Preface

In 1993, the authors published their book "Invariant Distances and Metrics in Complex Analysis", in which they discussed the state of affairs in the domain. In the meantime, some open questions mentioned in the book have been solved, more explicit formulas for various invariant functions on certain concrete domains were found (see also [Kob 1998]). Moreover, the classical Green function became important in studying Bergman completeness, which finally led to the result that any hyperconvex bounded domain is Bergman complete. Simultaneously, a new development started, namely the study of the Green functions with multipoles. This led to the creation of a lot of new objects. Recently, the surprising example of the symmetrized bidisc was found, which initiated a lot of new activities. The symmetrized bidisc is not biholomorphically equivalent to a convex domain but, nevertheless, its Carathéodory distance and its Lempert function coincide. Hence, it again becomes an interesting question for which domains these two objects are equal.

The main idea of this work is to describe what happened during the last 10 years in the area. The main source is our old book; in particular, if we quote a result before 1993 we refer the reader to our book and not to the original source. This kind of quotation seems to be easier for the authors as well as for the reader. As always, the authors have to apologize for their choice of the material they present. Of course, it reflects their personal taste.

At the end of the writing process a lot of our colleagues helped us to find typographical and mathematical errors in the manuscript and to essentially improve our presentation. We like to thank them all, in particular, thanks are due to our colleagues Z. Błocki, A. Edigarian, P. Jucha, N. Nikolov, H. Youssfi, P. Zapałowski, and W. Zwonek. Nevertheless, the authors are responsible for all the mistakes that remain.

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CHAPTER 1

Holomorphically invariant objects

1.1. Holomorphically contractible families of functions

Let us begin with the following definition of a holomorphically contractible family (cf. [J-P 1993, §4.1]).

DEFINITION 1.1.1. A family $(d_G)_G$ of functions $d_G : G \times G \to \mathbb{R}_+$ ([1]), where G runs over all domains $G \subset \mathbb{C}^n$ (with arbitrary $n \in \mathbb{N}$), is said to be *holomorphically contractible* if the following two conditions are satisfied:

(A) for the unit disc $E \subset \mathbb{C}$ we have

$$d_E(a,z) = m_E(a,z) := \left|\frac{z-a}{1-\bar{a}z}\right|, \qquad a, z \in E$$

(the function $m_E : E \times E \to [0,1)$ is called the *Möbius distance*),

(B) for any domains $G \subset \mathbb{C}^n$, $D \subset \mathbb{C}^m$, every holomorphic mapping $F : G \to D$ is a *contraction* with respect to d_G and d_D, i.e.

$$d_D(F(a), F(z)) \le d_G(a,z), \qquad a, z \in G. \tag{1.1.1}$$

Notice that there is also another version of the definition of the holomorphically invariant family in which the normalization condition (A) is replaced by the condition

(A′) $d_E = p_E$, where

$$p_E := \frac{1}{2}\log\frac{1+m_E}{1-m_E}$$

is the *Poincaré distance*.

Both definitions are obviously equivalent in the sense that $(d_G)_G$ satisfies (A, B) iff the family $(\tanh^{-1} d_G)_G$ satisfies (A′, B). In our opinion the normalization condition (A) is more handy in calculations.

The following contractible families seem to be the most important.

- *Möbius pseudodistance*:

$$\begin{aligned} c_G^*(a,z) &:= \sup\{m_E(f(a), f(z)) : f \in \mathcal{O}(G,E)\} \\ &= \sup\{|f(z)| : f \in \mathcal{O}(G,E),\ f(a) = 0\}, \qquad (a,z) \in G \times G; \end{aligned}$$

the function $c_G := \tanh^{-1} c_G^*$ is called the *Carathéodory pseudodistance*.

([1]) $A_+ := \{x \in A : x \ge 0\}$ $(A \subset \mathbb{R})$, $A_+^n := (A_+)^n$, e.g. $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{Z}_+ = \{0, 1, \dots\}$, $\mathbb{R}_+^n$, $\mathbb{Z}_+^n$.

- *Higher order Möbius function*:

$$m_G^{(k)}(a,z) := \sup\{|f(z)|^{1/k} : f \in \mathcal{O}(G,E), \operatorname{ord}_a f \geq k\}, \quad (a,z) \in G \times G,\ k \in \mathbb{N},$$

where $\operatorname{ord}_a f$ denotes the order of zero of f at a.

- *Pluricomplex Green function*:

$$g_G(a,z) := \sup\{u(z) : u : G \to [0,1), \log u \in \mathcal{PSH}(G), \\ \exists_{C=C(u,a)>0}\ \forall_{w\in G} : u(w) \leq C\|w-a\|\}, \quad (a,z) \in G \times G,$$

where $\mathcal{PSH}(G)$ denotes the family of all functions plurisubharmonic on G (and $\| \ \|$ is the Euclidean norm in $\mathbb{C}^n$) (2).

- *Lempert function*:

$$\begin{aligned}\widetilde{k}_G^*(a,z) &:= \inf\{m_E(\lambda,\mu) : \lambda,\mu \in E,\ \exists_{\varphi\in\mathcal{O}(E,G)} : \varphi(\lambda)=a,\ \varphi(\mu)=z\} \\ &= \inf\{\mu \in [0,1) : \exists_{\varphi\in\mathcal{O}(E,G)} : \varphi(0)=a,\ \varphi(\mu)=z\}, \quad (a,z) \in G \times G.\end{aligned}$$

It is well known that

$$c_G^* = m_G^{(1)} \leq m_G^{(k)} \leq g_G \leq \widetilde{k}_G^*,$$

and for any holomorphically contractible family $(d_G)_G$ we have

$$c_G^* \leq d_G \leq \widetilde{k}_G^*, \tag{1.1.2}$$

i.e. the Möbius family is minimal and the Lempert family is maximal.

Put $\widetilde{k}_G := \tanh^{-1} \widetilde{k}_G^*$. The pseudodistance

$$k_G := \sup\{d : d : G \times G \to \mathbb{R}_+ \text{ is a pseudodistance with } d \leq \widetilde{k}_G\}$$

is called the *Kobayashi pseudodistance*; cf. [J-P 1993, Ch. 3]. Observe that $(k_G)_G$ satisfies (A′, B).

Notice that one can consider conditions weaker than (B), for instance:

(B′) condition (1.1.1) holds for every injective holomorphic mapping $F : G \to D$;
(B″) condition (1.1.1) holds for every biholomorphic mapping $F : G \to D$.

For example:

- The family $(H_G^*)_G$ of *Hahn functions*

$$\begin{aligned}&H_G^*(a,z) := \inf\{m_E(\lambda,\mu) : \exists_{\varphi\in\mathcal{O}(E,G)} :\ \varphi \text{ is injective}, \varphi(\lambda)=a,\ \varphi(\mu)=z\} \\ &= \inf\{\mu \in [0,1) : \exists_{\varphi\in\mathcal{O}(E,G)} :\ \varphi \text{ is injective}, \varphi(0)=a,\ \varphi(\mu)=z\}, \quad (a,z) \in G \times G,\end{aligned}$$

satisfies (A, B′). Obviously, $\widetilde{k}_G^* \leq H_G^*$.

- The family $(b_G)_G$ of *Bergman pseudodistances* (see §3.5) satisfies (A, B″).

REMARK 1.1.2. The notion of the holomorphically contractible family $(d_G)_G$ (Definition 1.1.1) may be extended to the case where G runs through all connected complex manifolds, complex analytic sets, or even complex spaces. In particular, one can define the Möbius pseudodistance c_M^*, the Lempert function $\widetilde{k}_M^*$ (defined to be 1 for pairs of points

(2) The function $g_G : G \times G \to [0,1)$ is upper semicontinuous (cf. [Jar-Pfl 1995b]). For relations between the pluricomplex and classical Green functions in the unit ball see [Car 1997]. For a different pluricomplex Green function see [Ceg 1995], [Edi-Zwo 1998a].

for which there is no analytic disc passing through them), and the Kobayashi pseudodistance k_M for an arbitrary connected complex analytic set M. The following elementary example points out some new problems appearing in this case.

Let $M := \{(z,w) \in E^2 : z^2 = w^3\}$ be the Neil parabola. M is a connected one-dimensional analytic subset of E^2 with $\operatorname{Reg} M = M_* = M \setminus \{(0,0)\}$ (3). The set M has a global bijective holomorphic parametrization

$$E \ni \lambda \xrightarrow{p} (\lambda^3, \lambda^2) \in M.$$

- The mapping $q := p^{-1}$ is holomorphic on M_* and continuous on M. Note that $q(z,w) = z/w$, $(z,w) \in M_*$, $q(0,0) = 0$.
- The mapping $q|_{M_*} : M_* \to E_*$ is biholomorphic. Thus
$$c^*_{M_*}((a,b),(z,w)) = m_{E_*}(q(a,b), q(z,w)) = m_E(q(a,b), q(z,w)),$$
$$\widetilde{k}^*_{M_*}((a,b),(z,w)) = \widetilde{k}^*_{E_*}(q(a,b), q(z,w)), \qquad (a,b),(z,w) \in M_*.$$
- For any $\varphi \in \mathcal{O}(E, M)$ there exists a $\psi \in \mathcal{O}(E,E)$ such that $\varphi = p \circ \psi$. Hence
$$\widetilde{k}^*_M((a,b),(z,w)) = m_E(q(a,b), q(z,w)), \quad (a,b),(z,w) \in M.$$
- For any $f \in \mathcal{O}(M, E)$ the holomorphic function $h := f \circ p : E \to E$ satisfies $h'(0) = 0$. Conversely, for any $h \in \mathcal{O}(E,E)$ with $h'(0) = 0$ the function $f := h \circ q$ is holomorphic on M. Hence
$$c^*_M((a,b),(z,w)) = \sup\{|h(q(z,w))| : h \in \mathcal{O}(E,E),\ h(q(a,b)) = 0,\ h'(0) = 0\},$$
$$(a,b),(z,w) \in M.$$

It is a little surprising that, despite the elementary description, $\boxed{?}$ an effective formula for c^*_M is not known. $\boxed{?}$ One can prove that for any $\lambda_0 \in E_*$, we have

$$\begin{aligned}&\sup\{|h| : h \in \mathcal{O}(E,E),\ h(\lambda_0) = 0,\ h'(0) = 0\}\\ &\qquad = \sup\{|B| : B \text{ is a Blaschke product of order} \le 3,\ B(\lambda_0) = 0,\ B'(0) = 0\}.\end{aligned}$$

1.2. Holomorphically contractible families of pseudometrics

Parallel to the category of holomorphically contractible families of functions (in the sense of Definition 1.1.1) one studies holomorphically contractible families of pseudometrics (cf. [J-P 1993, §4.1]).

DEFINITION 1.2.1. A family $(\delta_G)_G$ of $\mathbb{C}$-*pseudometrics* $\delta_G : G \times \mathbb{C}^n \to \mathbb{R}_+$, $G \subset \mathbb{C}^n$,

$$\delta_G(a; \lambda X) = |\lambda| \delta_G(a; X), \qquad a \in G,\ X \in \mathbb{C}^n,\ \lambda \in \mathbb{C},$$

where G runs over all domains $G \subset \mathbb{C}^n$, is said to be *holomorphically contractible* if the following two conditions are satisfied:

(A) $\delta_E(a; X) = \gamma_E(a; X) := \dfrac{|X|}{1 - |a|^2}$, $a \in E$, $X \in \mathbb{C}$,

(3) $\operatorname{Reg} M$ denotes the set of all *regular points* of M, wheras $A_* := A \setminus \{0\}$ ($A \subset \mathbb{C}^n$), $A^n_* := (A_*)^n$, e.g. E_*, $\mathbb{C}_*$, $(\mathbb{Z}^n_+)_*$, $\mathbb{C}^n_*$.

(B) for any domains $G \subset \mathbb{C}^n$, $D \subset \mathbb{C}^m$ and for every holomorphic mapping $F : G \to D$ we have

$$\delta_D(F(a); F'(a)(X)) \le \delta_G(a; X), \quad (a, X) \in G \times \mathbb{C}^n. \tag{1.2.3}$$

The following holomorphically contractible families of pseudometrics correspond to the holomorphically contractible families of functions from §1.1.

- *Carathéodory–Reiffen pseudometric*:

$$\gamma_G(a; X) := \sup\{|f'(a)(X)| : f \in \mathcal{O}(G, E),\ f(a) = 0\}, \qquad (a, X) \in G \times \mathbb{C}^n;$$

we have

$$\gamma_G(a; X) = \lim_{\mathbb{C}_* \ni \lambda \to 0} \frac{1}{|\lambda|}\, c^*_G(a, a + \lambda X) = \lim_{\substack{z', z'' \to a \\ \frac{z' - z''}{\|z' - z''\|} \to X}} \frac{c^*_G(z', z'')}{\|z' - z''\|},$$
$$(a, X) \in G \times \mathbb{C}^n,\ \|X\| = 1.$$

- *Higher order Reiffen pseudometric*:

$$\gamma^{(k)}_G(a; X) := \sup\left\{\left|\frac{1}{k!}\, f^{(k)}(a)(X)\right|^{1/k} : f \in \mathcal{O}(G, E), \operatorname{ord}_a f \ge k\right\},$$
$$(a, X) \in G \times \mathbb{C}^n,\ k \in \mathbb{N};$$

we have

$$\gamma^{(k)}_G(a; X) = \lim_{\mathbb{C}_* \ni \lambda \to 0} \frac{1}{|\lambda|}\, m^{(k)}_G(a, a + \lambda X), \qquad (a, X) \in G \times \mathbb{C}^n,$$

and if G is biholomorphic to a bounded domain, then

$$\gamma^{(k)}_G(a; X) = \lim_{\substack{z', z'' \to a \\ \frac{z' - z''}{\|z' - z''\|} \to X}} \frac{m^{(k)}_G(z', z'')}{\|z' - z''\|}, \qquad (a, X) \in G \times \mathbb{C}^n,\ \|X\| = 1.$$

- *Azukawa pseudometric*:

$$A_G(a; X) := \limsup_{\mathbb{C}_* \ni \lambda \to 0} \frac{1}{|\lambda|}\, g_G(a, a + \lambda X), \qquad (a, X) \in G \times \mathbb{C}^n;$$

if G is a bounded hyperconvex domain, then

$$A_G(a; X) = \lim_{\substack{z', z'' \to a \\ \frac{z' - z''}{\|z' - z''\|} \to X}} \frac{g_G(z', z'')}{\|z' - z''\|}, \qquad (a, X) \in G \times \mathbb{C}^n,\ \|X\| = 1$$

(cf. [Zwo 2000c, Corollary 4.4]).

- *Kobayashi–Royden pseudometric*:

$$\varkappa_G(a; X) := \inf\{\alpha \ge 0 : \exists_{\varphi \in \mathcal{O}(E, G)} : \varphi(0) = a,\ \alpha\varphi'(0) = X\}, \qquad (a, X) \in G \times \mathbb{C}^n;$$

if G is taut, then

$$\varkappa_G(a; X) = \lim_{\mathbb{C}_* \ni \lambda \to 0} \frac{1}{|\lambda|}\, \widetilde{k}_G(a, a + \lambda X), \qquad (a, X) \in G \times \mathbb{C}^n$$

(cf. [Pan 1994]).

It is well known that $\gamma_G = \gamma_G^{(1)} \le \gamma_G^{(k)} \le A_G \le \varkappa_G$. Moreover, for any holomorphically contractible family of pseudometrics $(\delta_G)_G$ we have $\gamma_G \le \delta_G \le \varkappa_G$ for any G. Notice that (cf. [J-P 1993], [Jar-Pfl 1995b]):

- γ_G is Lipschitz continuous;
- $\gamma_G^{(k)}$ is upper semicontinuous; if $\gamma_G(a;X) > 0$, $(a,X) \in G \times (\mathbb{C}^n)_*$, then $\gamma_G^{(k)}$ is continuous (cf. [Nik 2000]); in particular, if G is bounded, then $\gamma_G^{(k)}$ is continuous;
- A_G is upper semicontinuous;
- $\varkappa_G$ is upper semicontinuous; if G is taut, then $\varkappa_G$ is continuous.

Similarly to the case of contractible functions, one can consider conditions weaker than (B), for example:

(B′) condition (1.2.3) holds for every injective holomorphic mapping $F : G \to D$;
(B″) condition (1.2.3) holds for every biholomorphic mapping $F : G \to D$.

For example:

- The family $(h_G)_G$ of *Hahn pseudometrics*

$$h_G(a;X) := \inf\{\alpha \ge 0 : \exists_{\varphi\in\mathcal{O}(E,G)} : \varphi \text{ is injective}, \varphi(0)=a, \alpha\varphi'(0)=X\}, \quad (a,X) \in G \times \mathbb{C}^n,$$

  satisfies (A, B′). Obviously, $\varkappa_G \le h_G$.
- The families of *Wu* and *Bergman pseudometrics* satisfy (A, B″) (see §§1.2.6, 3.5).

REMARK 1.2.2. (a) If $G \subset \mathbb{C}$, then $\widetilde{k}_G^* \equiv H_G^*$ iff $\varkappa_G \equiv h_G$ iff G is simply connected.
(b) $\widetilde{k}_G^* \equiv H_G^*$ and $\varkappa_G \equiv h_G$ for any domain $G \subset \mathbb{C}^n$ with $n \ge 3$ (cf. [Ove 1995]).
(c) Let $D_1, D_2 \subset \mathbb{C}$ be domains. Then (cf. [JarW 2000], [JarW 2001]):

- if at least one of the domains D_1, D_2 is simply connected or biholomorphic to $\mathbb{C}_*$, then $h_{D_1\times D_2} \equiv \varkappa_{D_1\times D_2}$ and $H_{D_1\times D_2}^* \equiv \widetilde{k}_{D_1\times D_2}^*$;
- otherwise $h_{D_1\times D_2} \not\equiv \varkappa_{D_1\times D_2}$ and $H_{D_1\times D_2}^* \not\equiv \widetilde{k}_{D_1\times D_2}^*$; see also [Choi 1998].

1.2.1. Inner pseudodistances. For a domain $G \subset \mathbb{C}^n$ let $\mathfrak{D}(G)$ be the family of all pseudodistances $\varrho : G \times G \to \mathbb{R}_+$ such that

$$\forall_{a\in G}\ \exists_{M,r>0} : \varrho(z,w) \le M\|z-w\|, \qquad z,w \in \mathbb{B}(a,r) \subset G,$$

where $\mathbb{B}(a,r)$ denotes the Euclidean ball with center at a and radius r. Notice that for any holomorphically contractible family of pseudodistances $(d_G)_G$ (with the normalization condition (A) or (A′)), we have $d_G \in \mathfrak{D}(G)$ for any G. Let $\mathcal{F}$ be one of the following three families of curves in $\mathbb{C}^n$:

- $\mathcal{F}_{in}$:= the family of all curves,
- $\mathcal{F}_i$:= the family of all rectifiable curves (in the Euclidean sense),
- $\mathcal{F}_{ic}$:= the family of all piecewise $\mathcal{C}^1$-curves.

For any $\varrho \in \mathfrak{D}(G)$ we define the *inner pseudodistance for ϱ with respect to the family $\mathcal{F}$*:

$$\varrho^{\mathcal{F}}(a,z) := \inf\{L_\varrho(\alpha) : \alpha : [0,1] \to G,\ \alpha(0)=a,\ \alpha(1)=z,\ \alpha\in\mathcal{F}\}, \qquad (a,z) \in G\times G,$$

where $L_\varrho(\alpha)$ is the *ϱ-length of α*:

$$L_\varrho(\alpha) := \sup\Big\{\sum_{j=1}^{N} \varrho(\alpha(t_{j-1}), \alpha(t_j)) : 0 = t_0 < t_1 < \cdots < t_N = 1,\ N \text{ arbitrary}\Big\}.$$

Note that:

- $\varrho^{\mathcal{F}} \in \mathfrak{D}(G)$,
- $\varrho^{\mathcal{F}} \ge \varrho$,
- $L_{\varrho^{\mathcal{F}}}(\alpha) = L_\varrho(\alpha)$ for any $\alpha \in \mathcal{F}$,
- $(\varrho^{\mathcal{G}})^{\mathcal{F}} = \varrho^{\mathcal{F}}$ for $\mathcal{F} \subset \mathcal{G}$,
- $(\varrho^{\mathcal{F}})^{\mathcal{F}} = \varrho^{\mathcal{F}}$,
- $\varrho^{\mathcal{F}} = (\tanh \varrho)^{\mathcal{F}}$.

We put:

- $\varrho^{in} := \varrho^{\mathcal{F}_{in}}$ (cf. [Rin 1961]),
- $\varrho^{i} := \varrho^{\mathcal{F}_{i}}$ (cf. [J-P 1993]),
- $\varrho^{ic} := \varrho^{\mathcal{F}_{ic}}$ (cf. [Ven 1989]).

Note that $\varrho \le \varrho^{in} \le \varrho^{i} \le \varrho^{ic}$. We say that ϱ is *inner* if $\varrho = \varrho^{ic}$ (in particular, $\varrho = \varrho^{in} = \varrho^{i} = \varrho^{ic}$); see also [Bar 1995].

In particular, we introduce the *inner Carathéodory pseudodistance* $(c^i_G)_G$. It is known that:

- $c^i_G = c^{ic}_G$ for any G;
- $c^{in}_G = c^i_G = c^{ic}_G$ if G is biholomorphic to a bounded domain or $G \subset \mathbb{C}^1$; notice that $\boxed{?}$ in the general case the equality $c^{in}_G = c^i_G$ remains still open; $\boxed{?}$
- $c^i \ne c$; for instance $c_A \not\equiv c^i_A$ if $A \subset \mathbb{C}$ is an annulus (cf. [J-P 1993, Example 2.5.7], see also [Jar-Pfl 1993a]);
- $m^i_E = p_E = p^i_E$.

On the other hand, the Kobayashi pseudodistance is obviously inner, i.e. $k_G = k^{in}_G = k^i_G = k^{ic}_G$ for any G (cf. [J-P 1993, Proposition 3.3.1]).

If $(d_G)_G$ is a holomorphically contractible family of pseudodistances (with the normalization condition (A) or (A′)), then the families $(d^{in}_G)_G$, $(d^i_G)_G$, $(d^{ic}_G)_G$ are holomorphically contractible with the normalization condition (A′).

1.2.2. Integrated forms. The idea of inner pseudodistances is strictly connected with the idea of integrated forms from differential geometry. More precisely, for a domain $G \subset \mathbb{C}^n$, let $\mathfrak{M}(G, \mathbb{K})$ ($\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$) denote the space of all *$\mathbb{K}$-pseudometrics*

$$\eta : G \times \mathbb{C}^n \to \mathbb{R}_+, \quad \eta(a; tX) = |t|\eta(a; X), \qquad (a, X) \in G \times \mathbb{C}^n,\ t \in \mathbb{K},\ (^4)$$

such that

$$\forall_{a\in G}\ \exists_{M,r>0} : \eta(z; X) \le M\|X\|, \qquad z \in \mathbb{B}(a, r) \subset G,\ X \in \mathbb{C}^n.$$

(4) Notice that so far we have used only $\mathbb{C}$-pseudometrics (cf. Definition 1.2.1).

If $\eta \in \mathfrak{M}(G, \mathbb{K})$ is Borel measurable (e.g. $\eta \in \{\gamma_G, \gamma_G^{(k)}, A_G, \varkappa_G\}$), then we define the *integrated form of* η:

$$(\textstyle\int \eta)(a, z) := \inf\{L_\eta(\alpha) : \alpha : [0, 1] \to G,\ \alpha(0) = a,\ \alpha(1) = z,\ \alpha \in \mathcal{F}_{ic}\}, \qquad a, z \in G,$$

where $L_\eta(\alpha)$ is the *η-length of* α:

$$L_\eta(\alpha) := \int_0^1 \eta(\alpha(t); \alpha'(t))\, dt.$$

One can easily prove that $\int \eta \in \mathfrak{D}(G)$ and $(\int \eta)^{ic} = \int \eta$, i.e. $\int \eta$ is always inner.

1.2.3. Buseman pseudometric. Let $h : \mathbb{C}^n \to \mathbb{R}_+$ be such that:

- $h(\lambda X) = |\lambda| h(X)$, $X \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$,
- there exists a constant $M > 0$ for which $h(X) \le M\|X\|$, $X \in \mathbb{C}^n$.

Define the *Buseman seminorm for* h:

$$\widehat{h} := \sup\{q : q \text{ is a } \mathbb{C}\text{-seminorm},\ q \le h\};$$

note that $\widehat{h}$ is a $\mathbb{C}$-seminorm (in particular, $\widehat{h}$ is continuous) and $\widehat{h} \le h$ (cf. [J-P 1993, §4.3]).

If $\eta \in \mathfrak{M}(G, \mathbb{C})$ (cf. §1.2.2), then we define the *Buseman pseudometric associated to* η:

$$\widehat{\eta}(a; X) := (\eta(a; \cdot))\widehat{\ }(X), \qquad (a, X) \in G \times \mathbb{C}^n$$

(cf. [J-P 1993, §4.3]). In particular, we define the *Kobayashi–Buseman pseudometric* $\widehat{\varkappa}_G$.

Recall that:

- if η is upper semicontinuous, then so is $\widehat{\eta}$;
- if $(\delta_G)_G$ is a holomorphically contractible family of pseudometrics, then so is $(\widehat{\delta_G})_G$.

REMARK 1.2.3. (a) One can easily prove that if η is a continuous metric ($\eta(a; X) > 0$, $(a, X) \in G \times (\mathbb{C}^n)_*$), then so is $\widehat{\eta}$ (cf. the proof of Proposition 1.2.13(a)).

(b) The following example (due to W. Jarnicki) shows that if η is a continuous pseudometric, then $\widehat{\eta}$ need not be continuous. Let $\eta : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{R}_+$, $\eta(z; X) := \max\{0, |X_2| - \|z\|\,|X_1|\}$. Then η is a continuous pseudometric. Observe that $\eta(0; X) = |X_2| = \widehat{\eta}(0; X)$ and $\widehat{\eta}(z; \cdot) \equiv 0$ for $z \ne 0$ (in particular, $\widehat{\eta}$ is not continuous). Indeed, for $z \ne 0$ we have $\widehat{\eta}(z; (X_1, 0)) \le \eta(z; (X_1, 0)) = 0$ and

$$\begin{aligned}\widehat{\eta}(z; (0, X_2)) &\le \widehat{\eta}(z; (0, 2X_2)) \le \widehat{\eta}(z; (X_2/\|z\|, X_2)) + \widehat{\eta}(z; (-X_2/\|z\|, X_2)) \\ &\le \eta(z; (X_2/\|z\|, X_2)) + \eta(z; (-X_2/\|z\|, X_2)) = 0.\end{aligned}$$

1.2.4. Derivatives. It is natural to conjecture that

(∗) for any $\varrho \in \mathfrak{D}(G)$ (cf. §1.2.1) there exists a Borel measurable pseudometric $\eta = \eta(\varrho) \in \mathfrak{M}(G, \mathbb{K})$ such that $\varrho^{ic} = \int \eta$ and if $(d_G)_G$ is a holomorphically contractible family of pseudodistances, then $(\eta(d_G))_G$ is a contractible family of $\mathbb{K}$-pseudometrics.

REMARK 1.2.4. (a) Recall that $k_G = \int \varkappa_G = \int \widehat{\varkappa}_G$, where $\widehat{\varkappa}_G$ is the Kobayashi–Buseman pseudometric (§1.2.3); cf. [Ven 1996] for a generalization of the formula $k_G = \int \varkappa_G$ to the case of analytic spaces. Thus, we can take $\eta(k_G) := \varkappa_G$ or $\eta(k_G) := \widehat{\varkappa}_G$.

(b) Notice that in general $\varkappa_G$ is not determined by k_G; there exists a pseudoconvex Hartogs domain $G \subset \mathbb{C}^2$ such that $k_G \equiv 0$ and $\varkappa_G \not\equiv 0$ (cf. [J-P 1993, Example 3.5.10]).

(c) It is known that the problem $(*)$ has a positive solution in the category of so-called $\mathcal{C}^1$-*pseudodistances*, i.e. those pseudodistances $\varrho \in \mathfrak{D}(G)$ for which the limit

$$(\mathcal{D}\varrho)(a;X) = \lim_{\substack{\mathbb{C}_* \ni \lambda \to 0 \\ z \to a \\ Y \to X}} \frac{1}{|\lambda|}\, \varrho(z, z + \lambda Y)$$

exists for all $(a, X) \in G \times \mathbb{C}^n$ and the function

$$G \times \mathbb{C}^n \ni (a, X) \mapsto (\mathcal{D}\varrho)(a;X)$$

is continuous. If ϱ is a $\mathcal{C}^1$-pseudodistance, then

$$(\mathcal{D}\varrho)(a;X) = \lim_{\substack{z', z'' \to a \\ \frac{z'-z''}{\|z'-z''\|} \to X}} \frac{\varrho(z', z'')}{\|z' - z''\|}, \qquad (a, X) \in G \times \mathbb{C}^n,\ \|X\| = 1;$$

$\varrho^i = \varrho^{ic} = \int(\mathcal{D}\varrho)$, ϱ^i is a $\mathcal{C}^1$-pseudodistance, and $\mathcal{D}\varrho^i = \mathcal{D}\varrho$ (cf. [J-P 1993, Proposition 4.3.9]).

In particular, since c_G is a $\mathcal{C}^1$-pseudodistance, we have $\eta(c_G) = \eta(c_G^i) = \gamma_G$.

(d) M. Kobayashi proved in [KobM 2000] that if G is taut, then k_G is a $\mathcal{C}^1$-pseudodistance.

In the case $\mathbb{K} = \mathbb{C}$ the problem $(*)$ seems to be open (cf. [J-P 1993, remark after Theorem 4.3.10]). Surprisingly, in the case $\mathbb{K} = \mathbb{R}$, $(*)$ has the following complete solution. For $\varrho \in \mathfrak{D}(G)$ define

$$(\mathbb{D}\varrho)(a;X) := \limsup_{\mathbb{R}_* \ni t \to 0} \frac{1}{|t|}\, \varrho(a, a + tX), \qquad (a, X) \in G \times \mathbb{C}^n;$$

cf. [Ven 1989]. We say that $\mathbb{D}\varrho$ is the *weak derivative of* ϱ. One can prove that:

- $\mathbb{D}\varrho \in \mathfrak{M}(G, \mathbb{R})$, $\mathbb{D}\varrho$ is Borel measurable.
- $$(\mathbb{D}\varrho)(a;X) = \limsup_{\substack{\mathbb{R}_* \ni t \to 0 \\ Y \to X}} \frac{1}{|t|}\varrho(a, a + tY), \qquad (a, X) \in G \times \mathbb{C}^n,$$
  $$(\mathbb{D}\varrho)(a;X) = \limsup_{\substack{z \to a \\ \frac{z-a}{\|z-a\|} \to X}} \frac{\varrho(a, z)}{\|a - z\|}, \qquad (a, X) \in G \times \mathbb{C}^n,\ \|X\| = 1.$$
- $L_\varrho(\alpha) = L_{\mathbb{D}\varrho}(\alpha)$ for any piecewise $\mathcal{C}^1$-curve $\alpha : [0, 1] \to G$. In particular, $\varrho^{ic} = \int(\mathbb{D}\varrho)$.
- If $(d_G)_G$ is a holomorphically contractible family of pseudodistances, then $(\mathbb{D}d_G)_G$ is a holomorphically contractible family of $\mathbb{R}$-pseudometrics.
- $\int(\mathbb{D}k_G) = k_G$.

1.2.5. Complex geodesics. Recall that a holomorphic mapping $\varphi : E \to G$ (G is a domain in $\mathbb{C}^n$) is called a *complex geodesic* if $c_G^*(\varphi(\lambda'), \varphi(\lambda'')) = m_E(\lambda', \lambda'')$ for any $\lambda', \lambda'' \in E$.

Let $(d_G)_G$ be a holomorphically contractible family of functions. Fix a domain $G \subset \mathbb{C}^n$ and let $z_0', z_0'' \in G$, $z_0' \neq z_0''$. We say that $\varphi \in \mathcal{O}(E, G)$ is a *d_G-geodesic* for (z_0', z_0'') if there exist $\lambda_0', \lambda_0'' \in E$ such that $z_0' = \varphi(\lambda_0')$, $z_0'' = \varphi(\lambda_0'')$, and $d_G(z_0', z_0'') = m_E(\lambda_0', \lambda_0'')$. If φ is a d_G-geodesic for (z_0', z_0''), then $d_G(z_0', z_0'') = \widetilde{k}_G^*(z_0', z_0'')$. Obviously, any complex geodesic is a c_G^*-geodesic for (z_0', z_0'') with arbitrary $z_0', z_0'' \in \varphi(E)$, $z_0' \neq z_0''$.

Let $(\delta_G)_G$ be a holomorphically contractible family of pseudometrics. Let $z_0 \in G$, $X_0 \in \mathbb{C}_*^n$. We say that $\varphi \in \mathcal{O}(E, G)$ is a *δ_G-geodesic* for (z_0, X_0) if there exist $\lambda_0 \in E$, $\alpha_0 \in \mathbb{C}$ such that $z_0 = \varphi(\lambda_0)$, $X_0 = \alpha_0 \varphi'(\lambda_0)$, and $\delta_G(z_0; X_0) = \gamma_E(\lambda_0; \alpha_0)$. If φ is a δ_G-geodesic for (z_0, X_0), then $\delta_G(z_0; X_0) = \varkappa_G(z_0; X_0)$.

PROPOSITION 1.2.5 ([J-P 1993, Proposition 8.1.3]). *For a mapping $\varphi \in \mathcal{O}(E, G)$ the following conditions are equivalent*:

(i) $\exists_{\lambda_0', \lambda_0'' \in E, \lambda_0' \neq \lambda_0''} : c_G^*(\varphi(\lambda_0'), \varphi(\lambda_0'')) = m_E(\lambda_0', \lambda_0'')$, *i.e. φ is a complex c_G^*-geodesic for $(\varphi(\lambda_0'), \varphi(\lambda_0''))$*;

(ii) $\forall_{\lambda', \lambda'' \in E} : c_G^*(\varphi(\lambda'), \varphi(\lambda'')) = m_E(\lambda', \lambda'')$, *i.e. φ is a complex geodesic*;

(iii) $\forall_{\lambda \in E} : \gamma_G(\varphi(\lambda); \varphi'(\lambda)) = \gamma_E(\lambda; 1)$, *i.e. φ is a complex γ_G-geodesic for any pair $(\varphi(\lambda), \varphi'(\lambda))$*;

(iv) $\exists_{\lambda_0 \in E} : \gamma_G(\varphi(\lambda_0); \varphi'(\lambda_0)) = \gamma_E(\lambda_0; 1)$, *i.e. φ is a complex γ_G-geodesic for $(\varphi(\lambda_0), \varphi'(\lambda_0))$*.

Consequently, any complex c_G^*- or γ_G-geodesic φ is a complex geodesic. Moreover, φ is injective, proper, and regular. In particular, $\varphi(E)$ is a 1-dimensional complex submanifold of G.

PROPOSITION 1.2.6 ([J-P 1993, Proposition 8.1.5]). *Let $G \subset \mathbb{C}^n$ be a taut domain. Then the following conditions are equivalent*:

(i) $c_G^* = \widetilde{k}_G^*$ *and* $\gamma_G = \varkappa_G$ (5);

(ii) $c_G^* = \widetilde{k}_G^*$;

(iii) *for any $z_0', z_0'' \in G$, $z_0' \neq z_0''$, there exist $\varphi \in \mathcal{O}(E, G)$ and $f \in \mathcal{O}(G, E)$ such that $z_0', z_0'' \in \varphi(E)$ and $f \circ \varphi = \mathrm{id}_E$*;

(iv) *for any $z_0', z_0'' \in G$ there exist a holomorphic embedding $\varphi : E \to G$ and a holomorphic retraction $r : G \to \varphi(E)$ such that $z_0', z_0'' \in \varphi(E)$.*

Moreover, any holomorphic mapping $\varphi : E \to G$ satisfying (iii) *or* (iv) *is a complex geodesic. Conversely, for any complex geodesic φ there exists f (resp. r) such that* (iii) (*resp.* (iv)) *is satisfied.*

Recently, Proposition 1.2.5 was generalized in the following way in [EHHM 2003, Corollary 9].

(5) For example, G is a convex domain (cf. Lempert Theorem 8.2.1 in [J-P 1993]).

PROPOSITION 1.2.7. *A holomorphic mapping $\varphi : E \to G$ (G is a domain in $\mathbb{C}^n$) is a complex geodesic iff there exist $\lambda_0', \lambda_0'' \in E$, $\lambda_0' \neq \lambda_0''$, such that $c_G^i(\varphi(\lambda_0'), \varphi(\lambda_0'')) = p_E(\lambda_0', \lambda_0'')$.*

Proof. (Here we present a direct proof independent of [EHHM 2003].) Using a suitable automorphism of E, we may assume that $\lambda_0' = 0$ and $\lambda_0'' =: t_0 \in (0,1)$. Recall that $p_E^i = p_E$. Hence, for any $t \in [0, t_0]$, we have

$$\begin{aligned} p_E(0,t_0) &= p_E(0,t) + p_E(t,t_0) \\ &\geq c_G^i(\varphi(0),\varphi(t)) + c_G^i(\varphi(t),\varphi(t_0)) \geq c_G^i(\varphi(0),\varphi(t_0)) = p_E(0,t_0). \end{aligned}$$

Consequently, $c_G^i(\varphi(0),\varphi(t)) = p_E(0,t)$ for any $t \in [0,t_0]$. Let $t_k \searrow 0$ be such that

$$\frac{\varphi(t_k)-\varphi(0)}{\|\varphi(t_k)-\varphi(0)\|} \to X_0 \in \partial\mathbb{B}_n$$

(observe that $X_0 = \varphi'(0)/\|\varphi'(0)\|$ if $\varphi'(0) \neq 0$). Recall that $\mathcal{D}c_G^i = \gamma_G$. We get

$$1 = \gamma_E(0;1) = \lim_{k\to\infty} \frac{p_E(0,t_k)}{t_k} = \lim_{k\to\infty} \frac{c_G^i(\varphi(0),\varphi(t_k))}{t_k} = \gamma_G(\varphi(0);X_0)\|\varphi'(0)\|.$$

Hence $\varphi'(0) \neq 0$ and $1 = \gamma_G(\varphi(0);\varphi'(0))$, which, by Proposition 1.2.5, implies that φ is a complex geodesic. ∎

REMARK 1.2.8. (a) Complex geodesics were recently studied by many authors. For instance:

- in [Jar-Pfl 1995a] for convex *complex ellipsoids*

$$\mathbb{E}_p := \Big\{(z_1,\dots,z_n) \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^{2p_j} < 1\Big\},$$
$$p = (p_1,\dots,p_n),\ p_j \geq 1/2,\ j = 1,\dots,n\ (n \geq 2);\ (^6)$$

- in [Zwo 1997] to prove the following result showing that in the category of complex ellipsoids the symmetry of the pluripolar Green function $g_{\mathbb{E}_p}$ is a very rare phenomenon:

  THEOREM. *For a complex ellipsoid $\mathbb{E}_p$ the following conditions are equivalent*:

  (i) $k_{\mathbb{E}_p}(\lambda_1 b, \lambda_2 b) = p_E(\lambda_1,\lambda_2)$, $b \in \partial\mathbb{E}_p$, $\lambda_1,\lambda_2 \in E$;
  (ii) $g_{\mathbb{E}_p}(\lambda b, 0) = g_{\mathbb{E}_p}(0,\lambda b)$, $b \in \partial\mathbb{E}_p$, $\lambda \in E$;
  (iii) $g_{\mathbb{E}_p}$ *is symmetric*;
  (iv) $\mathbb{E}_p$ *is convex*;

- in [Vis 1999a], [Vis 1999b], [Vis 1999c] for some classes of convex Reinhardt domains;
- in [Pfl-You 2003] for the so-called *minimal ball*

$$\mathbb{M}_n := \{z = (z_1,\dots,z_n) \in \mathbb{C}^n : \|z\|^2 + |z_1^2 + \dots + z_n^2| < 1\}$$

  (which will be studied in §3.1).

