a σ-Dedekind complete Riesz space is a projection band.* ■

Suppose that E and F are Banach lattices. For each $T \in \mathcal{B}(E_{\mathbb{R}},F_{\mathbb{R}})$, we see that

$$\|T\| \le \|T_{\mathbb{C}}\| \le 2\|T\|,$$

and so $T_{\mathbb{C}} \in \mathcal{B}(E,F)$. Clearly each bounded linear operator from E to F has the form $S + \mathrm{i}T$, where $S,T \in \mathcal{B}(E_{\mathbb{R}},F_{\mathbb{R}})$.

1.3.3. Continuity, boundedness and completeness. We first define two properties related to order of the norm on a Banach lattice.

DEFINITION 1.21. Let $(E, \|\cdot\|)$ be a Banach lattice. The norm $\|\cdot\|$ is *order-continuous* if $\|x_\alpha\| \downarrow 0$ whenever (x_α) is a net in E such that $x_\alpha \downarrow 0$. The norm $\|\cdot\|$ is *σ-order-continuous* if $\|x_n\| \downarrow 0$ whenever (x_n) is a sequence in E such that $x_n \downarrow 0$.

Characterizations of order-continuous Banach lattices are given in [1, §2.3], [7, §12], and [55, §2.4]. For example, the spaces $L^p(\Omega)$ for $p \geq 1$ and Banach lattices which are reflexive as Banach spaces have order-continuous norms, but the norm $|\cdot|_K$ in $C(K)$ is order-continuous only if K is finite; the uniform norm on c_0 is order-continuous. Each Banach lattice with an order-continuous norm is Dedekind complete. The uniform norm on the Banach lattice $C(\mathbb{I})$ is not a σ-order-continuous norm; however, the uniform norm on the space $C([0, \omega_1])$ is σ-order-continuous, but not order-continuous. Suppose that K is Stonean and infinite. Then $C(K)$ is Dedekind complete, but the norm is not order-continuous.

Our final definitions in this area are the following. The terms 'monotonically complete' and 'Nakano property', are defined in [55, Definition 2.4.18(iii)], in [73], and in [7, Definition 14.10], but we have not seen the term 'monotonically bounded' in the literature.

DEFINITION 1.22. Let $(E, \|\cdot\|)$ be a Banach lattice. Then:

(i) E is *monotonically bounded* if every increasing net in $E^+_{[1]}$ is bounded above;
(ii) E is *monotonically complete* if every increasing net in $E^+_{[1]}$ has a supremum;
(iii) E has the *weak Nakano property* if there is a constant $K \geq 1$ such that, for every increasing, order-bounded net $(x_\alpha : \alpha \in A)$ in $E_\mathbb{R}$ and every $\varepsilon > 0$, the set $\{x_\alpha : \alpha \in A\}$ has an upper bound $u \in E_\mathbb{R}$ such that $\|u\| \leq K \sup_{\alpha \in A} \|x_\alpha\| + \varepsilon$;
(iv) E has the *weak σ-Nakano property* if there is a constant $K \geq 1$ such that, for every increasing, order-bounded sequence $(x_n : n \in \mathbb{N})$ in $E_\mathbb{R}$ and every $\varepsilon > 0$, the set $\{x_n : n \in \mathbb{N}\}$ has an upper bound $u \in E_\mathbb{R}$ such that $\|u\| \leq K \sup_{n \in \mathbb{N}} \|x_n\| + \varepsilon$;
(v) E has the *Nakano property* if it has the weak Nakano property with $K = 1$.

Trivially, every monotonically complete Banach lattice is monotonically bounded and Dedekind complete. A Banach lattice with an order-continuous norm has the Nakano property. We note the following result which is essentially [55, Proposition 2.4.19].

PROPOSITION 1.23. *A monotonically bounded Banach lattice has the weak Nakano property.* ■

A Banach lattice is said to be a *KB-space* if it is monotonically complete and has an order-continuous norm [2, p. 89]. Thus every KB-space is Dedekind complete, monotonically bounded, and has the Nakano property. The L^p spaces for $p \geq 1$ are examples of KB-spaces.

The Banach lattice c_0 is Dedekind complete and has the Nakano property, but it is not monotonically bounded because the increasing sequence $(\delta_1 + \cdots + \delta_n : n \in \mathbb{N})$ in $(c_{0,\mathbb{R}})_{[1]}$ has no upper bound, and hence c_0 is not monotonically complete.

Let K be a compact space. Then the Banach lattice $C(K)$ is monotonically complete if and only if it is Dedekind complete (if and only if K is Stonean), and so the Banach lattice $\ell^\infty \cong C(\beta\mathbb{N})$ is monotonically complete, but its norm is not order-continuous; $C(K)$ is always monotonically bounded; $C(K)$ has the Nakano property whenever K is Stonean. The Banach lattice $M(K)$ is monotonically complete.

EXAMPLE 1.24. For $K \geq 1$, the Banach lattice $(\ell^\infty, \|\cdot\|_K)$, where $\|\cdot\|_K$ is given by

$$\|(\alpha_n)\| = |(\alpha_n)|_{\mathbb{N}} + K \limsup_{n\to\infty} |\alpha_n| \qquad ((\alpha_n) \in \ell^\infty),$$

is monotonically complete and has the weak Nakano property, but not the Nakano property whenever $K > 1$. The Banach lattice $\ell^\infty((\ell^\infty, \|\cdot\|_K) : K \in \mathbb{N})$ is Dedekind complete, but it does not have the weak σ-Nakano property. ■

A Dedekind-complete lattice has the Nakano property if and only if the norm is a *Fatou norm*, in the sense of [1, p. 65] and [55, Definition 2.4.18]. In [2] and [5], a norm $\|\cdot\|$ on a Banach lattice E is said to be a *Levi norm* if $(E, \|\cdot\|)$ is monotonically complete.

1.3.4. Positive, regular, and order-bounded operators. Let E and F be real Banach lattices, and take $S, T \in \mathcal{L}(E, F)$. We define

$$S \leq T \quad \text{if} \quad Sx \leq Tx \qquad (x \in E^+).$$

Clearly, $(\mathcal{L}(E, F), \leq)$ is an ordered linear space.

DEFINITION 1.25. Let E and F be real Banach lattices, and consider $T \in \mathcal{L}(E, F)$. Then:

(i) T is *positive* if $T \geq 0$;
(ii) T is *regular* if $T = T_1 - T_2$, where T_1 and T_2 are positive operators;
(iii) T is *order-bounded* if $T(B)$ is an order-bounded subset of F for each order-bounded subset B of E.

The set of positive operators from E to F is closed under addition and multiplication by $\alpha \in \mathbb{R}^+$, and so it is a *cone*, denoted by $\mathcal{L}(E, F)^+$.

The book [7] is devoted to positive operators.

We shall (at least implicitly) use a basic theorem of Kantorovich [7, Theorem 1.7]: each additive map $T : E^+ \to F^+$ extends uniquely to a positive operator from E to F, and the unique extension T satisfies

$$Tx = T(x^+) - T(x^-) \qquad (x \in E).$$

Thus a positive operator T has been specified as soon as we know that $T : E^+ \to F^+$ is additive.

Let E be a σ-Dedekind complete Banach lattice. Then, for each $v \in E^+$, a projection P_v is defined by first setting

$$P_v(x) = \bigvee\{nv \wedge x : n \in \mathbb{N}\} \qquad (x \in E^+), \tag{1.27}$$

and then extending P_v by linearity to the whole of E; see [52, II, p. 8]. In this case, the map $P_v : E \to E$ is a positive linear projection with $\|P_v\| \leq 1$ for each $v \in E^+$. Note that $P_{|x|}(x) = x$ $(x \in E)$. In the special case where E is $L^p(\Omega)$ for $p \in [1, \infty]$ and a measure space Ω, the map P_v is just multiplication by the characteristic function of the set $\{t \in \Omega : v(t) \neq 0\}$.

The space of all regular operators from E to F is denoted by $\mathcal{L}_r(E,F)$. We see immediately that $(\mathcal{L}_r(E,F),\leq)$ is an ordered linear subspace of $(\mathcal{L}(E,F),\leq)$, with positive cone $\mathcal{L}(E,F)^+$.

The space of all order-bounded operators from E to F is denoted by $\mathcal{L}_b(E,F)$. Clearly $(\mathcal{L}_b(E,F),\leq)$ is an ordered linear subspace of $(\mathcal{L}(E,F),\leq)$ and

$$\mathcal{L}(E,F)^+ \subset \mathcal{L}_r(E,F) \subset \mathcal{L}_b(E,F) \subset \mathcal{L}(E,F).$$

Each order-bounded linear operator is continuous [1, p. 22], and so $\mathcal{L}_b(E,F) \subset \mathcal{B}(E,F)$. For this reason, we denote $\mathcal{L}(E,F)^+$, $\mathcal{L}_r(E,F)$, and $\mathcal{L}_b(E,F)$ by $\mathcal{B}(E,F)^+$, $\mathcal{B}_r(E,F)$, and $\mathcal{B}_b(E,F)$, respectively.

Now suppose that E and F are Banach lattices. In the case where $T : E_{\mathbb{R}} \to F_{\mathbb{R}}$ is a positive operator, we have $\|T_{\mathbb{C}}\| = \|T\|$ (but this is not necessarily true for all regular operators T [1, Exercise 9 of §3.2]). We shall use the following observation. Take $T \in \mathcal{B}(E,F)^+$. Then

$$\|T\| = \sup\{\|Tx\| : x \in E^+,\ \|x\| \leq 1\}. \tag{1.28}$$

An operator $S + \mathrm{i}T \in \mathcal{B}(E,F)$ is *regular* or *order-bounded* or *order-isometric* if both S and T are regular or order-bounded or order-isometric, respectively. Again, each order-bounded operator is continuous, and so we denote the spaces of all positive, all regular, and all order-bounded operators from E to F by $\mathcal{B}(E,F)^+$, $\mathcal{B}_r(E,F)$, and $\mathcal{B}_b(E,F)$, respectively. Thus we have

$$\mathcal{B}(E,F)^+ \subset \mathcal{B}_r(E,F) \subset \mathcal{B}_b(E,F) \subset \mathcal{B}(E,F).$$

We write $\mathcal{B}_r(E)$ and $\mathcal{B}_b(E)$ for $\mathcal{B}_r(E,E)$ and $\mathcal{B}_b(E,E)$, respectively.

An operator $T \in \mathcal{B}(E,F)$ is *order-continuous* if $Tx = \text{o-}\lim_\alpha T(x_\alpha)$ in F whenever $x = \text{o-}\lim_\alpha x_\alpha$ in E. By [3, Theorem 2.1], each such operator is order-bounded.

The following result is based on [72, §3].

PROPOSITION 1.26. *Let E and F be Banach lattices. Then, for each $T \in \mathcal{B}_b(E,F)$, there exists $c > 0$ such that, for each $v \in E^+$, there exists $w \in F^+$ with $T(\Delta_v) \subset \Delta_w$ and $\|w\| \leq c\|v\|$.*

Proof. Assume towards a contradiction that no such constant c exists. For each $n \in \mathbb{N}$, there exists $v_n \in E^+$ with $\|v_n\| = 1/2^n$ such that $\|w\| \geq n$ whenever $w \in F^+$ has the property that $|Tx| \leq w$ for each $x \in E$ with $|x| \leq v_n$. Take

$$v = \sum_{n=1}^{\infty} v_n \in E^+.$$

Then there exists $w_0 \in F^+$ such that $|Tx| \leq w_0$ whenever $x \in E$ with $|x| \leq v$. For each $n \in \mathbb{N}$, we have $v_n \leq v$, and so $|Tx| \leq w_0$ whenever $|x| \leq v_n$, whence $\|w_0\| \geq n$. This is the required contradiction. ∎

DEFINITION 1.27. Let E and F be Banach lattices, and let $T \in \mathcal{B}_b(E,F)$. Then the infimum of the constants c such that, for each $v \in E^+$, there exists $w \in F^+$ with $T(\Delta_v) \subset \Delta_w$ and $\|w\| \leq c\|v\|$, is denoted by $\|T\|_b$.

Now consider $T \in \mathcal{B}_r(E,F)$. The following definition is given in [55, Exercise 2.2.E2].

DEFINITION 1.28. Let E and F be Banach lattices. For $T \in \mathcal{B}_r(E,F)$, set

$$\|T\|_r = \inf\{\|S\| : S \in \mathcal{B}(E,F)^+,\ |Tz| \le S(|z|)\ (z \in E)\}.$$

PROPOSITION 1.29. *Let E and F be Banach lattices. Then:*

(i) *$\|\cdot\|_b$ is a norm on the space $\mathcal{B}_b(E,F)$ such that $\|T\|_b \ge \|T\|$ $(T \in \mathcal{B}_b(E,F))$, and $(\mathcal{B}_b(E,F), \|\cdot\|_b)$ is a Banach space;*

(ii) *$\|\cdot\|_r$ is a norm on $\mathcal{B}_r(E,F)$ with*

$$\|T\|_r \ge \|T\|_b \ge \|T\| \quad (T \in \mathcal{B}_r(E,F)),$$

and $(\mathcal{B}_r(E,F), \|\cdot\|_r)$ is a Banach space. ■

If $\mathcal{B}_r(E,F) = \mathcal{B}_b(E,F)$, then the norms $\|\cdot\|_r$ and $\|\cdot\|_b$ are equivalent on $\mathcal{B}_r(E,F)$, but examples in [72] shows that the norms are not necessarily equal in this case, and that, in general, the norms are not necessarily equivalent on $\mathcal{B}_r(E,F)$; Example 4.1 of [72] exhibits Banach lattices E and F and a compact, order-bounded operator $V : E \to F$ which is not even in the $\|\cdot\|_b$-closure of $\mathcal{B}_r(E,F)$. Examples with $\mathcal{B}_r(E,F) \subsetneq \mathcal{B}_b(E,F)$ and with $\mathcal{B}_b(E,F) \subsetneq \mathcal{B}(E,F)$ are given in [7, Examples 1.11 and 15.1]. An example given in [72, §2] shows that there may be operators in $\mathcal{B}_b(E,F)$ that are not even in the $\|\cdot\|$-closure of $\mathcal{B}_r(E,F)$.

The three clauses of the following theorem are taken from [9], from [1, Theorem 3.9] and [7, Theorem 15.3], and from [11], respectively.

THEOREM 1.30.

(i) *Let K be a compact space with weight smaller than the smallest strongly inaccessible cardinal. Then $\mathcal{B}_r(C(K)) = \mathcal{B}(C(K))$ if and only if K is Stonean.*

(ii) *Let Ω be a measure space. Then $\mathcal{B}_r(L^1(\Omega)) = \mathcal{B}(L^1(\Omega))$ and, further,*

$$\|T\|_r = \|T\| \quad (T \in \mathcal{B}(L^1(\Omega))).$$

(iii) *Let Ω be a measure space and take p with $1 < p < \infty$ such that $L^p(\Omega)$ is infinite-dimensional. Then $\mathcal{B}_r(L^p(\Omega))$ is not dense in $\mathcal{B}(L^p(\Omega))$, and $\|\cdot\|_r$ and $\|\cdot\|$ are not equivalent on $\mathcal{B}_r(L^p(\Omega))$.* ■

Let $T \in \mathcal{B}(E,F)^+$. Then

$$\|T\|_b = \|T\|_r = \|T\|. \tag{1.29}$$

We shall use the following standard theorem of F. Riesz and Kantorovich; see [1, Theorems 1.16, 1.32, 3.24, 3.25], [7, Theorems 1.10 and 1.13], [55, Propositions 1.3.6 and 2.2.6], and [67, Chapter 4, §1].

THEOREM 1.31. *Let E and F be real Banach lattices, with F Dedekind complete. Then $\mathcal{B}_r(E,F) = \mathcal{B}_b(E,F)$ is a Dedekind complete real Banach lattice for the lattice operations defined for $T \in \mathcal{B}_r(E,F)$ and $x \in E^+$ by*

$$T^+(x) = \sup\{Ty : y \in [0,x]\}, \qquad T^-(x) = \sup\{-Ty : y \in [0,x]\}.$$

Let $T_1, \dots, T_n \in \mathcal{B}_r(E,F)$ and $x \in E^+$. Then

$$(T_1 \vee \cdots \vee T_n)(x) = \bigvee\Big\{\sum_{i=1}^n T_i x_i : x_i \in E^+,\ x_1 + \cdots + x_n = x\Big\}. \tag{1.30}$$

Let E and F be Banach lattices, with F Dedekind complete. Then $\mathcal{B}_r(E,F) = \mathcal{B}_b(E,F)$ is a Dedekind complete Banach lattice, and

$$|T|(u) = \sup\{|Tz| : |z| \leq u\} \qquad (u \in E^+).$$

Further, $\|T\|_r = \|\,|T|\,\|$ and $|Tx| \leq |T|(|z|)$ $(z \in E)$ for $T \in \mathcal{B}_r(E,F)$. ■

1.3.5. The Banach algebra $\mathcal{B}_r(E)$. The following result is clear.

THEOREM 1.32. *Let E be a Banach lattice. Then $(\mathcal{B}_r(E), \|\cdot\|_r)$ and $(\mathcal{B}_b(E), \|\cdot\|_b)$ are unital Banach algebras.* ■

There appears to be surprisingly little about the Banach algebra $\mathcal{B}_r(E)$ in the literature; for example, it is not mentioned in [16]. There seems to be no mention of the Banach algebra $\mathcal{B}_b(E)$ at all.

DEFINITION 1.33. Let E be a Banach lattice, and take $T \in \mathcal{B}_b(E)$. The *order-spectrum*, $\sigma_o(T)$, of T is the spectrum of T with respect to the Banach algebra $(\mathcal{B}_b(E), \|\cdot\|_b)$. The corresponding *order-spectral radius* is denoted by $\nu_o(a)$.

Of course, $\sigma_o(T) \supset \sigma(T)$ and $\nu_o(T) \geq \nu(T)$ for each $T \in \mathcal{B}_b(E)$.

For a discussion of $\sigma_o(T)$ and $\nu_o(a)$, see [1, §7.4] and [55, §4.5]; in the latter source, and elsewhere, the order-spectrum is defined for $T \in \mathcal{B}_r(E)$ with respect to the Banach algebra $\mathcal{B}_r(E)$.

EXAMPLE 1.34. Let E be the Banach lattice $L^2(\mathbb{T})$, so that E is monotonically complete with order-continuous norm.

An example of Arendt [10] exhibits a positive, compact operator $T \in \mathcal{K}(E) \cap \mathcal{B}_r(E)$ (so that $\sigma(T) \subset \mathbb{R}$ is countable) such that $\sigma_o(T)$ contains the unit circle $\mathbb{T}$. The operator has the form

$$T_\mu : f \mapsto \mu \star f, \qquad L^2(\mathbb{T}) \to L^2(\mathbb{T}),$$

where μ is a certain singular measure on $\mathbb{T}$. Note the interesting fact that

$$\sigma_o(T_\mu) = \sigma_{M(\mathbb{T})}(\mu) \supsetneq \sigma(T_\mu).$$

It follows that there are compact operators on $L^2(\mathbb{T})$ which are not regular.

An example of Ando, which is discussed in [1, Example 7.36] and [55, p. 306], exhibits a Dedekind complete Banach lattice E with order-continuous norm and an operator $T \in \mathcal{B}_r(E)$ such that $\nu_o(T) > \nu(T)$. ■

1.3.6. Dual Banach lattices. Let E be a real Banach lattice, with dual space E'. Then E' is ordered by the requirement that $\lambda \in E'$ belongs to $(E')^+$ if and only if $\langle x, \lambda\rangle \geq 0$ $(x \in E^+)$, and then E' becomes a real Banach lattice with respect to the following definitions of $\lambda \vee \mu$ and $\lambda \wedge \mu$ for $\lambda, \mu \in E'$. In fact, $\lambda \vee \mu$ and $\lambda \wedge \mu$ are defined for $x \in E^+$ by

$$\begin{cases} \langle x, \lambda \vee \mu\rangle = \sup\{\langle y, \lambda\rangle + \langle z, \mu\rangle : y, z \in E^+,\ y + z = x\}, \\ \langle x, \lambda \wedge \mu\rangle = \inf\{\langle y, \lambda\rangle + \langle z, \mu\rangle : y, z \in E^+,\ y + z = x\}, \end{cases} \tag{1.31}$$

and then $\lambda \vee \mu$ and $\lambda \wedge \mu$ are extended to E'. The dual of a Banach lattice E is also a Banach lattice; this is the *dual Banach lattice* of E.

Let E be a real Banach lattice, and take $x \in E^+$ and $\lambda \in E'$. Then we have

$$\langle x,\, \lambda^+ \rangle = \sup\{\langle y,\, \lambda\rangle : 0 \le y \le x\}.$$

Let E be a Banach lattice. We note that

$$|\langle z,\, \lambda\rangle| \le \langle |z|,\, |\lambda|\rangle \qquad (z \in E,\ \lambda \in E'); \tag{1.32}$$

this is easily checked.

Let $(\lambda_\alpha : \alpha \in A)$ be a net in E', where E is a real Banach lattice, and suppose that $\lambda_\alpha \uparrow \lambda \in (E')^+$. Define $\mu(x) = \lim_\alpha \langle x,\, \lambda_\alpha\rangle$ $(x \in E)$. Then μ is a positive linear functional on E, and so $\mu \in E'$; $\lambda_\alpha \le \mu \le \lambda$ $(\alpha \in A)$, whence $\mu = \lambda$. It follows that

$$\langle x,\, \lambda_\alpha\rangle \uparrow \langle x,\, \lambda\rangle \qquad (x \in E^+). \tag{1.33}$$

A dual Banach lattice E' is monotonically complete and has the Nakano property; E' is always Dedekind complete, and so every band in E' is a projection band.

For example, let (Ω, μ) be a measure space, and take $E = L^p(\Omega, \mu)$, where $p \ge 1$, in the case where $E' = L^q(\Omega, \mu)$, where q is the conjugate index to p. Then the dual lattice operations on E' coincide with the given lattice operations on $L^q(\Omega, \mu)$.

Let K be a non-empty, locally compact space. Then $M(K) = C_0(K)'$ is a dual Banach lattice, and

$$(\mu \vee \nu)(S) = \sup\{\mu(S_1) + \nu(S_2)\}, \qquad (\mu \wedge \nu)(S) = \inf\{\mu(S_1) + \nu(S_2)\}$$

for $\mu, \nu \in M_{\mathbb{R}}(K)$ and a measurable subset S of K, where the supremum and infimum are taken over all ordered partitions (S_1, S_2) of S. Let $\mu, \nu \in M(K)$. Then $\mu \perp \nu$ in the Banach lattice $M(K)$ if and only if $|\mu| \wedge |\nu| = 0$, so that μ and ν are mutually singular in the classical sense of measures. We see that the following are equivalent:

(a) $\mu \perp \nu$;
(b) $\|\mu\| + \|\nu\| = \|\mu + \nu\| = \|\mu - \nu\|$;
(c) $\|\,|\mu| + |\nu|\,\| = \|\,|\mu| \vee |\nu|\,\|$.

For example, $M(K) = M_d(K) \oplus_\perp M_c(K)$ is a band decomposition.

We shall use the following proposition.

Proposition 1.35. *Let E be a Banach lattice, and take $x \in E^+$, $\lambda \in E'$, and $\varepsilon > 0$. Then there exists $z \in E$ such that*

$$|z| \le x \quad \textit{and} \quad \langle z,\, \lambda\rangle > \langle x,\, |\lambda|\rangle - \varepsilon.$$

Proof. We write $\lambda = \mu + \mathrm{i}\nu$, where $\mu, \nu \in (E_{\mathbb{R}})'$. By the definition, we have

$$|\lambda| = \bigvee\{\mu\cos\theta + \nu\sin\theta : 0 \le \theta \le 2\pi\},$$

and so there exist $\theta_1, \dots, \theta_n \in [0, 2\pi]$ such that

$$\langle x,\, (\mu\cos\theta_1 + \nu\sin\theta_1) \vee \cdots \vee (\mu\cos\theta_n + \nu\sin\theta_n)\rangle > \langle x,\, |\lambda|\rangle - \varepsilon.$$

By extending the definition in (1.31), we see that there exist $u_1, \dots, u_n \in E^+$ such that $u_1 + \cdots + u_n = x$ and

$$\langle u_1,\, \mu\cos\theta_1 + \nu\sin\theta_1\rangle + \cdots + \langle u_n,\, \mu\cos\theta_n + \nu\sin\theta_n\rangle > \langle x,\, |\lambda|\rangle - \varepsilon.$$

Thus

$$\sum_{j=1}^{n}\langle(\cos\theta_j)u_j,\,\mu\rangle+\sum_{j=1}^{n}\langle(\sin\theta_j)u_j,\,\nu\rangle>\langle x,\,|\lambda|\rangle-\varepsilon. \tag{1.34}$$

Set

$$w=\sum_{j=1}^{n}(\cos\theta_j-\mathrm{i}\sin\theta_j)u_j\in E.$$

Then (1.34) states that $\Re\,\langle w,\,\lambda\rangle>\langle x,\,|\lambda|\rangle-\varepsilon$, and so $|\langle w,\,\lambda\rangle|>\langle x,\,|\lambda|\rangle-\varepsilon$. For each $\theta\in[0,2\pi]$, we have

$$\sum_{j=1}^{n}(\cos\theta\cos\theta_j-\sin\theta\sin\theta_j)u_j=\sum_{j=1}^{n}\cos(\theta+\theta_j)u_j,$$

and hence

$$|w|=\sup\Big\{\sum_{j=1}^{n}\cos(\theta+\theta_j)u_j:0\le\theta\le2\pi\Big\}\le\sum_{j=1}^{n}u_j=x.$$

Finally, set $z=\zeta w$, where $\zeta\in\mathbb{T}$ is chosen to be such that $\zeta\langle w,\,\lambda\rangle=|\langle w,\,\lambda\rangle|$. Then $|z|=|w|\le x$ and $\langle z,\,\lambda\rangle>\langle x,\,|\lambda|\rangle-\varepsilon$, as required. ∎

Let $E=E_1\oplus_\perp\cdots\oplus_\perp E_n$ be a band decomposition of a Banach lattice E. Then the corresponding decomposition of E' is a band decomposition, so that

$$E'=E_1'\oplus_\perp\cdots\oplus_\perp E_n'. \tag{1.35}$$

However, in general, it is not true that every band decomposition of E' arises in this way.

1.3.7. *AL* and *AM* spaces. We now define some special types of Banach lattices.

Definition 1.36. A real Banach lattice $(E,\|\cdot\|)$ is: an *AL-space* if

$$\|x+y\|=\|x\|+\|y\|\quad\text{whenever}\quad x,y\in E^+\text{ with }x\wedge y=0;$$

an *AL_p-space* (for $p\ge1$) if

$$\|x+y\|^p=\|x\|^p+\|y\|^p\quad\text{whenever}\quad x,y\in E^+\text{ with }x\wedge y=0;$$

and an *AM-space* if

$$\|x\vee y\|=\max\{\|x\|,\|y\|\}\quad\text{whenever}\quad x,y\in E^+\text{ with }x\wedge y=0.$$

A Banach lattice is an *AL-space* or an *AL_p-space* or an *AM-space* if $E_{\mathbb{R}}$ has the appropriate property.

For example, each space of the form $L^p(\Omega,\mu)$, where (Ω,μ) is a measure space, is an AL_p-space, and each space $C_0(K)$, where K is a non-empty, locally compact space, is an AM-space.

Let E be a Banach lattice. Then E is an AL-space if and only if

$$\|x+y\|=\|x\|+\|y\|\qquad(x,y\in E^+), \tag{1.36}$$

and an AM-space if and only if

$$\|x\vee y\|=\max\{\|x\|,\|y\|\}\qquad(x,y\in E^+). \tag{1.37}$$

The following duality result is [7, Theorem 12.22], for example.

THEOREM 1.37. *Let E be a Banach lattice, with dual Banach lattice E'. Then E is an AL-space if and only if E' is an AM-space, and E is an AM-space if and only if E' is an AL-space.* ■

The following central representation theorem is proved in [1, Theorems 3.5 and 3.6], [7, Theorems 12.26 and 12.28], and [52, II. §1.b]. We shall call it *Kakutani's theorem*; detailed attributions for the various statements are given in [1].

THEOREM 1.38.

(i) *Take $p \geq 1$. A Banach lattice is an AL_p-space if and only if it is order-isometric to a Banach lattice of the form $L^p(\Omega, \mu)$, where (Ω, μ) is a measure space, and hence each AL_p-space has an order-continuous norm and is Dedekind complete.*
(ii) *A Banach lattice is an AM-space if and only if it is order-isometric to a closed sublattice of a space $C(K)$, where K is a compact space.* ■

COROLLARY 1.39. *Let (Ω, μ) be a measure space. Then there is an order-isomorphism θ from the dual space of $L^1(\Omega, \mu)$ onto $C(K)$ for some compact space K, and the restriction of θ to $L^\infty(\Omega, \mu)$ is the Gel'fand identification of $L^\infty(\Omega, \mu)$ with a C^*-subalgebra of $C(K)$.* ■

COROLLARY 1.40. *Let K be a non-empty, locally compact space. Then $M(K)$ is order-isometric to the space $L^1(\Omega, \mu)$ for some measure space (Ω, μ).* ■

We also mention a related result from [73]. Let E be a Banach lattice. Then E is an AM-space with the Nakano property if and only if E is order-isometric to $C_0(K)$ for some locally compact space K.

1.4. Summary. In Chapter 2, we shall begin with our axiomatic definitions of multi-normed spaces and of their relatives, the dual multi-normed spaces; we shall obtain some immediate consequences and some characterizations. In particular, we shall show that, of course, the concept of a 'dual multi-normed space' is dual to that of 'multi-normed space'. We shall give alternative characterizations of multi-normed spaces in terms of matrices and of tensor products, and we shall show that our notion of a multi-normed space coincides with that of spaces satisfying 'condition (P)' of Pisier.

In Chapter 3, we shall give the first examples of multi-normed spaces. These are the minimum and the maximum multi-norms associated with a fixed normed space E. The latter notion leads to a sequence $(\varphi_n^{\max}(E))$ that is intrinsic to E. We shall relate this sequence to some known sequences connected with the theory of absolutely summing operators; the background involving p-summing operators will be reviewed. We shall give various characterizations of the maximum multi-norm, and then calculate the sequence $(\varphi_n^{\max}(E))$ for a variety of examples, including the spaces ℓ^p.

In Chapter 4, we shall give several specific examples of multi-norms, including the (p, q)-multi-norm based on an arbitrary normed space, the Hilbert multi-norm based on a Hilbert space, and the standard q-multi-norm based on $L^p(\Omega)$ for $1 \leq p \leq q$. We shall compare these multi-norms, and determine in some cases when they are mutually equivalent. This chapter concludes with the definition of the lattice multi-norm based on a

Banach lattice; there is a representation theorem that shows that every multi-normed space is a sub-multi-normed space of such an example.

In Chapter 5, we shall extend our theory to cover some multi-topological linear spaces, and shall discuss the notion of multi-convergence in these spaces, concentrating on the case of multi-convergence in multi-normed spaces.

In Chapter 6, we shall develop a theory of multi-bounded subsets of a multi-normed space and of multi-bounded and multi-continuous linear operators between multi-Banach spaces based on E and F that is parallel to the classical theory of continuous and bounded linear operators between the Banach spaces E and F. For a monotonically bounded Banach lattice, a subset is multi-bounded with respect to the lattice multi-norm if and only if it is order-bounded. The space of multi-bounded operators $\mathcal{M}(E, F)$ is a Banach operator algebra in $\mathcal{B}(E, F)$, and can be given a natural multi-normed structure. Examples show that sometimes $\mathcal{M}(E, F)$ coincides with $\mathcal{B}(E, F)$, but can coincide with $\mathcal{N}(E, F)$, the space of nuclear operators from E to F. The multi-normed space based on $\mathcal{M}(E, F)$ is identified for various classes of Banach lattices.

In Chapter 7, our aim is to find a reasonable theory of 'multi-dual spaces': we require a multi-norm based on E', given a multi-norm based on a normed space E. We shall achieve this by first establishing a theory of direct sum decompositions of a normed space E with respect to a multi-norm based on E,and then by using the duals of these decompositions to generate a multi-norm based on E'.

1.5. History and acknowledgements. This work was commenced in 2005 when Maksim Polyakov, from Moscow, was a Marie-Curie Research Fellow at the University of Leeds.

Our motivation at that time was to seek to resolve some questions left open in [18]. In particular, we were concerned with the following question; for the definitions of the terms used, see [18]. Let G be a locally compact group, and let $L^1(G)$ be the group algebra of G. For each $p \in [1, \infty]$, the Banach space $L^p(G)$ is a Banach left $L^1(G)$-module in a natural way. We would like to know when these modules are injective in the appropriate category. For $p = \infty$, this holds for each locally compact group G [18, Theorem 2.4]; for $p = 1$, this holds if and only if G is discrete and amenable [18, Theorem 4.9]. Now suppose that $1 < p < \infty$. Then $L^p(G)$ is a dual Banach left $L^1(G)$-module, and so it follows from now standard results that $L^p(G)$ is injective whenever G is an amenable group. We conjectured that the converse is true. In [18, Theorem 5.12], we proved that, in the case where G is discrete and $\ell^p(G)$ is injective for some $p \in (1, \infty)$, the group G is at least 'pseudo-amenable'. No example of a pseudo-amenable group which is not amenable is known; since such a group cannot contain the free group on two generators, there are very few candidates for such a group. In fact this conjectured result has now been proved, and will be established (with other results) in [19].

We realised that the above question, and other related questions, can be reformulated in the language of what we call 'multi-Banach algebras', and we began to develop a theory of such algebras. This required a substantial background in a new theory of 'multi-normed spaces'; this new theory came to life in its own right, and it seems to be a useful framework in which many important concepts of functional analysis can be expressed, often generalizing known ideas to a wider situation.

Tragically, Maksim Polyakov died in Moscow in January 2006 when this project had just been commenced. I pay great tribute to this fine mathematician and colleague, and especially to his original ideas which underlie this work.

In due course, the project was continued by myself. Eventually it became apparent that the preliminary work on multi-normed spaces was so considerable that there should be one memoir devoted just to this topic; this is the present work. Thus this work was developed with particular applications in mind, but these applications will not be discussed here. The subsequent papers [19] and [20] will develop a theory of multi-normed spaces, with particular application to the theory of modules over the group algebras $L^1(G)$, where G is a locally compact group; I anticipate a future paper on 'multi-Banach algebras'.

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H. G. D., Lancaster, September, 2011

2. The axioms and some consequences

We shall now commence our study of multi-norms.

2.1. The axioms

2.1.1. Multi-norms. We begin with our definition of a multi-norm.

DEFINITION 2.1. Let $(E, \|\cdot\|)$ be a complex (respectively, real) normed space, and take $n \in \mathbb{N}$. A *multi-norm of level n* on $\{E^k : k \in \mathbb{N}_n\}$ is a sequence $(\|\cdot\|_k) = (\|\cdot\|_k : k \in \mathbb{N}_n)$ such that $\|\cdot\|_k$ is a norm on E^k for each $k \in \mathbb{N}_n$, such that $\|x\|_1 = \|x\|$ for each $x \in E$ (so that $\|\cdot\|_1$ is the initial norm), and such that the following Axioms (A1)–(A4) are satisfied for each $k \in \mathbb{N}_n$ with $k \geq 2$:

(A1) for each $\sigma \in \mathfrak{S}_k$ and $x \in E^k$, we have

$$\|A_\sigma(x)\|_k = \|x\|_k;$$

(A2) for each $\alpha_1, \dots, \alpha_k \in \mathbb{C}$ (respectively, each $\alpha_1, \dots, \alpha_k \in \mathbb{R}$) and $x \in E^k$, we have

$$\|M_\alpha(x)\|_k \leq \left(\max_{i \in \mathbb{N}_k} |\alpha_i|\right) \|x\|_k;$$

(A3) for each $x_1, \dots, x_{k-1} \in E$, we have

$$\|(x_1, \dots, x_{k-1}, 0)\|_k = \|(x_1, \dots, x_{k-1})\|_{k-1};$$

(A4) for each $x_1, \dots, x_{k-1} \in E$, we have

$$\|(x_1, \dots, x_{k-2}, x_{k-1}, x_{k-1})\|_k = \|(x_1, \dots, x_{k-2}, x_{k-1})\|_{k-1}.$$

In this case, $((E^k, \|\cdot\|_k) : k \in \mathbb{N}_n)$ is a multi-normed space of level n.

A *multi-norm* on $\{E^k : k \in \mathbb{N}\}$ is a sequence

$$(\|\cdot\|_k) = (\|\cdot\|_k : k \in \mathbb{N})$$

such that $(\|\cdot\|_k : k \in \mathbb{N}_n)$ is a multi-norm of level n for each $n \in \mathbb{N}$. In this case, $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is a *multi-normed space*.

We shall sometimes say that $(\|\cdot\|_k : k \in \mathbb{N})$ is a multi-norm *based on E*.

Let $(E, \|\cdot\|)$ be a normed space. Then Axiom (A1) says that A_σ is an isometry on $(E^k, \|\cdot\|_k)$ whenever $\sigma \in \mathfrak{S}_k$, and Axiom (A2) says that $\|M_\alpha\| \leq 1$ whenever $\alpha \in \overline{\mathbb{D}}^k$, where we regard M_α as a bounded linear operator on $(E^k, \|\cdot\|_k)$; in fact,

$$\|M_\alpha\| = \max_{i \in \mathbb{N}_k} |\alpha_i| \quad (\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n).$$

Note that Axioms (A1) and (A4) together say precisely that, for each $n \in \mathbb{N}$, the value of $\|(x_1, \dots, x_n)\|_n$ depends on only the set $\{x_1, \dots, x_n\}$.

2.1.2. Dual multi-norms. We shall also have some occasion to refer to a dual concept to that of a multi-norm. We give the definition just in the case where the index set is $\mathbb{N}$, but there is also an obvious definition of 'dual multi-normed space of level n'. The justification of the term 'dual multi-normed space' will be apparent in §5 of this chapter.

DEFINITION 2.2. Let $(E, \|\cdot\|)$ be a normed space. A *dual multi-norm* on $\{E^k : k \in \mathbb{N}\}$ is a sequence $(\|\cdot\|_k) = (\|\cdot\|_k : k \in \mathbb{N})$ such that $\|\cdot\|_k$ is a norm on E^k for each $k \in \mathbb{N}$, such that $\|x\|_1 = \|x\|$ for each $x \in E$, and such that Axioms (A1), (A2), (A3) and the following modified form of Axiom (A4) are satisfied for each $k \in \mathbb{N}$ with $k \geq 2$:

(B4) for each $x_1, \ldots, x_{k-1} \in E$, we have

$$\|(x_1, \ldots, x_{k-2}, x_{k-1}, x_{k-1})\|_k = \|(x_1, \ldots, x_{k-2}, 2x_{k-1})\|_{k-1}.$$

In this case, we say that $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a *dual multi-normed space.*

We sometimes say, in the above situation, that $(\|\cdot\|_k : k \in \mathbb{N})$ is a dual multi-norm *based on E.*

Suppose that the normed spaces $(E, \|\cdot\|)$ and $(E^2, \|\cdot\|_2)$ satisfy just the case $k = 2$ of both Axioms (A4) and (B4). Then

$$\|x\| = \|(x, x)\|_2 = 2\|x\| \quad (x \in E),$$

and so $E = \{0\}$. Thus we should stress that a dual multi-normed space is **not** a multi-normed space unless $E = \{0\}$.

2.1.3. Independence of the axioms. It is natural to ask whether the four Axioms (A1)–(A4) are independent. We give examples to show that this is indeed the case.

EXAMPLE 2.3. Let $(E, \|\cdot\|)$ be a non-zero normed space. We set $\|x\|_1 = \|x\|$ $(x \in E)$, and, for each $n \in \mathbb{N}$ with $n \geq 2$, set

$$\|(x_1, \ldots, x_n)\|_n = \max\{\|x_1\|, \|x_2\|/2, \ldots, \|x_n\|/2\} \quad ((x_1, \ldots, x_n) \in E^n).$$

Then it is immediately checked that $\|\cdot\|_n$ is a norm on E^n for each $n \in \mathbb{N}$, and that $(\|\cdot\|_n)$ is a sequence that satisfies Axioms (A2), (A3), and (A4) for each $n \in \mathbb{N}$. However, take $x \in E$ with $\|x\| = 1$. Then $\|(2x, 3x)\|_2 = 2$, but $\|(3x, 2x)\|_2 = 3$, and so $\|\cdot\|_2$ does not satisfy Axiom (A1). ■

EXAMPLE 2.4. In this example, we work with $E = \mathbb{C}$. For $z \in \mathbb{C}$, we set $\|z\|_1 = |z|$. Next, for $(z, w) \in \mathbb{C}^2$, set

$$r((z, w)) = \frac{1}{2}(|z - w| + |z + w|).$$

Then r is a norm on $\mathbb{C}^2$. Further, $r((z, z)) = r((z, 0)) = |z|$ $(z \in \mathbb{C})$ and also

$$r((z, w)) = r((w, z)) \geq \max\{|z|, |w|\} \quad ((z, w) \in \mathbb{C}^2).$$

Finally, for $n \in \mathbb{N}$ with $n \geq 2$, set

$$\|(z_1, \ldots, z_n)\|_n = \max\{r((z_i, z_j)) : i, j \in \mathbb{N}_n\} \quad ((z_1, \ldots, z_n) \in \mathbb{C}^n),$$

so that $\|(z, w)\|_2 = r((z, w))$ $((z, w) \in \mathbb{C}^2)$ and

$$\|(z_1, \ldots, z_n)\|_n \geq \max_{i \in \mathbb{N}_n} |z_i| \quad ((z_1, \ldots, z_n) \in \mathbb{C}^n).$$

It follows easily that $\|\cdot\|_n$ is a norm on $\mathbb{C}^n$ and that the sequence $(\|\cdot\|_n)$ satisfies Axioms (A1), (A3), and (A4) for each $n \in \mathbb{N}$.

However we *claim* that $\|\cdot\|_2$ does not satisfy Axiom (A2). Indeed,

$$\|(1, \mathrm{i})\|_2 = \frac{1}{2}(|1 - \mathrm{i}| + |1 + \mathrm{i}|) = \sqrt{2} > 1 = \|(1, 1)\|_2,$$

giving the claim.

Here is a similar example involving real spaces. Let $E = \mathbb{R}$, and define

$$\|(x_1, \dots, x_n)\|_n = \max\Big\{\max_{i \in \mathbb{N}_n} |x_i|,\ \max_{i,j \in \mathbb{N}_n} |x_i - x_j|\Big\}$$

for $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \mathbb{R}$. Then $(\|\cdot\|_n : n \in \mathbb{N})$ satisfies Axioms (A1), (A3), and (A4), but Axiom (A2) fails because $\|(1,1)\|_2 = 1$, whilst $\|(1,-1)\|_2 = 2$. ■

We now consider the independence of Axiom (A3). The following example shows that for multi-norms of level 2, (A3) is indeed independent of the other axioms. However we shall see below that Axiom (A3) follows from Axioms (A1), (A2), and (A4) for multi-norms on the whole family $\{E^n : n \in \mathbb{N}\}$.

Example 2.5. We again take $E = \mathbb{C}$, and set $\|z\|_1 = |z|$ $(z \in \mathbb{C})$. Set

$$\|(z, w)\|_2 = \frac{1}{2}(|z| + |w|) \qquad (z, w \in \mathbb{C}).$$

Then $\|\cdot\|_2$ is a norm on $\mathbb{C}^2$, and $\|\cdot\|_2$ satisfies Axioms (A1), (A2), and (A4) for $n = 2$. However $\|(1,0)\|_2 = 1/2 < 1 = \|1\|_1$, and so Axiom (A3) does not hold. ■

Example 2.6. Let $(E, \|\cdot\|)$ be a non-zero normed space. For each $n \in \mathbb{N}$, set

$$\|(x_1, \dots, x_n)\|_n = \Big(\sum_{j=1}^n \|x_j\|^p\Big)^{1/p} \qquad ((x_1, \dots, x_n) \in E^n),$$

where $p \geq 1$. Then it is immediately checked that, for each p, the function $\|\cdot\|_n$ is a norm on E^n, and that $(\|\cdot\|_n)$ is a sequence that satisfies Axioms (A1), (A2), and (A3) for each $n \in \mathbb{N}$, but $\|\cdot\|_2$ does not satisfy Axiom (A4).

We note that the sequence $(\|\cdot\|_n : n \in \mathbb{N})$ satisfies Axiom (B4) if and only if $p = 1$; in this latter case, $(\|\cdot\|_n : n \in \mathbb{N})$ is a dual multi-norm. ■

We shall now show that Axiom (A3) follows from the other axioms in the case where we have norms on the whole family $\{E^n : n \in \mathbb{N}\}$.

Proposition 2.7. *Let $(E, \|\cdot\|)$ be a normed space. Let $(\|\cdot\|_n : n \in \mathbb{N})$ be a sequence such that $\|\cdot\|_n$ is a norm on E^n for each $n \in \mathbb{N}$, such that $\|x\|_1 = \|x\|$ for each $x \in E$, and such that Axioms* (A1), (A2), *and* (A4) *are satisfied for each $n \in \mathbb{N}$. Then $(\|\cdot\|_n : n \in \mathbb{N})$ is a multi-norm on $\{E^n : n \in \mathbb{N}\}$.*

Proof. We must show that Axiom (A3) holds.

Let $n \in \mathbb{N}$, and take $x = (x_i) \in E^n$, say $\|x\|_n = 1$. Set

$$\alpha = \|(x_1, \dots, x_n, 0)\|_{n+1},$$

so that $0 < \alpha \leq 1$ by (A2) and (A4). For each $k \in \mathbb{N}$, we see that $x^{[k+1]} \in E^{(k+1)n}$ and that $\|x^{[k+1]}\|_{(k+1)n} = 1$ by (A1) and (A4). For $i \in \mathbb{N}_{n+1}$, let B_i be the subset

$$\{(i-1)k+1, \dots, ik\}$$

of $\mathbb{N}_{(n+1)k}$, and let Q_{B_i} be the projection onto the complement of B_i; by (A1) and (A4), we have $\|Q_{B_i}(x^{[k+1]})\|_{(k+1)n} = \alpha$. Further,

$$\sum_{i=1}^{k+1} Q_{B_i}(x^{[k+1]}) = kx^{[k+1]},$$

and so

$$k = k\|x^{[k+1]}\|_{(k+1)n} \leq \sum_{i=1}^{k+1} \|Q_{B_i}(x^{[k+1]})\|_{(k+1)n} = (k+1)\alpha,$$

whence $\alpha \geq k/(k+1)$. This holds for each $k \in \mathbb{N}$, and so $\alpha = 1$.

The result follows. ∎

Stanislav Shkarin has pointed out that Axiom (A3) also follows from Axioms (A1), (A2), and (B4), imposed on the family $\{(E^n, \|\cdot\|_n) : n \in \mathbb{N}\}$.

2.2. Elementary consequences of the axioms. The following are immediate consequences of the axioms for multi-normed and dual multi-normed spaces.

2.2.1. Results for special-norms. A sequence $(\|\cdot\|_k) = (\|\cdot\|_k : k \in \mathbb{N})$ such that $\|\cdot\|_k$ is a norm on E^k for each $k \in \mathbb{N}$, such that $\|x\|_1 = \|x\|$ for each $x \in E$, and such that just the Axioms (A1), (A2), and (A3) are satisfied is called a *special-norm* in [61], and $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is then a *special-normed space*. Thus multi-norms and dual multi-norms are examples of special-norms.

Initially in this subsection, we suppose that $(E, \|\cdot\|)$ is a complex normed space, that $n \in \mathbb{N}$, and that $(\|\cdot\|_k : k \in \mathbb{N}_n)$ is a special-norm. Thus our first results apply to both multi-normed spaces and to dual multi-normed spaces of level n.

Trivial modifications give entirely similar results when $(E, \|\cdot\|)$ is a real normed space.

LEMMA 2.8. *Let $k \in \mathbb{N}_n$, $x_1, \dots, x_k \in E$, and $\zeta_1, \dots, \zeta_k \in \mathbb{T}$. Then*

$$\|(\zeta_1 x_1, \dots, \zeta_k x_k)\|_k = \|(x_1, \dots, x_k)\|_k.$$

Proof. This is immediate from (A2). ∎

LEMMA 2.9. *Let $k \in \mathbb{N}_{n-1}$ and $x_1, \dots, x_{k+1} \in E$. Then*

$$\|(x_1, \dots, x_k)\|_k \leq \|(x_1, \dots, x_k, x_{k+1})\|_{k+1}.$$

Proof. We have

$$\begin{aligned} \|(x_1, \dots, x_k)\|_k &= \|(x_1, \dots, x_k, 0)\|_{k+1} && \text{by (A3)} \\ &\leq \|(x_1, \dots, x_k, x_{k+1})\|_{k+1} && \text{by (A2),} \end{aligned}$$

giving the result. ∎

LEMMA 2.10. *Let $j, k \in \mathbb{N}$ with $j + k \leq n$ and $x_1, \dots, x_j, y_1, \dots, y_k \in E$. Then*

$$\|(x_1, \dots, x_j, y_1, \dots, y_k)\|_{j+k} \leq \|(x_1, \dots, x_j)\|_j + \|(y_1, \dots, y_k)\|_k.$$

Proof. This is immediate from Axiom (A3). ∎

LEMMA 2.11. *Let $k \in \mathbb{N}_n$ and $x_1, \ldots, x_k \in E$. Then*

$$\max_{i \in \mathbb{N}_k} \|x_i\| \leq \|(x_1, \ldots, x_k)\|_k \leq \sum_{i=1}^{k} \|x_i\| \leq k \max_{i \in \mathbb{N}_k} \|x_i\|.$$

Proof. Set $x = (x_i)$. For $i \in \mathbb{N}_k$, we have $\|x_i\| = \|(0, \ldots, 0, x_i, 0, \ldots, 0)\|_k \leq \|x\|_k$ by (A1), (A2), and (A3), and so the stated inequalities follow. ■

It follows that any two special-norms on $\{E^k : k \in \mathbb{N}_n\}$ define the same topology on the space E^k for each $k \in \mathbb{N}_n$; the topology is the product topology.

COROLLARY 2.12. *Suppose that $(E, \|\cdot\|)$ is a Banach space. Then the normed space $(E^k, \|\cdot\|_k)$ is a Banach space for each $k \in \mathbb{N}_n$.* ■

As we remarked, the above results apply to both multi-normed spaces and to dual multi-normed spaces, and so, in the light of the above corollary, the following definition is reasonable.

DEFINITION 2.13. Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space (respectively, dual multi-normed space) for which $(E, \|\cdot\|)$ is a Banach space. Then $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is a *multi-Banach space* (respectively, *dual multi-Banach space*).

More generally, we can refer to *special-Banach spaces.*

2.2.2. Results for multi-norms. We now give some elementary lemmas that suppose, further, that the sequence $(\|\cdot\|_k : k \in \mathbb{N}_n)$ also satisfies Axiom (A4), and hence that $((E^k, \|\cdot\|_k) : k \in \mathbb{N}_n)$ is a multi-normed space of level n, where $n \in \mathbb{N}$.

LEMMA 2.14. *Let $k \in \mathbb{N}_n$ and $x \in E$. Then $\|(x, \ldots, x)\|_k = \|x\|$.*

Proof. This is immediate from (A4). ■

LEMMA 2.15. *Let $j, k \in \mathbb{N}_n$ and $x_1, \ldots, x_j, y_1, \ldots, y_k \in E$ be such that $\{x_1, \ldots, x_j\}$ is a subset of $\{y_1, \ldots, y_k\}$. Then*

$$\|(x_1, \ldots, x_j)\|_j \leq \|(y_1, \ldots, y_k)\|_k.$$

Proof. By Axioms (A1) and (A4), we may suppose that $j \leq k$ and that $x_i = y_i$ $(i \in \mathbb{N}_j)$. Now the result follows from Lemma 2.9. ■

LEMMA 2.16. *Let $k \in \{2, \ldots, n\}$ and $x_1, \ldots, x_k \in E$. Take $\alpha, \beta \in \mathbb{I}$ with $\alpha + \beta = 1$, and set $x = \alpha x_{k-1} + \beta x_k$. Then*

$$\|(x_1, \ldots, x_{k-2}, x, x)\|_k \leq \|(x_1, \ldots, x_{k-2}, x_{k-1}, x_k)\|_k.$$

Proof. Set $y = (x_1, \ldots, x_{k-2})$ and $A = \|(y, x_{k-1}, x_k)\|_k$. Then

$$(y, x, x) = \alpha^2(y, x_{k-1}, x_{k-1}) + \alpha\beta(y, x_{k-1}, x_k) + \alpha\beta(y, x_k, x_{k-1}) + \beta^2(y, x_k, x_k).$$

But $\|(y, x_k, x_{k-1})\|_k = A$ by (A1). Also $\|(y, x_{k-1}, x_{k-1})\|_k \leq A$ and $\|(y, x_k, x_k)\|_k \leq A$ by Lemma 2.15. Hence

$$\|(x_1, \ldots, x_{k-2}, x, x)\|_k \leq (\alpha + \beta)^2 A = A,$$

giving the result. ■

The following *inequality-of-roots* will be useful later.

Proposition 2.17. *Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space, and take $k \in \mathbb{N}$. Set $\zeta_k = \exp(2\pi \mathrm{i}/k)$. Then*

$$\|(x_1, \dots, x_k)\|_k \le \frac{1}{k} \sum_{j=1}^{k} \Big\| \sum_{m=1}^{k} \zeta_k^{jm} x_m \Big\| \qquad (x_1, \dots, x_k \in E). \tag{2.1}$$

Proof. We write ζ for ζ_k.

First note that

$$x_\ell = \frac{1}{k} \sum_{m=1}^{k} \sum_{j=1}^{k} \zeta^{j(m-\ell)} x_m \qquad (\ell \in \mathbb{N}_k)$$

because $\sum_{j=1}^{k} \zeta^{j(m-\ell)} = 0$ when $m \neq \ell$ and $\sum_{j=1}^{k} \zeta^{j(m-\ell)} = k$ when $m = \ell$. Thus

$$\|(x_1, \dots, x_k)\|_k \le \frac{1}{k} \sum_{j=1}^{k} \Big\| \Big(\sum_{m=1}^{k} \zeta^{j(m-1)} x_m, \dots, \sum_{m=1}^{k} \zeta^{j(m-k)} x_m \Big) \Big\|_k.$$

For $j \in \mathbb{N}_k$, set $y_j = \sum_{m=1}^{k} \zeta^{jm} x_m$ $(j \in \mathbb{N}_k)$. Then

$$\Big\| \Big(\sum_{m=1}^{k} \zeta^{j(m-1)} x_m, \dots, \sum_{m=1}^{k} \zeta^{j(m-k)} x_m \Big) \Big\|_k = \|(\zeta^{-j} y_j, \dots, \zeta^{-kj} y_j)\|_k.$$

But $\|(\zeta^{-j} y_j, \dots, \zeta^{-kj} y_j)\|_k = \|y_j\|$ $(j \in \mathbb{N}_k)$ by (A2) and Lemma 2.14, and so inequality (2.1) follows. ■

Corollary 2.18. *Let $E = \ell^r$, where $r \ge 1$, and let $(\|\cdot\|_n : n \in \mathbb{N})$ be a multi-norm based on E. Then*

$$\|(\delta_1, \dots, \delta_k)\|_k \le k^{1/r} \qquad (k \in \mathbb{N}).$$

Proof. In this case,

$$\Big\| \sum_{m=1}^{k} \zeta_k^{jm} \delta_m \Big\| = \|(\zeta_k^{j}, \dots, \zeta_k^{kj})\|_{\ell^r} = k^{1/r}$$

for each $j \in \mathbb{N}_k$, and so the result follows from the proposition. ■

2.2.3. Results for dual multi-norms. We now have some elementary lemmas about dual multi-normed spaces. In the remainder of this section, we suppose that $(E, \|\cdot\|)$ is a normed space and that $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a dual multi-normed space, and so the sequence $(\|\cdot\|_k : k \in \mathbb{N})$ satisfies Axioms (A1)–(A3) and Axiom (B4).

Lemma 2.19. *Let $k \in \mathbb{N}$ and $x_1, \dots, x_k \in E$. Then*

$$\|(x_1, \dots, x_{k-2}, x_{k-1} + x_k)\|_{k-1} \le \|(x_1, \dots, x_{k-2}, x_{k-1}, x_k)\|_k.$$

Proof. We have

$$\begin{aligned} \|(x_1, x_{k-2}, x_{k-1} + x_k)\|_{k-1} &= \|(x_1, \dots, x_{k-2}, (x_{k-1} + x_k)/2, (x_{k-1} + x_k)/2\|_k \quad \text{by (B4)} \\ &= \frac{1}{2} \|(x_1, \dots, x_{k-2}, x_{k-1}, x_k) + (x_1, \dots, x_{k-2}, x_k, x_{k-1})\|_k \\ &\le \|(x_1, \dots, x_{k-2}, x_{k-1}, x_k)\|_k \quad \text{by (A1)}, \end{aligned}$$

as required. ■

Lemma 2.20. *Let $k \in \mathbb{N}$ and $x_1, \ldots, x_k \in E$. Then*

$$\sup\{\|\zeta_1 x_1 + \cdots + \zeta_k x_k\| : \zeta_1, \ldots, \zeta_k \in \mathbb{T}\} \le \|(x_1, \ldots, x_k)\|_k.$$

Proof. This follows from Lemmas 2.8 and 2.19. ■

Lemma 2.21. *Let $m, n \in \mathbb{N}$ with $m \le n$, let $x \in E^n$, and let $y \in E^m$ be a coagulation of x. Then $\|y\|_m \le \|x\|_n$.*

Proof. This follows from Lemma 2.19. ■

Lemma 2.22. *Let $k \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$, and $x \in E$. Then*

$$\|(\alpha_1 x, \ldots, \alpha_k x)\|_k = \Big(\sum_{j=1}^k |\alpha_j|\Big)\|x\|.$$

Proof. By Lemma 2.11, we have

$$\|(\alpha_1 x, \ldots, \alpha_k x)\|_k \le \Big(\sum_{j=1}^k |\alpha_j|\Big)\|x\|.$$

But also

$$\begin{aligned}\|(\alpha_1 x, \ldots, \alpha_k x)\|_k &= \|(|\alpha_1| x, \ldots, |\alpha_k| x)\|_k \quad \text{by Lemma 2.8}\\ &\ge \Big\|\sum_{j=1}^k |\alpha_j| x\Big\| = \Big(\sum_{j=1}^k |\alpha_j|\Big)\|x\| \quad \text{by Lemma 2.19.}\end{aligned}$$

The result follows. ■

2.2.4. The family of multi-norms. We first have an elementary result.

Proposition 2.23. *Let $(E, \|\cdot\|)$ be a normed space. Take $n \in \mathbb{N}$, and let $(\|\cdot\|_k^1 : k \in \mathbb{N}_n)$ and $(\|\cdot\|_k^2 : k \in \mathbb{N}_n)$ be two multi-norms of level n on the family $\{E^k : k \in \mathbb{N}_n\}$. For $k \in \mathbb{N}_n$ and $x_1, \ldots, x_k \in E$, set*

$$\|(x_1, \ldots, x_k)\|_k = \max\{\|(x_1, \ldots, x_k)\|_k^1, \|(x_1, \ldots, x_k)\|_k^2\}.$$

Then $((E^k, \|\cdot\|_k) : k \in \mathbb{N}_n)$ is a multi-normed space of level n.

Proof. This is immediately checked. ■

We now define a family of multi-norms.