---

(6) Observe that $\mathbb{E}_{(1,\dots,1)}$ coincides with the open unit Euclidean ball $\mathbb{B}_n$. Moreover, $\mathbb{E}_p$ is convex if and only if $p_j \geq 1/2$, $j = 1,\dots,n$ (cf. [J-P 1993, §8.4]).

(b) Consider the following general problem: Given a bounded convex balanced domain $G \subset \mathbb{C}^n$ $(n \geq 2)$ with *Minkowski function* h_G ([7]), find conditions on $a, b \in G$ and $r, R \in (0,1)$ under which the Carathéodory ball $B_{c^*_G}(a,r) := \{z \in G : c^*_G(a,z) < r\}$ coincides with the norm ball $B_{h_G}(b,R) := \{z \in \mathbb{C}^n : h_G(z-b) < R\}$ ([8]). Since $c^*_G(0,\cdot) = h_G(\cdot)$, we always have

$$B_{c^*_G}(0,r) = B_{h_G}(0,r), \qquad r \in (0,1).$$

In the case where

$$G = \mathbb{E}_{p,\alpha} := \Big\{(z_1,\dots,z_n) \in \mathbb{C}^n : 2\alpha|z_1|^{p_1}|z_2|^{p_2} + \sum_{j=1}^n |z_j|^{2p_j} < 1\Big\},$$
$$p = (p_1,\dots,p_n) \in \mathbb{R}^n_{>0},\ \alpha \geq 0, \ (^9)$$

the problem was studied in:

[Sch 1993] (the case $n = 2$, $\alpha = 0$, $p_1 = p_2 = 1$),
[Sre 1995], [Zwo 1995] (the case $\alpha = 0$, $p_1 = \dots = p_n = 1$),
[Sch-Sre 1996] (the case $\alpha = 0$, $1 < p_1 = \dots = p_n \notin \mathbb{N}$),
[Zwo 1996], [Zwo 2000b] (the case $\alpha = 0$),
[Vis 1999a] (the general case).

The methods introduced by W. Zwonek and developed by B. Visintin are based on complex geodesics. The most general result is the following theorem from [Vis 1999a].

THEOREM. *Assume that* $\alpha \geq 0$ *and* $p \in \mathbb{R}^n_{>0}$ *are such that* $\mathbb{E}_{p,\alpha}$ *is convex. Then*

$$B_{c^*_{\mathbb{E}_{p,\alpha}}}(a,r) = B_{h_{\mathbb{E}_{p,\alpha}}}(b,R)$$

for some $a, b \in \mathbb{E}_{p,\alpha}$, $a \neq 0$, $r, R \in (0,1)$ *iff* $\alpha = 0$, $\{j \in \{1,\dots,n\} : a_j \neq 0\} = \{j_0\}$, $p_{j_0} = 1$, *and* $p_j = 1/2$ *for all* $j \neq j_0$.

REMARK 1.2.9. Let $G \subset \mathbb{C}^n$ be a domain and let $(z_0, X_0) \in G \times \mathbb{C}^n$. Recall that a mapping $\varphi \in \mathcal{O}(E,G)$ is called a $\varkappa_G$-geodesic for (z_0,X_0) if there are $\lambda_0 \in E$ and $\alpha_0 \in \mathbb{C}$ such that $\varphi(\lambda_0) = z_0$ and $\varkappa_G(z_0;X_0) = \gamma_E(\lambda_0;\alpha_0)$ (see Chapter VIII in [J-P 1993]). If G is taut, then for any pair $(z_0,X_0) \in G \times \mathbb{C}^n$ there is a $\varkappa_G$-geodesic for (z_0,X_0). If G is even convex, then any $\varkappa_G$-geodesic is a complex geodesic.

Now let $G = \mathbb{E}_p$, where $p = (p_1,\dots,p_n)$ with $p_j > 0$. Observe that $\mathbb{E}_p$ is not necessarily convex. Fix a pair $(z_0,X_0) \in \mathbb{E}_p \times (\mathbb{C}^n)_*$. Then any $\varkappa_{\mathbb{E}_p}$-geodesic φ for (z_0,X_0), where $\varphi_j \not\equiv 0$, $j = 1,\dots,n$, is necessarily of the following form (see [Pfl-Zwo 1996]):

$$\varphi_j(\lambda) = B_j(\lambda)\Big(a_j\frac{1-\overline{\alpha}_j\lambda}{1-\overline{\alpha}_0\lambda}\Big)^{1/p_j}, \qquad j = 1,\dots,n, \tag{1.2.4}$$

([7]) Observe the difference between the Hahn pseudometric $h_G : G \times \mathbb{C}^n \to \mathbb{R}_+$ and the Minkowski function $h_G : \mathbb{C}^n \to \mathbb{R}_+$; since the Hahn pseudometric will not be used in what follows, no confusion will arise. Notice also that under our assumptions h_G is a complex norm.

([8]) Recall that in the case of the unit disc we have

$$B_{m_E}(a,r) = \mathbb{B}\Big(\frac{a(1-r^2)}{1-r^2|a|^2}, \frac{r(1-|a|^2)}{1-r^2|a|^2}\Big), \qquad a \in E,\ r \in (0,1).$$

([9]) $A_{>0} := \{x \in A : x > 0\}$ $(A \subset \mathbb{R})$, $A^n_{>0} := (A_{>0})^n$, e.g. $\mathbb{R}_{>0}$, $\mathbb{R}^n_{>0}$. Observe that $\mathbb{E}_{p,0} = \mathbb{E}_p$.

where B_j is a Blaschke product and the complex numbers a_j, α_j satisfy

- $a_j \in \mathbb{C}_*$, $\alpha_j \in E$, $j = 1, \dots, n$, $\alpha_0 \in E$,
- $1 + |\alpha_0|^2 = \sum_{j=1}^n |a_j|^2(1 + |\alpha_j|^2)$,
- $\alpha_0 = \sum_{j=1}^n |a_j|^2 \alpha_j$.

Moreover, if $p_j \geq 1/2$, then $B_j \equiv 1$ or $B_j(\lambda) = (\lambda - \alpha_j)/(1 - \overline{\alpha}_j\lambda)$ with $|\alpha_j| < 1$.

Additionally, if $\alpha_j \in E$ for all $j = 1, \dots, n$, then either $B_j \equiv 1$ or $B_j(\lambda) = (\lambda - \alpha_j)/(1 - \overline{\alpha}_j\lambda)$ for all $j = 1, \dots, n$.

Using this result, the Kobayashi metric for the non-convex domain $\mathbb{E}_{(1,m)}$, $0 < m < 1/2$, is obtained. First observe that the mappings

$$\mathbb{E}_{(1,m)} \ni z \mapsto \left(\frac{z_1 - a}{1 - \overline{a}z_1}, \frac{e^{i\theta}(1 - |a|^2)^{1/(2m)} z_2}{(1 - \overline{a}z_1)^{1/m}}\right) \in \mathbb{E}_{(1,m)}, \qquad a \in E,\ \theta \in \mathbb{R},$$

are automorphisms. Therefore, to find $\varkappa_{\mathbb{E}_{(1,m)}}$, it suffices to calculate $\varkappa_{\mathbb{E}_{(1,m)}}((0,b);\cdot)$, $b \geq 0$. The easy part is given by the following formulas:

- $\varkappa_{\mathbb{E}_{(1,m)}}((0,0);X) = h_{\mathbb{E}_{(1,m)}}(X)$, where $h_{\mathbb{E}_{(1,m)}}$ denotes the Minkowski function of $\mathbb{E}_{(1,m)}$, $X \in \mathbb{C}^2$;
- $\varkappa_{\mathbb{E}_{(1,m)}}((0,b);X) = |X_2|/(1 - b^2)$, $b > 0$, $X_1 = 0$;
- $\varkappa_{\mathbb{E}_{(1,m)}}((0,b);X) = |X_1|/(1 - b^{2m})^{1/2}$, $b > 0$, $X_2 = 0$.

To discuss the remaining case ($b > 0$ and, without loss of generality, $X = (X_1, 1) \in \mathbb{C}^2_*$), we put

$$\nu := \nu(m,b,X) := \left(\frac{b|X_1|}{m}\right)^2.$$

Moreover, in the case $\nu \leq 1/4m(1-m)$ set

$$t := t(m,b,X) := \frac{2m^2\nu}{1 + 2m(m-1)\nu + \sqrt{1 + 4m(m-1)\nu}}.$$

Observe that then the function

$$\xi^{2m} - t\xi^{2m-2} - (1-t)b^{2m}, \qquad \xi \in \mathbb{R},$$

has exactly one zero $x = x(m,b,X)$ in the interval $(0,1)$. Now we are able to give the remaining formulas.

THEOREM. *Let* $m \in (0,1/2)$, $b > 0$, $X = (X_1, 1) \in \mathbb{C}^2$, $\nu = \nu(b,m,X)$, *and* $x = x(b,m,X)$ *if* $\nu \leq 1/4m(1-m)$. *Then*:

- *if* $\nu \leq 1$, *then*
$$\varkappa_{\mathbb{E}_{(1,m)}}((0,b);X) = \frac{m}{b}\,\frac{x^{2m-1}}{(1-m)x^{2m} + mx^{2m-2} - b^{2m}} =: \varkappa_1(\nu);$$
- *if* $\nu \geq 1/4m(1-m)$, *then*
$$\varkappa_{\mathbb{E}_{(1,m)}}((0,b);X) = \frac{m}{b}\,\frac{\sqrt{(1-b^{2m})\nu + b^{2m}}}{1 - b^{2m}} =: \varkappa_2(\nu);$$
- *if* $1 < \nu < 1/4m(1-m)$, *then*
$$\varkappa_{\mathbb{E}_{(1,m)}}((0,b);X) = \min\{\varkappa_1(\nu), \varkappa_2(\nu)\}.$$

The minimum in the last formula is equal to $\varkappa_1(\nu)$ *for* $\nu \le \nu_0$ *and equal to* $\varkappa_2(\nu)$ *for* $\nu > \nu_0$, *where*

$$\nu_0 := \frac{t_0}{(t_0(1-m)+m)^2}, \qquad t_0 := \frac{x_0^{2m}-b^{2m}}{x_0^{2m-2}-b^{2m}}$$

and x_0 *is the only solution in the interval* $(0,1)$ *of the following equation*:

$$\xi^{4m-2}(-1-2m+2m^2+b^{2m})+\xi^{2m}(1+(1-2m)b^{2m}) \\ +\xi^{2m-2}(1+(2m-1)b^{2m})-(1-m)^2\xi^{4m}-m^2\xi^{4m-2}-b^{2m}=0.$$

It turns out that there is a mapping $\varphi \in \mathcal{O}(E, \mathbb{E}_{(1,m)})$ of the form (1.2.4) which is not a $\varkappa_{\mathbb{E}_{(1,m)}}$-geodesic for $(\varphi(0), \varphi'(0))$. Moreover, for some $b > 0$ such that $(0,b) \in G$ the function $\varkappa_{\mathbb{E}_{(1,m)}}((0,b);(\cdot,1))$ is not differentiable on $\mathbb{C}$.

1.2.6. Wu pseudometric. The Wu pseudometric has been introduced by H. Wu in [Wu 1993] (and [Wu]). Its various properties have been studied in [Che-Kim 1996], [Che-Kim 1997], [Kim 1998], [Che-Kim 2003], [Jar-Pfl 2005], [Juc 2002].

Following [Jar-Pfl 2005], let us formulate the definition of the Wu pseudometric in an abstract setting. Let $h : \mathbb{C}^n \to \mathbb{R}_+$ be a $\mathbb{C}$-seminorm. Put:

$I = I(h) := \{X \in \mathbb{C}^n : h(X) < 1\}$ (I is convex),
$V = V(h) := \{X \in \mathbb{C}^n : h(X) = 0\} \subset I$ (V is a vector subspace of $\mathbb{C}^n$),
$U = U(h) :=$ the orthogonal complement of V with respect to the standard Hermitian scalar product $\langle z, w\rangle := \sum_{j=1}^n z_j\overline{w}_j$ in $\mathbb{C}^n$,
$I_0 := I \cap U$, $h_0 := h|_U$ (h_0 is a norm on U, $I = I_0 + V$).

For any *pseudo-Hermitian scalar product* $s : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ ([10]), let

$$q_s(X) := \sqrt{s(X,X)}, \quad X \in \mathbb{C}^n, \qquad \mathbb{E}(s) := \{X \in \mathbb{C}^n : q_s(X) < 1\}.$$

Consider the family $\mathcal{F}$ of all pseudo-Hermitian scalar products $s : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ such that $I \subset \mathbb{E}(s)$, equivalently, $q_s \le h$. In particular,

$$V \subset I = I_0 + V \subset \mathbb{E}(s) = \mathbb{E}(s_0) + V,$$

where $s_0 := s|_{U\times U}$ (note that $\mathbb{E}(s_0) = \mathbb{E}(s) \cap U$). Let $\mathrm{Vol}(s_0)$ denote the volume of $\mathbb{E}(s_0)$ with respect to the Lebesgue measure of U. Since I_0 is bounded, there exists an $s \in \mathcal{F}$ with $\mathrm{Vol}(s_0) < +\infty$. Observe that for any basis $e = (e_1, \dots, e_m)$ of U ($m := \dim_{\mathbb{C}} U$) we have

$$\mathrm{Vol}(s_0) = \frac{C(e)}{\det S},$$

where $C(e) > 0$ is a constant (independent of s) and $S = S(s_0)$ denotes the matrix representation of s_0 in the basis e, i.e. $S_{j,k} := s(e_j, e_k)$, $j, k = 1, \dots, m$. In particular, if $U = \mathbb{C}^m \times \{0\}^{n-m}$ and $e = (e_1, \dots, e_m)$ is the canonical basis, then $C(e) = \Lambda_{2m}(\mathbb{B}_m)$, where Λ_{2m} denotes the Lebesgue measure in $\mathbb{C}^m$. We are interested in finding an $s \in \mathcal{F}$ for which $\mathrm{Vol}(s_0)$ is minimal, equivalently, $\det S(s_0)$ is maximal.

([10]) That is: $s(\cdot, w) : \mathbb{C}^n \to \mathbb{C}$ is $\mathbb{C}$-linear for any $w \in \mathbb{C}^n$; $s(z,w) = \overline{s(w,z)}$ for any $z, w \in \mathbb{C}^n$; $s(z,z) \ge 0$ for any $z \in \mathbb{C}^n$ (if $s(z,z) > 0$ for any $z \in (\mathbb{C}^n)_*$, then s is a *Hermitian scalar product*).

Observe that, if s has the above property with respect to h (i.e. the volume of $\mathbb{E}(s_0)$ is minimal), then, for any $\mathbb{C}$-linear isomorphism $L : \mathbb{C}^n \to \mathbb{C}^n$, the scalar product

$$\mathbb{C}^n \times \mathbb{C}^n \ni (X, Y) \overset{L(s)}{\longmapsto} s(L(X), L(Y)) \in \mathbb{C}$$

has the analogous property with respect to $h \circ L$. In particular, this permits us to reduce the situation to the case where $U = \mathbb{C}^m \times \{0\}^{n-m}$ and next to assume that $m = n$ (by restricting all the above objects to $\mathbb{C}^m \simeq \mathbb{C}^m \times \{0\}^{n-m}$).

LEMMA 1.2.10. *There exists exactly one element $s^h \in \mathcal{F}$ such that*

$$\operatorname{Vol}(s_0^h) = \min\{\operatorname{Vol}(s_0) : s \in \mathcal{F}\} < +\infty.$$

Proof ([Wu], [Wu 1993]). We may assume $U(h) = \mathbb{C}^n$. First we prove that the set $\mathcal{F}$ is compact. It is clear that $\mathcal{F}$ is closed. To prove that $\mathcal{F}$ is bounded, observe that

$$|s(e_j, e_k)| \le \sqrt{s(e_j, e_j)s(e_k, e_k)} = q_s(e_j)q_s(e_k) \le h(e_j)h(e_k), \qquad s \in \mathcal{F},\ j, k = 1, \dots, n,$$

where $e_1, \dots, e_n$ is the canonical basis in $\mathbb{C}^n$. Consequently, the entries of the matrix $S(s)$ are bounded (by a constant independent of s).

Recall that

$$\operatorname{Vol}(s) = \frac{\Lambda_{2n}(\mathbb{B}_n)}{\det S(s)}.$$

Now, using compactness of $\mathcal{F}$, we see that there exists an $s^h \in \mathcal{F}$ such that

$$\operatorname{Vol}(s^h) = \min\{\operatorname{Vol}(s) : s \in \mathcal{F}\} < +\infty.$$

It remains to show that s^h is uniquely determined. Suppose that $s', s'' \in \mathcal{F}$, $s' \ne s''$, are both minimal and let S', S'' denote the matrix representation of s', s'', respectively. We know that $\mu := \det S' = \det S''$ is maximal (with respect to any basis $(e_1, \dots, e_n)$) in the class $\mathcal{F}$. Take a basis $e_1, \dots, e_n$ such that the matrix $A := S''(S')^{-1}$ is diagonal and let $d_1, \dots, d_n$ be the diagonal elements. Note that $1 = \det A = d_1 \cdots d_n$ and that for at least one $j \in \{1, \dots, n\}$ we have $d_j \ne 1$. Put $s := \frac{1}{2}(s' + s'')$. Then $s \in \mathcal{F}$. Let $S = S(s)$ be the matrix representation of s. We have

$$\begin{aligned}\det S &= \frac{1}{2^n}\det(S' + S'') = \frac{1}{2^n}\det(\mathbb{I}_n + A)\det S' = \frac{1 + d_1}{2} \cdots \frac{1 + d_n}{2}\,\mu \\ &> \sqrt{d_1 \cdots d_n}\,\mu = \mu;\end{aligned}$$

this is a contradiction ($\mathbb{I}_n$ denotes the unit matrix). ∎

Put $\widehat{s}^h := m \cdot s^h$ ($m := \dim U(h)$), $\mathbb{W}h := q_{\widehat{s}^h}$ ($\mathbb{W}h(X) = \sqrt{ms^h(X, X)}$, $X \in \mathbb{C}^n$). Obviously, $\mathbb{W}h \le \sqrt{m}h$ and $\mathbb{W}h \equiv \sqrt{m}h$ iff $h = q_s$ for some pseudo-Hermitian scalar product s. For instance, $\mathbb{W}\|\ \| = \sqrt{n}\|\ \|$, where $\|\ \|$ is the Euclidean norm in $\mathbb{C}^n$. Moreover, $\mathbb{W}(\mathbb{W}h) \equiv \sqrt{m}\mathbb{W}h$.

REMARK 1.2.11. Assume that $U(h) = \mathbb{C}^n$. Let $L : \mathbb{C}^n \to \mathbb{C}^n$ be a $\mathbb{C}$-linear isomorphism such that $|\det L| = 1$ and $h \circ L = h$. Then $\operatorname{Vol}(s^h) = \operatorname{Vol}(L(s^h))$ and hence $s^h = L(s^h)$, i.e. $s^h(X, Y) = s^h(L(X), L(Y))$, $X, Y \in \mathbb{C}^n$.

PROPOSITION 1.2.12. (a) $h \le \mathbb{W}h \le \sqrt{m}h$.

(b) *If* $h(X) := \max\{h_1(X_1), h_2(X_2)\}$, $X = (X_1, X_2) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$, *then*

$$\widehat{s}^h(X, Y) = \widehat{s}^{h_1}(X_1, Y_1) + \widehat{s}^{h_2}(X_2, Y_2), \qquad X = (X_1, X_2),\ Y = (Y_1, Y_2) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}.$$

In particular,

$$\mathbb{W}h(X) = ((\mathbb{W}h_1(X_1))^2 + (\mathbb{W}h_2(X_2))^2)^{1/2}, \qquad X = (X_1, X_2) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}.$$

Proof ([Wu], [Wu 1993]). (a) Using a suitable $\mathbb{C}$-linear isomorphism we may reduce the situation to the case where:

- $U = \mathbb{C}^n$,
- $s^h(X, Y) = \langle X, Y \rangle$, $X, Y \in \mathbb{C}^n$,
- $\min\{\|X\| : h(X) = 1\} = \|X_*\| = a > 0$, $X_* = (0, \dots, 0, a) \in \partial I$; in particular, since I is a balanced convex domain, $I \subset \{(X', X_m) \in \mathbb{C}^{m-1} \times \mathbb{C} : |X_m| < a\}$.

We only need to show that $a \geq 1/\sqrt{n}$. Suppose that $a < 1/\sqrt{n}$ and let $0 < b < 1$ be such that $a^2 + b^2 = 1$. Put $c := a/b$. Note that $(n-1)c^2 < 1$. Let $L : \mathbb{C}^n \to \mathbb{C}^n$ be the $\mathbb{C}$-linear isomorphism

$$L(X) := (c\sqrt{n-1}\, X', X_n), \qquad X = (X', X_n) \in \mathbb{C}^{n-1} \times \mathbb{C}.$$

Obviously, $s^{h \circ L^{-1}} = L^{-1}(s^h)$, so

$$\operatorname{Vol}(s^{h \circ L^{-1}}) = \Lambda_{2n}(\mathbb{B}_n) |\det L|^2 = \Lambda_{2n}(\mathbb{B}_n)(c\sqrt{n-1})^{2(n-1)}.$$

On the other hand, $L(I) \subset \mathbb{B}(a\sqrt{n}) \subset \mathbb{C}^n$. Indeed, for $X = (X', X_n)$ we have

$$\begin{aligned}\|L(X)\|^2 &= (n-1)c^2\|X'\|^2 + |X_n|^2 = (n-1)c^2\|X\|^2 + (1 - (n-1)c^2)|X_n|^2 \\ &< (n-1)c^2 + (1 - (n-1)c^2)a^2 = a^2 + (1 - a^2)(n-1)(a^2/b^2) = na^2 < 1.\end{aligned}$$

Consequently, $\operatorname{Vol}(s^{h \circ L^{-1}}) \leq \Lambda_{2n}(\mathbb{B}_n)(a\sqrt{n})^{2n}$. Thus, using the above inequality, we get

$$(a\sqrt{n-1}/b)^{2(n-1)} \leq (a\sqrt{n})^{2n}.$$

Put $f(t) := t(1-t)^{n-1}$, $0 \leq t \leq 1$. Then

$$f(a^2) = a^2(1 - a^2)^{n-1} \geq \frac{1}{n}\left(1 - \frac{1}{n}\right)^{n-1} = f(1/n),$$

a contradiction (because f is strictly increasing in the interval $[0, 1/n]$ and $a^2 < 1/n$).

(b) We may assume that $U(h_j) = \mathbb{C}^{n_j}$, $j = 1, 2$. Put

$$s_*(X, Y) := \frac{n_1}{n_1 + n_2} s^{h_1}(X_1, Y_1) + \frac{n_2}{n_1 + n_2} s^{h_2}(X_2, Y_2),$$
$$X = (X_1, X_2), Y = (Y_1, Y_2) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}.$$

We only need to prove that $\det S(s^h) = \det S(s_*)$ (all matrix representations are taken in the canonical bases of $\mathbb{C}^{n_1}$ and $\mathbb{C}^{n_2}$, respectively). Let $s := s^h$. Since

$$I(h) = I(h_1) \times I(h_2) \subset \mathbb{E}(s_*),$$

we get $\det S(s) \geq \det S(s_*)$.

Let $L : \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \to \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$ be the isomorphism $L(X_1, X_2) := (X_1, -X_2)$. Then $h \circ L = h$ and, consequently, $s = L(s)$ (Remark 1.2.11), i.e.

$$s(X, Y) = s(L(X), L(Y)), \qquad X, Y \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}.$$

Hence $s((X_1, X_2), (Y_1, Y_2)) = 0$ if $(X_2 = 0$ and $Y_1 = 0)$ or $(X_1 = 0$ and $Y_2 = 0)$. Indeed,

$$\begin{aligned}s((X_1, 0), (0, Y_2)) &= s(L(X_1, 0), L(0, Y_2)) = s((X_1, 0), (0, -Y_2)) \\ &= s((X_1, 0), -(0, Y_2)) = -s((X_1, 0), (0, Y_2)).\end{aligned}$$

Consequently,

$$s(X,Y) = s_1(X_1,Y_1) + s_2(X_2,Y_2), \qquad X = (X_1,X_2), Y = (Y_1,Y_2) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2},$$

where s_j is a Hermitian scalar product in $\mathbb{C}^{n_j}$, $j = 1,2$. It is clear that $I(h_j) \subset \mathbb{E}(s_j)$, $j = 1,2$. Let $c_j \le 1$ be the minimal number such that $I(h_j) \subset \mathbb{E}(c_j^{-2}s_j)$, $j = 1,2$. Assume that $X_j^0 \in \partial I(h_j)$ is such that $s_j(X_j^0, X_j^0) = c_j^2$, $j = 1,2$. In particular, $q_s(X_1^0, X_2^0) \le 1$, so $c_1^2 + c_2^2 \le 1$. We have

$$\det S(s^{h_j}) \ge c_j^{-2n_j} \det S(s_j), \qquad j = 1,2,$$

and, therefore,

$$\begin{aligned}\det S(s) &= \det S(s_1) \det S(s_2) \le c_1^{2n_1} c_2^{2n_2} \det S(s^{h_1}) \det S(s^{h_2})\\ &\le c_1^{2n_1}(1-c_1^2)^{n_2} \det S(s^{h_1}) \det S(s^{h_2})\\ &\le \Big(\frac{n_1}{n_1+n_2}\Big)^{n_1}\Big(\frac{n_2}{n_1+n_2}\Big)^{n_2} \det S(s^{h_1}) \det S(s^{h_2}) = \det S(s_*),\end{aligned}$$

since the maximum of the function $f(t) = t^{n_1}(1-t)^{n_2}$, $0 \le t \le 1$, is attained at $t = n_1/(n_1+n_2)$. ∎

For a domain $G \subset \mathbb{C}^n$ and $\eta \in \mathfrak{M}(G,\mathbb{C})$ (cf. §1.2.1), we define the *Wu pseudometric*

$$(\mathbb{W}\eta)(a;X) := (\mathbb{W}\widehat{\eta}(a;\cdot))(X), \qquad (a,X) \in G \times \mathbb{C}^n,$$

where $\widehat{\eta}$ is the Buseman pseudometric associated to η (cf. §1.2.3). Observe that $\mathbb{W}\eta \in \mathfrak{M}(G,\mathbb{C})$.

Recall that a Borel measurable metric $\eta \in \mathfrak{M}(G,\mathbb{C})$ is said to be *complete* if any $\int\eta$-Cauchy sequence is convergent to a point from G (cf. [J-P 1993, §7.3]).

PROPOSITION 1.2.13. (a) *If $\eta \in \mathfrak{M}(G,\mathbb{C})$ is a continuous metric, then so is $\mathbb{W}\eta$ (cf. Example* 1.2.15*).*

(b) *If $\eta \in \mathfrak{M}(G,\mathbb{C})$ is a continuous complete metric, then so is $\mathbb{W}\eta$.*

(c) *If $(\delta_G)_G$ is a holomorphically contractible family of pseudometrics, then:*

- *for any biholomorphic mapping $F : G \to D$, $G, D \subset \mathbb{C}^n$, we have*

$$(\mathbb{W}\delta_D)(F(z); F'(z)(X)) = (\mathbb{W}\delta_G)(z;X), \qquad (z,X) \in G \times \mathbb{C}^n;$$

- *for any holomorphic mapping $F : G \to D$, $G \subset \mathbb{C}^{n_1}$, $D \subset \mathbb{C}^{n_2}$, we have*

$$(\mathbb{W}\delta_D)(F(z); F'(z)(X)) \le \sqrt{n_2}(\mathbb{W}\delta_G)(z;X), \qquad (z,X) \in G \times \mathbb{C}^{n_1},$$

but, for example, the family $(\mathbb{W}\varkappa_G)_G$ is not holomorphically contractible (cf. Example 1.2.14*).*

In the case $\eta = \varkappa_G$, the above properties (a)–(c) were formulated (without proof) in [Wu], [Wu 1993].

Proof. (a) Fix a point $z_0 \in G \subset \mathbb{C}^n$. Let $s_z := s^{\widehat{\eta}(z;\cdot)}$, $z \in G$. We are going to show that $s_z \to s_{z_0}$ as $z \to z_0$. By our assumptions, there exist $r > 0$, $c > 0$ such that

$$\eta(z;X) \ge c\|X\|, \qquad z \in \mathbb{B}(z_0,r) \subset G,\ X \in \mathbb{C}^n.$$

In particular, the sets

$$I_z := \{X \in \mathbb{C}^n : \widehat{\eta}(z;X) < 1\}, \qquad z \in \mathbb{B}(z_0, r),$$

are contained in the ball $\mathbb{B}(C)$ with $C := 1/c$. Moreover,

$$|\widehat{\eta}(z;X) - \widehat{\eta}(z_0;X)| \le \varphi(z)\|X\|, \qquad X \in \mathbb{C}^n,$$

where $\varphi(z) \to 0$ when $z \to z_0$. Hence

$$(1 + C\varphi(z))^{-1} I_z \subset I_{z_0} \subset (1 + C\varphi(z)) I_z, \qquad z \in \mathbb{B}(z_0, r),$$

and, consequently,

$$\begin{aligned} I_{z_0} &\subset (1 + C\varphi(z))\mathbb{E}(s_z) = \mathbb{E}((1 + C\varphi(z))^{-2} s_z), \\ I_z &\subset (1 + C\varphi(z))\mathbb{E}(s_{z_0}) = \mathbb{E}((1 + C\varphi(z))^{-2} s_{z_0}), \qquad z \in \mathbb{B}(z_0, r). \end{aligned} \tag{1.2.5}$$

Hence,

$$\begin{aligned} \operatorname{Vol}(s_{z_0}) &\le \operatorname{Vol}((1 + C\varphi(z))^{-2} s_z) = (1 + C\varphi(z))^{2n} \operatorname{Vol}(s_z), \\ \operatorname{Vol}(s_z) &\le \operatorname{Vol}((1 + C\varphi(z))^{-2} s_{z_0}) = (1 + C\varphi(z))^{2n} \operatorname{Vol}(s_{z_0}), \qquad z \in \mathbb{B}(z_0, r). \end{aligned}$$

Thus $\operatorname{Vol}(s_z) \to \operatorname{Vol}(s_{z_0})$ as $z \to z_0$.

Take a sequence $z_\nu \to z_0$. Since

$$|s_{z_\nu}(e_j, e_k)| \le \eta(z_\nu; e_j)\eta(z_\nu; e_k), \qquad j, k = 1, \dots, n,\ \nu \in \mathbb{N},$$

we may assume that $s_{z_\nu} \to s_*$, where s_* is a pseudo-Hermitian scalar product. We already know that $\operatorname{Vol}(s_*) = \operatorname{Vol}(s_{z_0})$. Moreover, by (1.2.5), $I_{z_0} \subset \mathbb{E}(s_*)$. Consequently, the uniqueness of s_{z_0} implies that $s_* = s_{z_0}$.

(b) Recall that $\int\eta = \int\widehat{\eta}$ (cf. [J-P 1993, Proposition 4.3.5(b)]). By (a), $\mathbb{W}\eta$ is a continuous metric. In particular, the distance $\int(\mathbb{W}\eta)$ is well defined. By Proposition 1.2.12(a) we get

$$\int\widehat{\eta} \le \int(\mathbb{W}\eta),$$

which directly implies the required result.

(c) Recall that the family $(\widehat{\delta}_G)_G$ is holomorphically contractible (cf. §1.2.3). If F is biholomorphic, then the result is obvious because for any $z \in G$, the mapping $F'(z)$ is a $\mathbb{C}$-linear isomorphism and $\widehat{\delta}_D(F(z); F'(z)(X)) = \widehat{\delta}_G(z;X)$, $X \in \mathbb{C}^n$.

In the general case, using Proposition 1.2.12(a), we get

$$\begin{aligned} (\mathbb{W}\delta_D)(F(z); F'(z)(X)) &\le \sqrt{n_2}\,\widehat{\delta}_D(F(z); F'(z)(X)) \\ &\le \sqrt{n_2}\,\widehat{\delta}_G(z;X) \le \sqrt{n_2}\,(\mathbb{W}\delta_G)(z;X), \qquad (z,X) \in G \times \mathbb{C}^{n_1}. \ \blacksquare \end{aligned}$$

EXAMPLE 1.2.14. Let $G_\varepsilon := \{(z_1, z_2) \in \mathbb{B}_2 : |z_1| < \varepsilon\}$, $0 < \varepsilon < 1/\sqrt{2}$. Recall that $\varkappa_{\mathbb{B}_2}(0;X) = \|X\|$ and $\varkappa_{G_\varepsilon}(0;X) = \max\{\|X\|, |X_1|/\varepsilon\}$, $X = (X_1, X_2)$. Then

$$(\mathbb{W}\varkappa_{G_\varepsilon})(0;(X_1,X_2)) = \sqrt{\frac{|X_1|^2}{\varepsilon^2} + \frac{|X_2|^2}{1-\varepsilon^2}}, \qquad X = (X_1, X_2) \in \mathbb{C}^2.$$

In particular,

$$(\mathbb{W}\varkappa_{\mathbb{B}_2})(0;(0,1)) = \sqrt{2} > \frac{1}{\sqrt{1-\varepsilon^2}} = (\mathbb{W}\varkappa_{G_\varepsilon})(0;(0,1)).$$

Consequently, the family $(\mathbb{W}\varkappa_G)_G$ is not contractible with respect to inclusions.

We point out that Proposition 1.2.13(a) gives us the continuity of $\mathbb{W}\eta$ only in the case where η is a continuous metric. The following Example 1.2.15 shows that if η is only upper semicontinuous, then $\mathbb{W}\eta$ need not be upper semicontinuous. We do not know whether $\mathbb{W}\eta$ is upper semicontinuous in the case where η is a continuous pseudometric. Observe that the upper semicontinuity (or at least Borel measurability) of $\mathbb{W}\eta$ appears in a natural way when one defines $\int(\mathbb{W}\eta)$. In the case where $\eta = \varkappa_G$, the upper semicontinuity of $\mathbb{W}\varkappa_G$ is claimed for instance in [Wu 1993] (Theorem 1), [Che-Kim 1996] (Proposition 2), [Juc 2002] (Theorem 0), but so far there is no proof.

$\boxed{?}$ Let $\eta \in \{\gamma_G^{(k)}, A_G, \varkappa_G\}$. Is $\mathbb{W}\eta$ upper semicontinuous? $\boxed{?}$

EXAMPLE 1.2.15. There is an upper semicontinuous metric η such that $\mathbb{W}\eta$ is not upper semicontinuous.

Indeed, let $\eta : \mathbb{B}_2 \times \mathbb{C}^2 \to \mathbb{R}_+$, $\eta(z;X) := \|X\|$ for $z \neq 0$, and $\eta(0;X) := \max\{\|X\|, |X_1|/\varepsilon\}$, $X = (X_1, X_2) \in \mathbb{C}^2$ ($\varepsilon > 0$ small). Then $(\mathbb{W}\eta)(z;X) = \sqrt{2}\|X\|$ for $z \neq 0$, and (by Example 1.2.14) $\{X \in \mathbb{C}^2 : (\mathbb{W}\eta)(0;X) < 1\} \not\subset \mathbb{B}(1/\sqrt{2})$, so $\mathbb{W}\eta$ is not upper semicontinuous.

EXAMPLE 1.2.16. There exists a bounded domain $G \subset \mathbb{C}^2$ such that $\mathbb{W}\varkappa_G$ is not continuous (see Proposition 2 in [Che-Kim 1996], where such a continuity is claimed).

Indeed, let $D \subset \mathbb{C}^2$ be a domain such that (cf. [J-P 1993, Example 3.5.10]):

- there exists a dense subset $M \subset \mathbb{C}$ such that $(M \times \mathbb{C}) \cup (\mathbb{C} \times \{0\}) \subset D$;
- $\varkappa_D(z; (0,1)) = 0$, $z \in A := M \times \mathbb{C}$;
- there exists a point $z^0 \in D \setminus A$ such that $\varkappa_D(z^0; X) \geq c\|X\|$, $X \in \mathbb{C}^2$, where $c > 0$ is a constant.

For $R > 0$ let $D_R := \{z = (z_1, z_2) \in D : |z_j - z_j^0| < R,\ j = 1,2\}$. It is known that $\varkappa_{D_R} \searrow \varkappa_D$ as $R \nearrow +\infty$. Observe that $z^0 \in D_R$ and

$$\varkappa_{D_R}(z^0; X) \geq \varkappa_D(z^0; X) \geq c\|X\|, \qquad X \in \mathbb{C}^2.$$

Hence, by Proposition 1.2.12(a), $(\mathbb{W}\varkappa_{D_R})(z^0; X) \geq c\|X\|$, $X \in \mathbb{C}^2$. In particular,

$$(\mathbb{W}\varkappa_{D_R})(z^0; (0,1)) \geq c.$$

Fix a sequence $M \ni z_k \to z_1^0$. Note that $\{z_k\} \times (z_2^0 + RE) \subset D_R$, which implies that $\varkappa_{D_R}((z_k, z_2^0); (0,1)) \leq 1/R$, $k = 1, 2, \dots$. In particular,

$$(\mathbb{W}\varkappa_{D_R})((z_k, z_2^0); (0,1)) \leq \sqrt{2}\varkappa_{D_R}((z_k, z_2^0); (0,1)) \leq \sqrt{2}/R, \qquad k = 1, 2, \dots.$$

Now it is clear that if $R > \sqrt{2}/c$, then

$$\limsup_{k \to +\infty} (\mathbb{W}\varkappa_{D_R})((z_k, z_2^0); (0,1)) \leq \sqrt{2}/R < c \leq (\mathbb{W}\varkappa_{D_R})(z^0; (0,1)),$$

which shows that for $G := D_R$ the pseudometric $\mathbb{W}\varkappa_G$ is not continuous.

REMARK 1.2.17. We point out the role played in the definition of $\mathbb{W}$ by the factor $\sqrt{m}$. Put $\widetilde{\mathbb{W}}h := q_{s^h}$, $\widetilde{\mathbb{W}}\eta(a;X) = (\widetilde{\mathbb{W}}\widehat{\eta}(a;\cdot))(X)$, $(a, X) \in G \times \mathbb{C}^n$. Let $D \subset \mathbb{C}^2$ and $D \ni z_k \to z_0 \in D$ be such that:

- $\varkappa_D(z_k; \cdot)$ is not a metric (in particular, $m(k) := \dim U(\widehat{\varkappa}_D(z_k; \cdot)) \leq 1$, $k \in \mathbb{N}$),
- $\varkappa_D(z_0; \cdot)$ is a metric (take, for instance, the domain D from Example 1.2.16).

Put $G := D \times E \subset \mathbb{C}^3$. Then

$$(\widetilde{\mathbb{W}}\varkappa_G)^2((z_k,0);(0,1)) = s^{\varkappa_G((z_k,0);\cdot)}((0,1),(0,1)) = \frac{1}{m(z_k)+1} \geq \frac{1}{2}, \qquad k \in \mathbb{N},$$
$$(\widetilde{\mathbb{W}}\varkappa_G)^2((z_0,0);(0,1)) = s^{\varkappa_G((z_0,0);\cdot)}((0,1),(0,1)) = \frac{1}{m(z_0)+1} = \frac{1}{3},$$

and, therefore, $\widetilde{\mathbb{W}}\varkappa_G$ is not upper semicontinuous at $((z_0,0),(0,1))$ (the example is due to W. Jarnicki).

REMARK 1.2.18. The Wu metric in complex ellipsoids $\mathbb{E}_{(1,m)}$ was studied in [Che-Kim 1996] ($m \geq 1/2$) and [Che-Kim 1997] ($0 < m < 1/2$). In a recent paper [Che-Kim 2003] the same authors proved the following two results.

Let $G := \mathbb{B}_n \cap U$, where U is open in $\mathbb{C}^n$. Then there exists a neighborhood V of $\partial G \cap \partial \mathbb{B}_n$ such that $\mathbb{W}\varkappa_G = \mathbb{W}\varkappa_{\mathbb{B}_n}$ in $V \cap G$.

Let $p = (p_1,\dots,p_n) \in \mathbb{N}^n$, $p_j \geq 2$, $j = 1,\dots,n$. Then any strongly pseudoconvex point $a \in \partial\mathbb{E}_p$ has a neighborhood V such that $\mathbb{W}\varkappa_{\mathbb{E}_p}$ is a Kähler metric with constant negative curvature in $V \cap \mathbb{E}_p$.

1.2.7. Regularity of contractible pseudodistances and pseudometrics. Let us mention a few new results related to different regularity properties of contractible objects.

• Let $(G_j)_{j=1}^{\infty}$ be a sequence of domains in $\mathbb{C}^n$ such that $G_{j+1} \subset\subset G_j$ and $\bigcap_{j=1}^{\infty} G_j = \overline{G}$, where G is a domain in $\mathbb{C}^n$. It is an open question to find conditions under which $c_{G_j} \to c_G$ or $k_{G_j} \to k_G$. M. Kobayashi in [KobM 2002] proved the following two results:

(a) If G is strongly pseudoconvex, then $c_{G_j} \to c_G$ locally uniformly.
(b) If G is a bounded domain such that every point $b \in \partial G$ admits a *weak peak function* (i.e. a function f holomorphic in a neighborhood of $\overline{G}$ such that $f(b) = 1$ and $|f| < 1$ on G), then $k_{G_j} \to k_G$ locally uniformly.

• The behavior of the Bergman, Carathéodory, and Kobayashi metrics on a smooth bounded pseudoconvex domain $G \subset \mathbb{C}^n$ near a boundary point of finite type, where the Levi form of ∂G has at least $n-2$ positive eigenvalues, was studied in [Cho 1995].

The behavior of the Kobayashi metric near boundary points of exponentially flat infinite type in bounded domains in $\mathbb{C}^2$ was studied in [Lee 2001].

Lower and upper non-tangential bounds for the Carathéodory metric of a smooth bounded pseudoconvex domain $G \subset \mathbb{C}^n$ near an *h-extendible* boundary point (a boundary point is said to be h-extendible if its Catlin multitype coincides with its D'Angelo type) were proved in [Nik 1997] and [Nik 1999].

Some localization theorems for contractible functions and metrics were proved in [Nik 2002].

• Let G be a strongly pseudoconvex balanced domain with $\mathcal{C}^\infty$ (resp. real-analytic) boundary. Then there is an open neighborhood $U = U(0) \subset G$ such that $\varkappa_G$ is $\mathcal{C}^\infty$ (resp. real analytic) on $U \times (\mathbb{C}^n)_*$ (cf. [Pan 1993]).

• Let $D_m := \mathbb{E}_{(1,m)} \times (\mathbb{C}^2)_* \subset \mathbb{C}^4$, $m > 0$. It was proved in [Ma 1995] that:

(a) $\varkappa_{\mathbb{E}_{(1,m)}} \in \mathcal{C}^2(D_m)$ for $m \geq 1$;
(b) $\varkappa_{\mathbb{E}_{(1,m)}}$ is piecewise $\mathcal{C}^3$ on D_m and $\varkappa_{\mathbb{E}_{(1,m)}} \notin \mathcal{C}^3(D_m)$ for $m \geq 3/2$.

• Let $G, D \subset \mathbb{C}^n$ be domains and let $a \in G$, $b \in D$. We say that a holomorphic mapping $F : G \to D$ with $F(a) = b$ is *Carathéodory extremal* if

$$|\det F'(a)| = \sup\{|\det \Phi'(a)| : \Phi \in \mathcal{O}(G, D),\ \Phi(a) = b\}.$$

In the cases:

$G = \mathbb{B}_n$, $D = \mathbb{E}_{m,p}$, $a = b = 0$,
$G = \mathbb{E}_{m,p}$, $D = \mathbb{B}_n$, $a = b = 0$,

where

$$\mathbb{E}_{m,p} := \Big\{z \in \mathbb{C}^{m_1} \times \cdots \times \mathbb{C}^{m_k} : \sum_{j=1}^{k} \|z_j\|^{2p_j} < 1\Big\},$$
$$m = (m_1, \dots, m_k) \in \mathbb{N}^k,\ m_1 + \cdots + m_k = n,\ p = (p_1, \dots, p_k) \in \mathbb{R}^n_{>0},$$

the Carathéodory extremal mappings are characterized in [Ma 1997].

1.3. Effective formulas for elementary Reinhardt domains

For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n_*$ and $c \in \mathbb{R}$ put

$D_{\alpha,c} := \{z \in \mathbb{C}^n : |z_1|^{\alpha_1} \cdots |z_n|^{\alpha_n} < e^c$ and $(\forall_{j\in\{1,\dots,n\}} : \alpha_j < 0 \Rightarrow z_j \neq 0)\}$, $D_\alpha := D_{\alpha,0}$;

$D_{\alpha,c}$ is called an *elementary Reinhardt domain*. We say that $D_{\alpha,c}$ is of *rational type* if $\alpha \in \mathbb{R} \cdot \mathbb{Z}^n$. The domain $D_{\alpha,c}$ is of *irrational type* if it is not of rational type. Without loss of generality we may assume that $\alpha_1, \dots, \alpha_k < 0$ and $\alpha_{k+1}, \dots, \alpha_n > 0$ for some $k \in \{0, \dots, n\}$. If $k < n$, then we put $t_k := \min\{\alpha_{k+1}, \dots, \alpha_n\}$. Let

$$V_0 := \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_1 \cdots z_n = 0\}.$$

For $\alpha \in \mathbb{Z}^n$ and $r \in \mathbb{N}$, put $\Phi(z) := z^\alpha$,

$$\Phi_{(r)}(a)(X) := \sum_{\beta\in\mathbb{Z}^n_+,\, |\beta|=r} \frac{1}{\beta!} D^\beta \Phi(a) X^\beta, \qquad a \in D_\alpha,\ X \in \mathbb{C}^n.$$

To simplify notation, for $z \in D_\alpha$, write $|z^\alpha| := |z_1|^{\alpha_1} \cdots |z_n|^{\alpha_n}$ (observe that this notation agrees with the standard one if $\alpha \in \mathbb{Z}^n$).

The following effective formulas for holomorphically contractible functions and pseudometrics on D_α are known.

THEOREM 1.3.1 ([J-P 1993, §4.4], [Pfl-Zwo 1998], [Zwo 1999a], [Zwo 2000a]). *Let $a = (a_1, \dots, a_n) \in D_\alpha$. Assume that $a_1 \cdots a_s \neq 0$, $a_{s+1} = \cdots = a_n = 0$ for some $s \in \{k+1, \dots, n\}$. Put $r := \alpha_{s+1} + \cdots + \alpha_n$ if $s < n$, and $r := 1$ if $s = n$. For $z \in D_\alpha$ and $X \in \mathbb{C}^n$ consider the following four cases.*

(1) $k < n$, D_α *is of rational type* (*we may assume that* $\alpha \in \mathbb{Z}^n$ *and* $\alpha_1, \dots, \alpha_n$ *are relatively prime*). *Then*:

$$\begin{aligned}
&c^*_{D_\alpha}(a,z) = m_E(a^\alpha, z^\alpha),\\
&g^*_{D_\alpha}(a,z) = (m_E(a^\alpha, z^\alpha))^{1/r},\\
&k_{D_\alpha}(a,z) = \min\{p_E(\zeta_1,\zeta_2) : \zeta_1,\zeta_2 \in E,\ a^\alpha = \zeta_1^{t_k},\ z^\alpha = \zeta_2^{t_k}\},\\
&\widetilde{k}_{D_\alpha}(a,z) = \begin{cases} \min\{p_E(\zeta_1,\zeta_2) : \zeta_1,\zeta_2 \in E,\ a^\alpha = \zeta_1^{t_k},\ z^\alpha = \zeta_2^{t_k}\}, & s = n,\ z \notin V_0,\\ p_E(0, |z^\alpha|^{1/r}), & s < n,\end{cases}\\
&\gamma_{D_\alpha}(a;X) = \gamma_E\left(a^\alpha; a^\alpha \sum_{j=1}^n \frac{\alpha_j X_j}{a_j}\right),\\
&A_{D_\alpha}(a;X) = (\gamma_E(a^\alpha; \Phi_{(r)}(a)(X)))^{1/r},\\
&\varkappa_{D_\alpha}(a;X) = \begin{cases} \gamma_E\left((a^\alpha)^{1/t_k}; (a^\alpha)^{1/t_k} \dfrac{1}{t_k} \displaystyle\sum_{j=1}^n \frac{\alpha_j X_j}{a_j}\right), & s = n,\\ (|a_1|^{\alpha_1} \cdots |a_s|^{\alpha_s} |X_{s+1}|^{\alpha_{s+1}} \cdots |X_n|^{\alpha_n})^{1/r}, & s < n.\end{cases}
\end{aligned}$$

(2) $k < n$, D_α *is of irrational type* (*we may assume that* $t_k = 1$). *Then*:

$$\begin{aligned}
&c^*_{D_\alpha} \equiv m^{(l)}_{D_\alpha} \equiv 0, \qquad l \in \mathbb{N},\\
&g_{D_\alpha}(a,z) = \begin{cases} 0, & s = n,\\ |z^\alpha|^{1/r}, & s < n,\end{cases}\\
&k_{D_\alpha}(a,z) = p_E(|a^\alpha|, |z^\alpha|),\\
&\widetilde{k}_{D_\alpha}(a,z) = \begin{cases} p_E(|a^\alpha|, |z^\alpha|), & s = n,\ z \notin V_0,\\ p_E(0, |z^\alpha|^{1/r}), & s < n,\end{cases}\\
&\gamma_{D_\alpha} \equiv \gamma^{(l)}_{D_\alpha} \equiv 0, \qquad l \in \mathbb{N},\\
&A_{D_\alpha}(a;X) = \begin{cases} 0, & s = n,\\ (|a_1|^{\alpha_1} \cdots |a_s|^{\alpha_s} |X_{s+1}|^{\alpha_{s+1}} \cdots |X_n|^{\alpha_n})^{1/r}, & s < n,\end{cases}\\
&\varkappa_{D_\alpha}(a;X) = \begin{cases} \gamma_E\left(|a^\alpha|; |a^\alpha| \displaystyle\sum_{j=1}^n \frac{\alpha_j X_j}{a_j}\right), & s = n,\\ (|a_1|^{\alpha_1} \cdots |a_s|^{\alpha_s} |X_{s+1}|^{\alpha_{s+1}} \cdots |X_n|^{\alpha_n})^{1/r}, & s < n.\end{cases}
\end{aligned}$$

(3) $k = n$, D_α *is of rational type* (*we may assume that* $\alpha \in \mathbb{Z}^n$ *and* $\alpha_1, \dots, \alpha_n$ *are relatively prime*). *Then*:

$$\begin{aligned}
&c^*_{D_\alpha}(a,z) = g_{D_\alpha}(a,z) = m_E(a^\alpha, z^\alpha),\\
&k_{D_\alpha}(a,z) = \widetilde{k}_{D_\alpha}(a,z) = k_{E_*}(a^\alpha, z^\alpha),\\
&\gamma_{D_\alpha}(a;X) = A_{D_\alpha}(a;X) = \gamma_{E_*}\left(a^\alpha; a^\alpha \sum_{j=1}^n \frac{\alpha_j X_j}{a_j}\right),\\
&\varkappa_{D_\alpha}(a;X) = \varkappa_{E_*}\left(a^\alpha; a^\alpha \sum_{j=1}^n \frac{\alpha_j X_j}{a_j}\right).
\end{aligned}$$

(4) $k = n$, D_α *is of irrational type. Then*:

$$c^*_{D_\alpha} \equiv m^{(l)}_{D_\alpha} \equiv g_{D_\alpha} \equiv 0, \qquad l \in \mathbb{N},$$
$$k_{D_\alpha}(a,z) = \widetilde{k}_{D_\alpha}(a,z) = k_{E_*}(|a^\alpha|, |z^\alpha|),$$
$$\gamma_{D_\alpha} \equiv \gamma^{(l)}_{D_\alpha} \equiv A_{D_\alpha} \equiv 0, \qquad l \in \mathbb{N},$$
$$\varkappa_{D_\alpha}(a;X) = \varkappa_{E_*}\Big(|a^\alpha|; |a^\alpha| \sum_{j=1}^n \frac{\alpha_j X_j}{a_j}\Big).$$

Moreover, if $\alpha \in \mathbb{N}^n$ *and* $\alpha_1, \dots, \alpha_n$ *are relatively prime, then*

$$m^{(l)}_{D_\alpha}(a,z) = (m_E(a^\alpha, z^\alpha))^{(1/l)\lceil l/r \rceil}, \text{ (11)}$$
$$\gamma^{(l)}_{D_\alpha}(a;X) = \begin{cases} (\gamma_E(a^\alpha; \Phi_{(r)}(a)(X)))^{1/r} & \textit{if } r \textit{ divides } l, \\ 0, & \textit{otherwise}, \end{cases} \qquad l \in \mathbb{N}.$$

REMARK 1.3.2. The formulas in Theorem 1.3.1 led W. Zwonek [Zwo 1998] to a negative answer to a question posed by S. Kobayashi (cf. Remark 3.3.8(b) in [J-P 1993]). Let $(\alpha_1, \dots, \alpha_{n-1}, -1) \in \mathbb{R}^n$, $\alpha_j < 0$, such that D_α is of irrational type. Then the mapping

$$\psi : \mathbb{C}^{n-1} \times E_* \to D_\alpha, \qquad \psi(z_1, \dots, z_n) := (e^{-z_1}, \dots, e^{-z_{n-1}}, e^{-(\alpha_1 z_1 + \dots + \alpha_{n-1} z_{n-1})}/z_n),$$

is a holomorphic covering. Fix points $z \in D_\alpha$ and $w := (z_1, \dots, z_{n-1}, |z_n|) \in D_\alpha$. Then, by the formula of S. Kobayashi (see Theorem 3.3.7 in [J-P 1993]), the product property for the Kobayashi pseudodistance, and Theorem 1.3.1, we have

$$0 = k_{D_\alpha}(w,z) = \inf_{l_1, \dots, l_{n-1} \in \mathbb{Z}} \Big\{ k_{E_*}\Big(\frac{|z_1|^{\alpha_1} \cdots |z_{n-1}|^{\alpha_{n-1}}}{|z_n|} \exp\Big(i \sum_{j=1}^{n-1} \arg(z_j)\alpha_j\Big),$$
$$\frac{|z_1|^{\alpha_1} \cdots |z_{n-1}|^{\alpha_{n-1}}}{z_n} \exp\Big(i \sum_{j=1}^{n-1} (\arg(z_j) + 2l_j\pi)\alpha_j\Big)\Big)\Big\}.$$

Assuming that the infimum is attained implies that there are $l_1, \dots, l_{n-1} \in \mathbb{Z}$ such that

$$\frac{\arg(z_n)}{2\pi} + \sum_{j=1}^{n-1} l_j \alpha_j \in \mathbb{Z}.$$

This is, in general, impossible. Just take a z_n in such a way that $(\arg(z_n))/2\pi$ does not belong to the $\mathbb{Q}$-linear subspace of $\mathbb{R}$ which is spanned by $1, \alpha_1, \dots, \alpha_{n-1}$. Recall that $\mathbb{R}$ is infinite-dimensional as a $\mathbb{Q}$-vector space.

REMARK 1.3.3. The Wu pseudometric for D_α (with $\alpha \in \mathbb{R}^n_*$) was investigated by P. Jucha in [Juc 2002]. He proved that

$$(\mathbb{W}\varkappa_{D_\alpha})(a;X) = \begin{cases} \sqrt{n}\varkappa_{D_\alpha}(a;X) & \text{if } \#J(a) \le 1, \\ 0 & \text{if } \#J(a) \ge 2, \end{cases} \qquad (a,X) \in D_\alpha \times \mathbb{C}^n,$$

where $J(a) := \{j \in \{1, \dots, n\} : a_j = 0\}$, $a \in D_\alpha$.

(11) $\lceil x \rceil := \inf\{m \in \mathbb{Z} : m \ge x\}$.

REMARK 1.3.4. (a) Observe that if $\alpha \in \mathbb{N}^n$, $\alpha_1, \dots, \alpha_n$ are relatively prime, and $t_0 = 1$, then

$$c^*_{D_\alpha} \equiv \tanh k^*_{D_\alpha} \leq g_{D_\alpha}, \quad \tanh k^*_{D_\alpha} \not\equiv g_{D_\alpha}, \quad \gamma_{D_\alpha} \leq \varkappa_{D_\alpha}, \gamma_{D_\alpha} \not\equiv \varkappa_{D_\alpha}.$$

(b) For a domain $G \subset \mathbb{C}^n$ define the following relation $\mathcal{R}$:

$$a\mathcal{R}b : \Leftrightarrow \; k_G(a,b) = 0, \qquad a, b \in G.$$

In [Kob 1976], S. Kobayashi asked whether the quotient $G/\mathcal{R}$ always has a complex structure. From Theorem 1.3.1 we see that if D_α is of irrational type with $k = 0$, then $D_\alpha/\mathcal{R} \approx [0,1)$. This gives a simple example of a very regular domain for which the answer to the above question is "No".

1.4. The converse to the Lempert theorem

First recall the fundamental Lempert theorem saying that if a domain $G \subset \mathbb{C}^n$ is strongly linearly convex, then $c^*_G \equiv \widetilde{k}^*_G$ (cf. [J-P 1993, Miscellanea C]; recall that any strongly convex domain is strongly linearly convex). Notice that if for a domain $G \subset \mathbb{C}^n$ we have $c^*_G \equiv \widetilde{k}^*_G$, then, by (1.1.2), all holomorphically contractible families coincide on G. Moreover, if G is taut and $c^*_G \equiv \widetilde{k}^*_G$, then $\gamma_G \equiv \varkappa_G$ (cf. Proposition 1.2.6) and, consequently, all holomorphically contractible families of pseudometrics coincide on G.

Note that in the case of convex domains $G \subset \mathbb{C}^n$, the equality $c^*_G \equiv \widetilde{k}^*_G$ may be also proved using functional analysis methods; cf. [Mey 1997].

Let $\mathfrak{L}_n$ be the class of all domains $G \subset \mathbb{C}^n$ with $c^*_G \equiv \widetilde{k}^*_G$. It is clear that $\mathfrak{L}_n$ is invariant under biholomorphic mappings. Moreover, if a domain $G \subset \mathbb{C}^n$ may be *exhausted* by domains from $\mathfrak{L}_n$ (i.e. $G = \bigcup_{i \in I} G_i$, $G_i \in \mathfrak{L}_n$, $i \in I$, and for any compact $K \Subset G$ there exists an $i_0 \in I$ with $K \subset G_{i_0}$), then $G \in \mathfrak{L}_n$.

Indeed, we only need to prove that

$$d_G = \inf\{d_{G_i} : i \in I\}, \qquad d \in \{c^*, \widetilde{k}^*\}.$$

Write $G = \bigcup_{k=1}^\infty G'_k$, where $G'_k \Subset G$ is a domain with $G'_k \subset G'_{k+1}$, $k \in \mathbb{N}$. For $k \in \mathbb{N}$ let $i(k) \in I$ be such that $G'_k \subset G_{i(k)}$. Then $d_{G'_k} \searrow d_G$ (cf. [J-P 1993, Propositions 2.5.1(a), 3.3.5(a)]). Hence

$$d_G \leq \inf\{d_{G_i} : i \in I\} \leq \inf\{d_{G_{i(k)}} : k \in \mathbb{N}\} \leq \inf\{d_{G'_k} : k \in \mathbb{N}\} = d_G.$$

For example, if $G = \bigcup_{k=1}^\infty G_k$ with $G_k \subset G_{k+1}$, $G_k \in \mathfrak{L}_n$, $k \in \mathbb{N}$, then $G \in \mathfrak{L}_n$. In particular, any convex domain belongs to $\mathfrak{L}_n$.

For more than 20 years the following conjecture was open.

Any bounded pseudoconvex domain $G \in \mathfrak{L}_n$ may be exhausted by domains biholomorphic to convex domains ([12]).

For instance, it is unknown whether the strongly linearly convex domain

$$G = \{z \in \mathbb{C}^n : \|z\|^2 + (\mathrm{Re}(z_1^2))^2 < 1\}$$

([12]) Observe that there are unbounded pseudoconvex domains $G \in \mathfrak{L}_n$ which cannot be exhausted by convex domains, e.g. $G = \mathbb{C}_*$ ($c^*_{\mathbb{C}_*} \equiv \widetilde{k}^*_{\mathbb{C}_*} \equiv 0$).

may be exhausted by domains biholomorphic to convex domains; even more, it is not known whether G is biholomorphic to a convex domain (cf. [J-P 1993, Example C.3]).

The first counterexample was recently constructed in a series of papers by J. Agler, C. Costara, and N. J. Young [Agl-You 2001], [Agl-You 2004], [Cos 2004a], [Cos 2003], [Cos 2004b], also [Ham-Seg 2003], when they investigated the 2×2-spectral Nevanlinna–Pick problem. Let

$$\begin{aligned}
&\pi : \mathbb{C}^2 \to \mathbb{C}^2, \quad \pi(\lambda_1, \lambda_2) := (\lambda_1 + \lambda_2, \lambda_1\lambda_2),\\
&\mathbb{G}_2 := \pi(E^2) = \{(\lambda_1 + \lambda_2, \lambda_1\lambda_2) : \lambda_1, \lambda_2 \in E\}, \quad \sigma_2 := \pi((\partial E)^2) \subset \partial\mathbb{G}_2,\\
&\varDelta_2 := \{(\lambda, \lambda) : \lambda \in E\}, \quad \varSigma_2 := \pi(\varDelta_2) = \{(2\lambda, \lambda^2) : \lambda \in E\},\\
&h_a(\lambda) := \frac{\lambda - a}{1 - \overline{a}\lambda}, \quad a \in E,\ \lambda \in \mathbb{C} \setminus \{1/\overline{a}\},\\
&F_a(s,p) := \frac{2ap - s}{2 - as}, \quad a \in \overline{E},\ (s,p) \in (\mathbb{C} \setminus \{2/a\}) \times \mathbb{C}.
\end{aligned}$$

Note that π is proper, $\pi|_{E^2} : E^2 \to \mathbb{G}_2$ is proper, and $\pi|_{E^2 \setminus \varDelta_2} : E^2 \setminus \varDelta_2 \to \mathbb{G}_2 \setminus \varSigma_2$ is a holomorphic covering. The domain $\mathbb{G}_2$ is called the *symmetrized bidisc*. One can prove (cf. Lemma 1.4.2) that $|s| < 2$ and $|F_a| < 1$ on $\mathbb{G}_2$ and that (Remark 1.4.5) $\mathbb{G}_2$ is hyperconvex.

THEOREM 1.4.1 ([13]). *We have*

$$\begin{aligned}
c^*_{\mathbb{G}_2}((s_1,p_1),(s_2,p_2)) &= \widetilde{k}^*_{\mathbb{G}_2}((s_1,p_1),(s_2,p_2))\\
&= \max\{m_E(F_z(s_1,p_1), F_z(s_2,p_2)) : z \in \overline{E}\}\\
&= \max\{m_E(F_z(s_1,p_1),\ F_z(s_2,p_2)) : z \in \partial E\},\ (s_1,p_1),(s_2,p_2) \in \mathbb{G}_2.
\end{aligned}$$

Moreover, $\mathbb{G}_2$ *cannot be exhausted by domains biholomorphic to convex domains.*

The proof will be given after auxiliary lemmas.

LEMMA 1.4.2 ([Agl-You 2004]). *For* $(s,p) \in \mathbb{C}^2$, *the following conditions are equivalent*:

(i) $(s,p) \in \mathbb{G}_2$;
(ii) $|s - \overline{s}p| + |p|^2 < 1$;
(iii) $|s| < 2$, $|s - \overline{s}p| + |p|^2 < 1$;
(iv) $\left|\dfrac{2zp - s}{2 - zs}\right| < 1$, $z \in \overline{E}$ (*i.e.* $|F_z(s,p)| < 1$, $z \in \overline{E}$);
(v) $\left|\dfrac{2p - \overline{z}s}{2 - zs}\right| < 1$, $z \in \overline{E}$;
(vi) $2|s - \overline{s}p| + |s^2 - 4p| + |s|^2 < 4$.

In particular, $F_a \in \mathcal{O}(\mathbb{G}_2, E)$ $(a \in \overline{E})$.

Proof. Observe that $(s,p) \in \mathbb{G}_2$ iff both roots of the polynomial $f(z) = z^2 - sz + p$ belong to E. By the Cohn criterion (cf. [Rah-Sch 2002]), $f^{-1}(0) \subset E$ iff $|p| < 1$ and the root of the polynomial

---

([13]) Special thanks are due to C. Costara and N. J. Young for sending us their preprints, without which this section would never have been written.

$$g(z) := \frac{1}{z}(f(z) - pz^2\overline{f(1/\overline{z})}) = (1-|p|^2)z - (s - p\overline{s})$$

belongs to E. Thus (i)⇔(ii).

The equivalence (iv)⇔(v) follows from the maximum principle:

$$\max\left\{\left|\frac{2zp-s}{2-zs}\right| : z \in \overline{E}\right\} = \max\left\{\left|\frac{2zp-s}{2-zs}\right| : z \in \partial E\right\}$$
$$= \max\left\{\left|\frac{2p-\overline{z}s}{2-zs}\right| : z \in \partial E\right\} = \max\left\{\left|\frac{2p-\overline{z}s}{2-zs}\right| : z \in \overline{E}\right\}.$$

Observe that (iv) with $z = 0$ gives $|s| < 2$ and, moreover,

$$\max\left\{\left|\frac{2p-\overline{z}s}{2-zs}\right|^2 : z \in \partial E\right\} = \max\left\{\frac{4|p|^2 + |s|^2 - 4\operatorname{Re}(zp\overline{s})}{4+|s|^2 - 4\operatorname{Re}(zs)} : z \in \partial E\right\}.$$

Thus (iv)⇒(iii)⇒(ii)⇒(iv).

Observe that for $|s| < 2$ the mapping $\overline{E} \ni z \mapsto F_z(s,p)$ maps $\overline{E}$ onto $\overline{\mathbb{B}}(a,r) \subset \mathbb{C}$ with $a := 2(\overline{s}p - s)/(4 - |s|^2)$, $r := (|s^2 - 4p|)/(4 - |s|^2)$. We have $\mathbb{B}(a,r) \subset E$ iff $|a| + r < 1$. Thus (vi)⇔(iv). ∎

LEMMA 1.4.3 ([Agl-You 2004]). *For* $(s,p) \in \mathbb{C}^2$, *the following conditions are equivalent*:

(i) $(s,p) \in \overline{\mathbb{G}}_2$;
(ii) $|s| \le 2$, $|s - \overline{s}p| + |p|^2 \le 1$;
(iii) $\left|\dfrac{2zp-s}{2-zs}\right| \le 1$, $z \in \overline{E}$ ([14]);
(iv) $\left|\dfrac{2p-\overline{z}s}{2-zs}\right| \le 1$, $z \in \overline{E}$ ([15]).

Notice that the condition $|s-\overline{s}p|+|p|^2 \le 1$ does not imply that $(s,p) \in \overline{\mathbb{G}}_2$ (e.g. $(s,p) = (5/2, 1)$).

Proof. Using Lemma 1.4.2 we see that (i)⇒(ii). Moreover, (ii)⇔(iii)⇔(iv). It remains to observe that if (ii) is satisfied and $s \neq p\overline{s}$, then $(ts, p) \in \mathbb{G}_2$, $0 < t < 1$. If (ii) is satisfied and $s = p\overline{s}$, then $(ts, t^2p) \in \mathbb{G}_2$, $0 < t < 1$. ∎

COROLLARY 1.4.4. *For* $(s,p) \in \mathbb{C}^2$, *the following conditions are equivalent*:

(i) $(s,p) \in \sigma_2$;
(ii) $s = \overline{s}p$, $|p| = 1$, *and* $|s| \le 2$.

In particular, $|F_a| = 1$ *on* σ_2 *for* $a \in \partial E$ ([16]).

REMARK 1.4.5. Let

$$\varrho(s,p) := \max\{\max\{|\lambda_1|, |\lambda_2|\} : (\lambda_1,\lambda_2) \in \pi^{-1}(s,p)\}, \qquad (s,p) \in \mathbb{C}^2.$$

([14]) Notice that if $(s,p) \in \overline{\mathbb{G}}_2$ and $|s| = 2$, then $(s,p) = (2\eta, \eta^2)$ for some $\eta \in \partial E$ and, consequently, $(2zp - s)/(2 - zs) = -\eta$, which implies that the function $z \mapsto (2zp-s)/(2-zs)$ has no essential singularities.

([15]) As above, notice that if $(s,p) = (2\eta, \eta^2)$ for some $\eta \in \partial E$, then $(2p - \overline{z}s)/(2 - \overline{zs}) = \eta^2$ and, consequently, the function $z \mapsto (2p - \overline{z}s)/(2 - \overline{zs})$ has no essential singularities.

([16]) Notice that for $(s,p) \in \sigma_2$, $|s| < 2$, we have $F_a(s,p) = ap(2 - \overline{as})/(2 - as)$.

Then ϱ is a continuous plurisubharmonic function such that

$$\varrho(\lambda s, \lambda^2 p) = |\lambda|\varrho(s,p), \qquad (s,p) \in \mathbb{C}^2,\ \lambda \in \mathbb{C},$$

and

$$\mathbb{G}_2 = \{(s,p) \in \mathbb{C}^2 : \varrho(s,p) < 1\}, \qquad \overline{\mathbb{G}}_2 = \{(s,p) \in \mathbb{C}^2 : \varrho(s,p) \le 1\}.$$

In particular, $\mathbb{G}_2$ is hyperconvex.

The maximum principle for plurisubharmonic functions gives the following result.

LEMMA 1.4.6. *Let $\varphi : \overline{E} \to \mathbb{C}^2$ be a continuous mapping, holomorphic in E, such that $\varphi(\partial E) \subset \overline{\mathbb{G}}_2$. Then $\varphi(\overline{E}) \subset \overline{\mathbb{G}}_2$. If $\varphi(E) \cap \mathbb{G}_2 \neq \emptyset$, then $\varphi(E) \subset \mathbb{G}_2$. If $\varphi(E) \cap \mathbb{G}_2 = \emptyset$, then $\varphi(E) \subset \partial\mathbb{G}_2$.*

REMARK 1.4.7. If $f \in \mathcal{O}(E^2)$ is symmetric, then the relation $F(\pi(\lambda_1, \lambda_2)) = f(\lambda_1, \lambda_2)$ defines a function $F \in \mathcal{O}(\mathbb{G}_2)$. In particular, if $h \in \mathcal{O}(E, E)$, then the relation $H_h(\pi(\lambda_1, \lambda_2)) = \pi(h(\lambda_1), h(\lambda_2))$ defines a holomorphic mapping $H_h : \mathbb{G}_2 \to \mathbb{G}_2$ with $H_h(\Sigma_2) \subset \Sigma_2$.

Observe that if $h \in \operatorname{Aut}(E)$, then $H_h \in \operatorname{Aut}(\mathbb{G}_2)$, $H_h^{-1} = H_{h^{-1}}$, $H_h(\Sigma_2) = \Sigma_2$, and $H_h(\sigma_2) = \sigma_2$. In particular, if $h(\lambda) := \tau\lambda$ for some $\tau \in \partial E$, then we get the "rotation" $R_\tau(s,p) := H_h(s,p) = (\tau s, \tau^2 p)$.

REMARK 1.4.8. For any point $(s_0, p_0) = (2a, a^2) \in \Sigma_2$ we get $H_{h_a}(s_0, p_0) = (0,0)$.

LEMMA 1.4.9. *σ_2 is the Shilov (and Bergman) boundary of $\mathbb{G}_2$.*

Proof. It is clear that the modulus of any function $f \in \mathcal{C}(\overline{\mathbb{G}}_2) \cap \mathcal{O}(\mathbb{G}_2)$ attains its maximum on σ_2. We have to prove that σ_2 is minimal. First observe that the function $f_0(s,p) := s + 2$ is a peak function at $(2,1) \in \sigma_2$. Take any other point $(s_0, p_0) = \pi(\lambda_1^0, \lambda_2^0) \in \sigma_2$. The case $\lambda_1^0 = \lambda_2^0$ reduces (via a rotation R_τ) to the previous one. Thus assume that $\lambda_1^0 \neq \lambda_2^0$. We are going to find a Blaschke product B of order 2 such that

$$\{\lambda \in \partial E : B(\lambda) = 1\} = \{\lambda_1^0, \lambda_2^0\}.$$

Suppose for a moment that such a B is already constructed. Then $f_0 \circ H_B$ is a peak function for (s_0, p_0).

We turn to the construction of B. Using a rotation we may reduce the proof to the case $\lambda_2^0 = \overline{\lambda_1^0}$. Then, using the fact that the mapping $(-1,1) \ni a \mapsto 2a/(1+a^2) \in (-1,1)$ is bijective, we see that there exists an $a \in (-1,1)$ such that $h_a(\lambda_1^0) = -h_a(\overline{\lambda_1^0})$. Finally, we take $B := \tau h_a^2$ with $\tau \in \partial E$ such that $\tau h_a^2(\lambda_1^0) = 1$. ■

LEMMA 1.4.10. *The domain $\mathbb{G}_2$ cannot be exhausted by domains biholomorphic to convex domains.*

Proof. This is a generalization of the proof of [Cos 2004a] due to A. Edigarian [Edi 2003a]. First observe that $\overline{\mathbb{G}}_2$ is not convex: for example, $(2,1), (2i,-1) \in \overline{\mathbb{G}}_2$, but $(1+i, 0) \notin \overline{\mathbb{G}}_2$. Consequently, $\mathbb{G}_2$ is also not convex.

Suppose that $\mathbb{G}_2 = \bigcup_{i \in I} G_i$, where each domain G_i is biholomorphic to a convex domain and for any compact $K \Subset \mathbb{G}_2$ there exists an $i_0 \in I$ with $K \subset G_{i_0}$. For any $0 < \varepsilon < 1$ take an $i = i(\varepsilon) \in I$ such that $\{(s,p) \in \mathbb{C}^2 : \varrho(s,p) \le 1 - \varepsilon\} \subset G_{i(\varepsilon)}$ and let $f_\varepsilon = (g_\varepsilon, h_\varepsilon) : G_{i(\varepsilon)} \to D_\varepsilon$ be a biholomorphic mapping onto a convex domain $D_\varepsilon \subset \mathbb{C}^n$ with $f_\varepsilon(0,0) = (0,0)$ and $f'_\varepsilon(0,0) = \mathbb{I}_2$.

Take arbitrary two points $(s_1,p_1),(s_2,p_2)\in\mathbb{C}^2$ and put

$$C:=\max\{\varrho(s_1,p_1),\varrho(s_2,p_2)\}.$$

Our aim is to prove that $\varrho(t(s_1,p_1)+(1-t)(s_2,p_2))\le C$, $t\in[0,1]$, which in particular shows that $\mathbb{G}_2$ is convex, a contradiction.

Observe that for $|\lambda|<(1-\varepsilon)/C$ we have $\varrho(\lambda s_j,\lambda^2p_j)=|\lambda|\varrho(s_j,p_j)<1-\varepsilon$, $j=1,2$. Consequently, for any $t\in[0,1]$, the mapping $\varphi_{\varepsilon,t}:\mathbb{B}((1-\varepsilon)/C)\to\mathbb{G}_2$,

$$\varphi_{\varepsilon,t}(\lambda)=(\psi_{\varepsilon,t}(\lambda),\chi_{\varepsilon,t}(\lambda)):=f_\varepsilon^{-1}(tf_\varepsilon(\lambda s_1,\lambda^2p_1)+(1-t)f_\varepsilon(\lambda s_2,\lambda^2p_2)),$$

is well defined. We have $\varphi_{\varepsilon,t}(0)=(0,0)$ and

$$\psi'_{\varepsilon,t}(0)=ts_1+(1-t)s_2,\qquad \chi'_{\varepsilon,t}(0)=0$$

and

$$\frac{1}{2}\,\chi''_{\varepsilon,t}(0)=tp_1+(1-t)p_2+\mu_\varepsilon t(1-t)(s_1-s_2)^2,\quad\text{where}\quad \mu_\varepsilon:=\frac{1}{2}\,\frac{\partial^2h_\varepsilon}{\partial s^2}(0,0).$$

Define $\varPhi_{\varepsilon,t}:\mathbb{B}((1-\varepsilon)/C)\to\mathbb{C}^2$ by

$$\varPhi_{\varepsilon,t}(\lambda):=\begin{cases}(\lambda^{-1}\psi_{\varepsilon,t}(\lambda),\lambda^{-2}\chi_{\varepsilon,t}(\lambda)), & \lambda\neq0,\\ (ts_1+(1-t)s_2,tp_1+(1-t)p_2+\mu_\varepsilon t(1-t)(s_1-s_2)^2), & \lambda=0.\end{cases}$$

Then $\varPhi_{\varepsilon,t}$ is holomorphic and, by the maximum principle, we get

$$\varrho(\varPhi_{\varepsilon,t}(0))\le\limsup_{s\to(1-\varepsilon)/C}\max_{|\lambda|=s}\varrho(\varPhi_{\varepsilon,t}(\lambda))=\limsup_{s\to(1-\varepsilon)/C}\frac{1}{s}\max_{|\lambda|=s}\varrho(\varphi_{\varepsilon,t}(\lambda))\le\frac{C}{1-\varepsilon},$$

that is,

$$\varrho(ts_1+(1-t)s_2,tp_1+(1-t)p_2+\mu_\varepsilon t(1-t)(s_1-s_2)^2)\le\frac{C}{1-\varepsilon}.$$

We only need to prove that $\mu_\varepsilon\to0$.