Definition 2.24. Let $(E, \|\cdot\|)$ be a normed space. Then $\mathcal{E}_E$ is the family of all multi-norms based on E. Let $(\|\cdot\|_k^1 : k \in \mathbb{N})$ and $(\|\cdot\|_k^2 : k \in \mathbb{N})$ belong to $\mathcal{E}_E$. Then

$$(\|\cdot\|_k^1 : k \in \mathbb{N}) \le (\|\cdot\|_k^2 : k \in \mathbb{N})$$

if

$$\|(x_1, \ldots, x_k)\|_k^1 \le \|(x_1, \ldots, x_k)\|_k^2 \quad (x_1, \ldots, x_k \in E,\ k \in \mathbb{N}).$$

Further, the multi-norm $(\|\cdot\|_k^2 : k \in \mathbb{N})$ *dominates* the multi-norm $(\|\cdot\|_k^1 : k \in \mathbb{N})$, written

$$(\|\cdot\|_k^1 : k \in \mathbb{N}) \preccurlyeq (\|\cdot\|_k^2 : k \in \mathbb{N}),$$

if there is a constant $C > 0$ such that

$$\|(x_1, \ldots, x_k)\|_k^1 \le C\|(x_1, \ldots, x_k)\|_k^2 \quad (x_1, \ldots, x_k \in E,\ k \in \mathbb{N}). \tag{2.2}$$

The two multi-norms $(\|\cdot\|_k^1 : k \in \mathbb{N})$ and $(\|\cdot\|_k^2 : k \in \mathbb{N})$ are *equivalent*, written

$$(\|\cdot\|_k^1 : k \in \mathbb{N}) \cong (\|\cdot\|_k^2 : k \in \mathbb{N}),$$

if each dominates the other.

It is clear that $(\mathcal{E}_E, \leq)$ is a partially ordered set; by Proposition 2.23, each pair of elements has an upper bound. We shall see in Proposition 3.10 that $(\mathcal{E}_E, \leq)$ is a Dedekind-complete lattice.

There is an entirely similar ordering of, and notion of equivalence for, the family of dual multi-norms on $\{E^k : k \in \mathbb{N}_n\}$.

In [20], we shall explore when various specific multi-norms are mutually equivalent, and sometimes calculate the best constant C in (2.2).

2.2.5. Standard constructions. We now give some standard constructions that generate new multi-normed spaces from old ones. Analogous constructions also generate new dual multi-normed spaces.

Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space, and let F be a closed linear subspace of E. For $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in E$, define

$$\|(x_1 + F, \ldots, x_n + F)\|_n = \inf\{\|(y_1, \ldots, y_n)\|_n : y_i \in x_i + F \ (i \in \mathbb{N}_n)\},$$

so that $\|\cdot\|_n$ is a norm on $(E/F)^n$.

PROPOSITION 2.25. *Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space.*

(i) *Let F be a linear subspace of E. Then $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ is a multi-normed space.*
(ii) *Let F be a closed linear subspace of E. Then $(((E/F)^n, \|\cdot\|_n) : n \in \mathbb{N})$ is a multi-normed space.*

Proof. These are easily checked; to show that each norm $\|\cdot\|_n$ on $(E/F)^n$ satisfies (A4), we use Lemma 2.16. ∎

We say that $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ and $(((E/F)^n, \|\cdot\|_n) : n \in \mathbb{N})$ are a *multi-normed subspace* and a *multi-normed quotient space*, respectively, of the multi-normed space $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$.

PROPOSITION 2.26. *Let F be a 1-complemented subspace of a normed space E, and suppose that $(\|\cdot\|_n : n \in \mathbb{N})$ is a multi-norm on $\{F^n : n \in \mathbb{N}\}$. Then there is a multi-norm $(|\!|\!|\cdot|\!|\!|_n : n \in \mathbb{N})$ on $\{E^n : n \in \mathbb{N}\}$ such that $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ is a multi-normed subspace of $((E^n, |\!|\!|\cdot|\!|\!|_n) : n \in \mathbb{N})$.*

Proof. Let $P : E \to F$ be a projection onto F with $\|P\| = 1$, and set

$$|\!|\!|(x_1, \ldots, x_n)|\!|\!|_n = \max\{\|x_1\|, \ldots, \|x_n\|, \|(Px_1, \ldots, Px_n)\|_n\}$$

for $x_1, \ldots, x_n \in E$. Then the sequence $(|\!|\!|\cdot|\!|\!|_n : n \in \mathbb{N})$ has the required properties, as is easily checked. ∎

PROPOSITION 2.27. *Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space, and let $k \in \mathbb{N}$. Set $F = E^k$ and $\|\cdot\|_F = \|\cdot\|_k$. Then $(F, \|\cdot\|_F)$ is a normed space, and $((F^n, \|\cdot\|_{nk}) : n \in \mathbb{N})$ is a multi-normed space.*

Proof. Let $y_1, \dots, y_n \in F$, say $y_i = (x_{i,1}, \dots, x_{i,k})$ $(i \in \mathbb{N}_n)$. Then

$$\|(y_1, \dots, y_n)\|_n = \|(x_{1,1}, \dots, x_{1,k}, \dots, x_{n,1}, \dots, x_{n,k})\|_{nk},$$

and $((F^n, \|\cdot\|_{nk}) : n \in \mathbb{N})$ is clearly a multi-normed space. ■

Let $\{((E_\alpha^n, \|\cdot\|_n^\alpha) : n \in \mathbb{N}) : \alpha \in A\}$ be a family of multi-normed spaces, defined for each α in a non-empty index set A (perhaps finite). Then we consider the following spaces.

First, for $n \in \mathbb{N}$ and $(x_\alpha^1), \dots, (x_\alpha^n) \in \ell^\infty(E_\alpha)$, set

$$\|((x_\alpha^1), \dots, (x_\alpha^n))\|_n = \sup\{\|(x_\alpha^1, \dots, x_\alpha^n)\|_n^\alpha : \alpha \in A\}.$$

PROPOSITION 2.28. *The space $((\ell^\infty(E_\alpha)^n, \|\cdot\|_n) : n \in \mathbb{N})$ is a multi-normed space.*

Proof. This is immediately checked. ■

Take p with $1 \le p < \infty$. For $n \in \mathbb{N}$ and $(x_\alpha^1), \dots, (x_\alpha^n) \in \ell^p(E_\alpha)$, we define

$$\|((x_\alpha^1), \dots, (x_\alpha^n))\|_n = \Big(\sum_\alpha (\|(x_\alpha^1, \dots, x_\alpha^n)\|_n^\alpha)^p\Big)^{1/p}.$$

PROPOSITION 2.29. *The space $((\ell^p(E_\alpha))^n, \|\cdot\|_n) : n \in \mathbb{N})$ is a multi-normed space.*

Proof. We must show that $\|((x_\alpha^1), \dots, (x_\alpha^n))\|_n$, as defined above, is finite in each case. Indeed,

$$\Big(\sum_\alpha (\|(x_\alpha^1, \dots, x_\alpha^n)\|_n^\alpha)^p\Big)^{1/p} \le \Big(\sum_\alpha (\|x_\alpha^1\|^\alpha + \dots + \|x_\alpha^n\|_n^\alpha)^p\Big)^{1/p}$$

by Lemma 2.11, and so, by Minkowski's inequality,

$$\|((x_\alpha^1), \dots, (x_\alpha^n))\|_n \le \Big(\sum_\alpha (\|x_\alpha^1\|^\alpha)^p\Big)^{1/p} + \dots + \Big(\sum_\alpha (\|x_\alpha^n\|^\alpha)^p\Big)^{1/p},$$

and the right-hand side is finite.

The triangle inequality for $\|\cdot\|_n$ also follows from Minkowski's inequality, and the remainder is easy to check. ■

In particular, let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ and $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ be multi-normed spaces. Set $G = E \oplus F$. For $n \in \mathbb{N}$, define $\|\cdot\|_n$ on G^n by taking

$$\|(x_1 + y_1, \dots, x_n + y_n)\|_n$$

to be either

$$\max\{\|(x_1, \dots, x_n)\|_n, \|(y_1, \dots, y_n)\|_n\} \quad \text{or} \quad \|(x_1, \dots, x_n)\|_n + \|(y_1, \dots, y_n)\|_n$$

for $x_1, \dots, x_n \in E$ and $y_1, \dots, y_n \in F$. Then $((G^n, \|\cdot\|_n) : n \in \mathbb{N})$ is a multi-normed space, denoted by

$$(((E \oplus_\infty F)^n, \|\cdot\|_n) : n \in \mathbb{N}) \quad \text{or} \quad (((E \oplus_1 F)^n, \|\cdot\|_n) : n \in \mathbb{N}),$$

respectively.

2.3. Theorems on duality. In this section, we shall justify the term 'dual multi-normed space'.

2.3.1. Special-normed spaces. Let $(E, \|\cdot\|)$ be a normed space, let $k \in \mathbb{N}$, and let $\|\cdot\|_k$ be any norm on the space E^k. As before, the dual norm on the space $(E')^k$ is denoted by $\|\cdot\|_k'$, so that, explicitly,

$$\|(\lambda_1, \dots, \lambda_k)\|_k' = \sup\Big\{\Big|\sum_{j=1}^{k}\langle x_j,\, \lambda_j\rangle\Big| : \|(x_1, \dots, x_k)\|_k \le 1\Big\}$$

for $\lambda_1, \dots, \lambda_k \in E'$, taking the supremum over $x_1, \dots, x_k \in E$.

Now let $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a special-normed space. Then it follows from Lemma 2.11 and Axiom (A3) that each norm $\|\cdot\|_k$ satisfies (1.8) and (1.9) (with $\|\cdot\|_k$ for $|\!|\!|\cdot|\!|\!|$), and so $((E^k)', \|\cdot\|_k')$ is linearly homeomorphic to $(E')^k$ (with the product topology from E'). Thus we have defined a sequence $(\|\cdot\|_k' : k \in \mathbb{N})$ such that $\|\cdot\|_k'$ is a norm on $(E')^k$ for each $k \in \mathbb{N}$. Clearly $\|\lambda\|_1' = \|\lambda\|'$ for each $\lambda \in E'$.

Proposition 2.30. *Let $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a special-normed space. Then it also holds that $(((E')^k, \|\cdot\|_k') : k \in \mathbb{N})$ is a special-Banach space.*

Proof. It is clear that Axioms (A1) and (A2) for $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ imply, respectively, that (A1) and (A2) hold for $(((E')^k, \|\cdot\|_k') : k \in \mathbb{N})$.

Take $k \ge 2$ and $\lambda_1, \dots, \lambda_{k-1} \in E'$. For each $x_1, \dots, x_k \in E$, it follows from Lemma 2.9 that $\|(x_1, \dots, x_{k-1})\|_{k-1} \le \|(x_1, \dots, x_{k-1}, x_k)\|_k$, and so

$$\|(\lambda_1, \dots, \lambda_{k-1}, 0)\|_k' \ge \|(\lambda_1, \dots, \lambda_{k-1})\|_{k-1}'.$$

Thus $(\|\cdot\|_k' : k \in \mathbb{N})$ satisfies (A3). ∎

2.3.2. Multi-normed and dual multi-normed spaces. We now establish the duality that we are seeking. Throughout, $(E, \|\cdot\|)$ and $(F, \|\cdot\|)$ are normed spaces.

Theorem 2.31. *Let $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-normed space. Then*

$$(((E')^k, \|\cdot\|_k') : k \in \mathbb{N})$$

is a dual multi-Banach space.

Proof. By Proposition 2.30, it suffices to show that $(((E')^k, \|\cdot\|_k') : k \in \mathbb{N})$ satisfies (B4).

Fix $\lambda_1, \dots, \lambda_{k-1} \in E'$, and set

$$A = \|(\lambda_1, \dots, \lambda_{k-2}, \lambda_{k-1}, \lambda_{k-1})\|_k', \qquad B = \|(\lambda_1, \dots, \lambda_{k-2}, 2\lambda_{k-1})\|_{k-1}'.$$

Take $\varepsilon > 0$.

First choose $(x_1, \dots, x_k) \in (E^k, \|\cdot\|_k)_{[1]}$ with

$$\Big|\sum_{j=1}^{k-2}\langle x_j,\, \lambda_j\rangle + \langle x_{k-1},\, \lambda_{k-1}\rangle + \langle x_k,\, \lambda_{k-1}\rangle\Big| > A - \varepsilon.$$

Set $x = (x_{k-1} + x_k)/2$, so that it follows from Lemma 2.16 and (A4) that we have $(x_1, \dots, x_{k-2}, x) \in (E^{k-1}, \|\cdot\|_{k-1})_{[1]}$, and hence

$$B \ge \Big|\sum_{j=1}^{k-2}\langle x_j,\, \lambda_j\rangle + \langle x,\, 2\lambda_{k-1}\rangle\Big| = \Big|\sum_{j=1}^{k-2}\langle x_j,\, \lambda_j\rangle + \langle x_{k-1},\, \lambda_{k-1}\rangle + \langle x_k,\, \lambda_{k-1}\rangle\Big| > A - \varepsilon.$$

Second, choose $(x_1, \dots, x_{k-1}) \in (E^{k-1}, \|\cdot\|_{k-1})_{[1]}$ with

$$\Big|\sum_{j=1}^{k-2} \langle x_j,\, \lambda_j\rangle + \langle x_{k-1},\, 2\lambda_{k-1}\rangle\Big| > B - \varepsilon.$$

Then $(x_1, \dots, x_{k-1}, x_{k-1}) \in (E^k, \|\cdot\|_k)_{[1]}$ by (A4), and so

$$A \geq \Big|\sum_{j=1}^{k-2} \langle x_j,\, \lambda_j\rangle + \langle x_{k-1},\, 2\lambda_{k-1}\rangle\Big| > B - \varepsilon.$$

The above two inequalities hold for each $\varepsilon > 0$, and so $A = B$.

Thus the sequence $(\|\cdot\|'_k : k \in \mathbb{N})$ satisfies Axiom (B4), and hence we have shown that $(((E')^k, \|\cdot\|'_k) : k \in \mathbb{N})$ is a dual multi-Banach space. ∎

DEFINITION 2.32. Let $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-normed space. Then

$$(((E')^k, \|\cdot\|'_k) : k \in \mathbb{N})$$

is the *dual multi-Banach space* of the space $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$.

THEOREM 2.33. *Let $((F^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a dual multi-normed space. Then*

$$(((F')^k, \|\cdot\|'_k) : k \in \mathbb{N})$$

is a multi-Banach space.

Proof. It suffices to show that $(((E')^k, \|\cdot\|'_k) : k \in \mathbb{N})$ satisfies Axiom (A4).

Fix $\lambda_1, \dots, \lambda_{k-1} \in F'$, and set

$$A = \|(\lambda_1, \dots, \lambda_{k-2}, \lambda_{k-1}, \lambda_{k-1})\|'_k, \qquad B = \|(\lambda_1, \dots, \lambda_{k-2}, \lambda_{k-1})\|'_{k-1}.$$

Take $\varepsilon > 0$.

First choose $(x_1, \dots, x_k) \in (F^k, \|\cdot\|_k)_{[1]}$ with

$$\Big|\sum_{j=1}^{k-1} \langle x_j,\, \lambda_j\rangle + \langle x_k,\, \lambda_{k-1}\rangle\Big| > A - \varepsilon.$$

Then $(x_1, \dots, x_{k-2}, x_{k-1} + x_k) \in (F^{k-1}, \|\cdot\|_{k-1})_{[1]}$ by Lemma 2.19, and so

$$B \geq \Big|\sum_{j=1}^{k-2} \langle x_j,\, \lambda_j\rangle + \langle x_{k-1} + x_k,\, \lambda_{k-1}\rangle\Big| > A - \varepsilon.$$

Second, choose $(x_1, \dots, x_{k-1}) \in (F^{k-1}, \|\cdot\|_{k-1})_{[1]}$ with

$$\Big|\sum_{j=1}^{k-1} \langle x_j,\, \lambda_j\rangle\Big| > B - \varepsilon.$$

Then $(x_1, \dots, x_{k-1}, 0) \in (F^k, \|\cdot\|_k)_{[1]}$ by (A3), and so $A > B - \varepsilon$.

It follows that $A = B$, and so the sequence $(\|\cdot\|'_k : k \in \mathbb{N})$ satisfies Axiom (A4). Thus $(((F')^k, \|\cdot\|'_k) : k \in \mathbb{N})$ is a multi-Banach space. ∎

Let $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-normed space. Then, for each $k \in \mathbb{N}$, the norm on $(E'')^k$ which is the dual norm to $\|\cdot\|'_k$ on $(E')^k$ is temporarily denoted by $\|\cdot\|''_k$. It is clear from Theorems 2.31 and 2.33 that $(((E'')^k, \|\cdot\|''_k) : k \in \mathbb{N})$ is a multi-Banach space.

Of course the embedding of each space $(E^k, \|\cdot\|_k)$ into $((E'')^k, \|\cdot\|_k'')$ is an isometry of normed spaces, and so we can write $\|\cdot\|_k$ consistently for $\|\cdot\|_k''$ on $(E^k)''$. Thus we have the following conclusion.

THEOREM 2.34. *Let $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-normed space. Then*

$$((E^k, \|\cdot\|_k) : k \in \mathbb{N})$$

is a multi-normed subspace of the multi-Banach space $(((E'')^k, \|\cdot\|_k) : k \in \mathbb{N})$. ■

2.4. Reformulations of the axioms. In this section, we shall give some reformulations of the axioms for a multi-normed space $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$.

2.4.1. Multi-norms and matrices. Again, let E be a linear space, and suppose that $m, n \in \mathbb{N}$. We have remarked that $\mathbb{M}_{m,n}$ acts as a map from E^n to E^m in the obvious way; in particular, E^n is a left $\mathbb{M}_n$-module. Our reformulation requires these actions to be 'Banach' actions, so that, for each $m, n \in \mathbb{N}$, we have

$$\|a \cdot x\|_m \leq \|a\| \, \|x\|_n \quad (x \in E^n, \, a \in \mathbb{M}_{m,n}),$$

where we recall that $\|a\|$ is an abbreviation of $\|a : \ell_n^\infty \to \ell_m^\infty\|$. In particular, E^n is a Banach left $\mathbb{M}_n$-module. See [16] for a discussion of the theory of Banach left A-modules over a Banach algebra A.

We first give some preliminary notions. Let $m, n \in \mathbb{N}$, and let

$$a = (a_{ij}) \in \mathbb{M}_{m,n}.$$

Then a is a *row-special* matrix if, for each $i \in \mathbb{N}_m$, there is at most one non-zero term, say $a_{i,j(i)}$, in the i^{th} row, the term $a_{i,j(i)}$ being in the $j(i)^{\text{th}}$ column.

We *claim* that each $a = (a_{ij}) \in \mathbb{M}_{m,n}$ can be written as

$$a = \sum_{r=1}^{k} a_r,$$

where $a_1, \dots, a_k$ are row-special matrices in $\mathbb{M}_{m,n}$ and

$$\|a\| = \sum_{r=1}^{k} \|a_r\|.$$

To prove this claim, we may suppose that $a \neq 0$. For each $i \in \mathbb{N}_m$ such that the i^{th} row of a is non-zero, choose $j(i) \in \mathbb{N}_n$ to be the maximum number $j \in \mathbb{N}_n$ such that $a_{ij} \neq 0$, and set

$$c_i = a_{i,j(i)} \quad (i \in \mathbb{N}_n),$$

taking $c_i = 0$ when the i^{th} row of a is zero. Then choose $i_0 \in \mathbb{N}_n$ such that

$$|c_{i_0}| = \min\{|c_i| : c_i \neq 0, \, i \in \mathbb{N}_m\}.$$

Finally, define a matrix $b \in \mathbb{M}_{m,n}$ by setting

$$b_{i,j(i)} = \frac{c_i}{|c_i|} |c_{i_0}| \quad (i \in \mathbb{N}_m),$$

(with $b_{i,j} = 0$ $(j \in \mathbb{N}_n)$ whenever the i^{th} row of a is zero), and setting $b_{r,s} = 0$ whenever $(r, s) \neq (i, j(i))$ for any $i \in \mathbb{N}_n$. The matrix b is row-special. Further, we can see from

(1.15) that $\|b\| = |c_{i_0}|$. The coefficients of the matrix $a - b$ are the same as those of a, save that, for each $i \in \mathbb{N}_n$ for which the i^{th} row of a is non-zero, the coefficient $a_{i,j(i)}$ has been replaced by

$$a_{i,j(i)}\left(1 - \frac{|c_{i_0}|}{|c_i|}\right) = c_i\left(1 - \frac{|c_{i_0}|}{|c_i|}\right),$$

and so $\sum_{j=1}^n |a_{ij}|$ is replaced by $\sum_{j=1}^n |a_{ij}| - |c_{i_0}| \geq 0$, and $a_{i_0,j(i_0)}$ becomes 0. Note that no zero term in the matrix (a_{ij}) is changed. It follows immediately that $\|a - b\| = \|a\| - |c_{i_0}|$, and so $\|a - b\| + \|b\| = \|a\|$.

We continue to decompose $a - b$ in a similar way; after at most mn steps, the process must terminate, and then we have the claimed representation of the matrix a.

THEOREM 2.35. *Let $(E, \|\cdot\|)$ be a normed space, and take $N \in \mathbb{N}$. Suppose that, for each $n \in \mathbb{N}_N$, $\|\cdot\|_n$ is a norm on the space E^n and, further, that $\|x\|_1 = \|x\|$ $(x \in E)$. Then the following are equivalent:*

(a) *$(\|\cdot\|_n : n \in \mathbb{N}_N)$ is a multi-norm of level N on $\{E^n : n \in \mathbb{N}_N\}$;*
(b) *$\|a \cdot x\|_m \leq \|a\| \, \|x\|_n$ for each row-special matrix $a \in \mathbb{M}_{m,n}$, each $x \in E^n$, and each $m, n \in \mathbb{N}_N$;*
(c) *$\|a \cdot x\|_m \leq \|a\| \, \|x\|_n$ for each $a \in \mathbb{M}_{m,n}$, each $x \in E^n$, and each $m, n \in \mathbb{N}_N$.*

Proof. (a)⇒(b) Suppose that $(\|\cdot\|_n : n \in \mathbb{N}_N)$ is a multi-norm of level N on the family $\{E^n : n \in \mathbb{N}_N\}$, and let a be a row-special matrix, of the form specified above. Then, for each $x \in E^n$, we have the following, where we take $a_{i,j(i)} = 0$ when the i^{th} row of a is zero:

$$\begin{aligned} \|a \cdot x\|_m &= \|(a_{1,j(1)}x_{j(1)}, \dots, a_{m,j(m)}x_{j(m)})\|_m \\ &\leq \max\{|a_{1,j(1)}|, \dots, |a_{m,j(m)}|\}\|(x_1, \dots, x_n)\|_n \quad \text{by Lemma 2.15} \\ &= \|a\| \, \|x\|_n \quad \text{by (1.15)}, \end{aligned}$$

and so (b) holds.

(b)⇒(c) Let $a \in \mathbb{M}_{m,n}$, where $m, n \in \mathbb{N}_N$. Then $a = \sum_{r=1}^k a_r$, where $a_1, \dots a_k$ are row-special matrices in $\mathbb{M}_{m,n}$ and $\|a\| = \sum_{r=1}^k \|a_r\|$, as in the decomposition given above. For each $x \in E^n$, we have

$$\|a \cdot x\|_m \leq \|a_1 \cdot x\|_n + \cdots + \|a_k \cdot x\|_n \leq (\|a_1\| + \cdots + \|a_k\|)\|x\|_n = \|a\| \, \|x\|_n,$$

as required.

(c)⇒(b) This is immediate.

(b)⇒(a) We must show that Axioms (A1)–(A4) of Definition 2.1 are satisfied. Let $k \in \mathbb{N}_N$ with $k \geq 2$.

Let $x \in E^k$. By taking a to be, first, a suitable matrix in $\mathbb{M}_k$ with exactly one non-zero term equal to 1 in each row, so that a corresponds to a given permutation in $\mathfrak{S}_k$, and, second, a diagonal matrix with diagonal terms $\alpha_1, \dots, \alpha_k \in \mathbb{C}$, we see that (A1) and (A2) follow immediately from (b).

Now take $x_1, \dots, x_{k-1} \in E$, and take $a \in \mathbb{M}_{k,k-1}$ to be the row-special matrix

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

It follows from (b) that $\|(x_1, \dots, x_{k-1}, 0)\|_k \leq \|(x_1, \dots, x_{k-1})\|_{k-1}$. Similarly, we see that $\|(x_1, \dots, x_{k-1})\|_{k-1} \leq \|(x_1, \dots, x_{k-1}, 0)\|_k$, and so (A3) holds.

Finally, take $a \in \mathbb{M}_k$ to be the row-special matrix

$$\begin{bmatrix} 1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

Then $\|a\| = 1$, and it follows from (b), (A2), and (A3) that

$$\|(x_1, \dots, x_{k-1}, x_{k-1})\|_k \leq \|(x_1, \dots, x_{k-1}, 0)\|_k = \|(x_1, \dots, x_{k-1})\|_{k-1}.$$

Similarly, $\|(x_1, \dots, x_{k-1})\|_{k-1} \leq \|(x_1, \dots, x_{k-1}, x_{k-1})\|_k$, and so (A4) holds.

We have shown that $(\|\cdot\|_n : n \in \mathbb{N}_N)$ is a multi-norm of level N on the family $\{E^n : n \in \mathbb{N}_N\}$, giving (a). ■

2.4.2. Dual multi-norms and matrices. Let $m, n \in \mathbb{N}$, and let $a = (a_{ij}) \in \mathbb{M}_{m,n}$. Then a is a *column-special* matrix if, for each $j \in \mathbb{N}_n$, there is at most one non-zero term in the j^{th} column. Clearly the transpose of a row-special matrix is a column-special matrix, and vice versa.

We *claim* that each $a = (a_{ij}) \in \mathbb{M}_{m,n}$ can be written as

$$a = \sum_{r=1}^{k} a_r,$$

where $a_1, \dots, a_k$ are column-special matrices in $\mathbb{M}_{m,n}$ and $\|a\| = \sum_{r=1}^{k} \|a_r\|$, where now $\|a\|$ is an abbreviation of $\|a : \ell_n^1 \to \ell_m^1\|$. This claim follows from an earlier remark by taking transposes.

The following theorem can be proved by a similar argument to that in Theorem 2.35. Indeed, the proof uses Lemma 2.21 and the above decomposition of matrices. For details, see [61, Theorem 4.6.4].

THEOREM 2.36. *Let $(E, \|\cdot\|)$ be a normed space, and take $N \in \mathbb{N}$. Suppose that, for each $n \in \mathbb{N}_N$, $\|\cdot\|_n$ is a norm on the spaces E^n and, further, that $\|x\|_1 = \|x\|$ $(x \in E)$. Then the following are equivalent:*

(a) $(\|\cdot\|_n : n \in \mathbb{N}_N)$ *is a dual multi-norm of level N on $\{E^n : n \in \mathbb{N}_N\}$;*
(b) $\|a \cdot x\|_m \leq \|a : \ell_n^1 \to \ell_m^1\| \, \|x\|_n$ *for each column-special $a \in \mathbb{M}_{m,n}$, each $x \in E^n$, and each $m, n \in \mathbb{N}_N$;*

(c) $\|a \cdot x\|_m \leq \|a : \ell_n^1 \to \ell_m^1\| \, \|x\|_n$ *for each* $a \in \mathbb{M}_{m,n}$*, each* $x \in E^n$*, and each* $m, n \in \mathbb{N}_N$*.* ∎

As remarked in [61], the above two characterizations of multi-normed spaces and of dual multi-normed spaces together give an alternative proof of Theorems 2.31 and 2.33.

2.4.3. Generalizations. Consideration of Theorems 2.35 and 2.36 suggest a further generalization of the notions of multi-norms and dual multi-norms. The following is [61, Definition 4.3.1].

DEFINITION 2.37. Let $(E, \|\cdot\|)$ be a normed space, and take $p \in [1, \infty]$. A *type-p multi-norm* on $\{E^n : n \in \mathbb{N}\}$ is a sequence $(\|\cdot\|_n : n \in \mathbb{N})$ such that $\|\cdot\|_n$ is a norm on E^n for each $n \in \mathbb{N}$, such that $\|x\|_1 = \|x\|$ for each $x \in E$, and such that

$$\|a \cdot x\|_m \leq \|a : \ell_n^p \to \ell_m^p\| \, \|x\|_n$$

for each matrix $a \in \mathbb{M}_{m,n}$, each $x \in E^n$, and each $m, n \in \mathbb{N}$.

Thus a multi-norm is a type-∞ multi-norm and a dual multi-norm is a type-1 multi-norm in the sense of the above definition. A type-p multi-norm is a special-norm in the above sense.

For example, fix $p \in [1, \infty]$, let $E = \mathbb{C}$, and take the ℓ^p-norm on E^n for each $n \in \mathbb{N}$. Then we obtain a type-p multi-norm. Further, a short calculation involving the matrices

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{in } \mathbb{M}_2,$$

shows that this example is not a type-q multi-norm for any $q \in [1, \infty]$ save for $q = p$. Thus the classes prescribed by type-p multi-norms are distinct for different values of p.

EXAMPLE 2.38. Let E be a Banach space, and take $p \in [1, \infty]$. For $n \in \mathbb{N}$, define

$$\|(x_1, \dots, x_n)\|_n = \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p} \qquad (x_1, \dots, x_n \in E),$$

and consider the sequence $(\|\cdot\|_n : n \in \mathbb{N})$. In the case where $p = 1$, we obtain a dual multi-norm, and in the case where $p = \infty$, we obtain a multi-norm based on E. Now take $p \in (1, \infty)$. Then it follows from [47, §4] that $(\|\cdot\|_n : n \in \mathbb{N})$ is a type-p multi-norm if and only if E is isometrically isomorphic to a subspace of a quotient of an L^p-space. ∎

The following is [61, Lemmas 4.3.2 and 4.3.3].

PROPOSITION 2.39. *Let E be a normed space. Suppose that $(\|\cdot\|_n : n \in \mathbb{N})$ is a type-p multi-norm on $\{E^n : n \in \mathbb{N}\}$, where $p \in [1, \infty]$. Take $n \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{C}$. Then*

$$\|(x_1, \dots, x_{n-1}, \alpha x_n, \beta x_n)\|_{n+1} = \|(x_1, \dots, x_{n-1}, \gamma x_n\|_n$$

for $x_1, \dots, x_n \in E$, where $\gamma = (|\alpha|^p + |\beta|^p)^{1/p}$. In particular,

$$\|(x_1, \dots, x_{n-1}, x_n, x_n)\|_{n+1} = \|(x_1, \dots, x_{n-1}, 2^{1/p} x_n)\|_n$$

for $x_1, \dots, x_n \in E$. ∎

The following result, from [61], generalizes those of §2.3.

THEOREM 2.40. *Let E be a normed space, and take $p \in [1, \infty]$. Then the dual of a type-p multi-norm on $\{E^n : n \in \mathbb{N}\}$ is a type-q multi-norm on $\{(E')^n : n \in \mathbb{N}\}$, where q is the conjugate index to p.* ■

2.4.4. Sequential norms. Let E be a Banach space. A somewhat similar notion to that of our multi-norms has already been defined; these are sequential norms on the family $\{E^n : n \in \mathbb{N}\}$; these norms were first defined and extensively studied in [49], and their definition and basic properties are summarized in [50].

Indeed, a *sequential norm* on $\{E^n : n \in \mathbb{N}\}$ is a sequence $(\|\cdot\|_n : n \in \mathbb{N})$ such that $\|\cdot\|_n$ is a norm on E^n for each $n \in \mathbb{N}$, such that $\|x\|_1 = \|x\|$ for each $x \in E$, and such that the following axioms are satisfied for each $m, n \in \mathbb{N}$:

(L1) $\|(x_1, \ldots, x_n, 0)\|_{n+1} = \|(x_1, \ldots, x_n)\|_n \ (x_1, \ldots, x_n \in E)$;

(L2) $\|(x_1, \ldots, x_m, y_1, \ldots, y_n)\|_{m+n}^2 = \|(x_1, \ldots, x_m)\|_m^2 + \|(y_1, \ldots, y_n)\|_n^2$

whenever $x_1, \ldots, x_m, y_1, \ldots, y_n \in E$;

(L3) $\|a \cdot x\|_m \leq \|a : \ell_n^2 \to \ell_m^2\| \, \|x\|_n \ (x \in E^n, \ a \in \mathbb{M}_{m,n})$.

The space E together with the sequential norm $(\|\cdot\|_n : n \in \mathbb{N})$ is called an *operator sequence space* over E.

It is clear that a sequential norm is a type-2 multi-norm, and so it satisfies our axioms (A1), (A2), and (A3). The above example, with $p = 2$, gives a sequential norm which is not a type-q multi-norm for any $q \in [1, \infty]$ save for $q = 2$. On the other hand, a multi-norm satisfies (L1), but it need not satisfy (L2). For example, let $E = \mathbb{C}$, and consider the multi-norm specified by

$$\|(\alpha, \beta)\|_2 = |\alpha| + |\beta| \quad (\alpha, \beta \in \mathbb{C}).$$

This is rarely equal to $(|\alpha|^2 + |\beta|^2)^{1/2}$, as required by (L2). In fact, a multi-norm never satisfies (L3) (unless $E = \{0\}$). For take $x \in E$ with $\|x\| = 1$, and take

$$a = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \in \mathbb{M}_2,$$

so that $\|(x, x)\|_2 = 1$ by Lemma 2.15 and hence $\|a\| = \sqrt{2}$, but $\|a \cdot x\|_2 = \|(2x, 0)\|_2 = 2$ by (A3). Thus (L3) fails.

2.4.5. Multi-norms and tensor norms. The following definition and theorem (with a proof) will be given in [19].

DEFINITION 2.41. Let $(E, \|\cdot\|)$ be a normed space. Then a norm $\|\cdot\|$ on the tensor product $c_0 \otimes E$ is a c_0-*norm* if

$$\|\delta_1 \otimes x\| = \|x\| \quad (x \in E)$$

and if $T \otimes I_E$ is a bounded linear operator on $(c_0 \otimes E, \|\cdot\|)$ with

$$\|T \otimes I_E\| \leq \|T\| \quad (T \in \mathcal{K}(c_0)).$$

In fact, each such c_0-norm $\|\cdot\|$ is a reasonable cross-norm, and so we have

$$\|z\|_\varepsilon \leq \|z\| \leq \|z\|_\pi \quad (z \in c_0 \otimes E).$$

It will also be noted in [19] that, for each such c_0-norm $\|\cdot\|$ on $c_0 \otimes E$, we have

$$\|T \otimes I_E\| = \|T\| \qquad (T \in \mathcal{B}(c_0)). \tag{2.3}$$

Let $\|\cdot\|$ be a c_0-norm on a space $c_0 \otimes E$, and take $n \in \mathbb{N}$. We define

$$\|(x_1, \dots, x_n)\|_n = \left\|\sum_{j=1}^{n} \delta_j \otimes x_j\right\| \qquad (x_1, \dots, x_n \in E). \tag{2.4}$$

For example, the injective norm $\|\cdot\|_\varepsilon$ on the tensor product $c_0 \otimes E$ is such that

$$\left\|\sum_{j=1}^{n} \delta_j \otimes x_j\right\|_\varepsilon = \max_{i \in \mathbb{N}_n} \|x_i\| \qquad (x_1, \dots, x_n \in E)$$

for each $n \in \mathbb{N}$, and it is easily seen that $\|\cdot\|_\varepsilon$ is a c_0-norm. It is also easily seen that the projective norm $\|\cdot\|_\pi$ is a c_0-norm. Thus $\|\cdot\|_\varepsilon$ and $\|\cdot\|_\pi$ are the minimum and maximum c_0-norms on $c_0 \otimes E$, respectively.

THEOREM 2.42. *Let E be a normed space. Then the family $\mathcal{E}_E$ of multi-norms based on E corresponds bijectively to the family of c_0-norms on $c_0 \otimes E$ via the above correspondence.* ∎

In fact, a more general theorem will be proved in [19, Theorem 3.5]. There is a similar characterization of dual multi-norms; one replaces 'c_0' by 'ℓ^1'; see [19].

Let E be a normed space. Then we have seen that there are two complementary approaches to the theory of multi-normed spaces: the 'coordinate approach' involving sequences $(\|\cdot\|_n : n \in \mathbb{N})$ of norms, where $\|\cdot\|_n$ is a norm on E^n for each $n \in \mathbb{N}$, and the 'non-coordinate approach' involving norms on the tensor product $c_0 \otimes E$. An analogous contrast appears in the well-known theory of *operator space theory*, or *quantum functional analysis*. The 'coordinate approach' to this theory involves sequences $(\|\cdot\|_n : n \in \mathbb{N})$ of norms, where $\|\cdot\|_n$ is a norm on $\mathbb{M}_n(E)$ for each $n \in \mathbb{N}$; the complementary 'non-coordinate approach' involves norms on $\mathcal{F}(L) \otimes E$, where $\mathcal{F}(L)$ denotes the space of finite-rank operators on a fixed separable Hilbert space L. The former approach predominates in the works [12, 28, 57, 60], for example; the latter approach predominates in the monograph [35] of Helemskii, and the Introduction to [35] contains a clear discussion of the contrasting strengths of the two approaches. We give some brief details of the two approaches.

DEFINITION 2.43. Let E be a linear space, and consider an assignment of norms $\|\cdot\|_n$ on $\mathbb{M}_n(E)$ for each $n \in \mathbb{N}$; these norms are called the *matrix norms*. An *abstract operator space* on E is a sequence $(\|\cdot\|_n : n \in \mathbb{N})$ of matrix norms such that:

(M1) $\|\alpha v \beta\|_n \leq \|\alpha\|\, \|v\|_m \|\beta\|$ for $m, n \in \mathbb{N}$, $\alpha \in \mathbb{M}_{n,m}$, $\beta \in \mathbb{M}_{m,n}$, and $v \in \mathbb{M}_m(E)$.
(M2) $\|v \oplus w\|_{m+n} = \max\{\|v\|_m, \|w\|_n\}$ for $m, n \in \mathbb{N}$, $v \in \mathbb{M}_m(E)$, and $w \in \mathbb{M}_n(E)$.

The following definition is taken from [35]. We set $\mathcal{F} = \mathcal{F}(L)$ for a fixed Hilbert space L, and note that $\mathcal{F} \otimes E$ is a bimodule (with operations denoted by $\cdot$) over $\mathcal{B}(L)$.

DEFINITION 2.44. Let E be a linear space. Then a *quantum norm* on E is a norm $\|\cdot\|$ on $\mathcal{F} \otimes E$ satisfying the following two conditions:

(R1) $\|T \cdot u\|, \|u \cdot T\| \leq \|T\| \, \|u\|$ whenever $T \in \mathcal{B}(L)$ and $u \in \mathcal{F} \otimes E$;

(R2) whenever $u, v \in \mathcal{F} \otimes E$ and there exist self-adjoint projections $P, Q \in \mathcal{B}(L)$ with $P \cdot u \cdot P = u$, with $Q \cdot v \cdot Q = v$, and with $PQ = 0$, then $\|u + v\| \leq \max\{\|u\|, \|v\|\}$.

It is shown in [35] that the family of quantum norms on E corresponds bijectively to the abstract operator space on E described in Definition 2.43.

Given an axiomatic theory one often wishes to find a 'concrete representation' of the objects defined by the theory. For example, the Gel'fand–Naimark theory gives a concrete representation of each abstractly-defined C^*-algebra as a self-adjoint, norm-closed subalgebra of the C^*-algebra $\mathcal{B}(H)$ for some Hilbert space H. The concrete representation of an abstract operator space is *Ruan's theorem*, which represents each such system as a closed subspace of $\mathcal{B}(H)$ for some Hilbert space H, the matricial norms being recovered in a canonical way.

After a first draft of this work was completed, the late Professor Nigel Kalton pointed out the memoir [53] of Marcolini Nhani; I am deeply grateful for this reference and for some valuable comments.

In fact, let E be a Banach space. Then a norm $\|\cdot\|$ on $c_0 \otimes E$ satisfies '*condition* (P)' of [53, §2, p. 12] if

$$\|(T \otimes I_E)(z)\| \leq \|T\| \, \|z\| \quad (z \in c_0 \otimes E, \, T \in \mathcal{B}(c_0)).$$

It is clear from our remarks that such norms are exactly the c_0-norms of Definition 2.41, and so the definition of a multi-normed space corresponds to the theory in [53] of norms on $c_0 \otimes E$ satisfying property (P).

As remarked in [53], this theory is a form of 'commutative counterpart' to that of operator space theory. Indeed, we obtain the Axiom (P) by replacing $\mathcal{F}$ by c_0 in the axiom (R1). However, our theory has no analogue of Axiom (R2), so, in that sense, it is more general.

The analogue of Ruan's theorem is Pisier's theorem, given as Théorème 2.1 in [53]; we shall describe this result in Theorem 4.56, below.

3. The minimum and maximum multi-norms

In this chapter, we shall first define a 'rate-of-growth' sequence $(\varphi_n(E))$ for each multi-normed space $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$, and then define two important examples of multi-norms for an arbitrary normed space E: these are the minimum and the maximum multi-norms. We shall investigate the duals of these multi-norms and the sequence $(\varphi_n^{\max}(E))$ corresponding to the maximum multi-norm, and relate them to p-summing constants.

3.1. An associated sequence

DEFINITION 3.1. Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space. For each $n \in \mathbb{N}$, set

$$\varphi_n(E) = \sup\{\|(x_1, \dots, x_n)\|_n : x_1, \dots, x_n \in E_{[1]}\}.$$

The sequence $(\varphi_n(E) : n \in \mathbb{N})$ is the *rate of growth* sequence for the multi-normed space.

Note that $\varphi_n(E)$ is not intrinsic to the initial normed space E; it depends on the multi-norm, and so, strictly, we should write $(\varphi_n(E^n, \|\cdot\|_n))$ instead of $(\varphi_n(E))$.

Suppose that two multi-norms are equivalent. Then their rate of growth sequences are similar. However, the converse to this is not true; see Proposition 4.29 below.

Clearly $(\varphi_n(E) : n \in \mathbb{N})$ is an increasing sequence in $\mathbb{R}$ for each multi-normed space $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$, and it follows from Lemma 2.11 that

$$1 \le \varphi_n(E) \le n \quad (n \in \mathbb{N})$$

and from Lemma 2.10 that

$$\varphi_{m+n}(E) \le \varphi_m(E) + \varphi_n(E) \quad (m, n \in \mathbb{N}).$$

Let F be a subspace of a normed space E, so that $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ is a multi-normed subspace of a multi-normed space $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$. Then clearly we have $\varphi_n(F) \le \varphi_n(E)$ $(n \in \mathbb{N})$.

3.2. The minimum multi-norm

3.2.1. Definitions. We first define the most obvious multi-norm.

DEFINITION 3.2. Let $(E, \|\cdot\|)$ be a normed space. For $k \in \mathbb{N}$, define $\|\cdot\|_k^{\min}$ on E^k by

$$\|(x_1, \dots, x_k)\|_k^{\min} = \max_{i \in \mathbb{N}_k} \|x_i\| \quad (x_1, \dots, x_k \in E).$$

It is immediate that $\|\cdot\|_k^{\min}$ is a norm on E^k for each $k \in \mathbb{N}$, and then that, for each $n \in \mathbb{N}$, the sequence $(\|\cdot\|_k^{\min} : k \in \mathbb{N}_n)$ is a multi-norm of level n. It follows that $((E^k, \|\cdot\|_k^{\min}) : k \in \mathbb{N})$ is a multi-normed space.

DEFINITION 3.3. Let $(E, \|\cdot\|)$ be a normed space. For each $n \in \mathbb{N}$, the sequence

$$(\|\cdot\|_k^{\min} : k \in \mathbb{N}_n)$$

is the *minimum multi-norm of level* n. The sequence $(\|\cdot\|_n^{\min} : n \in \mathbb{N})$ is the *minimum multi-norm.* The *rate of growth* of this multi-norm is $(\varphi_n^{\min}(E) : n \in \mathbb{N})$.

It follows immediately from this example that there is indeed a multi-norm based on each normed space $(E, \|\cdot\|)$. The terminology 'minimum' is justified by Lemma 2.11, given above. The minimum multi-norm corresponds to the injective norm on the tensor product $c_0 \otimes E$ via the correspondence of Chapter 2, §6.4; see [19].

Let $(\mathcal{E}_E, \leq)$ be the partially ordered family of multi-norms based on E, as in Definition 2.24. Then it is clear that the minimum multi-norm is the minimum element in $(\mathcal{E}_E, \leq)$.

More generally, take $n \in \mathbb{N}$, and let $((E^k, \|\cdot\|_k) : k \in \mathbb{N}_n)$ be a multi-normed space of level n on $\{E^k : k \in \mathbb{N}_n\}$. For $m > n$, define

$$\|(x_1, \dots, x_m)\|_m = \max\{\|(y_1, \dots, y_n)\|_n : y_1, \dots, y_n \in \{x_1, \dots, x_m\}\}$$

for $x_1, \dots, x_m \in E$. Then $((E^m, \|\cdot\|_m) : m \in \mathbb{N})$ is a multi-normed space. Thus a multi-norm of level n can be extended to a multi-norm, in an obvious sense.

The following result is immediate.

PROPOSITION 3.4. *Let E be a normed space, and let $(\|\cdot\|_n : n \in \mathbb{N})$ be a multi-norm based on E. Then $(\|\cdot\|_n : n \in \mathbb{N})$ is equal to the minimum multi-norm if and only if $\varphi_n(E) = 1$ $(n \in \mathbb{N})$, and it is equivalent to the minimum multi-norm if and only if $(\varphi_n(E) : n \in \mathbb{N})$ is bounded.* ■

Let E be a normed space, let $n \in \mathbb{N}$, and let $((E^k, \|\cdot\|_k) : k \in \mathbb{N}_n)$ be a multi-normed space of level n. Extend this multi-norm to the multi-normed space $((E^m, \|\cdot\|_m) : m \in \mathbb{N})$, as above. Then clearly $\varphi_m(E) = \varphi_n(E)$ $(m \geq n)$. Thus there are multi-norms which are equivalent to the minimum multi-norm, but are not equal to it, whenever $\varphi_2(E) > 1$.

Let $(E, \|\cdot\|)$ be a normed space. As noted above, there is a similar ordering of dual multi-norms on the family $\{E^n : n \in \mathbb{N}\}$. As in Example 2.6, the sequence $(\|\cdot\|_n : n \in \mathbb{N})$, where

$$\|(x_1, \dots, x_n)\|_n = \sum_{j=1}^{n} \|x_j\| \quad (x_1, \dots, x_n \in E),$$

is a dual multi-norm on $\{E^n : n \in \mathbb{N}\}$. It follows from Lemma 2.11 that this sequence $(\|\cdot\|_n : n \in \mathbb{N})$ is the *maximum dual multi-norm* on $\{E^n : n \in \mathbb{N}\}$.

Let E be a normed space. It is easily seen that the dual of the minimum multi-norm on $\{E^n : n \in \mathbb{N}\}$ is $(\|\cdot\|'_n : n \in \mathbb{N}_n)$, where $\|\cdot\|'_n$ is defined by

$$\|(\lambda_1, \dots, \lambda_n)\|'_n = \sum_{j=1}^{n} \|\lambda_j\| \quad (\lambda_1, \dots, \lambda_n \in E'),$$

and that the dual of the maximum dual multi-norm on $\{E^n : n \in \mathbb{N}\}$ is the sequence $(\|\cdot\|'_n : n \in \mathbb{N}_n)$, where

$$\|(\lambda_1, \dots, \lambda_n)\|'_n = \max\{\|\lambda_1\|, \dots, \|\lambda_n\|\} \quad (\lambda_1, \dots, \lambda_n \in E'),$$

and so, by Lemma 2.11, which applies to dual multi-norms, the following result is immediate.

PROPOSITION 3.5. *Let E be a normed space, and take $n \in \mathbb{N}$. Then:*

(i) *the dual of the minimum multi-norm on $\{E^n : n \in \mathbb{N}\}$ is the maximum dual multi-norm on $\{(E')^n : n \in \mathbb{N}\}$;*
(ii) *the dual of the maximum dual multi-norm on $\{E^n : n \in \mathbb{N}\}$ is the minimum multi-norm on $\{(E')^n : n \in \mathbb{N}\}$;*
(iii) *the second dual of the minimum multi-norm on $\{E^n : n \in \mathbb{N}\}$ is the minimum multi-norm on $\{(E'')^n : n \in \mathbb{N}\}$.* ■

3.2.2. Finite-dimensional spaces. We show the uniqueness of multi-norms based on finite-dimensional normed spaces.

PROPOSITION 3.6. *Let $n \in \mathbb{N}$. Then the minimum multi-norm of level n is the unique multi-norm of level n on $\{\mathbb{C}^k : k \in \mathbb{N}_n\}$.*

Proof. Let $(\|\cdot\|_k : k \in \mathbb{N}_n)$ be a multi-norm of level n on the family $\{\mathbb{C}^k : k \in \mathbb{N}_n\}$. Take $k \in \mathbb{N}_n$. By Lemma 2.15, we have $\|(1,\dots,1)\|_k = 1$. Now take $(\alpha_1,\dots,\alpha_k) \in \mathbb{C}^k$. By (A2), we have

$$\|(\alpha_1,\dots,\alpha_k)\|_k \le \Big(\max_{i\in\mathbb{N}_k}|\alpha_i|\Big)\|(1,\dots,1)\|_k = \max_{i\in\mathbb{N}_k}|\alpha_i|,$$

and, by Lemma 2.11,

$$\max_{i\in\mathbb{N}_k}|\alpha_i| \le \|(\alpha_1,\dots,\alpha_k)\|_k.$$

Thus

$$\|(\alpha_1,\dots,\alpha_k)\|_k = \max_{i\in\mathbb{N}_k}|\alpha_i|,$$

giving the result. ■

PROPOSITION 3.7. *Let $((E^n,\|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space such that E is finite-dimensional. Then $(\|\cdot\|_n : n \in \mathbb{N})$ is equivalent to the minimum multi-norm.*

Proof. Suppose that $\dim E = m$, and take $\{e_1,\dots,e_m\}$ to be a basis of E; we may suppose that $\|e_j\| = 1$ $(j \in \mathbb{N}_n)$. Set $e = (e_1,\dots,e_m) \in E^m$, so that $\|e\|_m \le m$.

There exists a constant $C > 0$ such that each $x \in E$ can be written uniquely as $x = \sum_{j=1}^m \alpha_j e_j$, with $\sum_{j=1}^m |\alpha_j| \le C\|x\|$.

Now take $n \in \mathbb{N}$ and $x_1,\dots,x_n \in E_{[1]}$, say $x_i = \sum_{j=1}^m \alpha_{i,j}e_j$ for $i \in \mathbb{N}_n$. Then $\sum_{j=1}^m |\alpha_{i,j}| \le C$ $(i \in \mathbb{N}_n)$. Set $a = (\alpha_{i,j}) \in \mathbb{M}_{n,m}$, so that

$$\|a : \ell^\infty_m \to \ell^\infty_n\| = \max_{i\in\mathbb{N}_n}\sum_{j=1}^m |\alpha_{i,j}| \le C.$$

Then, using Theorem 2.35, (a)⇒(c), we have

$$\|(x_1,\dots,x_n)\|_n = \|a\cdot e\|_n \le \|a : \ell^\infty_m \to \ell^\infty_n\|\,\|e\|_m \le Cm.$$

Thus $\varphi_n(E) \le Cm$ $(n \in \mathbb{N})$. By Proposition 3.4, $(\|\cdot\|_n : n \in \mathbb{N})$ is equivalent to the minimum multi-norm. ■

3.3. The maximum multi-norm. Let E be a normed space. The multi-norm based on E to be defined in this section, whilst natural, is much more interesting than the minimum multi-norm.

3.3.1. Existence of the maximum multi-norm. We first show that there is a maximum multi-norm.

DEFINITION 3.8. Let $(E, \|\cdot\|)$ be a normed space, take $n \in \mathbb{N}$, and suppose that

$$(|\!|\!|\cdot|\!|\!|_k : k \in \mathbb{N}_n)$$

is a multi-norm of level n on $\{E^k : k \in \mathbb{N}_n\}$. Then $(|\!|\!|\cdot|\!|\!|_k : k \in \mathbb{N}_n)$ is the *maximum multi-norm of level* n if

$$\|(x_1, \dots, x_k)\|_k \leq |\!|\!|(x_1, \dots, x_k)|\!|\!|_k \quad (x_1, \dots, x_k \in E,\ k \in \mathbb{N}_n)$$

whenever $(\|\cdot\|_k : k \in \mathbb{N}_n)$ is a multi-norm of level n on $\{E^k : k \in \mathbb{N}_n\}$.

We define the *maximum multi-norm* on the family $\{E^n : n \in \mathbb{N}\}$ similarly.

Let $n \in \mathbb{N}$. Then it is easy to see that there is a maximum multi-norm of level n on $\{E^k : k \in \mathbb{N}_n\}$ for each normed space $(E, \|\cdot\|)$. Indeed let $\{(\|\cdot\|_k^\alpha : k \in \mathbb{N}_n) : \alpha \in A\}$ be the (non-empty) family of all multi-norms of level n on $\{E^k : k \in \mathbb{N}_n\}$, and, for $k \in \mathbb{N}_n$, set

$$|\!|\!|(x_1, \dots, x_k)|\!|\!|_k = \sup_{\alpha \in A} \|(x_1, \dots, x_k)\|_k^\alpha \quad (x_1, \dots, x_k \in E).$$

It follows from Lemma 2.11 that the supremum is finite in each case, and then it is easily checked that the sequence $(|\!|\!|\cdot|\!|\!|_k : k \in \mathbb{N}_n)$ is a multi-norm of level n on $\{E^k : k \in \mathbb{N}_n\}$, and hence $(|\!|\!|\cdot|\!|\!|_k : k \in \mathbb{N}_n)$ is the maximum multi-norm of level n on $\{E^k : k \in \mathbb{N}_n\}$. Similarly this applies to multi-norms themselves.

DEFINITION 3.9. Let $(E, \|\cdot\|)$ be a normed space. We write

$$(\|\cdot\|_n^{\max} : n \in \mathbb{N})$$

for the maximum multi-norm on $\{E^n : n \in \mathbb{N}\}$.

Suppose that $m, n \in \mathbb{N}$ with $m \leq n$, and let $(\|\cdot\|_k^{\max} : k \in \mathbb{N}_n)$ be the maximum multi-norm of level n on $\{E^k : k \in \mathbb{N}_n\}$. Then it is immediate that $(\|\cdot\|_k^{\max} : k \in \mathbb{N}_m)$ is the maximum multi-norm of level m on $\{E^k : k \in \mathbb{N}_m\}$.

Let $(\mathcal{E}_E, \leq)$ be the partially ordered family of multi-norms on the family $\{E^n : n \in \mathbb{N}\}$ for a normed space E, as in Definition 2.24. It is clear that the maximum multi-norm is the maximum element in $(\mathcal{E}_E, \leq)$. The maximum multi-norm corresponds to the projective norm on the tensor product $c_0 \otimes E$ via the correspondence of Chapter 2, §6.4; see [19].

PROPOSITION 3.10. *Let E be a normed space. Then $(\mathcal{E}_E, \leq)$ is a Dedekind complete lattice.*

Proof. We know that $(\mathcal{E}_E, \leq)$ has a maximum and a minimum element. By Proposition 2.23, the maximum of each pair of elements $\mathcal{E}_E$ belongs to $\mathcal{E}_E$. It is now routine to check that the pointwise supremum of a non-empty set in $\mathcal{E}_E$ is the supremum of the set.

To see that each non-empty subset S in $\mathcal{E}_E$ has an infimum, consider the set T of multi-norms that lie under every element of S. This set T has a supremum, and this supremum is the infimum of S. ∎

Similarly, the family of dual multi-norms on $\{E^n : n \in \mathbb{N}\}$ is a Dedekind complete lattice.

3.3.2. The sequence $(\varphi_n^{\max}(E))$. We now define a key sequence associated to each normed space E.

DEFINITION 3.11. For $n \in \mathbb{N}$, set

$$\varphi_n^{\max}(E) = \sup\{\|(x_1, \ldots, x_n)\|_n^{\max} : x_1, \ldots, x_n \in E_{[1]}\}.$$

Thus the sequence $(\varphi_n^{\max}(E) : n \in \mathbb{N})$ is now intrinsic to the normed space $(E, \|\cdot\|)$; it is the *maximum rate of growth* of any multi-norm on $\{E^n : n \in \mathbb{N}\}$. We find it to be interesting to calculate this sequence for an arbitrary normed space E and for a variety of examples; we shall give some explicit calculations later.

Let E be a normed space with $\dim E \geq n$. Then

$$\varphi_n^{\max}(E) \leq \sup\{\varphi_n^{\max}(F) : \dim F \leq n\}, \tag{3.1}$$

where the supremum is taken over all subspaces F of E with $\dim F \leq n$. We shall see in Example 3.51 that we can have strict inequality in (3.1).

THEOREM 3.12. *Let E and F be Banach spaces, and let G be a λ-complemented subspace of E with G linearly homeomorphic to F. Then*

$$\varphi_n^{\max}(E) \geq \varphi_n^{\max}(F)/d(F,G)\lambda \quad (n \in \mathbb{N}).$$

Proof. There is a projection $P : E \to G$ with $\|P\| \leq \lambda$.

Set $C = d(F,G)$, and take $\varepsilon > 0$. Then there is a bijection $T \in \mathcal{B}(F,G)$ with $\|T\|\,\|T^{-1}\| < C + \varepsilon$.

Let $n \in \mathbb{N}$. Then there are elements $y_1, \ldots, y_n \in F_{[1]}$ and a multi-norm $(\|\cdot\|_k : k \in \mathbb{N})$ on $\{F^k : k \in \mathbb{N}\}$ such that

$$\|(y_1, \ldots, y_n)\|_n > \varphi_n^{\max}(F) - \varepsilon.$$

Set $Q = T^{-1} \circ P \in \mathcal{B}(E,F)$, so that

$$\|Q\|\,\|T\| \leq (C+\varepsilon)\|P\| \leq (C+\varepsilon)\lambda,$$

and then set

$$|||(x_1, \ldots, x_k)|||_k = \max\{\|x_1\|, \ldots, \|x_k\|, \|(Qx_1, \ldots, Qx_k)\|_k/\|Q\|\}$$

for each $k \in \mathbb{N}$ and $x_1, \ldots, x_k \in E$, so that $|||x|||_1 = \|x\|$ $(x \in E)$. Then we quickly see that $(|||\cdot|||_k : k \in \mathbb{N})$ is a multi-norm on the family $\{E^k : k \in \mathbb{N}\}$.

For $j \in \mathbb{N}_n$, set $z_j = Ty_j/\|T\| \in G_{[1]}$, so that $Qz_j = y_j/\|T\|$. Then

$$|||(z_1, \ldots, z_n)|||_n \geq \frac{\|(y_1, \ldots, y_n)\|_n}{\|Q\|\,\|T\|} \geq \frac{\varphi_n^{\max}(F) - \varepsilon}{(C+\varepsilon)\lambda}.$$

Thus $\varphi_n^{\max}(E) \geq (\varphi_n^{\max}(F) - \varepsilon)/(C+\varepsilon)\lambda$. This holds true for each $\varepsilon > 0$, and so the result follows. ■

COROLLARY 3.13. *Let E be a Banach space, and let F be a λ_F-complemented subspace of E. Then*

$$\varphi_n^{\max}(F) \leq \lambda_F \varphi_n^{\max}(E) \quad (n \in \mathbb{N}). \ ■$$

Corollary 3.14. *Let E and F be two linearly homeomorphic Banach spaces. Then*

$$\varphi_n^{\max}(F) \leq d(E,F)\varphi_n^{\max}(E) \quad (n \in \mathbb{N}). \blacksquare$$

The above corollary shows that, when we are seeking to calculate the sequence $(\varphi_n^{\max}(E) : n \in \mathbb{N})$ for a normed space E of dimension n, we may suppose that we have $(\ell_n^1)_{[1]} \subset E_{[1]} \subset (\ell_n^\infty)_{[1]}$ because E is isometrically isomorphic to a normed space F with this additional property.