Taking $t=1/2$ we get

$$\varrho\biggl(\frac{1}{2}\,(s_1+s_2),\frac{1}{2}\,(p_1+p_2)+\frac{1}{4}\,\mu_\varepsilon(s_1-s_2)^2\biggr)\le\frac{C}{1-\varepsilon}.$$

For $\alpha\in\partial E$, take $(s_1,p_1):=\pi(\alpha,-1)=(\alpha-1,-\alpha)$, $(s_2,p_2):=\pi(\alpha,1)=(\alpha+1,\alpha)$. Then $C=1$ and

$$\varrho(\alpha,\mu_\varepsilon)\le\frac{1}{1-\varepsilon}.$$

Hence $((1-\varepsilon)\alpha,(1-\varepsilon)^2\mu_\varepsilon)\in\overline{\mathbb{G}}_2$ and so, by Lemma 1.4.3,

$$|(1-\varepsilon)\alpha-(1-\varepsilon)^2\mu_\varepsilon(1-\varepsilon)\overline{\alpha}|+(1-\varepsilon)^4|\mu_\varepsilon|^2\le1,\qquad\alpha\in\partial E.$$

It follows that

$$(1-\varepsilon)+(1-\varepsilon)^3|\mu_\varepsilon|+(1-\varepsilon)^4|\mu_\varepsilon|^2\le1$$

and, finally, $|\mu_\varepsilon|\le\varepsilon/((1-\varepsilon)^3)\to0$. ■

LEMMA 1.4.11 ([Cos 2003], [Cos 2004b]). *Let $\varphi:E\to\mathbb{G}_2$ be a mapping of the form*

$$\varphi=(S,P)=\biggl(\frac{\widetilde{S}}{P_0},\frac{\widetilde{P}}{P_0}\biggr),\tag{1.4.6}$$

where P_0, $\widetilde{P}$, $\widetilde{S}$ *are polynomials of degree* ≤ 2 *with* $P_0^{-1}(0) \cap \overline{E} = \emptyset$. *Assume that* $\varphi(\partial E) \subset \sigma_2$ *and* $\varphi(\xi) = (2\eta, \eta^2)$ *for some* $\xi, \eta \in \partial E$. *Then* $h := F_{\overline{\eta}} \circ \varphi \in \operatorname{Aut}(E)$. *In particular*, *if* $a' := \varphi(\lambda')$, $a'' := \varphi(\lambda'')$, *then*

$$\begin{aligned} m_E(\lambda', \lambda'') &= m_E(h(\lambda'), h(\lambda'')) = m_E(F_{\overline{\eta}}(a'), F_{\overline{\eta}}(a'')) \\ &\leq \max\{m_E(F_z(a'), F_z(a'')) : z \in \partial E\} \\ &\leq \max\{m_E(F_z(a'), F_z(a'')) : z \in \overline{E}\} \leq c^*_{\mathbb{G}_2}(a', a'') \\ &\leq \widetilde{k}^*_{\mathbb{G}_2}(a', a'') = \widetilde{k}^*_{\mathbb{G}_2}(\varphi(\lambda'), \varphi(\lambda'')) \leq m_E(\lambda', \lambda''). \end{aligned}$$

Consequently, *the formulas from Theorem* 1.4.1 *hold for all* $(s_1, p_1), (s_2, p_2) \in \varphi(E)$, *and* φ *is a complex geodesic.*

Proof. Put

$$h := F_{\overline{\eta}} \circ \varphi = \frac{2\overline{\eta}P - S}{2 - \overline{\eta}S} = \frac{2\overline{\eta}\widetilde{P} - \widetilde{S}}{2P_0 - \overline{\eta}\widetilde{S}}.$$

First observe that $h(E) \subset E$ and $h(\partial E) \subset \partial E$. It is clear that h is a rational function of degree ≤ 2. Notice that $2\overline{\eta}P(\eta) - S(\eta) = 0 = 2P(\eta) - \overline{\eta}S(\eta)$. Consequently, h is a rational function of degree ≤ 1 and, therefore, h must be an automorphism of the unit disc. ■

LEMMA 1.4.12. *If* φ *satisfies the assumptions of Lemma* 1.4.11, *then for any* $g \in \operatorname{Aut}(E)$, *the mapping* $\psi := H_g \circ \varphi$ *satisfies the same assumptions.*

Proof. The only problem is to check that ψ has the form (1.4.6). Let $g = \tau h_a$ for some $\tau \in \partial E$, $a \in E$. Fix a λ and let $\varphi(\lambda) = (S(\lambda), P(\lambda)) = \pi(z_1, z_2)$. Then

$$\begin{aligned} \psi(\lambda) &= \pi(g(z_1), g(z_2)) = (\tau(h_a(z_1) + h_a(z_2)), \tau^2 h(z_1)h(z_2)) \\ &= \left(\tau\,\frac{(1+|a|^2)(z_1+z_2) - 2\overline{a}z_1z_2 - 2a}{1 - \overline{a}(z_1+z_2) + \overline{a}^2 z_1z_2}, \tau^2\,\frac{z_1z_2 - a(z_1+z_2) + a^2}{1 - \overline{a}(z_1+z_2) + \overline{a}^2 z_1z_2}\right). \end{aligned}$$

Consequently,

$$\begin{aligned} \psi &= \left(\tau\,\frac{(1+|a|^2)S - 2\overline{a}P - 2a}{1 - \overline{a}S + \overline{a}^2P}, \tau^2\,\frac{P - aS + a^2}{1 - \overline{a}S + \overline{a}^2P}\right) \\ &= \left(\tau\,\frac{(1+|a|^2)\widetilde{S} - 2\overline{a}\widetilde{P} - 2aP_0}{P_0 - \overline{a}\widetilde{S} + \overline{a}^2\widetilde{P}}, \tau^2\,\frac{\widetilde{P} - a\widetilde{S} + a^2P_0}{P_0 - \overline{a}\widetilde{S} + \overline{a}^2\widetilde{P}}\right). \end{aligned} \quad ■$$

Proof of Theorem 1.4.1. We already know (Lemma 1.4.10) that $\mathbb{G}_2$ cannot be exhausted by domains biholomorphic to convex domains.

STEP 1. First consider the case $s_1 = s_2 = 0$. Consider the embedding $E \ni \lambda \overset{\varphi}{\mapsto} (0, \lambda) \in \mathbb{G}_2$ and the projection $\mathbb{G}_2 \ni (s, p) \overset{F}{\mapsto} p \in E$. Then

$$\begin{aligned} m_E(p_1, p_2) &= \max\{m_E(zp_1, zp_2) : z \in \partial E\} = \max\{m_E(zp_1, zp_2) : z \in \overline{E}\} \\ &= m_E(F(0, p_1), F(0, p_2)) \leq c^*_{\mathbb{G}_2}((0, p_1), (0, p_2)) \\ &\leq \widetilde{k}^*_{\mathbb{G}_2}((0, p_1), (0, p_2)) = \widetilde{k}^*_{\mathbb{G}_2}(\varphi(p_1), \varphi(p_2)) \leq m_E(p_1, p_2), \end{aligned}$$

which completes the proof.

STEP 2. Assume that $s_1 = 0$, $s_2 \neq 0$. Let $t_0 \in (0,1)$ be defined by the formula

$$t_0 := \max\left\{ m_E\left(p_1, \frac{2p_2 - \overline{z}s_2}{2 - zs_2}\right) : z \in \overline{E}\right\} = m_E\left(p_1, \frac{2p_2 - \overline{\xi}s_2}{2 - \xi s_2}\right),$$

where $\xi \in \overline{E}$. Our aim is to construct a mapping $\varphi : E \to \mathbb{G}_2$ satisfying all the assumptions of Lemma 1.4.11 such that $\varphi(t_0) = (0, p_2)$, $\varphi(0) = (s_2, p_2)$.

First, we prove that

$$m_E\left(p_1, \frac{2p_2 - \overline{z}s_2}{2 - zs_2}\right) < t_0, \qquad z \in E,$$

and so $\xi \in \partial E$. Indeed, let $L : \mathbb{C} \to \mathbb{C}$, $L(z) := p_1 z - \overline{z}$. Then L is an $\mathbb{R}$-linear isomorphism. In particular, if $D := L(E)$, then $\partial D = L(\partial E)$. Observe that

$$t_0 = \max\{|\Phi(L(z))| : z \in \overline{E}\} = \max\{|\Phi(w)| : w \in \overline{D}\},$$

where

$$\Phi(w) := \frac{2(p_2 - p_1) + s_2 w}{2(1 - p_1\overline{p}_2) + \overline{s}_2 w}.$$

Note that $\Phi \not\equiv \text{const}$. Now, the required result follows easily from the maximum principle. In particular, $m_E(p_1, p_2) < t_0$.

Using the automorphism R_ξ, we may reduce the problem to the case $\xi = 1$.

STEP 3. Put

$$a_0 := F_1(s_2, p_2) = \frac{2p_2 - s_2}{2 - s_2} \in E$$

and let $h \in \operatorname{Aut}(E)$ be such that $h(t_0) = p_1$, $h(0) = a_0$. Let $\tau \in \partial E$ be such that $h(\tau) = 1$.

There exists a Blaschke product P of order 2 such that $P(t_0) = p_1$, $P(0) = p_2$, and $P(\tau) = 1$. Indeed, first observe that it suffices to find a Blaschke product Q of order 2 with $Q(t_0) = h_{p_2}(p_1) =: p_1'$, $Q(0) = 0$, and $Q(\tau) = h_{p_2}(1) =: \tau' \in \partial E$ (having such a Q we put $P := h_{-p_2} \circ Q$).

We have $Q(\lambda) = \lambda g(\lambda)$, where $g \in \operatorname{Aut}(E)$ is such that $g(t_0) = p_1'/t_0 =: a \in E$ (recall that $t_0 > m_E(p_1, p_2) = |p_1'|$) and $g(\tau) = \tau'/\tau =: \tau'' \in \partial E$. Define

$$g := h_{-a} \circ (\zeta \cdot h_{t_0}),$$

where $\zeta := h_a(\tau'')\overline{h_{t_0}(\tau)}$. Then $g(t_0) = h_{-a}(0) = a$ and $g(\tau) = h_{-a}(\zeta \cdot h_{t_0}(\tau)) = \tau''$.

STEP 4. Define

$$S := 2\frac{P - h}{1 - h}, \qquad \varphi := (S, P).$$

First observe that φ has the form (1.4.6). Indeed, let $P = \widetilde{P}/P_0$. The only problem is to show that $S = \widetilde{S}/P_0$ with $\widetilde{S}$ being a polynomial of degree ≤ 2. Let $h = \widetilde{h}/h_0$. Then

$$S = 2\,\frac{\widetilde{P}h_0 - \widetilde{h}P_0}{P_0(h_0 - \widetilde{h})}.$$

Since $h(\tau) = P(\tau) = 1$, the polynomial $\widetilde{P}h_0 - \widetilde{h}P_0$ is divisible by $\widetilde{h} - h_0$.

Observe that:

- $\varphi(t_0) = (S(t_0), P(t_0)) = \left(2\dfrac{P(t_0) - h(t_0)}{1 - h(t_0)}, p_1\right) = (0, p_1) = (s_1, p_1)$;
- $\varphi(0) = (S(0), P(0)) = \left(2\dfrac{p_2 - a_0}{1 - a_0}, p_2\right) = \left(2\dfrac{p_2 - (2p_2 - s_2)/(2 - s_2)}{1 - (2p_2 - s_2)/(2 - s_2)}, p_2\right)$

  $= (s_2, p_2)$;
- $F_1 \circ \varphi = \dfrac{2P - S}{2 - S} = \dfrac{2P - 2(P - h)/(1 - h)}{2 - 2(P - h)/(1 - h)} = h$;
- on ∂E we get

$$\overline{S}P = 2\,\frac{\overline{P} - \overline{h}}{1 - \overline{h}}P = 2\,\frac{1 - P/h}{1 - 1/h} = S.$$

We have

$$h = \frac{2P - S}{2 - S} = \frac{2\widetilde{P} - \widetilde{S}}{2P_0 - \widetilde{S}}.$$

Note that $\deg(2\widetilde{P} - \widetilde{S}) = 2$ or $\deg(2P_0 - \widetilde{S}) = 2$. Thus the polynomials $2\widetilde{P} - \widetilde{S}$, $2P_0 - \widetilde{S}$ must have a common zero, say z_0. We have $2\widetilde{P}(z_0) = \widetilde{S}(z_0) = 2P_0(z_0)$. Thus $P(z_0) = 1$, which implies that $z_0 \in \partial E$ and $S(z_0) = 2$.

Put $C := \max\{|S(\lambda)| : \lambda \in \partial E\}$ (we already know that $C \geq 2$). Define $\psi := (2S/C, P)$. Then ψ satisfies all the assumptions of Lemma 1.4.11 and, consequently, Theorem 1.4.1 holds for points from $\psi(E)$. In particular, there exists an $\eta \in \overline{E}$ such that

$$t_0 = m_E\left(\overline{\eta}p_1, \frac{2\overline{\eta}p_2 - 2s_2/C}{2 - \overline{\eta}2s_2/C}\right) = m_E\left(p_1, \frac{2p_2 - \eta 2s_2/C}{2 - \overline{\eta}2s_2/C}\right).$$

Hence, $C \leq 2$ and finally $C = 2$. Consequently, $\varphi = \psi$, which completes the proof of the theorem in the case $s_1 = 0$.

STEP 5. Now let $(s_1, p_1), (s_2, p_2) \in \mathbb{G}_2$ be arbitrary. Suppose that $s_1 = \lambda_1^0 + \lambda_2^0$ with $\lambda_1^0, \lambda_2^0 \in E$. One can easily prove that there exists an automorphism $g \in \operatorname{Aut}(E)$ such that $g(\lambda_1^0) + g(\lambda_2^0) = 0$. Then $H_g(s_1, p_1) = (0, p_1')$. Put $(s_2', p_2') := H_2(s_2, p_2)$. We have the following two cases:

- $s_2' = 0$. We already know that the mapping $\varphi = (0, h)$ with suitable $h \in \operatorname{Aut}(E)$ ($h(t_0) = p_1'$, $t_0 := m_E(p_1', p_2')$, $h(0) = p_2'$) is a complex geodesic for $(0, p_1')$ and $(0, p_2')$. By an argument as in the proof of Lemma 1.4.12, we easily conclude that if $g^{-1} = \tau h_a$, then

$$\psi := H_{g^{-1}} \circ \varphi = \left(\tau\,\frac{-2\overline{a}h - 2a}{1 + \overline{a}^2 h}, \tau^2\,\frac{h + a^2}{1 + \overline{a}^2 h}\right) = (\overline{\beta}q + \beta, q),$$

where

$$\beta := -\tau\,\frac{2a}{1 + |a|^2} \in E, \qquad q := \tau^2 h_{-a^2} \circ h \in \operatorname{Aut}(E).$$

For any $\alpha \in \partial E$ we have

$$F_\alpha \circ \psi = \alpha\,\frac{2q - \overline{\alpha}(\overline{\beta}q + \beta)}{2 - \alpha(\overline{\beta}q + \beta)} = \frac{2 - \overline{\alpha}\overline{\beta}}{2 + \alpha\beta}\,\frac{q - \overline{\alpha}\beta/(2 - \overline{\alpha}\overline{\beta})}{1 - \alpha\overline{\beta}/(2 + \alpha\beta)q} =: q_\alpha \in \operatorname{Aut}(E).$$

Hence

$$\begin{aligned}
t_0 &= m_E(q_\alpha(t_0), q_\alpha(0)) = m_E(F_\alpha(\psi(t_0)), F_\alpha(\psi(0)))\\
&\le \max\left\{m_E\left(\frac{2p_1 - \overline{z}s_1}{2 - zs_1}, \frac{2p_2 - \overline{z}s_2}{2 - zs_2}\right) : z \in \partial E\right\}\\
&= \max\{m_E(F_z(s_1,p_1), F_z(s_2,p_2)) : z \in \partial E\}\\
&\le \max\{m_E(F_z(s_1,p_1), F_z(s_2,p_2)) : z \in \overline{E}\}\\
&\le c^*_{\mathbb{G}_2}((s_1,p_1),(s_2,p_2)) \le \widetilde{k}^*_{\mathbb{G}_2}((s_1,p_1),(s_2,p_2)) = \widetilde{k}^*_{\mathbb{G}_2}(\psi(t_0),\psi(0)) \le t_0.
\end{aligned}$$

- $s_2' \neq 0$. We know that there exists a mapping $\varphi : E \to \mathbb{G}_2$ as in Lemma 1.4.11 such that $H_g(s_j, p_j) \in \varphi(E)$, $j = 1, 2$. It remains to observe that, by Lemma 1.4.12, the mapping $H_{g^{-1}} \circ \varphi$ also satisfies all the assumptions of Lemma 1.4.11. ■

COROLLARY 1.4.13.

$$\begin{aligned}
&c^*_{\mathbb{G}_2}((s_1,p_1),(s_2,p_2)) = \widetilde{k}^*_{\mathbb{G}_2}((s_1,p_1),(s_2,p_2))\\
&= \max\left\{\left|\frac{(s_1p_2 - p_1s_2)z^2 + 2(p_1 - p_2)z + s_2 - s_1}{(p_1\overline{s}_2 - s_1)z^2 + 2(1 - p_1\overline{p}_2)z + s_1\overline{p}_2 - \overline{s}_2}\right| : z \in \partial E\right\}, \qquad (s_1,p_1),(s_2,p_2) \in \mathbb{G}_2.
\end{aligned}$$

In particular,

$$\begin{aligned}
c^*_{\mathbb{G}_2}((0,0),(s,p)) &= \widetilde{k}^*_{\mathbb{G}_2}((0,0),(s,p))\\
&= \max\{|F_z(s,p)| : z \in \partial E\} = \frac{2|s - \overline{s}p| + |s^2 - 4p|}{4 - |s|^2}, \qquad (s,p) \in \mathbb{G}_2.
\end{aligned}$$

THEOREM 1.4.14 ([Jar-Pfl 2004], [Cos 2004b]).

$$\mathrm{Aut}(\mathbb{G}_2) = \{H_h : h \in \mathrm{Aut}(E)\}. \text{ (17)}$$

A characterization of $\mathrm{Aut}(\mathbb{G}_2)$ is also announced in [Agl-You 2004].

Proof. STEP 1. First observe that $\mathrm{Aut}(\mathbb{G}_2)$ does not act transitively on $\mathbb{G}_2$. Otherwise, by the Cartan classification theorem (cf. [Akh 1990], [Fuk 1965]), $\mathbb{G}_2$ would be biholomorphic to $\mathbb{B}_2$ or E^2, which is, by Theorem 1.4.1, impossible (18).

STEP 2. Next observe that $F(\Sigma_2) = \Sigma_2$ for every $F \in \mathrm{Aut}(\mathbb{G}_2)$. Indeed, let $V := \{F(0,0) : F \in \mathrm{Aut}(\mathbb{G}_2)\}$. By W. Kaups' theorem, V is a connected complex submanifold of $\mathbb{G}_2$ (cf. [Kau 1970]). We already know that $\Sigma_2 \subset V$ (Remark 1.4.8). Since $\mathrm{Aut}(\mathbb{G}_2)$ does not act transitively, we have $V \subsetneq \mathbb{G}_2$. Thus $V = \Sigma_2$.

Take a point $(s_0, p_0) = H_h(0,0) \in \Sigma_2$ with $h \in \mathrm{Aut}(E)$ (Remark 1.4.8 again). Then for every $F \in \mathrm{Aut}(\mathbb{G}_2)$, we get $F(s_0,p_0) = (F \circ H_h)(0,0) \in V = \Sigma_2$.

STEP 3. By Remark 1.4.8, we only need to show that every automorphism $F \in \mathrm{Aut}(\mathbb{G}_2)$ with $F(0,0) = (0,0)$ is equal to a "rotation" R_τ. Fix such an $F = (S, P)$.

(17) See Remark 1.4.17 for a more general result.

(18) Instead of invoking Theorem 1.4.1, one can also argue as follows: In the case $\mathbb{G}_2 \simeq \mathbb{B}_2$ we use the Remmert–Stein theorem (cf. [Nar 1971, p. 71]) saying that there is no proper holomorphic mapping $E^2 \to \mathbb{B}_2$. In the case $\mathbb{G}_2 \simeq E^2$ we use the characterization of proper holomorphic mappings $F : E^2 \to E^2$ (cf. [Nar 1971, p. 76]), saying that any such mapping has the form $F(z_1,z_2) = (F_1(z_1), F_2(z_2))$ up to a permutation of variables.

First observe that $F|_{\Sigma_2} \in \mathrm{Aut}(\Sigma_2)$. Hence the mapping

$$E \ni \lambda \mapsto (2\lambda, \lambda^2) \mapsto F(2\lambda, \lambda^2) \mapsto \frac{1}{2}\,\mathrm{pr}_s(F(2\lambda, \lambda^2)) \in E$$

must be a rotation, i.e. $F(2\lambda, \lambda^2) = (2\alpha\lambda, \alpha^2\lambda^2)$ for some $\alpha \in \partial E$. Taking $R_{1/\alpha} \circ F$ instead of F, we may assume that $\alpha = 1$. In particular, $F'(0,0)\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right] = \left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]$ and, therefore, $F'(0,0) = \left[\begin{smallmatrix}1 & b\\0 & d\end{smallmatrix}\right]$.

For $\tau \in \partial E$ put $G_\tau := F^{-1} \circ R_{1/\tau} \circ F \circ R_\tau \in \mathrm{Aut}(\mathbb{G}_2)$. Obviously, $G_\tau(0,0) = (0,0)$. Moreover, $G'_\tau(0,0) = \left[\begin{smallmatrix}1 & b(\tau-1)\\0 & 1\end{smallmatrix}\right]$. Let $G^n_\tau : \mathbb{G}_2 \to \mathbb{G}_2$ be the nth iterate of G_τ. We have $(G^n_\tau)'(0,0) = \left[\begin{smallmatrix}1 & nb(\tau-1)\\0 & 1\end{smallmatrix}\right]$. Using the Cauchy inequalities, we get

$$|nb(\tau - 1)| \le \text{const}, \qquad n \in \mathbb{N},\ \tau \in \partial E,$$

which implies that $b = 0$, i.e. $F'(0,0)$ is diagonal.

STEP 4. We have $G'_\tau(0,0) = \mathbb{I}_2$. Hence, by the Cartan theorem (cf. [Nar 1971, p. 66]), $G_\tau = \mathrm{id}$. Consequently, $R_\tau \circ F = F \circ R_\tau$, i.e.

$$(\tau S(s,p), \tau^2 P(s,p)) = (S(\tau s, \tau^2 p), P(\tau s, \tau^2 p)), \qquad (s,p) \in \mathbb{G}_2,\ \tau \in \partial E.$$

Hence $F(s,p) = (s, p + Cs^2)$. Since $F(2\lambda, \lambda^2) = (2\lambda, \lambda^2)$, we have $(2\lambda, \lambda^2 + 4C\lambda^2) = (2\lambda, \lambda^2)$, which immediately implies that $C = 0$, i.e. $F = \mathrm{id}$. ∎

From Theorem 1.4.1 we know that for any two points $(s_1, p_1), (s_2, p_2) \in \mathbb{G}_2$ there exists a complex geodesic φ with $(s_1, p_1), (s_2, p_2) \in \varphi(E)$ (cf. §1.2.5). Moreover, there exists an $\alpha \in \partial E$ such that $F_\alpha \circ \varphi \in \mathrm{Aut}(E)$. We have the following characterization of complex geodesics in $\mathbb{G}_2$ (cf. [Pfl-Zwo 2004]).

THEOREM 1.4.15. *Let* $\varphi = (S, P) : E \to \mathbb{G}_2$. *Then*:

(a) *If* $\#(\varphi(E) \cap \Sigma_2) \ge 2$, *then* φ *is a complex geodesic iff* $\varphi(\lambda) = (-2\lambda, \lambda^2)$ $(\lambda \in E)$ mod $\mathrm{Aut}(E)$. *In particular*, *if* φ *is a complex geodesic*, *then* $\varphi(E) = \Sigma_2$.

(b) *If* $\#(\varphi(E) \cap \Sigma_2) = 1$, *then* φ *is a complex geodesic iff* $\varphi(\lambda) = \pi(B(\sqrt{\lambda}), B(-\sqrt{\lambda}))$ $(\lambda \in E)$ mod $\mathrm{Aut}(E)$, *where* B *is a Blaschke product of order* ≤ 2 *with* $B(0) = 0$.

(c) *If* $\varphi(E) \cap \Sigma_2 = \emptyset$, *then* φ *is a complex geodesic iff* $\varphi = \pi(h_1, h_2)$, *where* $h_1, h_2 \in$ $\mathrm{Aut}(E)$ *are such that* $h_1 - h_2$ *has no zero in* E.

In particular, *any complex geodesic* $\varphi : E \to \mathbb{G}_2$ *extends holomorphically to* $\overline{E}$ *and* $\varphi(\partial E) \subset \sigma_2$.

A concrete description of all complex geodesics in $\mathbb{G}_2$ is also announced in [Agl-You 2004].

Proof. (a) Let $\varphi(\lambda) := (-2\lambda, \lambda^2)$, $\lambda \in E$. Then $\varphi(E) = \Sigma_2$. Consequently, by Lemma 1.4.11, φ is a complex geodesic.

Now, let $\varphi : E \to \mathbb{G}_2$ be a complex geodesic with $\varphi(\xi) = (2\mu, \mu^2) \in \Sigma_2$ $(\xi, \mu \in E)$. Taking $\varphi \circ h_{-\xi}$ instead of φ, we may assume that $\xi = 0$. Taking $H_{h_\mu} \circ \varphi$ instead of φ, we may assume that $\mu = 0$, i.e. $\varphi(0,0) = (0,0)$. By Theorem 1.4.1, there exists an $\alpha \in \partial E$ such that $F_\alpha \circ \varphi = (2\alpha P - S)/(2 - \alpha S) =: h \in \mathrm{Aut}(E)$. Taking $R_\alpha \circ \varphi$ instead of φ we may assume that $\alpha = 1$. Observe that h must be a rotation and, therefore, we may also assume that $h = \mathrm{id}$, i.e. $(2P - S)/(2 - S) = \mathrm{id}$. Thus $S(\lambda) = 2(P(\lambda) - \lambda)/(1 - \lambda)$, $\lambda \in E$.

Let $\eta \in E_*$ be such that $\varphi(\eta) \in \Sigma_2$. Then $S^2(\eta) = 4P(\eta)$, i.e.

$$\left(\frac{P(\eta)-\eta}{1-\eta}\right)^2 = P(\eta).$$

Hence $P(\eta) = \eta^2$ and so $S(\eta) = -2\eta$. The Schwarz lemma implies that $S(\lambda) = -2\lambda$, $\lambda \in E$, and finally, $P(\lambda) = \lambda^2$, $\lambda \in E$.

(b) Let $\varphi(\lambda) := \pi(B(\sqrt{\lambda}), B(-\sqrt{\lambda}))$, $\lambda \in E$, where B is a Blaschke product of order ≤ 2 with $B(0) = 0$.

In the case $B(\lambda) = \tau\lambda$, $\lambda \in E$ ($\tau \in \partial E$), we get $\varphi(\lambda) = (0, -\tau^2\lambda)$, $\lambda \in E$. Consequently, $F_a \circ \varphi \in \operatorname{Aut}(E)$ for any $a \in \partial E$ (cf. Step 5 of the proof of Theorem 1.4.1).

In the case $B(\lambda) = \tau\lambda h_b(\lambda)$, $\lambda \in E$ ($\tau \in \partial E$, $b \in E$), we get $\varphi(\partial E) \subset \sigma_2$ and

$$\varphi(\lambda) = \left(2\tau\lambda\frac{1-|b|^2}{1-\overline{b}^2\lambda}, \tau^2\lambda\frac{\lambda - b^2}{1-\overline{b}^2\lambda}\right), \qquad \lambda \in E.$$

To apply Lemma 1.4.11, we only need to observe that $\varphi(\xi) = (2\mu, \mu^2)$ for some $\xi, \mu \in \partial E$. The case $b = 0$ is obvious. In the case $b \neq 0$ take $\xi := b/\overline{b}$. Then $\varphi(\xi) = (2\tau\xi, \tau^2\xi^2)$.

Now, let $\varphi : E \to \mathbb{G}_2$ be a complex geodesic with $\#(\varphi(E) \cap \Sigma_2) = 1$. Then, as in (a), we may assume that $\varphi(0,0) = (0,0)$, $(2P - S)/(2 - S) = \mathrm{id}$. Observe that $\Delta(\lambda) := S^2(\lambda) - 4P(\lambda) \neq 0$ for $\lambda \in E_*$. Write $\Delta(\lambda) = \lambda^k\widetilde{\Delta}(\lambda)$, where $\widetilde{\Delta}(\lambda) \neq 0$, $\lambda \in E$. Define

$$B(\lambda) := \frac{1}{2}\Big(S(\lambda^2) + \lambda^k\sqrt{\widetilde{\Delta}(\lambda^2)}\Big), \qquad \lambda \in E.$$

Then

$$S^2(\lambda^2) - 4P(\lambda^2) = \Delta(\lambda^2) = \lambda^{2k}\widetilde{\Delta}(\lambda^2) = 4B^2(\lambda) - 4B(\lambda)S(\lambda^2) + S^2(\lambda^2), \qquad \lambda \in E,$$

which implies that

$$B(\lambda)S(\lambda^2) - B^2(\lambda) = P(\lambda^2) = B(-\lambda)S(\lambda^2) - B^2(-\lambda)$$

and, consequently,

$$(B(\lambda) - B(-\lambda))(S(\lambda^2) - (B(\lambda) + B(-\lambda))) = 0, \qquad \lambda \in E.$$

We have the following two cases:

(i) $S(\lambda^2) = B(\lambda) + B(-\lambda)$, $\lambda \in E$. Then

$$P(\lambda^2) = B(\lambda)S(\lambda^2) - B^2(\lambda) = B(\lambda)(B(\lambda) + B(-\lambda)) - B^2(\lambda) = B(\lambda)B(-\lambda), \qquad \lambda \in E.$$

Hence $\varphi(\lambda) = \pi(B(\sqrt{\lambda}), B(-\sqrt{\lambda}))$, $\lambda \in E$.

Fix a $t_0 \in (0,1)$. Let $(s_0, p_0) := \varphi(t_0^2) = \pi(B(t_0), B(-t_0)) = \pi(\lambda_1^0, \lambda_2^0)$. Suppose that there exists a function $f \in \mathcal{O}(E, E)$ such that $f(0) = 0$, $f(t_0) = \lambda_1^0$, $f(-t_0) = \lambda_2^0$, and $f(E) \Subset E$. Put $\psi := \pi(f, f) : E \to \mathbb{G}_2$ and observe that $\psi(0) = (0,0)$ and $\psi(t_0^2) = (s_0, p_0)$. Hence ψ would be a complex geodesic with $\psi(E) \Subset \mathbb{G}_2$, a contradiction.

Thus, the function B solves an extremal problem of 2-type in the sense of [Edi 1995]. Consequently, B must be a Blaschke product of order ≤ 2.

(ii) $B(\lambda) = B(-\lambda)$, $\lambda \in E$. Then there exists a function $B_1 \in \mathcal{O}(E, E)$ such that

$$B(\lambda) = B_1(\lambda^2) = \frac{1}{2}\Big(S(\lambda^2) + \lambda^k\sqrt{\widetilde{\Delta}(\lambda^2)}\Big), \qquad \lambda \in E.$$

Using the same argument, we reduce the proof to the case where there exists a $B_2 \in \mathcal{O}(E,E)$ such that

$$B_2(\lambda^2) = \frac{1}{2}\Big(S(\lambda^2) - \lambda^k\sqrt{\widetilde{\Delta}(\lambda^2)}\Big), \qquad \lambda \in E.$$

Hence $\varphi = \pi(B_1, B_2)$. Since $(2P - S)/(2 - S) = \mathrm{id}$, we get

$$2B_1(\lambda)B_2(\lambda) - (B_1(\lambda) + B_2(\lambda)) = \lambda(2 - (B_1(\lambda) + B_2(\lambda))), \qquad \lambda \in E,$$

which gives $-(B_1'(0) + B_2'(0)) = 2$. Consequently, by the Schwarz lemma, $B_1(\lambda) = B_2(\lambda) = -\lambda$, and finally $\varphi(\lambda) = (-2\lambda, \lambda^2)$, $\lambda \in E$. Thus, $\Delta \equiv 0$, a contradiction.

(c) Let $\varphi := \pi(h_1, h_2)$, where $h_1, h_2 \in \mathrm{Aut}(E)$ are such that $h_1 - h_2$ has no zero in E. Observe that φ satisfies (1.4.6) and $\varphi(\partial E) \subset \sigma_2$. To use Lemma 1.4.11 we only need to check that $\varphi(\xi) = (2\eta, \eta^2)$ for some $\xi, \eta \in \partial E$, i.e. $h_1(\xi) = h_2(\xi)$ for some $\xi \in \partial E$. Let $h_j = \tau_j h_{a_j}$ ($\tau_j \in \partial E$, $a_j \in E$), $j = 1, 2$. Then we have to find a root $z = \xi$ of the equation

$$\tau_1(z - a_1)(1 - \overline{a}_2 z) - \tau_2(z - a_2)(1 - \overline{a}_1 z) = A_2 z^2 + A_1 z + A_0 = 0$$

with $|\xi| = 1$. We have $A_2 = -\tau_1\overline{a}_2 + \tau_2\overline{a}_1$, $A_0 = -\tau_1 a_1 + \tau_2 a_2$. Observe that $|A_2| = |A_0|$. Since the equation $h_1 - h_2 = 0$ has no roots in E, we get $A_2 \neq 0$. Let z_1, z_2 be the roots of the above equation. We have $|z_1|, |z_2| \geq 1$ and $|z_1 z_2| = |A_0/A_2| = 1$. Thus $|z_1| = |z_2| = 1$.

Now, let $\varphi : E \to \mathbb{G}_2$ be a complex geodesic with $\varphi(E) \cap \Sigma_2 = \emptyset$. Then there exists a holomorphic mapping $\psi : E \to E^2$ with $\pi \circ \psi = \varphi$. Consequently, ψ must be a complex geodesic ($m_E(\lambda', \lambda'') = c^*_{\mathbb{G}_2}(\varphi(\lambda'), \varphi(\lambda'')) \leq c^*_{E^2}(\psi(\lambda'), \psi(\lambda'')) \leq m_E(\lambda', \lambda'')$). Hence, $\psi = (h_1, h_2)$, where $h_1, h_2 \in \mathcal{O}(E,E)$ and at least one of them is an automorphism. Assume that $h_1 \in \mathrm{Aut}(E)$.

Fix a $t_0 \in (0,1)$ and suppose that $m_E(h_2(0), h_2(t_0)) < t_0$. Let

$$\varrho := m_E(h_2(0), h_2(t_0))/t_0 \in (0,1).$$

There exists a $g \in \mathrm{Aut}(E)$ such that $g(0) = h_2(0)$, $g(\varrho t_0) = h_2(t_0)$. Put $f(\lambda) := g(\varrho\lambda)$, $\lambda \in \overline{E}$. Then $f(0) = h_2(0)$, $f(t_0) = h_2(t_0)$, and $f(E) \Subset E$. Put $\chi = (\chi_1, \chi_2) := \pi(h_1, f)$. Then $\chi(0) = \varphi(0)$ and $\chi(t_0) = \varphi(t_0)$. Thus χ is also a complex geodesic. Notice that by the Rouché theorem the function $h_1 - f$ has a zero in E. Hence $\chi(E) \cap \Sigma_2 \neq \emptyset$. In particular, in view of (a) and (b), $\chi(\partial E) \subset \sigma_2$. On the other hand $\chi_2 = h_1 f$, a contradiction.

Consequently, $m_E(h_2(0), h_2(t_0)) = t_0$, and therefore, $h_2 \in \mathrm{Aut}(E)$. ■

REMARK 1.4.16. With the help of Theorem 1.4.1, the Carathéodory and Kobayashi pseudodistances and the Lempert function were calculated for the following unbounded balanced domain

$$\Omega_2 := \{A \in \mathbb{C}(2 \times 2) : r(A) < 1\},$$

where $r(a)$ denotes the spectral radius of A; cf. [Cos 2004b].

REMARK 1.4.17. Let $n \geq 3$ and let $\pi_n : \mathbb{C}^n \to \mathbb{C}^n$,

$$\pi_n(\lambda_1, \dots, \lambda_n) := \Big(\sum_{1 \leq j_1 < \dots < j_k \leq n} \lambda_{j_1} \cdots \lambda_{j_k}\Big)_{k=1,\dots,n}.$$

Observe that π_n, $\pi_n|_{E^n}$ are proper. Put $\mathbb{G}_n := \pi_n(E^n)$. The domain $\mathbb{G}_n$ is called the *symmetrized n-disc*.

Recently, in [Edi-Zwo 2004], A. Edigarian and W. Zwonek proved the following result.

THEOREM. *Any proper holomorphic mapping $F : \mathbb{G}_n \to \mathbb{G}_n$ is of the form*

$$F(\pi_n(\lambda_1, \dots, \lambda_n)) = \pi_n(B(\lambda_1), \dots, B(\lambda_n)),$$

where B is a finite Blaschke product. In particular,

$$\mathrm{Aut}(\mathbb{G}_n) = \{H_h : h \in \mathrm{Aut}(E)\},$$

where $H_h(\pi_n(\lambda_1, \dots, \lambda_n)) = \pi_n(h(\lambda_1), \dots, h(\lambda_n))$.

It is an open question whether:

$\boxed{?}$ $c^*_{\mathbb{G}_n} \equiv \widetilde{k}^*_{\mathbb{G}_n}$, i.e. $\mathbb{G}_n \in \mathfrak{L}_n$; $\boxed{?}$

$\boxed{?}$ $\mathbb{G}_n$ cannot be exhausted by domains biholomorphic to convex domains. $\boxed{?}$

Moreover, one can conjecture that for any proper mapping $F : \mathbb{C}^n \to \mathbb{C}^n$ the domain $\mathbb{G} := F(E^n)$ belongs to $\mathfrak{L}_n$.

1.5. Generalized holomorphically contractible families

Observe that the Möbius and Lempert functions are obviously symmetric (c^*_G is even a pseudodistance). The higher Möbius functions and the Green function are in general not symmetric (cf. [J-P 1993, §4.2]). Their definitions distinguish one point (pole) at which we impose growth conditions. From that point of view it is natural to investigate objects with more general growth conditions. For instance, the Green function g_G may be generalized as follows.

DEFINITION 1.5.1. Let $G \subset \mathbb{C}^n$ be a domain and let $\boldsymbol{p} : G \to \mathbb{R}_+$ be a function. Define

$$g_G(\boldsymbol{p}, z) := \sup\{u(z) : u : G \to [0, 1),\ \log u \in \mathcal{PSH}(G),$$
$$\forall_{a \in G}\ \exists_{C = C(u,a) > 0}\ \forall_{w \in G} : u(w) \le C\|w - a\|^{\boldsymbol{p}(a)}\}, \qquad z \in G. \ (^{19})$$

The function $g_G(\boldsymbol{p}, \cdot)$ is called the *generalized pluricomplex Green function with poles (weights, pole function)* $\boldsymbol{p}$.

We have $g_G(\boldsymbol{0}, \cdot) \equiv 1$. Observe that if the set $|\boldsymbol{p}| := \{z \in G : \boldsymbol{p}(z) > 0\}$ is not pluripolar, then $g_G(\boldsymbol{p}, \cdot) \equiv 0$. Obviously, $g_G(\boldsymbol{p}, z) = 0$ for every $z \in |\boldsymbol{p}|$.

In the case where $\boldsymbol{p} = \chi_A =$ the characteristic function of a set $A \subset G$, we put $g_G(A, \cdot) := g_G(\chi_A, \cdot)$. Obviously, $g_G(\{a\}, \cdot) = g_G(a, \cdot)$, $a \in G$. In the case where the set $|\boldsymbol{p}|$ is finite, the function $g_G(\boldsymbol{p}, \cdot)$ was introduced by P. Lelong in [Lel 1989].

The definition of the generalized Green function may be formally extended to the case where $\boldsymbol{p} : G \to [0, +\infty]$. We put $g_G(\boldsymbol{p}, \cdot) :\equiv 0$ if there exists a $z_0 \in G$ with $\boldsymbol{p}(z_0) = +\infty$.

The generalized pluricomplex Green function was recently studied by many authors, e.g. [Car-Wie 2003], [Com 2000], [Edi 2002], [Edi-Zwo 1998b], [Jar-Jar-Pfl 2003], [Lár-Sig 1998b].

(^{19}) Here and in what follows, $0^0 := 1$. Observe that the condition is trivially satisfied at all points $a \in G$ with $\boldsymbol{p}(a) = 0$. The growth condition may be equivalently formulated as follows:

$$\forall_{a \in G : \boldsymbol{p}(a) > 0}\ \exists_{C, r > 0} : u(w) \le C\|w - a\|^{\boldsymbol{p}(a)}, \qquad w \in \mathbb{B}(a, r) \subset G.$$

Using similar ideas, one can generalize the Möbius function.

DEFINITION 1.5.2. Let $G \subset \mathbb{C}^n$ be a domain and let $\boldsymbol{p} : G \to \mathbb{Z}_+$ be a function. Define

$$m_G(\boldsymbol{p}, z) := \sup\{|f(z)| : f \in \mathcal{O}(G, E),\ \operatorname{ord}_a f \ge \boldsymbol{p}(a),\ a \in G\}, \qquad z \in G.$$

The function $m_G(\boldsymbol{p}, \cdot)$ is called the *generalized Möbius function with weights* $\boldsymbol{p}$.

We have $m_G(\boldsymbol{0}, \cdot) \equiv 1$. Notice that if the set $|\boldsymbol{p}|$ is not thin, then $m_G(\boldsymbol{p}, \cdot) \equiv 0$. As before, the definition may be formally extended to the case where $\boldsymbol{p} : G \to \mathbb{Z}_+ \cup \{+\infty\}$; $m_G(\boldsymbol{p}, \cdot) :\equiv 0$ if there exists a $z_0 \in G$ with $\boldsymbol{p}(z_0) = +\infty$. Similarly to the case of the generalized Green function, we put $m_G(A, \cdot) := m_G(\chi_A, \cdot)$ $(A \subset G)$, $m_G(a, \cdot) := m_G(\{a\}, \cdot)$ $(a \in G)$ ([20]).

Obviously, $m_G(a, \cdot) = c_G^*(a, \cdot)$, $a \in G$. More generally, $m_G(k\chi_{\{a\}}, \cdot) = [m_G^{(k)}(a, \cdot)]^k$. It is clear that $m_G(\boldsymbol{p}, \cdot) \le g_G(\boldsymbol{p}, \cdot)$ (for any function $\boldsymbol{p} : G \to \mathbb{Z}_+ \cup \{+\infty\}$). Properties of $g_G(\boldsymbol{p}, \cdot)$ and $m_G(\boldsymbol{p}, \cdot)$ will be presented in §1.6.

The above generalizations lead us to the following definition.

DEFINITION 1.5.3. A family $(d_G)_G$ of functions $d_G : \mathbb{R}_+^G \times G \to \mathbb{R}_+$, where $\mathbb{R}_+^G$ denotes the family of all functions $\boldsymbol{p} : G \to \mathbb{R}_+$, is said to be *a generalized holomorphically contractible family* (*g.h.c.f.*) if the following three conditions are satisfied:

(E) $\prod_{a \in E} [m_E(a, z)]^{\boldsymbol{p}(a)} \le d_E(\boldsymbol{p}, z) \le \inf_{a \in E} [m_E(a, z)]^{\boldsymbol{p}(a)}$, $(\boldsymbol{p}, z) \in \mathbb{R}_+^E \times E$ ([21]),

(H) for any $F \in \mathcal{O}(G, D)$ and $\boldsymbol{q} : D \to \mathbb{R}_+$, we have

$$d_D(\boldsymbol{q}, F(z)) \le d_G(\boldsymbol{q} \circ F, z), \qquad z \in G,$$

(M) for any $\boldsymbol{p}, \boldsymbol{q} : G \to \mathbb{R}_+$, if $\boldsymbol{p} \le \boldsymbol{q}$, then $d_G(\boldsymbol{q}, \cdot) \le d_G(\boldsymbol{p}, \cdot)$.

If in the above definition one considers only integer-valued weights (as in the case of the generalized Möbius function), then we get the definition of a *generalized holomorphically contractible family with integer-valued weights*.

Put $d_G(A, \cdot) := d_G(\chi_A, \cdot)$ $(A \subset G)$, $d_G(a, \cdot) := d_G(\{a\}, \cdot)$ $(a \in G)$.

One can prove that the generalized Green and Möbius functions are g.h.c.f. in the sense of the above definition; cf. §1.6. In the context of the inequalities (1.1.2), it is natural to ask whether there exist minimal and maximal g.h.c.f. Put

$$\begin{aligned} d_G^{\min}(\boldsymbol{p}, z) &:= \sup\Big\{ \prod_{\mu \in f(G)} [m_E(\mu, f(z))]^{\sup \boldsymbol{p}(f^{-1}(\mu))} : f \in \mathcal{O}(G, E) \Big\} \ ([22]) \\ &= \sup\Big\{ \prod_{\mu \in f(G)} |\mu|^{\sup \boldsymbol{p}(f^{-1}(\mu))} : f \in \mathcal{O}(G, E),\ f(z) = 0 \Big\}, \\ d_G^{\max}(\boldsymbol{p}, z) = \widetilde{k}_G^*(\boldsymbol{p}, z) &:= \inf\{[\widetilde{k}_G^*(a, z)]^{\boldsymbol{p}(a)} : a \in G\} \\ &= \inf\{|\mu|^{\boldsymbol{p}(\varphi(\mu))} : \varphi \in \mathcal{O}(E, G),\ \varphi(0) = z,\ \mu \in E\}, \qquad z \in G. \end{aligned}$$

([20]) In the case of the unit disc this definition of m_E coincides with the previous one from Definition 1.1.1.

([21]) For $h : A \to [0, 1]$, we put $\prod_{a \in A} h(a) := \inf_{B \subset A,\ \#B < \infty} \prod_{a \in B} h(a)$.

([22]) Note that the product is 0 if $\sup \boldsymbol{p}(f^{-1}(\mu_0)) = +\infty$ for a $\mu_0 \in f(G)$.

We have $d_G^{\min}(\mathbf{0},\cdot) = d_G^{\max}(\mathbf{0},\cdot) \equiv 1$. Observe that $d_G^{\min}(k\chi_{\{a\}},\cdot) = [c_G^*(a,\cdot)]^k$ and $d_G^{\max}(k\chi_{\{a\}},\cdot) = [\widetilde{k}_G^*(a,\cdot)]^k$. Moreover, for $\emptyset \neq A \subset G$ we get

$$\begin{aligned} d_G^{\min}(A,z) &= \sup\Big\{ \prod_{\mu\in f(A)} m_E(\mu, f(z)) : f\in\mathcal{O}(G,E)\Big\} \\ &\geq \sup\{|f(z)| : f\in\mathcal{O}(G,E),\ f|_A = 0\} = m_G(A,z), \qquad z\in G. \ (^{23}) \end{aligned}$$

We extend formally the definitions of $d_G^{\min}(\boldsymbol{p},\cdot)$ and $d_G^{\max}(\boldsymbol{p},\cdot)$ to the case where $\boldsymbol{p} : G\to [0,+\infty]$; $d_G^{\min}(\boldsymbol{p},\cdot) = d_G^{\max}(\boldsymbol{p},\cdot) :\equiv 0$ if there exists a $z_0\in G$ with $\boldsymbol{p}(z_0) = +\infty$.

Directly from the definitions it follows that the systems $(d_G^{\min})_G$, $(d_G^{\max})_G$ satisfy (E) and (M) of Definition 1.5.3.

PROPOSITION 1.5.4 ([Jar-Jar-Pfl 2003]). *The systems $(d_G^{\min})_G$ and $(d_G^{\max})_G$ are g.h.c.f. Moreover, for any g.h.c.f. $(d_G)_G$ (with integer-valued weights) we have*

$$d_G^{\min}(\boldsymbol{p},\cdot) \leq d_G(\boldsymbol{p},\cdot) \leq d_G^{\max}(\boldsymbol{p},\cdot)$$

for any function $\boldsymbol{p} : G\to\mathbb{R}_+$ ($\boldsymbol{p} : G\to\mathbb{Z}_+$). In particular,

$$d_G^{\min}(\boldsymbol{p},\cdot) \leq g_G(\boldsymbol{p},\cdot) \leq d_G^{\max}(\boldsymbol{p},\cdot)$$

for any function $\boldsymbol{p} : G\to\mathbb{R}_+$ and

$$d_G^{\min}(\boldsymbol{p},\cdot) \leq m_G(\boldsymbol{p},\cdot) \leq g_G(\boldsymbol{p},\cdot) \leq d_G^{\max}(\boldsymbol{p},\cdot) \ (^{24})$$

for any function $\boldsymbol{p} : G\to\mathbb{Z}_+$.

Consequently, $d_G^{\min}(A,\cdot) = m_G(A,\cdot)$, $A\subset G$.

The function $d_G^{\min}$ (resp. $d_G^{\max}$) may be considered as a generalization of the Möbius function c_G^* (resp. Lempert function $\widetilde{k}_G^*$). The properties of $d_G^{\min}$ and $d_G^{\max}$ will be presented in §1.8.

Proof. STEP 1. If $(d_G)_G$ satisfies (H) and

$$(\mathrm{E}^+) \qquad d_E(\boldsymbol{p},\lambda) \leq d_E^{\max}(\boldsymbol{p},\lambda) = \inf\{[m_E(\mu,\lambda)]^{\boldsymbol{p}(\mu)} : \mu\in E\}, \qquad (\boldsymbol{p},\lambda)\in\mathbb{R}_+^E\times E,$$

then $d_G \leq d_G^{\max}$ for any G. The same remains true in the category of g.h.c.f. with integer-valued weights. Indeed,

$$\begin{aligned} d_G(\boldsymbol{p},z) &\overset{(\mathrm{H})}{\leq} \inf\{d_E(\boldsymbol{p}\circ\varphi,0) : \varphi\in\mathcal{O}(E,G),\ \varphi(0) = z\} \\ &\overset{(\mathrm{E}^+)}{\leq} \inf\{|\mu|^{\boldsymbol{p}(\varphi(\mu))} : \varphi\in\mathcal{O}(E,G),\ \varphi(0) = z,\ \mu\in E\} \\ &= d_G^{\max}(\boldsymbol{p},z), \qquad (\boldsymbol{p},z)\in\mathbb{R}_+^G\times G. \end{aligned}$$

(^{23}) Proposition 1.5.4 states that in fact $d_G^{\min}(A,\cdot) = m_G(A,\cdot)$.

(^{24}) Notice that in general $d_G^{\min}(\boldsymbol{p},\cdot) \underset{\not\equiv}{\leq} m_G(\boldsymbol{p},\cdot) \underset{\not\equiv}{\leq} g_G(\boldsymbol{p},\cdot) \underset{\not\equiv}{\leq} d_G^{\max}(\boldsymbol{p},\cdot)$; cf. Examples 1.7.19, 1.7.20.

STEP 2. The system $(d_G^{\max})_G$ is a g.h.c.f. Indeed, to prove (H) let $F : G \to D$ be holomorphic and let $\boldsymbol{q} : D \to \mathbb{R}_+$. Then

$$\begin{aligned} d_D^{\max}(\boldsymbol{q}, F(z)) &= \inf\{[\widetilde{k}_D^*(b, F(z))]^{\boldsymbol{q}(b)} : b \in D\} \\ &\le \inf\{[\widetilde{k}_D^*(F(a), F(z))]^{\boldsymbol{q}(F(a))} : a \in G\} \\ &\le \inf\{[\widetilde{k}_G^*(a, z)]^{\boldsymbol{q}(F(a))} : a \in G\} = d_G^{\max}(\boldsymbol{q} \circ F, z), \qquad z \in G. \end{aligned}$$

STEP 3. If $(d_G)_G$ satisfies (H), (M), and

$$(\mathrm{E}^-) \qquad \prod_{\mu \in E} [m_E(\mu, \lambda)]^{\boldsymbol{p}(\mu)} \le d_E(\boldsymbol{p}, \lambda), \qquad (\boldsymbol{p}, \lambda) \in \mathbb{R}_+^E \times E,$$

then $d_G^{\min} \le d_G$ for any G. The same remains true in the category of g.h.c.f. with integer-valued weights. Indeed,

$$\begin{aligned} d_G(\boldsymbol{p}, z) &\overset{(\mathrm{M})}{\ge} \sup\{d_G(\boldsymbol{q} \circ f, z) : f \in \mathcal{O}(G, E),\, \boldsymbol{q} \in \mathbb{R}_+^E,\, f(z) = 0,\, \boldsymbol{p} \le \boldsymbol{q} \circ f\} \\ &\overset{(\mathrm{H})}{\ge} \sup\{d_E(\boldsymbol{q}, 0) : f \in \mathcal{O}(G, E),\, \boldsymbol{q} \in \mathbb{R}_+^E,\, f(z) = 0,\, \boldsymbol{p} \le \boldsymbol{q} \circ f\} \\ &\overset{(\mathrm{E}^-)}{\ge} \sup\Big\{\prod_{\mu \in E} |\mu|^{\boldsymbol{q}(\mu)} : f \in \mathcal{O}(G, E),\, \boldsymbol{q} \in \mathbb{R}_+^E,\, f(z) = 0,\, \boldsymbol{p} \le \boldsymbol{q} \circ f\Big\} \\ &\ge \sup\Big\{\prod_{\mu \in f(G)} |\mu|^{\sup \boldsymbol{p}(f^{-1}(\mu))} : f \in \mathcal{O}(G, E),\, f(z) = 0\Big\} \\ &= d_G^{\min}(\boldsymbol{p}, z), \qquad (\boldsymbol{p}, z) \in \mathbb{R}_+^G \times G. \end{aligned}$$

STEP 4. The system $(d_G^{\min})_G$ is a g.h.c.f. Indeed, to prove (H) let $F : G \to D$ be holomorphic and let $\boldsymbol{q} : D \to \mathbb{R}_+$. Then

$$\begin{aligned} d_D^{\min}(\boldsymbol{q}, F(z)) &= \sup\Big\{\prod_{\mu \in g(D)} [m_E(\mu, g(F(z))]^{\sup \boldsymbol{q}(g^{-1}(\mu))} : g \in \mathcal{O}(D, E)\Big\} \\ &\overset{f = g \circ F}{\le} \sup\Big\{\prod_{\mu \in f(G)} [m_E(\mu, f(z))]^{\sup (\boldsymbol{q} \circ F)(f^{-1}(\mu))} : f \in \mathcal{O}(G, E)\Big\} \\ &= d_G^{\min}(\boldsymbol{q} \circ F, z), \qquad z \in G. \ \blacksquare \end{aligned}$$

1.6. Properties of the generalized Möbius and Green functions

Directly from Definitions 1.5.1 and 1.5.2 we get the following elementary properties of the generalized Möbius and Green functions (cf. [Jar-Jar-Pfl 2003]).

REMARK 1.6.1. (a) $m_G(k\boldsymbol{p}, \cdot) \ge [m_G(\boldsymbol{p}, \cdot)]^k$, $k \in \mathbb{N}$ ([25]); $g_G(k\boldsymbol{p}, \cdot) = [g_G(\boldsymbol{p}, \cdot)]^k$, $k > 0$.

(b) If $\boldsymbol{p} \le \boldsymbol{q}$, then $m_G(\boldsymbol{p}, \cdot) \ge m_G(\boldsymbol{q}, \cdot)$ and $g_G(\boldsymbol{p}, \cdot) \ge g_G(\boldsymbol{q}, \cdot)$, i.e. both systems $(m_G)_G$, $(g_G)_G$ satisfy condition (M) from Definition 1.5.3. In particular, if $A \subset B \subset G$, then $m_G(A, \cdot) \ge m_G(B, \cdot)$ and $g_G(A, \cdot) \ge g_G(B, \cdot)$.

([25]) Notice that in general $m_G(k\boldsymbol{p}, \cdot) \not\equiv [m_G(\boldsymbol{p}, \cdot)]^k$; for instance, if $P \subset \mathbb{C}$ is an annulus, then $m_P(k\chi_{\{a\}}, \cdot) \not\equiv [m_P(a, \cdot)]^k$, $k \ge 2$; cf. [J-P 1993, Proposition 5.5].

(c)
$$m_G(\boldsymbol{p},\cdot)m_G(\boldsymbol{q},\cdot) \le m_G(\boldsymbol{p}+\boldsymbol{q},\cdot) \le \min\{m_G(\boldsymbol{p},\cdot),\, m_G(\boldsymbol{q},\cdot)\},$$
$$g_G(\boldsymbol{p},\cdot)g_G(\boldsymbol{q},\cdot) \le g_G(\boldsymbol{p}+\boldsymbol{q},\cdot) \le \min\{g_G(\boldsymbol{p},\cdot),\, g_G(\boldsymbol{q},\cdot)\}.$$

In particular,
$$m_G(\boldsymbol{p},\cdot) \le g_G(\boldsymbol{p},\cdot) \le \inf_{a\in G}[g_G(a,\cdot)]^{\boldsymbol{p}(a)} \le \inf_{a\in G}[\widetilde{k}_G^*(a,\cdot)]^{\boldsymbol{p}(a)} = d_G^{\max}(\boldsymbol{p},\cdot).$$

If $|\boldsymbol{p}|$ is finite, then
$$m_G(\boldsymbol{p},\cdot) \ge \prod_{a\in|\boldsymbol{p}|}[m_G(a,\cdot)]^{\boldsymbol{p}(a)}, \qquad g_G(\boldsymbol{p},\cdot) \ge \prod_{a\in|\boldsymbol{p}|}[g_G(a,\cdot)]^{\boldsymbol{p}(a)}$$
(cf. Proposition 1.6.5(a)).

(d) $g_G(\boldsymbol{p},z) = \sup\{u(z) : u : G \to [0,1),\ \log u \in \mathcal{PSH}(G),\ u \le \inf_{a\in G}[g_G(a,\cdot)]^{\boldsymbol{p}(a)}\}$, $z \in G$.

(e) Let $G \subset \mathbb{C}^n$, $D \subset \mathbb{C}^m$ be domains and let $F : G \to D$ be holomorphic. Then for any function $\boldsymbol{q} : D \to \mathbb{Z}_+$ (resp. $\boldsymbol{q} : D \to \mathbb{R}_+$) we have
$$m_D(\boldsymbol{q}, F(z)) \le m_G(\boldsymbol{q}_F, z) \le m_G(\boldsymbol{q}\circ F, z),$$
$$g_D(\boldsymbol{q}, F(z)) \le g_G(\boldsymbol{q}_F, z) \le g_G(\boldsymbol{q}\circ F, z), \qquad z \in G,$$
where
$$\boldsymbol{q}_F(a) := \boldsymbol{q}(F(a))\,\mathrm{ord}_a(F - F(a)), \qquad a \in G. \text{ (}^{26}\text{)}$$

Thus both systems $(m_G)_G$ and $(g_G)_G$ satisfy condition (H) from Definition 1.5.3. In particular,
$$m_D(B, F(z)) \le m_G(F^{-1}(B), z), \quad g_D(B, F(z)) \le g_G(F^{-1}(B), z), \qquad B \subset D,\ z \in G.$$

(f) $\log m_G(\boldsymbol{p},\cdot) \in \mathcal{C}(G) \cap \mathcal{PSH}(G)$, $\log g_G(\boldsymbol{p},\cdot) \in \mathcal{PSH}(G)$ (we can argue as in the one-pole case; cf. [J-P 1993, Proposition 4.2.11, Lemma 4.2.3]).

(g) If $\boldsymbol{p} \not\equiv 0$, then for any $z_0 \in G$ there exists an *extremal function for* $m_G(\boldsymbol{p}, z_0)$, i.e. a function $f_{z_0} \in \mathcal{O}(G, E)$ with $\mathrm{ord}_a f_{z_0} \ge \boldsymbol{p}(a)$, $a \in G$, and $m_G(\boldsymbol{p}, z_0) = |f_{z_0}(z_0)|$.

(h) If $G_k \nearrow G$ and $\boldsymbol{p}_k \nearrow \boldsymbol{p}$, then $m_{G_k}(\boldsymbol{p}_k,\cdot) \searrow m_G(\boldsymbol{p},\cdot)$, $g_{G_k}(\boldsymbol{p}_k,\cdot) \searrow g_G(\boldsymbol{p},\cdot)$. Indeed, the case of the generalized Möbius function follows from a Montel argument (based on (g)). In the case of the generalized Green function first recall that $g_{G_k}(a,\cdot) \searrow g_G(a,\cdot)$, $a \in G$; cf. [J-P 1993, Proposition 4.2.7(a)]. Let $u_k := g_{G_k}(\boldsymbol{p}_k,\cdot)$. Then $\log u_k \in \mathcal{PSH}(G_k)$ (by (f)) and $g_G(\boldsymbol{p},\cdot) \le u_{k+1} \le u_k$ on G_k (by (b) and (e)). Let $u := \lim_{k\to+\infty} u_k$. Obviously, $u \ge g_G(\boldsymbol{p},\cdot)$ and $\log u \in \mathcal{PSH}(G)$. Moreover, since $u_k \le [g_{G_k}(a,\cdot)]^{\boldsymbol{p}_k(a)}$, $a \in G_k$, we easily conclude that $u \le [g_G(a,\cdot)]^{\boldsymbol{p}(a)}$, $a \in G$. Hence, by (d), $u = g_G(\boldsymbol{p},\cdot)$.

(i) Let $P \subset G$ be a relatively closed pluripolar set such that $\boldsymbol{p} = 0$ on P. Then $g_{G\setminus P}(\boldsymbol{p},\cdot) = g_G(\boldsymbol{p},\cdot)$ on $G \setminus P$ (cf. [J-P 1993, Proposition 4.2.7(c)]).

PROPOSITION 1.6.2 ([Jar-Jar-Pfl 2003]). $g_G(\boldsymbol{p},\cdot) = \inf\{g_G(\boldsymbol{q},\cdot) : \boldsymbol{q} \le \boldsymbol{p},\ \#|\boldsymbol{q}| < +\infty\}$.

Proof. Let $u := \inf\{g_G(\boldsymbol{q},\cdot) : \boldsymbol{q} \le \boldsymbol{p},\ \#|\boldsymbol{q}| < +\infty\}$. Obviously $u \ge g_G(\boldsymbol{p},\cdot)$. To prove the opposite inequality we only need to show that $\log u$ is plurisubharmonic. Observe that

(26) Observe that in the case where $F \equiv \text{const} = b$ we have $\boldsymbol{q}_F \equiv +\infty$ if $\boldsymbol{q}(b) > 0$ and $\boldsymbol{q}_F \equiv 0$ if $\boldsymbol{q}(b) = 0$.

$g_G(\max\{\boldsymbol{q}_1, \dots, \boldsymbol{q}_N\}, \cdot) \le \min\{g_G(\boldsymbol{q}_1, \cdot), \dots, g_G(\boldsymbol{q}_N, \cdot)\}$. Thus we only need the general result given below.

LEMMA 1.6.3. *Let $(v_i)_{i\in A} \subset \mathcal{PSH}(\Omega)$ ($\Omega \subset \mathbb{C}^n$) be such that for any $i_1, \dots, i_N \in A$ there exists an $i_0 \in A$ such that $v_{i_0} \le \min\{v_{i_1}, \dots, v_{i_N}\}$. Then $v := \inf_{i\in A} v_i \in \mathcal{PSH}(\Omega)$.*

Proof. It suffices to consider the case $n = 1$. Take a disc $\mathbb{B}(a,r) \Subset \Omega$, $\varepsilon > 0$, and a continuous function $w \in \mathcal{C}(\partial\mathbb{B}(a,r))$ such that $w \ge v$ on $\partial\mathbb{B}(a,r)$. We want to show that

$$v(a) \le \frac{1}{2\pi}\int_0^{2\pi} w(a + re^{i\theta})\, d\theta + \varepsilon.$$

For any point $b \in \partial\mathbb{B}(a,r)$ there exists an $i = i(b) \in A$ such that $v_i(b) < w(b) + \varepsilon$. Hence there exists an open arc $I = I(b) \subset \partial\mathbb{B}(a,r)$ with $b \in I$ such that $v_i(\lambda) < w(\lambda) + \varepsilon$, $\lambda \in I$. By a compactness argument, we find $b_1, \dots, b_N \in \partial\mathbb{B}(a,r)$ such that $\partial\mathbb{B}(a,r) = \bigcup_{j=1}^N I(b_j)$. By assumption, there exists an $i_0 \in A$ such that $v_{i_0} \le \min\{v_{i(b_1)}, \dots, v_{i(b_N)}\}$. Then

$$v(a) \le v_{i_0}(a) \le \frac{1}{2\pi}\int_0^{2\pi} v_{i_0}(a + re^{i\theta})\, d\theta \le \frac{1}{2\pi}\int_0^{2\pi} w(a + re^{i\theta})\, d\theta + \varepsilon. \ \blacksquare$$

PROPOSITION 1.6.4. *For any function $\boldsymbol{p} : G \to \mathbb{Z}_+$ we get*

$$m_G(\boldsymbol{p}, \cdot) = \inf\{m_G(\boldsymbol{q}, \cdot) : \boldsymbol{q} : G \to \mathbb{Z}_+,\ \boldsymbol{q} \le \boldsymbol{p},\ \#|\boldsymbol{q}| < +\infty\}.$$

Proof. The case where $|\boldsymbol{p}|$ is finite is trivial. The case where the set $|\boldsymbol{p}|$ is countable follows from Remark 1.6.1(h). In the general case let $A_k := \{a \in G : \boldsymbol{p}(a) = k\}$ and let B_k be a countable (or finite) dense subset of A_k, $k \in \mathbb{Z}_+$. Put $B := \bigcup_{k=0}^\infty B_k$, $\boldsymbol{p}' := \boldsymbol{p}\cdot\chi_B$. Then $\boldsymbol{p}' \le \boldsymbol{p}$, the set $|\boldsymbol{p}'|$ is at most countable, and $m_G(\boldsymbol{p}, \cdot) \equiv m_G(\boldsymbol{p}', \cdot)$. Consequently, the problem reduces to the countable case. $\blacksquare$

PROPOSITION 1.6.5. (a)

$$m_G(\boldsymbol{p}, \cdot) \ge \prod_{a\in G} [m_G(a, \cdot)]^{\boldsymbol{p}(a)}, \qquad g_G(\boldsymbol{p}, \cdot) \ge \prod_{a\in G} [g_G(a, \cdot)]^{\boldsymbol{p}(a)}.$$

(b) *If $G \subset \mathbb{C}$, then*

$$g_G(\boldsymbol{p}, z) = \prod_{a\in G} [g_G(a, z)]^{\boldsymbol{p}(a)}, \qquad z \in G.$$

In particular,

$$d_E^{\min}(\boldsymbol{p}, z) = m_E(\boldsymbol{p}, z) = g_E(\boldsymbol{p}, z) = \prod_{a\in E} [m_E(a, z)]^{\boldsymbol{p}(a)}, \qquad z \in E.$$

Notice that the formula in (b) is not true for $G \subset \mathbb{C}^n$, $n \ge 2$; cf. Example 1.7.17.

Proof. (a) Use Remark 1.6.1(c) and Propositions 1.6.2, 1.6.4.

(b) By Proposition 1.6.2 we may assume that the set $|\boldsymbol{p}|$ is finite. Let

$$u := \prod_{a\in|\boldsymbol{p}|} [g_G(a, \cdot)]^{\boldsymbol{p}(a)}.$$

By (a) we only need to show that $g_G(\boldsymbol{p}, \cdot) \le u$. Now, by Remark 1.6.1(h), we may assume that $G \Subset \mathbb{C}$ is regular with respect to the Dirichlet problem. Then the function $\log u$

is subharmonic on G and harmonic on $G \setminus |\boldsymbol{p}|$. The function $v := \log g_G(\boldsymbol{p}, \cdot) - \log u$ is locally bounded from above in G and $\limsup_{z\to\zeta} v(z) \le 0$, $\zeta \in \partial G$. Consequently, v extends to a subharmonic function on G and, by the maximum principle, $v \le 0$ on G, i.e. $g_G(\boldsymbol{p}, \cdot) \le u$ on G. ∎

PROPOSITION 1.6.6 ([Edi-Zwo 1998b], [Lár-Sig 1998b]). *Let $G, D \subset \mathbb{C}^n$ be domains and let $F : G \to D$ be a proper holomorphic mapping.*

(a) *Let $\boldsymbol{q} : D \to \mathbb{R}_+$. Assume that* $\det F'(a) \neq 0$, $a \in F^{-1}(|\boldsymbol{q}|)$. *Then*

$$g_D(\boldsymbol{q}, F(z)) = g_G(\boldsymbol{q}_F, z) = g_G(\boldsymbol{q} \circ F, z), \qquad z \in G.$$

In particular, if $B \subset D$ is such that $\det F'(a) \neq 0$, $a \in F^{-1}(B)$, *then*

$$g_D(B, F(z)) = g_G(F^{-1}(B), z), \qquad z \in G.$$

(b) *Assume that D is convex. Then for any point $b \in D$ such that* $\det F'(a) \neq 0$, $a \in F^{-1}(b)$, *we have*

$$m_D(b, F(z)) = m_G(F^{-1}(b), z), \qquad z \in G.$$

Notice that (a) may be false if $\det F'(a) = 0$ for some $a \in F^{-1}(|\boldsymbol{q}|)$ (cf. Example 1.7.4). Moreover, (b) need not be true if D is not convex (cf. Example 1.7.7).

For the behavior of the pluricomplex Green function under coverings see [Azu 1995], [Azu 1996].

Proof. (a) We only need to show $g_D(\boldsymbol{q}, F(z)) \ge g_G(\boldsymbol{q} \circ F, z)$, $z \in G$; cf. Remark 1.6.1(e). Put $S := \{z \in G : \det F'(z) = 0\}$, $\Sigma := F(S)$. It is well known that

$$F|_{G\setminus F^{-1}(\Sigma)} : G \setminus F^{-1}(\Sigma) \to D \setminus \Sigma$$

is a holomorphic covering. Let N denote its multiplicity. Let $u : G \to [0, 1)$ be a logarithmically plurisubharmonic function such that

$$u(z) \le C(a)\|z - a\|^{\boldsymbol{q}(F(a))}, \qquad a, z \in G.$$

Define

$$v(w) := \max\{u(z) : z \in F^{-1}(w)\}, \qquad w \in D.$$

Since F is proper, $\log v \in \mathcal{PSH}(D)$ (cf. [Kli 1991, Proposition 2.9.26]). Take a $b \in D$ with $\boldsymbol{q}(b) > 0$ (recall that $b \notin \Sigma$) and let $F^{-1}(b) = \{a_1, \dots, a_N\}$ ($a_j \neq a_k$ for $j \neq k$). There exist open neighborhoods $U_1, \dots, U_N, V$ of $a_1, \dots, a_N, b$, respectively, such that $F|_{U_j} : U_j \to V$ is biholomorphic, $j = 1, \dots, N$. Let $g_j := (F|_{U_j})^{-1}$, $j = 1, \dots, N$. Shrinking the neighborhoods if necessary, we may assume that there is a constant $M > 0$ such that $\|g_j(w) - a_j\| \le M\|w - b\|$, $w \in V$. Then, for $w \in V$, we get

$$\begin{aligned} v(w) &= \max\{u \circ g_j(w) : j = 1, \dots, N\} \\ &\le \max\{C(a_j)\|g_j(w) - a_j\|^{\boldsymbol{q}(b)} : j = 1, \dots, N\} \\ &\le \max\{C(a_j) : j = 1, \dots, N\} M^{\boldsymbol{q}(b)}\|w - b\|^{\boldsymbol{q}(b)}. \end{aligned}$$

Consequently, $g_D(\boldsymbol{q}, \cdot) \ge v$ and, therefore, $g_D(\boldsymbol{q}, F(z)) \ge v(F(z)) \ge u(z)$, $z \in G$, which gives the required inequality.

(b) By Remark 1.6.1(e) we only need to check the inequality "$\geq$". Since D is convex, the Lempert theorem implies that $m_D(b,\cdot) = g_D(b,\cdot)$ (cf. [J-P 1993, Theorem 8.2.1]). Hence, by (a) we get

$$m_D(b, F(z)) = g_D(b, F(z)) = g_G(F^{-1}(b), z) \geq m_G(F^{-1}(b), z), \qquad z \in G. \ \blacksquare$$

1.7. Examples

EXAMPLE 1.7.1 ([Car-Ceg-Wik 1999]). Let

$$T := \{(z_1, z_2) \in E_* \times E : |z_2| < |z_1|\}$$

be the Hartogs triangle. Let $\boldsymbol{p} : T \to \mathbb{R}_+$. Consider the biholomorphism

$$E_* \times E \ni (z_1, z_2) \overset{F}{\longmapsto} (z_1, z_1 z_2) \in T.$$

The set $E^2 \setminus (E_* \times E)$ is pluripolar. Hence, by Remark 1.6.1(e, i),

$$g_T(\boldsymbol{p}, F(z)) = g_{E_*\times E}(\boldsymbol{p} \circ F, z) = g_{E^2}(\boldsymbol{p}', z), \qquad z \in E_* \times E,$$

where $\boldsymbol{p}' := \boldsymbol{p} \circ F$ on $E_* \times E$ and $\boldsymbol{p}' := 0$ on $\{0\} \times E$. In particular,

$$g_T(a, z) = \max\{m_E(a_1, z_1),\, m_E(a_2/a_1, z_2/z_1)\}, \qquad a = (a_1, a_2),\ z = (z_1, z_2) \in T.$$

EXAMPLE 1.7.2. For any non-empty sets $A_1, \dots, A_n \subset E$ we have

$$\begin{aligned} &m_{E^n}(A_1 \times \cdots \times A_n, z) \\ &\qquad = g_{E^n}(A_1 \times \cdots \times A_n, z) = \max\{m_E(A_1, z_1), \dots, m_E(A_n, z_n)\} \\ &\qquad = \max\Big\{\prod_{a_j \in A_j} m_E(a_j, z_j) : j = 1, \dots, n\Big\}, \qquad z = (z_1, \dots, z_n) \in E^n. \end{aligned}$$

In particular, for any non-empty set $A \subset E$ we have

$$\begin{aligned} m_{E^n}(A \times \{0\}^{n-1}, z) &= g_{E^n}(A \times \{0\}^{n-1}, z) \\ &= \max\{m_E(A, z_1), |z_2|, \dots, |z_n|\}, \qquad z = (z_1, \dots, z_n) \in E^n \end{aligned}$$

(cf. Example 1.7.17). Indeed, by Propositions 1.6.2, 1.6.4 we may assume that $A_1, \dots, A_n$ are finite. Let

$$F_j(\lambda) := \prod_{a \in A_j} \frac{\lambda - a}{1 - \overline{a}\lambda}, \qquad \lambda \in E,\ j = 1, \dots, n,$$

be the corresponding Blaschke products. The mapping

$$E^n \ni (z_1, \dots, z_n) \overset{F}{\longmapsto} (F_1(z_1), \dots, F_n(z_n)) \in E^n$$

is proper. Moreover, $\det F'(z) = F_1'(z_1) \cdots F_n'(z_n) \neq 0$ for $z \in A_1 \times \cdots \times A_n$. Consequently, by Proposition 1.6.6,

$$\begin{aligned} m_{E^n}(A_1 \times \cdots \times A_n, z) &= g_{E^n}(A_1 \times \cdots \times A_n, z) \\ &= g_{E^n}(0, F(z)) = \max\{|F_j(z_j)| : j = 1, \dots, n\} \\ &= \max\{m_E(A_1, z_1), \dots, m_E(A_n, z_n)\}, \qquad z = (z_1, \dots, z_n) \in E^n. \end{aligned}$$

EXAMPLE 1.7.3. Recall that for $p = (p_1, \dots, p_n) \in \mathbb{R}^n_{>0}$ $(n \geq 2)$, we put

$$\mathbb{E}_p := \Big\{(z_1, \dots, z_n) \in \mathbb{C}^n : \sum_{j=1}^{n} |z_j|^{2p_j} < 1\Big\}.$$

Fix $(\nu_1, \dots, \nu_n) \in \mathbb{N}^n$. The mapping

$$\mathbb{B}_n \ni (z_1, \dots, z_n) \overset{F}{\longmapsto} (z_1^{\nu_1}, \dots, z_n^{\nu_n}) \in \mathbb{E}_{(1/\nu_1, \dots, 1/\nu_n)}$$

is proper. Let $(a_1, \dots, a_n) \in \mathbb{B}_n$ be such that $a_j^{\nu_j - 1} \neq 0$, $j = 1, \dots, n$, and let

$$A := F^{-1}(F(a)) = \{(\varepsilon_1 a_1, \dots, \varepsilon_n a_n) : \varepsilon_j \in \sqrt[\nu_j]{1},\ j = 1, \dots, n\}.$$

Then, by Proposition 1.6.6,

$$g_{\mathbb{B}_n}(A, z) = g_{\mathbb{E}_{(1/\nu_1, \dots, 1/\nu_n)}}(F(a), F(z)), \qquad z \in \mathbb{B}_n;$$

roughly speaking, the multi-pole pluricomplex Green function for the Euclidean ball is expressed by the standard one-pole pluricomplex Green function for an ellipsoid.