3.4. Summing norms

3.4.1. Introduction. We shall see below that the calculation of the $(\varphi_n^{\max}(E) : n \in \mathbb{N})$ for certain normed spaces $(E, \|\cdot\|)$ involves some summing operators and p-summing norms. For this reason, we make some preliminary remarks on these norms. For much more information, including considerable history, see [26, 31, 39, 66, 71, 74], for example. Some remarks that we make will not actually be used, and are given to establish some background.

The first definition slightly extends [39, p. 24].

Definition 3.15. Let E be a normed space, let $x_1, \ldots, x_n \in E$, and take $p \geq 1$. Then

$$\mu_{p,n}(x_1, \ldots, x_n) = \sup\left\{ \left(\sum_{j=1}^n |\langle x_j, \lambda \rangle|^p \right)^{1/p} : \lambda \in E'_{[1]} \right\}.$$

Then $\mu_{p,n}$ is the *weak p-summing norm* on E^n.

Let E be a normed space, and take $p \geq 1$ and $n \in \mathbb{N}$. We see that $\mu_{p,n}(x_1, \ldots, x_n) \leq 1$ if and only if $\|(\langle x_j, \lambda \rangle : j \in \mathbb{N}_n)\|_{\ell_n^p} \leq 1$ for each $\lambda \in E'_{[1]}$. It is clear that each $\mu_{p,n}$ is a norm on the space E^n, and indeed $(E^n, \mu_{p,n})$ is a Banach space whenever E is a Banach space. We shall write $\ell_n^p(E)^w$ for the space $(E^n, \mu_{p,n})$.

The sequences $x = (x_j)$ for which there is a constant $C \geq 0$ such that

$$\mu_{p,n}(x_1, \ldots, x_n) \leq C \quad (n \in \mathbb{N})$$

are the *weakly p-summable sequences* in E [66, p. 134]; the least such constant C is a norm on the space of these sequences. These norms are denoted by $\|\cdot\|_p^{\mathrm{weak}}$ in [26, p. 32] and by $\|\cdot\|_p^w$ in [66, (6.4)]. We shall write $\ell^p(E)^w$ for the space of weakly p-summable sequences in E.

Clearly $\mu_{p,n}(0, \ldots, 0, x_j, 0, \ldots, 0) = \|x_j\|$ and

$$\max\{\|x_i\| : i \in \mathbb{N}_n\} \leq \mu_{p,n}(x_1, \ldots, x_n) \leq \left(\sum_{j=1}^n \|x_j\|^p \right)^{1/p} \tag{3.2}$$

for $x_1, \ldots, x_n \in E$, and so $\mu_{p,n}$ satisfies (1.8) and (1.9). Now let $T \in \mathcal{B}(E)$. Then clearly

$$\mu_{p,n}(Tx_1, \ldots, Tx_n) \leq \|T\| \mu_{p,n}(x_1, \ldots, x_n) \quad (x_1, \ldots, x_n \in E). \tag{3.3}$$

Further, we have

$$\mu_{p,n}(x_1, \ldots, x_n) \geq \mu_{q,n}(x_1, \ldots, x_n) \quad (x_1, \ldots, x_n \in E)$$

whenever $1 \leq p \leq q$.

THEOREM 3.16. *Let E be a normed space, and take $p \geq 1$. Then $(\mu_{p,n} : n \in \mathbb{N})$ is a type-p multi-norm.*

Proof. It is easily checked that $(\mu_{p,n} : n \in \mathbb{N})$ satisfies Axioms (A1)–(A3), and so the sequence $(\mu_{p,n} : n \in \mathbb{N})$ is a special-norm.

Take $m, n \in \mathbb{N}$, and then take $x = (x_1, \dots, x_n) \in E^n$ and $a \in \mathbb{M}_{m,n}$. Set $y = a \cdot x$ so that $y \in E^m$, and consider $\lambda \in E'_{[1]}$; we write

$$u_\lambda = (\langle x_j,\, \lambda\rangle : j \in \mathbb{N}_n) \in \ell^p_n \quad \text{and} \quad v_\lambda = (\langle y_i,\, \lambda\rangle : i \in \mathbb{N}_m) \in \ell^p_m.$$

Then $v_\lambda = a \cdot u_\lambda$, and so

$$\|v_\lambda\|_{\ell^p_m} \leq \|a : \ell^p_n \to \ell^p_m\| \, \|u_\lambda\|_{\ell^p_n}.$$

It follows that $\mu_{p,m}(y) \leq \|a : \ell^p_n \to \ell^p_m\| \mu_{p,n}(x)$, and hence $(\mu_{p,n} : n \in \mathbb{N})$ is a type-p multi-norm. ∎

It follows from Proposition 2.39 that

$$\mu_{p,n+1}(x_1, \dots, x_{n-1}, \alpha x_n, \beta x_n) = \mu_{p,n}(x_1, \dots, x_{n-1}, \gamma x_n)$$

for $x_1, \dots, x_n \in E$ and $n \in \mathbb{N}$, where $\gamma = (|\alpha|^p + |\beta|^p)^{1/p}$.

Suppose that F is a subspace of a normed space E, and take elements $x_1, \dots, x_n \in F$. Then, by the Hahn–Banach theorem, the value of $\mu_{p,n}(x_1, \dots, x_n)$ is the same, whether it be evaluated with respect to either F or E. In particular, the restriction of the weak p-summing norm defined on $(E'')^n$ to the subspace E^n agrees with the weak p-summing norm defined on this space.

Let E be a normed space, and take $p > 1$; the conjugate index to p is denoted by q. By [39, p. 26] and [66, (6.4)], it follows that, for each $n \in \mathbb{N}$ and $x_1, \dots, x_n \in E$, we have

$$\mu_{p,n}(x_1, \dots, x_n) = \sup\Big\{\Big\|\sum_{j=1}^n \zeta_j x_j\Big\| : \sum_{j=1}^n |\zeta_j|^q \leq 1\Big\}. \tag{3.4}$$

(Here, and later, we think of E as a complex normed space and take $\zeta_1, \dots, \zeta_n \in \mathbb{C}$; in the case where E is a real normed space, we must take $\zeta_1, \dots, \zeta_n \in \mathbb{R}$.) Similarly, by [39, 2.2], we have

$$\mu_{1,n}(x_1, \dots, x_n) = \sup\Big\{\Big\|\sum_{j=1}^n \zeta_j x_j\Big\| : \zeta_1, \dots, \zeta_n \in \mathbb{T}\Big\}. \tag{3.5}$$

(In the real case, the numbers $\zeta_1, \dots, \zeta_n$ range over the finite set $\{\pm 1\}$.)

Take $p \geq 1$ with conjugate index q. For each $x = (x_1, \dots, x_n) \in E^n$, define

$$T_x : (\zeta_1, \dots, \zeta_n) \mapsto \sum_{j=1}^n \zeta_j x_j, \quad \ell^q_n \to E.$$

Then $T_x \in \mathcal{B}(\ell^q_n, E)$, and it follows from (3.4) and (3.5) that $\mu_{p,n}(x_1, \dots, x_n) = \|T_x\|$. Further, the map $x \mapsto T_x$, $(E^n, \mu_{p,n}) \to \mathcal{B}(\ell^q_n, E)$, is an isometric isomorphism, and so, as in [26, Proposition 2.2],

$$(E^n, \mu_{p,n}) = \ell^p_n(E)^w \cong \mathcal{B}(\ell^q_n, E) \cong (\ell^p_n \otimes E, \|\cdot\|_\pi)'. \tag{3.6}$$

Let E be a normed space, and take $p \geq 1$. By [39, p. 26], we have

$$\mu_{p,n}(\lambda_1, \dots, \lambda_n) = \sup\Big\{ \Big(\sum_{j=1}^{n} |\langle x, \lambda_j \rangle|^p \Big)^{1/p} : x \in E_{[1]} \Big\} \tag{3.7}$$

for $\lambda_1, \dots, \lambda_n \in E'$. In particular,

$$\mu_{1,n}(\lambda_1, \dots, \lambda_n) = \sup\Big\{ \sum_{j=1}^{n} |\langle x, \lambda_j \rangle| : x \in E_{[1]} \Big\}. \tag{3.8}$$

PROPOSITION 3.17. *Let E be a normed space, and take $p \geq 1$ and $n \in \mathbb{N}$. Then the weak p-summing norm on $(E'')^n$ is the second dual of the weak p-summing norm on E^n.*

Proof. We write $\mu_{p,n}$ and $\nu_{p,n}$ for the weak p-summing norms on E^n and $(E'')^n$, respectively. Take $\Lambda = (\Lambda_1, \dots, \Lambda_n) \in (E'')^n$. By (3.7),

$$\nu_{p,n}(\Lambda) = \sup\Big\{ \Big(\sum_{j=1}^{n} |\langle \Lambda_j,\, \lambda \rangle|^p \Big)^{1/p} : \lambda \in (E')^n_{[1]} \Big\}.$$

Take a net (x_α) in E^n such that $x_\alpha \to \Lambda$ in the weak-$*$ topology on $(E'')^n$. For each $\lambda \in (E')^n$, we have $\langle x_\alpha,\, \lambda \rangle \to \langle \Lambda,\, \lambda \rangle$, and so $\nu_{p,n}(x_\alpha) \to \nu(\Lambda)$. Since $\nu_{p,n} \mid E^n = \mu_{p,n}$, it follows that $\nu_{p,n}(\Lambda) = \mu''_{p,n}(\Lambda)$. This gives the result. ∎

Recall that we take $\gamma = \max\{|\alpha|, |\beta|\}$ for $\gamma = (|\alpha|^q + |\beta|^q)^{1/q}$ in the special case where $q = \infty$.

PROPOSITION 3.18. *Let E be a normed space. Take $n \in \mathbb{N}$, $p \geq 1$, and $\alpha, \beta \in \mathbb{C}$, and set $\gamma = (|\alpha|^q + |\beta|^q)^{1/q}$, where q is the conjugate index to p. Then*

$$\mu_{p,n}(x_1, \dots, x_{n-1}, \alpha x_n + \beta x_{n+1}) \leq \mu_{p,n+1}(x_1, \dots, x_{n-1}, \gamma x_n, \gamma x_{n+1})$$

for each $x_1, \dots, x_{n+1} \in E$.

Proof. Take $\lambda \in E'$. Then we have

$$|\langle \alpha x_n + \beta x_{n+1},\, \lambda \rangle| \leq \gamma (|\langle x_n,\, \lambda \rangle|^p + |\langle x_{n+1},\, \lambda \rangle|^p)^{1/p}$$

by Hölder's inequality, and so

$$|\langle \alpha x_n + \beta x_{n+1},\, \lambda \rangle|^p \leq |\langle \gamma x_n,\, \lambda \rangle|^p + |\langle \gamma x_{n+1},\, \lambda \rangle|^p.$$

The result follows from (3.7). ∎

THEOREM 3.19. *Let $(E, \|\cdot\|)$ be a normed space. Then $(\mu_{1,n} : n \in \mathbb{N})$ is a dual multi-norm on $\{E^n : n \in \mathbb{N}\}$, and*

$$\mu_{1,n}(x_1, \dots, x_n) \leq \|(x_1, \dots, x_n)\|_n \qquad (x_1, \dots, x_n \in E) \tag{3.9}$$

whenever $(\|\cdot\|_n : n \in \mathbb{N})$ is a dual multi-norm on $\{E^n : n \in \mathbb{N}\}$.

Proof. It is immediate that $(\mu_{1,n} : n \in \mathbb{N})$ also satisfies Axiom (B4), and so $(\mu_{1,n} : n \in \mathbb{N})$ is a dual multi-norm. Inequality (3.9) follows from (3.5) and Lemma 2.20. ∎

Thus $(\mu_{1,n} : n \in \mathbb{N})$ is the minimum dual multi-norm on $\{E^n : n \in \mathbb{N}\}$.

Clause (i) of the following proposition concerning a specific normed space is given in [39, 2.6]; clause (ii) follows because, for each measure space Ω, the dual space to $L^1(\Omega)$ is order-isometric to a space $C(K)$ for some compact space K, as noted in Corollary 1.39.

Proposition 3.20. *Let $n \in \mathbb{N}$ and take $p \geq 1$.*

(i) *Let K be a compact space. Then*

$$\mu_{p,n}(f_1, \dots, f_n) = \left| \sum_{i=1}^{n} |f_i|^p \right|_K^{1/p} \qquad (f_1, \dots, f_n \in C(K)).$$

(ii) *Let Ω be a measure space. Then*

$$\mu_{p,n}(\lambda_1, \dots, \lambda_n) = \left\| \sum_{i=1}^{n} |\lambda_i|^p \right\|^{1/p} \qquad (\lambda_1, \dots, \lambda_n \in L^1(\Omega)'). \ \blacksquare$$

3.4.2. Summing constants. The following definition of certain important constants is given explicitly in [24], extending one in [26, p. 56], [31, §16.3], [39, p. 33], and [66, §6.3].

Definition 3.21. Let E and F be normed spaces, and take $n \in \mathbb{N}$ and $p, q \in [1, \infty)$ with $p \leq q$. Then the (q,p)-*summing constants* of the operator $T \in \mathcal{B}(E, F)$ are the numbers

$$\pi_{q,p}^{(n)}(T) := \sup\Big\{ \Big(\sum_{j=1}^{n} \|Tx_j\|^q \Big)^{1/q} : x_1, \dots, x_n \in E,\ \mu_{p,n}(x_1, \dots, x_n) \leq 1 \Big\}.$$

Further, $\pi_{q,p}^{(n)}(E) = \pi_{q,p}^{(n)}(I_E)$; these are the (q,p)-*summing constants* of the normed space E. We write $\pi_p^{(n)}(T)$ for $\pi_{p,p}^{(n)}(T)$ and $\pi_p^{(n)}(E)$ for $\pi_{p,p}^{(n)}(E)$.

Let E be a normed space, and take $n \in \mathbb{N}$. For each $p \geq 1$, it follows that

$$\pi_p^{(n)}(E) = \sup\Big\{ \Big(\sum_{j=1}^{n} \|x_j\|^p \Big)^{1/p} : \mu_{p,n}(x_1, \dots, x_n) \leq 1 \Big\}, \tag{3.10}$$

where the supremum is taken over $x_1, \dots, x_n \in E$. In particular,

$$\pi_1^{(n)}(E) = \sup\Big\{ \sum_{j=1}^{n} \|x_j\| : \Big\| \sum_{j=1}^{n} \zeta_j x_j \Big\| \leq 1 \ (\zeta_1, \dots, \zeta_n \in \mathbb{T}) \Big\}, \tag{3.11}$$

where again the supremum is taken over $x_1, \dots, x_n \in E$.

Clearly, in each case,

$$\|T\| \leq \pi_{q,p}^{(n)}(T) \leq n\|T\|,$$

so that $1 \leq \pi_{q,p}^{(n)}(E) \leq n$, and $(\pi_{q,p}^{(n)}(T) : n \in \mathbb{N})$ is an increasing sequence. Also, each $\pi_{q,p}^{(n)}$ is a norm on $\mathcal{B}(E, F)$. Suppose that E is a closed subspace of a Banach space F. Then it is clear that $\pi_{q,p}^{(n)}(E) \leq \pi_{q,p}^{(n)}(F)$.

Let E and F be normed spaces. Then these norms are closely related to the standard (q,p)-*summing norms*. Indeed, for $T \in \mathcal{B}(E, F)$, we set

$$\pi_{q,p}(T) = \sup\{\pi_{q,p}^{(n)}(T) : n \in \mathbb{N}\} = \lim_{n \to \infty} \pi_{q,p}^{(n)}(T) \in [0, \infty].$$

In the case where $\pi_{q,p}(T) < \infty$, the operator T is said to be (*absolutely*) (q,p)-*summing*; the set of these operators is denoted by $\Pi_{q,p}(E, F)$. We shall write $\pi_p(E)$ for $\pi_p(I_E)$, $\pi_p(T)$ for $\pi_{p,p}(T)$, and $(\Pi_p(E, F), \pi_p)$ for $(\Pi_{p,p}(E, F), \pi_{p,p})$, etc. It is clear that $(\Pi_{q,p}(E, F), \pi_{q,p})$ is a Banach space whenever F is a Banach space, and indeed it is a component of an *operator ideal*. There are extensive studies of these ideals in [26, 39, 41, 58, 66, 71, 74], for example.

Elements of $\Pi_1(E,F)$ are also called the (*absolutely*) *summing operators*; they are characterized by the property that the series $\sum_{j=1}^{\infty} Tx_j$ converges absolutely in F whenever $\sum_{j=1}^{\infty} x_j$ converges weakly unconditionally in E.

We shall use the following results about the norms $\pi_p^{(n)}$.

PROPOSITION 3.22. *Let E be a normed space, and take $n \in \mathbb{N}$. Then:*

(i) $\pi_2^{(n)}(T) \le \pi_1^{(n)}(T)$ *and* $\pi_2(T) \le \pi_1(T)$ *for each* $T \in \mathcal{B}(E)$*;*
(ii) $\pi_2(E) = \sqrt{n}$ *whenever* $\dim E = n$*;*
(iii) $\pi_1(E) \ge \sqrt{n}$ *whenever* $\dim E \ge n$*;*
(iv) $\pi_p^{(n)}(E) = \pi_p^{(n)}(E'')$ *for each* $p \ge 1$.

Proof. Clause (i) is a small variation of [39, 3.3, p. 32], and (ii) is [39, Proposition 5.13, p. 62] and [26, Theorem 4.17]. Clearly (iii) follows from (i) and (ii).

Clause (iv) is essentially [39, Proposition 17.4, p. 157]; it follows from the Principle of Local Reflexivity, Proposition 1.4. ∎

There have been studies of the relationship of the numbers $\pi_{q,p}^{(n)}(T)$, and especially when suitable multiples bound $\pi_{q,p}(T)$. For a summary, see [71, Chapter 4]; further results are given in [24] and [40]. We shall use the following result of Szarek [68, Theorem 3].

THEOREM 3.23. *There is a universal constant $C > 0$ such that, for each $n \in \mathbb{N}$, each Banach spaces E and F with* $\dim E = n$*, and each $T \in \mathcal{B}(E,F)$, there exists $k \in \mathbb{N}$ with $k \le n \log n$ such that*

$$\pi_1(T) \le C\pi_1^{(k)}(T). \blacksquare$$

In the following corollary, C is the constant of the above theorem.

COROLLARY 3.24. *Let $n \in \mathbb{N}$, and let F be a normed space such that* $\dim F \ge n$*. Then*

$$\pi_1^{(n)}(F) \ge \frac{1}{C}\sqrt{\left[\frac{n}{\log n}\right]}.$$

Proof. Set $m = [n/\log n]$, and take a subspace E of F with $\dim E = m$; let T be the embedding of E into F.

We have $\pi_1^{(n)}(T) = \pi_1^{(n)}(E) \le \pi_1^{(n)}(F)$, and it follows from (i) and (ii) of Proposition 3.22 that $\sqrt{m} = \pi_2(T) \le \pi_1(T)$. By Theorem 3.23, there exists $k \in \mathbb{N}$ with $k \le m \log m$ and $\pi_1(T) \le C\pi_1^{(k)}(T)$. But

$$m \log m \le \frac{n}{\log n} \cdot (\log n - \log\log n) < n,$$

and so $k \le n$. Thus

$$\pi_1^{(k)}(T) \le \pi_1^{(n)}(T).$$

By combining the various inequalities, we obtain the result. ∎

For a similar result involving $\pi_p^{(n)}(F)$ for $p \ge 1$, see [42].

3.4.3. Related constants. We now introduce two constants related to $\pi_1^{(n)}(E)$ that will be referred to later. Recall that S_E denotes the unit sphere of a normed space E.

DEFINITION 3.25. Let E be a normed space, and take $n \in \mathbb{N}$. Then

$$\overline{\pi}_1^{(n)}(E) = \sup\Big\{\Big(\sum_{j=1}^{n} \|x_j\|\Big) : \|x_1\| = \cdots = \|x_n\|,\ \mu_{1,n}(x_1, \dots, x_n) \le 1\Big\}$$

and

$$c_n(E) = \inf\{\sup\{\|\zeta_1 x_1 + \cdots + \zeta_n x_n\| : \zeta_1, \dots, \zeta_n \in \mathbb{T}\} : x_1, \dots, x_n \in S_E\}.$$

In particular, $c_1(E) = 1$ and

$$c_2(E) = \inf\Big\{\sup_{\zeta \in \mathbb{T}}\{\|x_1 + \zeta x_2\|\} : x_1, x_2 \in S_E\Big\}. \tag{3.12}$$

We see that $(c_n(E) : n \in \mathbb{N})$ is an increasing sequence in $[1, \infty)$. Let $n \in \mathbb{N}$. Then it follows from (3.5) that

$$c_n(E) = \inf\{\mu_{1,n}(x_1, \dots, x_n) : x_1, \dots, x_n \in S_E\}.$$

Clearly, $\overline{\pi}_1^{(n)}(E) \le \pi_1^{(n)}(E)$ and $\overline{\pi}_1^{(n)}(E) \cdot c_n(E) = n$, and so

$$\pi_1^{(n)}(E) \cdot c_n(E) \ge n \quad (n \in \mathbb{N}). \tag{3.13}$$

We first make a simple remark. Let $(E, \|\cdot\|)$ be a normed space. Suppose that $x_1, \dots, x_n \in E$ are such that

$$\sup\{\|\zeta_1 x_1 + \cdots + \zeta_n x_n\| : \zeta_1, \dots, \zeta_n \in \mathbb{T}\} = C$$

(so that $C \in \mathbb{R}^+$), and take $t \ge 1$. Then we *claim* that

$$\sup\{\|\zeta_1 t x_1 + \zeta_2 x_2 + \cdots + \zeta_n x_n\| : \zeta_1, \dots, \zeta_n \in \mathbb{T}\} \ge C.$$

Indeed, take $\varepsilon > 0$ and $\zeta_1, \dots, \zeta_n \in \mathbb{T}$ with $\|\zeta_1 x_1 + \zeta_2 x_2 + \cdots + \zeta_n x_n\| > C - \varepsilon$. Set $y = \zeta_1 x_1 + \zeta_2 x_2 + \cdots + \zeta_n x_n$ and $z = \zeta_1 x_1 - \zeta_2 x_2 - \cdots - \zeta_n x_n$, so that $\|z\| \le C$. Then

$$2(\zeta_1 t x_1 + \cdots + \zeta_n x_n) = (t+1)y + (t-1)z,$$

and so

$$2\|\zeta_1 t x_1 + \cdots + \zeta_n x_n\| \ge (t+1)(C - \varepsilon) - (t-1)C = 2C - (t-1)\varepsilon,$$

from which the claim follows. It follows that

$$c_n(E) \le \sup\{\|\zeta_1 t_1 x_1 + \cdots + \zeta_n t_n x_n\| : \zeta_1, \dots, \zeta_n \in \mathbb{T}\} \tag{3.14}$$

for each $x_1, \dots, x_n \in S_E$ and $t_1, \dots, t_n \ge 1$.

PROPOSITION 3.26. *Let $(E, \|\cdot\|)$ and $(F, \|\cdot\|)$ be Banach spaces, and let G be a closed subspace of E with G linearly homeomorphic to F. Then*

$$c_n(E) \le c_n(F) d(F, G) \quad (n \in \mathbb{N}).$$

Proof. Set $C = d(F, G)$, and take $\varepsilon > 0$. Then there is a bijection $T \in \mathcal{B}(F, G)$ with $\|T\|\,\|T^{-1}\| < C + \varepsilon$.

Let $n \in \mathbb{N}$. Then there are elements $y_1, \dots, y_n \in S_F$ such that

$$\|\zeta_1 y_1 + \cdots + \zeta_n y_n\| < c_n(F) + \varepsilon \quad (\zeta_1, \dots, \zeta_n \in \mathbb{T}).$$

Set $x_j = Ty_j/\|Ty_j\|$ $(j \in \mathbb{N}_n)$. Then $x_1, \dots, x_n \in S_E$. For $j \in \mathbb{N}_n$, set $t_j = \|T^{-1}\|\,\|Ty_j\|$, so that $t_j \geq 1$. By (3.14), we have

$$\begin{aligned} c_n(E) &\leq \sup\{\|\zeta_1 t_1 x_1 + \dots + \zeta_n t_n x_n\| : \zeta_1, \dots, \zeta_n \in \mathbb{T}\} \\ &= \|T^{-1}\| \sup\{\|\zeta_1 T y_1 + \dots + \zeta_n T y_n\| : \zeta_1, \dots, \zeta_n \in \mathbb{T}\} \\ &\leq (C + \varepsilon)(c_n(F) + \varepsilon). \end{aligned}$$

This holds true for each $\varepsilon > 0$, and so the result follows. ■

3.4.4. Orlicz property. The following definition is given in [39, p. 43] and [74, Remark II.D.7].

DEFINITION 3.27. A Banach space $(E, \|\cdot\|)$ has the *Orlicz property with constant C* if $\pi_{2,1}(E) = C$ is finite, so that

$$C := \sup\Big\{\Big(\sum_{j=1}^n \|x_j\|^2\Big)^{1/2} : x_1, \dots, x_n \in E,\ \mu_{1,n}(x_1, \dots, x_n) \leq 1\Big\} < \infty.$$

Clearly $C \geq 1$ in each case. It is shown in [26, Corollary 11.17] and [39, p. 69] that every Banach space 'of cotype 2' has the Orlicz property. We remark that, by [26, Theorem 14.5], an infinite-dimensional Banach space E with the Orlicz property is 'of cotype q' for each $q > 2$, but an example of Talagrand [69] shows that there is a Banach lattice E with the Orlicz property such that E is not of cotype 2.

THEOREM 3.28. *Let E be a Banach space such that E has the Orlicz property with constant C. Then*

$$\pi_1^{(n)}(E) \leq C\sqrt{n}, \qquad \sqrt{n} \leq C c_n(E) \qquad (n \in \mathbb{N}).$$

Proof. Let $n \in \mathbb{N}$ and $x_1, \dots, x_n \in E$. Then

$$\sum_{j=1}^n \|x_j\| \leq \sqrt{n}\Big(\sum_{j=1}^n \|x_j\|^2\Big)^{1/2}$$

by the Cauchy–Schwarz inequality. Now suppose that $\mu_{1,n}(x_1, \dots, x_n) \leq 1$. Then it follows from Definition 3.27 that

$$\sum_{j=1}^n \|x_j\| \leq C\sqrt{n},$$

and so the result follows from (3.10) and (3.13). ■

In particular, $(\pi_1^{(n)}(E)) = O(\sqrt{n})$ for each Banach space E of cotype 2.

The following theorem of Orlicz [56] can be regarded as the historical beginning of the study of summing operators; a proof is given in [26, Corollary 11.7(a)] and [74, Theorem II.D.6].

THEOREM 3.29. *Let (Ω, μ) be a measure space, and take $q \in [1, 2]$. Then the Banach space $L^q(\Omega, \mu)$ has cotype 2, and hence the Orlicz property.* ■

The Orlicz constant associated with the space ℓ^q (for $1 \leq q \leq 2$) is denoted by C_q. We know that $C_2 = 1$ [39, 3.25] and that $C_1 \leq \sqrt{2}$ [39, 7.6].

Corollary 3.30. *Let $q \in [1,2]$. Then*

$$\pi_1^{(n)}(\ell^q) \le C_q\sqrt{n} \quad (n \in \mathbb{N}).$$

In particular, $\pi_1^{(n)}(\ell^2) \le \sqrt{n}$ $(n \in \mathbb{N})$.

Proof. This follows from Theorems 3.28 and 3.29. ∎

3.4.5. Specific spaces. We shall also use the following specific calculations involving the spaces ℓ^p, where $p \ge 1$. Note that, for $n \in \mathbb{N}$, always $\pi_1^{(n)}(\ell_n^p) \le \pi_1^{(n)}(\ell^p) \le \pi_1(\ell^p)$.

Proposition 3.31. *Let $n \in \mathbb{N}$. Then:*

(i) *for each $q \in [1,2]$, we have $\sqrt{n} \le \pi_1^{(n)}(\ell_n^q)$;*
(ii) $\sqrt{n} \le \pi_1^{(n)}(\ell_n^1) \le \pi_1^{(n)}(\ell^1) = \pi_1(\ell_n^1) \le \sqrt{2n}$;
(iii) $\sqrt{n} = \pi_1^{(n)}(\ell_n^2) = \pi_1^{(n)}(\ell^2) \le \pi_1(\ell_n^2) \le (2/\sqrt{\pi})\sqrt{n}$;
(iv) $\pi_1^{(n)}(\ell_n^\infty) = \pi_1^{(n)}(\ell^\infty) = n$;
(v) *for each $q \in [2,\infty)$, we have*

$$\sqrt{n} \le n^{1-1/q} \le \pi_1^{(n)}(\ell_n^q) \le \pi_1^{(n)}(\ell^q).$$

Proof. (i) Take $\zeta = \exp(2\pi \mathrm{i}/n)$ and then set $f_i = (\zeta^i, \zeta^{2i}, \dots, \zeta^{ni})$ for $i \in \mathbb{N}_n$. Then we have $\|\zeta_1 f_1 + \dots + \zeta_n f_n\| \le n^{1/2+1/q}$ by Lemma 1.1(ii). But $\sum_{i=1}^n \|f_i\| = n^{1+1/q}$, and so $\pi_1^{(n)}(\ell_n^q) \ge \sqrt{n}$ by (3.11).

(ii) We have $\pi_1^{(n)}(\ell^1) = \pi_1(\ell_n^1) \le \sqrt{2n}$ by [39, 7.18 and 7.12].

(iii) We have $\pi_1^{(n)}(\ell^2) = \sqrt{n}$ by [39, 3.9] and $\pi_1(\ell_n^2) \le (2/\sqrt{\pi})\sqrt{n}$ by [39, 8.10].

(iv) By taking $x_j = \delta_j$ $(j \in \mathbb{N}_n)$ in (3.11), we see that $\pi_1^{(n)}(\ell_n^\infty) \ge n$; certainly $\pi_1^{(n)}(\ell^\infty) \le n$.

(v) Let $q \in [2,\infty)$. By taking $x_j = \delta_j/n^{1/q}$ $(j \in \mathbb{N}_n)$ in equation (3.11), we see that $\pi_1^{(n)}(\ell_n^q) \ge n^{1-1/q}$. ∎

We note that the precise value of $\pi_1(\ell_n^2)$ is given in [39, 8.10], and that $\pi_1(\ell_n^2) > \sqrt{n}$ for $n \ge 2$; the results are due to Gordon [33]. We also remark that the following estimates (and more general estimates) are contained in [33, Theorem 5]; we shall not use the results. (The results in [33] are for real-valued spaces, but the analogous results follow for our complex-valued spaces, with a possible change in the implicit constants.)

Proposition 3.32.

(i) *Take q with $2 \le q < \infty$. Then $\pi_1^{(n)}(\ell_n^q) \sim n^{1-1/q}$ as $n \to \infty$.*
(ii) *Take q with $1 \le q \le 2$. Then $\pi_1^{(n)}(\ell_n^q) \sim n^{1/2}$ as $n \to \infty$.* ∎

It would be interesting to find the exact values of $\pi_1^{(n)}(\ell_m^p)$ for each $m, n \in \mathbb{N}$ and $p \in [1,\infty]$. Towards this, take q to be the conjugate index to p, and let $\lambda_1, \dots, \lambda_n \in \ell_m^q$, say $\lambda_j = (\lambda_{1j}, \dots, \lambda_{mj})$ $(j = 1, \dots, n)$. Then set $\Lambda = (\lambda_{ij} : i \in \mathbb{N}_m,\ j \in \mathbb{N}_n)$, an $m \times n$-matrix, so that $\Lambda \in \mathbb{M}_{m,n}$. Following Feng and Tonge in [29] (but replacing their p and q by u and v, where $1 \le u, v \le \infty$), we define

$$|\Lambda|_{u,v} = \Big(\sum_{j=1}^n \Big(\sum_{i=1}^m |\lambda_{ij}|^u\Big)^{v/u}\Big)^{1/v}$$

and

$$\|\Lambda\|_{u,v} = \max\{\|\Lambda x\|_v : \|x\|_u \le 1\}.$$

By (3.7), the condition that $\mu_{1,n}(\lambda_1, \dots, \lambda_n) \le 1$ is just the condition that $\|\Lambda\|_{p,1} \le 1$. The number $\sum_{j=1}^{n} \|\lambda_j\|$ is just $|\Lambda|_{q,1}$. Thus $\pi_1^{(n)}(\ell_m^p)$ is the least constant d such that

$$|\Lambda|_{q,1} \le d\|\Lambda\|_{p,1}$$

for each $\Lambda \in \mathbb{M}_{m,n}$. The determination of such a d is exactly a special case of the question addressed in [29, Problem 1, (4)]; unfortunately, this is a case that is left open in [29].

More generally, Feng and Tonge study in [29], for fixed $m, n \in \mathbb{N}$, the constant

$$d_{m,n}(u, v, r, s) = \sup\{|\Lambda|_{u,v} : \Lambda \in \mathbb{M}_{m,n},\ \|\Lambda\|_{r,s} \le 1\};$$

this number was determined in the case where $u = v \ge 2$ for most (but not all) choices of $r, s \in [1, \infty)$. We see that the above argument shows that

$$d_{m,n}(u, v, r, s) = \pi_{v,r'}^{(n)}(I : \ell_m^s \to \ell_m^u),$$

where I is the identity map and r' is the conjugate index to r.

3.5. Characterizations of the maximum multi-norm

3.5.1. Characterizations in terms of weak summing norms. We now give some alternative descriptions of the maximum multi-norm; these remarks will be used to give some calculations of the maximum rate of growth for certain Banach spaces E.

Let E be a normed space, and take $n \in \mathbb{N}$. Then we set

$$S_n = \{(\zeta_1 x, \dots, \zeta_n x) \in E^n : \zeta_1, \dots, \zeta_n \in \mathbb{T},\ x \in S_E\}$$

and $K_n = \overline{\mathrm{co}}(S_n)$, the closed convex hull of S_n, so that K_n is absolutely convex and absorbing. Then the Minkowski functional, temporarily called p_n, of K_n is a norm on E^n. Since $A_\sigma(K_n) = K_n$ for each $\sigma \in \mathfrak{S}_n$ and $M_\alpha(K_n) \subset K_n$ for each $\alpha \in \overline{\mathbb{D}}^n$, the norm p_n satisfies Axioms (A1) and (A2). Now let n vary in $\mathbb{N}$, so that we obtain a sequence $(p_n : n \in \mathbb{N})$ of norms. This sequence clearly satisfies (A3) and (A4), and so $(p_n : n \in \mathbb{N})$ is a multi-norm on $\{E^n : n \in \mathbb{N}\}$. Further, let $(\|\cdot\|_n : n \in \mathbb{N})$ be any multi-norm on $\{E^n : n \in \mathbb{N}\}$, and let B_n be the closed unit ball of $(E^n, \|\cdot\|_n)$. Then we see that $K_n \subset B_n$, and so $(p_n : n \in \mathbb{N})$ is the maximum multi-norm on $\{E^n : n \in \mathbb{N}\}$. We conclude that the closed unit ball of $(E^n, \|\cdot\|_n^{\max})$ is the set K_n.

The first characterization of $\|\cdot\|_n^{\max}$ follows easily from the Hahn–Banach theorem; in the proof, we temporarily write p_n for $\|\cdot\|_n^{\max}$, q_n for the dual norm to p_n, and we write $\mu_{1,n}$ for the weak 1-summing norm on $(E')^n$.

Theorem 3.33. *Let E be a normed space, and take $n \in \mathbb{N}$. Then*

$$\|(x_1, \dots, x_n)\|_n^{\max} = \sup\Big\{\Big|\sum_{j=1}^{n} \langle x_j,\, \lambda_j\rangle\Big| : \mu_{1,n}(\lambda_1, \dots, \lambda_n) \le 1\Big\}$$

for each $x_1, \dots, x_n \in E$, where the supremum is taken over $\lambda_1, \dots, \lambda_n \in E'$. Further, the dual of $\|\cdot\|_n^{\max}$ is $\mu_{1,n}$.

Proof. Let $x_1, \dots, x_n \in E$, and set $x = (x_1, \dots, x_n)$. By the Hahn–Banach theorem,

$$\|x\|_n^{\max} = \sup\{|\langle x, \lambda\rangle| : q_n(\lambda) \le 1\},$$

where $\langle x, \lambda\rangle = \sum_{j=1}^n \langle x_j, \lambda_j\rangle$ for $\lambda = (\lambda_1, \dots, \lambda_n) \in (E')^n$, as in (1.10). However it is clear that $q_n(\lambda) \le 1$ if and only if $|\langle y, \lambda\rangle| \le 1$ $(y \in S_n)$, and so $q_n(\lambda) \le 1$ if and only if

$$\Big|\sum_{j=1}^n \langle \zeta_j y, \lambda_j\rangle\Big| \le 1 \quad (\zeta_1, \dots, \zeta_n \in \mathbb{T},\ y \in S_E).$$

This latter occurs if and only if $\sum_{j=1}^n |\langle y, \lambda_j\rangle| \le 1$ for each $y \in E_{[1]}$.

Further, $q_n(\lambda) \le 1$ if and only if

$$\Big|\sum_{j=1}^n \langle y, \zeta_j \lambda_j\rangle\Big| \le 1 \quad (\zeta_1, \dots, \zeta_n \in \mathbb{T},\ y \in S_E),$$

and this occurs if and only if $\mu_{1,n}(\lambda_1, \dots, \lambda_n) \le 1$. Hence $q_n = \mu_{1,n}$.

The result follows. ∎

Thus we can confirm from Theorem 2.31 that $(((E')^n, \mu_{1,n}) : n \in \mathbb{N})$ is a dual multi-Banach space, as already noted in Theorem 3.19. For a related result, see Theorem 4.4.

Corollary 3.34. *Let $E = \ell^r$, where $r \ge 1$. Then*

$$\|(\delta_1, \dots, \delta_n)\|_n^{\max} = n^{1/r} \quad (n \in \mathbb{N}).$$

Proof. By Corollary 2.18, $\|(\delta_1, \dots, \delta_n)\|_n^{\max} \le n^{1/r}$ $(n \in \mathbb{N})$.

The conjugate index to r is s. Take $\lambda_j = \delta_j \in E'$ $(j \in \mathbb{N}_n)$. By (3.5),

$$\mu_{1,n}(\delta_1, \dots, \delta_n) = \sup\{\|(\zeta_1, \dots, \zeta_n)\|_{\ell^s} : \zeta_1, \dots, \zeta_n \in \mathbb{T}\} = n^{1/s} \quad (n \in \mathbb{N}),$$

and so $\|(\delta_1, \dots, \delta_n)\|_n^{\max} \ge n/n^{1/s} = n^{1/r}$ $(n \in \mathbb{N})$. ∎

Corollary 3.35. *Let E be a normed space, and take $n \in \mathbb{N}$. Then*

$$\begin{aligned}\varphi_n^{\max}(E) &= \sup\Big\{\sum_{j=1}^n \|\lambda_j\| : \mu_{1,n}(\lambda_1, \dots, \lambda_n) \le 1\Big\}\\ &= \sup\Big\{\sum_{j=1}^n \|\lambda_j\| : \Big\|\sum_{j=1}^n \zeta_j \lambda_j\Big\| \le 1\ (\zeta_1, \dots, \zeta_n \in \mathbb{T})\Big\},\end{aligned}$$

where the supremum is taken over $\lambda_1, \dots, \lambda_n \in E'$, and so

$$\varphi_n^{\max}(E) = \pi_1^{(n)}(E') \ge n/c_n(E').$$

Proof. Take $\lambda_1, \dots, \lambda_n \in E'$. Then

$$\sup\Big\{\Big|\sum_{j=1}^n \langle x_j, \lambda_j\rangle\Big| : x_1, \dots, x_n \in E_{[1]}\Big\} = \sum_{j=1}^n \|\lambda_j\|,$$

and so the first equality holds. The final remark follow from (3.10) and (3.13). ∎

Corollary 3.36. *Let E be a normed space, and take $n \in \mathbb{N}$. Then*

$$\varphi_n^{\max}(E') = \pi_1^{(n)}(E) \quad \text{and} \quad \varphi_n^{\max}(E) = \varphi_n^{\max}(E'').$$

Proof. These follow from Proposition 3.22(iv) and Corollary 3.35. ∎

Corollary 3.37. *There is a constant $C > 0$ such that, for each $n \in \mathbb{N}$ and each normed space E with* $\dim E \geq n$*, we have*

$$\varphi_n^{\max}(E) \geq \frac{1}{C}\sqrt{\left[\frac{n}{\log n}\right]}.$$

Proof. Since $\dim E \geq n$, we have $\dim E' \geq n$, and so this follows from Corollaries 3.24 and 3.35. ∎

We do not know if the factor '$\log n$' is required in the above theorem; we shall see in Theorem 3.58 that it is not required in the case where the space E is infinite-dimensional.

3.5.2. The dual of the minimum dual multi-norm. Let $(E, \|\cdot\|)$ be a normed space. Then we have seen that $(\mu_{1,n} : n \in \mathbb{N})$ is the minimum dual multi-norm on $\{E^n : n \in \mathbb{N}\}$, and so $(\mu_{1,n}' : n \in \mathbb{N})$ is a multi-norm on $\{(E')^n : n \in \mathbb{N}\}$. We ask if it is the maximum multi-norm. To see that this is the case, take $\lambda_1, \dots, \lambda_n \in E'$, and set $\lambda = (\lambda_1, \dots, \lambda_n)$. By Theorem 3.33, we have

$$\|\lambda\|_n^{\max} = \sup\Big\{\Big|\sum_{j=1}^n \langle \Lambda_j,\, \lambda_j\rangle\Big| : \Lambda_1, \dots, \Lambda_n \in E'',\, \mu_{1,n}(\Lambda_1, \dots, \Lambda_n) \leq 1\Big\}.$$

On the other hand, we have

$$\mu_{1,n}'(\lambda) = \sup\Big\{\Big|\sum_{j=1}^n \langle x_j,\, \lambda_j\rangle\Big| : x_1, \dots, x_n \in E,\, \mu_{1,n}(x_1, \dots, x_n) \leq 1\Big\},$$

where we recall that the restriction of $\mu_{1,n}$ defined on $(E'')^n$ to E^n is just $\mu_{1,n}$ defined on E^n. Clearly, $\mu_{1,n}'(\lambda) \leq \|\lambda\|_n^{\max}$. The reverse inequality follows from the Principle of Local Reflexivity.

Theorem 3.38. *Let E be a normed space, and take $n \in \mathbb{N}$. For each $\lambda \in (E')^n$, we have* $\mu_{1,n}'(\lambda) = \|\lambda\|_n^{\max}$.

Proof. Take $\varepsilon > 0$. Then there exists $\Lambda = (\Lambda_1, \dots, \Lambda_n) \in (E'')^n$ with $\mu_{1,n}(\Lambda) \leq 1$ and

$$\Big|\sum_{j=1}^n \langle \Lambda_j, \lambda_j\rangle\Big| \geq \|\lambda\|_n^{\max} - \varepsilon.$$

Set $X = \mathrm{lin}\{\Lambda_1, \dots, \Lambda_n\}$ and $Y = \mathrm{lin}\{\lambda_1, \dots, \lambda_n\}$, so that X and Y are finite-dimensional subspaces of E'' and E', respectively. By the Principle of Local Reflexivity, Theorem 1.4, there is an injective, bounded linear map $S : X \to E$ with $\|S\| < 1 + \varepsilon$ and with $\langle S(\Lambda_j),\, \lambda_j\rangle = \langle \Lambda_j,\, \lambda_j\rangle$ $(j \in \mathbb{N}_n)$. Set $x = (S(\Lambda_1), \dots, S(\Lambda_n)) \in E^n$. Then it follows from (3.3) that $\mu_{1,n}(x) \leq (1 + \varepsilon)\mu_{1,n}(\Lambda)$ (with T taken to be $S : X \to S(X)$), and so $\mu_{1,n}(x) \leq 1 + \varepsilon$. Now we have

$$\mu_{1,n}'(\lambda) \geq \frac{1}{1+\varepsilon}\Big|\sum_{j=1}^n \langle S(\Lambda_j),\, \lambda_j\rangle\Big| = \frac{1}{1+\varepsilon}\Big|\sum_{j=1}^n \langle \Lambda_j,\, \lambda_j\rangle\Big| \geq \frac{1}{1+\varepsilon}(\|\lambda\|_n^{\max} - \varepsilon).$$

This holds true for each $\varepsilon > 0$, and so $\mu_{1,n}'(\lambda) \geq \|\lambda\|_n^{\max}$.

Thus $\|\lambda\|_n^{\max} = \mu_{1,n}'(\lambda)$, as required. ∎

THEOREM 3.39. *Let E be a normed space. Then $(\mu'_{1,n} : n \in \mathbb{N})$ is the maximum multi-norm on the family $\{(E')^n : n \in \mathbb{N}\}$.* ■

In summary, we have the following. Let E be a normed space. Then the minimum and maximum multi-norms based on E are $(\|\cdot\|_n^{\min} : n \in \mathbb{N})$ and $(\|\cdot\|_n^{\max} : n \in \mathbb{N})$, respectively. The dual of these multi-norms are the maximum and minimum dual multi-norms, respectively, on the family $\{(E')^n : n \in \mathbb{N}\}$, and the latter is exactly the multi-norm $(\mu_{1,n} : n \in \mathbb{N})$. Combining these remarks, we have the following consequence.

COROLLARY 3.40. *Let E be a normed space. Then the second dual of the maximum multi-norm $(\|\cdot\|_n^{\max} : n \in \mathbb{N})$ based on E is the maximum multi-norm based on E''.* ■

3.5.3. Characterizations in terms of projective norms. Our second characterization of the maximum multi-norm involves a projective norm.

DEFINITION 3.41. Let E be a linear space. A subset S of E is *one-dimensional* if $S \subset \mathbb{C}x$ for some $x \in E$. A family $\{y_1, \dots, y_m\}$ in E has an *elementary representation* if there exist $n \in \mathbb{N}$ and $x_{ij} \in E$ for $i \in \mathbb{N}_m$ and $j \in \mathbb{N}_n$ with

$$y_i = \sum_{j=1}^{n} x_{ij} \quad (i \in \mathbb{N}_m)$$

and such that $\{x_{ij} : i \in \mathbb{N}_m\}$ of E is one-dimensional for each $j \in \mathbb{N}_n$.

Each family $\{y_1, \dots, y_m\}$ in the linear space E has at least one elementary representation. Indeed, each such family has a representation of the form

$$y_i = \sum_{j=1}^{n} \alpha_{ij} x_j \quad (i \in \mathbb{N}_m), \tag{3.15}$$

where $n \in \mathbb{N}$, $\alpha_{ij} \in \mathbb{C}$ $(i \in \mathbb{N}_m,\ j \in \mathbb{N}_n)$, and $x_j \in E$ $(j \in \mathbb{N}_n)$ have the property that $\|x_1\| = \cdots = \|x_n\| = 1$.

Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space. Take $k \in \mathbb{N}$, and suppose that $\{x_1, \dots, x_k\}$ is a one-dimensional set in E. Then clearly

$$\|(x_1, \dots, x_k)\|_k = \max\{\|x_1\|, \dots, \|x_k\|\}.$$

Now let $\{y_1, \dots, y_m\}$ be a family in E with the elementary representation of (3.15). Then

$$\begin{aligned}\|(y_1, \dots, y_m)\|_m &= \Big\|\sum_{j=1}^{n} (\alpha_{1j}x_j, \dots, \alpha_{mj}x_j)\Big\|_m \le \sum_{j=1}^{n} \|(\alpha_{1j}x_j, \dots, \alpha_{mj}x_j)\|_m \\ &= \sum_{j=1}^{n} \max\{|\alpha_{ij}| : i \in \mathbb{N}_m\},\end{aligned}$$

and so

$$\|(y_1, \dots, y_m)\|_m \le \|\!|\!|(y_1, \dots, y_m)|\!|\!\|_m, \tag{3.16}$$

where

$$\|\!|\!|(y_1, \dots, y_m)|\!|\!\|_m = \inf\Big\{\sum_{j=1}^{n} \max\{|\alpha_{ij}| : i \in \mathbb{N}_m\}\Big\} \tag{3.17}$$

and the infimum is taken over all elementary representations as specified in equation (3.15) of the family $\{y_1, \dots, y_m\}$.

THEOREM 3.42. *Let E be a normed space. Then the above sequence $(\| \cdot \|_n : n \in \mathbb{N})$ is the maximum multi-norm on $\{E^n : n \in \mathbb{N}\}$, and, for each $m \in \mathbb{N}$, we have*

$$\varphi_m^{\max}(E) = \sup\Big\{\inf\Big\{\sum_{j=1}^{n} \max\{|\alpha_{ij}| : i \in \mathbb{N}_m\}\Big\} : y_1, \dots, y_m \in E_{[1]}\Big\},$$

where the infimum is taken over all elementary representations of the form

$$y_i = \sum_{j=1}^{n} \alpha_{ij} x_j \qquad (i \in \mathbb{N}_m)$$

for which $n \in \mathbb{N}$, $\alpha_{ij} \in \mathbb{C}$ ($i \in \mathbb{N}_m$, $j \in \mathbb{N}_n$), and $x_j \in E_{[1]}$ ($j \in \mathbb{N}_n$).

Proof. It is clear from (3.16) that it is sufficient to show that $(\| \cdot \|_n : n \in \mathbb{N})$ is a multi-norm on $\{E^n : n \in \mathbb{N}\}$. However it is easily checked that $\| \cdot \|_n$ is a norm on E^n for each $n \in \mathbb{N}$, that $\| \cdot \|_1$ is the initial norm on E, and that Axioms (A1), (A2), and (A4) are satisfied. It follows that $(\| \cdot \|_n : n \in \mathbb{N})$ is indeed a multi-norm. ∎

We can re-express the above evaluation of $\| \cdot \|_m^{\max}$ as follows. In the statement, π denotes the projective norm on the space $\ell_m^\infty \otimes E$. More general versions of the following theorem will be given in [19].

THEOREM 3.43. *Let E be a normed space, and take $m \in \mathbb{N}$. Then*

$$(E^m, \| \cdot \|_m^{\max}) \cong (\ell_m^\infty \otimes E, \| \cdot \|_\pi).$$

Proof. Let $m \in \mathbb{N}$, and take $\{\delta_1, \dots, \delta_m\}$ to be the standard basis of ℓ_m^∞. Then the map

$$T : (y_1, \dots, y_n) \mapsto \sum_{i=1}^{m} \delta_i \otimes y_i, \qquad E^m \to \ell_n^\infty \otimes E,$$

is a linear bijection. Let $y_i = \sum_{j=1}^n \alpha_{ij} x_j$ be the elementary representation of y_i for $i \in \mathbb{N}_m$, as in (3.15), where $\|x_1\| = \cdots = \|x_n\| = 1$, and set $z = T(y_1, \dots, y_m)$. Then

$$z = \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} \alpha_{ij} \delta_i\Big) \otimes x_j,$$

and every representation of z as an element of $\ell_m^\infty \otimes E$ has this form. By (3.17), we have

$$\|z\|_\pi = \inf\Big\{\sum_{j=1}^{n} \max\{|\alpha_{ij}| : i \in \mathbb{N}_m\}\Big\} = \|(y_1, \dots, y_m)\|_m^{\max}.$$

This shows that T is an isometry. ∎

The following is related to (3.6).

COROLLARY 3.44. *Let E be a normed space. Then*

$$((E')^n, \mu_{1,n}) = \ell_n^p(E')^w \cong \mathcal{B}(E, \ell_n^1) \cong \mathcal{B}(\ell_n^\infty, E') \qquad (n \in \mathbb{N}).$$

Proof. Let $n \in \mathbb{N}$. By Theorem 3.33, the dual space to $(E^n, \| \cdot \|_n^{\max})$ is $((E')^n, \mu_{1,n})$. By (1.5), the dual space of $(\ell_n^\infty \otimes E, \| \cdot \|_\pi)$ is the Banach space $\mathcal{B}(E, \ell_n^1) \cong \mathcal{B}(\ell_n^\infty, E')$. ∎

3.6. The function $\varphi_n^{\max}$ for some examples. We shall calculate the value of $\varphi_n^{\max}(E)$ for some standard Banach spaces E; sometimes we shall use elementary means, even if more general theorems are available.

3.6.1. The spaces ℓ^p. In the following examples, $p \in [1, \infty]$, and q is the conjugate index to p. Take $n \in \mathbb{N}$. Then ℓ_n^p is 1-complemented in ℓ^p, and so it follows from Corollary 3.13 that $\varphi_n^{\max}(\ell^p) \geq \varphi_n^{\max}(\ell_n^p)$.

EXAMPLE 3.45. Let $n \in \mathbb{N}$. Then we have $\pi_1^{(n)}(\ell_n^\infty) = \pi_1^{(n)}(\ell^\infty) = n$ by Proposition 3.31(iv), and so, by Corollary 3.35,

$$\varphi_n^{\max}(\ell_n^1) = \varphi_n^{\max}(\ell^1) = n.$$

The maximum multi-norm on the family $\{(\ell^1)^n : n \in \mathbb{N}\}$ will be calculated in Theorem 4.23. ■

EXAMPLE 3.46. Let $n \in \mathbb{N}$, and take $q > 1$. Set $F = \ell_n^q$. By the choice $\lambda_j = \delta_j \in S_F$ for $j \in \mathbb{N}_n$, we see that $c_n(\ell_n^q) \leq n^{1/q}$. Now take $p > 1$. Then $(\ell_n^p)' = \ell_n^q$, whence

$$\varphi_n^{\max}(\ell^p) \geq \varphi_n^{\max}(\ell_n^p) \geq n/n^{1/q}$$

by Corollary 3.35, and so $\varphi_n^{\max}(\ell^p) \geq n^{1/p}$. ■

We make a trivial preliminary remark: for $\zeta \in \mathbb{T}$ and $q \geq 1$, we have

$$|1+\zeta|^q + |1-\zeta|^q \leq \max\{2 \cdot 2^{q/2}, 2^q\}. \tag{3.18}$$

EXAMPLE 3.47. Let $F = \ell_2^q$, where $q \geq 1$. We choose

$$\lambda_1 = (1,1)/2^{1/q} \quad \text{and} \quad \lambda_2 = (1,-1)/2^{1/q},$$

so that $\lambda_1, \lambda_2 \in S_F$. By (3.18), $\sup\{\|\zeta_1\lambda_1 + \zeta_2\lambda_2\| : \zeta_1, \zeta_2 \in \mathbb{T}\} \leq \max\{\sqrt{2}, 2^{1/p}\}$.

Now suppose that $p \geq 2$, so that $2^{1/p} \leq \sqrt{2}$. Then $c_2(F) \leq \sqrt{2}$, and so, by Corollary 3.35, we have

$$\varphi_2^{\max}(\ell^p) \geq \varphi_2^{\max}(\ell_2^p) \geq \sqrt{2}. \tag{3.19}$$

By Examples 3.45 and 3.46, this inequality also holds for $p \in [1, 2]$, and so (3.19) holds for all $p \geq 1$. ■

EXAMPLE 3.48. Let $n \in \mathbb{N}$. We have noted in Proposition 3.31(iii) the equality

$$\pi_1^{(n)}(\ell_n^2) = \pi_1^{(n)}(\ell^2) = \sqrt{n},$$

and hence, by Corollary 3.35, we have

$$\varphi_n^{\max}(\ell^2) = \varphi_n^{\max}(\ell_n^2) = \sqrt{n}.$$

We wish to obtain this result directly from our definitions.

Let $E = \ell^2$, so that $E' = E$, and we write F for E'; the usual inner product on E is denoted by $[\,\cdot\,,\,\cdot\,]$. Let $(\|\cdot\|_n : n \in \mathbb{N})$ be any multi-norm on $\{E^n : n \in \mathbb{N}\}$, and take $n \in \mathbb{N}$. For $x_1, \dots, x_n \in E_{[1]}$ and $\zeta = \exp(2\pi\mathrm{i}/n)$, we have

$$\sum_{j=1}^n \Big\| \sum_{m=1}^n \zeta^{jm} x_m \Big\|^2 = \sum_{j=1}^n \sum_{m=1}^n [\zeta^{jm} x_m, \zeta^{jm} x_m] = \sum_{j=1}^n \sum_{m=1}^k \|x_m\|^2 \leq k^2,$$

and so, by Hölder's inequality, we have

$$\sum_{j=1}^{n}\Big\|\sum_{m=1}^{n}\zeta^{jm}x_m\Big\| \le k^{1/2}\Big(\sum_{j=1}^{n}\Big\|\sum_{m=1}^{n}\zeta^{jm}x_m\Big\|^2\Big)^{1/2}.$$

Hence

$$\frac{1}{n}\sum_{j=1}^{n}\Big\|\sum_{m=1}^{n}\zeta^{jm}x_m\Big\| \le n^{1/2}.$$

It follows from Proposition 2.17 that $\|(x_1,\dots,x_n)\|_n \le n^{1/2}$, and thus we have $\varphi_n^{\max}(E) \le n^{1/2}$.

By Example 3.46, $\varphi_n^{\max}(E) \ge n^{1/2}$, and so $\varphi_n^{\max}(\ell^2) = n^{1/2}$.

It now follows from Corollary 3.35 that $c_n(\ell^2) \ge n^{1/2}$, and so, by Example 3.46, we have $c_n(\ell^2) = c_n(\ell_n^2) = n^{1/2}$. ■

Example 3.49. Let $n \in \mathbb{N}$, and take $F = \ell_n^q$, where $q \in [1,2]$.

Let $\zeta = \exp(2\pi \mathrm{i}/n)$, and then set

$$\lambda_j = \frac{1}{n^{1/q}}(\zeta^j,\zeta^{2j},\dots,\zeta^{nj}) \in S_F \quad (j \in \mathbb{N}_n).$$

For each $\zeta_1,\dots,\zeta_n \in \mathbb{T}$, we have $\|\zeta_1\lambda_1+\cdots+\zeta_n\lambda_n\| \le \sqrt{n}$ by Lemma 1.1(ii), and so $c_n(\ell^q) \le c_n(\ell_n^q) \le \sqrt{n}$.

Now take p with $2 \le p < \infty$, so that $q \in (1,2]$. Set $E = \ell_n^p$ and $F = E' = \ell_n^q$. By Corollaries 3.13 and 3.35, $\varphi_n^{\max}(\ell^p) \ge \varphi_n^{\max}(\ell_n^p) \ge \sqrt{n}$. By Corollaries 3.30 and 3.35, $\varphi_n^{\max}(\ell^p) \le C_q\sqrt{n}$, where C_q is the Orlicz constant for ℓ^q, and so, again by Corollary 3.35, $c_n(\ell^q) \ge \sqrt{n}/C_q$.

In particular, we have shown that

$$\sqrt{n} \le \varphi_n^{\max}(\ell_n^p) \le \varphi_n^{\max}(\ell^p) \le C_q\sqrt{n} \quad (n \in \mathbb{N})$$

whenever $2 \le p < \infty$. ■

Example 3.50. Let $n \in \mathbb{N}$. As in Example 3.49, $c_n(\ell_n^1) \le \sqrt{n}$, and so, by Corollary 3.35, $\varphi_n^{\max}(\ell_n^\infty) \ge \sqrt{n}$. Thus it follows from Proposition 3.31(ii) and Corollary 3.36 that

$$\sqrt{n} \le \varphi_n^{\max}(\ell_n^\infty) \le \varphi_n^{\max}(\ell^\infty) \le \sqrt{2n}. \ ■$$

The above two results are in accord with the estimates of Gordon given in Proposition 3.32.