Notice that for some special cases the function $g_{\mathbb{E}_{(1/\nu_1, \dots, 1/\nu_n)}}(F(a), F(\cdot))$ may be effectively calculated. For example, let $n = 2$, $\nu_1 = 1$, $\nu_2 = 2$, $a = (0, s)$ $(s \in (0, 1))$. Then $A = \{(0, -s), (0, s)\}$ and

$$g_{\mathbb{B}_2}(\{(0, -s), (0, s)\}, (z_1, z_2)) = g_{\mathbb{E}_{(1,1/2)}}((0, s^2), (z_1, z_2^2))$$
$$= \begin{cases} \left(1 - \dfrac{(1 - s^2)(1 - |z_1|^2 - |z_2|^2)}{|1 - s z_2|^2}\right)^{1/2} & \text{if } s|z_1| \geq |z_2 - s|, \\ \left(1 - \dfrac{(1 - s^2)(1 - |z_1|^2 - |z_2|^2)}{|1 + s z_2|^2}\right)^{1/2} & \text{if } s|z_1| \geq |z_2 + s|, \\ \left(\dfrac{2(1 - s^2 \operatorname{Re} z_2^2)|z_1|^2 + |s^2 - s^2|z_1|^2 - z_2^2|^2 + \sqrt{\Delta}}{2|1 - s^2 z_2^2|^2}\right)^{1/2} & \\ & \text{if } s|z_1| < \min\{|z_2 - s|, |z_2 + s|\}, \end{cases}$$

where

$$\Delta := -4|z_1|^4 (s^2 \operatorname{Im} z_2^2)^2 + 4|z_1|^2 (1 - s^2 \operatorname{Re} z_2^2)|s^2 - s^2|z_1|^2 - z_2^2|^2 + |s^2 - s^2|z_1|^2 - z_2^2|^4;$$

cf. [Edi-Zwo 1998b] (see also [Com 2000] for a different approach). We would like to point out that even $\boxed{?}$ for the case $|\boldsymbol{p}| = \{a_1, a_2\}$, $\boldsymbol{p}(a_1) \neq \boldsymbol{p}(a_2)$, a formula for $g_{\mathbb{B}_n}(\boldsymbol{p}, \cdot)$ is not known. $\boxed{?}$

EXAMPLE 1.7.4. Let $\mathbb{B}_2 \ni (z_1, z_2) \overset{F}{\longmapsto} (z_1, z_2^2) \in \mathbb{E}_{(1,1/2)}$, $a := (0, 0)$. Then $\det F'(0) = 0$ and $g_{\mathbb{B}_2}(0, \cdot) \not\equiv [g_{\mathbb{E}_{(1,1/2)}}(0, F(\cdot))]^s$ for any $s > 0$. Indeed,

$$g_{\mathbb{B}_2}((0, 0), (z_1, z_2)) = h_{\mathbb{B}_2}(z_1, z_2) = \sqrt{|z_1|^2 + |z_2|^2},$$
$$g_{\mathbb{E}_{(1,1/2)}}((0, 0), (z_1, z_2)) = h_{\mathbb{E}_{(1,1/2)}}(z_1, z_2) = \frac{|z_2| + \sqrt{4|z_1|^2 + |z_2|^2}}{2},$$

where h_D is the Minkowski function. In particular, for small $t > 0$, we get

$$g_{\mathbb{B}_2}((0, 0), (t, t)) = t\sqrt{2}, \qquad g_{\mathbb{E}_{(1,1/2)}}((0, 0), (t, t^2)) \approx t,$$

which implies the required result.

EXAMPLE 1.7.5 ([JarW 2004]). Let $p = (p_1, \dots, p_n) \in \mathbb{R}^n_{>0}$, $\mathbb{E} := \mathbb{E}_p$. Put

$$A = A_{\mathbb{E},k} := \{z \in \mathbb{E} : z_1 \cdots z_k = 0\}, \qquad k = 1, \dots, n.$$

Our aim is to find effective formulae for $m_{\mathbb{E}}(A_{\mathbb{E},k}, z)$ and $g_{\mathbb{E}}(A_{\mathbb{E},k}, z)$, where $z=(z_1, \dots, z_n) \in \mathbb{E}$. It is clear that we may assume that

$$p_1|z_1|^{2p_1} \le \dots \le p_k|z_k|^{2p_k}. \ (^{27})$$

Put

$$q_s := \sum_{j=1}^{s} \frac{1}{2p_j}, \ r_s(z) := 1 - \sum_{j=s+1}^{n} |z_j|^{2p_j}, \ c_s(z) := r_s(z)/q_s, \ s = 1, \dots, k \ (r_n = 1),$$

$$d = d(z) := \max\{s \in \{1, \dots, k\} : 2p_s|z_s|^{2p_s} \le c_s(z)\}, \ (^{28})$$

$$R_{\mathbb{E}}(A, z) := \prod_{j=1}^{d} |z_j| \Big(\frac{2p_j}{c_d(z)}\Big)^{1/2p_j} = \Big(q_d^{q_d} \prod_{j=1}^{d} (2p_j)^{1/2p_j}\Big) \frac{|z_1 \cdots z_d|}{(1 - \sum_{j=d+1}^{n} |z_j|^{2p_j})^{q_d}}.$$

Then:

(a) $g_{\mathbb{E}}(A, z) = R_{\mathbb{E}}(A, z)$;
(b) $m_{\mathbb{E}}(A, z) = g_{\mathbb{E}}(A, z) = R_{\mathbb{E}}(A, z)$ if $p_j \ge 1/2$, $j = d+1, \dots, n$;
(c) $m_{\mathbb{E}}(A, z) = g_{\mathbb{E}}(A, z) = R_{\mathbb{E}}(A, z)$ for $k = 1$, $n = 2$, $p_2 \ge 1/2$;
(d) $m_{\mathbb{E}}(A, z) \ne g_{\mathbb{E}}(A, z)$ if there exists a $j_0 \in \{k+1, \dots, n\}$ with $p_{j_0} < 1/2$, $|z_l| \ne 0$ small enough, $l = 1, \dots, k, j_0$, $z_l = 0$, $l = k+1, \dots, j_0 - 1, j_0 + 1, \dots, n$;
(e) $m_{\mathbb{E}}(A, z) = g_{\mathbb{E}}(A, z) = R_{\mathbb{E}}(A, z)$ for $k = n = 2$, $p_1 \le p_2$, and either $p_2 \ge 1/2$ or $8p_1 + 4p_2(1 - p_2) > 1$.

$\boxed{?}$ It is an open question whether $m_{\mathbb{E}}(A, z) = g_{\mathbb{E}}(A, z) = R_{\mathbb{E}}(A, z)$ if $p_j \ge 1/2$, $j = k+1, \dots, n$ (with arbitrary n and k). $\boxed{?}$

Proof of (a). STEP 1. We have $m_{E^n}(A_{E^n,k}, \zeta) = g_{E^n}(A_{E^n,k}, \zeta) = |\zeta_1 \cdots \zeta_k|$, $\zeta \in E^n$, where $A_{E^n,k} := \{\zeta \in E^n : \zeta_1 \cdots \zeta_k = 0\}$. Indeed, it is clear that $|\zeta_1 \cdots \zeta_k| \le m_{E^n}(A_{E^n,k}, \zeta) \le g_{E^n}(A_{E^n,k}, \zeta)$. It remains to prove that $u(\zeta) := g_{E^n}(A_{E^n,k}, \zeta) \le |\zeta_1 \cdots \zeta_k|$, $\zeta \in E^n$. We proceed by induction on k (with arbitrary n and logarithmically plurisubharmonic function $u : E^n \to [0, 1)$ such that $u(\zeta) \le C(a)\|\zeta - a\|$, $a \in A_{E^n,k}$, $\zeta \in E^n$).

For $k = 1$ the inequality follows from the Schwarz type lemma for logarithmically subharmonic functions $u(\cdot, \zeta_2, \dots, \zeta_n)$.

For $k > 1$ we first apply the case $k = 1$ and get $u(\zeta_1, \dots, \zeta_n) \le |\zeta_1|$, $\zeta \in E^n$. Next we apply the inductive assumption to the functions $u(\zeta_1, \cdot)/|\zeta_1|$, $\zeta_1 \in E_*$.

STEP 2. Consider the mapping

$$E^d \ni (\zeta_1, \dots, \zeta_d) \mapsto \Big(\zeta_1 \Big(\frac{c_d(z)}{2p_1}\Big)^{1/2p_1}, \dots, \zeta_d \Big(\frac{c_d(z)}{2p_d}\Big)^{1/2p_d}, z_{d+1}, \dots, z_n\Big) \in \mathbb{E}.$$

Using the holomorphic contractivity and Step 1, we get $m_{\mathbb{E}}(A, z) \le g_{\mathbb{E}}(A, z) \le R_{\mathbb{E}}(A, z)$.

---

(27) In particular, if $p_1 = \dots = p_k$, then the condition simply means that $|z_1| \le \dots \le |z_k|$.
(28) Observe that $z_{d+1} \cdots z_k \ne 0$.

STEP 3. We have $g_{\mathbb{E}}(A,z) \ge R_{\mathbb{E}}(A,z)$. Indeed, we may assume that $z_1 \cdots z_d \neq 0$. First consider the case $d = k = n$. Put $f(\zeta) := q_n^{q_n} \prod_{j=1}^n \zeta_j (2p_j)^{1/2p_j}$, $\zeta \in \mathbb{E}$. Observe that $|f(z)| = R_{\mathbb{E}}(A,z)$ and

$$|f(\zeta)| \le q_n^{q_n} \Big(\frac{\sum_{j=1}^n |\zeta_j|^{2p_j}}{q_d} \Big)^{q_n} < 1, \qquad \zeta \in \mathbb{E}. \text{ (}^{29}\text{)}$$

Thus $g_{\mathbb{E}}(A,z) \ge m_{\mathbb{E}}(A,z) \ge R_{\mathbb{E}}(A,z)$.

Now assume that $d < n$. Put $\mathbb{E}' := \mathbb{E}_{(p_{d+1},\dots,p_n)}$. Observe that we only need to find a logarithmically plurisubharmonic function $v : \mathbb{E}' \to [0,1)$, $v \not\equiv 0$, such that

- $v(\zeta') \le |\zeta_j|$, $\zeta' = (\zeta_{d+1},\dots,\zeta_n) \in \mathbb{E}'$, $j = d+1,\dots,k$ ([30]),
- the mapping $\mathbb{E}' \ni \zeta' \to v(\zeta') r_d^{q_d}(\zeta') \in \mathbb{R}_+$ attains its maximum for $\zeta' = (z_{d+1},\dots,z_n)$ ([31]).

Indeed, suppose that such a v is already constructed and let M be the maximal value of the function $\mathbb{E}' \ni \zeta' \mapsto v(\zeta') r_d^{q_d}(\zeta')$. Put

$$u(\zeta) := \frac{q_d^{q_d}}{M} \Big(\prod_{j=1}^d |\zeta_j| (2p_j)^{1/2p_j} \Big) v(\zeta'), \qquad \zeta = (\zeta_1,\dots,\zeta_n) = (\zeta_1,\dots,\zeta_d,\zeta').$$

Then $\log u \in \mathcal{PSH}(\mathbb{E})$ and $u(\zeta) \le C(a)|\zeta_j| \le C(a)\|\zeta - a\|$ for any $\zeta \in \mathbb{E}$ and $a \in A$ with $a_j = 0$, where $j \in \{1,\dots,k\}$. Moreover, for $\zeta \in \mathbb{E}$ we have

$$u(\zeta) \le \frac{q_d^{q_d}}{M} \Big(\frac{\sum_{j=1}^d |\zeta_j|^{2p_j}}{q_d} \Big)^{q_d} v(\zeta') = \frac{1}{M} \Big(\frac{\sum_{j=1}^d |\zeta_j|^{2p_j}}{r_d(\zeta')} \Big)^{q_d} v(\zeta') r_d^{q_d}(\zeta') < 1.$$

Consequently, $u : \mathbb{E} \to [0,1)$ and, therefore,

$$g_{\mathbb{E}}(A,z) \ge u(z) = \frac{1}{M} R_{\mathbb{E}}(A,z) v(z') r_d^{q_d}(z') = R_{\mathbb{E}}(A,z).$$

STEP 4 (Construction of the function v). We may assume that $z_{d+1},\dots,z_n \ge 0$. For $\alpha = (\alpha_{d+1},\dots,\alpha_n) \in \mathbb{R}_+^{n-d}$ define

$$v_\alpha(\zeta') := \Big(\prod_{j=d+1}^k |\zeta_j|^{1+\alpha_j} \Big) \Big(\prod_{j=k+1}^n |\zeta_j|^{\alpha_j} \Big).$$

Obviously $v : \mathbb{E}' \to [0,1)$, $\log v \in \mathcal{PSH}(\mathbb{E}')$, and $v(\zeta') \le |\zeta_j|$, $\zeta' \in \mathbb{E}'$, $j = d+1,\dots,k$. It is enough to find an α such that the function $\mathbb{E}' \cap \mathbb{R}_+^{n-d} \ni t' \overset{\varphi_\alpha}{\longmapsto} v_\alpha(t') r_d^{q_d}(t')$ attains its maximum at $t' = z'$. In particular,

$$\frac{\partial \varphi_\alpha}{\partial t_j}(z') = 0, \qquad j = d+1,\dots,n.$$

Hence

([29]) We have used the following elementary inequality:

$$\prod_{j=1}^d a_j^{w_j} \le \Big(\frac{\sum_{j=1}^d w_j a_j}{\sum_{j=1}^d w_j} \Big)^{\sum_{j=1}^d w_j}, \qquad a_1,\dots,a_d \ge 0,\ w_1,\dots,w_d > 0.$$

([30]) Notice that this condition is empty if $d = k$.

([31]) $r_d(\zeta') = 1 - \sum_{j=d+1}^n |\zeta_j|^{2p_j}$.

$$0 = 1 + \alpha_j - 2p_j q_d \frac{z_j^{2p_j}}{r_d(z')}, \qquad j = d+1, \dots, k,$$
$$0 = \alpha_j - 2p_j q_d \frac{z_j^{2p_j}}{r_d(z')}, \qquad j = k+1, \dots, n,$$

which gives formulas for $\alpha_{d+1}, \dots, \alpha_n$. To prove that there are no other points like this, rewrite the above equations in the form

$$r_d(z') = \frac{2p_j q_d z_j^{2p_j}}{1 + \alpha_j}, \qquad j = d+1, \dots, k,$$
$$r_d(z') = \frac{2p_j q_d z_j^{2p_j}}{\alpha_j}, \qquad j = k+1, \dots, n.$$

The left side is decreasing in any of the variables $z_{d+1}, \dots, z_n$, while the right sides are increasing. Thus, at most one common zero is allowed.

It remains to check whether $\alpha_j \geq 0$, $j = d+1, \dots, n$. Obviously, $\alpha_j \geq 0$, $j = k+1, \dots, n$. In the remaining cases, using the definition of the number d, we have

$$\alpha_j = \frac{2p_j q_d z_j^{2p_j} - r_d(z')}{r_d(z')} \geq 0, \qquad j = d+1, \dots, k. \ \blacksquare$$

Proof of (b). By the proof of (a), we only have to check whether $m_{\mathbb{E}}(A, z) \geq R_{\mathbb{E}}(a, z)$ in the case where $d < n$. First observe that it is sufficient to find a function $h \in \mathcal{O}(\mathbb{E}')$, $h \not\equiv 0$, such that:

- $h(\zeta') = 0$ if $\zeta_{d+1} \cdots \zeta_k = 0$,
- the function $\mathbb{E}' \ni \zeta' \mapsto |h(\zeta')| r_d^{q_d}(\zeta') \in \mathbb{R}_+$ attains its maximum for $\zeta' = (z_{d+1}, \dots, z_n)$.

Indeed, suppose that such an h is already constructed and let M be the maximal value of the function $\mathbb{E}' \ni \zeta' \mapsto |h(\zeta')| r_d^{q_d}(\zeta')$. Put

$$f(\zeta) := \frac{q_d^{q_d}}{M} \Big(\prod_{j=1}^{d} \zeta_j (2p_j)^{1/2p_j} \Big) h(\zeta'), \qquad \zeta \in \mathbb{E}.$$

Obviously $f(\zeta) = 0$ for $\zeta \in A$. Similarly to (a) we prove that $|f| < 1$ and $|f(z)| = R_{\mathbb{E}}(A, z)$. Thus $m_{\mathbb{E}}(A, z) \geq |f(z)| = R_{\mathbb{E}}(A, z)$.

To construct h assume that $z_{d+1}, \dots, z_n \geq 0$ and define

$$h_\alpha(\zeta') := \Big(\prod_{j=d+1}^{k} \zeta_j e^{\alpha_j \zeta_j} \Big) \Big(\prod_{j=k+1}^{n} e^{\alpha_j \zeta_j} \Big),$$

where $\alpha = (\alpha_{d+1}, \dots, \alpha_n) \in \mathbb{R}_+^{n-d}$. It is enough to find an α such that the function $\mathbb{E}' \cap \mathbb{R}_+^{n-d} \ni t' \mapsto h_\alpha(t') r_d^{q_d}(t)$ attains its maximum at $t' = (z_{d+1}, \dots, z_n)$. Considering partial derivatives results in the following equations:

$$0 = \frac{1}{z_j} + \alpha_j - 2p_j q_d \frac{z_j^{2p_j - 1}}{r_d(z')}, \qquad j = d+1, \dots, k,$$
$$0 = \alpha_j - 2p_j q_d \frac{z_j^{2p_j - 1}}{r_d(z')}, \qquad j = k+1, \dots, n.$$

We continue as in the proof of (a). $\blacksquare$

Proof of (c). Assertion (c) follows directly from (b). ∎

Proof of (d). STEP 1. Suppose that $m_{\mathbb{E}}(A,z) = R_{\mathbb{E}}(A,z)$. Let $f \in \mathcal{O}(\mathbb{E}, E)$ be such that $f|_A \equiv 0$ and $|f(z)| = R_{\mathbb{E}}(A,z)$ (cf. [Jar-Jar-Pfl 2003, Property 2.5]). Put

$$h(\zeta') := \frac{\partial^d f}{\partial z_1 \dots \partial z_d}(0,\zeta'), \qquad \zeta' \in \mathbb{E}'.$$

We have $h(\zeta') = 0$ if $\zeta_{d+1}\cdots\zeta_k = 0$. For $\zeta' \in \mathbb{E}'$ consider the mapping

$$E^d \ni (\xi_1,\dots,\xi_d) \overset{\iota_{\zeta'}}{\longmapsto} \left(\xi_1\left(\frac{c_d(\zeta')}{2p_1}\right)^{1/2p_1},\dots,\xi_d\left(\frac{c_d(\zeta')}{2p_d}\right)^{1/2p_d},\zeta'\right) \in \mathbb{E}.$$

Applying the Schwarz lemma to the mapping $f \circ \iota_{\zeta'}$, $\zeta' \in \mathbb{E}'$, we get

$$|h(\zeta')| r_d^{q_d}(\zeta') \le q_d^{q_d}\left(\prod_{j=1}^d (2p_j)^{1/2p_j}\right), \quad \zeta' \in \mathbb{E}', \quad |h(z')| r_d^{q_d}(z') = q_d^{q_d}\left(\prod_{j=1}^d (2p_j)^{1/2p_j}\right).$$

Thus we have constructed a mapping h as in the proof of (b) (consequently, the equality $m_{\mathbb{E}}(A,z) = R_{\mathbb{E}}(A,z)$ is equivalent to the existence of the mapping h).

STEP 2. For any $p \in (0,1)$ and $q > 0$ there exists $c = c(p,q) \in (0,1)$ such that for any function $f \in \mathcal{O}(E)$ if the function $E \ni \lambda \mapsto |f(\lambda)|(1-|\lambda|^p)^q$ attains its maximum at $\lambda_0 \neq 0$, then $|\lambda_0| \ge c$. Indeed, let

$$\varphi(t) := \frac{1}{(1-t^p)^q}, \qquad t \in [0,1).$$

Observe that there exists a $b \in (0,1)$ such that φ is strictly concave on $[0,b)$. Moreover,

$$\lim_{t\to 0+} \frac{\varphi(t)-\varphi(0)}{t} = +\infty.$$

Consequently, there exists a $c \in (0,b)$ such that

$$\varphi(0) + \frac{b}{c}(\varphi(c)-\varphi(0)) > \varphi(b) + 2.$$

Suppose that $f \in \mathcal{O}(E)$ is such that the function $E \ni \lambda \mapsto |f(\lambda)|/\varphi(|\lambda|)$ attains its maximum at $\lambda_0 \neq 0$ with $|\lambda_0| < c$. We may assume that $|f(\lambda_0)| = \varphi(|\lambda_0|)$. Consider the function

$$[0,b] \ni t \overset{\psi}{\longmapsto} |f(0)| + \frac{t}{|\lambda_0|}|f(\lambda_0) - f(0)|.$$

From $\psi(0) = |f(0)| \le \varphi(0) = 1$, $\psi(|\lambda_0|) \ge \varphi(|\lambda_0|)$, and the convexity condition we get

$$\begin{aligned}\psi(b) &= |f(0)| + \frac{b}{|\lambda_0|}|f(\lambda_0) - f(0)| \\ &\ge \varphi(0) + \frac{b}{|\lambda_0|}|\varphi(|\lambda_0|) - \varphi(0)| \ge \varphi(0) + \frac{b}{c}|\varphi(c)-\varphi(0)| > \varphi(b) + 2.\end{aligned}$$

The Schwarz lemma and the maximum principle imply that there exists a $\lambda_* \in E$ with $|\lambda_*| = b$ and

$$\frac{|f(\lambda_*) - f(0)|}{|\lambda_*|} \ge \frac{|f(\lambda_0) - f(0)|}{|\lambda_0|}.$$

This means that

$$\begin{aligned}|f(\lambda_*)| \geq& |f(\lambda_*) - f(0)| - |f(0)| = |f(0)| + |f(\lambda_*) - f(0)| - 2|f(0)| \\ \geq& \psi(b) - 2|f(0)| > \varphi(b) + 2 - 2|f(0)| \geq \varphi(b) = \varphi(|\lambda_*|),\end{aligned}$$

a contradiction.

STEP 3. We may assume that $p_{k+1} < 1/2$. Assume that $0 < |z_j| < \varepsilon$, $j = 1, \dots, k+1$, $z_j = 0$, $j = k+2, \dots, n$, with $0 < \varepsilon < c(2p_{k+1}, q_k)$. Observe that $d(z) = k$ provided ε is small enough. Let h be as in Step 1. Then the mapping

$$E \ni \lambda \mapsto |h(\lambda, 0, \dots, 0)|(1 - |\lambda|^{2p_{k+1}})^{q_k}$$

attains its maximum at $\lambda = z_{k+1}$, which contradicts Step 2. ∎

Proof of (e). See [JarW 2004]. ∎

EXAMPLE 1.7.6. Let $P = P(R) := \{z \in \mathbb{C} : 1/R < |z| < R\}$ $(R > 1)$. Put $q := 1/R^2$ and let

$$\begin{gathered}\varPi(a, z) = \varPi_R(a, z) := \frac{\prod_{\nu=1}^{\infty}(1 - (z/a)q^{2\nu})(1 - (a/z)q^{2\nu})}{\prod_{\nu=1}^{\infty}(1 - azq^{2\nu-1})(1 - (1/az)q^{2\nu-1})}, \\ f(a, z) = f_R(a, z) := (1 - z/a)\varPi(a, z), \qquad 1/R < a < R,\ z \in P.\end{gathered}$$

Using the same methods as in the proof of Proposition 5.5 in [J-P 1993], one can prove that for any function $\boldsymbol{p} : P \to \mathbb{Z}_+$ such that $|\boldsymbol{p}| = \{a_1, \dots, a_N\}$ is finite, if $a_j = |a_j|e^{i\varphi_j}$, $|a_j| = R^{1-2s_j}$, $s_j \in (0,1)$, $j = 1, \dots, N$, then we get

$$m_P(\boldsymbol{p}, z) = \frac{f(b, -|z|)}{|Rz|^{\ell}} \prod_{j=1}^{N} |f(|a_j|, e^{-i\varphi_j} z)|^{k_j}, \qquad z \in P,$$

where

- $\ell = \ell(\boldsymbol{p}) := \lceil s_1 + \dots + s_N \rceil$,
- $b = b(\boldsymbol{p}) := R^{1-2(l-(s_1+\dots+s_N))}$,
- $f(R, \cdot) :\equiv 1$.

EXAMPLE 1.7.7. If D is not convex, then Proposition 1.6.6(b) need not be true. Indeed, let $P(R)$, $\varPi_R$, and f_R be as Example 1.7.6. Consider $F : P(R) \to P(R^2)$, $F(z) := z^2$, and suppose that $m_{P(R^2)}(1, z^2) = m_{P(R)}(\{-1, +1\}, z)$, $z \in P(R)$, $R > 1$. Then, using Example 1.7.6, we get

$$\frac{f_{R^2}(1, -|z|^2)}{R^2|z|^2} |f_{R^2}(1, z^2)| = \frac{1}{R|z|} |f_R(1, z) f_R(1, -z)|, \qquad z \in P(R).$$

Consequently,

$$\frac{1}{R|z|}(1 + |z|^2)\varPi_{R^2}(1, -|z|^2)|(1 - z^2)\varPi_{R^2}(1, z^2)| = |(1 - z)\varPi_R(1, z)(1 + z)\varPi_R(1, -z)|,$$

and hence

$$\frac{1}{R|z|}(1 + |z|^2)\varPi_{R^2}(1, -|z|^2)|\varPi_{R^2}(1, z^2)| = |\varPi_R(1, z)\varPi_R(1, -z)|, \qquad z \in P(R),$$

a contradiction (at least for large R) (take $z = 1$ and then let $R \to +\infty$).

REMARK 1.7.8. Let $G \subset \mathbb{C}^n$ and assume that $\boldsymbol{p} : G \to \mathbb{R}_+$, $|\boldsymbol{p}| = \{a_1, \dots, a_N\}$. Directly from the definition of the function $d_G^{\min}$ we get the following estimate:

$$\begin{aligned} d_G^{\min}(\boldsymbol{p}, z) &= \sup\Big\{ \prod_{j=1}^{s} [m_E(\mu_j, f(z))]^{\max \boldsymbol{p}(B_j)} : s \in \mathbb{N},\ \mu_1, \dots, \mu_s \in E,\ \mu_j \neq \mu_k\ (j \neq k), \\ &\qquad B_1 \cup \dots \cup B_s = |\boldsymbol{p}|,\ B_j \cap B_k = \emptyset\ (j \neq k),\ \exists_{f \in \mathcal{O}(G,E)} : f|_{B_j} \equiv \mu_j,\ j = 1, \dots, s \Big\} \\ &\le \sup\Big\{ \prod_{j=1}^{s} [m_G(B_j, z)]^{\max \boldsymbol{p}(B_j)} : s \in \mathbb{N},\ B_1 \cup \dots \cup B_s = |\boldsymbol{p}|,\ B_j \cap B_k = \emptyset\ (j \neq k) \Big\} \\ &=: d'_G(\boldsymbol{p}, z), \qquad z \in G. \end{aligned}$$

Recall that in the case where $\boldsymbol{p} = \chi_A$ we have $d_G^{\min}(\chi_A, \cdot) \equiv m_G(A, \cdot) \equiv d'_G(\chi_A, \cdot)$ (cf. Proposition 1.5.4).

In particular, if $N = 2$, $|\boldsymbol{p}| = \{a, b\}$, $\alpha = \boldsymbol{p}(a) \ge \boldsymbol{p}(b) = \beta$, then we get

$$\begin{aligned} d_G^{\min}(\alpha\chi_{\{a\}} + \beta\chi_{\{b\}}, z) &\le \max\{[m_G(a, z)]^\alpha [m_G(b, z)]^\beta,\ [m_G(\{a, b\}, z)]^\alpha\} \\ &= d'_G(\alpha\chi_{\{a\}} + \beta\chi_{\{b\}}, z), \qquad z \in G. \end{aligned}$$

Notice that in general $d_G^{\min}(\alpha\chi_{\{a\}} + \beta\chi_{\{b\}}, \cdot) \not\equiv d'_G(\alpha\chi_{\{a\}} + \beta\chi_{\{b\}}, \cdot)$. In fact, let $G = P$ be an annulus as in Example 1.7.6. Take $1/R < a, b < R$, $a \neq b$, $ab \neq 1$, $\boldsymbol{p} := 2\chi_{\{a\}} + \chi_{\{b\}}$. We are going to show that there exists a $z \in P$ such that

$$\begin{aligned} &[m_P(a, z)]^2 m_P(b, z) > [m_P(\{a, b\}, z)]^2, \\ &\qquad d_P^{\min}(\boldsymbol{p}, z) < [m_P(a, z)]^2 m_P(b, z) \\ &\qquad\qquad = \max\{[m_P(a, z)]^2 m_P(b, z),\ [m_P(\{a, b\}, z)]^2\} = d'_P(\boldsymbol{p}, z). \end{aligned}$$

First observe that there are points z (near b) such that

$$[m_P(a, z)]^2 m_P(b, z) > [m_P(\{a, b\}, z)]^2.$$

For, using the effective formula from Example 1.7.6, we have:

$$\begin{aligned} [m_P(a, z)]^2 m_P(b, z) &= \left(\frac{f(1/a, -|z|)}{R|z|} |f(a, z)| \right)^2 \frac{f(1/b, -|z|)}{R|z|} |f(b, z)|, \\ [m_P(\{a, b\}, z)]^2 &= \left(\frac{f(c, -|z|)}{(R|z|)^\ell} |f(a, z) f(b, z)| \right)^2, \end{aligned}$$

with $c = c(a, b)$ and $\ell = \ell(a, b)$ as in Example 1.7.6. Consequently, we only need to find a $z \in P \setminus \{a, b\}$ such that

$$\left(\frac{f(1/a, -|z|)}{R|z|} \right)^2 \frac{f(1/b, -|z|)}{R|z|} > \left(\frac{f(c, -|z|)}{(R|z|)^\ell} \right)^2 |f(b, z)|.$$

Observe that at $z = b$ the right hand side of the above formula is zero while the left hand side is strictly positive. Thus, by continuity, we can easily find the required z, say z_0.

Let $\varphi \in \mathcal{O}(P, E)$ be an extremal function for $d_P^{\min}(\boldsymbol{p}, z_0)$ (Proposition 1.8.4). We may assume that $\varphi(a) = 0$. There are two cases:

(a) $\varphi(b) = 0$. Then $d_P^{\min}(\boldsymbol{p}, z_0) = |\varphi(z_0)|^2 \le [m_P(\{a, b\}, z_0)]^2 < d'_P(\boldsymbol{p}, z_0)$.

(b) $\varphi(b) \neq 0$. Then $d_P^{\min}(\boldsymbol{p}, z_0) = |\varphi(z_0)|^2 m_E(\varphi(b), \varphi(z_0)) \leq [m_P(a, z_0)]^2 m_P(b, z_0) = d'_P(\boldsymbol{p}, z_0)$. The equality $d_P^{\min}(\boldsymbol{p}, z_0) = d'_P(\boldsymbol{p}, z_0)$ would imply that φ is simultaneously extremal for $m_P(a, z_0)$ and $m_P(b, z_0)$. Using Robinson's lemma ([J-P 1993, Lemma 5.6]), we know that such extremal functions are uniquely determined up to rotations. Hence

$$\varphi(z) = e^{i\theta_a} \frac{f(1/a, -e^{-i\varphi_0} z)}{Rz} f(a, z) = e^{i\theta_b} \frac{f(1/b, -e^{-i\varphi_0} z)}{Rz} f(b, z),$$

where $\varphi_0 \in \arg z_0$, a contradiction (the two sides have different zeros).

EXAMPLE 1.7.9 ([Jar-Pfl 1999a]). Let F be a primitive polynomial of n complex variables, i.e. $F \in \mathcal{P}(\mathbb{C}^n)$ is a polynomial which cannot be represented in the form $F = f \circ Q$, where f is a polynomial of one complex variable of degree ≥ 2 and $Q \in \mathcal{P}(\mathbb{C}^n)$. Notice that a monomial z^α ($\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$) is primitive iff the numbers $\alpha_1, \dots, \alpha_n$ are relatively prime.

One can prove (cf. [Cyg 1992]) that there exists a finite set $S \subset \mathbb{C}$ such that for any $b \in \mathbb{C} \setminus S$:

- the fiber $F^{-1}(b)$ is connected,
- $F'(a) \neq 0$ for $a \in F^{-1}(b)$.

Thus, if $b \notin S$, then the fiber $F^{-1}(b)$ is a connected $(n-1)$-dimensional algebraic manifold. In particular, for any $b \notin S$, the fiber $F^{-1}(b)$ has the plurisubharmonic Liouville property, i.e. any plurisubharmonic function $u : F^{-1}(b) \to [-\infty, 0)$ is constant (cf. [Jar-Pfl 1999a, Proposition 6]).

Put $r(a) := \operatorname{ord}_a(F - F(a))$. Let $D \subset \mathbb{C}$ be a domain. Put $G := F^{-1}(D)$. Then G is a domain. Indeed, since the set $F^{-1}(S)$ is thin, it suffices to prove that the set $G_0 := F^{-1}(D \setminus S)$ is connected. Suppose $G_0 = U_1 \cup U_2$, where U_1, U_2 are non-empty disjoint open sets. Put $B_j := \{w \in D \setminus S : F^{-1}(w) \subset U_j\}$, $j = 1, 2$. Since the fibers over points from $D \setminus S$ are connected, we conclude that B_1, B_2 are disjoint and $B_j = F(U_j)$, $j = 1, 2$. In particular, B_j is open, non-empty, and $D \setminus S = B_1 \cup B_2$, a contradiction.

(a) Let $\boldsymbol{p} : G \to \mathbb{R}_+$ be such that $|\boldsymbol{p}|$ is finite. Then

$$g_G(\boldsymbol{p}, z) = g_D(\boldsymbol{p}^F, F(z)), \quad z \in G, \tag{1.7.7}$$

where

$$\boldsymbol{p}^F(b) := \max\left\{\frac{\boldsymbol{p}(a)}{r(a)} : a \in |\boldsymbol{p}| \cap F^{-1}(b)\right\}, \quad b \in D.$$

In particular,

$$g_G(a, z) = [g_D(F(a), F(z))]^{1/r(a)}, \quad a, z \in G.$$

(b) Let $\boldsymbol{p} : G \to \mathbb{Z}_+$ be such that $|\boldsymbol{p}|$ is finite. Then

$$m_G(\boldsymbol{p}, z) = m_D(\boldsymbol{p}', F(z)), \quad z \in G, \tag{1.7.8}$$

where

$$\boldsymbol{p}'(b) := \max\left\{\left\lceil\frac{\boldsymbol{p}(a)}{r(a)}\right\rceil : a \in |\boldsymbol{p}| \cap F^{-1}(b)\right\}, \quad b \in D.$$

In particular,

$$m_G(k\chi_{\{a\}}, z) = m_D(k'\chi_{\{F(a)\}}, F(z)), \qquad a, z \in G,\ k \in \mathbb{N},$$

where $k' := \lceil k/r(a) \rceil$.

Indeed, in both cases the inequalities "$\geq$" follow from Remark 1.6.1(e).

To prove the opposite inequality in (a), take a logarithmically plurisubharmonic function $u : G \to [0,1)$ such that

$$u(z) \leq C(a)\|z - a\|^{\boldsymbol{p}(a)}, \qquad a, z \in G.$$

For any $b \in D \setminus S$, the function $u|_{F^{-1}(b)}$ is constant. Consequently, there exists a logarithmically subharmonic function $\widetilde{u} : D \setminus S \to [0,1)$ such that $u = \widetilde{u} \circ F$ on $G \setminus F^{-1}(S)$. The function $\widetilde{u}$ extends to a logarithmically subharmonic function on D (the extended function will be denoted by the same symbol $\widetilde{u}$). By the identity principle for plurisubharmonic functions we get $u = \widetilde{u} \circ F$ in G. We want to show that

$$\widetilde{u}(w) \leq \text{const}(b)|w - b|^{\boldsymbol{p}^F(b)}, \qquad b, w \in D; \tag{1.7.9}$$

then $\widetilde{u} \leq g_D(\boldsymbol{p}^F, \cdot)$ and hence $u = \widetilde{u} \circ F \leq g_D(\boldsymbol{p}^F, F)$, which gives the required inequality.

Fix a $b \in D$ with $\boldsymbol{p}^F(b) > 0$, and let $a \in F^{-1}(b)$ be such that $\boldsymbol{p}(a) > 0$. Observe that there exist $\varepsilon > 0$, $\delta > 0$, and $M > 0$ such that $\mathbb{B}(b, \varepsilon) \subset D$ and

$$\forall_{w \in \mathbb{B}(b,\varepsilon)}\ \exists_{z(w) \in \mathbb{B}(a,\delta)} : F(z(w)) = w,\ \|z(w) - a\|^{r(a)} \leq M|w - b|.$$

Indeed, let $X \in \mathbb{C}^n$, $\|X\| = 1$, be such that $\operatorname{ord}_0(\varphi - b) = r(a)$, where $\varphi(\lambda) := F(a + \lambda X)$. Then $|\varphi(\lambda) - b| \geq (1/M)|\lambda|^{r(a)}$ for $|\lambda| < \delta$ (where $M > 0$, $\delta > 0$) and φ is open. Consequently, $\varphi(\mathbb{B}(\delta)) \supset \mathbb{B}(b, \varepsilon)$ for some $\varepsilon > 0$. Thus for any $w \in \mathbb{B}(b, \varepsilon)$ there exists a $\lambda(w) \in \mathbb{B}(\delta)$ such that $z(w) := a + \lambda(w)X$ satisfies all the required conditions.

Consequently,

$$\widetilde{u}(w) = u(z(w)) \leq C(a)\|z(w) - a\|^{\boldsymbol{p}(a)} \leq C(a)M^{\boldsymbol{p}(a)/r(a)}|w - b|^{\boldsymbol{p}(a)/r(a)}, \qquad w \in \mathbb{B}(b, \varepsilon).$$

Since the set of all $a \in F^{-1}(b)$ with $\boldsymbol{p}(a) > 0$ is finite, we get (1.7.9).