Example 3.51. This example shows that strict inequality can arise in (3.1).

Indeed, take $n \in \mathbb{N}$, and consider $E = \ell^\infty$, so that $\varphi_n^{\max}(E) \le \sqrt{2n}$ by Example 3.50. By [6, Theorem 2.5.7], each separable Banach space is isometrically isomorphic to a closed subspace of ℓ^∞, and so we can regard ℓ^1 as a closed subspace of E. However, by Example 3.45, we know that $\varphi_n^{\max}(\ell_n^1) = \varphi_n^{\max}(\ell^1) = n$. Thus $F := \ell_n^1$ is a closed subspace of E with $\dim F = n$ and

$$\varphi_n^{\max}(E) \le \sqrt{2n} < n = \varphi_n^{\max}(F)$$

for $n \ge 3$. ■

The next result refers to the Banach–Mazur distance $d(F,\ell_n^2)$ for a normed space F with $\dim F = n$.

PROPOSITION 3.52. *Let E be a Banach space. Then*

$$\varphi_n^{\max}(E) \leq \sqrt{n}\,\sup\{d(F,\ell_n^2) : F \subset E, \dim F = n\} \qquad (n \in \mathbb{N}).$$

Proof. This follows from (3.1), Corollary 3.14, and Example 3.48. ∎

EXAMPLE 3.53. Let $p \in [1,\infty]$, and take $n \in \mathbb{N}$. By [74, Corollary III.B.9], we have

$$d(F,\ell_n^2) \leq n^{|1/p-1/2|} \qquad (n \in \mathbb{N}) \tag{3.20}$$

whenever F is a subspace of ℓ^p with $\dim F = n$.

Now suppose that $p \in [1,2]$. By (3.20), $d(F,\ell_n^2) \leq n^{1/p-1/2}$ whenever F is a subspace of ℓ^p with $\dim F = n$, and so $\varphi_n^{\max}(\ell^p) \leq n^{1/p}$ by Proposition 3.52. By Example 3.46, $\varphi_n^{\max}(\ell_n^p) \geq n^{1/p}$, and so we see that

$$\varphi_n^{\max}(\ell^p) = \varphi_n^{\max}(\ell_n^p) = n^{1/p} \qquad (n \in \mathbb{N}).$$

This is a sharpening of the result of Gordon contained in Proposition 3.32.

It now follows from Corollary 3.35 and Example 3.46 that for $q \geq 2$ we have

$$c_n(\ell^q) = c_n(\ell_n^q) = n^{1/q} \qquad (n \in \mathbb{N}). \ ∎$$

We summarize some results of this section; again, q is the conjugate index to $p \in [1,\infty]$ and C_q is the Orlicz constant for ℓ^q, where $q \in [1,2]$.

THEOREM 3.54. *Let $n \in \mathbb{N}$. Then:*

(i) *for $p \in [1,2]$, we have $\varphi_n^{\max}(\ell^p) = \varphi_n^{\max}(\ell_n^p) = n^{1/p}$;*
(ii) *for $p \in [2,\infty]$, we have $\sqrt{n} \leq \varphi_n^{\max}(\ell_n^p) \leq \varphi_n^{\max}(\ell^p) \leq C_q\sqrt{n}$.* ∎

3.6.2. The spaces L^p. We now consider, more briefly, spaces denoted by $L^p := L^p(\Omega,\mu)$ for a measure space (Ω,μ). Throughout, we shall suppose that L^p is infinite-dimensional, and so, for each $n \in \mathbb{N}$, there exist pairwise-disjoint, measurable subsets $X_1,\dots,X_n$ of Ω with $0 < \mu(X_i) < \infty$ $(i \in \mathbb{N}_n)$; we may suppose that Ω is σ-finite. We shall determine the rate of growth of the sequence $(\varphi_n^{\max}(L^p) : n \in \mathbb{N})$.

THEOREM 3.55. *Let $n \in \mathbb{N}$. Then:*

(i) *for $p \in [1,2]$, we have $\varphi_n^{\max}(L^p) = n^{1/p}$ $(n \in \mathbb{N})$;*
(ii) *for $p \in [2,\infty]$, we have $\varphi_n^{\max}(L^p) \sim \sqrt{n}$ as $n \to \infty$.*

Proof. Take $p \in [1,\infty]$, with conjugate index q. Fix $n \in \mathbb{N}$, and take measurable subsets $X_1,\dots,X_n$ of Ω with $0 < \mu(X_i) < \infty$ $(i \in \mathbb{N}_n)$. For each $i \in \mathbb{N}_n$, set $\chi_i = \chi_{X_i}/\mu(X_i)^{1/q}$ when $q < \infty$ and $\chi_i = \chi_{X_i}$ when $q = \infty$, so that $\|\chi_i\| = 1$ in $L^q = (L^p)'$ for each $p \in [1,\infty]$. Clearly,

$$\|\zeta_1\chi_1 + \cdots + \zeta_n\chi_n\|_{L^q} = \|(\zeta_1,\dots,\zeta_n)\|_{\ell^q} \qquad (\zeta_1,\dots,\zeta_n \in \mathbb{C}).$$

It follows immediately that $c_n(L^q) \leq n^{1/q}$ when $q \geq 2$ and $c_n(L^q) \leq \sqrt{n}$ when $q \in [1,2]$. By Corollary 3.35, $\varphi_n^{\max}(L^p) \geq n^{1/p}$ when $p \in [1,2]$ and $\varphi_n^{\max}(L^p) \geq \sqrt{n}$ when $p \in [2,\infty]$.

Again by [74, Corollary III.B.9], we have

$$d(F,\ell_n^2) \leq n^{|1/p-1/2|} \qquad (n \in \mathbb{N})$$

whenever F is a subspace of L^p with $\dim F = n$.

For $p \in [1,2]$, it follows from Proposition 3.52 that $\varphi_n^{\max}(L^p) \leq n^{1/p}$, and thus we have shown that $\varphi_n^{\max}(L^p) = n^{1/p}$.

For $p \in [2, \infty]$, L^q has the Orlicz property, and so $\pi_1^{(n)}(L^q) \leq C\sqrt{n}$ for a constant $C > 0$. By Corollary 3.35, $\varphi_n^{\max}(L^p) \leq C\sqrt{n}$, and so we have $\varphi_n^{\max}(L^p) \sim \sqrt{n}$. ∎

3.6.3. The spaces $C(K)$. The calculation of the maximum rate of growth of the spaces $C(K)$ is rather easy.

Theorem 3.56. *Let K be an infinite, compact space. For each $n \in \mathbb{N}$, we have*

$$\sqrt{n} \leq \varphi_n^{\max}(C(K)) \leq \sqrt{2n}.$$

Proof. Take $n \in \mathbb{N}$. There exist $f_1, \dots, f_n \in C(K)^+$ with $|f_1|_K = \cdots = |f_n|_K = 1$ and such that $f_i f_j = 0$ for $i, j \in \mathbb{N}_n$ with $i \neq j$. The map

$$(\zeta_1, \dots, \zeta_n) \mapsto \sum_{j=1}^{n} \zeta_j f_j, \quad \ell_n^\infty \to C(K),$$

is an isometry onto a closed subspace of $C(K)$, and so, by Example 3.50, we have $\varphi_n^{\max}(C(K)) \leq \varphi_n^{\max}(\ell_n^\infty) \leq \sqrt{2n}$.

There exist $\mu_1, \dots, \mu_n \in M(K)^+$ such that $\|\mu_1\| = \cdots = \|\mu_n\| = 1$ and with pairwise-disjoint supports. The map

$$(\zeta_1, \dots, \zeta_n) \mapsto \sum_{j=1}^{n} \zeta_j \mu_j, \quad \ell_n^1 \to M(K),$$

is an isometry onto a closed subspace of $M(K)$, and so $c_n(M(K)) \leq c_n(\ell_n^1)$ by Proposition 3.26. By Example 3.50, $c_n(\ell_n^1) \leq \sqrt{n}$, and so, by Corollary 3.35, $\varphi_n^{\max}(C(K)) \geq \sqrt{n}$.

The result follows. ∎

3.6.4. A lower bound for $\varphi_n^{\max}(E)$. We shall now establish that $\varphi_n^{\max}(E) \geq \sqrt{n}$ for each $n \in \mathbb{N}$ and each infinite-dimensional Banach space E (cf. Corollary 3.37). Since $\varphi_n^{\max}(\ell^2) = \sqrt{n}$ $(n \in \mathbb{N})$, this is the best-possible lower bound. For this, we shall use the following famous *theorem of Dvoretzky*, sometimes called the *theorem on almost spherical sections*; for proofs and discussions, see [6, §12.3], [26, Chapter 19], or [59, Chapter 4].

Theorem 3.57. *For each $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists $m = m(n, \varepsilon)$ in $\mathbb{N}$ such that, for each normed space F with $\dim F \geq m$, there is an n-dimensional subspace L of F such that $d(L, \ell_n^2) < 1 + \varepsilon$.* ∎

Theorem 3.58. *Let E be an infinite-dimensional normed space. Then*

$$\varphi_n^{\max}(E) \geq \sqrt{n} \quad (n \in \mathbb{N}).$$

Proof. Fix $n \in \mathbb{N}$, and take $\varepsilon > 0$.

By Theorem 3.57, there is an n-dimensional subspace L of E' with $d(L, \ell_n^2) < 1 + \varepsilon$. By Proposition 3.26,

$$c_n(E') \leq c_n(\ell_n^2) d(L, \ell_n^2).$$

As in Example 3.48, $c_n(\ell_n^2) = \sqrt{n}$. Thus $c_n(E') \leq (1 + \varepsilon)\sqrt{n}$. This holds true for each $\varepsilon > 0$, and so $c_n(E') \leq \sqrt{n}$.

By Corollary 3.35, $\varphi_n^{\max}(E) \geq \sqrt{n}$. ∎

Corollary 3.59. *Let E be an infinite-dimensional normed space. Then the maximum multi-norm is not equivalent to the minimum multi-norm.* ∎

4. Specific examples of multi-norms

In this chapter, we shall give some specific examples of multi-normed spaces.

4.1. The (p, q)-multi-norm

4.1.1. Definition. Let $(E, \|\cdot\|)$ be a normed space, and take p, q such that $1 \le p, q < \infty$. Again we shall sometimes write p' and q' for the conjugate indices of p and q, respectively.

For each $n \in \mathbb{N}$ and $x = (x_1, \dots, x_n) \in E^n$, we define

$$\|x\|_n^{(p,q)} = \sup\Big\{\Big(\sum_{i=1}^n |\langle x_i,\, \lambda_i\rangle|^q\Big)^{1/q} : \mu_{p,n}(\lambda_1, \dots, \lambda_n) \le 1\Big\}, \tag{4.1}$$

taking the supremum over $\lambda_1, \dots, \lambda_n \in E'$. It is clear that $\|\cdot\|_n^{(p,q)}$ is a norm on E^n.

It is convenient for calculations to see that, for a constant $C \ge 0$, we have $\|x\|_n^{(p,q)} \le C$ if and only if

$$\Big(\sum_{i=1}^n |\langle x_i,\, \lambda_i\rangle|^q\Big)^{1/q} \le C \sup\Big\{\Big(\sum_{i=1}^n |\langle y,\, \lambda_i\rangle|^p\Big)^{1/p} : y \in E_{[1]}\Big\} \tag{4.2}$$

for all $\lambda_1, \dots, \lambda_n \in E'$; this is immediate from (3.7).

Theorem 4.1. *Let E be a normed space. Suppose that $p, q \in [1, \infty)$. Then the sequence $(\|\cdot\|_n^{(p,q)} : n \in \mathbb{N})$ is a special-norm based on E; it is a multi-norm when $1 \le p \le q < \infty$.*

Proof. It is clear that $(\|\cdot\|_n^{(p,q)} : n \in \mathbb{N})$ satisfies Axioms (A1)–(A3), and so it is a special-norm; we shall verify that the sequence satisfies Axiom (A4) when $1 \le p \le q < \infty$.

Take $n \in \mathbb{N}$, let $x_1, \dots, x_n \in E$, and set

$$x = (x_1, \dots, x_{n-1}, x_n, x_n) \in E^{n+1}.$$

By Lemma 2.9, it suffices to show that $\|x\|_{n+1}^{(p,q)} \le \|(x_1, \dots, x_n)\|_n^{(p,q)}$.

Take $\varepsilon > 0$. Then there exist elements $\lambda_1, \dots, \lambda_{n+1} \in E'$ such that

$$\mu_{p,n+1}(\lambda_1, \dots, \lambda_{n+1}) \le 1$$

and such that

$$\Big(\sum_{i=1}^{n-1} |\langle x_i,\, \lambda_i\rangle|^q + |\langle x_n,\, \lambda_n\rangle|^q + |\langle x_n,\, \lambda_{n+1}\rangle|^q\Big)^{1/q} > \|x\|_{n+1}^{(p,q)} - \varepsilon.$$

Since $(\ell_2^q)' = \ell_2^{q'}$, there exist $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^{q'} + |\beta|^{q'} \le 1$ and

$$|\langle x_n,\, \lambda_n\rangle|^q + |\langle x_n,\, \lambda_{n+1}\rangle|^q = \langle x_n,\, \alpha\lambda_n + \beta\lambda_{n+1}\rangle^q.$$

Set $\gamma = |\alpha|^{p'} + |\beta|^{p'}$; since $q' \le p'$, we have $\gamma \le 1$. By Proposition 3.18,

$$\mu_{p,n}(\lambda_1, \dots, \lambda_{n-1}, \alpha\lambda_n + \beta\lambda_{n+1}) \le \mu_{p,n+1}(\lambda_1, \dots, \lambda_{n-1}, \gamma\lambda_n, \gamma\lambda_{n+1}),$$

and so, since $\mu_{p,n+1}$ satisfies (A2),

$$\mu_{p,n}(\lambda_1, \dots, \lambda_{n-1}, \alpha\lambda_n + \beta\lambda_{n+1}) \le \max\{1, \gamma\}\mu_{p,n+1}(\lambda_1, \dots, \lambda_{n+1}) \le 1.$$

Hence

$$\begin{aligned}\|(x_1, \dots, x_n)\|_n^{(p,q)} &\ge \Big(\sum_{i=1}^{n-1} |\langle x_i,\, \lambda_i\rangle|^q + \langle x_n,\, \alpha\lambda_n + \beta\lambda_{n+1}\rangle^q\Big)^{1/q} \\ &= \Big(\sum_{i=1}^{n-1} |\langle x_i,\, \lambda_i\rangle|^q + |\langle x_n,\, \lambda_n\rangle|^q + |\langle x_n,\, \lambda_{n+1}\rangle|^q\Big)^{1/q} > \|x\|_{n+1}^{(p,q)} - \varepsilon.\end{aligned}$$

This holds true for each $\varepsilon > 0$, and so the result follows. ■

DEFINITION 4.2. Let E be a normed space, and take $p, q \in [1, \infty)$. Then $(\|\cdot\|_n^{(p,q)} : n \in \mathbb{N})$ is the *(p,q)-special-norm based on E*; it is the *(p,q)-multi-norm* when $1 \le p \le q < \infty$. The rate of growth of this multi-norm is denoted by $(\varphi_n^{(p,q)}(E) : n \in \mathbb{N})$.

Let E be a normed space, take $1 \le p \le q < \infty$, and take $x_1, \dots, x_n \in E$. Suppose that F is a closed subspace of E with $x_1, \dots, x_n \in E$. Then the value of $\|(x_1, \dots, x_n)\|_n^{(p,q)}$ might depend on the space F to which $x_1, \dots, x_n$ belong. To indicate this, we (temporarily) write $(\|\cdot\|_{n,F}^{(p,q)})$ for the (p,q)-special-norm based on F.

PROPOSITION 4.3. *Let E be a normed space, let F be a closed subspace of E, and suppose that $p, q \in [1, \infty)$. Let $n \in \mathbb{N}$ and $x = (x_1, \dots, x_n) \in F^n$. Then $\|x\|_{n,F}^{(p,q)} \ge \|x\|_{n,E}^{(p,q)}$. In the case where F is 1-complemented in E, $\|x\|_{n,F}^{(p,q)} = \|x\|_{n,E}^{(p,q)}$.*

Proof. Take $\lambda_1, \dots, \lambda_n \in E'$. By (3.7), $\mu_{p,n}(\lambda_1|F, \dots, \lambda_n|F) \le \mu_{p,n}(\lambda_1, \dots, \lambda_n)$, and so $\|x\|_{n,F}^{(p,q)} \ge \|x\|_{n,E}^{(p,q)}$.

Now suppose that F is 1-complemented in E, so that there is a projection $P : E \to F$ with $\|P\| = 1$. For $\lambda_1, \dots, \lambda_n \in F'$, we have

$$|\langle y,\, P'\lambda_j\rangle| = |\langle Py,\, \lambda_j\rangle| \quad (y \in E_{[1]}).$$

Since $Py \in F_{[1]}$, it follows from (3.7) that $\|x\|_{n,F}^{(p,q)} \le \|x\|_{n,E}^{(p,q)}$. Hence $\|x\|_{n,F}^{(p,q)} = \|x\|_{n,E}^{(p,q)}$. ■

The following result is a generalization of Corollary 3.35; it follows by the same argument.

THEOREM 4.4. *Let E be a normed space. Suppose that $1 \le p \le q < \infty$ and $n \in \mathbb{N}$. Then $\varphi_n^{(p,q)}(E) = \pi_{q,p}^{(n)}(E')$.* ■

Indeed, it is explained in [20] that the (p,q)-multi-norm based on a normed space E corresponds via the correspondence of §2.4.5 to the norm induced on the space $c_0 \otimes E$ by embedding $c_0 \otimes E$ into $\Pi_{q,p}(E', c_0)$. The (p,p)-multi-norm corresponds to the *Chevet–Saphar* norm, d_p, on the tensor product $c_0 \otimes E$; for a discussion of the Chevet–Saphar norm and related norms on tensor products, see [23] and [66, §6.2].

4.1.2. Relations between (p, q)-multi-norms. Take $n \in \mathbb{N}$. Clearly, for each fixed $p \geq 1$ and $q_1 \geq q_2 \geq p$, we have $\|\cdot\|_n^{(p,q_1)} \leq \|\cdot\|_n^{(p,q_2)}$, and, for each fixed $q \geq 1$ and $p_1 \leq p_2 \leq q$, we have $\|\cdot\|_n^{(p_1,q)} \leq \|\cdot\|_n^{(p_2,q)}$. In fact, $\|\cdot\|^{(p,p)}$ is also a decreasing function of p on the interval $[1, \infty)$; this is not immediately obvious, but is given by the following calculation, which is essentially that of page 134 of [66]. A more general result is given in [26, Theorem 10.4]. Thus the maximum among these norms is $\|\cdot\|_n^{(1,1)}$.

THEOREM 4.5. *Let E be a normed space, and suppose that $1 \leq p \leq q < \infty$. Then*

$$\|x\|_n^{(p,p)} \geq \|x\|_n^{(q,q)} \qquad (x \in E^n)$$

for each $n \in \mathbb{N}$.

Proof. We may suppose that $p < q$.

Take $n \in \mathbb{N}$ and $x = (x_1, \ldots, x_n) \in E^n$, and set $C = \|x\|_n^{(p,p)}$. Then

$$A := \Big(\sum_{i=1}^{n} |\langle x_i, \lambda_i \rangle|^q\Big)^{1/p} = \Big(\sum_{i=1}^{n} |\langle x_i, \alpha_i\lambda_i \rangle|^p\Big)^{1/p},$$

where $\alpha_i = |\langle x_i, \lambda_i \rangle|^{(q-p)/p}$ for $i \in \mathbb{N}_n$. By (3.7) and (4.2), we have

$$\Big(\sum_{i=1}^{n} |\langle x_i, \alpha_i\lambda_i \rangle|^p\Big)^{1/p} \leq C \sup\Big\{\Big(\sum_{i=1}^{n} |\langle y, \alpha_i\lambda_i \rangle|^p\Big)^{1/p} : y \in E_{[1]}\Big\}.$$

However

$$\sum_{i=1}^{n} |\langle y, \alpha_i\lambda_i \rangle|^p = \sum_{i=1}^{n} |\langle x_i, \lambda_i \rangle|^{q-p} |\langle y, \lambda_i \rangle|^p.$$

By Hölder's inequality with conjugate exponents $q/(q-p)$ and q/p, the right-hand side of the above equation is at most

$$\Big(\sum_{i=1}^{n} |\langle x_i, \lambda_i \rangle|^q\Big)^{(q-p)/q} \cdot \Big(\sum_{i=1}^{n} |\langle y, \lambda_i \rangle|^q\Big)^{p/q}.$$

Hence we have

$$A \leq C \sup\Big\{\Big(\sum_{i=1}^{n} |\langle x_i, \lambda_i \rangle|^q\Big)^{(q-p)/pq} \cdot \Big(\sum_{i=1}^{n} |\langle y, \lambda_i \rangle|^q\Big)^{1/q} : y \in E_{[1]}\Big\}.$$

Note that $(1/p) - (q-p)/pq = 1/q$, and so, by an appropriate division, we see that

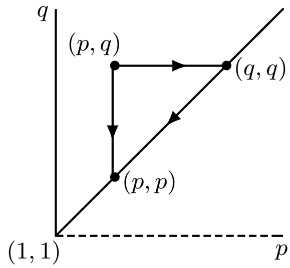
$$\Big(\sum_{i=1}^{n} |\langle x_i, \lambda_i \rangle|^q\Big)^{1/q} \leq C \sup\Big\{\Big(\sum_{i=1}^{n} |\langle y, \lambda_i \rangle|^q\Big)^{1/q} : y \in E_{[1]}\Big\}.$$

Thus $\|x\|_n^{(q,q)} \leq C = \|x\|_n^{(p,p)}$, as required. ∎

By Theorem 3.33, we have the following result.

THEOREM 4.6. *Let E be a normed space. Then $\|\cdot\|_n^{(1,1)} = \|\cdot\|_n^{\max}$ for $n \in \mathbb{N}$, and so $(\|\cdot\|_n^{(1,1)} : n \in \mathbb{N})$ is the maximum multi-norm based on E.* ∎

The relations between the multi-norms $(\|x\|_n^{(p,p)} : n \in \mathbb{N})$ can be illustrated in the following diagram, where the arrows indicate increasing multi-norms in the ordering $\leq$:



PROPOSITION 4.7. *Let E be a normed space, and suppose that $1 \le p \le q < \infty$. Then $\varphi_n^{(p,q)}(E) \le n^{1/q}$ for $n \in \mathbb{N}$.*

Proof. Consider $\lambda_1, \dots, \lambda_n \in E'$ with $\mu_{p,n}(\lambda_1, \dots, \lambda_n) \le 1$. Then $\|\lambda_i\| \le 1$ $(n \in \mathbb{N})$. Now take $x_1, \dots, x_n \in E_{[1]}$. Then $|\langle x_i, \lambda_i \rangle| \le 1$ $(i \in \mathbb{N}_n)$, and so $\|(x_1, \dots, x_n)\|_n^{(p,q)} \le n^{1/q}$. The result follows. ∎

EXAMPLE 4.8. Let $E = \ell^r$, where $r \ge 1$, so that $E' = \ell^s$, where $s = r'$.

Fix p, q with $1 \le p \le q < \infty$, and take $n \in \mathbb{N}$. We shall calculate $\|f\|_n^{(p,q)}$, where $f = (\delta_1, \dots, \delta_n) \in E^n$. Set $u = p'$.

By Proposition 4.7, $\|f\|_n^{(p,q)} \le n^{1/q}$.

Now consider the choice $\lambda_i = \delta_i$ $(i \in \mathbb{N}_n)$, and set $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n$. Then $\mu_{p,n}(\lambda_1, \dots, \lambda_n) = \sup\{\|\zeta\|_{\ell^s} : \|\zeta\|_{\ell^u} \le 1\}$. In the case where $u \le s$, i.e., $p \ge r$, we have $\|\zeta\|_{\ell^s} \le \|\zeta\|_{\ell^u}$, and so $\mu_{p,n}(\lambda_1, \dots, \lambda_n) \le 1$. Hence $\|f\|_n^{(p,q)} \ge n^{1/q}$.

This implies that

$$\|(\delta_1, \dots, \delta_n)\|_n^{(p,q)} = n^{1/q} \quad \text{whenever} \quad p \ge r.$$

A similar calculation gives the same conclusion in the case where $r = 1$.

We conclude that two multi -norms $(\|\cdot\|_n^{(p_1,q_1)})$ and $(\|\cdot\|_n^{(p_2,q_2)})$ based on ℓ^r are not equivalent whenever $p_1, p_2 \ge r$ and $q_1 \ne q_2$.

It follows that

$$\|(\delta_1, \dots, \delta_n)\|_n^{(p,q)} \le n^{1/q} \quad \text{whenever} \quad q \ge r.$$

However, we know from Corollary 3.34 that

$$\|(\delta_1, \dots, \delta_n)\|_n^{\max} = n^{1/r} \qquad (n \in \mathbb{N}),$$

and so the multi-norm $(\|\cdot\|_n^{(p,q)})$ is not equivalent to $(\|\cdot\|_n^{\max})$ whenever $q > r$. Further, $\|(\delta_1, \dots, \delta_n)\|_n^{(p,p)} = n^{1/r}$ $(n \in \mathbb{N})$ whenever $p \in [1, r]$. ∎

The general question of the equivalence of the two multi-norms

$$(\|\cdot\|_n^{(p_1,q_1)} : n \in \mathbb{N}) \quad \text{and} \quad (\|\cdot\|_n^{(p_2,q_2)} : n \in \mathbb{N})$$

on the spaces $L^r(\Omega)$ will be addressed in [20].

THEOREM 4.9. *Let E and F be isomorphic Banach spaces such that $d(E,F) \le C$, and suppose that $1 \le p \le q < \infty$. Then*

$$\varphi_n^{(p,q)}(E) \le C\varphi_n^{(p,q)}(F) \qquad (n \in \mathbb{N}).$$

Proof. Take $\varepsilon > 0$. Then there exists a linear bijection $T : E \to F$ with $\|T\| < C + \varepsilon$ and $\|S\| = 1$, where $S = T^{-1} : F \to E$. We have $\|S'\| = 1$.

Take $n \in \mathbb{N}$ and $x_1, \dots, x_n \in E_{[1]}$. Suppose that $\lambda = (\lambda_1, \dots, \lambda_n) \in (E')^n$ with $\mu_{p,n}(\lambda) \le 1$. By (3.3), $\mu_{p,n}(S'\lambda_1, \dots, S'\lambda_n) \le 1$, and so

$$\Big(\sum_{i=1}^n |\langle x_i,\, \lambda_i\rangle|^q\Big)^{1/q} = \Big(\sum_{i=1}^n |\langle Tx_i,\, S'\lambda_i\rangle|^q\Big)^{1/q} \le (C+\varepsilon)\varphi_n^{(p,q)}(F).$$

Thus $\varphi_n^{(p,q)}(E) \le (C+\varepsilon)\varphi_n^{(p,q)}(F)$. This holds true for each $\varepsilon > 0$, and so the result follows. ■

4.1.3. Duality theory. Let E be a normed space, and take $p, q \in [1, \infty)$. For $n \in \mathbb{N}$ and $\lambda = (\lambda_1, \dots, \lambda_n) \in (E')^n$, the formula for $\|\lambda\|_n^{(p,q)}$ is

$$\|\lambda\|_n^{(p,q)} = \sup\Big\{\Big(\sum_{i=1}^n |\langle \Lambda_i,\, \lambda_i\rangle|^q\Big)^{1/q} : \mu_{p,n}(\Lambda_1, \dots, \Lambda_n) \le 1\Big\},$$

taking the supremum over $\Lambda_1, \dots, \Lambda_n \in E''$. In fact, there is a simpler formula for $\|\lambda\|_n^{(p,q)}$; the proof, from the Principle of Local Reflexivity, of the following proposition is almost identical to that of Theorem 3.38, and is omitted.

PROPOSITION 4.10. *Let E be a normed space, and take $p, q \in [1, \infty)$. For each $n \in \mathbb{N}$ and $\lambda \in (E')^n$, we have*

$$\|\lambda\|_n^{(p,q)} = \sup\Big\{\Big(\sum_{i=1}^n |\langle x_i,\, \lambda_i\rangle|^q\Big)^{1/q} : \mu_{p,n}(x_1, \dots, x_n) \le 1\Big\},$$

taking the supremum over $x_1, \dots, x_n \in E$. ■

4.1.4. The dual of the (p,q)-special-norm. In this section, we shall determine the dual of the the special-norm $(\|\cdot\|_n^{(p,q)} : n \in \mathbb{N})$ based on $(E')^n$, following remarks of Paul Ramsden.

Let E be a Banach space, and fix r, s with $1 \le r < \infty$ and $1 < s \le \infty$. The conjugate index to r is r'. For each $n \in \mathbb{N}$ and $x = (x_1, \dots, x_n) \in E^n$, we set

$$|\!|\!| x |\!|\!|_n^{(r,s)} = \inf\Big\{\sum_{k=1}^m \|\alpha_k\|_s \cdot \mu_{r,n}(y_k)\Big\},$$

where the infimum is taken over all representations

$$x = \sum_{k=1}^m M_{\alpha_k}(y_k)$$

for which $\alpha_1, \dots, \alpha_m \in \mathbb{C}^n$, $y_1, \dots, y_m \in E^n$, and $m \in \mathbb{N}$. It is clear that $|\!|\!| \cdot |\!|\!|_n^{(r,s)}$ is a norm on E^n.

The following is 'dual' to the proof of Theorem 4.1, and will also follow from Theorem 4.13, below, and so the direct proof is omitted.

THEOREM 4.11. *Let E be a normed space, and take $r, s \in [1, \infty]$ with $1 \le r \le s' < \infty$. Then the sequence $(|\!|\!| \cdot |\!|\!|_n^{(r,s)} : n \in \mathbb{N})$ is a dual multi-norm based on E.* ■

DEFINITION 4.12. Let E be a normed space, and take $r, s \in [1, \infty]$ with $1 \le r \le s' < \infty$. Then $(\|\!| \cdot |\!\|_n^{(r,s)} : n \in \mathbb{N})$ is the *(r,s)-dual multi-norm based on E.*

Let E be a normed space, and take $p, q \in [1, \infty)$. For $n \in \mathbb{N}$ and $x = (x_1, \dots, x_n) \in E^n$, define an embedding

$$\nu_E(x) : (\lambda_1, \dots, \lambda_n) \mapsto (\langle x_1, \lambda_1 \rangle, \dots, \langle x_n, \lambda_n \rangle), \qquad \ell_n^p(E')^w \to \ell_n^q.$$

Then $\nu_E(x) : \ell_n^p(E')^w \to \ell_n^q$ is a bounded linear map, and we have $\|\nu_E(x)\| = \|x\|_n^{(p,q)}$, and so we have an isometric embedding

$$\nu_E : (E^n, \|\cdot\|_n^{(p,q)}) \to \mathcal{B}(\ell_n^p(E')^w, \ell_n^q). \tag{4.3}$$

Now take r, s with $1 \le r < \infty$ and $1 < s \le \infty$. Then there is a continuous linear surjection $\theta_E : \ell_n^r(E)^w \,\widehat{\otimes}\, \ell_n^s \to E^n$ such that

$$\theta_E(x \otimes \alpha) = (\alpha_1 x_1, \dots, \alpha_n x_n)$$

whenever $x = (x_1, \dots, x_n) \in E^n$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \ell_n^s$. Thus there is an isometric isomorphism of Banach spaces

$$(\ell_n^r(E)^w \,\widehat{\otimes}\, \ell_n^s)/\ker \theta_E \cong (E^n, \|\!| \cdot |\!\|_n^{(r,s)}).$$

THEOREM 4.13. *Let E be a Banach space, and take p, q, r, s such that $1 \le p, q, r < \infty$ and $1 < s \le \infty$. Then there are isometric isomorphisms:*

(i) $(E^n, \|\cdot\|_n^{(p,q)})' \cong ((E')^n, \|\!| \cdot |\!\|_n^{(p,q')})$;

(ii) $(E^n, \|\!| \cdot |\!\|_n^{(r,s)})' \cong ((E')^n, \|\cdot\|_n^{(r,s')})$.

Proof. (i) It is easily checked that the following diagram commutes:

$$\begin{tikzcd}
(\ell_n^p(E')^w \,\widehat{\otimes}\, \ell_n^{q'})'' \arrow[r, "\nu_E'"] & (E')^n \\
\ell_n^p(E')^w \,\widehat{\otimes}\, \ell_n^{q'}. \arrow[u] \arrow[ur, "\theta_{E'}"'] &
\end{tikzcd}$$

Hence we have isometric isomorphisms of Banach spaces

$$(E^n, \|\cdot\|_n^{(p,q)})' \cong (\ell_n^p(E')^w \,\widehat{\otimes}\, \ell_n^{q'})''/\ker \nu_E' \cong (\ell_n^p(E')^w \,\widehat{\otimes}\, \ell_n^{q'})/\ker \theta_{E'} \cong ((E')^n, \|\!| \cdot |\!\|_n^{(p,q')}).$$

(ii) Similarly, the following diagram commutes:

$$\begin{tikzcd}
(E')^n \arrow[r, "\theta_E'"] \arrow[dr, "\nu_{E'}"'] & \mathcal{B}(\ell_n^r(E)^w, \ell_n^{s'}) \\
& \mathcal{B}(\ell_n^r(E'')^w, \ell_n^{s'}) \arrow[u, "j : T \mapsto T|E^n"']
\end{tikzcd}$$

Hence there is an isometric isomorphism

$$(E^n, \|\!| \cdot |\!\|_n^{(r,s)})' \cong \operatorname{im} \theta_E' = \operatorname{im}(j \circ \nu_{E'}).$$

By Proposition 4.10, there is an isometric isomorphism

$$\operatorname{im}(j \circ \nu_{E'}) \cong \operatorname{im} \nu_{E'} \cong ((E')^n, \|\cdot\|_n^{(r,s')}),$$

and so the result follows. ■

Thus, in the case where $1 \le p \le q < \infty$, the dual of the multi-norm $(\|\cdot\|_n^{(p,q)} : n \in \mathbb{N})$ based on E is the dual multi-norm $(\|\!|\cdot\|\!|_n^{(p,q')} : n \in \mathbb{N})$ based on E'.

The following corollary resolves a 'second dual question' for the (p,q)-multi-norm (defined when $1 \le p \le q < \infty$).

Corollary 4.14. *Let E be a Banach space, and take $p, q \in [1, \infty)$. Then*

$$(E^n, \|\cdot\|_n^{(p,q)})'' \cong ((E'')^n, \|\cdot\|_n^{(p,q)}). \ \blacksquare$$

4.1.5. Multi-norms on Hilbert spaces. We now consider an example which involves Hilbert spaces. It will lead to an alternative description of the $(2,2)$-multi-norm based on a Hilbert space.

Let $(H, \|\cdot\|)$ be a Hilbert space. (Basic facts about Hilbert spaces were recalled in §1.2.6.) For each family $\mathbf{H} = \{H_1, \dots, H_n\}$, where $n \in \mathbb{N}$ and each H_j is a closed subspace of H and $H = H_1 \oplus_\perp \cdots \oplus_\perp H_n$, set

$$r_{\mathbf{H}}((x_1, \dots, x_n)) = (\|P_1 x_1\|^2 + \cdots + \|P_n x_n\|^2)^{1/2} = \|P_1 x_1 + \cdots + P_n x_n\| \tag{4.4}$$

for $x_1, \dots, x_n \in H$, where $P_i : H \to H_i$ for $i \in \mathbb{N}_n$ is the orthogonal projection, and then set

$$\|(x_1, \dots, x_n)\|_n^H = \sup_{\mathbf{H}} r_{\mathbf{H}}((x_1, \dots, x_n)) \quad (x_1, \dots, x_n \in H), \tag{4.5}$$

where the supremum is taken over all such families $\mathbf{H}$. (We allow the possibility that $H_j = \{0\}$ and $P_j = 0$ for some $j \in \mathbb{N}_n$.)

The following result is easily checked.

Theorem 4.15. *Let H be a Hilbert space. Then $(\|\cdot\|_n^H : n \in \mathbb{N})$ is a multi-norm on the family $\{H^n : n \in \mathbb{N}\}$.* $\blacksquare$

For example, let $H = \ell^2$, and take $n \in \mathbb{N}$ and $\beta_1, \dots, \beta_n \in \mathbb{C}$. Then

$$\|(\beta_1 \delta_1, \dots, \beta_n \delta_n)\|_n^H = \left(\sum_{j=1}^n \beta_j^2 \right)^{1/2}.$$

Definition 4.16. Let $(H, \|\cdot\|)$ be a Hilbert space. Then the *Hilbert multi-norm* based on H is the multi-norm $(\|\cdot\|_n^H : n \in \mathbb{N})$ defined above. The rate of growth of this multi-norm is denoted by $(\varphi_n^H(H) : n \in \mathbb{N})$.

The following results are based on remarks of Hung Le Pham.

Proposition 4.17. *Let H be a Hilbert space, take $n \in \mathbb{N}$, and $x_1, \dots, x_n \in H$. Then*

$$\|(x_1, \dots, x_n)\|_n^H = \sup\{|\alpha_1[e_1, x_1] + \cdots + \alpha_n[e_n, x_n]|\}, \tag{4.6}$$

taking the supremum over orthonormal sets $\{e_1, \dots, e_n\}$ in H and $(\alpha_1, \dots, \alpha_n) \in (\ell_n^2)_{[1]}$.

Proof. Set $A = \|(x_1, \dots, x_n)\|_n^H$ and $B = \sup\{|\alpha_1[e_1, x_1] + \cdots + \alpha_n[e_n, x_n]|\}$.

Given $\varepsilon > 0$, let $\{P_1, \dots, P_n\}$ be an orthogonal family of projections such that $\|P_1 x_1\|^2 + \cdots + \|P_n x_n\|^2 > A^2 - \varepsilon$. It follows from (1.14) that there is an orthonormal set $\{e_1, \dots, e_n\}$ in H such that

$$\|P_1 x_1\|^2 + \cdots + \|P_n x_n\|^2 = [e_1, x_1]^2 + \cdots + [e_n, x_n]^2.$$

Set $\alpha_j = [e_j, x_j]/(A+\varepsilon)$ ($j \in \mathbb{N}_n$). Then

$$\sum_{j=1}^n |\alpha_j|^2 \leq 1 \quad \text{and} \quad \sum_{j=1}^n |\alpha_j [e_j, x_j]| > \frac{A^2 - \varepsilon}{A+\varepsilon}.$$

Thus $B \geq (A^2 - \varepsilon)/(A+\varepsilon)$. Since this holds true for each $\varepsilon > 0$, we have $B \geq A$.

Conversely, given an orthonormal set $\{e_1, \dots, e_n\}$ in H, there is an orthogonal decomposition $H = H_1 \oplus_\perp \cdots \oplus_\perp H_n$ such that $e_j \in H_j$ ($j \in \mathbb{N}_n$), and then

$$|[e_j, x]| \leq \|P_j x\| \quad (x \in H,\ j \in \mathbb{N}_n).$$

Take $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ such that $\sum_{j=1}^n |\alpha_j|^2 \leq 1$. Then

$$\sum_{j=1}^n |\alpha_j [e_j, x_j]| \leq \sum_{j=1}^n |\alpha_j| \|P_j x\| \leq \Big(\sum_{j=1}^n \|P_j x\|^2 \Big)^{1/2} \leq A$$

by the Cauchy–Schwarz inequality. Hence $B \leq A$.

The result follows. ■

In the following result, D_n denotes the family of all orthonormal n-tuples of elements in H, and the closure of $\mathrm{co}(D_n)$ is taken in the weak-$*$ topology on H^n.

PROPOSITION 4.18. *Take $n \in \mathbb{N}$, and let H be a Hilbert space with $\dim H \geq n$. Then the closed unit ball of $(H^n, \mu_{2,n})$ is equal to $\overline{\mathrm{co}}(D_n)$.*

Proof. We write B_n for $(H^n, \mu_{2,n})_{[1]}$, and we identify H with $\ell^2(I)$, for an index set I; we may suppose that $\mathbb{N}_n$ is a subset of I.

It is clear from (3.4) that $D_n \subset B_n$, and so $\overline{\mathrm{co}}(D_n) \subset B_n$.

For the converse, take $x = (x_1, \dots, x_n) \in B_n$. Define $S : H \to H$ by setting

$$S(\delta_i) = x_i \quad (i \in \mathbb{N}_n), \qquad S(\delta_i) = 0 \quad (i \in I \setminus \mathbb{N}_n).$$

Since $\mu_{2,n}(x) \leq 1$, we see that $\|S\| \leq 1$, and so $S \in \overline{\mathrm{co}}(\mathcal{U}(\mathcal{B}(H))$ by the Russo–Dye theorem. Hence $(x_1, \dots, x_n) = (S(\delta_1), \dots, S(\delta_n)) \in \overline{\mathrm{co}}(D_n)$, as required. ■

THEOREM 4.19. *Let H be an infinite-dimensional Hilbert space. Then*

$$\|(x_1, \dots, x_n)\|_n^H = \|(x_1, \dots, x_n)\|_n^{(2,2)} \quad (x_1, \dots, x_n \in H)$$

for each $n \in \mathbb{N}$.

Proof. This follows from the two previous propositions. ■

Thus the Hilbert multi-norm and the $(2,2)$-multi-norm on ℓ^2 are equal. It is natural to ask if these multi-norms are also equal to the maximum multi-norm on ℓ^2; in fact, they are equivalent to the maximum multi-norm, but not equal to it [20].

A more general version of following result will be proved in [20].

THEOREM 4.20. *Let H be an infinite-dimensional Hilbert space. Then the following multi-norms based on H are mutually equivalent:*

(a) *the Hilbert multi-norm ($\|\cdot\|_n^H$);*
(b) *the maximum multi-norm ($\|\cdot\|_n^{\max}$);*
(c) *the (p,p)-multi-norm ($\|\cdot\|_n^{(p,p)}$) for $p \in [1,2]$.*

For the above multi-norms, the rate of growth is equivalent to ($\sqrt{n} : n \in \mathbb{N}$).

Further, the (p,p)*-multi-norm and the* (q,q)*-multi-norms based on* H *are not equivalent whenever* $p \neq q$ *and* $\max\{p,q\} > 2$. ∎

4.2. Standard q-multi-norms. We shall now construct some multi-norms based on the Banach spaces $L^p(\Omega,\mu)$ and $M(K)$. We begin with the spaces $L^p(\Omega,\mu)$.

4.2.1. Definition. Let (Ω,μ) be a measure space. For each $p \in [1,\infty)$, we consider the Banach space $E = L^p(\Omega,\mu)$, with the norm

$$\|f\| = \left(\int_\Omega |f|^p\right)^{1/p} = \left(\int_\Omega |f|^p \,\mathrm{d}\mu\right)^{1/p} \qquad (f \in E),$$

as in §1.2.7. For a measurable subset X of Ω, we write r_X for the seminorm on E specified by

$$r_X(f) = \|f\chi_X\| = \left(\int_X |f|^p\right)^{1/p} \qquad (f \in E),$$

where we again suppress in the notation the dependence on p. (We take $r_X(f) = 0$ when $X = \emptyset$.)

Now take $q \geq p$; we shall define a multi-norm based on E that depends on q.

Take $n \in \mathbb{N}$. For each ordered partition $\mathbf{X} = (X_1,\dots,X_n)$ of Ω into measurable subsets and each $f_1,\dots,f_n \in E$, we set

$$\begin{aligned} r_{\mathbf{X}}((f_1,\dots,f_n)) &= (r_{X_1}(f_1)^q + \cdots + r_{X_n}(f_n)^q)^{1/q} \\ &= \left(\left(\int_{X_1} |f_1|^p\right)^{q/p} + \cdots + \left(\int_{X_n} |f_n|^p\right)^{q/p}\right)^{1/q}, \end{aligned}$$

so that $r_{\mathbf{X}}$ is a seminorm on E^n and

$$r_{\mathbf{X}}((f_1,\dots,f_n)) \leq (\|f_1\|^q + \cdots + \|f_n\|^q)^{1/q}.$$

Finally, we define

$$\|(f_1,\dots,f_n)\|_n^{[q]} = \sup_{\mathbf{X}} r_{\mathbf{X}}((f_1,\dots,f_n)) \qquad (f_1,\dots,f_n \in E), \tag{4.7}$$

where the supremum is taken over all such ordered partitions $\mathbf{X}$. Then $\|\cdot\|_n^{[q]}$ is a norm on E^n.

In the case where $q = p$, we have

$$\|(f_1,\dots,f_n)\|_n^{[p]} = \sup_{\mathbf{X}} \|f_1 | X_1 + \cdots + f_n | X_n\| \qquad (f_1,\dots,f_n \in E). \tag{4.8}$$

In the case where $q \geq p$ and $f_1,\dots,f_n \in E$ have disjoint support, we have

$$\|(f_1,\dots,f_n)\|_n^{[q]} = (\|f_1\|^q + \cdots + \|f_n\|^q)^{1/q}; \tag{4.9}$$

if, further, $q = p$, then

$$\|(f_1,\dots,f_n)\|_n^{[p]} = \|f_1 + \cdots + f_n\|. \tag{4.10}$$

It is easily checked that $(\|\cdot\|_n^{[q]} : n \in \mathbb{N})$ is a multi-norm based on E: indeed, Axioms (A1), (A2), and (A3) are immediate, and Axiom (A4) follows because

$$(\alpha^p + \beta^p)^{1/p} \geq (\alpha^q + \beta^q)^{1/q} \qquad (\alpha,\beta \in \mathbb{R}^+)$$

whenever $p \leq q$. Further, for each $n \in \mathbb{N}$, we have

$$\|(f_1, \dots, f_n)\|_n^{[q]} \leq (\|f_1\|^q + \cdots + \|f_n\|^q)^{1/q} \quad (f_1, \dots, f_n \in E). \tag{4.11}$$

Definition 4.21. Let Ω be a measure space, and take $p \geq 1$. Then, for each $q \geq p$, the *standard q-multi-norm* based on $L^p(\Omega)$ is the multi-norm $(\|\cdot\|_n^{[q]} : n \in \mathbb{N})$. The rate of growth of this multi-norm is denoted by $(\varphi_n^{[q]}(L^p(\Omega)) : n \in \mathbb{N})$.

At this point, it appears that the definition of the standard q-multi-norm based on $L^p(\Omega)$ depends on the concrete representation of $L^p(\Omega)$ as a Banach space of functions. We would wish that, if $L^p(\Omega_1)$ and $L^p(\Omega_2)$ are isometrically order-isomorphic Banach lattices, then the corresponding standard q-multi-norms based on $L^p(\Omega_1)$ and on $L^p(\Omega_2)$ are equal. We shall see in Theorem 4.36 that this is indeed the case; see also Theorem 4.37.

It follows from (4.11) that $\varphi_n^{[q]}(L^p(\Omega)) \leq n^{1/q}$.

We may consider these multi-norms $(\|\cdot\|_n^{[q]} : n \in \mathbb{N})$ as a function of q when $q \in [p, \infty)$; clearly, for each $n \in \mathbb{N}$, the norms $\|\cdot\|_n^{[q]}$ decrease as q increases, and so the maximum multi-norm among these multi-norms is $(\|\cdot\|_n^{[p]} : n \in \mathbb{N})$.

There is an equivalent way of defining the norm $\|(f_1, \dots, f_n)\|_n^{[q]}$ for $f_1, \dots, f_n \in L^p(\Omega)$ in the special case where $q = p$. Indeed, set $f = |f_1| \vee \cdots \vee |f_n|$, so that

$$f(x) = \max\{|f_1(x)|, \dots, |f_n(x)|\} \quad (x \in \Omega).$$

Then we see immediately that

$$\|(f_1, \dots, f_n)\|_n^{[p]} = \|f\| = \left(\int_\Omega (|f_1| \vee \cdots \vee |f_n|)^p\right)^{1/p}. \tag{4.12}$$

In particular, in the case where $E = \ell^p$, we have

$$\|(f_1, \dots, f_n)\|_n^{[p]} = \left(\sum_{j=1}^\infty (|f_1(j)| \vee \cdots \vee |f_n(j)|)^p\right)^{1/p}. \tag{4.13}$$

[To see that the formula $\|(f_1, \dots, f_n)\|_n^{[q]} = \|f\|$ is correct only when $q = p$, consider the case where X_1 and X_2 are disjoint subsets of $\mathbb{N}$ of cardinalities m and n, respectively, and let f_j be the characteristic function of X_j for $j = 1, 2$. By (4.7),

$$(\|(f_1, f_2)\|_2^{[q]})^q = m^{q/p} + n^{q/p}$$

whereas $\|f\|^q = (m+n)^{q/p}$, and we have $m^{q/p} + n^{q/p} = (m+n)^{q/p}$ for all $m, n \in \mathbb{N}$ if and only if $q = p$.]

Suppose that $1 \leq p \leq q < \infty$, and set $E = \ell^p$. Take $n \in \mathbb{N}$, and consider the elements $\delta_1, \dots, \delta_n \in E_{[1]}$. Let $\mathbf{X} = (X_1, \dots, X_n)$ be an ordered partition of $\mathbb{N}$; suppose, in fact, that $i \in X_i$ $(i \in \mathbb{N}_n)$. For each $q \geq p$, we have $r_{\mathbf{X}}((\delta_1, \dots, \delta_n)) = n^{1/q}$, and so $\|(\delta_1, \dots, \delta_n)\|_n^{[q]} \geq n^{1/q}$. It follows that

$$\varphi_n^{[q]}(\ell^p) = n^{1/q} \quad (n \in \mathbb{N}). \tag{4.14}$$

In particular, taking $q = p$, we see that $\varphi_n^{\max}(\ell^p) \geq \varphi_n^{[p]}(\ell^p) = n^{1/p}$ for $n \in \mathbb{N}$, so recovering a result of Example 3.46.

Let $n \in \mathbb{N}$, and let (α_i) be a fixed element of $\mathbb{C}^n$. Set $x_i = \alpha_i \delta_i$ $(i \in \mathbb{N}_n)$. Then we now have

$$\|(x_1, \dots, x_n)\|_n^{[q]} = (|\alpha_1|^q + \cdots + |\alpha_n|^q)^{1/q} \quad (n \in \mathbb{N}). \tag{4.15}$$

Thus $(E^n, \|\cdot\|_n^{[q]})$ contains ℓ_n^q as a closed subspace.

There does not seem to be an accessible, explicit formula for the dual of the standard q-multi-norm based on $L^p(\Omega)$ in the general case where $q \geq p$. Let $(\|\!|\cdot\|\!|_n^{[s]} : n \in \mathbb{N})$ denote the dual multi-norm, based on $L^r(\Omega)$, to the standard q-multi-norm based on $L^p(\Omega)$; here r and s are the conjugate indices to p and q, respectively, so that we have $1 < s \leq r < \infty$. Then we have an estimate

$$\|\!|(\lambda_1, \dots, \lambda_n)\|\!|_n^{[s]} \leq \inf_{\mathbf{X}} \Big\{ \sum_{k=1}^n \Big(\sum_{j=1}^n \|\lambda_{j+k-1} \mid X_j\|_{\ell^r}^s \Big)^{1/s} \Big\}$$

for $\lambda_1, \dots, \lambda_n \in L^r(\Omega)$ and $n \in \mathbb{N}$, where the infimum is taken over all ordered partitions $\mathbf{X} = (X_1, \dots, X_n)$ of Ω into measurable subsets. For the special case where $q = p$, see Example 4.47, below; unfortunately, the above estimate does not give the 'correct' value even in this special case.

4.2.2. A comparison of multi-norms. Suppose that $1 \leq p \leq q < \infty$. We have defined the (p,q)-multi-norm $(\|\cdot\|_n^{(p,q)} : n \in \mathbb{N})$ and the standard q-multi-norm $(\|\cdot\|_n^{[q]} : n \in \mathbb{N})$ based on $E := L^p(\Omega)$, where Ω is a measure space. We shall now show that

$$(\|\cdot\|_n^{[q]} : n \in \mathbb{N}) \leq (\|\cdot\|_n^{(p,q)} : n \in \mathbb{N})$$

in $\mathcal{E}_E$ in the notation of Definition 2.24.

THEOREM 4.22. *Let (Ω, μ) be a measure space, and suppose that $1 \leq p \leq q < \infty$. Then*

$$\|(f_1, \dots, f_n)\|_n^{[q]} \leq \|(f_1, \dots, f_n)\|_n^{(p,q)} \qquad (f_1 \dots, f_n \in L^p(\Omega, \mu),\ n \in \mathbb{N}).$$

Proof. We set $r = p'$, the conjugate index to p. Take $n \in \mathbb{N}$ and $f_1 \dots, f_n \in L^p(\Omega)$, and then suppose that $\mathbf{X} = (X_1, \dots, X_n)$ is an ordered partition of Ω. There exist elements $\lambda_1, \dots, \lambda_n \in L^r(\Omega)$ such that $\operatorname{supp} \lambda_i \subset X_i$, such that $\|\lambda_i\|_{L^r} = 1$, and such that we have $\langle f_i, \lambda_i \rangle = \|f_i | X_i\|_{L^p}$ for $i \in \mathbb{N}_n$. For each $\zeta_1, \dots, \zeta_n \in \mathbb{C}$, we have

$$\Big\| \sum_{i=1}^n \zeta_i \lambda_i \Big\|_{L^r} = \Big(\sum_{i=1}^n |\zeta_i|^r \Big)^{1/r},$$

and so, by (3.4),

$$\mu_{p,n}(\lambda_1, \dots, \lambda_n) = \sup \Big\{ \Big\| \sum_{i=1}^n \zeta_i \lambda_i \Big\|_{L^r} : \sum_{i=1}^n |\zeta_i|^r \leq 1 \Big\} \leq 1.$$

Thus

$$r_{\mathbf{X}}((f_1, \dots, f_n)) = \Big(\sum_{i=1}^n \|f_i | X_i\|_{L^p}^q \Big)^{1/q} = \Big(\sum_{i=1}^n \langle f_i, \lambda_i \rangle^q \Big)^{1/q} \leq \|(f_1, \dots, f_n)\|_n^{(p,q)}.$$

This holds for each ordered partition $\mathbf{X}$ of Ω, and so the result follows. ∎

4.2.3. Maximality. The following result was pointed out by Paul Ramsden; a more general version will be given in Theorem 4.54(i), below.

THEOREM 4.23. *Let Ω be a measure space. Then the standard 1-multi-norm and the maximum multi-norm based on $L^1(\Omega)$ are equal.*

Proof. Set $E = L^1(\Omega)$. Fix $n \in \mathbb{N}$, take $f_1, \dots, f_n \in E$, and set $f = |f_1| \vee \cdots \vee |f_n|$ in E. For $\lambda_1, \dots, \lambda_n \in E'$, it follows from Proposition 3.20(ii) that

$$\Big|\sum_{j=1}^{n} \langle f_j, \lambda_j \rangle\Big| \le \sum_{j=1}^{n} |\langle f_j, \lambda_j \rangle| \le \sum_{j=1}^{n} \langle |f_j|, |\lambda_j| \rangle \le \Big\langle f, \sum_{j=1}^{n} |\lambda_j| \Big\rangle$$
$$\le \|f\| \Big\|\sum_{j=1}^{n} |\lambda_j|\Big\| = \|(f_1, \dots, f_n)\|_n^{[1]} \mu_{1,n}(\lambda_1, \dots, \lambda_n).$$

Hence $\|(f_1, \dots, f_n)\|_n^{\max} \le \|(f_1, \dots, f_n)\|_n^{[1]}$ by Theorem 3.33, giving the result. ∎

PROPOSITION 4.24. *Suppose that $1 \le p < q < \infty$. Then the (p, q)-multi-norm is not equivalent to the maximum multi-norm on ℓ^p.*

Proof. By Proposition 4.7, $\varphi_n^{(1,q)}(\ell^1) \le n^{1/q}$ $(n \in \mathbb{N})$.

Suppose that $p \in [1, 2]$. By Theorem 3.54(i), $\varphi_n^{\max}(\ell^p) = n^{1/p}$ $(n \in \mathbb{N})$. Since there is no constant $C > 0$ such that $n^{1/p} \le C n^{1/q}$ $(n \in \mathbb{N})$, the two multi-norms are not equivalent.

Suppose that $p \in [2, \infty)$. By Theorem 3.54(ii), $\varphi_n^{\max}(\ell^p) \sim n^{1/2}$. Since there is no constant $C > 0$ such that $n^{1/2} \le C n^{1/q}$ $(n \in \mathbb{N})$, the two multi-norms are not equivalent. ∎

4.2.4. Equality of two multi-norms on $L^1(\Omega)$. The first result of this section is similar to that given in Proposition 4.18.

Let (Ω, μ) be a measure space, and set $E = L^1(\Omega, \mu)$. Then there is a compact space K such that E' is order-isometric to $C(K)$; $F := L^\infty(\Omega, \mu)$ is a C^*-subalgebra of $C(K)$. For $n \in \mathbb{N}$, the weak-$*$ topology on $(E')^n$ as the dual of E^n is denoted by σ_n. In the following result, $\overline{\mathrm{co}}(S)$ denotes the σ_n-closure of the convex hull of a subset S of $(E')^n$.

For each $n \in \mathbb{N}$, let D_n be the set of elements $(\lambda_1, \dots, \lambda_n)$ in F^n such that the subsets $\operatorname{supp} \lambda_1, \dots, \operatorname{supp} \lambda_n$ of Ω are pairwise disjoint. Since D_n is balanced, $\mathrm{co}(D_n)$ is also balanced, and hence absolutely convex.

LEMMA 4.25. *Let $n \in \mathbb{N}$. Then $(F^n, \mu_{1,n})_{[1]} = \overline{\mathrm{co}}(D_n)$.*

Proof. Write B_n for the closed unit ball $(F^n, \mu_{1,n})_{[1]}$. Clearly we have $D_n \subset B_n$, and so $\overline{\mathrm{co}}(D_n) \subset B_n$.

Assume towards a contradiction that there exists

$$\lambda = (\lambda_1, \dots, \lambda_n) \in B_n \setminus \overline{\mathrm{co}}(D_n).$$

By the Hahn–Banach separation theorem, there exists $f = (f_1, \dots, f_n) \in E^n$ such that $\sum_{i=1}^{n} \langle f_i, \lambda_j \rangle > 1$, but $|\sum_{i=1}^{n} \langle f_i, \mu_i \rangle| \le 1$ $(\mu_1, \dots, \mu_n \in D_n)$. By the definition of the standard 1-multi-norm $(\|\cdot\|_n^{[1]})$ on E, we have

$$\|f\|_n^{[1]} = \sup\Big\{\sum_{i=1}^{n} \|f_i | X_i\| : \mathbf{X} = (X_1, \dots, X_n)\Big\},$$

where the supremum is taken over all ordered partitions $\mathbf{X}$ of Ω. For $i \in \mathbb{N}_n$, we have $\|f_i | X_i\| = \sup\{|\langle f_i, \mu_i \rangle| : \mu_i \in L^1(X_i)'_{[1]}\}$, and so

$$\|f\|_n^{[1]} = \sup\Big\{\Big|\sum_{i=1}^{n} \langle f_i, \mu_i \rangle\Big| : \mu_1, \dots, \mu_n \in D_n\Big\} \le 1,$$

whereas $\|f\|_n^{(1,1)} \geq \sum_{i=1}^n \langle f_i,\, \lambda_j\rangle > 1$. However $\|f\|_n^{[1]} = \|f\|_n^{\max}$ by Theorem 4.23, and so $\|f\|_n^{\max} < \|f\|_n^{[1]}$, a contradiction. ∎

THEOREM 4.26. *Let Ω be a measure space, and take $q \geq 1$. Then the standard q-multi-norm and the $(1,q)$-multi-norm based on $L^1(\Omega)$ are equal.*

Proof. Set $E = L^1(\Omega,\mu)$, and take $n \in \mathbb{N}$ and $f = (f_1,\dots,f_n) \in E^n$. By replacing Ω by $\bigcup_{i=1}^n \operatorname{supp} f_i$, we may suppose that Ω is σ-finite and hence that $F = E'$ in the notation of Lemma 4.25. Then

$$\|f\|_n^{(1,q)} = \sup\Big\{\Big(\sum_{i=1}^n |\langle f_i,\, \lambda_i\rangle|^q\Big)^{1/q} : \mu_{p,n}(\lambda_1,\dots,\lambda_n) \leq 1\Big\}$$

and

$$\|f\|_n^{[q]} = \sup\Big\{\Big(\sum_{i=1}^n |\langle f_i,\, \lambda_i\rangle|^q\Big)^{1/q} : (\lambda_1,\dots,\lambda_n) \in D_n\Big\},$$

taking the supremum over all $\lambda_1,\dots,\lambda_n \in E'$ in each case. By Lemma 4.25, the two suprema are equal. ∎

4.2.5. Equivalence of multi-norms on ℓ^p. We now ask when various multi-norms based on the spaces ℓ^p are equivalent.

Take p,q such that $1 \leq p \leq q < \infty$, and set $E = \ell^p$. Then we know that

$$\|f\|_n^{[q]} \leq \|f\|_n^{(p,q)} \leq \|f\|_n^{\max} \qquad (f \in E^n)$$

for each $n \in \mathbb{N}$. We ask whether $(\|\cdot\|_n^{[q]})$ is equivalent to $(\|\cdot\|_n^{(p,q)})$, and whether $(\|\cdot\|_n^{(p,q)})$ is equivalent to $(\|\cdot\|_n^{\max})$, where each multi-norm is based on ℓ^p.

First suppose that $p = 1$. Then we saw in Theorem 4.23 that the answer to both these questions is 'yes' when also $q = 1$ (with equality of norms). In the case where $q > 1$, the $(1,q)$-multi-norm is not equivalent to the maximum multi-norm by Proposition 4.24. However, by Theorem 4.26, $\|f\|_n^{[q]} = \|f\|_n^{(1,q)}$ for $f \in (\ell^1)^n$, $n \in \mathbb{N}$, and all $q \geq 1$. Thus we have complete answers when $p = 1$, and so we shall now consider the case where $p > 1$.

We shall show first that $(\|\cdot\|_n^{[q]})$ is not equivalent to $(\|\cdot\|_n^{(p,q)})$ on ℓ^p in certain cases when $p > 1$.

THEOREM 4.27. *Take p,q such that $1 < p \leq q < \infty$. Suppose that either $2 \leq p \leq q$ or that $1 < p < 2$ and $p \leq q < p/(2-p)$. Then the multi-norms $(\|\cdot\|_n^{[q]} : n \in \mathbb{N})$ and $(\|\cdot\|_n^{(p,q)} : n \in \mathbb{N})$ based on ℓ^p are not equivalent.*

Proof. The conjugate index to p is denoted by r.

Assume towards a contradiction that the two multi-norms are equivalent, so that there exists $C > 0$ such that

$$\|(f_1,\dots,f_k)\|_k^{(p,q)} \leq C\|(f_1,\dots,f_k)\|_k^{[q]}$$

for each $k \in \mathbb{N}$ and each $f_1,\dots,f_k \in \ell^p$.

Fix $k \in \mathbb{N}$. For $i \in \mathbb{N}_k$, take

$$f_i = \sum_{j=1}^k \zeta^{-ij}\delta_j = (\zeta^{-i}, \zeta^{-2i},\dots,\zeta^{-ki},0,0,\dots) \in \ell^p,$$

where $\zeta = \exp(2\pi \mathrm{i}/k)$, and set $f = (f_1,\dots,f_k)$.

For each ordered partition $\mathbf{X} = (X_1, \dots, X_k)$ of $\mathbb{N}_k$, we have

$$r_{\mathbf{X}}((f_1, \dots, f_k)) \le (|X_1|^{q/p} + \cdots + |X_k|^{q/p})^{1/q} \le k^{1/p},$$

and so $\|f\|_k^{[q]} = k^{1/p}$.

Now take $\lambda = (\lambda_1, \dots, \lambda_k)$, where

$$\lambda_i = \sum_{j=1}^{k} \zeta^{ij}\delta_j = (\zeta^i, \zeta^{2i}, \dots, \zeta^{ki}, 0, 0, \dots) \in \ell^r.$$

As in Lemma 1.1, we set $z_i = \sum_{j=1}^{k} \zeta_j \zeta^{ij}$ $(i \in \mathbb{N}_k)$, so that

$$\Big\|\sum_{i=1}^{k} \zeta_i \lambda_i\Big\|_{\ell^r} = \Big(\sum_{i=1}^{k} |z_i|^r\Big)^{1/r}.$$

It follows from (3.4) that

$$\mu_{2,k}(\lambda) = \sup\Big\{\Big(\sum_{i=1}^{k} |z_i|^r\Big)^{1/r} : \sum_{i=1}^{k} |\zeta_i|^2 \le 1\Big\}.$$

In the case where $2 \le p \le q$, we have $\mu_{p,k}(\lambda) \le \mu_{2,k}(\lambda)$, and so, by Lemma 1.1(i), $\mu_{p,k}(\lambda) \le k^{1/r}$. Hence

$$\|f\|_k^{(p,q)} \ge \frac{1}{k^{1/r}}\Big(\sum_{i=1}^{k} |\langle f_i,\, \lambda_i\rangle|^q\Big)^{1/q} = \frac{1}{k^{1/r}}(k \cdot k^q)^{1/q} = k^{1/p+1/q}.$$

We conclude that $k^{1/p+1/q} \le Ck^{1/p}$ for each $k \in \mathbb{N}$, a contradiction.

In the case where $1 < p < 2$, so that $r > 2$, it follows from (1.2) that

$$\Big(\sum_{i=1}^{k} |\zeta_i|^2\Big)^{1/2} \le k^{1/2-1/r}$$

whenever $\sum_{i=1}^{k} |\zeta_i|^r \le 1$, and so, using Lemma 1.1(i) again,

$$\Big\|\sum_{i=1}^{k} \zeta_i \lambda_i\Big\|_{\ell^r} \le \Big\|\sum_{i=1}^{k} \zeta_i \lambda_i\Big\|_{\ell^2} = \Big(\sum_{i=1}^{k} |z_i|^2\Big)^{1/2} \le k^{1/2} \cdot k^{1/2-1/r} = k^{1/p}.$$

Thus $\mu_{p,k}(\lambda) \le k^{1/p}$, and so

$$\|f\|_k^{(p,p)} \ge \frac{1}{k^{1/p}}\Big(\sum_{i=1}^{k} |\langle f_i,\, \lambda_i\rangle|^q\Big)^{1/q} = \frac{1}{k^{1/p}}(k \cdot k^q)^{1/q} = k^{1+1/q-1/p}.$$

We conclude that $k^{1+1/q-1/p} \le Ck^{1/p}$ for each $k \in \mathbb{N}$. Thus $1 + 1/q \le 2/p$, and so $q \ge p/(2-p)$, again a contradiction of an hypothesis.

Thus the two multi-norms are not equivalent in the cases stated. ∎

We do not know if the two multi-norms are equivalent in the case where $1 < p < 2$ and $q \ge p/(2-p)$. This point, and more general ones, will be discussed in [20].

Corollary 4.28. *Let $p \ge 1$. Then the two multi-norms*

$$(\|\cdot\|_n^{[p]} : n \in \mathbb{N}) \quad \text{and} \quad (\|\cdot\|_n^{\max} : n \in \mathbb{N})$$

based on ℓ^p are equivalent if and only if $p = 1$. ∎

We noted in §3.1 that the rates of growth of two equivalent multi-norms are similar. The next result, taken together with Corollary 4.28, shows that the converse statement is not true.

PROPOSITION 4.29. *Take $p \geq 1$ and $n \in \mathbb{N}$. Then:*

(i) $\varphi_n^{[p]}(\ell^p) = n^{1/p}$;
(ii) $\varphi_n^{\max}(\ell^p) = n^{1/p}$ *when* $p \in [1,2]$ *and* $\varphi_n^{\max}(\ell^p) \sim \sqrt{n}$ *when* $p \in [2,\infty)$.

Thus, for $p \in (1,2]$, we have $(\varphi_n^{[p]}(\ell^p)) \sim (\varphi_n^{\max}(\ell^p))$, but the multi-norms $(\|\cdot\|_n^{[p]})$ and $(\|\cdot\|_n^{\max})$ based on ℓ^p are not equivalent.

Proof. This follows from (4.14), Theorem 3.54, and Corollary 4.28. ∎

There remains the question whether the two multi-norms $(\|\cdot\|_n^{(p,p)})$ and $(\|\cdot\|_n^{\max})$ based on ℓ^p are equivalent. We know from Theorem 4.26 that they are equivalent in the case where $p = 1$, and, as we remarked in Theorem 4.20, they are equivalent in the case where $p = 2$. The question for other values of p will be resolved in [20].

4.2.6. The spaces $M(K)$. Throughout this section, K is a non-empty, locally compact space. For $q \geq 1$, we shall define the *standard q-multi-norm* based on $M(K)$ in essentially the same way as above.

Take $q \geq 1$. For each ordered partition $\mathbf{X} = (X_1, \dots, X_n)$ of K into (Borel) measurable subsets and each $\mu_1, \dots, \mu_n \in M(K)$, we set

$$r_{\mathbf{X}}((\mu_1, \dots, \mu_n)) = (\|\mu_1 | X_1\|^q + \cdots + \|\mu_n | X_n\|^q)^{1/q},$$

so that $r_{\mathbf{X}}$ is a seminorm on $M(K)^n$ and

$$r_{\mathbf{X}}((\mu_1, \dots, \mu_n)) \leq (\|\mu_1\|^q + \cdots + \|\mu_n\|^q)^{1/q} \quad (\mu_1, \dots, \mu_n \in M(K)).$$

Finally, we define

$$\|(\mu_1, \dots, \mu_n)\|_n^{[q]} = \sup_{\mathbf{X}} r_{\mathbf{X}}((\mu_1, \dots, \mu_n)) \quad (\mu_1, \dots, \mu_n \in M(K)),$$

where the supremum is taken over all such ordered partitions $\mathbf{X}$. Then $\|\cdot\|_n^{[q]}$ is a norm on $M(K)^n$, and it is again easily checked that $(\|\cdot\|_n^{[q]} : n \in \mathbb{N})$ is a multi-norm on $\{M(K)^n : n \in \mathbb{N}\}$.

DEFINITION 4.30. Let K be a non-empty, locally compact space. For each $q \geq 1$, the *standard q-multi-norm* based on $M(K)$ is the multi-norm $(\|\cdot\|_n^{[q]} : n \in \mathbb{N})$, with rate of growth $(\varphi_n^{[q]}(M(K)) : n \in \mathbb{N})$.

We shall see in Theorem 4.37 that the standard q-multi-norm on a space of the form $M(K)$ is a property of the Banach space $M(K)$.

THEOREM 4.31. *Let K be a non-empty, locally compact space. Then the standard 1-multi-norm $(\|\cdot\|_n^{[1]} : n \in \mathbb{N})$ based on $M(K)$ is given by*

$$\|(\mu_1, \dots, \mu_n)\|_n^{[1]} = \|\, |\mu_1| \vee \cdots \vee |\mu_n| \,\| \quad (\mu_1, \dots, \mu_n \in M(K)). \tag{4.16}$$

Proof. Take $\mu_1, \dots, \mu_n \in M(K)$, and set $\mu = |\mu_1| \vee \cdots \vee |\mu_n| \in M(K)$.

For each ordered partition $\mathbf{X} = (X_1, \ldots, X_n)$ of K, we have

$$\|\mu_1 | X_1\| + \cdots + \|\mu_n | X_n\| = \sum_{i=1}^{n} |\mu_i|(X_i) \leq \sum_{i=1}^{n} \mu(X_i) = \|\mu\|.$$

Thus $\|(\mu_1, \ldots, \mu_n)\|_n^{[1]} \leq \|\, |\mu_1| \vee \cdots \vee |\mu_n|\, \|$.

For the opposite inequality, we shall show that, for each $n \geq 2$ and $\mu_1, \ldots, \mu_n \in M(K)$, there is an ordered partition $\mathbf{X} = (X_1, \ldots, X_n)$ of K such that

$$\|\mu\| = \|\mu_1 | X_1\| + \cdots + \|\mu_n | X_n\|.$$

Consider first the case where $n = 2$ and $\mu_1, \mu_2 \in M(K)$. Let $P = X_1$ and $N = X_2$ be the measurable subsets of K associated with $|\mu_1| - |\mu_2|$ in the Hahn decomposition. (See page 22.) Then

$$\begin{aligned}\|\, |\mu_1| \vee |\mu_2|\, \| &= (|\mu_1| \vee |\mu_2|)(X_1) + (|\mu_1| \vee |\mu_2|)(X_2) \\ &= |\mu_1|(X_1) + |\mu_2|(X_2) = \|\mu_1 | X_1\| + \|\mu_2 | X_2\|,\end{aligned}$$

and so (X_1, X_2) is the required partition.

The result for a general $n \in \mathbb{N}$ follows by an easy induction. ∎

We shall see in Theorem 4.54(i) that the standard 1-multi-norm based on $M(K)$ is the maximum multi-norm on $M(K)$.

Recall that the topology of a Stonean space has a basis consisting of clopen subsets; the space $\beta\mathbb{N}$ is a Stonean space.

Proposition 4.32. *Let K be a Stonean space, and take $q \geq 1$. Then, for each $n \in \mathbb{N}$ and $\mu_1, \ldots, \mu_n \in M(K)$, we have*

$$\|(\mu_1, \ldots, \mu_n)\|_n^{[q]} = \sup(\|\mu_1 | K_1\|^q + \cdots + \|\mu_n | K_n\|^q)^{1/q},$$

taking the supremum over all ordered partitions $(K_1, \ldots, K_n)$ of K into clopen subspaces.

Proof. Clearly,

$$\|(\mu_1, \ldots, \mu_n)\|_n^{[q]} \geq (\|\mu_1 | K_1\|^q + \cdots + \|\mu_n | K_n\|^q)^{1/q}$$

for each such ordered partition $(K_1, \ldots, K_n)$.

Now fix $\varepsilon > 0$, and choose an ordered partition $\mathbf{X} = (X_1, \ldots, X_n)$ of K into measurable subsets such that

$$r_{\mathbf{X}}((\mu_1, \ldots, \mu_n)) > \|(\mu_1, \ldots, \mu_n)\|_n^{[q]} - \varepsilon.$$

Set $\mu = |\mu_1| + \cdots + |\mu_n|$. Since μ is regular, there exists a family $\{L_1, \ldots, L_n\}$ of clopen subsets of K such that $\mu(L_i \triangle X_i) < \varepsilon$ $(i \in \mathbb{N}_n)$. Set $K_1 = L_1$ and $K_i = L_i \setminus (L_1 \cup \cdots \cup L_{i-1})$ for $i = 2, \ldots, n$, so that $(K_1, \ldots, K_n)$ is an ordered partition of K into clopen subspaces. Then

$$\mu(K_i \triangle X_i) < \varepsilon + \sum_{j=1}^{i-1} \mu(L_i \cap L_j) < 2n\varepsilon \quad (i = 2, \ldots, n),$$

where in the last inequality we also use the fact that $L_i \cap L_j = \emptyset$ when $j < i$. Thus we see that

$$r_{\mathbf{X}}((\mu_1, \dots, \mu_n)) < \Big(\sum_{i=1}^{n} (\|\mu_i | K_i\| + 2n\varepsilon)^q\Big)^{1/q},$$

and hence that

$$\|(\mu_1, \dots, \mu_n)\|_n^{[q]} < \Big(\sum_{i=1}^{n} \|\mu_i | K_i\|^q\Big)^{1/q} + O(\varepsilon) \quad \text{as } \varepsilon \searrow 0.$$

The result follows. ■

4.2.7. The Schauder multi-norm. We now give an example related to the standard p-multi-norm on ℓ^p.

Let $(E, \|\cdot\|)$ be a Banach space. A series $\sum_{n=1}^{\infty} x_n$ in E is said to *converge unconditionally* if the series $\sum_{n=1}^{\infty} \varepsilon_n x_n$ converges in E whenever $\varepsilon_n \in \{1, -1\}$ $(n \in \mathbb{N})$. This is equivalent to the requirement that $\sum_{n=1}^{\infty} x_{\sigma(n)}$ converges in E for each $\sigma \in \mathfrak{S}_{\mathbb{N}}$.

Now suppose that E has a Schauder basis $\{e_n : n \in \mathbb{N}\}$, so that each $x \in E$ has a unique expansion in the form

$$x = \sum_{n=1}^{\infty} \alpha_n e_n,$$

where $\alpha_n \in \mathbb{C}$ $(n \in \mathbb{N})$. The basis $\{e_n : n \in \mathbb{N}\}$ is an *unconditional basis* if, for each $x \in E$, the corresponding series $\sum_{n=1}^{\infty} \alpha_n e_n$ converges unconditionally. The standard basis of ℓ^p (for $p \geq 1$) and of c_0 is unconditional in the appropriate Banach space. We note that the Banach spaces $L^p(\mathbb{I})$ have an unconditional basis whenever $p > 1$, but that the Banach spaces $L^1(\mathbb{I})$ and $C(\mathbb{I})$ do not have an unconditional basis.

For details of these and related results about unconditional bases, see [6, §3.1], [52, I, §1.c], or [74, §II.D], for example.

We now define

$$\Big|\!\Big|\!\Big| \sum_{n=1}^{\infty} \alpha_n e_n \Big|\!\Big|\!\Big| = \sup\Big\{\Big\| \sum_{n=1}^{\infty} \alpha_n \beta_n e_n \Big\| : |\beta_n| \leq 1 \ (n \in \mathbb{N})\Big\}.$$

As in [52, I, p. 19], $|\!|\!| \cdot |\!|\!|$ is a norm on E such that

$$\|x\| \leq |\!|\!| x |\!|\!| \leq C\|x\| \quad (x \in E)$$

for some constant $C \geq 1$. The original norm is 1-*unconditional* if the modified norm coincides with the original one. In the case where $E = \ell^p$ for $p \geq 1$, the usual norm is 1-unconditional.

Now suppose that $\|\cdot\|$ is a 1-unconditional norm on E. For each non-empty subset S of $\mathbb{N}$, define

$$P_S : \sum_{n=1}^{\infty} \alpha_n e_n \mapsto \sum_{n \in S} \alpha_n e_n, \qquad E \to E,$$

so that $\|P_S\| = 1$. Let $\mathbf{S} = (S_1, \dots, S_n)$ be an ordered partition of $\mathbb{N}$, say into infinite subsets of $\mathbb{N}$, and define

$$r_{\mathbf{S}}((x_1, \dots, x_n)) = \|P_{S_1}(x_1) + \cdots + P_{S_n}(x_n)\| \quad (x_1, \dots, x_n \in E),$$

and then set

$$\|(x_1,\dots,x_n)\|_n = \sup_{\mathbf{S}} r_{\mathbf{S}}((x_1,\dots,x_n)) \quad (x_1,\dots,x_n \in E),$$

where the supremum is taken over all such ordered partitions $\mathbf{S}$. It is again easily checked that $(\|\cdot\|_n : n \in \mathbb{N})$ is a multi-norm on $\{E^n : n \in \mathbb{N}\}$.

In the case where $E = \ell^p$, these norms are exactly the standard p-multi-norms on ℓ^p of §4.2.1.

DEFINITION 4.33. Let $(E, \|\cdot\|)$ be a Banach space with a 1-unconditional norm. Then the *Schauder multi-norm* on $\{E^n : n \in \mathbb{N}\}$ is the multi-norm defined above.

In particular, let $E = \mathbb{C}^k$ for some $k \in \mathbb{N}$, and let $\|\cdot\|$ be a norm on E such that

$$\|(\zeta_1 z_1,\dots,\zeta_k z_k)\| = \|(z_1,\dots,z_k)\| \quad (z_1,\dots,z_k \in \mathbb{C},\ \zeta_1,\dots,\zeta_k \in \mathbb{T}).$$

Then we can generate a Schauder multi-norm on $\{E^n : n \in \mathbb{N}\}$.

4.2.8. Abstract q-multi-norms. We now give a more abstract version of the standard q-multi-norm on the space $L^p(\Omega)$, where Ω is a measure space. This subsection is based on discussions with Hung Le Pham.

Let E be a σ-Dedekind complete Banach lattice. Recall from (1.27) that, for each $v \in E^+$, there is a certain positive linear projection P_v with $\|P_v\| \le 1$. Now take $q \ge 1$ and $n \in \mathbb{N}$. For each $v = (v_1,\dots,v_n) \in E^n$ with $|v_i| \wedge |v_j| = 0$ for $i,j \in \mathbb{N}_n$ with $i \neq j$, set

$$r_v((x_1,\dots,x_n)) = \Big(\sum_{i=1}^n \|P_{|v_i|}x_i\|^q\Big)^{1/q} \quad (x_1,\dots,x_n \in E).$$

Next define

$$\|(x_1,\dots,x_n)\|_n^{[q]} = \sup_v r_v((x_1,\dots,x_n)) \quad (x_1,\dots,x_n \in E),$$

where the supremum is taken over all $v = (v_i) \in E^n$ with $|v_i| \wedge |v_j| = 0$ for $i,j \in \mathbb{N}_n$ with $i \neq j$.

Let $n \in \mathbb{N}$, and take $q \ge 1$. Then it is obvious that $\|\cdot\|_n^{[q]}$ is a norm on E^n. Since $P_{|x|}(x) = x$ $(x \in E)$, we have $\|x\|_1^{[q]} = \|x\|$ $(x \in E)$. Moreover, we see that

$$\|(x_1,\dots,x_n)\|_n^{[q]} = \sup\Big(\sum_{i=1}^n \|y_i\|^q\Big)^{1/q} \quad (x_1,\dots,x_n \in E), \tag{4.17}$$

where the supremum is taken over $y_1,\dots,y_n \in E^+$ with $y_i \le |x_i|$ $(i \in \mathbb{N}_n)$ and $y_i \wedge y_j = 0$ for $i,j \in \mathbb{N}_n$ with $i \neq j$.

The following is clear.

THEOREM 4.34. *Let $(E, \|\cdot\|)$ be a σ-Dedekind complete Banach lattice, and take $q \ge 1$. Then $(\|\cdot\|_n^{[q]} : n \in \mathbb{N})$ is a special-norm; it is a multi-norm if and only if*

$$\|x+y\|^q \ge \|x\|^q + \|y\|^q \quad \text{for } x,y \in E \quad \text{with} \quad |x| \wedge |y| = 0.$$

In the case where E is an AL_p-space and $q \ge p$, $(\|\cdot\|_n^{[q]} : n \in \mathbb{N})$ is a multi-norm on $\{E^n : n \in \mathbb{N}\}$. ∎

Definition 4.35. Let $(E, \|\cdot\|)$ be an AL_p-space, and take $q \geq p$. Then $(\|\cdot\|_n^{[q]} : n \in \mathbb{N})$ is the *abstract q-multi-norm based on E.*

For example, suppose that $p \geq 1$ and $E = L^p(\Omega)$ for a measure space Ω and $q \geq p$, or that $E = M(K)$ for a non-empty, locally compact space K and $q \geq 1$. Then the abstract q-multi-norm $(\|\cdot\|_n^{[q]} : n \in \mathbb{N})$ is precisely the standard q-multi-norm of Definition 4.21 or 4.30. Thus the following theorem follows easily from (4.17).

Theorem 4.36. *Let E be the Banach lattice $L^p(\Omega)$ for a measure space Ω and $p \geq 1$, or the Banach lattice $M(K)$ for a non-empty, locally compact space K. Suppose that $q \geq p$ or $q \geq 1$, respectively. Then the standard q-multi-norm on E does not depend on the particular realization of E as an L^p-space or as a space of measures; it depends on only the norm and the lattice structures of E.* ∎

In fact, more can be said. Let E be an AL_p-space, and take $q \geq p$. In Definition 4.35, we defined the abstract q-multi-norm based on E. We shall now show that the abstract q-multi-norms based on two AL_p-spaces which are just isometrically isomorphic are equal whenever $p \neq 2$.

The first result is a special case of Theorem 4.39, given below, but we give a separate short proof.

Theorem 4.37. *Let Ω be a measure space, and take $q \geq 1$. Then the standard q-multi-norm on $L^1(\Omega)$ is determined by the Banach-space structure of $L^1(\Omega)$.*

Proof. By Theorem 4.26, the standard q-multi-norm and the $(1, q)$-multi-norm based on $L^1(\Omega)$ are equal. However, the $(1, q)$-multi-norm is determined by the Banach-space structure of $L^1(\Omega)$. ∎

We shall now consider the 'second dual question' (cf. Corollary 4.14): we should like the second dual of the abstract q-multi-norm on $L^p(\Omega)$ or $M(K)$ to be the abstract q-multi-norm on the second dual of the respective space. In the case of $L^p(\Omega)$ for $p > 1$, this is immediate because $L^p(\Omega)$ is then a reflexive Banach space, and so it suffices to consider the spaces $L^1(\Omega)$ and $M(K)$, which are AL-spaces.

Theorem 4.38. *Let E be an AL-space. For each $q \geq 1$, the second dual of the abstract q-multi-norm based on E is the abstract q-multi-norm based on E''.*

Proof. The standard q-multi-norm on the second dual of an AL-space is the same whether the second dual be considered as a measure space or as an L^1-space, and is equal to the abstract q-multi-norm by Theorem 4.36; by Theorem 4.26, it is the $(1, q)$-multi-norm, say on a space $L^1(\Omega)$. Also by Theorem 4.26, the standard q-multi-norm on E is the $(1, q)$-multi-norm. Thus the result follows from Corollary 4.14. ∎

We now extend Theorem 4.37 to multi-norms based on $L^p(\Omega)$ when $p \neq 2$. In the following result $(\|\cdot\|_n^{[q]} : n \in \mathbb{N})$ denotes the abstract q-multi-norm based on both E and F.

Theorem 4.39. *Take p, q with $1 \leq p \leq q < \infty$ and $p \neq 2$. Suppose that both E and F are AL_p-spaces and that $U : E \to F$ is an isometric isomorphism. Then*

$$U^{(n)} : (E^n, \|\cdot\|_n^{[q]}) \to (F^n, \|\cdot\|_n^{[q]})$$

is an isometry for each $n \in \mathbb{N}$.

Proof. By Theorem 1.38(i), we may suppose that $E = L^p(\Omega_1)$ and $F = L^p(\Omega_2)$, where Ω_1 and Ω_2 are measure spaces.

Fix $n \in \mathbb{N}$. In our setting, we have

$$\|(f_1, \dots, f_n)\|_n^{[q]} = \sup\Big(\sum_{i=1}^n \|p_i \cdot f_i\|^q\Big)^{1/q} \qquad (f_1, \dots, f_n \in L^p(\Omega_1)),$$

where the supremum is taken over the collection, say $C_{n,E}$, of all tuples $(p_1, \dots, p_n)$ of disjoint projections in $L^\infty(\Omega_1)$ and $p_i \cdot f_i$ is the $L^\infty(\Omega_1)$-module product in E; a similar formula holds for elements in F^n.

By Lamperti's theorem, Theorem 1.11 (which applies because $p \neq 2$), we see that U has the form

$$U : f \mapsto h \cdot T_\sigma f, \qquad L^p(\Omega_1) \to L^p(\Omega_2),$$

where $h : \Omega_2 \to \mathbb{C}$ and $T_\sigma \in \mathcal{B}(L^p(\Omega_1), L^p(\Omega_2))$ is induced by a regular set isomorphism σ. Note that T_σ extends to a $*$-isomorphism from the algebra of all measurable functions on Ω_1 (modulo null functions), and so T_σ restricts to a $*$-isomorphism from $L^\infty(\Omega_1)$ to $L^\infty(\Omega_2)$. For each $p \in L^\infty(\Omega_1)$ and $f \in L^p(\Omega_1)$, we have

$$T_\sigma(p) \cdot Uf = T_\sigma(p)h \cdot T_\sigma f = hT_\sigma(pf),$$

and so $U(p \cdot f) = T_\sigma(p) \cdot Uf$. Hence, for each $n \in \mathbb{N}$ and $f_1, \dots, f_n \in L^p(\Omega_1)$, we have

$$\begin{aligned}
\|(Uf_1, \dots, Uf_n)\|_n^{[q]} &= \sup_{(q_i) \in C_{n,F}} \Big(\sum_{i=1}^n \|q_i \cdot Uf_i\|^q\Big)^{1/q} \\
&= \sup_{(p_i) \in C_{n,E}} \Big(\sum_{i=1}^n \|U(p_i \cdot f_i)\|^q\Big)^{1/q} \\
&= \sup_{(p_i) \in C_{n,E}} \Big(\sum_{i=1}^n \|p_i \cdot f_i\|^q\Big)^{1/q} = \|(f_1, \dots, f_n)\|_n^{[q]},
\end{aligned}$$

and so $U^{(n)}$ is an isometry, as required. ■

THEOREM 4.40. *Let E be an AL_p-space, where $p \geq 1$ and $p \neq 2$. Then, for each $q \geq p$, the abstract q-multi-norms based on E depends on only the Banach space E, and not on its lattice structure.* ■

4.3. Lattice multi-norms. We now define a 'lattice multi-norm' based on a Banach lattice. Basic facts about Banach lattices were recalled in §1.3.

4.3.1. Multi-norms and Banach lattices. We define a multi-norm and a dual multi-norm naturally connected with a Banach lattice.

DEFINITION 4.41. Let $(E, \|\cdot\|)$ be a Banach lattice. For $n \in \mathbb{N}$ and $x_1, \dots, x_n \in E$, set

$$\|(x_1, \dots, x_n)\|_n^L = \|\,|x_1| \vee \cdots \vee |x_1|\,\|, \qquad \|(x_1, \dots, x_n)\|_n^{DL} = \|\,|x_1| + \cdots + |x_1|\,\|.$$

THEOREM 4.42. *Let $(E, \|\cdot\|)$ be a Banach lattice. Then the sequence $(\|\cdot\|_n^L : n \in \mathbb{N})$ is a multi-norm based on E, and $(\|\cdot\|_n^{DL} : n \in \mathbb{N})$ is a dual multi-norm based on E*

Proof. This is immediately checked. ■

Definition 4.43. Let $(E, \|\cdot\|)$ be a Banach lattice. Then $(\|\cdot\|_n^L : n \in \mathbb{N})$ is the *lattice multi-norm* based on $\{E^n : n \in \mathbb{N}\}$ and $(\|\cdot\|_n^{DL} : n \in \mathbb{N})$ is the *dual lattice multi-norm* based on $\{E^n : n \in \mathbb{N}\}$. The rate of growth of the lattice multi-norm is denoted by $(\varphi_n^L(E) : n \in \mathbb{N})$.

Theorem 4.44. *Let $(E, \|\cdot\|)$ be a Banach lattice. Then the dual of the lattice multi-norm on $\{E^n : n \in \mathbb{N}\}$ is the dual lattice multi-norm on $\{(E')^n : n \in \mathbb{N}\}$.*

Proof. Let $(\|\cdot\|_n^L : n \in \mathbb{N})$ be the lattice multi-norm on the family $\{E^n : n \in \mathbb{N}\}$.

For $n \in \mathbb{N}$, write $\|\cdot\|_n'$ for the dual norm to $\|\cdot\|_n^L$ on $(E')^n$. We must prove that

$$\|(\lambda_1, \dots, \lambda_n)\|_n' = \|\,|\lambda_1| + \cdots + |\lambda_n|\,\| \quad (\lambda_1, \dots, \lambda_n \in E'). \tag{4.18}$$

Indeed, take $\lambda_1, \dots, \lambda_n \in E'$, and write $\lambda = |\lambda_1| + \cdots + |\lambda_n| \in E'$.

Suppose that $x_1, \dots, x_n \in E$ with $\|(x_1, \dots, x_n)\|_n^L \le 1$, and set

$$x = |x_1| \vee \cdots \vee |x_n|,$$

so that $\|x\| \le 1$. Using (1.32), we see that

$$|\langle (x_1, \dots, x_n),\, (\lambda_1, \dots, \lambda_n)\rangle| \le \sum_{j=1}^n |\langle x_j,\, \lambda_j\rangle| \le \sum_{j=1}^n \langle |x_j|,\, |\lambda_j|\rangle \le \langle x,\, \lambda\rangle,$$

and hence that $\|(\lambda_1, \dots, \lambda_n)\|_n' \le \|\lambda\|$.

Given $\varepsilon > 0$, there exists $x \in E^+$ with $\|x\| = 1$ and $\langle x,\, \lambda\rangle > \|\lambda\| - \varepsilon$. It follows from Proposition 1.35 that, for each $j \in \mathbb{N}_n$, there exists $y_j \in E$ with $|y_j| \le x$ and $\langle y_j,\, \lambda\rangle > \langle x,\, |\lambda|\rangle - \varepsilon$. We have $|y_1| \vee \cdots \vee |y_n| \le x$, and so

$$\|(y_1, \dots, y_n)\|_n^L = \|\,|y_1| \vee \cdots \vee |y_n|\,\| \le \|x\| \le 1.$$

Also,

$$\begin{aligned} |\langle (y_1, \dots, y_n),\, (\lambda_1, \dots, \lambda_n)\rangle| &= \left|\sum_{j=1}^n \langle y_j,\, \lambda_j\rangle\right| \ge \sum_{j=1}^n \langle x,\, |\lambda_j|\rangle - n\varepsilon \\ &= \langle x,\, |\lambda|\rangle - n\varepsilon > \|\lambda\| - (n+1)\varepsilon, \end{aligned}$$

and so $\|(\lambda_1, \dots, \lambda_n)\|_n' \ge \|\lambda\| - (n+1)\varepsilon$. This holds true for each $\varepsilon > 0$, and so $\|(\lambda_1, \dots, \lambda_n)\|_n' \ge \|\lambda\|$.

Thus (4.18) holds. ∎

Theorem 4.45. *Let $(E, \|\cdot\|)$ be a Banach lattice. Then the dual of the dual lattice multi-norm on $\{E^n : n \in \mathbb{N}\}$ is the lattice multi-norm on $\{(E')^n : n \in \mathbb{N}\}$.*

Proof. This is similar to the above proof. ∎

Corollary 4.46. *Let $(E, \|\cdot\|)$ be a Banach lattice. Then the second dual of the lattice multi-norm on $\{E^n : n \in \mathbb{N}\}$ is the lattice multi-norm on $\{(E'')^n : n \in \mathbb{N}\}$.* ∎

Example 4.47. Let Ω be a measure space, take $p \ge 1$, and let E be the Banach lattice $L^p(\Omega)$. Then the corresponding lattice multi-norm $\{(E^n, \|\cdot\|_n) : n \in \mathbb{N}\}$ is given by

$$\|(f_1, \dots, f_n)\|_n^L = \left(\int_\Omega (|f_1| \vee \cdots \vee |f_n|)^p\right)^{1/p} = \|(f_1, \dots, f_n)\|_n^{[p]},$$

where we are using (4.12). Thus the lattice multi-norm and the standard p-multi-norm based on E coincide.

It follows that the dual of the standard p-multi-norm based on $L^p(\Omega)$ is given by

$$|||(\lambda_1,\dots,\lambda_n)|||_n^{[r]} = \|\,|\lambda_1|+\cdots+|\lambda_n|\,\|_{L^r(\Omega)}$$

for $\lambda_1,\dots,\lambda_n \in L^r(\Omega)$ and $n\in\mathbb{N}$, where $r=p'$. ■

EXAMPLE 4.48. Let K be a non-empty, locally compact space, so that the Banach space $(M(K),\|\cdot\|)$ is a Banach lattice. Then the corresponding lattice multi-norm based on $M(K)$ is just the standard 1-multi-norm; for this, see Theorem 4.31. ■

DEFINITION 4.49. Let $(E,\|\cdot\|)$ be a Banach lattice. Then a multi-norm $(\|\cdot\|_n : n\in\mathbb{N})$ on $\{E^n : n\in\mathbb{N}\}$ is *compatible with the lattice structure* if, for each $n\in\mathbb{N}$, we have

$$\|(x_1,\dots,x_n)\|_n \le \|(y_1,\dots,y_n)\|_n$$

whenever $|x_i|\le|y_i|$ in $E_{\mathbb{R}}$ for each $i\in\mathbb{N}_n$.

PROPOSITION 4.50. *Let $(E,\|\cdot\|)$ be a Banach lattice. Then the lattice multi-norm is the maximum multi-norm which is compatible with the lattice structure.*

Proof. Certainly the lattice multi-norm $(\|\cdot\|_n^L : n\in\mathbb{N})$ is compatible with the lattice structure. Let $(\|\cdot\|_n : n\in\mathbb{N})$ be any multi-norm which is compatible with the lattice structure. Take $n\in\mathbb{N}$ and $x_1,\dots,x_n\in E$, and set $x=|x_1|\vee\cdots\vee|x_n|$. Then

$$\|(x_1,\dots,x_n)\|_n \le \|(x,\dots,x)\|_n = \|x\| = \|(x_1,\dots,x_n)\|_n^L,$$

and so the lattice multi-norm is the maximal norm with this property. ■

PROPOSITION 4.51. *Let $(E,\|\cdot\|)$ be a Banach lattice, let $n\in\mathbb{N}$, and suppose that*

$$E = E_1 \oplus_\perp \cdots \oplus_\perp E_n.$$

Then

$$\|(x_1,\dots,x_n)\|_n^L = \|\,|x_1|+\cdots+|x_n|\,\| = \|x_1+\cdots+x_n\|$$

whenever $x_j\in E_j$ for $j\in\mathbb{N}_n$.

Proof. This follows immediately from (1.26). ■

Thus the lattice multi-norm and the dual lattice multi-norm coincide on elements $(x_1,\dots,x_n)\in E^n$ such that $x_j\in E_j$ for $j\in\mathbb{N}_n$.

The following result is easily checked.

PROPOSITION 4.52. *Let E be a Banach lattice, and let F be a closed subspace which is an order-ideal in E. Then the multi-norm defined by the Banach lattice E/F coincides with the quotient multi-norm.* ■

There is one circumstance in which we can identify the lattice multi-norm as the maximum multi-norm.

PROPOSITION 4.53. *Let $(E,\|\cdot\|)$ be a Banach lattice, and take $n\in\mathbb{N}$. Then*

$$\mu_{1,n}(x_1,\dots,x_n) \le \|\,|x_1|+\cdots+|x_n|\,\| \qquad (x_1,\dots,x_n\in E).$$

Further, suppose that E is an AM-space. Then

$$\mu_{1,n}(x_1,\dots,x_n) = \|\,|x_1| + \cdots + |x_n|\,\| \quad (x_1,\dots,x_n \in E),$$

and the dual $(\mu'_{1,n} : n \in \mathbb{N})$ is equal to the maximum multi-norm based on E'.

Proof. The first part of the proposition follows immediately from (3.5) (and also from Theorems 3.19 and 4.42); see also [39, 18.4].

To show that $(\mu'_{1,n} : n \in \mathbb{N})$ is equal to the maximum multi-norm based on E', we must show that their respective dual norms are equal on the family $\{(E'')^n : n \in \mathbb{N}\}$. By Theorem 3.33, the dual of the maximum multi-norm on $\{(E')^n : n \in \mathbb{N}\}$ is the weak 1-summing norm on $\{(E'')^n : n \in \mathbb{N}\}$, and, by Proposition 3.17, the latter norm is $\mu''_{1,n}$. Thus the last clause follows. ∎

Theorem 4.54. *Let $(E, \|\cdot\|)$ be a Banach lattice.*

(i) *Suppose that E is an AL-space. Then the lattice multi-norm is the maximum multi-norm based on E.*
(ii) *Suppose that E is an AM-space. Then the lattice multi-norm is the minimum multi-norm based on E.*

Proof. (i) By Theorem 4.44, the dual of the lattice multi-norm based on E is the dual lattice multi-norm based on E'. The dual of the maximum multi-norm based on E is $(\mu_{1,n} : n \in \mathbb{N})$. By Theorem 1.37, E' is an *AM*-space, and so, by Proposition 4.53, the latter two multi-norms are equal on the family $\{(E')^n : n \in \mathbb{N}\}$. Thus the result follows.

(ii) Using (1.37), we see that

$$\begin{aligned}\|(x_1,\dots,x_n)\|_n^L &= \|\,|x_1| \vee \cdots \vee |x_n|\,\| = \max\{\|\,|x_1|\,\|,\dots,\|\,|x_n|\,\|\}\\ &= \max\{\|x_1\|,\dots,\|x_n\|\} = \|(x_1,\dots,x_n)\|_n^{\min}\end{aligned}$$

for each $n \in \mathbb{N}$ and $x_1,\dots,x_n \in E$. Thus the lattice multi-norm is the minimum multi-norm on $\{E^n : n \in \mathbb{N}\}$. ∎

The following corollary gives a different proof of Theorem 4.23.

Corollary 4.55. *Let Ω be a measure space. Then the standard 1-multi-norm based on $L^1(\Omega)$ is the maximum multi-norm.*

Proof. This follows from Example 4.47 and Theorem 4.54(i). ∎

4.3.2. A representation theorem. The following theorem gives a general *representation theorem for multi-normed spaces.* It shows a universal property of the lattice multi-norms of this section; the result follows from a theorem of Pisier stated as [53, Théorème 2.1] and translated into our notation via Theorem 2.42.

Theorem 4.56. *Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-Banach space. Then there is a Banach lattice X and an isometric embedding $J : E \to X$ such that*

$$\|(Jx_1,\dots,Jx_n)\|_n^L = \|(x_1,\dots,x_n)\|_n \quad (x_1,\dots,x_n \in E).$$

for each $n \in \mathbb{N}$. ∎

Thus our multi-normed spaces are the 'sous-espace de trellis' of [53, Définition 3.1].

As noted in [53, p. 18], a lattice multi-norm corresponding to the minimum multi-norm is easily described. Indeed, let E be a Banach space, and set $K = (E'_{[1]}, \sigma(E', E))$, a compact space, so that $C(K)$ is a Banach lattice. Then the map

$$J : x \mapsto \iota(x) \mid K, \quad E \to C(K),$$

is an isometry, and

$$\|(Jx_1, \dots, Jx_n)\|_n^L = \|(x_1, \dots, x_n)\|_n^{\min} \quad (x_1, \dots, x_n \in E).$$

A description of a lattice multi-norm corresponding to the maximum multi-norm is also given in [53, Proposition 3.1]. Indeed, let E be a Banach space, and then set $\Gamma = \mathcal{B}(E, \ell^1)_{[1]}$, so that $\ell^\infty(\Gamma, \ell^1)$ is a Banach lattice. Then the map

$$J : x \mapsto (Tx : T \in \Gamma), \quad E \to \ell^\infty(\Gamma, \ell^1),$$

is an isometry, and

$$\|(Jx_1, \dots, Jx_n)\|_n^L = \|(x_1, \dots, x_n)\|_n^{\max} \quad (x_1, \dots, x_n \in E).$$

4.4. Summary. We collect here summary descriptions of the main multi-norms that we have defined, their dual multi-norms, and their rates of growth.

1. The *minimum multi-norm* $((E^n, \|\cdot\|_n^{\min}) : n \in \mathbb{N})$ based on a normed space E is defined by
$$\|(x_1, \dots, x_n)\|_n^{\min} = \max_{i \in \mathbb{N}_n} \|x_i\| \quad (x_1, \dots, x_n \in E).$$
The dual multi-norm is the maximum dual multi-norm based on E'. The rate of growth of the minimum multi-norm is given by $\varphi_n^{\min}(E) = 1$ $(n \in \mathbb{N})$.
2. The *maximum multi-norm* based on a normed space E is denoted by
$$((E^n, \|\cdot\|_n^{\max}) : n \in \mathbb{N}).$$
The dual multi-norm is $(\mu_{1,n} : n \in \mathbb{N})$, where $\mu_{1,n}$ is the weak 1-summing norm on $(E')^n$. The rate of growth of the maximum multi-norm (for E infinite dimensional) satisfies
$$\sqrt{n} \le \varphi_n^{\max}(E) = \pi_1^{(n)}(E') \le n \quad (n \in \mathbb{N}),$$
and both bounds can be attained. For example, we have the following.
Let L^p be an infinite-dimensional measure space. Then:
$$\begin{aligned} &\varphi_n^{\max}(L^p) = n^{1/p} \ (n \in \mathbb{N}) && \text{for } p \in [1, 2]; \\ &\varphi_n^{\max}(L^p) \sim \sqrt{n} \quad \text{as } n \to \infty && \text{for } p \in [2, \infty]. \end{aligned}$$
Let K be an infinite compact space. Then:
$$\sqrt{n} \le \varphi_n^{\max}(C(K)) \le \sqrt{2n} \quad (n \in \mathbb{N}).$$
3. Let E be a normed space. For $1 \le p \le q < \infty$, the *(p,q)-multi-norm* based on E is denoted by $((E^n, \|\cdot\|_n^{(p,q)}) : n \in \mathbb{N})$. The dual multi-norm based on E' is
$$((E')^n, \|\!|\cdot|\!\|_n^{(p,q')}) : n \in \mathbb{N}).$$
The rate of growth of the (p,q)-multi-norm satisfies
$$\varphi_n^{(p,q)}(E) = \pi_{q,p}^{(n)}(E') \le n^{1/q} \quad (n \in \mathbb{N}),$$
and the upper bound can be attained.

4. Fix $p \in [1, \infty)$, and take $q \geq p$. For a measure space L^p, the *standard q-multi-norm* based on L^p is denoted by $(\|\cdot\|_n^{[q]} : n \in \mathbb{N})$. We have
$$\|(f_1, \ldots, f_n)\|_n^{[q]} \leq \|(f_1, \ldots, f_n)\|_n^{(p,q)} \leq \|(f_1, \ldots, f_n)\|_n^{\max}$$
for all $f_1 \ldots, f_n \in L^p$ and $n \in \mathbb{N}$. The rate of growth of the standard q-multi-norm satisfies $\varphi_n^{[q]}(L^p) = n^{1/q}$ $(n \in \mathbb{N})$.
5. The *Hilbert multi-norm* based on a Hilbert space H is denoted by $(\|\cdot\|_n^H : n \in \mathbb{N})$. This multi-norm is equal to the $(2, 2)$-multi-norm, and is equivalent to the (p, p)-multi-norm for $p \in [1, 2]$ and to the maximum multi-norm. The rate of growth of this multi-norm (for infinite-dimensional H) is given by $\varphi_n^{\min}(H) = \sqrt{n}$ $(n \in \mathbb{N})$.
6. The *lattice multi-norm* based on a Banach lattice E is denoted by $(\|\cdot\|_n^L : n \in \mathbb{N})$; it is defined by
$$\|(x_1, \ldots, x_n)\|_n^L = \|\,|x_1| \vee \cdots \vee |x_1|\,\| \quad (x_1, \ldots, x_n \in E,\, n \in \mathbb{N}).$$
The dual multi-norm based on E' is the dual lattice multi-norm $(\|\cdot\|_n^{DL} : n \in \mathbb{N})$; it is defined by
$$\|(x_1, \ldots, x_n)\|_n^{DL} = \|\,|x_1| + \cdots + |x_1|\,\| \quad (x_1, \ldots, x_n \in E,\, n \in \mathbb{N}).$$
For an AL-space, the lattice multi-norm is the maximum multi-norm based on E, and, for an AM-space, it is the minimum multi-norm based on E. The rate of growth of this multi-norm is denoted by $(\varphi_n^L(E) : n \in \mathbb{N})$.

5. Multi-topological linear spaces and multi-norms

5.1. Basic sets

5.1.1. Topological linear spaces. Let E be a linear space. A *local base* of E is a family $\mathfrak{B}$ of non-empty, balanced, absorbing subsets of E such that:

(i) for each $B \in \mathfrak{B}$, there exists $C \in \mathfrak{B}$ with $C + C \subset B$;
(ii) for each $B_1, B_2 \in \mathfrak{B}$, there exists $C \in \mathfrak{B}$ with $C \subset B_1 \cap B_2$;
(iii) for each $B \in \mathfrak{B}$ and $x \in B$, there exists $C \in \mathfrak{B}$ with $x + C \subset B$.

A subset B of a topological linear space is *bounded* if, for each neighbourhood U of 0 in E, there exists $\alpha > 0$ with $B \subset \beta U$ $(\beta > \alpha)$.

Let E be a topological linear space. Then E has a local base $\mathfrak{B}$ consisting of all the balanced neighbourhoods of 0; in this case, each neighbourhood of 0 contains a member of $\mathfrak{B}$ (and then the open sets of F are precisely the unions of translates of members of $\mathfrak{B}$). Conversely, let $\mathfrak{B}$ be a local base of E. Then there is a unique topology τ on E such that (E, τ) is topological linear space and $\mathfrak{B}$ is a local base for τ at 0. The topological linear space is Hausdorff if and only if $\bigcap\{B : B \in \mathfrak{B}\} = \{0\}$.

For details of these remarks, see [65], for example.

5.1.2. Multi-topological linear spaces. Let E be a linear space, and consider the space $E^{\mathbb{N}}$, also a linear space; a generic element of $E^{\mathbb{N}}$ is $x = (x_i) = (x_i : i \in \mathbb{N})$. Define $\iota : x \mapsto (x)$, $E \to E^{\mathbb{N}}$, so that $\iota(E)$ is a linear subspace of $E^{\mathbb{N}}$.

For a non-empty subset S of $\mathbb{N}$, we define P_S, Q_S on $E^{\mathbb{N}}$ essentially as in §1.1.4. We also define A_σ and M_α in $\mathcal{L}(E^{\mathbb{N}})$ for $\sigma \in \mathfrak{S}_{\mathbb{N}}$ and $\alpha = (\alpha_i) \in \overline{\mathbb{D}}^{\mathbb{N}}$ by

$$A_\sigma((x_i)) = (x_{\sigma(i)}), \quad M_\alpha((x_i)) = (\alpha_i x_i) \quad ((x_i) \in E^{\mathbb{N}}).$$

Finally we define the *amalgamation* $x \amalg y$ of two elements $x = (x_i)$ and $y = (y_i)$ of $E^{\mathbb{N}}$ as the element

$$x \amalg y = (x_1, y_1, x_2, y_2, x_3, y_3, \dots)$$

of $E^{\mathbb{N}}$. Let $k \in \mathbb{N}$. The amalgamation of k copies of $x \in E^{\mathbb{N}}$ is denoted by $x \amalg_k x$, so that

$$x \amalg_k x = (\overbrace{x_1, \dots, x_1}^{k}, \overbrace{x_2, \dots, x_2}^{k}, \dots).$$

Definition 5.1. Let E be a linear space, and let F be a linear subspace of $E^{\mathbb{N}}$ with $\iota(E) \subset F$. A subset B of F is *basic* if:

(T1) $A_\sigma(B) = B$ for each $\sigma \in \mathfrak{S}_{\mathbb{N}}$;
(T2) $M_\alpha(B) \subset B$ for each $\alpha \in \overline{\mathbb{D}}^{\mathbb{N}}$;

(T3) for each $x \in F$, $x \in B$ if and only if $x \amalg x \in B$;
(T4) for each $x \in F$, $x \in B$ if and only if $P_{\mathbb{N}_n}(x) \in B$ $(n \in \mathbb{N})$.

Let τ be a topology on F. Then E is a *multi-topological linear space* (with respect to (F, τ)) if (F, τ) is a Hausdorff topological linear space with a local base $\mathfrak{B}$ consisting of basic sets, each a neighbourhood of 0.

It may be that $F = E^{\mathbb{N}}$ in the above definition, but we allow greater generality for the sake of future applications.

Let E be a multi-topological linear space with respect to (F, τ). For each $x \in E$, we have $\iota_1(x) := (x, 0, 0, 0, \dots) \in F$, and so τ induces a topology called τ_E on E such that a subset U of E belongs to τ_E if and only if $\iota_1(U)$ is relatively τ-open in F. It is clear that (E, τ_E) is a topological linear space.

Let E be such a multi-topological linear space, let B be a basic set in (F, τ) that is a neighbourhood of 0, and take $x \in F$. Since the set B is absorbing, there exists $\beta > 0$ such that $x \in \beta B$. It follows easily from the definitions that $A_\sigma(x) \in F$ for each $\sigma \in \mathfrak{S}_{\mathbb{N}}$, that $M_\alpha(x) \in F$ for each $\alpha \in \overline{\mathbb{D}}^{\mathbb{N}}$, that $x \amalg x \in F$, and that $P_{\mathbb{N}_n}(x) \in F$ for each $n \in \mathbb{N}$.

PROPOSITION 5.2. *Let E be a multi-topological linear space with respect to (F, τ), and let B be a basic set in F.*

(i) *Take $x \in F$ and $k \in \mathbb{N}$. Then $x \in B$ if and only if $x \amalg_k x \in B$.*
(ii) *Take $x \in F$. Then $x \in B$ if and only if $x \amalg 0 \in B$.*
(iii) *Take $(x_i) \in F$. Then $(x_i) \in B$ if and only if $(0, x_1, x_2, x_3, \dots) \in B$.*
(iv) *Take $x, y \in B$. Then $x \amalg y \in B + B$.*
(v) *Take $x = (x_i) \in B$, and let (k_n) be strictly increasing in $\mathbb{N}$. Then $(x_{k_n}) \in B$.*
(vi) *Take $x = (x_i) \in B$, and suppose that y is a sequence that contains finitely many occurrences of each x_i in any order. Then $y \in B$.*

Proof. (i) Take $n \in \mathbb{N}$ such that $2^j \geq k$. By (T3), $x \in B$ if and only if $x \amalg_{2^j} x \in B$. Take $m \geq k$. By (T4), $x \amalg_m x \in B$ if and only if $P_{\mathbb{N}_n}(x \amalg_m x) \in B$ $(n \in \mathbb{N})$. By (T2) and (T1), this holds if and only if $P_{\mathbb{N}_n}(x \amalg_k x) \in B$ $(n \in \mathbb{N})$. By (T4) again, this holds if and only if $x \amalg_k x \in B$. The result follows.

(ii) Suppose that $x \in B$. Then $x \amalg x \in B$ by (T3), and then $x \amalg 0 \in B$ by (T2). Suppose that $x \amalg 0 \in B$. Then it follows from (T4) that $P_{\mathbb{N}_{2n}}(x \amalg 0) \in B$ $(n \in \mathbb{N})$. By (T1), $P_{\mathbb{N}_n}(x) \in B$ $(n \in \mathbb{N})$, and so $x \in B$ by (T4).

(iii) This is immediate from (T1) and (T4).

(iv) By (ii), $x \amalg 0, y \amalg 0 \in B$. By (T1), $0 \amalg y \in B$. Thus

$$x \amalg y = x \amalg 0 + 0 \amalg y \in B + B.$$

(v) By (T2), $(0, \dots, 0, x_{k_1}, 0, \dots, 0, x_{k_2}, 0, \dots) \in B$. By (T1), we have $(x_{k_n}) \amalg 0 \in B$. By (ii), $(x_{k_n}) \in B$.

(vi) Suppose that y contains k_i copies of x_i for $i \in \mathbb{N}$. Take $n \in \mathbb{N}$, and then set $m = \max\{k_1, \dots, k_n\}$. By (i), $x \amalg_m x \in B$. By (T4), $P_{\mathbb{N}_n}(x \amalg_m x) \in B$. By (T2) and (T1), $P_{\mathbb{N}_n}(y) \in B$. But this holds for each $n \in \mathbb{N}$, and so $y \in B$ by (T4). ∎

5.2. Multi-null sequences

5.2.1. Convergence. Let E be a multi-topological linear space. We can define a notion of convergence in E as follows.

DEFINITION 5.3. Let E be a multi-topological linear space with respect to (F, τ) such that (F, τ) has a local base $\mathfrak{B}$ of basic subsets of F, and let (x_i) be a sequence in E. Then

$$\operatorname*{Lim}_{i\to\infty} x_i = 0 \quad \text{in } E$$

if, for each $B \in \mathfrak{B}$, there exists $n_0 \in \mathbb{N}$ such that $(x_n, x_{n+1}, x_{n+2}, \dots) \in B$ $(n \geq n_0)$. Such sequences (x_i) are the *multi-null sequences* in E. Further, let $x \in E$. Then

$$\operatorname*{Lim}_{i\to\infty} x_i = x \quad \text{in } E$$

if $(x_i - x)$ is a multi-null sequence in E; the sequence (x_i) is *multi-convergent to x*.

The collections of multi-convergent and multi-null sequences in E are denoted by $c_m(E)$ and $c_{m,0}(E)$, respectively.

Let E be a multi-topological linear space with respect to (F, τ). Clearly, each multi-null sequence in E is a null sequence in (E, τ_E), where τ_E was described above. Further, let (x_i) be a sequence such that $\lim_{i\to\infty} x_i = 0$ in (E, τ_E). Then there is a subsequence (x_{k_i}) of (x_i) such that $\operatorname{Lim}_{i\to\infty} x_{k_i} = 0$. Let S be a subset of E. One might define the 'multi-closure' of S to be the set of elements x in E such that there exists a multi-null sequence (x_i) contained in S with $\operatorname{Lim}_{i\to\infty} x_i = x$; however, the above remark shows that this multi-closure coincides with the closure of S in (E, τ_E).

The four axioms specified above have an immediate and natural interpretation in terms of this convergence. Thus: (T1) states that each permutation of a multi-null sequence is a multi-null sequence; (T2) states that $M_\alpha(x)$ is a multi-null sequence whenever $\alpha = (\alpha_i)$ is a bounded sequence in $\mathbb{C}$ and x is a multi-null sequence; (T3) states that $x \amalg x$ is a multi-null sequence if and only if x is a multi-null sequence. Axiom (T4) is a 'Cauchy criterion' for multi-null sequences. A sequence $(x_i) \in E^{\mathbb{N}}$ is a *multi-Cauchy sequence* if, for each $B \in \mathfrak{B}$, there exists $n_0 \in \mathbb{N}$ such that

$$(x_m, x_{m+1}, \dots, x_n, 0, 0, \dots) \in B \quad (n \geq m \geq n_0).$$

By (T4), a sequence is a multi-null sequence if and only if it is a multi-Cauchy sequence.

We shall see shortly that the notion of a multi-null sequence can depend on the choice of the space F.

PROPOSITION 5.4. *Let E be a multi-topological linear space.*

(i) *Each subsequence of a multi-null sequence in E is itself a multi-null sequence.*

(ii) *Let $\alpha, \beta \in \mathbb{C}$, and let $(x_i), (y_i) \in E^{\mathbb{N}}$ be such that*

$$\operatorname*{Lim}_{i\to\infty} x_i = x \quad \text{and} \quad \operatorname*{Lim}_{i\to\infty} y_i = y$$

in E. Then $\operatorname{Lim}_{i\to\infty}(\alpha x_i + \beta y_i) = \alpha x + \beta y$ in E.

(iii) *The collections $c_m(E)$ and $c_{m,0}(E)$ are linear subspaces of $E^{\mathbb{N}}$.*

Proof. These are immediately checked. ∎

5.2.2. Multi-normed spaces. We now investigate the relation between multi-topological linear spaces and multi-normed spaces.

Definition 5.5. Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space, and suppose that $x = (x_i) \in E^{\mathbb{N}}$. Then

$$\operatorname{Sup} x = \operatorname{Sup}(x_i) = \sup\{\|(x_{k_1}, \dots, x_{k_n})\|_n : k_1, \dots, k_n \in \mathbb{N},\, n \in \mathbb{N}\}.$$

In fact, it follows from (A1), (A4), and Lemma 2.9 that $(\|(x_1, x_2, \dots, x_n)\|_n : n \in \mathbb{N})$ is an increasing sequence and that

$$\operatorname{Sup} x = \sup\{\|(x_1, x_2, \dots, x_n)\|_n : n \in \mathbb{N}\} = \lim_{n\to\infty} \|(x_1, x_2, \dots, x_n)\|_n. \tag{5.1}$$

Define

$$F = \{x \in E^{\mathbb{N}} : \operatorname{Sup} x < \infty\}. \tag{5.2}$$

For each $\varepsilon > 0$, set

$$B_\varepsilon = \{x \in F : \operatorname{Sup} x < \varepsilon\},$$

and set $\mathfrak{B} = \{B_\varepsilon : \varepsilon > 0\}$.