In the situation of (b) let $f \in \mathcal{O}(G, E)$ be such that $\operatorname{ord}_a f \geq \boldsymbol{p}(a)$, $a \in G$. For any $b \in D \setminus S$ the function $f|_{F^{-1}(b)}$ must be constant. Hence there exists a function $\widetilde{f} \in \mathcal{O}(D \setminus S, E)$ such that $f = \widetilde{f} \circ F$. Using the Riemann removable singularity theorem, we extend $\widetilde{f}$ holomorphically to the whole D. Take $b \in D$ and $a \in F^{-1}(a)$ such that $\boldsymbol{p}(a) > 0$. Then, using the same argument as in (a), we get

$$|\widetilde{f}(w)| = |f(z(w))| \leq \text{const}\,\|z(w) - a\|^{\operatorname{ord}_a f} \leq \text{const}\,\|z(w) - a\|^{\boldsymbol{p}(a)} \leq |w - b|^{\boldsymbol{p}(a)/r(a)}$$

for w in a neighborhood of b. Consequently, $\operatorname{ord}_b \widetilde{f} \geq \boldsymbol{p}'(b)$, $b \in G$, and, therefore, $|f(z)| = |\widetilde{f}(F(z))| \leq m_D(\boldsymbol{p}', F(z))$, $z \in G$.

The case where $F(z) = z^\alpha$ and $D = E$ was studied in [J-P 1993, §4.4] (one pole), and [Edi-Zwo 1999] (many poles).

$\boxed{?}$ Are formulas (1.7.7), (1.7.8) true for arbitrary $\boldsymbol{p}$ $\boxed{?}$

In the special case where $F(z) = z^\alpha$ we get the following example.

EXAMPLE 1.7.10. Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $G := \{z \in \mathbb{C}^n : |z^\alpha| < 1\}$. Assume that $\alpha_1, \dots, \alpha_n$ are relatively prime. Then for any function $\boldsymbol{p} : G \to \mathbb{R}_+$ (resp. $\boldsymbol{p} : G \to \mathbb{Z}_+$)

such that $|\boldsymbol{p}|$ is finite, the following formulae are true:

$$g_G(\boldsymbol{p}, z) = m_E(\boldsymbol{p}^{z^\alpha}, z^\alpha), \qquad m_G(\boldsymbol{p}, z) = m_E(\boldsymbol{p}', z^\alpha), \qquad z \in G,$$

where

$$\boldsymbol{p}^{z^\alpha}(b) := \sup\left\{\frac{\boldsymbol{p}(a)}{r(a)} : a \in G,\ a^\alpha = b\right\}, \text{ (32)}$$

$$\boldsymbol{p}'(b) := \sup\left\{\left\lceil\frac{\boldsymbol{p}(a)}{r(a)}\right\rceil : a \in G,\ a^\alpha = b\right\}, \qquad b \in E.$$

Example 1.7.9 may be extended in the following way.

EXAMPLE 1.7.11 ([Jar-Pfl 1999a, Theorem 1]). Let

$$\alpha_j = (\alpha_{j,1}, \dots, \alpha_{j,n}) \in (\mathbb{Z}_+^n)_*, \qquad j = 1, \dots, m \le n-1,\ m \ge 2,$$

be such that $\operatorname{rank} A = m$, where $A := [\alpha_{j,k}]$. Assume that $A\mathbb{Z}^n = \mathbb{Z}^m$ (33). Let $F = (F_1, \dots, F_m) : \mathbb{C}^n \to \mathbb{C}^m$ be given by the formula

$$F_j(z) := z^{\alpha_j}, \qquad j = 1, \dots, m.$$

Define

$$G := F^{-1}(E^m) = \{z \in \mathbb{C}^n : |z^{\alpha_j}| < 1,\ j = 1, \dots, m\}.$$

Let $\boldsymbol{p} : G \to \mathbb{R}_+$ (resp. $\boldsymbol{p} : G \to \mathbb{Z}_+$) be such that $|\boldsymbol{p}|$ is finite and for any $a \in |\boldsymbol{p}|$ we have $\operatorname{rank} F'(a) = m$ (in particular, $r(a) = 1$). Then

(1.7.10) $$g_G(\boldsymbol{p}, z) = g_{E^m}(\boldsymbol{p}^F, F(z)), \quad m_G(\boldsymbol{p}, z) = m_{E^m}(\boldsymbol{p}^F, F(z)) \quad z \in G, \text{ (34)}$$

where

$$\boldsymbol{p}^F(b) := \max\{\boldsymbol{p}(a) : a \in |\boldsymbol{p}| \cap F^{-1}(b)\}, \qquad b \in E^m.$$

In particular, if $\boldsymbol{p} = \chi_{\{a\}}$, then

$$g_G(a, z) = g_{E^m}(F(a), F(z)), \quad m_G(a, z) = m_{E^m}(F(a), F(z)), \qquad z \in G.$$

Indeed, put $V_0 := \{w = (w_1, \dots, w_m) \in \mathbb{C}^m : w_1 \cdots w_m = 0\}$ and observe that for any $w \in E^m \setminus V_0$ the fiber $V_w := F^{-1}(w)$ is connected (35). For (the proof is due to W. Zwonek), let $w = (u_1 e^{2\pi i\theta_1}, \dots, u_m e^{2\pi i\theta_m})$. Take arbitrary two points $a, b \in F^{-1}(w)$, $a = (r_1 e^{2\pi i\varphi_1}, \dots, r_n e^{2\pi i\varphi_n})$, $b = (s_1 e^{2\pi i\psi_1}, \dots, s_n e^{2\pi i\psi_n})$. We have $F(r) = F(s) = u$, $A\varphi = \theta \bmod \mathbb{Z}^m$, $A\psi = \theta \bmod \mathbb{Z}^m$. We have to find a curve $\gamma : [0,1] \to F^{-1}(w)$ such that $\gamma(0) = a$, $\gamma(1) = b$. Write

$$\gamma(t) = (R_1(t) e^{2\pi i(\varphi_1 + \sigma_1(t))}, \dots, R_n(t) e^{2\pi i(\varphi_n + \sigma_n(t))}),$$

where $R : [0,1] \to \mathbb{R}_+^n$ is continuous, $\sigma : [0,1] \to \mathbb{R}^n$ is such that the mapping $t \mapsto (e^{2\pi i\sigma_1(t)}, \dots, e^{2\pi i\sigma_n(t)})$ is continuous, $F(R(t)) = u$, $A\sigma(t) = 0 \bmod \mathbb{Z}^m$, $t \in [0,1]$, $R(0) = r$, $R(1) = s$, $\sigma(0) = 0 \bmod \mathbb{Z}^n$, $\sigma(1) = \psi - \varphi \bmod \mathbb{Z}^n$.

(32) Observe that $r(a) = 1$ if $a_1 \dots a_n \neq 0$, and $r(a) =$ the sum of those α_i for which $a_i = 0$ if $a_1 \cdots a_n = 0$.

(33) One can prove that $A\mathbb{Z}^n = \mathbb{Z}^m$ iff the greatest common divisor of all determinants of $m \times m$ submatrices of A equals 1.

(34) Notice that the formula (1.7.10) may not be true if $\operatorname{rank} F'(a) \le m - 1$; cf. Example 1.7.13.

(35) In fact, one can prove that V_w is connected iff $A\mathbb{Z}^n = \mathbb{Z}^m$.

Note that the set $\{x \in \mathbb{R}^n_+ : F(x) = u\}$ is connected. Hence we can easily find an R with the required properties. To find a σ it would be sufficient to know that the set

$$T := \{x \in \mathbb{R}^n : Ax \in \mathbb{Z}^m\}/\mathrm{mod}\,\mathbb{Z}^n$$

is connected. Since $A\mathbb{Z}^n = \mathbb{Z}^m$, we get $T = \{x \in \mathbb{R}^n : Ax = 0\}/\mathrm{mod}\,\mathbb{Z}^n$, which directly implies that T is connected (because $A^{-1}(0)$ is connected).

The inequalities "$\geq$" in (1.7.10) follow from Remark 1.6.1(e).

For the proof of "$\leq$" in the case of a generalized Green function, let $u : G \to [0,1)$ be such that $\log u \in \mathcal{PSH}(G)$ and $u(z) \leq C(a)\|z-a\|^{\boldsymbol{p}(a)}$ for any $a, z \in G$. For any $w \in E^m \setminus V_0$, since V_w is a connected algebraic set, the function $u|_{V_w}$ is constant. Hence there exists a logarithmically plurisubharmonic function $v : E^m \setminus V_0 \to [0,1)$ such that $u = v \circ F$ on $G \setminus F^{-1}(V_0)$. By the Riemann type extension theorem for plurisubharmonic functions, v extends to a logarithmically plurisubharmonic function on E^m. By the identity principle for plurisubharmonic functions we get $u = v \circ F$ in G.

Fix a $b \in E^m$ with $\boldsymbol{p}^F(b) > 0$ and let $a \in F^{-1}(b)$ be such that $\boldsymbol{p}(a) > 0$. By our assumption ($\operatorname{rank} F'(a) = m$) there exists an m-dimensional vector subspace $L \subset \mathbb{C}^n$ such that the mapping $L \ni z \overset{g}{\mapsto} F(a+z)$ is biholomorphic in a neighborhood of $0 \in L$. Then $\|g(z) - b\| \geq (1/M)\|z\|$, $z \in L \cap \mathbb{B}(\delta)$ (for some $M, \delta > 0$) and $g(L \cap \mathbb{B}(\delta)) \supset \mathbb{B}(b, \varepsilon)$ (cf. Example 1.7.9). Hence, for any $w \in \mathbb{B}(b,\varepsilon)$ there exists a $z(w) \in L \cap \mathbb{B}(\delta)$ such that $F(a + z(w)) = w$. Finally,

$$\begin{aligned} v(w) &= v(F(a+z(w))) = u(a+z(w)) \\ &\leq C(a)\|z(w)\|^{\boldsymbol{p}(a)} \leq C(a)M\|g(z(w)) - b\|^{\boldsymbol{p}(a)} = C(a)M\|w-b\|^{\boldsymbol{p}(a)}, \end{aligned}$$

and, consequently, $v(w) \leq \widetilde{C}(b)\|w-b\|^{\boldsymbol{p}^F(b)}$, $b, w \in E^m$. Hence

$$u(z) = v(F(z)) \leq g_{E^m}(\boldsymbol{p}^F, F(z)), \qquad z \in G,$$

which implies that $g_G(\boldsymbol{p}, z) \leq g_{E^m}(\boldsymbol{p}^F, F(z))$, $z \in G$.

In the case of the generalized Möbius function we use an analogous argument (as in Example 1.7.9).

REMARK 1.7.12. Example 1.7.11 may be extended to more general mappings F and domains G. The condition $\operatorname{rank} F'(a) = m$, $a \in |\boldsymbol{p}|$, may also be weakened; cf. [Jar-Pfl 1999a].

The cases which are not covered by Example 1.7.11 are in general much more difficult (even when $\boldsymbol{p} = \chi_{\{a\}}$).

EXAMPLE 1.7.13 ([Jar-Pfl 1999a, Proposition 3]). Let $\alpha_j = (\alpha_{j,1}, \dots, \alpha_{j,n}) \in (\mathbb{Z}^n_+)_*$, $j = 1, \dots, m$, $m \geq 2$, $A := [\alpha_{j,k}]$,

$$G := \{z \in \mathbb{C}^n : |z^{\alpha_j}| < 1,\ j = 1, \dots, m\}. \text{ (}^{36}\text{)}$$

Fix an $a \in G$ with $a^{\alpha_j} = 0$, $j = 1, \dots, m$. Assume that $a = (a_1, \dots, a_s, 0, \dots, 0)$ with $a_1 \cdots a_s \neq 0$ and $1 \leq s \leq n-1$. Put

(36) We do not assume that $m \leq n-1$, $\operatorname{rank} A = m$, $A\mathbb{Z}^n = \mathbb{Z}^m$.

$$\widetilde{A} := [\alpha_{j,k}]_{\substack{j=1,\dots,m\\ k=s+1,\dots,n}} = \begin{bmatrix}\beta_1\\ \vdots\\ \beta_m\end{bmatrix}.$$

Notice that $r_j := \operatorname{ord}_a z^{\alpha_j} = |\beta_j| > 0$, $j = 1,\dots,m$. Then the following conditions are equivalent:

(i) $\operatorname{rank} A = \operatorname{rank} \widetilde{A}$;
(ii) $g_G(a,z) = \max\{|z^{\alpha_j}|^{1/r_j} : j = 1,\dots,m\}$, $z \in G$; (37)
(iii) $g_G(a,z) = \sup\{|z^\alpha|^{1/r} : \alpha \in (\mathbb{Z}_+)^n,\ |\xi^\alpha| < 1 \text{ for } \xi \in G,\ r = \operatorname{ord}_a \xi^\alpha > 0\}$, $z \in G$;
(iv) $g_G(a,(z',\lambda z'')) = |\lambda| g_G(a,z)$, $z = (z',z'') \in G \subset \mathbb{C}^s \times \mathbb{C}^{n-s}$, $\lambda \in \overline{E}$;
(v) $\limsup_{\theta\to 0+} \frac{1}{\theta} m_G^{(r+1)}(a,(z',\theta z'')) < +\infty$, $(z',z'') \in G \subset \mathbb{C}^s \times \mathbb{C}^{n-s}$.

Indeed, to prove (i)⇒(ii), let

$$L(z) := g_G(a,z), \quad R(z) := \max\{|z^{\alpha_j}|^{1/r_j} : j = 1,\dots,m\}, \qquad z \in G.$$

The inequality $L \ge R$ follows from the definition of g_G. To prove that $L \le R$ it suffices to show that $L(z) \le R(z)$ for any $z \in G_0 := G \cap ((\mathbb{C}_*)^s \times \mathbb{C}^{n-s})$.

By (i), for any $k = 1,\dots,s$, the system of equations

$$\alpha_{j,s+1}x_{s+1} + \dots + \alpha_{j,n}x_n = -\alpha_{j,k}, \qquad j = 1,\dots,m,$$

has a rational solution $(Q_{s+1,k}/\mu_k,\dots,Q_{n,k}/\mu_k)$ with $Q_{s+1,k},\dots,Q_{n,k} \in \mathbb{Z}$, $\mu_k \in \mathbb{N}$. Put $Q_{k,k} := \mu_k$ and $Q_{j,k} := 0$, $j,k = 1,\dots,s$, $j \ne k$. Then

$$\alpha_{j,1}Q_{1,k} + \dots + \alpha_{j,n}Q_{n,k} = 0, \qquad j = 1,\dots,m,\ k = 1,\dots,s. \tag{1.7.11}$$

Let $Q_j := (Q_{j,1},\dots,Q_{j,s}) \in \mathbb{Z}^s$, $j = 1,\dots,n$. Define $\Phi : (\mathbb{C}_*)^s \times \mathbb{C}^{n-s} \to (\mathbb{C}_*)^s \times \mathbb{C}^{n-s}$,

$$\Phi(\xi,\eta) := (\xi^{Q_1},\dots,\xi^{Q_s},\xi^{Q_{s+1}}\eta_1,\dots,\xi^{Q_n}\eta_{n-s}) = (\xi_1^{\mu_1},\dots,\xi_s^{\mu_s},\xi^{Q_{s+1}}\eta_1,\dots,\xi^{Q_n}\eta_{n-s}),$$
$$(\xi,\eta) = (\xi_1,\dots,\xi_s,\eta_1,\dots,\eta_{n-s}) \in (\mathbb{C}_*)^s \times \mathbb{C}^{n-s}.$$

Observe that Φ is surjective. Indeed, for $z = (z_1,\dots,z_n) \in (\mathbb{C}_*)^s \times \mathbb{C}^{n-s}$, take an arbitrary $\xi_j \in (z_j)^{1/\mu_j}$, $j = 1,\dots,s$, and define $\eta_j := z_{s+j}/\xi^{Q_{s+j}}$, $j = 1,\dots,n-s$.

If $z = \Phi(\xi,\eta)$, then by (1.7.11) we get

$$z^{\alpha_j} = \xi^{\alpha_{j,1}Q_1+\dots+\alpha_{j,n}Q_n}\eta^{\beta_j} = \eta^{\beta_j}, \qquad j = 1,\dots,m. \tag{1.7.12}$$

Let $D := \{\eta \in \mathbb{C}^{n-s} : |\eta^{\beta_j}| < 1,\ j = 1,\dots,m\}$. Using (1.7.12) we get the equality $\Phi((\mathbb{C}_*)^s \times D) = G_0$. Fix a $\xi_0 \in (\mathbb{C}_*)^s$ such that $a = \Phi(\xi_0,0)$. Then, for any $z = \Phi(\xi,\eta) \in G_0$, we have

$$\begin{aligned} g_G(a,z) &= g_G(\Phi(\xi_0,0),\Phi(\xi,\eta)) \\ &\le g_{(\mathbb{C}_*)^s\times D}((\xi_0,0),(\xi,\eta)) = g_D(0,\eta) = h_D(\eta) \\ &= \max\{|\eta^{\beta_j}|^{1/r_j} : j = 1,\dots,m\} = \max\{|z^{\alpha_j}|^{1/r_j} : j = 1,\dots,m\}. \end{aligned}$$

The implications (ii)⇒(iii)⇒(iv)⇒(v) are trivial.

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(37) Note that (ii) gives an effective formula for $g_G(a,\cdot)$.

(v)$\Rightarrow$(i). Suppose that $\operatorname{rank}\widetilde{A} < \operatorname{rank} A$. We may assume that

$$2 \le t := \operatorname{rank} A = \operatorname{rank}\begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_t\end{bmatrix}, \qquad \operatorname{rank}\begin{bmatrix}\beta_1\\ \vdots \\ \beta_t\end{bmatrix} < t.$$

Then there exist $c_1,\dots,c_t \in \mathbb{Z}$ such that $c_1\beta_1+\dots+c_t\beta_t = 0$ and $|c_1|+\dots+|c_t| > 0$. To simplify notation, assume that $c_1,\dots,c_u \ge 0$, $c_{u+1},\dots,c_t < 0$ for some $1 \le u \le t-1$. Let

$$d := a^{c_1\alpha_1+\dots+c_t\alpha_t}, \qquad r := c_1r_1+\dots+c_ur_u = -(c_{u+1}r_{u+1}+\dots+c_tr_t),$$
$$f(z) := \frac{z^{c_1\alpha_1+\dots+c_u\alpha_u} - dz^{-(c_{u+1}\alpha_{u+1}+\dots+c_t\alpha_t)}}{1+|d|}, \qquad z \in G.$$

Observe that $f \in \mathcal{O}(G,E)$, $\operatorname{ord}_a f \ge r+1$, and $f \not\equiv 0$ (because $\alpha_1,\dots,\alpha_t$ are linearly independent). Fix a $b=(b',b'') \in G \subset \mathbb{C}^s \times \mathbb{C}^{n-s}$ with $f(b) \ne 0$. Notice that $f(b',\theta b'') = \theta^r f(b)$, $0 \le \theta \le 1$. Thus we get

$$\frac{1}{\theta}\, m_G^{(r+1)}(a,(b',\theta b'')) \ge \frac{1}{\theta}\,|f(b',\theta b'')|^{1/(r+1)} = \theta^{-1/(r+1)}|f(b)|^{1/(r+1)} \xrightarrow[\theta\to 0+]{} +\infty,$$

a contradiction.

EXAMPLE 1.7.14. Let $n=3$, $m=2$, $\alpha_1 := (1,1,0)$, $\alpha_2 := (1,0,1)$, $F(z) = (z_1z_2, z_1z_3)$,

$$G = \{(z_1,z_2,z_3) \in \mathbb{C}^3 : |z_1z_2| < 1,\ |z_1z_3| < 1\}.$$

Observe that:

- $\operatorname{rank} A = 2$, $A\mathbb{Z}^3 = \mathbb{Z}^2$,
- $r(a) = 1$ iff $a \ne 0$,
- $r(0) = 2$, and
- $\operatorname{rank} F'(a) = 2$ iff $a_1 \ne 0$.

(a) It is well known that

$$g_G(0,z) = h_G(z) = \max\{|z_1z_2|^{1/2},\ |z_1z_3|^{1/2}\}, \qquad z \in G.$$

Moreover, one can prove (see (c)) that

$$m_G^{(2p)}(0,z) = g_G(0,z) = \max\{|z_1z_2|^{1/2},\ |z_1z_3|^{1/2}\}, \qquad p \in \mathbb{N},$$
$$m_G^{(2p+1)}(0,z) = \max\{|z_1z_2|^{(p+1)/(2p+1)}, |z_1z_3|^{(p+1)/(2p+1)}\}, \qquad p \in \mathbb{Z}_+,\ z \in G.$$

(b) By Example 1.7.11, if $a_1 \ne 0$, then

$$m_G^{(k)}(a,z) = g_G(a,z) = m_{E^2}(F(a),F(z))$$
$$= \max\{m_E(a_1a_2, z_1z_2), m_E(a_1a_3, z_1z_3)\}, \qquad z \in G,\ k \in \mathbb{N}.$$

(c) By Example 1.7.13, if $a_3 \ne 0$, then

$$m_G^{(2p)}((0,0,a_3),z) = g_G((0,0,a_3),z) = \max\{|z_1z_2|^{1/2},\ |z_1z_3|\}, \qquad z \in G,\ p \in \mathbb{N}.$$

Moreover,

$$m_G^{(2p+1)}(0,0,a_3),z)$$
$$= \max\{|z_1z_2|^{(p+1)/(2p+1)},\ |(z_1z_2)^p z_1z_3|^{1/(2p+1)}, |z_1z_3|\}, \qquad z \in G,\ p \in \mathbb{Z}_+.$$

Indeed, the inequality "$\geq$" is obvious. Thus we have to show that

$$\begin{aligned}(1.7.13)\qquad L(z) &:= m_G^{(2p+1)}((0,0,a_3),z)\\ &\leq \max\{|z_1z_2|^{(p+1)/(2p+1)}, |(z_1z_2)^p z_1z_3|^{1/(2p+1)}, |z_1z_3|\} =: R(z).\end{aligned}$$

The inequality is clearly true if $z_1 = 0$ (because $\{0\}\times\mathbb{C}^2 \subset G$). Take $z = (z_1,z_2,z_3) \in G$ with $z_1 \neq 0$. Then

$$R(z) = \begin{cases} |z_1z_2|^{(p+1)/(2p+1)} & \text{if } |z_2| \geq |z_3|,\\ |(z_1z_2)^p z_1z_3|^{1/(2p+1)} & \text{if } |z_1z_3^2| \leq |z_2| \leq |z_3|,\\ |z_1z_3| & \text{if } |z_2| \leq |z_1z_3^2|.\end{cases}$$

Using standard arguments, we reduce the proof of (1.7.13) to the cases where $|z_2| = |z_1z_3^2|$ or $|z_2| = |z_3|$.

If $|z_2| = |z_1z_3^2|$, then we have $L(z) \leq g_G(a,z) = |z_1z_3| = R(z)$.

If $|z_2| = |z_3|$, then

$$(1.7.14)\qquad L(z) \leq g_G(a,z) = |z_1z_2|^{1/2}.$$

Take an arbitrary $f \in \mathcal{O}(G,E)$ with $\operatorname{ord}_a f \geq 2p+1$. We know that $f(z) = \widetilde{f}(z_1z_2, z_1z_3)$, where $\widetilde{f} \in \mathcal{O}(E^2,E)$ (cf. Example 1.7.11). Inequality (1.7.14) shows that

$$|\widetilde{f}(\lambda, e^{i\theta}\lambda)| \leq |\lambda|^{p+1/2}, \qquad \lambda \in E,\ \theta \in \mathbb{R}.$$

Hence

$$|\widetilde{f}(\lambda, e^{i\theta}\lambda)| \leq |\lambda|^{p+1}, \qquad \lambda \in E,\ \theta \in \mathbb{R},$$

and therefore, if $|z_2| = |z_3|$, then

$$L(z) \leq |z_1z_2|^{(p+1)/(2p+1)} = R(z).$$

(d) Similar formulae hold at points $(0,a_2,0)$ with $a_2 \neq 0$.

(e) In the case $a_2a_3 \neq 0$, by Example 1.7.13, we already know that

$$g_G((0,a_2,a_3),z) \underset{\not\equiv}{\geq} \max\{|z_1z_2|, |z_1z_3|\}, \qquad z \in G.$$

One can prove that

$$g_G((0,a_2,a_3),z) \geq m_G^{(k)}(a,z) \underset{\not\equiv}{\geq} \max\left\{|z_1z_2|, |z_1z_3|, \left|\frac{a_3z_1z_2 - a_2z_1z_3}{|a_2|+|a_3|}\right|^{\lceil k/2\rceil/k}\right\}, \qquad z \in G.$$

$\boxed{?}$ It seems that effective formulae for $m_G^{(k)}((0,a_2,a_3),\cdot)$ and $g_G((0,a_2,a_3),\cdot)$ are not known. $\boxed{?}$

Let us mention that necessarily

$$\begin{aligned} m_G^{(k)}((0,a_2,a_3),(e^{i\varphi}z_1, e^{i\psi}z_2, e^{i\psi}z_3)) &= m_G^{(k)}((0,a_2,a_3),z),\\ g_G((0,a_2,a_3),(e^{i\varphi}z_1, e^{i\psi}z_2, e^{i\psi}z_3)) &= g_G((0,a_2,a_3),z), \qquad z \in G,\ \varphi,\psi \in \mathbb{R},\end{aligned}$$

and

$$m_G^{(k)}(a,z) = g_G((0,a_2,a_3),z) = \max\{|z_1z_2|, |z_1z_3|\}, \qquad z \in G \cap \{a_3z_2 - a_2z_3 = 0\}.$$

As in the one-pole case (cf. [J-P 1993, Proposition 4.2.7(h)]), the generalized Green function may be characterized in terms of the Monge–Ampère operator $(dd^c\,\cdot)^n$.

THEOREM 1.7.15 ([Lel 1989]). *Let $G \subset \mathbb{C}^n$ be a bounded hyperconvex domain. Assume that the set $|\boldsymbol{p}|$ is finite. Then the function $u := g_G(\boldsymbol{p}, \cdot)$ is a unique solution of the following problem*:

$$\begin{cases} u \in \mathcal{C}(\overline{G}, [0,1]), \qquad \log u \in \mathcal{PSH}(G), \\ u = 1 \qquad \textit{on } \partial G, \\ \forall_{a \in |\boldsymbol{p}|}\ \exists_{C(a)>0}\ \forall_{z \in G} : u(z) \le C(a)\|z-a\|^{\boldsymbol{p}(a)}, \\ (dd^c \log u)^n = 0 \qquad \textit{in } G \setminus |\boldsymbol{p}|.\ ({}^{38}) \end{cases}$$

The proof is beyond the scope of this article.

REMARK 1.7.16. (a) Recall that even in the case of the single pole, the Green function g_G is not symmetric. Thus one can for instance ask whether for a bounded hyperconvex domain $G \subset \mathbb{C}^n$ we have $\lim_{a \to b} g_G(a,z) = 1$ for arbitrary $b \in \partial G$ and $z \in G$. The question is also interesting from the point of view of the boundary behavior of the Bergman function.

D. Coman [Com 1998] proved that if G is a bounded domain with a plurisubharmonic peak function ϱ at a point $b \in \partial G$ (i.e. $\varrho \in \mathcal{PSH}(G) \cap \mathcal{C}(\overline{G})$, $\varrho(b) = 0$, and $\varrho(z) < 0$, $z \in \overline{G} \setminus \{b\}$) such that ϱ is Hölder continuous at b, and

$$\max\left\{\frac{\log|\varrho(z)|}{\log|z-b|} : z \in \overline{G},\ r \le |z-b| \le 1/2\right\} = O\left(\log\log\frac{1}{r}\right), \qquad r \to 0,$$

then $\lim_{a \to b} \inf_{z \in K} g_G(a,z) = 1$ for any compact $K \Subset \overline{G} \setminus \{b\}$. In particular, the result is true in the case where G is a pseudoconvex domain with smooth boundary and b is of finite type.

G. Herbort [Her 2000] proved that if G is a bounded hyperconvex domain with a Hölder continuous bounded plurisubharmonic exhaustion function, then for any $K \Subset G$ and $b \in \partial G$ we have $\lim_{a \to b} \inf_{z \in K} g(a,z) = 1$. In particular, the result holds if G is a bounded pseudoconvex domain with $\mathcal{C}^2$ boundary.

(b) Let G be a bounded *strictly hyperconvex* domain, i.e. there exist a domain $U \subset \mathbb{C}^n$, $G \Subset U$, and a function $\varrho \in \mathcal{PSH}(U) \cap \mathcal{C}(U)$ such that $G = \{z \in U : \varrho(z) < 0\}$. S. Nivoche [Niv 1994], [Niv 1995], [Niv 2000] proved that in this case, for every $a \in G$, we have

$$g_G(a,z) = \lim_{k \to +\infty} m_G^{(k)}(a,z) = \sup_{k \in \mathbb{N}} m_G^{(k)}(a,z), \qquad z \in G,$$

$$A_G(a;X) = \lim_{k \to +\infty} \gamma_G^{(k)}(a;X) = \sup_{k \in \mathbb{N}} \gamma_G^{(k)}(a;X), \qquad X \in \mathbb{C}^n \setminus P,$$

where $P \subset \mathbb{C}^n$ is pluripolar; in fact, $P = \emptyset$ as was shown by N. Nikolov [Nik 2000].

Observe that for an elementary Reinhardt domain D_α of irrational type all the $m_{D_\alpha}^{(k)}$'s vanish and $m_{D_\alpha}^{(k)} \not\to g_{D_\alpha}$; cf. Theorem 1.3.1.

(c) In the case $n = 1$ the above result was generalized by N. Nikolov and W. Zwonek in [Nik-Zwo 2004a, Theorem 2]. They proved that if $G \subset \mathbb{C}$ is a domain for which the

(38) Recall that for a locally bounded function $v \in \mathcal{PSH}(D)$ ($D \subset \mathbb{C}^n$) we have $(dd^c v)^n = 0$ iff v is *maximal*, i.e. for any domain $D_0 \Subset D$ and for any function $v_0 \in \mathcal{PSH}(D_0)$ upper semicontinuous on $\overline{D}_0$, if $v_0 \le v$ on ∂D_0, then $v_0 \le v$ in D_0; cf. [J-P 1993, Appendix, §MA].

set of one-point connected components of $\mathbb{C} \setminus G$ is polar, then

$$g_G = \sup_{k \in \mathbb{N}} m_G^{(k)}, \qquad A_G = \sup_{k \in \mathbb{N}} \gamma_G^{(k)}.$$

Moreover, they gave an example of a hyperconvex domain $G \subset \mathbb{C}$ for which the above equalities do not hold.

(d) Recently E. A. Poletsky [Pol 2002] proved the following important theorem. Let $G \subset \mathbb{C}^n$ be a bounded strictly hyperconvex domain and let u be a negative plurisubharmonic function on G with zero boundary values, i.e. $\liminf_{z \to \zeta} u(z) = 0$, $\zeta \in \partial G$. Then there exist functions $\boldsymbol{p}_k : G \to \mathbb{R}_+$, $|\boldsymbol{p}_k|$ finite, $k = 1, 2, \dots$, such that $\log g_G(\boldsymbol{p}_k, \cdot) \to u$ in $L^1(G)$. Moreover, if u is continuous and $\psi \in \mathcal{C}_0((-\infty, 0])$, then

$$\int_G \psi(u(z))(dd^c \log g_G(\boldsymbol{p}_k, \cdot))^n \to \int_G \psi(u(z))(dd^c u)^n.$$

EXAMPLE 1.7.17 ([Car-Wie 2003]). Let $\boldsymbol{p} : E^n \to \mathbb{R}_+$ be such that

$$|\boldsymbol{p}| = \{a_1, \dots, a_N\} \subset E \times \{0\}^{n-1}.$$

Put $a_j = (c_j, 0, \dots, 0)$, $k_j := \boldsymbol{p}(a_j)$, $j = 1, \dots, N$, and assume that $k_1 \geq \dots \geq k_N$. Then

$$g_{E^n}(\boldsymbol{p}, z) = \prod_{j=1}^{N} u_j^{k_j - k_{j+1}}(z), \qquad z \in E^n,$$

where $k_{N+1} := 0$ and

$$\begin{aligned} u_j(z) :&= \max\{m_E(c_1, z_1) \cdots m_E(c_j, z_1), |z_2|, \dots, |z_n|\} \\ &= \max\{m_E(\{c_1, \dots, c_j\}, z_1), |z_2|, \dots, |z_n|\} \\ &= m_{E^n}(\{a_1, \dots, a_j\}, z), \qquad j = 1, \dots, N. \end{aligned}$$

Moreover, if $k_1, \dots, k_N \in \mathbb{N}$, then $m_{E^n}(\boldsymbol{p}, \cdot) = g_{E^n}(\boldsymbol{p}, \cdot)$.

The result extends easily to the case where $|\boldsymbol{p}| = \{a_1, \dots, a_N\} \subset E \times \{c\}^{n-1} \subset E \times E^{n-1}$. Observe that if $k_1 = \dots = k_N = 1$, then the above formula coincides with that from Example 1.7.2.

Indeed, let $u := \prod_{j=1}^{N} u_j^{k_j - k_{j+1}}$. Notice that u is continuous on $\overline{E}^n$, $\log u$ is plurisubharmonic, and $u = 1$ on $\partial(E^n)$. Take $1 \leq s \leq N$ and $z = (z_1, \dots, z_n)$ in a small neighborhood of a_s. Then for $j = s, \dots, N$ we get

$$u_j(z) \leq \max\{\text{const}\, |z_1 - c_s|, |z_2|, \dots, |z_n|\} \leq \text{const}\, \|z - a_s\|.$$

Consequently,

$$u(z) \leq \text{const} \prod_{j=s}^{N} u_j^{k_j - k_{j+1}}(z) \leq \text{const} \prod_{j=s}^{N} \|z - a_s\|^{k_j - k_{j+1}} = \text{const}\, \|z - a_s\|^{k_s}.$$

Thus $g_{E^n}(\boldsymbol{p}, \cdot) \geq u$. To prove the opposite inequality we consider first the case $n = 2$. By Theorem 1.7.15, we only need to verify that the function $\log u$ is maximal on $E^2 \setminus \{a_1, \dots, a_N\}$. Fix a point $b = (b_1, b_2) \in E^2 \setminus \{a_1, \dots, a_N\}$. Observe that the functions $\log u_j(z)$, $j = 1, \dots, N$, are maximal on $E^2 \setminus \{a_1, \dots, a_N\}$ (cf. [Kli 1991, Example 3.1.2]).

It is clear that there exists at most one $j_0 \in \{1, \dots, N\}$ such that

$$m(c_1, b_1) \cdots m(c_{j_0}, b_1) = |b_2|.$$

Consequently, all the functions $\log u_j$ with $j \neq j_0$ are pluriharmonic near b. Since $\log u = \sum_{j=1}^{N}(k_j - k_{k+1}) \log u_j$, we easily conclude that $\log u$ is maximal near b.

Now, consider the general case $n \geq 3$. Take a point

$$b = (b_1, \dots, b_n) \in E^n \setminus \{a_1, \dots, a_N\}$$

and let $\max\{|b_2|, \dots, |b_n|\} = |b_{s_0}|$. If $b_{s_0} = 0$ (i.e. $b_2 = \cdots = b_n = 0$), then consider the mapping $E \ni \lambda \overset{F}{\mapsto} (\lambda, 0, \dots, 0) \in E^n$ and use Remark 1.6.1(e):

$$g_{E^n}(\boldsymbol{p}, b) = g_{E^n}(\boldsymbol{p}, F(b_1)) \leq g_E(\boldsymbol{p} \circ F, b_1) = \prod_{j=1}^{N} [m_E(c_j, b_1)]^{k_j - k_{j+1}} = u(b).$$

If $b_{s_0} \neq 0$, then let $q_s := b_s / b_{s_0} \in \overline{E}$, $s = 2, \dots, n$. Consider the mapping

$$E^2 \ni (\lambda, \xi) \overset{F}{\mapsto} (\lambda, q_1\xi, \dots, q_n\xi) \in E^n.$$

Using Remark 1.6.1(e) and the case $n = 2$, we get

$$\begin{aligned} g_{E^n}(\boldsymbol{p}, b) &= g_{E^n}(\boldsymbol{p}, F(b_1, b_{s_0})) \\ &\leq g_{E^2}(\boldsymbol{p} \circ F, (b_1, b_{s_0})) = \prod_{j=1}^{N} [\max\{m_E(c_1, b_1) \cdots m(c_j, b_1), |b_{s_0}|\}]^{k_j - k_{j+1}} = u(b). \end{aligned}$$

EXAMPLE 1.7.18. Using Proposition 1.6.6 and Theorem 1.4.1 we get

$$\begin{aligned} g_{E^2}(\{(a,b),(b,a)\},(z,w)) &= c^*_{\mathbb{G}_2}((a+b, ab), (z+w, zw)) \\ &= \max\left\{ m_E\left(\frac{2\alpha ab - (a+b)}{2 - \alpha(a+b)}, \frac{2\alpha zw - (z+w)}{2 - \alpha(z+w)} \right) : \alpha \in \partial E \right\}, \\ &\qquad (a,b), (z,w) \in E^2,\ a \neq b. \end{aligned} \tag{1.7.15}$$

Notice that the above case is not covered by Example 1.7.17.

For any points $(a_1, b_1), (a_2, b_2) \in E^2$ with $m_E(a_1, a_2) = m_E(b_1, b_2) > 0$ there exists an $h \in \operatorname{Aut}(E)$ such that $h(a_1) = b_2$, $h(a_2) = b_1$, and consequently, formula (1.7.15) may be easily extended to such pairs of points.

$\boxed{?}$ In the case where $0 < m_E(a_1, a_2) \neq m_E(b_1, b_2) > 0$, an effective formula for $g_{E^2}(\{(a_1, b_1), (a_2, b_2)\}, \cdot)$ is still unknown. $\boxed{?}$

Recall that by the Lempert theorem, if $G \subset \mathbb{C}^n$ is convex, then $c^*_G = \widetilde{k}^*_G$ and, consequently, all holomorphically contractible families coincide on G. The following example shows that this is not true in the category of generalized holomorphically contractible families.

EXAMPLE 1.7.19 (due to W. Zwonek). Let

$$D := \{(z, w) \in \mathbb{C}^2 : |z| + |w| < 1\}, \qquad A_t := \{(t, \sqrt{t}), (t, -\sqrt{t})\}, \quad 0 < t \ll 1.$$

Then

$$m_D(A_t, (0,0)) < g_D(A_t, (0,0)) < d_D^{\max}(A_t, (0,0))$$

for small t.

Indeed, let $G := \{(z,w) \in \mathbb{C}^2 : |z| + \sqrt{|w|} < 1\}$ and let $F : D \to G$, $F(z,w) := (z, w^2)$. Note that F is proper and locally biholomorphic in a neighborhood of A_t. Moreover, $A_t = F^{-1}(t,t)$. Using Proposition 1.6.6, we conclude that $g_D(A_t, (0,0)) = g_G((t,t),(0,0))$.

Observe that $m_D(A_t,(0,0)) = m_G((t,t),(0,0))$. In fact, the inequality "$\geq$" follows from (H) (applied to F). The opposite inequality may be proved as follows. Let $f \in \mathcal{O}(D,E)$ be such that $f|_{A_t} = 0$. Define

$$\widetilde{f}(z,w) := \tfrac{1}{2}(f(z,\sqrt{w}) + f(z,-\sqrt{w})), \qquad (z,w) \in G.$$

Note that $\widetilde{f}$ is well defined, $|\widetilde{f}| < 1$, $\widetilde{f}(t,t) = 0$, $\widetilde{f}$ is continuous, and $\widetilde{f}$ is holomorphic on $D \cap \{w \neq 0\}$. In particular, $\widetilde{f}$ is holomorphic on D. Consequently, $|f(0,0)| = |\widetilde{f}(0,0)| \leq m_G((t,t),(0,0))$.