Theorem 5.6. *Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space, and let F and $\mathfrak{B}$ be as above. Then F is a linear subspace of $E^{\mathbb{N}}$ with $\iota(E) \subset F$, and E is a multi-topological linear space with respect to (F, τ), where (F, τ) has $\mathfrak{B}$ as a local base. Further, each set B_ε is convex and bounded.*

Proof. It is clear that F is a linear subspace of $E^{\mathbb{N}}$; by Lemma 2.14, $\iota(E) \subset F$.

We shall show that $\mathfrak{B}$ is a local base at 0 in F. Given $\varepsilon > 0$, we have $B_{\varepsilon/2} + B_{\varepsilon/2} \subset B_\varepsilon$. Given $\varepsilon_1, \varepsilon_2 > 0$, we have $B_\varepsilon \subset B_{\varepsilon_1} \cap B_{\varepsilon_2}$ for $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Given $\varepsilon > 0$ and $x \in B_\varepsilon$, we have $\operatorname{Sup} x < \varepsilon$, and then $x + B_\eta \subset B_\varepsilon$ for $\eta = \varepsilon - \operatorname{Sup} x$. Thus $\mathfrak{B}$ is a local base at 0 in F, and so $\mathfrak{B}$ defines a topology τ such that (F, τ) is a topological linear space. Since $\bigcap\{B_\varepsilon : \varepsilon > 0\} = \{0\}$, the topology τ is Hausdorff.

It is clear that Axioms (T1), (T2), and (T4) are satisfied. Suppose that $x \in B_\varepsilon$, where $\varepsilon > 0$, and take $k_1, \dots, k_n \in \mathbb{N}$. Then

$$\|((x \amalg x)_{k_1}, \dots, (x \amalg x)_{k_n})\|_n = \|(x_{j_1}, \dots, x_{j_m})\|_m$$

for some $m \in \mathbb{N}_n$ and $j_1, \dots, j_m \in \mathbb{N}$ by (A1) and (A4), and so we have $x \amalg x \in B_\varepsilon$; the converse is immediate, and so (T3) is satisfied. Thus each B_ε is a basic set in F.

Clearly each set B_ε is convex and bounded. ∎

Definition 5.7. The topology τ defined on F in the above theorem is that *specified by* the multi-normed space $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$.

In the future, we shall regard (F, τ) as the space specified by a multi-normed space $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ without explicit mention. We now interpret the concept of 'multi-null sequence' in the above situation.

Theorem 5.8. *Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space. Take $(x_i) \in E^{\mathbb{N}}$. Then (x_i) is a multi-null sequence in E if and only if, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$\sup_{k\in\mathbb{N}} \|(x_{n+1}, \dots, x_{n+k})\|_k < \varepsilon \qquad (n \ge n_0).$$

Proof. This is again immediate. ∎

Let (x_i) be a sequence in E with $\mathrm{Lim}_{i\to\infty} x_i = x$. It follows that

$$\lim_{n\to\infty} \sup_{k\in\mathbb{N}} \|(x_{n+1},\dots,x_{n+k})\|_k = \|x\|. \tag{5.3}$$

EXAMPLE 5.9. Let (α_i) be a fixed element of $\mathbb{C}^{\mathbb{N}}$, and set

$$x_i = \alpha_i \delta_i \quad (i \in \mathbb{N}).$$

(i) Let E be one of the Banach spaces ℓ^p (for $p \geq 1$) or c_0, and take $(\|\cdot\|_n^{\min} : n \in \mathbb{N})$ to be the minimum multi-norm on $\{E^n : n \in \mathbb{N}\}$. Then it follows immediately that (x_i) is a multi-null sequence in E if and only if $\lim_{i\to\infty} \alpha_i = 0$, i.e., if and only if $(\alpha_i) \in c_0$. This is independent of the choice of the space E.

(ii) Let $E = \ell^p$ (where $p \geq 1$), and let $(\|\cdot\|_n^{[p]} : n \in \mathbb{N})$ be the standard p-multi-norm based on $\{E^n : n \in \mathbb{N}\}$. Then it follows from (4.15) that (x_i) is a multi-null sequence in E if and only if

$$\lim_{n\to\infty} \left(\sum_{i=n}^{\infty} |\alpha_i|^p\right)^{1/p} = 0,$$

i.e., if and only if $(\alpha_i) \in \ell^p$.

(iii) We now see, by comparing examples (i) and (ii), that the multi-null sequences in a multi-normed space based on a Banach space E depend on the multi-norm that we are considering. ∎

PROPOSITION 5.10. *Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space. Then the following are equivalent:*

(a) *each null sequence in $(E, \|\cdot\|)$ is a multi-null sequence;*
(b) *the multi-norm $(\|\cdot\|_n : n \in \mathbb{N})$ is equivalent to the minimum multi-norm;*
(c) *there is a topology σ on E such that the multi-null sequences are precisely the convergent sequences in (E, σ).*

Proof. Here $(\varphi_n(E) : n \in \mathbb{N})$ is the rate of growth sequence for the multi-normed space $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$.

(a)⇒(b) Assume towards a contradiction that $\limsup_{n\to\infty} \varphi_n(E) = \infty$. Then, for each $n \in \mathbb{N}$, there exists $m_n \in \mathbb{N}$ such that $\varphi_{m_n}(E) > n$, and so there exist $x_{1,n}, \dots, x_{m_n,n} \in E_{[1/n]}$ with $\|(x_{1,n}, \dots, x_{m_n,n})\|_{m_n} \geq 1$. The sequence

$$(x_{1,1}, \dots, x_{m_1,1}, x_{1,2}, \dots, x_{m_2,2}, \dots, x_{1,n}, \dots, x_{m_n,n}, \dots)$$

is a null sequence in $(E, \|\cdot\|)$, but it is not a multi-null sequence. This is a contradiction of (a). Thus $(\varphi_n(E) : n \in \mathbb{N})$ is bounded, and so, by Proposition 3.4, $(\|\cdot\|_n : n \in \mathbb{N})$ is equivalent to the minimum multi-norm.

(b)⇒(a) Suppose that $\sup\{\varphi_n(E) : n \in \mathbb{N}\} \leq C$. Then

$$\|(x_{n+1}, \dots, x_{n+k})\|_k \leq C \max\{\|x_{n+1}\|, \dots, \|x_{n+k}\|\} \quad (n, k \in \mathbb{N}),$$

and so each null sequence in $(E, \|\cdot\|)$ is a multi-null sequence.

(a)⇒(c) This is trivial.

(c)⇒(a) Assume towards a contradiction that (a) fails. Then there is a null sequence (x_i) in $(E, \|\cdot\|)$ such that (x_i) is not a multi-null sequence. By (c), (x_i) is not convergent in (E, σ), and so there is a σ-neighbourhood U of 0 in E and a subsequence (x_{i_j}) of (x_i)

such that $x_{i_j} \notin U$ $(j \in \mathbb{N})$. There is a subsequence (y_n) of (x_{i_j}) with $\|y_n\| \leq 1/n^2$ $(n \in \mathbb{N})$, and then (y_n) is a multi-null sequence in E. However $y_n \notin U$ $(n \in \mathbb{N})$, and so (y_n) is not convergent in (E, σ), a contradiction of (c). ∎

Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space such that E is finite-dimensional. Then it follows from Proposition 3.7 that the equivalent conditions of the above proposition are satisfied.

Let K be a compact space. Then the multi-null sequences in $C(K)$ for the lattice multi-norm based on $C(K)$ are just the usual null sequences.

Recall that a topological linear space E is a *locally convex space* if and only if there is a local base consisting of convex sets. By [65, Theorem 1.14(b)], each neighbourhood of zero in such a space contains a balanced, convex neighbourhood of 0. The following result shows that the topology of a locally convex space is determined by a class of multi-null sequences.

PROPOSITION 5.11. *Let E be a locally convex space.*

(i) *Let V be a convex, balanced neighbourhood of* 0 *in E. Then $V^{\mathbb{N}}$ is a basic subset of $E^{\mathbb{N}}$.*
(ii) *Let $\mathfrak{B}$ be the family of sets in $E^{\mathbb{N}}$ of the form $V^{\mathbb{N}}$, where V is a convex, balanced neighbourhood of* 0 *in E. Then there is a topology τ on E such that $E^{\mathbb{N}}$ is a multi-topological linear space with respect to (E, τ), and (E, τ) has $\mathfrak{B}$ as a local base.*

Proof. (i) This is immediate.

(ii) It is clear that the specified family $\mathfrak{B}$ is a local base at 0 for E consisting of basic sets. There is a unique topology τ on E such that (E, τ) is topological linear space and $\mathfrak{B}$ is a local base for τ at 0. The topology τ is Hausdorff because $\bigcap\{B : B \in \mathfrak{B}\} = \{0\}$. Thus $E^{\mathbb{N}}$ is a multi-topological linear space with respect to (E, τ). ∎

We now seek a version for multi-topological linear spaces of *Kolmogorov's theorem* for topological linear spaces: this states that a topological linear space E is normable if and only if 0 has a convex, bounded neighbourhood [65, Theorem 1.39].

THEOREM 5.12. *Let E be a multi-topological linear space with respect to (F, τ). Then the topology τ is specified by a multi-normed space if and only if there is a basic set which is a convex, bounded neighbourhood of* 0 *in F.*

Proof. Suppose that τ is specified by a multi-normed space. Then each set B_ε given above is a basic subset of F which is a convex, bounded neighbourhood of 0.

Conversely, suppose that B is a basic subset of F which is a convex, bounded neighbourhood of 0 in F. By [65, Theorem 1.14(b)], we may suppose that B is balanced.

Let $n \in \mathbb{N}$, and take $x_1, \dots, x_n \in E$ so that $(x_1, \dots, x_n, 0, \dots) \in F$. We define

$$\|(x_1, \dots, x_n)\|_n = p_B((x_1, \dots, x_n, 0, \dots)) \quad (x_1, \dots, x_n \in E),$$

where p_B is the Minkowski functional of B. Clearly $\|\cdot\|_n$ is a seminorm on E^n. Suppose that $(x_1, \dots, x_n, 0, \dots) \neq 0$ in F. Since (F, τ) is a Hausdorff space, there is a neighbourhood V of 0 in F such that $(x_1, \dots, x_n, 0, \dots) \notin V$. Since B is bounded, there exists

$\alpha > 0$ such that $B \subset \beta V$ $(\beta > \alpha)$. Since $x \notin (1/\beta)B$ $(\beta > \alpha)$, we have

$$p_B((x_1, \dots, x_n, 0, \dots)) > 1/\alpha > 0.$$

Thus $\|\cdot\|_n$ is a norm on E^n.

Set $\|x\| = \|x\|_1$ $(x \in E)$. Then $(E, \|\cdot\|)$ is a normed space.

We shall now show that $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is a multi-normed space.

It is immediate that Axioms (A1), (A2), and (A3) are satisfied. Let $x_1, \dots, x_n \in E$. By Proposition 5.2(vi), $(x_1, \dots, x_n, 0, \dots) \in B$ if and only if $(x_1, \dots, x_n, x_n, 0, \dots) \in B$, and so Axiom (A4) is satisfied.

Consider the family $\mathfrak{B} = \{\alpha B : \alpha > 0\}$. By [65, Theorem 1.15(c)], $\mathfrak{B}$ is a local base for the topological linear space (F, τ). Let σ be the topology on F defined by the multi-norms $(\|\cdot\|_n : n \in \mathbb{N})$ as in Theorem 5.6, and take $x \in B$. Then, by (T4), $P_{\mathbb{N}_n}(x) \in B$ $(n \in \mathbb{N})$, and so, by (5.1), $\operatorname{Sup} x \leq 1$, whence $\tau \subset \sigma$. Let $x \in F$ with $\operatorname{Sup} x < 1$. Then $x \in B$, and so $\sigma \subset \tau$. Thus $\tau = \sigma$. It also follows that $F = \bigcup\{\alpha B : \alpha > 0\}$, and so, by (T4), F is exactly the space specified in (5.2) in terms of the multi-norms.

This completes the proof. ∎

Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space. We have seen in Proposition 5.10 that multi-null sequences in E are the null sequences for a topology on E only in special cases. We generalize this remark.

PROPOSITION 5.13. *Let E be a multi-topological linear space with respect to (F, τ), and suppose that τ has a countable base of neighbourhoods of* 0 *in F. Then* either *the multi-null sequences in E are exactly the null sequences in (E, τ_E),* or *there is no topology σ on E such that the multi-null sequences in E are exactly the null sequences in (E, σ).*

Proof. We first note the following. Let (U_n) be a countable base at 0 for the topology τ on F. Then there is a countable base (V_n) at 0 for the topology τ on F such that

$$V_n \supset V_{n+1} + V_{n+2} + \cdots + V_{n+k} \quad (n, k \in \mathbb{N}).$$

Indeed set $V_1 = U_1$, and inductively choose V_n to be a neighbourhood of 0 in (F, τ) such that $V_{n+1} \subset (U_{n+1} \cap V_n)$ and $V_{n+1} + V_{n+1} \subset V_n$ for each $n \in \mathbb{N}$.

We next note that each null sequence (x_i) in (E, τ_E) has a subsequence (x_{i_k}) which is multi-null. Indeed we choose the sequence $(i_k : k \in \mathbb{N})$ inductively so that, for each $k \in \mathbb{N}$, we have $i_{k+1} > i_k$ and $(x_{i_k}, 0, 0, \dots) \in V_k$. That (x_{i_k}) is a multi-null sequence follows from Axiom (T4).

The result now follows essentially as before. ∎

5.2.3. Multi-null sequences and order-convergence. Let E be a Banach lattice, as in §1.3, and let (x_n) be a sequence in E. Recall that (x_n) is order-null if and only if there is a sequence (u_n) in E^+ such that $u_n \downarrow 0$ and $|x_n| \leq u_n$ $(n \in \mathbb{N})$. The lattice multi-norm on $\{E^n : n \in \mathbb{N}\}$ was defined for each $n \in \mathbb{N}$ in Definition 4.41 by the formula

$$\|(x_1, \dots, x_n)\|_n^L = \| \, |x_1| \vee \cdots \vee |x_n| \, \| \quad (x_1, \dots, x_n \in E).$$

We shall consider multi-null sequences with respect to this multi-norm.

THEOREM 5.14. *Let E be a Banach lattice. Then each multi-null sequence in E is order-null in E.*

Proof. Let (x_n) be a multi-null sequence in E. Then, for each $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that

$$\| \, |x_{n_k}| \vee |x_{n_k+1}| \vee \cdots \vee |x_n| \, \| < 2^{-k} \quad (n \geq n_k);$$

we may suppose that the sequence $(n_k : k \in \mathbb{N})$ is strictly increasing. Set

$$I_k = \{n_k, \dots, n_{k+1} - 1\} \subset \mathbb{N} \quad (k \in \mathbb{N}),$$

and, for $k \in \mathbb{N}$, define

$$y_k = |x_{n_k}| \vee |x_{n_k+1}| \vee \cdots \vee |x_{n_{k+1}-1}|,$$

so that $\|y_k\| \leq 2^{-k}$ and the series $\sum_{k=n}^{\infty} y_k$ is convergent in E for each $n \in \mathbb{N}$. Set

$$u_n = \sum_{j=k}^{\infty} y_j \quad \text{for each } n \in I_k.$$

For $n \in I_k$, we have $|x_n| \leq y_k \leq u_n$. Also, $0 \leq u_{n+1} \leq u_n$ $(n \in \mathbb{N})$. Suppose that $u \in E$ with $0 \leq u \leq u_n$ $(n \in \mathbb{N})$. Then $0 \leq \|u\| \leq \|u_n\| \leq 2^{-k+1}$ $(n \geq n_k)$, and so $u = 0$ and $u_n \downarrow 0$. This implies that (x_n) is an order-null sequence in E. ∎

We wish to determine when the converse of the above theorem holds.

THEOREM 5.15. *Let $(E, \|\cdot\|)$ be a Banach lattice. Then each order-null sequence in E is multi-null in E if and only if the norm is σ-order-continuous.*

Proof. Suppose that each order-null sequence is multi-null, and let (x_n) be a sequence in E with $x_n \downarrow 0$. Then (x_n) is order-null, and hence multi-null. Certainly this implies that $\|x_n\| \downarrow 0$, and so the norm is σ-order-continuous.

Conversely, suppose that the norm is σ-order-continuous, and let (x_n) be an order-null sequence. Then there exists a sequence (u_n) in E^+ with $|x_n| \leq u_n$ $(n \in \mathbb{N})$ and $u_n \downarrow 0$. By hypothesis, we have $\|u_n\| \downarrow 0$, and now

$$\| \, |x_n| \vee \cdots \vee |x_{n+k}| \, \|_k \leq \|u_n \vee \cdots \vee u_{n+k}\| = \|u_n\| \quad (n, k \in \mathbb{N}),$$

so that $\lim_{n\to\infty} \sup_{k\in\mathbb{N}} \| \, |x_n| \vee \cdots \vee |x_{n+k}| \, \|_k = 0$. Hence (x_n) is multi-null. ∎

For example, multi-null and order-null sequences coincide in each Banach lattice $L^p(\Omega)$ for $p \geq 1$ (when this space has the lattice multi-norm, which, by Example 4.47, is equal to the standard p-multi-norm) based on $L^p(\Omega)$ and on the space $C([0, \omega_1])$ (when this space has the minimum multi-norm).

6. Multi-bounded sets and multi-bounded operators

The theory of Banach spaces gains great strength from the facts that, for each Banach spaces E and F, a linear operator from E to F is continuous if and only if it is bounded, and that the collection of all bounded linear operators from E to F is itself a Banach space. Our aim in this chapter is to establish analogous results for multi-normed spaces.

6.1. Definitions and basic properties. We first define multi-bounded sets in multi-topological linear spaces (which were defined in Definition 5.1).

6.1.1. Multi-bounded sets

DEFINITION 6.1. Let E be a multi-topological linear space with respect to (F, τ). A subset B of E is *multi-bounded* if $B^{\mathbb{N}}$ is a bounded set in the topological linear space (F, τ).

We denote the family of multi-bounded sets in E by $\mathcal{MB}(E)$, suppressing in the notation the role of F.

Let $B, C \in \mathcal{MB}(E)$ and $\alpha, \beta \in \mathbb{C}$. Then it is immediate from the definition that $B \cup C, \alpha B + \beta C \in \mathcal{MB}(E)$; each compact set is multi-bounded; the absolutely convex hull of a multi-bounded set is multi-bounded.

PROPOSITION 6.2. *Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space, and let B be a subset of E. Then B is multi-bounded in E if and only if*

$$\sup\{\|(x_1, \dots, x_n)\|_n : x_1, \dots, x_n \in B,\, n \in \mathbb{N}\} < \infty.$$

Proof. This is immediate from our earlier results. ∎

COROLLARY 6.3. *Let E be a normed space, and consider two multi-norms based on E such that the multi-norms are equivalent. Then the families of multi-bounded sets with respect to the two multi-norms are equal.* ∎

DEFINITION 6.4. Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space, and let $B \in \mathcal{MB}(E)$. Then

$$c_B = \sup\{\|(x_1, \dots, x_n)\|_n : x_1, \dots, x_n \in B,\, n \in \mathbb{N}\};$$

c_B is the *multi-bound* of a multi-bounded set B.

PROPOSITION 6.5. *Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space.*

(i) *A finite subset $B = \{x_1, \dots, x_k\}$ in E is multi-bounded, with $c_B = \|(x_1, \dots, x_k)\|_k$.*
(ii) *Suppose that $B \subset E$ is multi-bounded. Then $C := \mathrm{aco}(B)$ is multi-bounded, with $c_C = c_B$.*

Proof. (i) This is immediate from Lemma 2.15.

(ii) Take $y_1, \dots, y_m \in C$. Then clearly there exist $n \in \mathbb{N}$, $a = (\alpha_{ij}) \in \mathbb{M}_{m,n}$, and $x = (x_1, \dots, x_n) \in B$ such that

$$\sum_{j=1}^{n} |\alpha_{ij}| \leq 1 \quad \text{and} \quad y_i = \sum_{j=1}^{n} \alpha_{ij} x_j$$

for $i \in \mathbb{N}_m$. By (1.15), $\|a : \ell_n^\infty \to \ell_m^\infty\| \leq 1$, and so, by Theorem 2.35, (a)$\Rightarrow$(c), we have $\|(y_1, \dots, y_m)\|_m = \|a \cdot x\|_m \leq \|x\|_n \leq c_B$, and so $c_C \leq c_B$. Thus $c_C = c_B$. ∎

Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space, and let (x_n) be a sequence in E. Then we see that the set $\{x_n : n \in \mathbb{N}\}$ is multi-bounded if and only if

$$\sup_{n \in \mathbb{N}} \|(x_1, \dots, x_n)\|_n = \lim_{n \to \infty} \|(x_1, \dots, x_n)\|_n < \infty;$$

in this case, (x_n) is a *multi-bounded sequence*. It follows from (5.3) that each multi-convergent sequence in E is multi-bounded.

6.1.2. Multi-bounded sets for lattice multi-norms. Let E be a Banach lattice. The lattice multi-norm $(\|\cdot\|_n^L : n \in \mathbb{N})$ based on E was defined in Definition 4.43.

PROPOSITION 6.6. *Let E be a Banach lattice. Then each order-bounded subset of E is multi-bounded with respect to the lattice multi-norm.*

Proof. Suppose that B is order-bounded in E, so that there exists $y \in E^+$ such that $|x| \leq y$ $(x \in B)$. Let $n \in \mathbb{N}$, and choose $x_1, \dots, x_n \in B$; define

$$x = |x_1| \vee \cdots \vee |x_n|,$$

so that $x \leq y$. Then $\|(x_1, \dots, x_n)\|_n^L = \|x\| \leq \|y\|$. Thus we see that $B \in \mathcal{MB}(E)$ (with $c_B \leq \|y\|$). ∎

PROPOSITION 6.7. *Let E be a Banach lattice. For each pairwise-disjoint, multi-bounded sequence (x_i) in E and each null sequence (α_i), the series $\sum_{i=1}^\infty \alpha_i x_i$ converges in E.*

Proof. Set $c = \sup\{\| \, |x_1| \vee \cdots \vee |x_n| \, \| : n \in \mathbb{N}\}$. For each $\varepsilon > 0$, take $i_0 \in \mathbb{N}$ such that $|\alpha_i| < \varepsilon$ $(i \geq i_0)$. Now take $m, n \in \mathbb{N}$ with $i_0 \leq m < n$. Then, using equation (1.26), we have

$$\Big\| \sum_{i=m}^{n} \alpha_i x_i \Big\| = \| \, |\alpha_m| \, |x_m| \vee \cdots \vee |\alpha_n| \, |x_n| \, \| \leq \varepsilon c,$$

and so

$$\Big(\sum_{i=1}^{n} \alpha_i x_i : n \in \mathbb{N} \Big)$$

is Cauchy, and hence convergent, in E. ∎

A 'monotonically bounded Banach lattice' was defined in Definition 1.22(i).

THEOREM 6.8. *Let E be a monotonically bounded Banach lattice. Then a subset of E is order-bounded if and only if it is multi-bounded.*

Proof. It follows from Proposition 6.6 that we must show just that a multi-bounded set in E is order-bounded.

Let B be a multi-bounded subset of E, and let $\mathcal{F} = \mathcal{P}_f(B)$, the family of finite subsets of B, so that $\mathcal{F}$ is a directed set when ordered by inclusion. For each $F \in \mathcal{F}$, set

$$y_F = \max\{|x| : x \in F\}.$$

Then $\{y_F : F \in \mathcal{F}\}$ is an increasing net in $E_{\mathbb{R}}$. Since B is multi-bounded, the net $\{y_F : F \in \mathcal{F}\}$ is bounded in $(E, \|\cdot\|)$, and so, since E is monotonically bounded, there exists $y \in E$ with $y_F \leq y$ $(F \in \mathcal{F})$. Thus y is an upper bound for B, and so B is order-bounded. ■

In particular, take $E = C(K)$, where K is a compact space, and let $\{E^n : n \in \mathbb{N}\}$ have the minimum multi-norm, which is the lattice multi-norm from the Banach lattice E. Then the multi-bounded sets and the order-bounded sets coincide, and these are just the $\|\cdot\|$-bounded subsets of E. On the other hand, let $B = \{e_n : n \in \mathbb{N}\} \subset c_0$. Then B is multi-bounded, but not order-bounded, in c_0.

Now let $E = L^p(\Omega)$, where Ω is a measure space and $p \geq 1$, and let the family $\{E^n : n \in \mathbb{N}\}$ have the standard p-multi-norm, which, as we noted in Example 4.47, is the lattice multi-norm from the Banach lattice E. Then the multi-bounded sets and the order-bounded sets coincide.

Further, let K be a compact space. Then again the multi-bounded sets for the standard 1-multi-norm based on $M(K)$ and the order-bounded sets of $M(K)$ coincide; this follows from Theorem 4.31.

6.1.3. Multi-bounded operators. The above notion of a multi-bounded set leads immediately to the definition of a multi-bounded operator.

Definition 6.9. Let E and F be multi-topological linear spaces, and let $T \in \mathcal{L}(E, F)$. Then T is a *multi-bounded operator* if

$$T(B) \in \mathcal{MB}(F) \quad (B \in \mathcal{MB}(E)).$$

The collection of multi-bounded linear maps from E to F is denoted by $\mathcal{M}(E, F)$. We write $\mathcal{M}(E)$ for $\mathcal{M}(E, E)$ in the case where E and F are equal as multi-topological linear spaces.

Proposition 6.10. *Let E, F, and G be multi-topological linear spaces. Then:*

(i) *$\mathcal{M}(E, F)$ is a linear subspace of $\mathcal{L}(E, F)$;*
(ii) *$T \circ S \in \mathcal{M}(E, G)$ whenever $S \in \mathcal{M}(E, F)$ and $T \in \mathcal{M}(F, G)$.*

Proof. This is immediate from a remark above. ■

Proposition 6.11. *Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ and $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ be two multi-normed spaces, and let $T \in \mathcal{M}(E, F)$. Then*

$$\sup\{c_{T(B)} : B \in \mathcal{MB}(E) \textit{ with } c_B \leq 1\} < \infty.$$

Proof. Assume towards a contradiction that the specified supremum is infinite. Then, for each $n \in \mathbb{N}$, there exists $B_n \in \mathcal{MB}(E)$ such that $c_{B_n} \leq 1/n^2$, but $c_{T(B_n)} > n$, and there exist $x_{1,n}, \ldots, x_{k_n,n} \in B_n$ such that $\|(x_{1,n}, \ldots, x_{k_n,n})\|_{k_n} < 1/n^2$ and

$$\|(Tx_{1,n}, \ldots, Tx_{k_n,n})\|_{k_n} > n. \tag{6.1}$$

Consider the subset

$$B := \{x_{1,1}, \ldots, x_{k_1,1}, x_{1,2}, \ldots, x_{k_2,2}, \ldots, x_{1,n}, \ldots, x_{k_n,n}, \ldots\}$$

of E. Set $K_n = \sum_{i=1}^n k_i$ for $n \in \mathbb{N}$. For each $y_1, \ldots, y_m \in B$, there exists $n \in \mathbb{N}$ such that

$$\{y_1, \ldots, y_m\} \subset \{x_{1,1}, \ldots, x_{k_1,1}, x_{1,2}, \ldots, x_{k_2,2}, \ldots, x_{1,n}, \ldots, x_{k_n,n}\},$$

and so, by Lemmas 2.15 and 2.10,

$$\begin{aligned}\|(y_1, \ldots, y_m)\|_m &\leq \|(x_{1,1}, \ldots, x_{k_1,1}, x_{1,2}, \ldots, x_{k_2,2}, \ldots, x_{1,n}, \ldots, x_{k_n,n})\|_{K_n} \\ &\leq \sum_{j=1}^n \|(x_{1,j}, \ldots, x_{k_j,j})\|_{k_j} \leq \sum_{j=1}^n \frac{1}{j^2} < \infty.\end{aligned}$$

This shows that $B \in \mathcal{MB}(E)$. Thus there exists $M > 0$ such that

$$\|(Ty_1, \ldots, Ty_m)\|_m \leq M \quad (y_1, \ldots, y_m \in B,\, m \in \mathbb{N}).$$

But this contradicts (6.1).

Thus the result holds. ■

The above proposition shows that the following definition of $\|T\|_{mb}$ always gives a number in $\mathbb{R}^+$.

Definition 6.12. Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ and $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ be two multi-normed spaces, and let $T \in \mathcal{M}(E, F)$. Then

$$\|T\|_{mb} = \sup\{c_{T(B)} : B \in \mathcal{MB}(E) \text{ with } c_B \leq 1\}.$$

The map T is a *multi-contraction* if $\|T\|_{mb} \leq 1$, and T is a *multi-isometry* if T is an isometry onto a closed subspace $T(E)$ of F and if, further, $T \in \mathcal{M}(E, T(E))$ and $T^{-1} \in \mathcal{M}(T(E), E)$ are both multi-contractions.

Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ and $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ be two multi-normed spaces, and let $T \in \mathcal{M}(E, F)$. Then it is immediately clear that $T \in \mathcal{B}(E, F)$ and that $\|T\| \leq \|T\|_{mb}$. More generally, for each $n \in \mathbb{N}$, we have

$$\|(Tx_1, \ldots, Tx_n)\|_n \leq \|T\|_{mb}\|(x_1, \ldots, x_n)\|_n \quad (x_1, \ldots, x_n \in E). \tag{6.2}$$

Indeed, for $n \in \mathbb{N}$, set

$$p_n(T) = \sup\{\|(Tx_1, \ldots, Tx_n)\|_n : \|(x_1, \ldots, x_n)\|_n \leq 1\}.$$

Then $(p_n(T) : n \in \mathbb{N})$ is an increasing sequence with

$$\|T\|_{mb} = \lim_{n\to\infty} p_n(T).$$

Explicitly, we have

$$\|T\|_{mb} = \sup_n \sup\left\{\frac{\|(Tx_1, \ldots, Tx_n)\|_n}{\|(x_1, \ldots, x_n)\|_n} : (x_1, \ldots, x_n) \neq 0\right\} < \infty, \tag{6.3}$$

and so T is multi-bounded if and only if $\|T\|_{mb} = \sup_{n\in\mathbb{N}} \|T^{(n)}\| < \infty$, where $T^{(n)}$ is the n^{th} amplification of T.

We have noted in Theorem 2.42 that multi-norms correspond to c_0-norms on $c_0 \otimes E$. Now take $T \in \mathcal{B}(E, F)$. Then T is multi-bounded if and only if $I_{c_0} \otimes T$ is bounded as a map from $c_0 \otimes E$ to $c_0 \otimes F$ (when these spaces have the c_0-norms corresponding to the respective multi-norms), and then $\|T\|_{mb} = \|I_{c_0} \otimes T\|$. Thus our multi-bounded operators are the same as the 'opérateurs réguliers' of [53, Définition 3.2]. For further details, see [19].

6.1.4. Multi-continuous operators. We shall now show that the multi-bounded operators on multi-normed spaces are exactly the 'multi-continuous' ones, mirroring the fact that an operator on a normed space is continuous if and only if it is bounded.

DEFINITION 6.13. Let E_1 and E_2 be multi-topological linear spaces with respect to (F_1, τ_1) and (F_2, τ_2), respectively. Then $T \in \mathcal{L}(E_1, E_2)$ is *multi-continuous* if (Tx_i) is a multi-null sequence in E_2 whenever (x_i) is a multi-null sequence in E_1.

The following result is taken from [17], where some applications are given.

THEOREM 6.14. *Let* $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ *and* $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ *be two multi-normed spaces. Then a linear map from* E *to* F *is multi-continuous if and only if it is multi-bounded.*

Proof. Suppose that $T \in \mathcal{L}(E,F)$ is multi-bounded, and let (x_i) be a multi-null sequence in E. Then, by Theorem 5.8, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{k\in\mathbb{N}} \|(x_{n+1}, \ldots, x_{n+k})\|_k < \varepsilon \quad (n \geq n_0).$$

But now

$$\sup_{k\in\mathbb{N}} \|(Tx_{n+1}, \ldots, Tx_{n+k})\|_k \leq \|T\|_{mb}\varepsilon \quad (n \geq n_0),$$

and so, by Theorem 5.8 again, (Tx_i) is a multi-null sequence in F. Thus T is multi-continuous.

Suppose that $T \in \mathcal{L}(E,F)$ is not multi-bounded. Then there exists a subset B of E such that B is multi-bounded in E, but $T(B)$ is not multi-bounded in F. For each $n \in \mathbb{N}$, there exist $x_{1,n}, \ldots, x_{k_n,n} \in B$ such that

$$\|(x_{1,n}, \ldots, x_{k_n,n})\|_{k_n} < \frac{1}{n^2} \quad \text{and} \quad \|(Tx_{1,n}, \ldots, Tx_{k_n,n})\|_{k_n} > 1.$$

We may suppose that $k_n \geq n$ for each $n \in \mathbb{N}$. Consider the sequence

$$y = (x_{1,1}, \ldots, x_{k_1,1}, x_{1,2}, \ldots, x_{k_2,2}, \ldots, x_{1,n}, \ldots, x_{k_n,n}, \ldots).$$

We *claim* that y is a multi-null sequence in E. Indeed, take $\varepsilon > 0$. Then there exists $j \in \mathbb{N}$ such that $\sum_{i=j}^{\infty} 1/i^2 < \varepsilon$, and then

$$\|(x_{1,j}, \ldots, x_{k_j,j}, \ldots, x_{1,j+n}, \ldots, x_{k_{j+n},j+n})\|_{k_j+\cdots+k_{j+n}} \leq \varepsilon \quad (n \in \mathbb{N}),$$

giving the claim. However (Ty_i) is clearly not a multi-null sequence in F. Thus T is not multi-continuous. ∎

6.2. The space $\mathcal{M}(E,F)$

6.2.1. The normed space $\mathcal{M}(E,F)$. We shall recognize $\mathcal{M}(E,F)$ as a normed space of operators.

Let E and F be normed spaces. Recall that the spaces $\mathcal{F}(E,F)$ and $\mathcal{N}(E,F)$ of finite-rank and nuclear operators were defined in Chapter 1, §1.2.1.

THEOREM 6.15. *Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ and $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ be multi-normed spaces, with F a Banach space. Then*

$$(\mathcal{M}(E,F), \|\cdot\|_{mb})$$

is a Banach space. Further:

(i) *$y_0 \otimes \lambda_0 \in \mathcal{M}(E,F)$ with $\|y_0 \otimes \lambda_0\|_{mb} = \|y_0\|\,\|\lambda_0\| = \|y_0 \otimes \lambda_0\|$ for each $\lambda_0 \in E'$ and $y_0 \in F$;*

(ii) *$\mathcal{N}(E,F) \subset \mathcal{M}(E,F)$, and the natural embedding is a contraction.*

Proof. It is immediate that $(\mathcal{M}(E,F), \|\cdot\|_{mb})$ is a normed space.

Let (T_k) be a Cauchy sequence in $(\mathcal{M}(E,F), \|\cdot\|_{mb})$. Then there exists $T \in \mathcal{B}(E,F)$ such that $\|T_k - T\| \to 0$ as $k \to \infty$. Take $\varepsilon > 0$. Then there exists $k_0 \in \mathbb{N}$ such that $\|T_j - T_k\|_{mb} < \varepsilon$ $(j,k \geq k_0)$. It follows from equation (6.3) that $T - T_k \in \mathcal{M}(E,F)$ and $\|T - T_k\|_{mb} \leq \varepsilon$ for each $j \geq k_0$. Thus $T_k \to T$ with respect to $\|\cdot\|_{mb}$, and so $(\mathcal{M}(E,F), \|\cdot\|_{mb})$ is a Banach space.

(i) Let $\lambda_0 \in E'$ and $y_0 \in F$, and set $T = y_0 \otimes \lambda_0$. For each $n \in \mathbb{N}$ and $x_1, \dots, x_n \in E$, we have

$$\begin{aligned}\|(Tx_1, \dots, Tx_n)\|_n &\leq \max\{|\langle x_j, \lambda_0\rangle| : j \in \mathbb{N}_n\}\|(y_0, \dots, y_0)\|_n \\ &\leq \|y_0\|\,\|\lambda_0\| \max\{\|x_j\| : j \in \mathbb{N}_n\} \\ &\leq \|y_0\|\,\|\lambda_0\|\,\|(x_1, \dots, x_n)\|_n,\end{aligned}$$

and so

$$\|y_0 \otimes \lambda_0\| \leq \|y_0 \otimes \lambda_0\|_{mb} \leq \|y_0\|\,\|\lambda_0\| = \|y_0 \otimes \lambda_0\|.$$

It follows that $y_0 \otimes \lambda_0 \in \mathcal{M}(E,F)$ with $\|y_0 \otimes \lambda_0\|_{mb} = \|y_0 \otimes \lambda_0\|$, and hence we have $\mathcal{F}(E,F) \subset \mathcal{M}(E,F)$.

(ii) Let $T \in \mathcal{N}(E,F)$. Then clearly $T \in \mathcal{M}(E,F)$ with $\|T\|_{mb} \leq \nu(T)$, so that the natural embedding is a contraction. ∎

We shall see in Example 6.25, below, that the 'minimum' case for which we have $\mathcal{N}(E,F) = \mathcal{M}(E,F)$ can occur.

THEOREM 6.16. *Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space. Then $(\mathcal{M}(E), \|\cdot\|_{mb})$ is a unital Banach operator algebra.* ∎

The following result was pointed out by Matt Daws; the result is also essentially contained in [53, Remarque, p. 20].

THEOREM 6.17. *Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ and $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ be two multi-normed spaces. Suppose that the multi-norm based on F is the minimum multi-norm, or that the multi-norm based on E is the maximum multi-norm. Then*

$$\mathcal{M}(E,F) = \mathcal{B}(E,F) \quad \textit{and} \quad \|T\|_{mb} = \|T\| \qquad (T \in \mathcal{B}(E,F)).$$

Proof. First, suppose that the multi-norm based on F is the minimum multi-norm. We take $T \in \mathcal{B}(E,F)$ and $B \in \mathcal{MB}(E)$. Since

$$\|(Tx_1, \dots, Tx_n)\|_n = \max_{i \in \mathbb{N}_n} \|Tx_i\| \qquad (n \in \mathbb{N}),$$

it is clear that $c_{T(B)} \leq \|T\| c_B$. It follows that $T \in \mathcal{M}(E,F)$ and that $\|T\|_{mb} \leq \|T\|$. But always $\|T\| \leq \|T\|_{mb}$, and so we have $\|T\| = \|T\|_{mb}$, as required.

Second, suppose that the multi-norm based on E is the maximum multi-norm. We take $T \in \mathcal{B}(E,F)_{[1]}$, and define

$$|||(x_1,\dots,x_n)|||_n = \max\{\|(x_1,\dots,x_n)\|_n,\ \|(Tx_1,\dots,Tx_n)\|_n\}$$

for $x_1,\dots,x_n \in E$. It is easy to check that $((E^n, |||\cdot|||_n) : n \in \mathbb{N})$ is a multi-normed space and that

$$|||x||| = \max\{\|x\|,\ \|Tx\|\} = \|x\| \quad (x \in E).$$

Since the multi-norm based on E is the maximum multi-norm, it follows that

$$\|(Tx_1,\dots,Tx_n)\|_n \leq \|(x_1,\dots,x_n)\|_n \quad (x_1,\dots,x_n \in E)$$

for each $n \in \mathbb{N}$, and so $T \in \mathcal{M}(E,F)$ with $\|T\|_{mb} \leq 1$. This shows that we have $\mathcal{M}(E,F) = \mathcal{B}(E,F)$, and also that $\|T\|_{mb} = \|T\|$ for each $T \in \mathcal{B}(E,F)$. ∎

6.2.2. A multi-norm based on $\mathcal{M}(E,F)$. We shall now see that there is a natural multi-normed structure based on $\mathcal{M}(E,F)$.

DEFINITION 6.18. Let $((E^n,\|\cdot\|_n) : n \in \mathbb{N})$ and $((F^n,\|\cdot\|_n) : n \in \mathbb{N})$ be two multi-normed spaces, and let $n \in \mathbb{N}$ and $T_1,\dots,T_n \in \mathcal{M}(E,F)$. Then

$$\|(T_1,\dots,T_n)\|_n^{mb} = \sup\{c_{T_1(B)\cup\cdots\cup T_n(B)} : B \in \mathcal{MB}(E) \text{ with } c_B \leq 1\}.$$

Let $T \in \mathcal{M}(E,F)$. Then, by the definition, $\|T\|_1^{mb}$ is exactly $\|T\|_{mb}$. We have a somewhat more explicit formula for $\|(T_1,\dots,T_n)\|_n^{mb}$.

PROPOSITION 6.19. *Let $((E^n,\|\cdot\|_n) : n \in \mathbb{N})$ and $((F^n,\|\cdot\|_n) : n \in \mathbb{N})$ be two multi-normed spaces, and let $n \in \mathbb{N}$ and $T_1,\dots,T_n \in \mathcal{M}(E,F)$. Then*

$$\|(T_1,\dots,T_n)\|_n^{mb} = \sup \|(T_i x_j : i \in \mathbb{N}_n,\ j \in \mathbb{N}_k)\|_{nk}, \tag{6.4}$$

where the supremum is taken over $x_1,\dots,x_k \in E$ with $\|(x_1,\dots,x_k)\|_k \leq 1$.

Proof. Denote the left- and right-hand sides of (6.4) by a and b, respectively.

Take $x_1,\dots,x_k \in E$ with $\|(x_1,\dots,x_k)\|_k \leq 1$, and set $B = \{x_1,\dots,x_k\}$. Then $c_B \leq 1$ and $\{T_i x_j : i \in \mathbb{N}_n,\ j \in \mathbb{N}_k\} \subset T_1(B)\cup\cdots\cup T_n(B)$. Since $c_{T_1(B)\cup\cdots\cup T_n(B)} \leq a$, we have $\|(T_i x_j : i \in \mathbb{N}_n,\ j \in \mathbb{N}_k)\|_{nk} \leq a$. Hence $b \leq a$.

Take $\varepsilon > 0$. Then there exists a set B in E such that $c_B \leq 1$ and

$$c_{T_1(B)\cup\cdots\cup T_n(B)} \geq a - \varepsilon,$$

and there exist $k_1,\dots,k_n \in \mathbb{N}$ and $x_{1,i},\dots,x_{k_i,i} \in B$ for $i \in \mathbb{N}_n$ such that

$$\|(T_i x_{r,i} : r \in \mathbb{N}_{k_i},\ i \in \mathbb{N}_n)\|_k > c_{T_1(B)\cup\cdots\cup T_n(B)} - \varepsilon,$$

where $k = k_1 + \cdots + k_n$. Let $x_1,\dots,x_k$ be a listing of the elements $x_{r,i}$. By Lemma 2.15,

$$\|(T_i x_j : i \in \mathbb{N}_n,\ j \in \mathbb{N}_k)\|_{nk} \geq \|(T_i x_{r,i} : r \in \mathbb{N}_{k_i},\ i \in \mathbb{N}_n)\|_k,$$

and so $b > a - 2\varepsilon$. This holds true for each $\varepsilon > 0$, and so $b \geq a$. ∎

THEOREM 6.20. *Let $((E^n,\|\cdot\|_n) : n \in \mathbb{N})$ and $((F^n,\|\cdot\|_n) : n \in \mathbb{N})$ be two multi-normed spaces. Then $\|\cdot\|_n^{mb}$ is a norm on the linear space $\mathcal{M}(E,F)^n$, and*

$$((\mathcal{M}(E,F)^n, \|\cdot\|_n^{mb}) : n \in \mathbb{N})$$

is a multi-normed space with $\|T\|_1^{mb} = \|T\|_{mb}$*; it is a multi-Banach space in the case where* F *is a Banach space.*

Proof. This now follows easily. ■

DEFINITION 6.21. The multi-norm $(\|\cdot\|_n^{mb} : n \in \mathbb{N})$ is the *multi-bounded multi-norm* based on $\mathcal{M}(E,F)$.

THEOREM 6.22. *Let* $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ *and* $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ *be two multi-normed spaces, with* $E \neq \{0\}$*. Then the multi-bounded multi-norm based on* $\mathcal{M}(E,F)$ *is the minimum multi-norm if and only if the multi-norm based on* F *is the minimum multi-norm.*

Proof. Suppose that the multi-norm based on F is the minimum multi-norm.

Let $n \in \mathbb{N}$ and $T_1, \dots, T_n \in \mathcal{B}(E,F)$. For $k \in \mathbb{N}$, take $x_1, \dots, x_k \in E$ such that $\|(x_1, \dots, x_k)\|_k \leq 1$. Then $\|x_j\| \leq 1$ $(j \in \mathbb{N}_k)$, and so $\|T_i x_j\| \leq \|T_i\|$ $(i \in \mathbb{N}_n,\ j \in \mathbb{N}_k)$. It follows from (6.4) that

$$\|(T_1, \dots, T_n)\|_n^{mb} \leq \max_{i \in \mathbb{N}_n} \|T_i\|.$$

By Theorem 6.17, $\|T_i\|_{mb} = \|T_i\|$ $(i \in \mathbb{N}_n)$, and hence $(\|\cdot\|_n^{mb} : n \in \mathbb{N})$ is the minimum multi-norm based on $\mathcal{M}(E,F)$.

Conversely, suppose that

$$\|(T_1, \dots, T_n)\|_n^{mb} = \max_{i \in \mathbb{N}_n} \|T_i\|$$

whenever $T_1, \dots, T_n \in \mathcal{B}(E,F)$ and $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$, and take $y_1, \dots, y_n \in F$. Since $E \neq \{0\}$, there exist $x_0 \in E$ and $\lambda_0 \in E'$ with $\|x_0\| = \|\lambda_0\| = \langle x_0,\, \lambda_0 \rangle = 1$. For $i \in \mathbb{N}_n$ define $T_i = y_i \otimes \lambda_0$, so that $T_i \in \mathcal{M}(E,F)$ and $\|T_i\|_{mb} = \|T_i\| = \|y_i\|$ $(i \in \mathbb{N}_n)$ by Theorem 6.15(i). From (6.4),

$$\|(T_1 x_1, \dots, T_n x_n)\|_n \leq \|(T_1, \dots, T_n)\|_n^{mb}.$$

Hence

$$\|(y_1, \dots, y_n)\|_n \leq \max_{i \in \mathbb{N}_n} \|T_i\| = \max_{i \in \mathbb{N}_n} \|y_i\| = \|(y_1, \dots, y_n)\|_n^{\min}.$$

It follows that $(\|\cdot\|_n : n \in \mathbb{N}) \leq (\|\cdot\|_n^{\min} : n \in \mathbb{N})$, and so $(\|\cdot\|_n : n \in \mathbb{N})$ is the minimum multi-norm based on F. ■

COROLLARY 6.23. *Let* $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ *and* $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ *be two multi-normed spaces, with* F *a finite-dimensional space. Then the multi-bounded multi-norm based on* $\mathcal{M}(E,F)$ *is equivalent to the minimum multi-norm.*

Proof. By Proposition 3.7, the multi-bounded multi-norm is equivalent to the minimum multi-norm based on F, and so this follows by a slight variation of the above proof. ■

In particular, we see that $\mathcal{M}(E,\mathbb{C}) = E'$, and so the multi-bounded multi-norm based on $\mathcal{M}(E,\mathbb{C})$ is just the minimum multi-normed space $((E')^n, \|\cdot\|_n^{\min} : n \in \mathbb{N})$. We shall discuss in Chapter 7 a different way of constructing multi-norms based on dual spaces.

6.3. Examples. We give some specific examples of the Banach spaces $\mathcal{M}(E,F)$ and the Banach algebras $\mathcal{M}(E)$.

6.3.1. Algebras of operators. Let E and F be normed spaces. The linear space of compact operators on a normed space E is denoted by $\mathcal{K}(E)$, as in §1.2.1.

In the first example, we shall show that it may be that $\mathcal{K}(E) \not\subset \mathcal{M}(E)$, and hence that $\mathcal{M}(E) \subsetneq \mathcal{B}(E)$.

EXAMPLE 6.24. Let H be the Hilbert space $\ell^2(\mathbb{N})$, with the standard 2-multi-norm $(\|\cdot\|_n^{[2]} : n \in \mathbb{N})$ based on H of Definition 4.21. As before, $(\delta_n : n \in \mathbb{N})$ is the standard basis of H; the inner product in H is denoted by $[\,\cdot\,,\,\cdot\,]$.

Consider the system of vectors $(x_r^s : r \in \mathbb{N}_s, s \in \mathbb{N})$ in H defined as follows: $x_r^s(k) = 0$ except when $k \in \{2^{s-1}, \dots, 2^s - 1\}$; at the 2^{s-1} numbers k in the set $\{2^{s-1}, \dots, 2^s - 1\}$, $x_r^s(k) = \pm 1/\sqrt{2^{s-1}}$, the values ± 1 being chosen so that $[x_{r_1}^s, x_{r_2}^s] = 0$ when $r_1, r_2 \in \mathbb{N}_s$ and $r_1 \neq r_2$. Such a choice is clearly possible. Then

$$S := \{x_r^s : r \in \mathbb{N}_s, s \in \mathbb{N}\}$$

is an orthonormal set in H. Order the set S as (y_n) by using the lexicographic order on the pairs (s, r) (so that $y_1 = x_1^1$, $y_2 = x_1^2$, $y_3 = x_2^2$, $y_4 = x_1^3$, etc.).

Let $(\alpha_i) \in \ell^\infty$. We define an operator T by setting

$$Tx_r^s = \alpha_s \delta_n \quad \text{when} \quad x_r^s = y_n;$$

clearly T extends by linearity and continuity to become an operator in $\mathcal{B}(H)$. It is also clear that, in the case where $(\alpha_i) \in c_0$, we have $T \in \mathcal{K}(H)$.

For $k \in \mathbb{N}$, set $N_k = \sum_{i=1}^k i = k(k+1)/2$. We see that $\|(y_1, y_2, \dots, y_{N_k})\|_{N_k}^{[2]} = \sqrt{k}$. However

$$\|(Ty_1, Ty_2, \dots, Ty_{N_k})\|_{N_k}^{[2]} = \|(\alpha_1\delta_1, \alpha_2\delta_2, \alpha_2\delta_3, \alpha_3\delta_4, \dots, \alpha_k\delta_{N_k})\|_{N_k}^{[2]} = \Big(\sum_{i=1}^k i|\alpha_i|^2\Big)^{1/2}.$$

Now take $\gamma \in (0, 1/2)$, and set $\alpha_i = i^{-\gamma}$ $(i \in \mathbb{N})$, so that $(\alpha_i) \in c_0$ and $T \in \mathcal{K}(H)$. Then

$$\sum_{i=1}^k i|\alpha_i|^2 = \sum_{i=1}^k i^{1-2\gamma} \geq \int_1^k t^{1-2\gamma}\,\mathrm{d}t \geq \frac{1}{2-2\gamma}(k^{2-2\gamma} - 1).$$

Thus

$$\frac{\|(Ty_1, Ty_2, \dots, Ty_{N_k})\|_{N_k}}{\|(y_1, y_2, \dots, y_{N_k})\|_{N_k}} \geq ck^{(1-2\gamma)/2}$$

for a constant $c > 0$. Since $\gamma < 1/2$, we have $T \notin \mathcal{M}(H)$. Thus $\mathcal{K}(H) \not\subset \mathcal{M}(H)$. In particular, $\mathcal{M}(H) \subsetneq \mathcal{B}(H)$. However $\mathcal{M}(H) \not\subset \mathcal{K}(H)$ because $I_H \in \mathcal{M}(H)$.

Now consider the Hilbert multi-norm $(\|\cdot\|_n^H : n \in \mathbb{N})$ based on H. By Theorem 4.20, the Hilbert multi-norm is equivalent to the maximum multi-norm $(\|\cdot\|_n^{\max} : n \in \mathbb{N})$ based on H, and so it follows from Theorem 6.17 that $\mathcal{M}(H) = \mathcal{B}(H)$ in this case. ■

EXAMPLE 6.25. In this example, we shall show that the inclusion $\mathcal{N}(E,F) \subset \mathcal{M}(E,F)$ given in Theorem 6.15(ii) is best possible.

One might guess that a form of Banach's isomorphism theorem would hold for multi-bounded operators. This would assert that $T^{-1} \in \mathcal{M}(F,E)$ whenever both

$$((E^n, \|\cdot\|_n) : n \in \mathbb{N}) \quad \text{and} \quad ((F^n, \|\cdot\|_n) : n \in \mathbb{N})$$

are multi-normed spaces, $T \in \mathcal{M}(E,F)$, and T is a bijection. However we shall show that this is not the case; this will also be shown, in stronger form, in Examples 6.30 and 6.39, below.

Let $E = \ell^1$. Then $((E^n, \|\cdot\|_n^{[1]}) : n \in \mathbb{N})$ is a multi-normed space, where we are writing $(\|\cdot\|_n^{[1]} : n \in \mathbb{N})$ for the standard 1-multi-norm of Definition 4.21. In this case,

$$\|(\delta_1, \dots, \delta_n)\|_n^{[1]} = n \quad (n \in \mathbb{N}),$$

as in (4.15). By Example 4.47, $(\|\cdot\|_n^{[1]} : n \in \mathbb{N})$ coincides with the lattice multi-norm $(\|\cdot\|_n^{L} : n \in \mathbb{N})$ on the Banach lattice E. However, also let $F = \ell^1$, and consider the minimum multi-norm $(\|\cdot\|_n^{\min} : n \in \mathbb{N})$ based on F, so that

$$\|(\delta_1, \dots, \delta_n)\|_n^{\min} = 1 \quad (n \in \mathbb{N}).$$

Since $\|\cdot\|_n^{\min} \leq \|\cdot\|_n^{[1]}$ $(n \in \mathbb{N})$, the identity map I_E on E, regarded as map from E to F belongs to $\mathcal{M}(E,F)$ (and I_E is a multi-contraction). However the above two equations show that $I_E^{-1} : F \to E$ is not multi-bounded.

Indeed, by Theorem 6.17, $\mathcal{M}(E,F) = \mathcal{B}(E,F)$ and, by Theorem 6.22, the multi-bounded multi-norm based on $\mathcal{M}(E,F)$ is the minimum multi-norm.

We shall now identify $\mathcal{M}(F,E)$. Take $T \in \mathcal{M}(F,E)$. The unit ball $F_{[1]}$ of F is multi-bounded, and so $T(F_{[1]})$ is multi-bounded in E. Since the Banach lattice ℓ^1 is monotonically bounded, it follows from Theorem 6.8 that $T(F_{[1]})$ is order-bounded in ℓ^1, and so there exists $x = (x_i) \in \ell^1$ with

$$|(Ty)_i| \leq x_i \quad (i \in \mathbb{N})$$

for each $y \in F_{[1]}$; further, $\sum_{i=1}^{\infty} x_i \geq \|T\|_{mb}$. Take $i \in \mathbb{N}$, let $\pi_i : z \mapsto z_i\delta_i$ be the rank-one operator on ℓ^1, and set $T_i = \pi_i \circ T = (\delta_i \otimes T')(\delta_i)$. For each $y \in F_{[1]}$, we have

$$|\langle y,\, T'(\delta_i)\rangle| = |\langle Ty,\, \delta_i\rangle| = |(Ty)_i| \leq x_i,$$

and so $\|T'(\delta_i)\| \leq x_i$, whence $\nu(T_i) = \|T'(\delta_i)\|\,\|\delta_i\| \leq x_i$. Clearly, we have $T = \sum_{i=1}^{\infty} T_i$, and hence $\nu(T) \leq \sum_{i=1}^{\infty} x_i < \infty$. Thus $T \in \mathcal{N}(F,E)$.

In summary, in this case we have

$$\mathcal{M}(E,F) = \mathcal{B}(E,F) \quad \text{and} \quad \mathcal{M}(F,E) = \mathcal{N}(F,E). \ \blacksquare$$

6.3.2. Partition multi-norms. We present an example that was suggested by Michael Elliott.

Take $p \geq 1$, and consider $\ell^p = \ell^p(\mathbb{N})$; the norm on ℓ^p is denoted by $\|\cdot\|$.

Definition 6.26. For each partition Π of $\mathbb{N}$ and $n \in \mathbb{N}$, set

$$\|(f_1, \dots, f_n)\|_n^{\Pi} = \left(\sum \left\{ \max_{k \in \mathbb{N}_n} \|f_k \mid P\|^p : P \in \Pi \right\}\right)^{1/p} \quad (f_1, \dots, f_n \in \ell^p).$$

It is easy to check that $\|\cdot\|_n^{\Pi}$ is a norm on $(\ell^p)^n$ for each $n \in \mathbb{N}$ and that

$$(((\ell^p)^n, \|\cdot\|_n^{\Pi}) : n \in \mathbb{N})$$

is a multi-normed space.

By taking Π to be the singleton $\{\mathbb{N}\}$, we see that we obtain the minimum multi-norm $(\|\cdot\|_n^{\min} : n \in \mathbb{N})$ as an example; by taking Π to be the collection of singletons $\{n\}$ in $\mathbb{N}$, we obtain the lattice multi-norm $(\|\cdot\|_n^{L} : n \in \mathbb{N})$ based on ℓ^p.

Definition 6.27. For each partition Π of $\mathbb{N}$, the above multi-norm $(\|\cdot\|_n^{\Pi} : n \in \mathbb{N})$ is the *partition multi-norm* based on ℓ^p.

For $\sigma \in \mathfrak{S}_{\mathbb{N}}$ and $S \subset \mathbb{N}$, we set $\sigma(S) = \{\sigma(n) : n \in S\}$, and we define

$$T_\sigma : f \mapsto f \circ \sigma, \qquad \ell^p \to \ell^p,$$

so that $T_\sigma : \ell^p \to \ell^p$ is an isometry.

Let Π be a partition of $\mathbb{N}$, and take $\sigma \in \mathfrak{S}_{\mathbb{N}}$. Define the sets

$$\Pi_\sigma(P) = \{Q \in \Pi : \sigma(Q) \cap P \neq \emptyset\} \quad \text{and} \quad \Pi_\sigma^{-1}(P) = \{Q \in \Pi : \sigma(P) \cap Q \neq \emptyset\}$$

for each $P \in \Pi$, so that $\Pi_\sigma^{-1}(P) = \Pi_{\sigma^{-1}}(P)$ and $\sigma(P)$ is contained in the pairwise-disjoint union of the family $\Pi_\sigma^{-1}(P)$ of subsets of $\mathbb{N}$.

Lemma 6.28. *Let Π be a partition of $\mathbb{N}$, and take $\sigma \in \mathfrak{S}_{\mathbb{N}}$. Then*

$$\|(T_\sigma f) \mid P\| \leq \left(\sum\{\|f \mid Q\|^p : Q \in \Pi_\sigma^{-1}(P)\}\right)^{1/p} \qquad (f \in \ell^p,\ P \in \Pi).$$

Proof. Take $f \in \ell^p$. Then

$$\begin{aligned}\|(T_\sigma f) \mid P\|^p &= \sum_{n \in P} |f(\sigma(n))|^p \leq \sum\Big\{\sum_{m \in Q} |f(m)|^p : Q \in \Pi_\sigma^{-1}(P)\Big\} \\ &= \sum\{\|f \mid Q\|^p : Q \in \Pi_\sigma^{-1}(P)\},\end{aligned}$$

giving the stated result. ∎

Let Π be a partition of $\mathbb{N}$, and take $\sigma \in \mathfrak{S}_{\mathbb{N}}$. Then we define

$$m_\sigma = \sup\{|\Pi_\sigma(P)| : P \in \Pi\},$$

so that $m_\sigma \in \mathbb{N} \cup \{\infty\}$.

Theorem 6.29. *Let Π be a partition of $\mathbb{N}$, and consider the multi-norm $(\|\cdot\|_n^{\Pi} : n \in \mathbb{N})$ based on ℓ^p, where $p \geq 1$. Take $\sigma \in \mathfrak{S}_{\mathbb{N}}$. Then $T_\sigma : \ell^p \to \ell^p$ is multi-bounded with respect to this multi-norm if and only if $m_\sigma < \infty$; in this latter case, $\|T\|_{mb} = m_\sigma^{1/p}$.*

Proof. Suppose that $m_\sigma < \infty$. Take $n \in \mathbb{N}$ and $f_1, \dots, f_n \in \ell^p$. Then

$$\begin{aligned}\left(\|(T_\sigma f_1, \dots, T_\sigma f_n)\|_n^{\Pi}\right)^p &= \sum_{P \in \Pi} \max_{k \in \mathbb{N}_n} \|(T_\sigma f_k) \mid P\|^p \\ &\leq \sum_{P \in \Pi} \max_{k \in \mathbb{N}_n} \sum_{Q \in \Pi_\sigma^{-1}(P)} \|f_k \mid Q\|^p \qquad \text{by Lemma 6.28} \\ &\leq \sum_{P \in \Pi} \sum_{Q \in \Pi_\sigma^{-1}(P)} \max_{k \in \mathbb{N}_n} \|f_k \mid Q\|^p \\ &= \sum_{Q \in \Pi} \sum_{P \in \Pi_\sigma(Q)} \max_{k \in \mathbb{N}_n} \|f_k \mid Q\|^p \\ &\leq \sum_{Q \in \Pi} |\Pi_\sigma(Q)| \max_{k \in \mathbb{N}_n} \|f_k \mid Q\|^p \\ &\leq m_\sigma \left(\|(f_1, \dots, f_n)\|_n^{\Pi}\right)^p,\end{aligned}$$

and so $T_\sigma \in \mathcal{M}(\ell^p)$ with $\|T\|_{mb} \leq m_\sigma^{1/p}$.

We continue to suppose that $m_\sigma < \infty$, say $k = m_\sigma \in \mathbb{N}$. Then there exists $P \in \Pi$ and k pairwise-disjoint sets $Q_1, \dots, Q_k \in \Pi_\sigma(P)$. For each $j \in \mathbb{N}_k$, choose $n_j \in Q_j$ with $\sigma(n_j) \in P$, and set $f_j = \delta_{\sigma(n_j)}$. Then

$$\|(f_1, \dots, f_k)\|_k^\Pi = \max_{j \in \mathbb{N}_k} \|f_j \mid P\| = 1$$

and $T_\sigma f_j = \delta_{n_j}$, so that $\|(T_\sigma f_j) \mid Q_j\| = 1$ and $\|(T_\sigma f_j) \mid Q\| = 0$ for $Q \in \Pi$ with $Q \neq Q_j$. Thus $\|(T_\sigma f_1, \dots, T_\sigma f_n)\|_n^\Pi = k^{1/p}$.

It follows that $\|T\|_{mb} = m_\sigma^{1/p}$ in the case where $m_\sigma < \infty$.

In the case where we have $m_\sigma = \infty$, the argument of the last paragraph shows that, for each $k \in \mathbb{N}$, there exist $f_1, \dots, f_k \in \ell^p$ such that $\|(f_1, \dots, f_k)\|_k^\Pi = 1$ and $\|(T_\sigma f_1, \dots, T_\sigma f_k)\|_k^\Pi = k^{1/p}$, and so T is not multi-bounded. ∎

The next example shows a failure of the 'Banach isomorphism theorem for multi-normed spaces' in the special case where the two multi-norms are equal.

Example 6.30. Let Π be a partition of $\mathbb{N}$ into infinitely many infinite subsets, say $P_1, P_2, \dots$, where the sets P_j are distinct. Take

$$Q_0 = P_1 \cup P_3 \cup P_5 \cup \cdots \quad \text{and} \quad Q_j = P_{2j} \quad (j \in \mathbb{N}),$$

so that $\{Q_0, Q_1, Q_2, \dots\}$ is also a partition of $\mathbb{N}$ into infinite sets. For each $k \in \mathbb{N}$, let $\sigma_k : P_k \to Q_{k-1}$ be a bijection, and define $\sigma \in \mathfrak{S}_\mathbb{N}$ by setting $\sigma(n) = \sigma_k(n)$ when $n \in P_k$.

Consider the partition multi-norm $(\|\cdot\|_n^\Pi : n \in \mathbb{N})$ based on ℓ^1, and set $T = T_\sigma$ in the above notation. For each $P_i \in \Pi$, we have $\Pi_\sigma(P_i) = \{P_i\}$, and so $|\Pi_\sigma(P_i)| = 1$. By Theorem 6.29, $T \in \mathcal{M}(\ell^1)$ with $\|T\|_{mb} = 1$. On the other hand, $T^{-1} = T_{\sigma^{-1}} \in \mathcal{B}(\ell^1)$, and

$$\Pi_{\sigma^{-1}}(P_1) = \{P_j : Q_0 \cap P_j \neq \emptyset\} = \{P_1, P_3, P_5, \dots\},$$

an infinite set, so that T^{-1} is not multi-bounded. ∎

6.4. Multi-bounded operators on Banach lattices. Our next aim is to identify the space $\mathcal{M}(E, F)$ of multi-bounded operators in the case where E and F are Banach lattices. Throughout this section, we are taking the lattice multi-norms $(\|\cdot\|_n^L : n \in \mathbb{N})$ of Definition 4.43 as the multi-norms on both of the families $\{E^n : n \in \mathbb{N}\}$ and $\{F^n : n \in \mathbb{N}\}$.

6.4.1. Multi-bounded and order-bounded operators. Let E and F be Banach lattices. Recall that the space $\mathcal{B}_b(E, F)$ of order-bounded operators from E to F and the norm $\|T\|_b$ of $T \in \mathcal{B}_b(E, F)$ were defined in §1.3.4. In this subsection, we shall compare $\mathcal{B}_b(E, F)$ with $\mathcal{M}(E, F)$.

Theorem 6.31. *Let E and F be Banach lattices. Then each order-bounded operator T from E to F is multi-bounded, and $\|T\|_{mb} \leq \|T\|_b$.*

Proof. Let $T \in \mathcal{B}_b(E, F)$, and suppose that $B \in \mathcal{MB}(E)$. Now take $x_1, \dots, x_n \in B$, and set $v = |x_1| \vee \cdots \vee |x_n|$, so that $\|v\| \leq c_B$. For each $x \in \Delta_v$ and $\varepsilon > 0$, there exists $w \in F$ such that

$$|Tx| \leq w \quad \text{and} \quad \|w\| < \|T\|_b \|v\| + \varepsilon.$$

For $i \in \mathbb{N}_n$, we have $|Tx_i| \leq w$, and so $|Tx_1| \vee \cdots \vee |Tx_n| \leq w$. Thus

$$\|(Tx_1, \ldots, Tx_n)\|_n^L = \| \, |Tx_1| \vee \cdots \vee |Tx_n| \, \| \leq \|w\| < \|T\|_b c_B + \varepsilon .$$

This holds true for each $\varepsilon > 0$, and so $T \in \mathcal{B}_b(E, F)$ with $c_{T(B)} \leq \|T\|_b c_B$. Thus $T \in \mathcal{M}(E, F)$ with $\|T\|_{mb} \leq \|T\|_b$, as claimed. ∎

COROLLARY 6.32. *Let E and F be Banach lattices, and let $T \in \mathcal{B}(E, F)^+$. Then*

$$\|T\|_r = \|T\|_b = \|T\|_{mb} = \|T\| .$$

Proof. Always $\|T\| \leq \|T\|_{mb}$. By the theorem, $\|T\|_{mb} \leq \|T\|_b$. But $\|T\|_b = \|T\|_r = \|T\|$ for positive operators T by (1.29). ∎

The present formulation of the following result is due to Michael Elliott.

THEOREM 6.33. *Let E and F be Banach lattices.*

(i) *Suppose that F is monotonically bounded. Then $\mathcal{B}_b(E, F) = \mathcal{M}(E, F)$ and $\| \cdot \|_{mb}$ and $\| \cdot \|_b$ are equivalent on $\mathcal{B}_b(E, F)$.*

(ii) *Suppose that F has the weak Nakano property. Then $\| \cdot \|_{mb}$ and $\| \cdot \|_b$ are equivalent on $\mathcal{B}_b(E, F)$, with equality of norms when F has the Nakano property.*

(iii) *Suppose that F is monotonically bounded and has the Nakano property. Then*

$$\mathcal{B}_b(E, F) = \mathcal{M}(E, F) \quad \textit{and} \quad \|T\|_{mb} = \|T\|_b \quad (T \in \mathcal{B}_b(E, F)).$$

(iv) *Suppose that F is monotonically bounded and Dedekind complete. Then*

$$\mathcal{B}_r(E, F) = \mathcal{B}_b(E, F) = \mathcal{M}(E, F)$$

and $\| \cdot \|_{mb}$ and $\| \cdot \|_r$ are equivalent on $\mathcal{B}_r(E, F)$, with equality of norms when F has the Nakano property.

Proof. Let $T \in \mathcal{B}(E, F)$. Suppose that $T \in \mathcal{B}_b(E, F)$. Then it follows from Theorem 6.31 that $T \in \mathcal{M}(E, F)$ with $\|T\|_{mb} \leq \|T\|_b$.

(i) Suppose that $T \in \mathcal{M}(E, F)$, and take an order-bounded subset B of E. By Proposition 6.6, $B \in \mathcal{MB}(E)$, and so $T(B) \in \mathcal{MB}(F)$. Since F is monotonically bounded, it follows from Theorem 6.8 that $T(B)$ is order-bounded, and so $T \in \mathcal{B}_b(E, F)$. Thus $\mathcal{M}(E, F) = \mathcal{B}_b(E, F)$. Since $(\mathcal{M}(E, F), \| \cdot \|_{mb})$ and $(\mathcal{B}_b(E, F), \| \cdot \|_b)$ are Banach spaces with their norms dominating the operator norm, the equivalence of the norms follows from the closed graph theorem.

(ii) Suppose that $T \in \mathcal{B}_b(E, F)$. Fix $\varepsilon > 0$, and take $v \in E^+_{[1]}$.

The set $B := \Delta_v$ is order-bounded, and so $B \in \mathcal{MB}(E)$ with $c_B \leq 1$, as in Proposition 6.6. Take $\mathcal{F} = \mathcal{P}_f(T(B))$, and set $y_S = \bigvee \{|y| : y \in S\}$ for $S \in \mathcal{F}$. Then $\{y_S : S \in \mathcal{F}\}$ is an increasing net in F^+ such that $\|y_S\| \leq \|T\|_{mb}$ $(S \in \mathcal{F})$.

The set $T(B)$ is order-bounded, and so $\{y_S : S \in \mathcal{F}\}$ is order-bounded. Take $\varepsilon > 0$. Since F has the weak Nakano property, there exist $K \geq 1$ and $u \in F_{\mathbb{R}}$ such that

$$y_S \leq u \quad (S \in \mathcal{F}) \quad \text{and} \quad \|u\| \leq K \sup_{S \in \mathcal{F}} \|y_S\| + \varepsilon \leq K \|T\|_{mb} + \varepsilon .$$

It follows that $\|T\|_b \leq K\|T\|_{mb} + \varepsilon$.

This holds true for each $\varepsilon > 0$, and so $\|T\|_b \leq K\|T\|_{mb}$. The result follows.

(iii) This follows immediately from (i) and (ii).

(iv) By Theorem 1.31, $\mathcal{B}_b(E,F) = \mathcal{B}_r(E,F)$ and $\|T\|_r = \|T\|_b$ for each $T \in \mathcal{B}_b(E,F)$, and so the result follows from (i) and (iii). ∎

Corollary 6.34. *Let E be a Banach lattices, and let $F = L^p(\Omega)$ for a measure space Ω and $p \geq 1$. Then $\mathcal{B}_r(E,F) = \mathcal{B}_b(E,F) = \mathcal{M}(E,F)$ and*

$$\|T\|_{mb} = \|T\|_r = \|T\|_b = \|\,|T|\,\| \qquad (T \in \mathcal{B}_r(E,F)).$$

Proof. The hypotheses on F in Theorem 6.33(iv) are satisfied by every monotonically complete Banach lattice with order-continuous norm, and hence by the lattices $L^p(\Omega)$. ∎

In the case where $E = F = L^p(\Omega)$, $p > 1$, and $L^p(\Omega)$ is infinite-dimensional, it follows from Theorem 1.30(iii) that $\mathcal{M}(E,F)$ is not dense in $\mathcal{B}(E,F)$. We are grateful to Anthony Wickstead for the following remarks. First, let $p,q \in [1,\infty]$. Then $\mathcal{B}_r(\ell^p,\ell^q) \neq \mathcal{B}(\ell^p,\ell^q)$ whenever either $p > 1$ or $q < \infty$, and so, in the latter case, $\mathcal{M}(\ell^p,\ell^q) \neq \mathcal{B}(\ell^p,\ell^q)$. Second, suppose that $1 \leq q < p < \infty$. Then it follows from Pitt's theorem [6, Theorem 2.1.4] that $\mathcal{K}(\ell^p,\ell^q) = \mathcal{B}(\ell^p,\ell^q)$, and so $\mathcal{M}(\ell^p,\ell^q) \subsetneq \mathcal{K}(\ell^p,\ell^q)$ in this case.

The following easy example shows that 'monotonically bounded' is not redundant in Theorem 6.33, (i), (iii), and (iv).

Example 6.35. Take $E = c = c_0 \oplus \mathbb{C}1$, where $1 = (1,1,\dots)$, and $F = c_0$. Then F is Dedekind complete and has the Nakano property, but it is not monotonically bounded. By Theorem 4.54(ii), the lattice multi-norm on the AM-space F is the minimum multi-norm, and so, by Theorem 6.17, we have $\mathcal{M}(E,F) = \mathcal{B}(E,F)$ and $\|T\|_{mb} = \|T\|$ $(T \in \mathcal{B}(E,F))$.

Consider the map

$$T : \alpha + z1 \mapsto \alpha, \qquad E \to F.$$

Then $T \in \mathcal{B}(E,F)$ with $\|T\| = 2$, but T is not order-bounded. For set $\alpha_n = \sum_{i=n}^{\infty} \delta_i \in E$, so that $\{\alpha_n : n \in \mathbb{N}\}$ is order-bounded. However, $|T(\alpha_n)| = \sum_{i=1}^{n-1} \delta_i \in E$, so that the set $\{T\alpha_n : n \in \mathbb{N}\}$ is not order-bounded in F. ∎

The following example, also due to Michael Elliott, shows that 'has the weak Nakano property' is not redundant in Theorem 6.33(ii), even when F is Dedekind complete.

Let (R_n) denote the sequence of Rademacher functions on $\mathbb{I}$. Thus

$$R_1 = \chi_{[0,1/2]} - \chi_{(1/2,1]}, \qquad R_2 = \chi_{[0,1/4]} - \chi_{(1/4,1/2]} + \chi_{(1/2,3/4]} - \chi_{(3/4,1]},$$

etc.; we regard these functionals as elements of the dual space $L^\infty(\mathbb{I})$ of $L^1(\mathbb{I})$.

We *claim* that, for each $f \in L^1(\mathbb{I})$, the sequence $(\langle f,\, R_n\rangle : n \in \mathbb{N})$ is a null sequence. Indeed, first suppose that $f = \chi_{[a,b]}$ for $0 \leq a \leq b \leq 1$. Then $|\langle f,\, R_n\rangle| \leq 1/2^{n-1}$ $(n \in \mathbb{N})$, so that the claim holds in this case. Hence it holds for each simple function f, and then for each $f \in L^1(\mathbb{I})$ because the simple functions are dense in $L^1(\mathbb{I})$. It follows from Proposition 6.7 that

$$\sum_{n=1}^{\infty} |\langle f,\, R_n\rangle| y_n$$

is convergent in E for each pairwise-disjoint, multi-bounded sequence (y_n) in a Banach lattice.

LEMMA 6.36. *Let E be the Banach lattice $L^1(\mathbb{I})$, and let F be any Banach lattice. Suppose that (y_n) is a pairwise-disjoint, multi-bounded sequence in F^+, and define*

$$T : f \mapsto \sum_{n=1}^{\infty} \langle f,\, R_n \rangle y_n, \qquad E \to F.$$

Then $T \in \mathcal{M}(E, F)$ with $\|T\|_{mb} \leq \|(y_n)\|_{mb}$.

Proof. Note that, using (1.26), we have

$$\|y_1 + \cdots + y_n\| = \|y_1 \vee \cdots \vee y_n\| = \|(y_1, \dots, y_n)\|_n^L \quad (n \in \mathbb{N}),$$

and so

$$\|(y_n)\|_{mb} = \sup\{\|y_1 + \cdots + y_n\| : n \in \mathbb{N}\}.$$

Let $B \in \mathcal{MB}(E)$ with $c_B \leq 1$, so that $B \subset E_{[1]}$, and take $\{z_1, \dots, z_k\}$ to be a finite subset of $T(B)$. For each $j \in \mathbb{N}_k$, choose $f_j \in B$ with $Tf_j = z_j$, and then, for $n \in \mathbb{N}$, define the numbers

$$s_n = |\langle f_1, R_n \rangle| \vee \cdots \vee |\langle f_k, R_n \rangle|, \qquad t_n = |\langle f_1, R_n \rangle| + \cdots + |\langle f_k, R_n \rangle|.$$

Fix $\varepsilon > 0$. For each $j \in \mathbb{N}_k$, there exists $i \in \mathbb{N}$ such that

$$\left\| \sum_{n=i}^{\infty} |\langle f_j,\, R_n \rangle| y_n \right\| < \frac{\varepsilon}{k} \quad (j \in \mathbb{N}_k),$$

and so

$$\left\| \sum_{n=i}^{\infty} s_n y_n \right\| \leq \left\| \sum_{n=i}^{\infty} t_n y_n \right\| < \varepsilon.$$

However, $|z_1| \vee \cdots \vee |z_k| = |Tf_1| \vee \cdots \vee |Tf_k| = \sum_{n=1}^{\infty} s_n y_n$, and so

$$\|\, |z_1| \vee \cdots \vee |z_k| \,\| \leq \left\| \sum_{n=1}^{i-1} y_n \right\| + \varepsilon = \|(y_1, \dots, y_{i-1})\|_{i-1}^L + \varepsilon \leq \|(y_n)\|_{mb} + \varepsilon.$$

Thus

$$\|(z_1, \dots, z_k)\|_k^L \leq \sup\{\|y_1 + \cdots + y_n\| : n \in \mathbb{N}\} + \varepsilon.$$

This holds true for each $\varepsilon > 0$, and so

$$\|(z_1, \dots, z_k)\|_k^L \leq \sup\{\|y_1 + \cdots + y_n\| : n \in \mathbb{N}\}.$$

Hence the result follows. ∎

THEOREM 6.37. *Let E be the Banach lattice $L^1(\mathbb{I})$, and let F be any Dedekind complete lattice. Then F has the weak σ-Nakano property if and only if $\|\cdot\|_{mb}$ is equivalent to $\|\cdot\|_r$ on $\mathcal{B}_r(E, F)$.*

Proof. Recall from Theorem 1.31 that $\mathcal{B}_r(E, F) = \mathcal{B}_b(E, F)$ and that $\|T\|_b = \|T\|_r$ for $T \in \mathcal{B}_r(E, F)$. Thus, when F has the weak Nakano property, the norms $\|\cdot\|_{mb}$ and $\|\cdot\|_r$ are equivalent on $\mathcal{B}_r(E, F)$ by Theorem 6.33(ii); since E is separable, a trivial variation of the argument shows this when F has just the weak σ-Nakano property.

Conversely, suppose that $K \geq 1$ with $\|T\|_r \leq K\|T\|_{mb}$ $(T \in \mathcal{B}_r(E, F))$.

Let (x_n) be an increasing, order-bounded sequence in $E_{\mathbb{R}}$. Since F is Dedekind complete, the set $\{x_n : n \in \mathbb{N}\}$ has a supremum, say y.

Define $y_1 = x_1$ and $y_n = x_n - x_{n-1}$ for $n \geq 2$, so that (y_n) is pairwise-disjoint and $y_1+\cdots+y_n = x_n$ ($n \in \mathbb{N}$). The sequence (y_n) is order-bounded, and so, by Proposition 6.6, (y_n) is multi-bounded. Thus Lemma 6.36 applies to the sequence (y_n) and the operator T defined in that lemma. The operator T is bounded above by the positive operator

$$S : f \mapsto \langle f,\, 1\rangle y, \qquad E \to F,$$

and so $T \in \mathcal{B}_r(E, F)$; clearly $S = |T|$, so that $\|T\|_r = \|S\| = \|y\|$.

By Lemma 6.36, $T \in \mathcal{M}(E, F)$ with $\|T\|_{mb} \leq \sup_{n\in\mathbb{N}} \|x_n\|$. It follows that

$$\|y\| \leq K \sup_{n\in\mathbb{N}} \|x_n\|,$$

and so F has the weak σ-Nakano property. ∎

An example of a Dedekind complete Banach lattice without the weak σ-Nakano property was given in Example 1.24.

We now note that, even in the case where E is a monotonically complete lattice with the Nakano property, it is not necessarily the case that every compact operator on E is multi-bounded.

EXAMPLE 6.38. Let $n \in \mathbb{N}$. Essentially as in [7, Example 16.6], there is $T_n \in \mathbb{M}_{2^n}(\mathbb{C})$ with $\|T_n\| = 1$ and $\|\,|T_n|\,\| = 2^{n/2}$ (where $\mathbb{C}^n$ has the Euclidean norm). Let E be the ℓ^2-sum of the spaces $(\mathbb{C}^n, \|\cdot\|_2)$ (not the c_0-sum given in [7]). Then E is a KB-space, and so satisfies the conditions on F in Theorem 6.33(iv). Let

$$T((x_n)) = (2^{-n/3}T_n x_n) \qquad ((x_n) \in E).$$

Then, as in [7], $T \in \mathcal{K}(E)$, but T is not regular. Thus $T \in \mathcal{K}(E) \setminus \mathcal{M}(E)$.

As remarked in [7, Example 16.6], a compact operator need not have a modulus, and a compact operator can have a modulus that is not compact (see also [4]). ∎

In Examples 6.25 and 6.30, we showed that the multi-bounded version of Banach's isomorphism theorem might fail. We now give another example of this failure; it applies even in the special case when we consider one Banach lattice and the lattice multi-norm.

EXAMPLE 6.39. Let E be the Banach lattice $L^2(\mathbb{T})$, and consider the lattice multi-norm based on E. By Corollary 6.34, $\mathcal{B}_r(E) = \mathcal{M}(E)$.

As in Example 1.34, there exists $T \in \mathcal{K}(E) \cap \mathcal{B}_r(E)$ with $\sigma_o(T) \supsetneq \sigma(T)$; choose $z \in \sigma_o(T) \setminus \sigma(T)$. Then $zI_E - T \in \mathcal{M}(E)$ and $zI_E - T$ is invertible in $\mathcal{B}(E)$, so that $zI_E - T : E \to E$ is a linear isomorphism. However, $zI_E - T$ is not invertible in the Banach algebra $\mathcal{M}(E)$. ∎

We now enquire when we have $\mathcal{M}(E, F) = \mathcal{B}(E, F)$.

THEOREM 6.40. *Let E and F be Banach lattices. Suppose that either E is an AL-space or that F is an AM-space. Then $\mathcal{M}(E, F) = \mathcal{B}(E, F)$ and $\|T\|_{mb} = \|T\|$ ($T \in \mathcal{B}(E, F)$).*

Proof. In the two cases, by Theorem 4.54, the lattice multi-norms based on E and F are the maximum and minimum multi-norms, respectively. The result now follows from Theorem 6.17. ∎

Corollary 6.41. *Let E and F be Banach lattices. Suppose that F is a Dedekind complete AM-space with an order-unit. Then*

$$\mathcal{B}_r(E,F) = \mathcal{B}_b(E,F) = \mathcal{M}(E,F) = \mathcal{B}(E,F)$$

and $\|T\|_r = \|T\|_b = \|T\|_{mb} = \|T\|$ $(T \in \mathcal{B}(E,F)$.

Proof. This follows from Theorems 6.33(iv) and 6.40, where we note that an AM-space with an order-unit is monotonically bounded and has the Nakano property whenever it is Dedekind complete. ■

6.4.2. The multi-bounded multi-norm. We shall now extend Theorem 6.33 by considering the multi-bounded multi-norm $(\|\cdot\|_n^{mb} : n \in \mathbb{N})$. We shall show that, for all Banach lattices E and suitable Banach lattices F, the multi-norm based on $\mathcal{M}(E,F)$ is not greater than the lattice multi-norm, with equality when E is the space ℓ^1. However, an example will show that these multi-norms are not necessarily equivalent when $E = \ell^p$ for $p > 1$.

We first note the following formula. Let E and F be Banach lattices, and take $T_1, \dots, T_n \in \mathcal{M}(E,F)$. Then it follows from (6.4) that

$$\|(T_1, \dots, T_n)\|_n^{mb} = \sup\left\{\left\|\bigvee\{|T_i x_j| : i \in \mathbb{N}_n,\, j \in \mathbb{N}_k\}\right\|\right\}, \tag{6.5}$$

where the supremum is taken over all $x_1, \dots, x_k \in E$ with $\|\,|x_1| \vee \cdots \vee |x_k|\,\| \le 1$.

Recall from Theorem 6.33(iv) that $\mathcal{B}_r(E,F) = \mathcal{B}_b(E,F) = \mathcal{M}(E,F)$, with equality of norms, whenever F is Dedekind complete, monotonically bounded, and has the Nakano property, and so $\mathcal{M}(E,F)$ is a Banach lattice with respect to the lattice multi-norm $(\|\cdot\|_n^L : n \in \mathbb{N})$ in this case.

Theorem 6.42. *Let E and F be Banach lattices such that F is Dedekind complete, monotonically bounded, and has the Nakano property. Let $T_1, \dots, T_n \in \mathcal{M}(E,F)$. Then*

$$\|(T_1, \dots, T_n)\|_n^{mb} \le \|\,|T_1| \vee \cdots \vee |T_n|\,\| = \|(T_1, \dots, T_n)\|_n^L. \tag{6.6}$$

Proof. We set $T = |T_1| \vee \cdots \vee |T_n|$. Take $x_1, \dots, x_k \in E$ and set $x = |x_1| \vee \cdots \vee |x_k|$, so that $\|x\| \le 1$. Since $|x_j| \le x$ $(j \in \mathbb{N}_k)$ and $|T_i| \le T$ $(i \in \mathbb{N}_n)$, it follows from Theorem 1.31 that

$$|T_i x_j| \le |T_i|(|x_j|) \le |T_i|(x) \le Tx \quad (i \in \mathbb{N}_n,\, j \in \mathbb{N}_k),$$

and so

$$\left\|\bigvee\{|T_i x_j| : i \in \mathbb{N}_n,\, j \in \mathbb{N}_k\}\right\| \le \|Tx\| \le \|T\|.$$

By (6.5), $\|(T_1, \dots, T_n)\|_n^{mb} \le \|T\|$, as required. ■

Theorem 6.43. *Let E be the Banach lattice ℓ^1, and suppose that F is a Dedekind complete, monotonically bounded Banach lattice with the Nakano property. Take $n \in \mathbb{N}$ and $T_1, \dots, T_n \in \mathcal{M}(E,F)$. Then*

$$\|(T_1, \dots, T_n)\|_n^{mb} = \|\,|T_1| \vee \cdots \vee |T_n|\,\|. \tag{6.7}$$

Proof. Set $T = |T_1| \vee \cdots \vee |T_n| \in \mathcal{B}(E,F)^+$. By Theorem 6.42, $\|(T_1, \dots, T_n)\|_n^{mb} \le \|T\|$; we must prove the opposite inequality.

We know that $\|T\| = \sup_{i\in\mathbb{N}} \|T(\delta_i)\|$. Take $i \in \mathbb{N}$. Then the only way that we can write δ_i as $f_1 + \cdots + f_n$, where $f_1, \dots, f_n \in (\ell^1)^+$ is to take $f_j = \alpha_j \delta_i$, where $\alpha_1, \dots, \alpha_n \in \mathbb{I}$ and $\alpha_1 + \cdots + \alpha_n = 1$. In this case,

$$|T_1|(f_1) + \cdots + |T_n|(f_n) = \alpha_1|T_1|(\delta_i) + \cdots + \alpha_n|T_n|(\delta_i) \leq |T_1|(\delta_i) \vee \cdots \vee |T_n|(\delta_i)$$

using Proposition 1.18(v), and so $T(\delta_i) = |T_1|(\delta_i) \vee \cdots \vee |T_n|(\delta_i)$ by (1.30). Thus we have $\|T\| = \sup_{i\in\mathbb{N}} \|\, |T_1|(\delta_i) \vee \cdots \vee |T_n|(\delta_i)\|$. However, by (6.5),

$$\|(T_1, \dots, T_n)\|_n^{mb} \geq \|\, |T_1|(\delta_i) \vee \cdots \vee |T_n|(\delta_i)\, \| \quad (i \in \mathbb{N}),$$

and so $\|T\| \leq \|(T_1, \dots, T_n)\|_n^{mb}$, as required. ∎

THEOREM 6.44. *Let E be an AM-space, and let F be an AL-space. Then the lattice multi-norm on $\mathcal{B}_b(E,F)$ is the maximum multi-norm.*

Proof. By [7, Exercise 15.3, p. 263], the Banach lattice $\mathcal{B}_b(E,F)$ is an AL-space. Thus the result follows from Theorem 4.54(i). ∎

EXAMPLE 6.45. We take $E = \ell^p$ and $F = \ell^q$, where $p, q \geq 1$. For $n \in \mathbb{N}$, set

$$e_n = \sum_{j=1}^{n} \delta_j = (1, \dots, 1, 0, \dots).$$

For $j \in \mathbb{N}_n$, we define $T_j : (\alpha_i) \mapsto \alpha_j e_n$, $E \to F$, so that $T_j \geq 0$ and

$$\|T_j\| = \|e_n\|_{\ell^q} = n^{1/q}.$$

Set $T = T_1 \vee \cdots \vee T_n$. Then, using (1.30), we see that

$$T(e_n) \geq \sum_{j=1}^{n} T_j(\delta_j) = n e_n,$$

and so $\|T\| \geq n \cdot n^{1/q-1/p} = n^{1+1/q-1/p}$.

Now take $x_1, \dots, x_k \in E$ with $\|\, |x_1| \vee \cdots \vee |x_k|\, \| \leq 1$. Then each component of each x_j has modulus at most 1, and so $|T_i x_j| \leq e_n$ for $i \in \mathbb{N}_n$ and $j \in \mathbb{N}_k$. By (6.5), $\|(T_1, \dots, T_n)\|_n^{mb} \leq \|e_n\|_{\ell^q} = n^{1/q}$, and so

$$\|T\| = \|(T_1, \dots, T_n)\|_n^{L} \geq n^{1-1/p} \|(T_1, \dots, T_n)\|_n^{mb}.$$

This shows that the multi-norms $(\|\cdot\|_n^{mb} : n \in \mathbb{N})$ and $(\|\cdot\|_n^{L} : n \in \mathbb{N})$ based on $\mathcal{B}_b(E,F)$ are not equivalent whenever $p > 1$. ∎

6.5. Extensions of multi-norms. In this section, we shall show how to take various extensions of multi-norms.

6.5.1. Definitions. Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space, and let $\mathcal{F}$ be a fixed family in $\mathcal{B}(E)_{[1]}$ such that $I_E \in \mathcal{F}$. Then we can define a multi-norm structure on $\{E^n : n \in \mathbb{N}\}$ by using $\mathcal{F}$: indeed, for $n \in \mathbb{N}$ and $x_1, \dots, x_n \in E$, set

$$|\!|\!|(x_1, \dots, x_n)|\!|\!|_n = \sup\{\|(Tx_1, \dots, Tx_n)\|_n : T \in \mathcal{F}\}. \tag{6.8}$$

We see that $((E^n, |\!|\!|\cdot|\!|\!|_n) : n \in \mathbb{N})$ is a multi-normed space and that

$$|\!|\!|x|\!|\!|_n \geq \|x\|_n \quad (x \in E^n,\ n \in \mathbb{N});$$

it is the *extension* of the given multi-norm by $\mathcal{F}$.

In particular, let us take $\mathcal{F}$ to be the family $\mathcal{B}(E)_{[1]}$ or the family of all isometric isomorphisms on E. The multi-normed structure that we obtain is the *balanced extension* or *isometric extension*, respectively.

DEFINITION 6.46. A multi-normed space $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is:

(i) *balanced* if $\|T\|_{mb} = \|T\|$ $(T \in \mathcal{B}(E))$;
(ii) *isometric* if $\|T\|_{mb} = 1$ for each isometric isomorphism $T \in \mathcal{B}(E)$.

Thus $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is balanced if and only if $(\mathcal{M}(E), \|\cdot\|_{mb})$ is isometrically isomorphic to $(\mathcal{B}(E), \|\cdot\|)$; since $\|T\| \leq \|T\|_{mb}$ $(T \in \mathcal{M}(E))$, this holds if and only if, for each $T \in \mathcal{B}(E)$ and $n \in \mathbb{N}$, we have

$$\|(Tx_1, \ldots, Tx_n)\|_n \leq \|T\| \, \|(x_1, \ldots, x_n)\|_n \quad (x_1, \ldots, x_n \in E).$$

Clearly a balanced multi-norm is isometric, and the balanced or isometric extension of a multi-norm is balanced or isometric, respectively.

6.5.2. Examples of balanced multi-normed spaces

EXAMPLE 6.47. Let E be any normed space, and let

$$(\|\cdot\|_n^{\min} : n \in \mathbb{N}) \quad \text{and} \quad (\|\cdot\|_n^{\max} : n \in \mathbb{N})$$

be the minimum and maximum multi-norms on the family $\{E^n : n \in \mathbb{N}\}$, respectively. Then it follows from Theorem 6.17 that both these multi-normed spaces are balanced. ■

EXAMPLE 6.48. Let E be a normed space, and take $1 \leq p \leq q < \infty$. Consider the (p,q)-multi-norm $(\|\cdot\|_n^{(p,q)} : n \in \mathbb{N})$ based on E. Take $n \in \mathbb{N}$.

For each $T \in \mathcal{B}(E)_{[1]}$, we have $\|T'\| \leq 1$, and so, by (3.3),

$$\mu_{p,n}(T'\lambda_1, \ldots, T'\lambda_n) \leq \mu_{p,n}(\lambda_1, \ldots, \lambda_n) \quad ((\lambda_1, \ldots, \lambda_n) \in (E')^n).$$

Let $(x_1, \ldots, x_n) \in E^n$. Since $|\langle Tx_i, \lambda_i\rangle| = |\langle x_i, T'\lambda_i\rangle|$ $(i \in \mathbb{N}_n)$, it follows from (4.1) that $\|(Tx_1, \ldots, Tx_n)\|_n^{(p,q)} \leq \|(x_1, \ldots, x_n)\|_n^{(p,q)}$. Thus $((E^n, \|\cdot\|_n^{(p,q)}) : n \in \mathbb{N})$ is a balanced multi-normed space. ■

The following result is a special case of [19, Proposition 7.3].

THEOREM 6.49. *Let (Ω, μ) be a measure space, and suppose that $1 \leq p \leq q < \infty$ and $L^p(\Omega, \mu)$ is infinite-dimensional. Then the balanced extension of the standard q-multi-norm based on $L^p(\Omega, \mu)$ is the (p,q)-multi-norm.* ■

6.5.3. Examples of isometric multi-normed spaces. We now consider when some examples of multi-normed spaces are isometric.

THEOREM 6.50. *Let (Ω, μ) be a measure space, and and suppose that $1 \leq p \leq q < \infty$ with $p \neq 2$. Then the standard q-multi-norm based on $L^p(\Omega, \mu)$ is isometric.*

Proof. Let U be an isometric isomorphism on $L^p(\Omega, \mu)$. Since $p \neq 2$, U has the form of (1.20), where σ is a regular set isomorphism on Ω and

$$\int_{\sigma(X)} |h|^p \, \mathrm{d}\mu_2 = \mu_1(X)$$

for each measurable subset X of Ω.

For $n \in \mathbb{N}$, let $\mathbf{X} = (X_1, \dots, X_n)$ be an ordered partition of Ω, and define

$$Y_j = \sigma^{-1}(X_j) \quad (j \in \mathbb{N}_n).$$

Then clearly $\mathbf{Y} = (Y_1, \dots, Y_n)$ is an ordered partition of Ω. For each $j \in \mathbb{N}_n$ and a measurable subset X of Ω, we have

$$\begin{aligned} \int_{X_j} |U\chi_X|^p &= \int_\Omega \chi_{X_j} |h|^p \chi_{\sigma(X)} = \int_\Omega |h|^p \chi_{\sigma(X\cap Y_j)} \\ &= \int_\Omega |U\chi_{X\cap Y_j}|^p = \int_\Omega \chi_{X\cap Y_j} = \int_{Y_j} \chi_X, \end{aligned}$$

and so $\int_{X_j} |Uf|^p = \int_{Y_j} |f|^p$ for all $f \in L^p(\Omega, \mu)$. Take $f_1, \dots, f_n \in L^p(\Omega, \mu)$. Then $r_{\mathbf{X}}((Uf_1, \dots, Uf_n)) = r_{\mathbf{Y}}((f_1, \dots, f_n))$.

It follows from the definition in (4.7) that

$$\|(Uf_1, \dots, Uf_n)\|_n^{[q]} = \|(f_1, \dots, f_n)\|_n^{[q]},$$

and hence we obtain an isometric multi-norm. ∎

Example 6.51. In this example, we shall show that the constraint that $p \neq 2$ in Theorem 6.50 is necessary.

Set $H = \ell^2$, and consider Example 6.24. In that example, we obtained an orthonormal subset $S = \{x_r^s : r \in \mathbb{N}_s, s \in \mathbb{N}\}$ of H. As before, enumerate S as a sequence (y_n), and now choose a sequence $T = (z_n)$ in H such that $S \cup T$ is an orthonormal basis of H. Define a bounded linear operator $U \in \mathcal{B}(H)$ by requiring that

$$Uy_n = \delta_{2n}, \quad Uz_n = \delta_{2n-1} \quad (n \in \mathbb{N}).$$

Clearly, U is an isometric isomorphism on H.

Consider the standard 2-multi-norm on $\{H^n : n \in \mathbb{N}\}$. As in Example 6.24 (in the elementary case where $\alpha_i = 1$ $(i \in \mathbb{N})$), U is not even a multi-bounded map with respect to this multi-norm. ∎

7. Orthogonality and duality

In this final chapter, we shall discuss a notion of orthogonality in multi-normed spaces; we are seeking a theory of orthogonality involving multi-norms that extends the classical notions of orthogonality in Hilbert spaces and Banach lattices to more general Banach spaces. These ideas will be used to define the multi-dual of a multi-normed space; our motivation is to try to establish a satisfactory duality theory for general multi-normed spaces.

A 'test question' for our approach is the following. Let $E = L^p(\Omega)$, where Ω is a measure space and $1 < p < \infty$, and let $\{E^n : n \in \mathbb{N}\}$ have the standard p-multi-norm $(\|\cdot\|_n^{[p]} : n \in \mathbb{N})$ of Definition 4.21. Let q be the conjugate index to p, and set $F = E' = L^q(\Omega)$. Then we expect that the 'multi-dual' of the multi-normed space $((E^n, \|\cdot\|_n^{[p]}) : n \in \mathbb{N})$ should be $((F^n, \|\cdot\|_n^{[q]}) : n \in \mathbb{N})$, and hence that

$$((E^n, \|\cdot\|_n^{[p]}) : n \in \mathbb{N})$$

is 'multi-reflexive'. We also expect that the 'multi-dual' of the lattice multi-norm on the multi-normed space $((E^n, \|\cdot\|_n^L) : n \in \mathbb{N})$, where E is a Banach lattice, will be the lattice multi-norm on $\{(E')^n : n \in \mathbb{N}\}$. We should formulate the notion of 'multi-dual' to achieve these aims. This seems to be not completely straightforward.

In this chapter, we consider Banach spaces over only the complex field.

7.1. Decompositions. We recall that the notion of a direct sum decomposition of a Banach space was given §1.2.3; this included the notion of a 'closed family of decompositions'.

7.1.1. Hermitian decompositions of a normed space. The first decomposition that we consider is essentially known, and does not involve multi-norms.

DEFINITION 7.1. Let $E = E_1 \oplus \cdots \oplus E_k$ be a direct sum decomposition of a normed space $(E, \|\cdot\|)$. Then the decomposition is *hermitian* if

$$\|\zeta_1 x_1 + \cdots + \zeta_k x_k\| \leq \|x_1 + \cdots + x_k\| \tag{7.1}$$

whenever $\zeta_1, \ldots, \zeta_k \in \overline{\mathbb{D}}$ and $x_1 \in E_1, \ldots, x_k \in E_k$.

In particular, we see that $\|\zeta_1 x_1 + \cdots + \zeta_k x_k\| = \|x_1 + \cdots + x_k\|$ when $\zeta_1, \ldots, \zeta_k \in \mathbb{T}$ and $x_1 \in E_1, \ldots, x_k \in E_k$. Further, it follows from a simple remark on page 68 that this condition implies that the decomposition is hermitian.

The reason for the above terminology (suggested by [44]) is the following. Suppose that $E = F \oplus G$ is a decomposition. Then the decomposition is hermitian if and only if

$\|\zeta x + y\| = \|x + y\|$ ($x \in F$, $y \in G$, $\zeta \in \mathbb{T}$). Let $P : E \to F$ be the projection. Then

$$\exp(\mathrm{i}\theta P)(x + y) = \mathrm{e}^{\mathrm{i}\theta} x + y \quad (x \in F,\ y \in G,\ \theta \in \mathbb{R}),$$

and so the decomposition is hermitian if and only if P is a hermitian operator.

We see that trivial decompositions are hermitian. For example, let us identify $\mathbb{C}^n$ as $\mathbb{C} \oplus \cdots \oplus \mathbb{C}$, and suppose that $\|\cdot\|$ is a norm on $\mathbb{C}^n$. Then this decomposition is a hermitian decomposition of $(\mathbb{C}^n, \|\cdot\|)$ if and only if $\|\cdot\|$ is a lattice norm on $\mathbb{C}^n$.

A decomposition $E = F \oplus G$ of a Banach space E is said to be an *M-decomposition* if $\|y + z\| = \max\{\|y\|, \|z\|\}$ and an *L-decomposition* if $\|y + z\| = \|y\| + \|z\|$ for all $y \in F$ and $z \in G$; in these cases, F and G are M- and *L-summands*, respectively. Clearly, M- and L- decompositions are hermitian. See [34] for a discussion of M- and L- decompositions.

There have been many generalized versions of 'orthogonality' in the theory of normed linear spaces; our concept of a hermitian decomposition $E = F \oplus G$ implies that we have $\|x - y\| = \|x + y\|$ for each $x \in F$ and $y \in G$; thus x and y are 'isosceles orthogonal' in the sense of [38, Definition 2.1]. Indeed, $\|x - ky\| = \|x + ky\|$ for each $x \in F$, $y \in G$, and $k \in \mathbb{C}$, and so x and y are 'orthogonal' in the sense of the early paper [62]. See also the notion of *h-summand* in [32].

Definition 7.2. Let $(E, \|\cdot\|)$ be a normed space. Then the family of all hermitian decompositions of E is $\mathcal{K}_{\mathrm{herm}}$.

It is clear that $\mathcal{K}_{\mathrm{herm}}$ is a closed family of direct sum decompositions. Let $(E, \|\cdot\|)$ be a normed space, and consider a family $\mathcal{K}$ of hermitian decompositions of E. Then the smallest closed family $\mathcal{L}$ of hermitian decompositions of E such that $\mathcal{L}$ contains $\mathcal{K}$ is the *hermitian closed family generated by* $\mathcal{K}$.

Example 7.3. Let B be the subset of $\mathbb{C}^2$ which is the absolutely convex hull of the set consisting of the three points $(1, 0)$, $(0, 1)$, and $(2, 2)$. Then B is the closed unit ball of a norm, say $\|\cdot\|$, on $\mathbb{C}^2$. Then the obvious direct sum decomposition

$$\mathbb{C}^2 = (\mathbb{C} \times \{0\}) \oplus (\{0\} \times \mathbb{C})$$

is not a hermitian decomposition of $(\mathbb{C}^2, \|\cdot\|)$. Indeed, $\|(2, 2)\| = 1$, but $\|(2, 0)\| = 2$. ■

Example 7.4. Let $E = \ell_2^p$, where $p \geq 1$. Then $E = (\mathbb{C} \times \{0\}) \oplus (\{0\} \times \mathbb{C})$ is a hermitian decomposition.

We consider which other non-trivial direct sum decompositions of E are hermitian. Indeed, for $\alpha \in \mathbb{C}$, set $E_\alpha = \{(z, w) \in \mathbb{C}^2 : w = \alpha z\}$. Then $E = E_\alpha \oplus E_\beta$ whenever $\alpha \neq \beta$, and every such decomposition has this form for some $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq \beta$, say $\alpha \neq 0$. Take $x_1 = (1, \alpha) \in E_\alpha$ and $x_2 = (\zeta, \beta\zeta) \in E_\beta$, where $\zeta \in \mathbb{C}$. Then $\|x_1 + x_2\| = \|x_1 - x_2\|$ only if

$$|1 + \zeta|^p + |\alpha + \beta\zeta|^p = |1 - \zeta|^p + |\alpha - \beta\zeta|^p \quad (\zeta \in \mathbb{C}). \tag{7.2}$$

Thus $\beta \neq 0$. In the case where $-\beta/\alpha \notin \mathbb{R}^+$, there exists $\zeta \in \mathbb{T}$ with $\Re\zeta > 0$ and $\Re(\beta\zeta/\alpha) > 0$, and then $|1 + \zeta| > |1 - \zeta|$ and $|\alpha + \beta\zeta| > |\alpha - \beta\zeta|$, a contradiction of (7.2). Thus we have $\beta = -\alpha r$ for some $r > 0$. For $t \in \mathbb{R}$ with $|t| < \min\{r, 1\}$, we have

$$(1 + t)^p - (1 - t)^p = |\alpha|^p((1 + rt)^p - (1 - rt)^p). \tag{7.3}$$

Suppose that $p \neq 1, 2$. Then, by equating the first and third derivatives at $t = 0$ of both sides of (7.3), we see that $|\alpha|^p r = |\alpha|^p r^3 = 1$, and so $r = |\alpha| = 1$, say $\alpha = \mathrm{e}^{\mathrm{i}\theta}$, and then $\beta = -\mathrm{e}^{\mathrm{i}\theta}$.

Suppose that $p = 1$. Then, from (7.3), $|\alpha| r = 1$, and so, from (7.2),

$$|\zeta + 1| - |\zeta - 1| = |\zeta + 1/r| - |\zeta - 1/r| \quad (\zeta \in \mathbb{C}).$$

By taking $\zeta = 1 + \mathrm{i}$, we see that this is only possible when $r = 1$, and again we have $\alpha = \mathrm{e}^{\mathrm{i}\theta}$, and then $\beta = -\mathrm{e}^{\mathrm{i}\theta}$.

Thus, for $p \neq 2$, we obtain a hermitian decomposition only if $\alpha = \mathrm{e}^{\mathrm{i}\theta}$ and $\beta = -\mathrm{e}^{\mathrm{i}\theta}$ for some $\theta \in [0, 2\pi)$. But finally take $x_1 = (1, \mathrm{e}^{\mathrm{i}\theta})$ and $x_2 = (1, -\mathrm{e}^{\mathrm{i}\theta})$. Then

$$\|x_1 + x_2\| = 2^p \neq 2 \cdot 2^{p/2} = \|x_1 + \mathrm{i}x_2\|,$$

and so there is no hermitian decomposition of this form.

Thus the only hermitian decompositions of $E = \ell_2^p$ for $p \geq 1$ and $p \neq 2$ are

$$E = (\mathbb{C} \times \{0\}) \oplus (\{0\} \times \mathbb{C}) \quad \text{and} \quad E = (\{0\} \times \mathbb{C}) \oplus (\mathbb{C} \times \{0\}).$$

A similar argument shows that this is also true for $E = \ell_2^\infty$.

Suppose that $p = 2$. Then, from (7.2), $\Re(\zeta) = -\Re(\overline{\alpha}\beta\zeta)$ for all $\zeta \in \mathbb{C}$, and so there exist $\theta \in [0, 2\pi)$ and $r > 0$ with $\alpha = r\mathrm{e}^{\mathrm{i}\theta}$ and $\beta = -\mathrm{e}^{\mathrm{i}\theta}/r$. Each decomposition corresponding to such a choice of α and β is hermitian.

More general results about hermitian decompositions of ℓ^p follow from Theorem 1.15 and [30, Theorem 5.2.13]. ∎

Let $E = E_1 \oplus \cdots \oplus E_k$ be a hermitian decomposition of a normed space E. Then the maps P_j are continuous, and $\|P_j\| = 1$ when $E_j \neq \{0\}$ (even in the case where $(E, \|\cdot\|)$ is not necessarily complete), and the maps $P_j' : E_j' \to E'$ are isometric embeddings. Again, $E' = E_1' \oplus \cdots \oplus E_k'$.

Proposition 7.5. *Let $E = E_1 \oplus \cdots \oplus E_k$ be a hermitian decomposition of a normed space $(E, \|\cdot\|)$. Then the decomposition $E' = E_1' \oplus \cdots \oplus E_k'$ is also hermitian.*

Proof. Let $\zeta_1, \ldots, \zeta_k \in \overline{\mathbb{D}}$ and $\lambda_i \in E_i'$ $(i \in \mathbb{N}_k)$. Then

$$\begin{aligned}
\|\zeta_1\lambda_1 + \cdots + \zeta_k\lambda_k\| &= \sup_{x \in E_{[1]}} |\langle x, \zeta_1\lambda_1\rangle + \cdots + \langle x, \zeta_k\lambda_k\rangle| \\
&= \sup_{x \in E_{[1]}} |\langle \zeta_1 P_1 x, \lambda_1\rangle + \cdots + \langle \zeta_k P_k x, \lambda_k\rangle| \\
&= \sup_{x \in E_{[1]}} |\langle \zeta_1 P_1 x + \cdots + \zeta_k P_k x, \lambda_1 + \cdots + \lambda_k\rangle|.
\end{aligned}$$

But

$$\|\zeta_1 P_1 x + \cdots + \zeta_k P_k x\| \leq \|P_1 x + \cdots + P_k x\| = \|x\| \leq 1 \quad (x \in E_{[1]}),$$

and it follows that $\|\zeta_1\lambda_1 + \cdots + \zeta_k\lambda_k\| \leq \|\lambda_1 + \cdots + \lambda_k\|$, giving the result. ∎

The above result also follows from [13, §9, Corollary 6(ii)], where it is stated that $P' \in \mathcal{B}(E')$ is hermitian if and only if $P \in \mathcal{B}(E)$ is hermitian.

Proposition 7.6. *Let $(E, \|\cdot\|)$ be a normed space, and let $k \in \mathbb{N}$. Suppose that E has two hermitian decompositions*

$$E = E_1 \oplus \cdots \oplus E_k = F_1 \oplus \cdots \oplus F_k.$$

For $j \in \mathbb{N}_k$, let $Q_j : E \to F_j$ be the natural projections. Then

$$\|Q_1x_1 + \cdots + Q_kx_k\| \le \|x_1 + \cdots + x_k\| \quad (x_1 \in E_1, \ldots, x_k \in E_k). \tag{7.4}$$

Proof. Set $\zeta = \exp(2\pi\mathrm{i}/k)$. Then we note that

$$Q_\ell = \frac{1}{k}\sum_{i=1}^{k}\sum_{j=1}^{k}\zeta^{j(i-\ell)}Q_i \quad (\ell \in \mathbb{N}_k).$$

Take $x_i \in E_i$ $(i \in \mathbb{N}_k)$. Then

$$\begin{aligned}\|Q_1x_1 + \cdots + Q_kx_k\| &= \frac{1}{k}\Big\|\sum_{\ell=1}^{k}\sum_{i=1}^{k}\sum_{j=1}^{k}\zeta^{j(i-\ell)}Q_ix_\ell\Big\| \\ &\le \frac{1}{k}\sum_{j=1}^{k}\Big\|\sum_{i=1}^{k}\sum_{\ell=1}^{k}\zeta^{j(i-\ell)}Q_ix_\ell\Big\| \\ &= \frac{1}{k}\sum_{j=1}^{k}\Big\|\Big(\sum_{i=1}^{k}\zeta^{ji}Q_i\Big)\Big(\sum_{\ell=1}^{k}\zeta^{-j\ell}x_\ell\Big)\Big\| \\ &= \frac{1}{k}\sum_{j=1}^{k}\Big\|\sum_{\ell=1}^{k}\zeta^{-j\ell}x_l\Big\|\end{aligned}$$

because the decomposition $E = F_1 \oplus \cdots \oplus F_k$ is hermitian, and so

$$\|Q_1x_1 + \cdots + Q_kx_k\| \le \frac{1}{k}\sum_{j=1}^{k}\Big\|\sum_{\ell=1}^{k}\zeta^{-j\ell}x_\ell\Big\| = \|x_1 + \cdots + x_k\|$$

because the decomposition $E = E_1 \oplus \cdots \oplus E_k$ is hermitian. Thus (7.4) follows. ∎

We now give some examples of hermitian decompositions of particular Banach spaces.

Theorem 7.7. *Let K be a compact space, and let $C(K) = E_1 \oplus \cdots \oplus E_k$ be a hermitian decomposition. Then there exist clopen subspaces $K_1, \ldots, K_k$ of K such that $E_j = C(K_j)$ $(j \in \mathbb{N}_k)$. In particular, in the case where K is connected, there are no non-trivial hermitian decompositions of $C(K)$.*

Proof. Take $j \in \mathbb{N}_k$, and let P_j be the projection of $C(K)$ onto E_j, so that P_j is a hemitian operator. By Theorem 1.14, there exists $h_j \in C_{\mathbb{R}}(K)$ with $P_jf = h_jf$ $(f \in C(K))$. Since $P_j = P_j^2$, we have $h_j = h_j^2$ in $C(K)$, and so h_j is the characteristic function of a subset, say K_j, of K. Clearly, K_j is clopen and $E_j = C(K_j)$. ∎

Proposition 7.8. *Take $p \in [1, \infty]$ with $p \neq 2$, and let $\ell^p = E_1 \oplus \cdots \oplus E_k$ be a hermitian decomposition. Then there exist subsets $S_1, \ldots, S_k$ of $\mathbb{N}$ such that $E_j = \ell^p(S_j)$ $(j \in \mathbb{N}_k)$.*

Proof. This follows similarly, now using Theorem 1.15. ∎

7.1.2. Small decompositions of multi-normed spaces. We now turn to decompositions of normed spaces E with respect to multi-norms based on E.

DEFINITION 7.9. Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space, let $k \in \mathbb{N}$,and let $E = E_1 \oplus \cdots \oplus E_k$ be a direct sum decomposition of E. Then the decomposition is *small* (with respect to the multi-norm) if

$$\|P_1 x_1 + \cdots + P_k x_k\| \leq \|(x_1, \ldots, x_k)\|_k \quad (x_1, \ldots, x_k \in E).$$

We shall see in Example 7.25 that the notion of a small decomposition of a normed space E depends on the multi-norm $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$, and is not intrinsic to the normed space E.

Clearly $\|P_j\| \leq 1$ $(j \in \mathbb{N}_n)$ for each such small decomposition.

PROPOSITION 7.10. *Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space, and suppose that $E = E_1 \oplus \cdots \oplus E_k$ is a small decomposition of E. Then the decomposition is hermitian. Further,*

$$\|(x_1, \ldots, x_k)\|_k = \|x_1 + \cdots + x_k\| \quad (x_1 \in E_1, \ldots, x_k \in E_k). \tag{7.5}$$

Proof. Take $\zeta_1, \ldots, \zeta_k \in \overline{\mathbb{D}}$ and $x_1 \in E_1, \ldots, x_k \in E_k$, and then set $x = x_1 + \cdots + x_k$. Clearly $P_j x = x_j$ $(j \in \mathbb{N}_k)$, and so

$$\begin{aligned}\|\zeta_1 x_1 + \cdots + \zeta_1 x_1\| &= \|P_1(\zeta_1 x) + \cdots + P_k(\zeta_k x)\| \leq \|(\zeta_1 x, \ldots, \zeta_k x)\|_k \\ &\leq \|(x, \ldots, x)\|_k = \|x\| = \|x_1 + \cdots + x_k\|,\end{aligned}$$

and so the decomposition is hermitian.

Now take $x_1 \in E_1, \ldots, x_k \in E_k$, and set $\zeta = \exp(2\pi\mathrm{i}/k)$. Then

$$\begin{aligned}\|x_1 + \cdots + x_k\| &= \|P_1 x_1 + \cdots + P_k x_k\| \leq \|(x_1, \ldots, x_k)\|_k \\ &\leq \frac{1}{k} \sum_{j=1}^{k} \Big\| \sum_{m=1}^{k} \zeta^{jm} x_m \Big\| \quad \text{by Proposition 2.17} \\ &\leq \max_{j \in \mathbb{N}_k} \Big\| \sum_{m=1}^{k} \zeta^{jm} x_m \Big\| \leq \|x_1 + \cdots + x_k\|,\end{aligned}$$

which gives the equality (7.5). ∎

EXAMPLE 7.11. Let $E = \ell^p(\mathbb{N})$, where $p \geq 1$, and consider the lattice multi-norm based on E, namely $(\|\cdot\|_n^L : n \in \mathbb{N})$; by Example 4.47, this is the standard p-multi-norm on E.

For $k \in \mathbb{N}$, take $(S_1, \ldots, S_k)$ to be an ordered partition of $\mathbb{N}$, and set $E_i = \ell^p(S_i)$ for $i \in \mathbb{N}_k$). Then it is clear that $E = E_1 \oplus \cdots \oplus E_k$ is a small decomposition with respect to the lattice multi-norm because

$$\|f_1 \mid S_1 + \cdots + f_k \mid S_1\| \leq \|\, |f_1| \vee \cdots \vee |f_k| \,\| = \|(f_1, \ldots, f_k)\|_k^L$$

for all $f_1, \ldots, f_k \in E$. The collection of all such decompositions is a closed family. ∎

The following remark will be generalized later, in Theorem 7.40.

PROPOSITION 7.12. *Let $(E, \|\cdot\|)$ be a normed space, and suppose that $E = E_1 \oplus E_2$ is a hermitian decomposition of E. For $x_1, x_2 \in E$, set*

$$\|(x_1, x_2)\|_2 = \max\{\|x_1\|, \|x_2\|, \|P_1 x_1 + P_2 x_2\|, \|P_1 x_2 + P_2 x_1\|\}.$$

Then $(\|\cdot\|, \|\cdot\|_2)$ *is a multi-norm of level* 2 *on* $\{E, E^2\}$, *and the direct sum decomposition* $E = E_1 \oplus E_2$ *is small with respect to this multi-norm.*

Proof. It is clear that $\|\cdot\|_2$ is a norm on E^2 and that $\|\cdot\|_2$ satisfies (A1); $\|\cdot\|_2$ satisfies (A2) because the decomposition is hermitian.

Let $x \in E$. Then $\|(x, 0)\|_2 = \|x\|$ because $\|P_1\|, \|P_2\| \leq 1$, so that (A3) holds, and $\|(x, x)\|_2 = \|x\|$ because $P_1x + P_2x = x$, so that (A4) holds. Thus $(\|\cdot\|, \|\cdot\|_2)$ is a multi-norm of level 2 on $\{E, E^2\}$.

Clearly the decomposition $E = E_1 \oplus E_2$ is small with respect to the multi-norm $(\|\cdot\|, \|\cdot\|_2)$. ∎

Definition 7.13. Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space. Then the family of all small decompositions of E is $\mathcal{K}_{\text{small}}$.

Proposition 7.14. *Let* $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ *be a multi-normed space. Then* $\mathcal{K}_{\text{small}}$ *is a closed family of direct sum decompositions.*

Proof. Clearly, Axiom (C1) of Definition 1.7 is satisfied, and (C3) is trivially satisfied.

Take $k \geq 3$, and let $E = E_1 \oplus \cdots \oplus E_k$ be a small decomposition. Take $x, x_3, \ldots, x_k \in E$. Then the projection of x with kernel $E_3 \oplus \cdots \oplus E_k$ onto the space $E_1 \oplus E_2$ is $P_1x + P_2x$, and so

$$\|(P_1x + P_2x) + P_3x_3 + \cdots + P_kx_k\| \leq \|(x, x, x_3, \ldots, x_k)\|_k = \|(x, x_3, \ldots, x_k)\|_{k-1}.$$

Hence $E = (E_1 \oplus E_2) \oplus E_3 \oplus \cdots \oplus E_k$ is a small decomposition of E, and so Axiom (C2) is satisfied.

Thus $\mathcal{K}_{\text{small}}$ is a closed family. ∎

7.1.3. Orthogonal decompositions of multi-normed spaces. We now move to consideration of orthogonal decompositions of multi-normed spaces. It will be seen later that such decompositions generalize various classical notions of orthogonality.

Let E be a linear space. We recall that a ‘coagulation’ of an element $(x_1, \ldots, x_n) \in E^n$ was defined on page 10.

Definition 7.15. Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space, let $k \in \mathbb{N}$, and let $E = \{E_1, \ldots, E_k\}$ be family of closed subspaces of E. Then $\{E_1, \ldots, E_k\}$ is an *orthogonal family* in E if, for each $x_1 \in E_1, \ldots, x_k \in E_k$ and each coagulation $(y_1, \ldots, y_j)$ of $(x_1, \ldots, x_k)$, we have

$$\|(y_1, \ldots, y_j)\|_j = \|(x_1, \ldots, x_k)\|_k.$$

A subset $\{x_1, \ldots, x_k\}$ of E is *orthogonal* if the family $\{\mathbb{C}x_1, \ldots, \mathbb{C}x_k\}$ of subspaces is an orthogonal family.

Again, the notion of an orthogonal family depends on the multi-norm structure; it is not intrinsic to the normed space E. The definition depends on only the set $\{E_1, \ldots, E_k\}$, and not on the ordering of the spaces $E_1, \ldots, E_k$.

For example, a trivial direct sum decomposition of E is orthogonal for any multi-normed space $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$; this follows from the basic Axiom (A3).

Let $\{E_1, \ldots, E_k\}$ be an orthogonal family of subspaces of E. Then certainly

$$\|(x_1, \ldots, x_k)\|_k = \|x_1 + \cdots + x_k\| \quad (x_1 \in E_1, \ldots, x_k \in E_k). \tag{7.6}$$

Indeed, suppose that $x_i \in E_i$ $(i \in \mathbb{N}_k)$. Then

$$\|(x_1, \dots, x_k)\|_k = \|\zeta_1 x_1 + \cdots + \zeta_k x_k\| \quad (\zeta_1, \dots, \zeta_k \in \mathbb{T}). \tag{7.7}$$

LEMMA 7.16. *Let* $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ *be a multi-normed space, let* $k \in \mathbb{N}$, *and let* $\{E_1, \dots, E_k\}$ *be an orthogonal family in* E. *Then:*

(i) *for* $i, j \in \mathbb{N}_k$ *with* $i \neq j$, *we have* $E_i \cap E_j = \{0\}$;
(ii) $\{E_1 \oplus E_2, E_3, \dots, E_k\}$ *is an orthogonal family in* E (*whenever* $k \geq 3$);
(iii) *for* $j \in \mathbb{N}_k$ *such that* $E_j \neq \{0\}$, *the norm of the projection from* $(E_1 \oplus \cdots \oplus E_k, \|\cdot\|)$ *onto* $(E_j, \|\cdot\|)$ *is* 1.

Proof. These are immediate. ∎

DEFINITION 7.17. Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space, let $k \in \mathbb{N}$, and let $E = E_1 \oplus \cdots \oplus E_k$ be a direct sum decomposition. Then the decomposition is *orthogonal* (with respect to the multi-norm of E) if $\{E_1, \dots, E_k\}$ is an orthogonal family.

We make the following remark, without proof.

Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space, and let $\mathcal{K}$ be a closed family of hermitian decompositions of E. Suppose that, for each decomposition $E = E_1 \oplus \cdots \oplus E_k$ in $\mathcal{K}$, we have

$$\|(x_1, \dots, x_k)\|_k \geq \|x_1 + \cdots + x_k\| \quad (x_1 \in E_1, \dots, x_k \in E_k).$$

Then each decomposition in $\mathcal{K}$ is orthogonal.

DEFINITION 7.18. Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space. Then the family of all orthogonal decompositions of E is $\mathcal{K}_{\mathrm{orth}}$.

PROPOSITION 7.19. *Let* $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ *be a multi-normed space. Then* $\mathcal{K}_{\mathrm{orth}}$ *is a closed family of direct sum decompositions.*

Proof. Clearly trivial direct sum decompositions of E are orthogonal, and so this follows from Lemma 7.16. ∎

THEOREM 7.20. *Let* $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ *be a multi-normed space. Then:*

(i) *each orthogonal decomposition of* E *is hermitian;*
(ii) *each small decomposition of* E *is orthogonal.*

Proof. (i) This is immediate from (7.7).

(ii) Let $E = E_1 \oplus \cdots \oplus E_k$ be a small decomposition of E, and then take elements $x_1 \in E_1, \dots, x_k \in E_k$. Suppose that $(y_1, \dots, y_j)$ is a coagulation of $(x_1, \dots, x_k)$, and let $\{S_j : j \in \mathbb{N}_k\}$ be a partition of $\mathbb{N}_n$ such that

$$y_j = \sum \{x_i : i \in S_j\} \quad (j \in \mathbb{N}_k).$$

Set $F_j = \bigoplus \{E_i : i \in S_j\}$ $(j \in \mathbb{N}_k)$. Then $E = F_1 \oplus \cdots \oplus F_j$ is a direct sum decomposition of E, and, by Proposition 7.14, it is a small decomposition of E. By (7.5), we have $\|(y_1, \dots, y_j)\|_j = \|y_1 + \cdots + y_j\|$ and $\|(x_1, \dots, x_k)\|_k = \|x_1 + \cdots + x_k\|$. But clearly $\|y_1 + \cdots + y_j\| = \|x_1 + \cdots + x_k\|$, and so $\|(y_1, \dots, y_j)\|_j = \|(x_1, \dots, x_k)\|_k$. Thus the decomposition is orthogonal. ∎

QUESTION. Let $(\|\cdot\|_n : n \in \mathbb{N})$ be multi-norm based on a Banach space E. We regret that we do not know whether every orthogonal decomposition with respect to this multi-norm is necessarily small. If this is not true in general, one could seek classes of multi-norms or of Banach spaces E for which it is true.

PROPOSITION 7.21. *Let E be a Banach space. Then every orthogonal decomposition of E with respect to the minimum multi-norm is small with respect to this multi-norm.*

Proof. Take $k \in \mathbb{N}$, and let $E = E_1 \oplus \cdots \oplus E_k$ be an orthogonal, and hence hermitian, decomposition of E. Then $\|P_j\| \leq 1$ $(j \in \mathbb{N}_k)$.

Take $x_1, \dots, x_k \in E$. Since the decomposition is orthogonal, we have

$$\|P_1x_1 + \cdots + P_kx_k\| = \|(P_1x_1, \dots, P_kx_k)\|_k^{\min},$$

and so

$$\|P_1x_1 + \cdots + P_kx_k\| \leq \max\{\|x_j\| : j \in \mathbb{N}_n\} = \|(x_1, \dots, x_k)\|_k^{\min}.$$

Hence the decomposition is small. ∎

PROPOSITION 7.22. *Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space. Let $k \in \mathbb{N}$, and let $E = E_1 \oplus \cdots \oplus E_k$ be an orthogonal decomposition of E, with corresponding projections $P_1, \dots, P_k$. Take $\lambda_1, \dots, \lambda_k \in E'$. Then*

$$\sup\Big\{\Big|\sum_{i=1}^k \langle P_i x,\, \lambda_i\rangle\Big| : x \in E_{[1]}\Big\} = \sup\Big\{\Big|\sum_{i=1}^k \langle x_i,\, \lambda_i\rangle\Big| : x_i \in E_i,\ \|(x_1, \dots, x_k)\|_k \leq 1\Big\}.$$

Proof. Let the left-hand and right-hand sides of the above equation be A and B, respectively.

Take $x \in E_{[1]}$. Then $P_i x \in E_i$ $(i \in \mathbb{N}_k)$, and so

$$\|(P_1x, \dots, P_kx)\|_k = \|P_1x + \cdots + P_kx\| = \|x\| \leq 1.$$

Thus $|\sum_{i=1}^k \langle P_i x,\, \lambda_i\rangle| \leq B$, and so $A \leq B$.

Take elements $x_i \in E_i$ $(i \in \mathbb{N}_k)$ with $\|(x_1, \dots, x_k)\|_k \leq 1$, and set $x = x_1 + \cdots + x_k$. Then, by equation (7.6), $x \in E_{[1]}$, and $P_i x = x_i$ $(i \in \mathbb{N}_k)$. Thus $|\sum_{i=1}^k \langle x_i,\, \lambda_i\rangle| \leq A$, and so $B \leq A$.

The result follows. ∎

PROPOSITION 7.23. *Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ and $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ be multi-normed spaces, let $k \in \mathbb{N}$, and let $E = E_1 \oplus \cdots \oplus E_k$ be an orthogonal decomposition of E. Then*

$$\|(Tx_1, \dots, Tx_k)\|_k \leq \|T\|\,\|(x_1, \dots, x_k)\|_k \tag{7.8}$$

for $x_1 \in E_1, \dots, x_k \in E_k$ and $T \in \mathcal{B}(E, F)$.

Proof. Take $x_j \in E_j$ for each $j \in \mathbb{N}_k$. By Proposition 2.17,

$$\|(Tx_1, \dots, Tx_k)\|_k \leq \frac{1}{k}\sum_{j=1}^k \Big\|\sum_{m=1}^k \zeta^{jm} T x_m\Big\|,$$

where $\zeta = \exp(2\pi \mathrm{i}/k)$. However,

$$\Big\|\sum_{m=1}^k \zeta^{jm} T x_m\Big\| \leq \|T\|\,\Big\|\sum_{m=1}^k \zeta^{jm} x_m\Big\| = \|T\|\,\|(x_1, \dots, x_k)\|_k$$

for each $j \in \mathbb{N}_k$ by (7.7), and now (7.8) follows. ∎

7.1.4. Elementary examples. We give four elementary examples involving hermitian, small, and orthogonal decompositions; further examples will be given later.

EXAMPLE 7.24. Let $E = \ell_2^p$, where $p \in [1, \infty]$; the norm on E is $\|\cdot\|$. Set $E_1 = \mathbb{C} \times \{0\}$ and $E_2 = \{0\} \times \mathbb{C}$, so that $E = E_1 \oplus E_2$ is a hermitian decomposition for each $p \in [1, \infty]$.

Let $(\|\cdot\|, \|\cdot\|_2^{\min})$ be the minimum multi-norm of level 2 on $\{E, E^2\}$.

First, suppose that $p < \infty$, and take $x_1 = (1, 0)$ and $x_2 = (0, 1)$, so that $x_1 \in E_1$ and $x_2 \in E_2$. Then $\|x_1 + x_2\| = \|(1, 1)\| = 2^{1/p}$, whereas

$$\|(x_1, x_2)\|_2^{\min} = \max\{\|x_1\|, \|x_2\|\} = 1;$$

since $2^{1/p} > 1$, the decomposition is not orthogonal. We conclude that there are hermitian decompositions that are not orthogonal with respect to a particular multi-norm.

Second, suppose that $p = \infty$, and let $(\|\cdot\|, \|\cdot\|_2)$ be any multi-norm of level 2 on $\{E, E^2\}$. Take $x_1 = (z_1, w_1)$ and $x_2 = (z_2, w_2)$ in E. Then

$$\|P_1 x_1 + P_2 x_2\| = \max\{|z_1|, |w_2|\},$$

whereas

$$\|(x_1, x_2)\|_2 \geq \|(x_1, x_2)\|_2^{\min} = \max\{|z_1|, |z_2|, |w_1|, |w_2|\} \geq \|P_1 x_1 + P_2 x_2\|,$$

and so the decomposition is small. ■

EXAMPLE 7.25. Let B be the subset of $\mathbb{C}^2$ which is the absolutely convex hull of the set consisting of the three points $x_1 = (1, 0)$, $x_2 = (0, 1)$, and $x_1 + x_2 = (1, 1)$. Then B is the closed unit ball of a norm, say $\|\cdot\|$, on $\mathbb{C}^2$. Again set $E_1 = \mathbb{C} \times \{0\}$ and $E_2 = \{0\} \times \mathbb{C}$, so that $E = E_1 \oplus E_2$. We have $x_1 \in E_1$ and $x_2 \in E_2$. Also $\|x_1 + x_2\| = 1$, but $\|x_1 - x_2\| = 2$, and so the decomposition is not hermitian.

Let $(\|\cdot\|_n : n \in \mathbb{N})$ be any multi-norm based on E. Then, by Theorem 7.20(i), the decomposition $E = E_1 \oplus E_2$ is not orthogonal with respect to this multi-norm because it is not hermitian.

Next set $F_1 = \{(z, z) : z \in \mathbb{C}\}$ and $F_2 = \{(z, -z) : z \in \mathbb{C}\}$. Then $E = F_1 \oplus F_2$ is a direct sum decomposition; say the projections onto F_1 and F_2 are Q_1 and Q_2, respectively. Simple geometrical considerations show that this decomposition is hermitian.

Let $(\|\cdot\|, \|\cdot\|_2^{\min})$ be the minimum multi-norm of level 2 on $\{E, E^2\}$, and now take $x_1 = (1, 1) \in F_1$ and $x_2 = (1/2, -1/2) \in F_2$, so that we have $\|x_1\| = \|x_2\| = 1$ and $\|(x_1, x_2)\|_2^{\min} = 1$. Further,

$$\|x_1 + x_2\| = \left\|\left(\frac{3}{2}, \frac{1}{2}\right)\right\| > 1.$$

Thus the decomposition $E = F_1 \oplus F_2$ is not orthogonal with respect to the minimum multi-norm. Again we see that there are hermitian decompositions that are not orthogonal with respect to a particular multi-norm.

As in Proposition 7.12, there is a multi-norm of level 2 on $\{E, E^2\}$ with respect to which the decomposition $E = F_1 \oplus F_2$ is small. ■

EXAMPLE 7.26. This example shows that we cannot determine the orthogonality of a set just by looking at pairs of elements in the set.

Let E be the space $\mathbb{C}^4$, with the norm $\|\cdot\|$ given by

$$\|(z_1,\dots,z_4)\| = \max\{|z_1|,\dots,|z_4|\} \quad (z_1,\dots,z_4 \in \mathbb{C}),$$

so that $E = \ell^\infty_4$. Then $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is a multi-normed space for the minimum multi-norm.

Set $f_1 = (1,0,0,1/2)$, $f_2 = (0,1,0,1/2)$, and $f_3 = (0,0,1,1/2)$. It is immediate that $\|f_1\| = \|f_2\| = \|f_3\| = 1$.

We *claim* that $\{f_1, f_2\}$ is orthogonal. Indeed take $\zeta_1, \zeta_2 \in \mathbb{C}$. Then we see that $\|(f_1, f_2)\|_2 = \max\{|\zeta_1|, |\zeta_2|\}$ and

$$\|\zeta_1 f_1 + \zeta_2 f_2\| = \max\{|\zeta_1|, |\zeta_2|, |(\zeta_1+\zeta_2)/2|\} = \max\{|\zeta_1|, |\zeta_2|\},$$

as required. Similarly, $\{f_1, f_3\}$ and $\{f_2, f_3\}$ are orthogonal. However, we calculate that $f_1 + f_2 + f_3 = (1,1,1,3/2)$, so that

$$\|f_1 + f_2 + f_3\| = 3/2 > 1 = \|(f_1, f_2, f_3)\|_3.$$

Thus $\{f_1, f_2, f_3\}$ is not orthogonal. ∎

EXAMPLE 7.27. Let $E = C(\mathbb{I})$, with the uniform norm $|\cdot|_{\mathbb{I}}$, and consider the minimum multi-norm based on E. We ask when $\{f_1, f_2\}$ is orthogonal. This is certainly the case whenever f_1 and f_2 have disjoint supports. However this may occur in other cases.

For example, define a function $f_1 \in E$ by requiring that $f_1(0) = 1$, that $f_1(1) = 0$, that $f_1(t_0) = 1/2$ for some $t_0 \in (0,1)$, and that f_1 be linear on $[0, t_0]$ and $[t_0, 1]$, and then set $f_2(t) = f_1(1-t)$ $(t \in \mathbb{I})$. Then it is easy to see that $\{f_1, f_2\}$ is orthogonal if and only if $t_0 \geq 1/2$.

Let K be a compact space, and let $f \in C(K)$ be such that $f(K) = \mathbb{I}$. Then $\{f, 1-f\}$ is an orthogonal set. ∎

7.1.5. Decompositions of the spaces $C(K)$. Throughout this subsection, K is a non-empty, compact space.

PROPOSITION 7.28. *Let $(\|\cdot\|_n : n \in \mathbb{N})$ be any multi-norm based on $C(K)$, and suppose that $\{K_1,\dots,K_k\}$ is a partition of K into clopen subspaces. Then the decomposition $C(K) = C(K_1) \oplus \cdots \oplus C(K_k)$ is small with respect to this multi-norm.*

Proof. We write $P_j : f \mapsto f \mid K_j$ for $j \in \mathbb{N}_n$. Let $f_1,\dots,f_k \in C(K)$. Then

$$\begin{aligned}|P_1 f_1 + \cdots + P_k f_k|_K &= \max\{|P_j f_j|_K : j \in \mathbb{N}_k\} \leq \max\{|f_j|_K : j \in \mathbb{N}_k\} \\ &= \|(f_1,\dots,f_k)\|_k^{\min} \leq \|(f_1,\dots,f_k)\|_k,\end{aligned}$$

and so the decomposition is small. ∎

The following theorem gives more information about decompositions of the space $C(K)$. Recall from Theorem 4.54(ii) that the lattice multi-norm based on $C(K)$ is just the minimum multi-norm.

THEOREM 7.29. *Let $C(K) = E_1 \oplus \cdots \oplus E_k$ be a direct sum decomposition of $C(K)$, and let $(\|\cdot\|_n : n \in \mathbb{N})$ be a multi-norm based on $C(K)$. Then the following are equivalent:*

(a) *$E_j = C(K_j)$ $(j \in \mathbb{N}_k)$ for some partition $\{K_1,\dots,K_k\}$ of K into clopen subspaces;*
(b) *the decomposition is small with respect to the lattice multi-norm;*

(c) *the decomposition is orthogonal with respect to the lattice multi-norm;*
(d) *the decomposition is hermitian.*

Proof. (a)⇒(b) This follows from Proposition 7.28.

(b)⇒(c)⇒(d) This follows from Theorem 7.20.

(d)⇒(a) This follows from Theorem 7.7. ∎

7.1.6. Decompositions of Hilbert spaces. Let H be a Hilbert space. Recall that the Hilbert multi-norm $(\|\cdot\|_n^H : n \in \mathbb{N})$ based on H was defined in Definition 4.16; orthogonal decompositions of H were defined in Chapter 1, §2.6. We again denote the inner product on H by $[\cdot\,,\,\cdot]$.

THEOREM 7.30. *Let H be a Hilbert space, and let $H = H_1 \oplus \cdots \oplus H_k$ be a direct sum decomposition of H. Then the following are equivalent:*

(a) *the decomposition is orthogonal;*
(b) *the decomposition is small with respect to the Hilbert multi-norm;*
(c) *the decomposition is orthogonal with respect to the Hilbert multi-norm;*
(d) *the decomposition is hermitian.*

Proof. (a)⇒(b)⇒(c)⇒(d) These are immediate from the definition of the Hilbert multi-norm and Theorem 7.20.

(d)⇒(a) First, let $H = H_1 \oplus H_2$ be a hermitian decomposition, with $H_1, H_2 \neq \{0\}$. We choose $x_1 \in H_1$ and $x_2 \in H_2$ with $\|x_1\| = \|x_2\| = 1$, and set $\zeta = [x_1, x_2]$, so that $|\zeta| \leq 1$. We shall show that $\zeta = 0$, and hence deduce that $H = H_1 \oplus H_2$ is an orthogonal decomposition.

We may suppose that $\zeta \leq 0$. Write $y = x_1 - \zeta x_2$, so that $[x_2, y] = 0$. Then

$$1 + \|y\|^2 = \|y - x_2\|^2 = \|x_1 - (1+\zeta)x_2\|^2 \leq \|x_1 + x_2\|^2$$

because $|1+\zeta| \leq 1$ and the decomposition is hermitian, and so

$$1 + \|y\|^2 \leq \|(1+\zeta)x_2 + y\|^2 = (1+\zeta)^2 + \|y\|^2 .$$

Thus $\zeta = 0$, giving the claim.

The general case follows by induction. ∎

7.1.7. Decompositions of lattices. Let E be a Banach lattice. We recall that a direct sum decomposition $E = E_1 \oplus \cdots \oplus E_k$ is a band decomposition, written

$$E = E_1 \oplus_\perp \cdots \oplus_\perp E_k ,$$

if each of $E_1, \dots, E_k$ is a band, and that $\|P_j\| \leq 1$ $(j \in \mathbb{N}_k)$ in this case.

THEOREM 7.31. *Let E be a Banach lattice. Then every band decomposition of E is small with respect to the lattice multi-norm.*

Proof. Suppose that $E = E_1 \oplus_\perp \cdots \oplus_\perp E_k$ is a band decomposition, and take elements $x_1, \dots, x_k \in E$. Then

$$\|P_1x_1 + \cdots + P_kx_k\| = \|\,|P_1x_1| \vee \cdots \vee |P_kx_k|\,\| \leq \|\,|x_1| \vee \cdots \vee |x_k|\,\|$$

by (1.23), and so $\|P_1x_1 + \cdots + P_kx_k\| \leq \|(x_1, \dots, x_k)\|_k^L$. Thus the decomposition is small with respect to the lattice multi-norm. ∎

Thus every band decomposition of a Banach lattice is orthogonal with respect to the lattice multi-norm. We enquire whether the converse to this statement holds. For example, take K to be a non-empty, locally compact space, and suppose that $M(K) = E \oplus F$ is an orthogonal decomposition with respect to the lattice multi-norm. Then it follows from remarks on page 35 that this is a band decomposition, and so the converse holds in this case; further, $M(K) = M_d(K) \oplus M_c(K)$ is an example of such a decomposition.

First we note that the above converse need not hold in the case when E is a *real* Banach lattice, as the following example shows.

EXAMPLE 7.32. Consider the space $E = \mathbb{R}^2$, with the ℓ^1-norm, so that E is a Banach lattice. Set

$$E_1 = \{(x,x) : x \in \mathbb{R}\}, \qquad E_2 = \{(x,-x) : x \in \mathbb{R}\}.$$

Then $E = E_1 \oplus E_2$ is a direct sum decomposition. We note that, for $x, y \in \mathbb{R}$, so that $(x,x) \in E_1$ and $(y,-y) \in E_2$, we have

$$\| \, |(x,x)| \wedge |(y,-y)| \, \| = \|(|x|,|x|) \wedge (|y|,|y|)\| = 2\max\{|x|,|y|\} \tag{7.9}$$

and

$$\|(x,x) + (y,-y)\| = |x+y| + |x-y| = 2\max\{|x|,|y|\}. \tag{7.10}$$

Hence $E = E_1 \oplus E_2$ is an orthogonal decomposition with respect to the lattice multi-norm.

However, it is not true that $|(x,x)| \wedge |(y,-y)| = 0$ for each $x, y \in \mathbb{R}$, and so $E = E_1 \oplus E_2$ is not a band decomposition. ■

However this leaves open the converse for (complex) Banach lattices. We are very grateful to the late Professor Nigel Kalton for responding to a question by proving the converse in this case; see [44, Theorem 4.2].

THEOREM 7.33. *Let $E = E_1 \oplus \cdots \oplus E_k$ be a direct sum decomposition of a Banach lattice E. Suppose that*

$$\|x_1 + \cdots + x_k\| = \| \, |x_1| \vee \cdots \vee |x_k| \, \| \qquad (x_j \in E_j,\ j \in \mathbb{N}_k).$$

Then the decomposition is a band decomposition. ■

The following theorem is now a consequence of Theorems 7.20(ii), 7.31, and 7.33.

THEOREM 7.34. *Let $E = E_1 \oplus \cdots \oplus E_k$ be a direct sum decomposition of a Banach lattice E. Then the following are equivalent:*

(a) *the decomposition is orthogonal with respect to the lattice multi-norm;*
(b) *the decomposition is small with respect to the lattice multi-norm;*
(c) *the decomposition is a band decomposition.* ■

It is not true that every hermitian decomposition of a Banach lattice is a band decomposition. For let $X = \ell_2^2$, and set

$$E = \{(z,z) : z \in \mathbb{C}\} \quad \text{and} \quad F = \{(w,-w) : w \in \mathbb{C}\},$$

so that $X = E \oplus F$. For $x = (z,z) \in E$ and $y = (w,-w) \in F$, we have

$$\|x + e^{i\theta}y\|^2 = 2(|z|^2 + |w|^2) \qquad (\theta \in [0, 2\pi)),$$

and so the decomposition is hermitian. However it is not a band decomposition.

In fact, in [44, Theorems 5.4 and 5.5], Kalton proved the following stronger and considerably deeper result.

THEOREM 7.35. *Let $E = F \oplus G$ be a direct sum decomposition of a Banach lattice E.*

(i) *Suppose that the decomposition is hermitian. Then*

$$\|x + y\| = \|(|x|^2 + |y|^2)^{1/2}\| \quad (x \in F,\ y \in G).$$

(ii) *Suppose that, for some $p \in [1, \infty)$ with $p \neq 2$, we have*

$$\|x + y\| = \|(|x|^p + |y|^p)^{1/p}\| \quad (x \in F,\ y \in G).$$

Then the decomposition is a band decomposition. ∎

7.1.8. Decompositions of L^p-spaces. We have seen that Theorem 7.33 does not extend to all real Banach lattices E. However, by an argument due to Hung Le Pham, it does extend to certain real Banach lattices

We first make a remark. Take $p \geq 1$. Then we have the inequality

$$\frac{1}{2}(|z + w|^p + |z - w|^p) \geq |z|^p \quad (z, w \in \mathbb{C}). \tag{7.11}$$

Now suppose that $|z| \geq |w|$. In the case where $p > 1$, equality holds in the above if and only if $w = 0$; in the case where $p = 1$, equality holds in the above if and only if $z = \alpha w$ for some $\alpha \in \mathbb{R}$.

PROPOSITION 7.36. *Let (Ω, μ) be a measure space, and take E to be $L^p(\Omega, \mu)$ or $L^p_{\mathbb{R}}(\Omega, \mu)$, where $p > 1$, or $L^1(\Omega, \mu)$. Suppose that $E = F \oplus G$ is an orthogonal decomposition with respect to the lattice multi-norm. Then $E = F \oplus G$ is a band decomposition.*

Proof. Take $f \in F$ and $g \in G$, and set $A = \{x \in \Omega : |f(x)| \geq |g(x)|\}$ and $B = \Omega \setminus A$, so that A and B are Borel measurable subsets of Ω. Since the decomposition is orthogonal, we have

$$\|\,|f| \vee |g|\,\| = \|f + g\| = \|f - g\|,$$

and so

$$\begin{aligned}
\|\,|f| \vee |g|\,\|^p &= \frac{1}{2}\|f + g\|^p + \frac{1}{2}\|f - g\|^p \\
&= \frac{1}{2}\int_\Omega (|f + g|^p + |f - g|^p)\,\mathrm{d}\mu \\
&= \left(\int_A + \int_B\right)\frac{1}{2}\Big(|f + g|^p + |f - g|^p\Big)\,\mathrm{d}\mu \\
&\geq \int_A |f|^p\,\mathrm{d}\mu + \int_B |g|^p\,\mathrm{d}\mu \quad \text{by (7.11)} \\
&= \|\,|f| \vee |g|\,\|^p.
\end{aligned}$$

In the case where $p > 1$, it follows that $g = 0$ almost everywhere on A and $f = 0$ almost everywhere on B, and so $|f| \wedge |g| = 0$.

Now suppose that $E = L^1(\Omega, \mu)$. Then $g(x) = \alpha(x)f(x)$ for almost all $x \in A$, where $\alpha(x) \in \mathbb{R}$ $(x \in A)$. By repeating the argument with g replaced by $\mathrm{i}g$ (which does not change the sets A and B), we see that $\mathrm{i}g(x) = \beta(x)f(x)$ for almost all $x \in A$, where

$\beta(x) \in \mathbb{R}$ $(x \in A)$. Thus again $g = 0$ almost everywhere on A and $f = 0$ almost everywhere on B. ∎

Let Ω be a σ-finite measure space, and take $p \geq 1$. Then $L^p(\Omega)$ has a weak order unit, say e. Suppose that $L^p(\Omega) = E_1 \oplus \cdots \oplus E_k$ is a band decomposition, with corresponding projections $P_1, \ldots, P_k$. Set $v_j = P_j e$ $(j \in \mathbb{N}_n)$. Then, as remarked on page 31, each P_{v_j} is just multiplication of elements of $L^p(\Omega)$ by the characteristic function of a measurable set, say S_j; since we have $P_j = P_{v_j}$, the range E_j of P_j is just $L^p(S_j)$. Thus each band decomposition of $L^p(\Omega)$ has the form $L^p(S_1) \oplus \cdots \oplus L^p(S_k)$ for a measurable partition $\{S_1, \ldots, S_k\}$ of Ω. This may not be true when Ω is not σ-finite. However the following result applies even when S is not countable.

COROLLARY 7.37. *Let S be a non-empty set, and take $p \geq 1$. Suppose that*

$$\ell^p(S) = E_1 \oplus \cdots \oplus E_k$$

is an orthogonal decomposition with respect to the standard p-multi-norm. Then there is a partition $\{S_1, \ldots, S_k\}$ of S such that $E_j = \ell^p(S_j)$ $(j \in \mathbb{N}_k)$.

Proof. By Example 4.47, the standard p-multi-norm is the lattice multi-norm.

The result follows from Proposition 7.8 and Theorem 7.20(i), and also by an easy direct version of the above argument. ∎

COROLLARY 7.38. *Let S be a non-empty set, and suppose that $1 \leq p < q$. Then there are no non-trivial decompositions of $\ell^p(S)$ which are orthogonal with respect to the standard q-multi-norm.*

Proof. Suppose that $\ell^p(S) = F \oplus G$ is an orthogonal decomposition with respect to the standard q-multi-norm. For $f \in F$ and $g \in G$, we have

$$\|\,|f| \vee |g|\,\| = \|(f,g)\|_2^{[p]} \geq \|(f,g)\|_2^{[q]} = \|f+g\| = \|f-g\|,$$

and so, by the argument in Proposition 7.36, $\|\,|f| \vee |g|\,\| = \|(f,g)\|_2^{[q]}$. Thus the decomposition is also orthogonal with respect to the standard p-multi-norm. By Corollary 7.37, there are subset S_F and S_G of S with $F = \ell^p(S_F)$ and $G = \ell^p(S_G)$.

Assume towards a contradiction that both S_F and S_G are non-empty, and take $s \in S_F$ and $t \in S_G$. Then

$$2^{1/p} = \|\delta_s + \delta_t\| = \|(\delta_s, \delta_t)\|_2^{[q]} \leq 2^{1/q},$$

a contradiction because $q > p$. Thus the decomposition is trivial. ∎

7.2. Multi-norms generated by closed families. We now discuss multi-norms that are generated by various closed families of direct sum decompositions of Banach spaces; this will lead to a theory of 'multi-duals' of multi-normed spaces.

7.2.1. Generation of multi-norms

DEFINITION 7.39. Let $(E, \|\cdot\|)$ be a normed space, and consider a closed family $\mathcal{K}$ of hermitian decompositions of E. For $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in E$, set

$$\|(x_1, \ldots, x_n)\|_n^{\mathcal{K}} = \sup\{\|P_1 x_1 + \cdots + P_n x_n\| : E = E_1 \oplus \cdots \oplus E_n\},$$

where the supremum is taken over all decompositions in $\mathcal{K}$ of length n.

THEOREM 7.40. *Let $(E, \|\cdot\|)$ be a normed space, and let $\mathcal{K}$ be a closed family of hermitian decompositions of E. Then $((E^n, \|\cdot\|_n^{\mathcal{K}}) : n \in \mathbb{N})$ is a multi-normed space, and each direct sum decomposition in $\mathcal{K}$ is small with respect to this multi-norm.*

Proof. Let $n \in \mathbb{N}$. Then it is clear that $\|\cdot\|_n$ is a seminorm on E^n. By considering the trivial decompositions in $\mathcal{K}$, we see that

$$\|(x_1, \dots, x_n)\|_n \geq \max\{\|x_1\|, \dots, \|x_n\|\} \quad (x_1, \dots, x_n \in E),$$

and so $\|\cdot\|_n$ is a norm on E^n.

It is now easy to see that $((E^n, \|\cdot\|_n^{\mathcal{K}}) : n \in \mathbb{N})$ is a multi-normed space; Axioms (A1), (A3), and (A4) hold because the family $\mathcal{K}$ is closed, and (A2) holds because all the decompositions in the family $\mathcal{K}$ are hermitian.

Take a decomposition $E = E_1 \oplus \cdots \oplus E_n$ in the family $\mathcal{K}$, and take $x_1, \dots, x_n \in E_n$. Then $\|P_1x_1 + \cdots + P_nx_n\| \leq \|(x_1, \dots, x_n)\|_n^{\mathcal{K}}$, and so the decomposition is small with respect to the multi-norm. ∎

DEFINITION 7.41. Let $(E, \|\cdot\|)$ be a normed space, and let $\mathcal{K}$ be a closed family of hermitian decompositions of E. Then the *multi-norm generated by* $\mathcal{K}$ is the multi-norm $(\|\cdot\|_n^{\mathcal{K}} : n \in \mathbb{N})$.

EXAMPLE 7.42. Let $\mathcal{K}$ be the family of all trivial decompositions of a Banach space E. Then the multi-norm generated by $\mathcal{K}$ is the minimum multi-norm. ∎

We now consider when the multi-norm generated by $\mathcal{K}_{\mathrm{herm}}$ is the maximum multi-norm.

EXAMPLE 7.43. (i) Let K be an infinite, connected compact space. By Theorem 7.29, the only decompositions of $C(K)$ in $\mathcal{K}_{\mathrm{herm}}$ are trivial, and so the multi-norm generated by $\mathcal{K}_{\mathrm{herm}}$ is the minimum multi-norm. By Corollary 3.59 (or Theorem 3.56), the minimum multi-norm is not equivalent to the maximum multi-norm.

(ii) Let $E = \ell^p$ with $p \neq 2$. By Proposition 7.8, each hermitian decomposition of E has the form $E = \ell^p(S_1) \oplus \cdots \oplus \ell^p(S_k)$, where $k \in \mathbb{N}$ and $\{S_1, \dots, S_k\}$ is a partition of $\mathbb{N}$. Thus the family $\mathcal{K}_{\mathrm{herm}}$ generates the standard p-multi-norm $(\|\cdot\|_n^{[p]} : n \in \mathbb{N})$. By Corollary 4.28, this multi-norm is equivalent to the maximum multi-norm if and only if $p = 1$ (with equality of multi-norms when $p = 1$).

(iii) Let H be a Hilbert space. The Hilbert multi-norm $(\|\cdot\|_n^H : n \in \mathbb{N})$ based on H was defined in Definition 4.16. It was shown in Theorem 7.30 that the following closed families are equal: (a) the family of all orthogonal decompositions; (b) $\mathcal{K}_{\mathrm{small}}$; (c) $\mathcal{K}_{\mathrm{orth}}$; (d) $\mathcal{K}_{\mathrm{herm}}$. Let these families be called $\mathcal{K}$. Then it is clear from the definition of the Hilbert multi-norm that $(\|\cdot\|_n^{\mathcal{K}} : n \in \mathbb{N}) = (\|\cdot\|_n^H : n \in \mathbb{N})$.

As we remarked on page 87, the Hilbert multi-norm is equivalent to the maximum multi-norm, but is not equal to it, whenever $\dim H$ is sufficiently large. ∎

Let $(E, \|\cdot\|)$ be a normed space, and let $\mathcal{K}$ and $\mathcal{L}$ be two closed families of hermitian decompositions of E with $\mathcal{K} \subset \mathcal{L}$. Then clearly

$$\|(x_1, \dots, x_n)\|_n^{\mathcal{K}} \leq \|(x_1, \dots, x_n)\|_n^{\mathcal{L}} \quad (x_1, \dots, x_n \in E,\, n \in \mathbb{N}),$$

and so $(\|\cdot\|_n^{\mathcal{K}} : n \in \mathbb{N}) \leq (\|\cdot\|_n^{\mathcal{L}} : n \in \mathbb{N})$ with respect to the ordering of $\mathcal{E}_E$ given in Definition 2.24.

The next example shows that two different families of decompositions may generate the same multi-norm.

Example 7.44. Let K be a compact space, and consider the lattice multi-norm based on $C(K)$; this is just the minimum multi-norm based on $C(K)$.

Let $\mathcal{K}$ be the family of trivial decompositions of $C(K)$, and let $\mathcal{L}$ be the family of decompositions of the form $C(K_1) \oplus \cdots \oplus C(K_k)$, where $\{K_1, \ldots, K_k\}$ is a partition of K into clopen subsets. By Theorem 7.29, $\mathcal{L} = \mathcal{K}_{\mathrm{small}} = \mathcal{K}_{\mathrm{orth}} = \mathcal{K}_{\mathrm{herm}}$. Thus $\mathcal{K} \subset \mathcal{L}$, and $\mathcal{K} \neq \mathcal{L}$ as soon as K is not connected. However the multi-norm generated by both $\mathcal{K}$ and $\mathcal{L}$ is the lattice multi-norm based on $C(K)$. ■

7.2.2. Orthogonality with respect to families

Definition 7.45. Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space, and let $\mathcal{K}$ be a closed family of small decompositions of E. Then the multi-norm is *orthogonal with respect to* $\mathcal{K}$ if

$$\|(x_1, \ldots, x_n)\|_n = \|(x_1, \ldots, x_n)\|_n^{\mathcal{K}} \tag{7.12}$$

for each $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in E$. The multi-norm is *orthogonal* if it is orthogonal with respect to $\mathcal{K}_{\mathrm{small}}$.

Thus, in this case, the given multi-norm $(\|\cdot\|_n : n \in \mathbb{N})$ is the multi-norm generated by $\mathcal{K}$.

Of course, it is automatically the case that

$$\|(x_1, \ldots, x_n)\|_n^{\mathcal{K}} \leq \|(x_1, \ldots, x_n)\|_n \quad (x_1, \ldots, x_n \in E,\, n \in \mathbb{N}).$$

We see that a multi-norm $(\|\cdot\|_n : n \in \mathbb{N})$ based on a normed space E is orthogonal if and only if, for each $n \in \mathbb{N}$, each $x_1, \ldots, x_n \in E$, and each $\varepsilon > 0$, there is a direct sum decomposition $E = E_1 \oplus \cdots \oplus E_n$ of E such that

$$\|(x_1, \ldots, x_n)\|_n - \varepsilon \leq \|P_1 x_1 + \cdots + P_n x_n\| \leq \|(x_1, \ldots, x_n)\|_n.$$

For example, it follows from Example 7.43(iii) that the Hilbert multi-norm based on a Hilbert space is orthogonal, and from Example 7.44 that the lattice multi-norm based on $C(K)$ is orthogonal. However, Example 7.50, below, will give an example of a lattice multi-norm that is not orthogonal.

7.2.3. Orthogonality and Banach lattices. Let E be a Banach lattice, and let $\mathcal{K}$ be the family of all band decompositions of E. Clearly $\mathcal{K}$ is a closed family of decompositions that are small with respect to the lattice multi-norm.

Theorem 7.46. *Let E a Banach lattice which is either an AM-space or σ-Dedekind complete. Then the lattice multi-norm based on E is orthogonal with respect to the family of band decompositions of E, and hence is the multi-norm generated by the band decompositions.*

Proof. We must show that, for each $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in E$, we have

$$\|\,|x_1| \vee \cdots \vee |x_n|\,\| = \sup \|P_1 x_1 + \cdots + P_n x_n\|, \tag{7.13}$$

where the supremum is taken over the band decompositions of length n. It is sufficient to suppose that $x_1, \dots, x_n \in E^+$, and we do this. By (1.24), it is sufficient to prove that

$$x \leq \sup\{P_1 x_1 + \cdots + P_n x_n\} \tag{7.14}$$

and that the supremum on the right is attained, where $x = x_1 \vee \cdots \vee x_n$.

In the case where E is an AM-space, the result follows by a slight variation of the argument in Example 7.44.

Now we consider the case where E is σ-Dedekind complete.

First suppose that $n = 2$, and set

$$y = (x_1 - x_2)^+ \quad \text{and} \quad z = -(x_1 - x_2)^-,$$

and let B_y be the band generated by y. By Proposition 1.20(ii) (which applies because E is σ-Dedekind complete), $E = B_y \perp B_y^{\perp}$; the projections onto B_y and $B_y^{\perp}$ are P_y and Q_y, respectively, say. We have $y = P_y(x_1 - x_2)$ and $z = -Q_y(x_1 - x_2)$, and so

$$P_y(x_1 \vee x_2) = P_y(x_2) + P_y((x_1 - x_2) \vee 0) = P_y(x_2) + (P_y(x_1 - x_2)) \vee 0) = P_y(x_2) + y.$$

It follows that $P_y(x_1 \vee x_2) = P_y x_1 \vee P_y x_2 \geq P_y x_2$, and so $P_y(x_1 \vee x_2) = P_y x_1$. Similarly, $Q_y(x_1 \vee x_2) = Q_y x_2$. Thus $x_1 \vee x_2 = P_y x_1 + Q_y x_2$. This establishes the result in the special case where $n = 2$.

The general case follows easily by induction. ∎

Corollary 7.47. *Let E be a σ-Dedekind complete Banach lattice. Then the lattice multi-norm based on E is the multi-norm generated by the family of all band decompositions of E.* ∎

Corollary 7.48. *Take $p \geq 1$. Then the multi-norm generated by the family of all decompositions of ℓ^p as $\ell^p(S_1) \oplus \cdots \oplus \ell^p(S_k)$, where $\{S_1, \dots, S_k\}$ is a partition of $\mathbb{N}$, is the standard p-multi-norm.*

Proof. By Theorem 7.34 and Corollary 7.37, the specified family is the family of all band decompositions of ℓ^p. By Corollary 7.47, this family generates the lattice multi-norm; by Example 4.47, this is the standard p-multi-norm. ∎

Corollary 7.49. *Let E be a Banach lattice with no non-trivial band decompositions. Then the lattice multi-norm based on E is orthogonal with respect to the family of band decompositions of E if and only if E is an AM-space.*

Proof. We must show that E is an AM-space whenever the lattice multi-norm is orthogonal with respect to the family $\mathcal{K}$ of band decompositions of E.

Take $x, y \in E^+$. Since there are only the two trivial band decompositions of length 2, we have

$$\|x \vee y\| = \|(x, y)\|_2^L = \|(x, y)\|_2^{\mathcal{K}} = \|(x, 0)\|_2^L \vee \|(0, y)\|_2^L = \|x\| \vee \|y\|.$$

By (1.37), this shows that E is an AM-space. ∎

Example 7.50. Consider the Banach space $C(\mathbb{I})$ with the norm $\|\cdot\|$ specified by

$$\|f\| = |f|_{\mathbb{I}} + |f(0)| \quad (f \in C(\mathbb{I})).$$

Then $(C(\mathbb{I}), \|\cdot\|)$ is a Banach lattice with no non-trivial band decompositions. However it is not an AM-space (take $f = 1/2$ and $g = Z$, so that $\|f\| = \|g\| = 1$ but $\|f \vee g\| = 3/2$), and so, by Corollary 7.49, the lattice multi-norm is not orthogonal with respect to the family of band decompositions. ∎

7.3. Multi-norms on dual spaces. We now consider how to form the 'multi-dual' of a multi-normed space.

Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space. It is tempting to regard $\mathcal{M}(E, \mathbb{C})$ as the 'multi-dual' of this space. However recall that $\mathcal{M}(E, \mathbb{C}) = E'$ when we regard $\mathbb{C}$ as having its unique multi-norm structure, and that, as a multi-normed space, $\mathcal{M}(E, \mathbb{C})$ has just the minimum multi-norm. Thus the approach of using this multi-normed space as a 'multi-dual' is not satisfactory.

A second temptation is to look at the family $(((E')^n, \|\cdot\|'_n) : n \in \mathbb{N})$ for a multi-normed space $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$, where $\|\cdot\|'_n$ is the dual of the norm $\|\cdot\|_n$. But this is an even worse failure: $(\|\cdot\|'_n : n \in \mathbb{N})$ is a dual multi-norm, not a multi-norm, on $\{(E')^n : n \in \mathbb{N}\}$.

We shall give a different approach, using the notion of orthogonal decompositions. We continue to use the notation of earlier sections.

7.3.1. The multi-dual space. Here we define our concept of a multi-dual space.

Let $(E, \|\cdot\|)$ be a normed space, and let $\mathcal{K}$ be a closed family of hermitian decompositions of E. As in Definition 7.41, $\mathcal{K}$ generates a multi-norm $(\|\cdot\|_n^{\mathcal{K}} : n \in \mathbb{N})$ based on E. We shall now define a multi-norm on $\{(E')^n : n \in \mathbb{N}\}$ in terms of $\mathcal{K}$. Recall that the dual $\mathcal{K}'$ of a closed family $\mathcal{K}$ of direct sum decompositions of E was defined in Definition 1.9.

Definition 7.51. Let $(E, \|\cdot\|)$ be a normed space, and let $\mathcal{K}$ be a closed family of hermitian decompositions of E. Then the multi-norm based on E' which is generated by $\mathcal{K}'$ is the *multi-dual multi-norm* to the multi-norm $(\|\cdot\|_n^{\mathcal{K}} : n \in \mathbb{N})$; it is denoted by

$$(\|\cdot\|_{n,\mathcal{K}}^{\dagger} : n \in \mathbb{N}).$$

The multi-normed space $(((E')^n, \|\cdot\|_{n,\mathcal{K}}^{\dagger}) : n \in \mathbb{N})$ is the *multi-dual space* (with respect to $\mathcal{K}$).

Let $\mathcal{K}$ be a closed family of hermitian decompositions of E. By Proposition 7.5, each member of $\mathcal{K}'$ is a hermitian decomposition of E', and so $(\|\cdot\|_{n,\mathcal{K}}^{\dagger} : n \in \mathbb{N})$ is indeed a multi-norm based on E' by Theorem 7.40. It is an orthogonal multi-norm.

For each $n \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_n \in E'$, we have

$$\|(\lambda_1, \dots, \lambda_n)\|_{n,\mathcal{K}}^{\dagger} = \sup \|P'_1\lambda_1 + \dots + P'_n\lambda_n\| = \sup \|\lambda_1 \circ P_1 + \dots + \lambda_n \circ P_n\|,$$

where the supremum is taken over all the decompositions $E = E_1 \oplus \dots \oplus E_n$ in the closed family $\mathcal{K}$.

Definition 7.52. Let $(\|\cdot\|_n : n \in \mathbb{N})$ be an orthogonal multi-norm based on a normed space E. Then the *multi-dual multi-norm based on* E' is the multi-norm generated by the family $(\mathcal{K}_{\mathrm{small}})'$.

In the above case, the multi-dual multi-norm is itself orthogonal, and so we generate multi-norms based on all the successive dual spaces of E.

PROPOSITION 7.53. *Let* $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ *be a multi-normed space, and let* $\mathcal{K}$ *be a closed family of orthogonal decompositions (with respect to the multi-norm) of* E. *Take* $\lambda_1, \ldots, \lambda_n \in E'$. *Then*

$$\|(\lambda_1, \ldots, \lambda_n)\|_{n,\mathcal{K}}^{\dagger} = \sup_{\mathcal{K}} \sup \left\{ \left| \sum_{i=1}^{n} \langle x_i,\, \lambda_i \rangle \right| : x_i \in E_i,\ \|(x_1, \ldots, x_n)\|_n \leq 1 \right\},$$

where the first supremum is taken over all decompositions $E = E_1 \oplus \cdots \oplus E_n$ *in* $\mathcal{K}$.

Each direct sum decomposition in $\mathcal{K}'$ *is small with respect to the multi-dual multi-norm* $(\|\cdot\|_{n,\mathcal{K}}^{\dagger} : n \in \mathbb{N})$.

Proof. This follows from Proposition 7.22. ∎

EXAMPLE 7.54. Let $\mathcal{K}$ be the family of trivial decompositions of a normed space E, so that the multi-norm generated by $\mathcal{K}$ is the minimum multi-norm based on E. Then $\mathcal{K}'$ is the the family of trivial decompositions of E', and the multi-dual multi-norm is the minimum multi-norm based on E'. ∎

EXAMPLE 7.55. Let K be a compact space, and consider the lattice multi-norm based on $C(K)$. Let $\mathcal{K}$ be the family of trivial decompositions of $C(K)$, and let

$$\mathcal{L} = \mathcal{K}_{\mathrm{small}} = \mathcal{K}_{\mathrm{orth}} = \mathcal{K}_{\mathrm{herm}}$$

be as in Example 7.44. Then both $\mathcal{K}$ and $\mathcal{L}$ generate the lattice multi-norm based on $C(K)$. However $\mathcal{K}'$ is the family of trivial decompositions of $C(K)' = M(K)$, and so the multi-dual multi-norm $(\|\cdot\|_{n,\mathcal{K}}^{\dagger} : n \in \mathbb{N})$ is the minimum multi-norm based on $M(K)$, whereas the multi-dual multi-norm $(\|\cdot\|_{n,\mathcal{L}}^{\dagger} : n \in \mathbb{N})$ is a strictly larger multi-norm based on $M(K)$ as soon as K is not connected. Indeed, in the case where K is a Stonean space, or, equivalently, when $C(K)$ is Dedekind complete, it follows from Proposition 4.32 that $(\|\cdot\|_{n,\mathcal{L}}^{\dagger} : n \in \mathbb{N})$ is the standard 1-multi-norm based on $M(K)$; by Proposition 4.31, this is the lattice multi-norm, and, by Theorem 4.54(i), it is the maximum multi-norm. ∎

EXAMPLE 7.56. Take $p \geq 1$, and let $E = \ell^p$. We again consider the standard p-multi-norm, $(\|\cdot\|_n^{[p]} : n \in \mathbb{N})$, based on E. By Example 4.47, this is the lattice multi-norm based on E.

Let $\mathcal{K}$ be the family of decompositions of the form

$$\ell^p(S_1) \oplus \cdots \oplus \ell^p(S_n),$$

where $\{S_1, \ldots, S_n\}$ is a partition of $\mathbb{N}$. By Theorem 7.34 and Corollary 7.37, we have $\mathcal{K} = \mathcal{K}_{\mathrm{small}} = \mathcal{K}_{\mathrm{orth}}$. Then it is clear that $\mathcal{K}$ generates the standard p-multi-norm on E, and so this multi-norm is orthogonal with respect to $\mathcal{K}$.

Suppose that $p > 1$. The conjugate index to p is q; set $F = \ell^q$. Clearly, $\mathcal{K}'$ is the family of decompositions of the form $\ell^q(S_1) \oplus \cdots \oplus \ell^q(S_k)$, where $\{S_1, \ldots, S_k\}$ is a partition of $\mathbb{N}$. Thus $\mathcal{K}'$ generates the standard q-multi-norm on E'. This shows that the multi-dual of $(((\ell^p)^n, \|\cdot\|_n^{[p]}) : n \in \mathbb{N})$ (with respect to $\mathcal{K}$) is $(((\ell^q)^n, \|\cdot\|_n^{[q]}) : n \in \mathbb{N})$, a fact that was one of the aims of our theory. ∎

Example 7.57. Let H be a Hilbert space. Let $\mathcal{K}$ be the family of orthogonal decompositions of H; by Theorem 7.30, $\mathcal{K} = \mathcal{K}_{\mathrm{small}} = \mathcal{K}_{\mathrm{orth}} = \mathcal{K}_{\mathrm{herm}}$. It is clear from the definition of the Hilbert multi-norm in (4.5) that the multi-norm generated by $\mathcal{K}$ is the Hilbert multi-norm, and that this multi-norm is orthogonal. It is immediate that the multi-dual of $((H^n, \|\cdot\|_n^H) : n \in \mathbb{N})$ (with respect to $\mathcal{K}$) is itself. ■

Let E be a Banach lattice, and let $\mathcal{K}$ be the family of band decompositions of E. By Theorem 7.34, $\mathcal{K} = \mathcal{K}_{\mathrm{small}} = \mathcal{K}_{\mathrm{orth}}$. We shall consider the multi-normed space

$$((E^n, \|\cdot\|_n^L) : n \in \mathbb{N}),$$

where $((\|\cdot\|_n^L) : n \in \mathbb{N})$ is the lattice multi-norm, and suppose that this multi-norm is generated by the family $\mathcal{K}$. We would like to know when the multi-dual (with respect to $\mathcal{K}$) of this multi-Banach space is $((E^n, \|\cdot\|_n^L) : n \in \mathbb{N})$, where $(\|\cdot\|_n^L : n \in \mathbb{N})$ is now the lattice multi-norm on E'. It follows from Example 7.55 that this is not always the case. However we have the following theorem.

Theorem 7.58. *Let E be a Dedekind complete Banach lattice. Then the lattice multi-norm based on E is generated by the family $\mathcal{K}$ of band decompositions, and the multi-dual with respect to $\mathcal{K}$ is the lattice multi-norm based on E'.*

Proof. It follows from Corollary 7.47 that lattice multi-norms based on E and E' are generated by $\mathcal{K}$ and by the family, say $\mathcal{L}$, of band decompositions of E', respectively.

We shall show that the lattice multi-norm based on E' is also generated by $\mathcal{K}'$. Take $n \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_n \in E'$. Then certainly

$$\|(\lambda_1, \dots, \lambda_n)\|_{n,\mathcal{K}}^{\dagger} \leq \|(\lambda_1, \dots, \lambda_n)\|_n^{\mathcal{L}}.$$

We shall show the reverse inequality. We prove the result in the case where $n = 2$; the general case follows by induction.

Thus take $\varepsilon > 0$, and let $E' = F_1 \oplus_\perp F_2$ be a band decomposition of E' such that

$$\|Q_1\lambda_1 + Q_2\lambda_2\| > \|(\lambda_1, \lambda_2)\|_2^{\mathcal{L}} - \varepsilon,$$

where Q_i is the projection on F_i. Thus there exist $\mu_1, \mu_2 \in E'$ such that $|\mu_i| \leq |\lambda_i|$ for $i = 1, 2$ and $\mu_1 \perp \mu_2$ and such that

$$\|\mu_1 + \mu_2\| > \|(\lambda_1, \lambda_2)\|_2^{\mathcal{L}} - \varepsilon.$$

For $i = 1, 2$, define $X_i = \{x \in E : \langle |x|, |\mu_i| \rangle = 0\}$. Then X_1 and X_2 are bands in E, and so, by Proposition 1.20(i), they are principal bands. Set $E_1 = X_1^\perp$ and $E_2 = X_2^\perp$, so that $E_1 \perp E_2$. It is clear that $\mu_1 \in E_1'$ and $\mu_2 \in E_2'$. By enlarging E_1 and E_2, if necessary, we may suppose that $E_1 \oplus E_2 = E$, and so $E = E_1 \oplus_\perp E_2$ is a band decomposition of E. Thus the decomposition $E' = E_1' \oplus_\perp E_2'$ belongs to $\mathcal{K}'$. It follows that

$$\|(\lambda_1, \lambda_2)\|_{n,\mathcal{K}}^{\dagger} > \|(\lambda_1, \lambda_2)\|_2^{\mathcal{L}} - \varepsilon.$$

This holds true for each $\varepsilon > 0$, and so the result follows. ■

7.3.2. Second dual spaces. Let $(E, \|\cdot\|)$ be a normed space, and let $\mathcal{K}$ be a closed family of hermitian decompositions of E. Then $\mathcal{K}$ and $\mathcal{K}'$ generate multi-norms on the two families $\{E^n : n \in \mathbb{N}\}$ and $\{(E')^n : n \in \mathbb{N}\}$, respectively. Similarly, the closed family $\mathcal{K}''$ of hermitian decompositions of E'' generates a multi-norm $(\|\cdot\|_{n,\mathcal{K}}^{\dagger\dagger} : n \in \mathbb{N})$ on $\{(E'')^n : n \in \mathbb{N}\}$.

The following result can be regarded as a multi-normed form of the Hahn–Banach theorem.

THEOREM 7.59. *Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space, let $\mathcal{K}$ be a closed family of small decompositions of E, and consider the multi-norm*

$$(\|\cdot\|_{n,\mathcal{K}}^{\dagger\dagger} : n \in \mathbb{N})$$

based on E''. Then the canonical embedding of E into E'' gives a multi-isometry if and only if the multi-normed space $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is orthogonal with respect to the family $\mathcal{K}$.

Proof. Let $x_1, \dots, x_k \in E$. Then

$$\|(x_1, \dots, x_k)\|_{k,\mathcal{K}}^{\dagger\dagger} = \sup \|P_1'' x_1 + \cdots + P_k'' x_k\|,$$

where the supremum is taken over all projections $P_1, \dots, P_k$ that arise from decompositions in $\mathcal{K}$. Since $P_i'' x_i = P_i x_i$ for $x_i \in E_i$ and $i \in \mathbb{N}_n$, it follows that the canonical embedding is a multi-isometry if and only if the multi-normed space $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is orthogonal with respect to the family $\mathcal{K}$. ■

DEFINITION 7.60. Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space, let $\mathcal{K}$ be a closed family of small decompositions of E. Then the space $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is *multi-reflexive with respect to* $\mathcal{K}$ if the canonical embedding of E into E'' (when the multi-norm based on E'' is taken to be $(\|\cdot\|_{n,\mathcal{K}}^{\dagger\dagger} : n \in \mathbb{N})$) is a multi-isometry that is a surjection.

Thus $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is multi-reflexive with respect to $\mathcal{K}$ if and only if E is a reflexive Banach space and $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is orthogonal with respect to the family $\mathcal{K}$.

EXAMPLE 7.61. Let E be a Banach lattice such that E is reflexive as a Banach space. Then E is Dedekind complete, and so, by Theorem 7.46, the lattice multi-norm is orthogonal with respect to the family $\mathcal{K}$ of band decompositions, and so the space E is multi-reflexive with respect to $\mathcal{K}$. ■

EXAMPLE 7.62. Take $p > 1$, and let $E = L^p(\Omega, \mu)$ for a measure space (Ω, μ), with the standard p-multi-norm. Then $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is multi-reflexive with respect to the family of all band decompositions of E. ■

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