Suppose that $m_D(A_{t_k},(0,0)) = g_D(A_{t_k},(0,0))$ for a sequence $t_k \searrow 0$. Then

$$\begin{aligned} g_G((t_k,t_k),(0,0)) = g_D(A_{t_k},(0,0)) = m_D(A_{t_k},(0,0)) = m_G((t_k,t_k),(0,0)) \\ \leq g_G((t_k,t_k),(0,0)), \qquad k = 1,2,\dots. \end{aligned}$$

Thus $m_G((t_k,t_k),(0,0)) = g_G((t_k,t_k),(0,0))$, $k = 1,2,\dots$. Consequently, using [J-P 1993, §2.5], and [Zwo 2000c, Corollary 4.4], (cf. §1.2), we conclude that

$$\gamma_G((0,0);(1,1)) = A_G((0,0);(1,1)),$$

where γ_G (resp. A_G) denotes the Carathéodory–Reiffen (resp. Azukawa) metric of G (cf. §1.2). Hence, by Propositions 4.2.7 and 2.2.1(d) from [J-P 1993], using the fact that D is the convex envelope of G, we get

$$2 = h_D(1,1) = \gamma_G((0,0);(1,1)) = A_G((0,0);(1,1)) = h_G(1,1) = \frac{2}{3-\sqrt{5}}, \text{ (39)}$$

a contradiction.

To see the inequality $g_D(A_t,(0,0)) < d_D^{\max}(A_t,(0,0))$, we may argue as follows. We know (cf. [Zwo 2000c, Corollary 4.5] (40)) that

$$g_D(A_t,(0,0)) = g_G((t,t),(0,0)) \approx g_G((0,0),(t,t)) = h_G(t,t) = \frac{2t}{3-\sqrt{5}}$$

for small $t > 0$. On the other hand,

$$\begin{aligned} d_D^{\max}(A_t,(0,0)) &= \min\{\widetilde{k}_D^*((t,-\sqrt{t}),(0,0)), \widetilde{k}_D^*((t,\sqrt{t}),(0,0))\} \\ &= \min\{h_D(t,-\sqrt{t}), h_D(t,\sqrt{t})\} = t + \sqrt{t}. \end{aligned}$$

It remains to observe that

$$\frac{2t}{3-\sqrt{5}} < t + \sqrt{t}$$

for small $t > 0$.

---

(39) Recall that h_D (resp. h_G) denotes the Minkowski function for D (resp. G).

(40) Let $G \subset \mathbb{C}^n$ be a bounded hyperconvex domain. Then

$$\lim_{\substack{z',z'' \to a \\ z' \neq z''}} \frac{g_G(z',z'')}{g_G(z'',z')} = 1, \qquad a \in G.$$

EXAMPLE 1.7.20. Let $G := E^2$, $a_- := (-1/2, 0)$, $a_+ := (1/2, 0)$, $b := (0, 1/3)$, $\boldsymbol{p} := 2\chi_{a_-} + \chi_{a_+}$. Then $d_{E^2}^{\min}(\boldsymbol{p}, b) \le d'_{E^2}(\boldsymbol{p}, b) < m_{E^2}(\boldsymbol{p}, b)$, where $d'_{E^2}(\boldsymbol{p}, \cdot)$ is defined in Remark 1.7.8. Recall that $d_{E^2}^{\min}(A, \cdot) \equiv m_{E^2}(A, \cdot)$ $(A \subset E^2)$ (Proposition 1.5.4).

Indeed, by Example 1.7.17,

$$m_{E^2}(\boldsymbol{p}, b) = u_1(b)u_2(b) = \max\left\{\tfrac{1}{2}, \tfrac{1}{3}\right\} \max\left\{\tfrac{1}{2} \cdot \tfrac{1}{2}, \tfrac{1}{3}\right\} = \tfrac{1}{2} \cdot \tfrac{1}{3} = \tfrac{1}{6}.$$

On the other hand,

$$\begin{aligned} d'_{E^2}(\boldsymbol{p}, b) &= \max\{[m_{E^2}(a_-, b)]^2 m_{E^2}(a_+, b), [m_{E^2}(\{a_-, a_+\}, b)]^2\} \\ &= \max\left\{\left[\max\{\tfrac{1}{2}, \tfrac{1}{3}\}\right]^2 \max\left\{\tfrac{1}{2}, \tfrac{1}{3}\right\}, \left[m_{E^2}\left(\left\{-\tfrac{1}{2}, \tfrac{1}{2}\right\} \times \{0\}, b\right)\right]^2\right\} \\ &= \max\left\{\tfrac{1}{8}, \left[\max\left\{\tfrac{1}{2} \cdot \tfrac{1}{2}, \tfrac{1}{3}\right\}\right]^2\right\} = \tfrac{1}{8}. \end{aligned}$$

1.8. Properties of $d_G^{\min}$ and $d_G^{\max}$

REMARK 1.8.1. If $D \subset \mathbb{C}^m$ is a Liouville domain (i.e. $\mathcal{O}(D, E) \simeq E$), then

$$d_{G\times D}^{\min}(\boldsymbol{p}, (z, w)) = d_G^{\min}(\boldsymbol{p}', z), \qquad (z, w) \in G \times D,$$

where $\boldsymbol{p}'(z) := \sup\{\boldsymbol{p}(z, w) : w \in D\}$, $z \in G$.

PROPOSITION 1.8.2. (a) *The functions* $d_G^{\min}(\boldsymbol{p}, \cdot)$ *and* $d_G^{\max}(\boldsymbol{p}, \cdot)$ *are upper semicontinuous.*

(b) *If* $\boldsymbol{p} : G \to \mathbb{Z}_+$, *then* $d_G^{\min}(\boldsymbol{p}, \cdot) \in \mathcal{C}(G)$.

Proof. (a) The case of $d_G^{\max}(\boldsymbol{p}, \cdot)$ is obvious. To prove the upper semicontinuity of $d_G^{\min}(\boldsymbol{p}, \cdot)$, fix a $z_0 \in G$ and suppose that $d_G^{\min}(\boldsymbol{p}, z_k) \to \alpha > \beta > d_G^{\min}(\boldsymbol{p}, z_0)$ for a sequence $z_k \to z_0$. Take functions $f_k \in \mathcal{O}(G, E)$, $k \in \mathbb{N}$, such that $f_k(z_k) = 0$ and $\prod_{\mu \in f_k(G)} |\mu|^{\sup \boldsymbol{p}(f_k^{-1}(\mu))} \to \alpha$. By a Montel argument we may assume that $f_k \to f_0$ locally uniformly in G with $f_0 \in \mathcal{O}(G, E)$, $f_0(z_0) = 0$. Since $\prod_{\mu \in f_0(G)} |\mu|^{\sup \boldsymbol{p}(f_0^{-1}(\mu))} < \beta$, we can find a finite set $A \subset G$ such that $f_0|_A$ is injective and $\prod_{a \in A} |f_0(a)|^{\boldsymbol{p}(a)} < \beta$. Consequently, $\prod_{a \in A} |f_k(a)|^{\boldsymbol{p}(a)} < \beta$ and $f_k|_A$ is injective for $k \gg 1$. Finally, $\prod_{\mu \in f_k(G)} |\mu|^{\sup \boldsymbol{p}(f_k^{-1}(\mu))} < \beta$, $k \gg 1$, a contradiction.

(b) In view of (a), it suffices to prove that for every $f \in \mathcal{O}(G, E)$ the function $u_f(z) := \prod_{\mu \in f(G)} [m_E(\mu, f(z))]^{\sup \boldsymbol{p}(f^{-1}(\mu))}$, $z \in G$, is continuous on G. Observe that $u_f(z) = \inf_M \{\prod_{\mu \in M} [m_E(\mu, f(z))]^{k_f(\mu)}\}$, where M runs over all finite sets $M \subset f(|\boldsymbol{p}|)$ such that $k_f(\mu) := \sup \boldsymbol{p}(f^{-1}(\mu)) < +\infty$, $\mu \in M$. Thus $u_f = \inf_M\{|h_M|\}$, where $h_M \in \mathcal{O}(G, E)$. Consequently, since the family $(h_M)_M$ is equicontinuous, the function u_f is continuous on G. ∎

EXAMPLE 1.8.3. Let $\boldsymbol{p} : E \times \mathbb{C} \to \mathbb{R}_+$, $\boldsymbol{p}(1/k, k) := 1/k^2$, $k = 2, 3, \dots$, and $\boldsymbol{p}(z, w) := 0$ otherwise. Notice that $|\boldsymbol{p}|$ is discrete. Then by Remark 1.8.1,

$$d_{E\times\mathbb{C}}^{\min}(\boldsymbol{p}, (z, w)) = d_E^{\min}(\boldsymbol{p}', z) = \prod_{k=2}^{\infty} [m_E(1/k, z)]^{1/k^2}, \qquad (z, w) \in E \times \mathbb{C}.$$

In particular, $d_{E\times\mathbb{C}}^{\min}(\boldsymbol{p}, \cdot)$ is discontinuous at $(0, w) \in E \times \mathbb{C} \setminus |\boldsymbol{p}|$.

PROPOSITION 1.8.4. *If $\#|\boldsymbol{p}| < +\infty$, then for any $z_0 \in G$ there exists an extremal function for $d_G^{\min}(\boldsymbol{p}, z_0)$, i.e. a function $f_{z_0} \in \mathcal{O}(G, E)$ with $f_{z_0}(z_0) = 0$ and*

$$\prod_{\mu \in f_{z_0}(G)} |\mu|^{\sup \boldsymbol{p}(f_{z_0}^{-1}(\mu))} = d_G^{\min}(\boldsymbol{p}, z_0).$$

Proof. Fix a $z_0 \in G$ and let $f_k \in \mathcal{O}(G, E)$, $f_k(z_0) = 0$, be such that

$$\alpha_k := \prod_{\mu \in f_k(G)} |\mu|^{\sup \boldsymbol{p}(f_k^{-1}(\mu))} \to \alpha := d_G^{\min}(\boldsymbol{p}, z_0).$$

Let $A_k \subset |\boldsymbol{p}|$ be such that $f_k|_{A_k}$ is injective, $f_k(A_k) = f_k(|\boldsymbol{p}|)$, and

$$\boldsymbol{p}(a) = \sup \boldsymbol{p}(f_k^{-1}(f_k(a))), \quad a \in A_k.$$

Thus $\alpha_k = \prod_{a \in A_k} |f_k(a)|^{\boldsymbol{p}(a)}$. We may assume that $A_k = B$ is independent of k and for any $a \in B$ the fiber $B_a := f_k^{-1}(f_k(a)) \cap |\boldsymbol{p}|$ is also independent of k. Moreover, we may assume that $f_k \to f_0$ locally uniformly in G. Then $f_0 \in \mathcal{O}(G, E)$, $f_0(z_0) = 0$, and $\prod_{a \in B} |f_0(a)|^{\boldsymbol{p}(a)} = \alpha$. Observe that $f_0(B) = f_0(|\boldsymbol{p}|)$. Let $B_0 \subset B$ be such that $f_0|_{B_0}$ is injective and $f_0(B_0) = f_0(B)$. We have

$$\alpha \geq \prod_{\mu \in f_0(|\boldsymbol{p}|)} |\mu|^{\sup \boldsymbol{p}(f_0^{-1}(\mu))} = \prod_{\mu \in f_0(B_0)} |\mu|^{\sup \boldsymbol{p}(f_0^{-1}(\mu))} = \prod_{a \in B_0} |f_0(a)|^{\max\{\boldsymbol{p}(b) : b \in B,\, f_0(b) = f_0(a)\}}$$
$$\geq \prod_{a \in B} |f_0(a)|^{\boldsymbol{p}(a)} = \alpha. \ \blacksquare$$

PROPOSITION 1.8.5. $\log d_G^{\min}(\boldsymbol{p}, \cdot) \in \mathcal{PSH}(G)$.

Proof. By virtue of Proposition 1.8.2(a), we only need to show that for any $f \in \mathcal{O}(G, E)$ the function $u_f(z) := \prod_{\mu \in f(G)} [m_E(\mu, f(z))]^{\sup \boldsymbol{p}(f^{-1}(\mu))}$, $z \in G$, is log-plurisubharmonic on G. The proof of Proposition 1.8.2 shows that $u_f = \inf_M v_M$, where v_M is a log–plurisubharmonic function given by the formula $v_M(z) := \prod_{\mu \in M} [m_E(\mu, f(z))]^{k_f(\mu)}$ and M runs over a family of finite sets. Observe that $v_{M_1 \cup M_2} \leq \min\{v_{M_1}, v_{M_2}\}$. It remains to apply Lemma 1.6.3. $\blacksquare$

PROPOSITION 1.8.6. *If $G_k \nearrow G$ and $\boldsymbol{p}_k \nearrow \boldsymbol{p}$, then*

$$d_{G_k}^{\min}(\boldsymbol{p}_k, z) \searrow d_G^{\min}(\boldsymbol{p}, z), \quad d_{G_k}^{\max}(\boldsymbol{p}_k, z) \searrow d_G^{\max}(\boldsymbol{p}, z), \qquad z \in G.$$

Proof. By (H) and (M) (Definition 1.5.3) the sequence is monotone and for the limit function u we have $u \geq d_G^{\min}(\boldsymbol{p}, \cdot)$ (resp. $u \geq d_G^{\max}(\boldsymbol{p}, \cdot)$). Fix a $z_0 \in G$.

In the case of the minimal family suppose that $u(z_0) > \alpha > d_G^{\min}(\boldsymbol{p}, z_0)$. Let $f_k \in \mathcal{O}(G_k, E)$ be such that $f_k(z_0) = 0$ and $\prod_{\mu \in f_k(G_k)} |\mu|^{\sup \boldsymbol{p}_k(f_k^{-1}(\mu))} \to u(z_0)$. By a Montel argument we may assume that $f_k \to f_0$ locally uniformly in G with $f_0 \in \mathcal{O}(G, E)$, $f_0(z_0) = 0$. Since $\prod_{\mu \in f_0(G)} |\mu|^{\sup \boldsymbol{p}(f_0^{-1}(\mu))} < \alpha$, we can find a finite set $A \subset G$ such that $f|_A$ is injective and $\prod_{a \in A} |f_0(a)|^{\boldsymbol{p}(a)} < \alpha$. Consequently, $\prod_{a \in A} |f_k(a)|^{\boldsymbol{p}_k(a)} < \alpha$ and $f_k|_A$ is injective for $k \gg 1$. Finally, $\prod_{\mu \in f_k(G_k)} |\mu|^{\sup \boldsymbol{p}_k(f_k^{-1}(\mu))} < \alpha$, $k \gg 1$, a contradiction.

In the case of the maximal family for any $a \in G$ and $\varepsilon > 0$ there exists a $k(a, \varepsilon) \in \mathbb{N}$ such that $z_0, a \in G_k$, $\widetilde{k}_{G_k}^*(a, z_0) \leq \widetilde{k}_G^*(a, z_0) + \varepsilon$, and $\boldsymbol{p}_k(a) \geq \boldsymbol{p}(a) - \varepsilon$ for $k \geq k(a, \varepsilon)$.

Hence

$$\begin{aligned}\inf_{k\in\mathbb{N}} d_{G_k}^{\max}(\boldsymbol{p}_k, z_0) &= \inf_{k\in\mathbb{N},\, a\in G_k} [\widetilde{k}_{G_k}^*(a, z_0)]^{\boldsymbol{p}_k(a)}\\ &\le \inf_{a\in G} \inf\{[\widetilde{k}_G^*(a, z_0)+\varepsilon]^{\boldsymbol{p}_k(a)} : 0<\varepsilon\ll 1,\ k\ge k(a,\varepsilon)\}\\ &\le \inf_{a\in G} \inf\{[\widetilde{k}_G^*(a, z_0)+\varepsilon]^{\boldsymbol{p}(a)-\varepsilon} : 0<\varepsilon\ll 1\} = d_G^{\max}(\boldsymbol{p}, z_0). \ \blacksquare\end{aligned}$$

EXAMPLE 1.8.7. Let $G := \{z \in \mathbb{C}^n : |z^\alpha| < 1\}$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ is such that $\alpha_1, \dots, \alpha_n$ are relatively prime. Then

$$d_G^{\min}(\boldsymbol{p}, z) = d_E^{\min}(\boldsymbol{p}', z^\alpha) = \prod_{\mu\in E} [m_E(\mu, z^\alpha)]^{\boldsymbol{p}'(\mu)}, \qquad z \in G,$$

where $\boldsymbol{p}'(\lambda) = \sup\{\boldsymbol{p}(a) : a^\alpha = \lambda\}$, $\lambda \in E$.

Indeed, it is known (cf. Example 1.7.9) that any function $f \in \mathcal{O}(G, E)$ has the form $f = \widetilde{f} \circ F$, where $F(z) := z^\alpha$ and $\widetilde{f} \in \mathcal{O}(E, E)$. Thus

$$\begin{aligned}d_G^{\min}(\boldsymbol{p}, z) &= \sup\Big\{ \prod_{\mu\in\widetilde{f}(F(G))} [m_E(\mu, \widetilde{f}(F(z)))]^{\sup \boldsymbol{p}(F^{-1}(\widetilde{f}^{-1}(\mu)))} : \widetilde{f} \in \mathcal{O}(E, E)\Big\}\\ &= \sup\Big\{ \prod_{\mu\in\widetilde{f}(E)} [m_E(\mu, \widetilde{f}(F(z)))]^{\sup \boldsymbol{p}'(\widetilde{f}^{-1}(\mu))} : \widetilde{f} \in \mathcal{O}(E, E)\Big\} = d_E^{\min}(\boldsymbol{p}', F(z)).\end{aligned}$$

1.9. Relative extremal function

DEFINITION 1.9.1. Let $G \subset \mathbb{C}^n$ be a domain. For $A \subset G$ the *relative extremal function of A in G* is given by the formula (cf. [Kli 1991, §4.5])

$$\omega_{A,G} := \sup\{u \in \mathcal{PSH}(G) : u \le 0,\ u|_A \le -1\}.$$

Let $\omega_{A,G}^*$ denote the upper semicontinuous regularization of $\omega_{A,G}$.

REMARK 1.9.2. Let $F : G \to D$ be a holomorphic mapping and let $A \subset G$, $B \subset D$ be such that $F(A) \subset B$. Then

$$\omega_{B,D}(F(z)) \le \omega_{A,G}(z), \qquad z \in G.$$

THEOREM 1.9.3 ([Edi 2001]). *Let $G \subset \mathbb{C}^n$ be a domain, let $\boldsymbol{p} : G \to \mathbb{R}_+$ be such that the set $\#|\boldsymbol{p}|$ is finite. Fix an $R > 0$ so small that*

- $\mathbb{B}(a, R^{1/\boldsymbol{p}(a)}) \Subset G$ *for any* $a \in |\boldsymbol{p}|$,
- $\mathbb{B}(a, R^{1/\boldsymbol{p}(a)}) \cap \mathbb{B}(b, R^{1/\boldsymbol{p}(b)}) = \emptyset$ *for any* $a, b \in |\boldsymbol{p}|$, $a \ne b$.

Let $A_r := \bigcup_{a\in|\boldsymbol{p}|} \mathbb{B}(a, r^{1/\boldsymbol{p}(a)})$, $0 < r < R$. Then

$$(\log (R/r))\omega_{A_r,G} \searrow \log g_G(\boldsymbol{p}, \cdot) \quad \textit{when } r \searrow 0.$$

Proof. Let

$$v_r := (\log (R/r))\omega_{A_r,G}, \qquad 0 < r < R.$$

STEP 1. We have $v_{r_1} \le v_{r_2}$ for $0 < r_1 < r_2$. Indeed, fix $0 < r_1 < r_2 < R$ and define

$$u := \frac{v_{r_1}}{\log (R/r_2)} = \frac{(\log (R/r_1))\omega_{A_{r_1},G}}{\log (R/r_2)}.$$

Then $u \in \mathcal{PSH}(G)$ and $u \le 0$. It suffices to show that $u \le -1$ on A_{r_2}. Fix an $a \in |\boldsymbol{p}|$. Let $k := \boldsymbol{p}(a)$. Take a $z \in \mathbb{B}(a, r_2^{1/k})$. Then (cf. [Kli 1991, Lemma 4.5.8]):

$$u(z) \le \frac{(\log{(R/r_1)})\omega_{\mathbb{B}(a,r_1^{1/k}),\,\mathbb{B}(a,R^{1/k})}(z)}{\log{(R/r_2)}} = \frac{\log{(R/r_1)}}{\log{(R/r_2)}}\left(\frac{\log^+{(\|z-a\|/R^{1/k})}}{\log{(R^{1/k}/r_1^{1/k})}} - 1\right) \le -1.$$

Let

$$v := \lim_{r\to 0+} v_r = \lim_{r\to 0+} (\log{(R/r)})\omega_{A_r,G}.$$

Note that $v \in \mathcal{PSH}(G)$.

STEP 2. We have $v_r \ge \log g_G(\boldsymbol{p},\cdot)$, $0 < r < R$. In particular, $v \ge \log g_G(\boldsymbol{p},\cdot)$. Indeed, fix $0 < r < R$ and let

$$u_r := \frac{\log g_G(\boldsymbol{p},\cdot)}{\log{(R/r)}}.$$

Then $u_r \in \mathcal{PSH}(G)$ and $u_r \le 0$. Fix $a \in |\boldsymbol{p}|$ and $z \in \mathbb{B}(a, r^{1/k})$ $(k := \boldsymbol{p}(a))$. Then

$$u_r(z) \le \frac{k \log g_{\mathbb{B}(a,R^{1/k})}(a,z)}{\log{(R/r)}} = \frac{k\log{(\|z-a\|/R^{1/k})}}{\log{(R/r)}} \le -1.$$

Thus $u_r \le \omega_{A_r,G}$.

STEP 3. We have $v \le \log g_G(\boldsymbol{p},\cdot)$. Indeed, it suffices to check the growth of v near every point $a \in |\boldsymbol{p}|$. Fix an $a \in |\boldsymbol{p}|$ and let $z \in \mathbb{B}(a, R^{1/k})$, $z \ne a$ $(k := \boldsymbol{p}(a))$. Let $0 < r < \|z-a\|^k$. Then

$$\begin{aligned} v(z) - k\log\|z-a\| &\le (\log{(R/r)})\omega_{A_r,G}(z) - k\log\|z-a\| \\ &\le (\log{(R/r)})\omega_{\mathbb{B}(a,r^{1/k}),\,\mathbb{B}(a,R^{1/k})}(z) - k\log\|z-a\| \\ &= (\log{(R/r)})\left(\frac{\log^+{(\|z-a\|/R^{1/k})}}{\log{(R^{1/k}/r^{1/k})}} - 1\right) - k\log\|z-a\| \\ &\le -\log R. \ \blacksquare \end{aligned}$$

1.10. Analytic discs method

From some general point of view the invariant objects we have studied so far may be divided into three groups:

(a) objects related to certain extremal problems concerning holomorphic mappings $f : G \to E$, e.g. $c_G^*(a,z)$, $m_G(a,z)$, $\gamma_G(a;X)$, $\gamma_G^{(k)}(a;X)$, $m_G(\boldsymbol{p},z)$, $d_G^{\min}(\boldsymbol{p},z)$;

(b) objects related to certain extremal problems concerning logarithmically plurisubharmonic functions $u : G \to [0,1)$, e.g. $g_G(a,z)$, $A_G(a;X)$, $g_G(\boldsymbol{p},z)$;

(c) objects related to certain extremal problems concerning analytic discs $\varphi : E \to G$, e.g. $\widetilde{k}_G^*(a,z)$, $H^*(a,z)$, $\varkappa_G(a;X)$, $h_G(a;X)$, $\widetilde{k}_G^*(\boldsymbol{p},z)$.

In the late eighties E. A. Poletsky invented and partially developed a general method which reduces in some sense problems of type (b) to (c). This method found various important applications, due mainly to A. Edigarian (cf. [Edi 2002] and the references given there) and E. A. Poletsky (cf. [Pol 1991], [Pol 1993], [Edi-Pol 1997]); see for instance §1.12. In the present section we are mainly inspired by the exposition of the analytic disc theory presented in [Lár-Sig 1998b] and [Edi 2002].

DEFINITION 1.10.1. Let $G \subset \mathbb{C}^n$ be a domain. By a *disc functional* (on G) we mean any function

$$\varXi : \mathcal{O}(\overline{E}, G) \to \overline{\mathbb{R}}.$$

The *envelope* of a disc functional $\varXi : \mathcal{O}(\overline{E}, G) \to \overline{\mathbb{R}}$ is the function $\boldsymbol{\mathcal{E}}_{\varXi} : G \to \overline{\mathbb{R}}$ defined by the formula

$$\boldsymbol{\mathcal{E}}_{\varXi}(z) := \inf\{\varXi(\varphi) : \varphi \in \mathcal{O}(\overline{E}, G),\ \varphi(0) = z\}, \qquad z \in G.$$

DEFINITION 1.10.2. The following four types of disc functionals play an important role in complex analysis:

- *Poisson functional*:

$$\varXi^{\boldsymbol{p}}_{\mathrm{Poi}}(\varphi) := \frac{1}{2\pi}\int_0^{2\pi} \boldsymbol{p}(\varphi(e^{i\theta}))\,d\theta, \qquad \varphi \in \mathcal{O}(\overline{E}, G),$$

  where $\boldsymbol{p} : G \to [-\infty, \infty)$ is an upper semicontinuous function ([41]).
- *Green functional*:

$$\varXi^{\boldsymbol{p}}_{\mathrm{Gre}}(\varphi) := \sum_{\lambda \in E_*} \boldsymbol{p}(\varphi(\lambda)) \log|\lambda|, \qquad \varphi \in \mathcal{O}(\overline{E}, G),\ \boldsymbol{p} : G \to \mathbb{R}_+.\ ([42])$$

- *Lelong functional*:

$$\varXi^{\boldsymbol{p}}_{\mathrm{Lel}}(\varphi) := \sum_{\lambda \in E_*} \boldsymbol{p}(\varphi(\lambda))\,\mathrm{ord}_\lambda(\varphi - \varphi(\lambda)) \log|\lambda|, \qquad \varphi \in \mathcal{O}(\overline{E}, G),\ \boldsymbol{p} : G \to \mathbb{R}_+.$$

- *Lempert functional*:

$$\varXi^{\boldsymbol{p}}_{\mathrm{Lem}}(\varphi) := \inf\{\boldsymbol{p}(\varphi(\lambda)) \log|\lambda| : \lambda \in E_*\}, \qquad \varphi \in \mathcal{O}(\overline{E}, G),\ \boldsymbol{p} : G \to \mathbb{R}_+.$$

Put $\boldsymbol{\mathcal{E}}^{\boldsymbol{p}}_{\mathrm{Lel}} := \boldsymbol{\mathcal{E}}_{\varXi^{\boldsymbol{p}}_{\mathrm{Lel}}}$, $\boldsymbol{\mathcal{E}}^{\boldsymbol{p}}_{\mathrm{Gre}} := \boldsymbol{\mathcal{E}}_{\varXi^{\boldsymbol{p}}_{\mathrm{Gre}}}$, $\boldsymbol{\mathcal{E}}^{\boldsymbol{p}}_{\mathrm{Poi}} := \boldsymbol{\mathcal{E}}_{\varXi^{\boldsymbol{p}}_{\mathrm{Poi}}}$, $\boldsymbol{\mathcal{E}}^{\boldsymbol{p}}_{\mathrm{Lem}} := \boldsymbol{\mathcal{E}}_{\varXi^{\boldsymbol{p}}_{\mathrm{Lem}}}$.

REMARK 1.10.3. (a) $\varXi^{\boldsymbol{p}}_{\mathrm{Lel}} \le \varXi^{\boldsymbol{p}}_{\mathrm{Gre}} \le \varXi^{\boldsymbol{p}}_{\mathrm{Lem}}$ and, consequently, $\boldsymbol{\mathcal{E}}^{\boldsymbol{p}}_{\mathrm{Lel}} \le \boldsymbol{\mathcal{E}}^{\boldsymbol{p}}_{\mathrm{Gre}} \le \boldsymbol{\mathcal{E}}^{\boldsymbol{p}}_{\mathrm{Lem}}$.

(b) Let $\varXi \in \{\varXi^{\boldsymbol{p}}_{\mathrm{Gre}}, \varXi^{\boldsymbol{p}}_{\mathrm{Lel}}, \varXi^{\boldsymbol{p}}_{\mathrm{Lem}}\}$. Then $\varXi(\varphi)$ is well defined for $\varphi \in \mathcal{O}(E, G)$. Moreover, $\boldsymbol{\mathcal{E}}_{\varXi}(z) = \inf\{\varXi(\varphi) : \varphi \in \mathcal{O}(E, G),\ \varphi(0) = z\}$, $z \in G$. Indeed, for $\varphi \in \mathcal{O}(E, G)$ let $\varphi_r(\lambda) := \varphi(r\lambda)$, $|\lambda| < 1/r$, $0 < r < 1$. Then $\varphi_r \in \mathcal{O}(\overline{E}, G)$, $\varphi_r(0) = \varphi(0)$, and $\varXi(\varphi) = \inf_{0<r<1} \varXi(\varphi_r)$.

(c) If $F : G \to D$ is holomorphic, then $\varXi^{\boldsymbol{q}} \circ F = \varXi^{\boldsymbol{q} \circ F}$, $\varXi \in \{\varXi_{\mathrm{Poi}}, \varXi_{\mathrm{Gre}}, \varXi_{\mathrm{Lem}}\}$, and $\varXi^{\boldsymbol{q}}_{\mathrm{Lel}} \circ F \le \varXi^{\boldsymbol{q} \circ F}_{\mathrm{Lel}}$, where $(\varXi \circ F)(\varphi) := \varXi(F \circ \varphi)$, $\varphi \in \mathcal{O}(\overline{E}, G)$.

(d) Let $F : G \to D$ be holomorphic and let $\varXi : \mathcal{O}(\overline{E}, D) \to \overline{\mathbb{R}}$ be a disc functional on D. Then the mapping $\varXi \circ F$ is a disc functional on G and $\boldsymbol{\mathcal{E}}_{\varXi} \circ F \le \boldsymbol{\mathcal{E}}_{\varXi \circ F}$. If, moreover, F is a covering, then $\boldsymbol{\mathcal{E}}_{\varXi} \circ F = \boldsymbol{\mathcal{E}}_{\varXi \circ F}$. Indeed, we only need to observe that if F is a covering, then for any disc $\psi \in \mathcal{O}(\overline{E}, D)$ with $\psi(0) = F(z)$ there exists a $\varphi \in \mathcal{O}(\overline{E}, G)$ such that $\varphi(0) = z$ and $F \circ \varphi = \psi$.

([41]) The Poisson functional may be defined for more general functions $\boldsymbol{p}$ — see [Edi 2002].
([42]) $\sum_{\lambda \in A} f(\lambda) := \inf_{\substack{B \subset A \\ \#B < +\infty}} \sum_{\lambda \in B} f(\lambda)$ $(f : A \to [-\infty, 0])$.

(e) Observe that $\widehat{k}_G(\boldsymbol{p}, z) := \boldsymbol{\mathcal{E}}^{\boldsymbol{p}}_{\text{Lem}}(z) = \log \widetilde{k}^*_G(\boldsymbol{p}, z) = \log d^{\max}_G(\boldsymbol{p}, z)$, $z \in G$ (cf. §1.5). In particular, the function $\widehat{k}_G(\boldsymbol{p}, \cdot)$ is upper semicontinuous. Moreover, if $\mathbb{B}(a, r) \subset G$, then for any $z \in \mathbb{B}(a, r)$ we get

$$\widehat{k}_G(\boldsymbol{p}, z) \le \log \widetilde{k}^*_G(a, z) \le \boldsymbol{p}(a) \log \widetilde{k}^*_{\mathbb{B}(a,r)}(a, z) = \boldsymbol{p}(a) \log \frac{\|z - a\|}{r}$$
$$= \boldsymbol{p}(a) \log \|z - a\| - \boldsymbol{p}(a) \log r.$$

(f) In the case where $\boldsymbol{p} = \chi_{\{a\}}$ we have:

$$\varXi^{\boldsymbol{p}}_{\text{Poi}}(\varphi) = \frac{1}{2\pi} \varLambda(\varphi^{-1}(a) \cap \partial E), \qquad \varXi^{\boldsymbol{p}}_{\text{Lel}}(\varphi) = \sum_{\lambda \in \varphi^{-1}(a) \cap E_*} \operatorname{ord}_\lambda(\varphi - a) \log |\lambda|,$$
$$\varXi^{\boldsymbol{p}}_{\text{Gre}}(\varphi) = \sum_{\lambda \in \varphi^{-1}(a) \cap E_*} \log |\lambda|, \qquad \varXi^{\boldsymbol{p}}_{\text{Lem}}(\varphi) = \inf\{\log |\lambda| : \lambda \in \varphi^{-1}(a) \cap E_*\},$$

where $\varLambda$ denotes the Lebesgue measure on ∂E ($\varLambda(\partial E) = 2\pi$).

1.10.1. Poisson functional. For any upper semicontinuous function $\boldsymbol{p} : G \to [-\infty, \infty)$ let

$$\mathcal{P}_{\boldsymbol{p}}(G) := \{u \in \mathcal{PSH}(G) : u \le \boldsymbol{p}\}, \qquad \widehat{\omega}_G(\boldsymbol{p}, z) := \sup\{u(z) : u \in \mathcal{P}_{\boldsymbol{p}}(G)\}.$$

The function $\widehat{\omega}_G(\boldsymbol{p}, \cdot)$ is called the *generalized relative extremal function with weights* $\boldsymbol{p}$. Observe that $\widehat{\omega}_G(\boldsymbol{p}, \cdot) \in \mathcal{P}_{\boldsymbol{p}}(G)$. Moreover, $\omega_{U,G} \equiv \widehat{\omega}_G(-\chi_U, \cdot)$ for any open set $U \subset G$, where $\omega_{U,G}$ is the relative extremal function (Definition 1.9.1).

REMARK 1.10.4. (a) $\boldsymbol{\mathcal{E}}^{\boldsymbol{p}}_{\text{Poi}} \le \boldsymbol{p}$.

(b) If $\boldsymbol{p}_k \searrow \boldsymbol{p}$, then $\varXi^{\boldsymbol{p}_k}_{\text{Poi}} \searrow \varXi^{\boldsymbol{p}}_{\text{Poi}}$ and $\boldsymbol{\mathcal{E}}^{\boldsymbol{p}_k}_{\text{Poi}} \searrow \boldsymbol{\mathcal{E}}^{\boldsymbol{p}}_{\text{Poi}}$.

PROPOSITION 1.10.5. *For any upper semicontinuous function $\boldsymbol{p} : G \to [-\infty, \infty)$ we have $\widehat{\omega}_G(\boldsymbol{p}, \cdot) \le \boldsymbol{\mathcal{E}}^{\boldsymbol{p}}_{\text{Poi}}$. Consequently, if $\boldsymbol{\mathcal{E}}^{\boldsymbol{p}}_{\text{Poi}} \in \mathcal{PSH}(G)$, then $\boldsymbol{\mathcal{E}}^{\boldsymbol{p}}_{\text{Poi}} \in \mathcal{P}_{\boldsymbol{p}}(G)$ and $\widehat{\omega}_G(\boldsymbol{p}, \cdot) \equiv \boldsymbol{\mathcal{E}}^{\boldsymbol{p}}_{\text{Poi}}$.*

Proof. For $u \in \mathcal{P}_{\boldsymbol{p}}(G)$ and $\varphi \in \mathcal{O}(\overline{E}, G)$ we have

$$u(\varphi(0)) \le \frac{1}{2\pi} \int_0^{2\pi} u(\varphi(e^{i\theta}))\, d\theta \le \frac{1}{2\pi} \int_0^{2\pi} \boldsymbol{p}(\varphi(e^{i\theta}))\, d\theta = \varXi^{\boldsymbol{p}}_{\text{Poi}}(\varphi). \ \blacksquare$$

LEMMA 1.10.6. *The function $\boldsymbol{\mathcal{E}}^{\boldsymbol{p}}_{\text{Poi}}$ is upper semicontinuous on G.*

Proof. By Remark 1.10.4(b), we may assume that $\boldsymbol{p} : G \to \mathbb{R}$ is continuous. Fix a $z_0 \in G$ and suppose that $\boldsymbol{\mathcal{E}}^{\boldsymbol{p}}_{\text{Poi}}(z_0) < A$. Then there exists a $\varphi_0 \in \mathcal{O}(\overline{E}, G)$ such that $\varphi_0(0) = z_0$ and $\varXi^{\boldsymbol{p}}_{\text{Poi}}(\varphi_0) < A$. Take $0 < r < \operatorname{dist}(\varphi_0(\overline{E}), \partial G)$. Then for $z \in \mathbb{B}(z_0, r)$ we get

$$\boldsymbol{\mathcal{E}}^{\boldsymbol{p}}_{\text{Poi}}(z) \le \varXi^{\boldsymbol{p}}_{\text{Poi}}(\varphi_0 + z - z_0) = \frac{1}{2\pi} \int_0^{2\pi} \boldsymbol{p}(\varphi_0(e^{i\theta}) + z - z_0)\, d\theta.$$

It is clear that the function

$$\mathbb{B}(z_0, r) \ni z \mapsto \frac{1}{2\pi} \int_0^{2\pi} \boldsymbol{p}(\varphi_0(e^{i\theta}) + z - z_0)\, d\theta$$

is continuous and smaller than A at $z = z_0$. Consequently, there exists $0 < \delta < r$ such that $\boldsymbol{\mathcal{E}}^{\boldsymbol{p}}_{\text{Poi}}(z) < A$, $z \in \mathbb{B}(z_0, \delta)$. $\blacksquare$

THEOREM 1.10.7. *For any upper semicontinuous function $\boldsymbol{p} : G \to [-\infty, \infty)$ we have $\boldsymbol{\mathcal{E}}^{\boldsymbol{p}}_{\mathrm{Poi}} \in \mathcal{PSH}(G)$. Consequently, by Proposition* 1.10.5,

$$\inf\Big\{\frac{1}{2\pi}\int_0^{2\pi} \boldsymbol{p}(\varphi(e^{i\theta}))\,d\theta : \varphi \in \mathcal{O}(\overline{E}, G),\ \varphi(0) = z\Big\}$$
$$= \boldsymbol{\mathcal{E}}^{\boldsymbol{p}}_{\mathrm{Poi}}(z) = \widehat{\omega}_G(\boldsymbol{p}, z) = \sup\{u(z) \in \mathcal{PSH}(G) : u \le \boldsymbol{p}\}, \qquad z \in G.$$

In particular, if $U \subset G$ is open, then

$$\omega_{U,G}(z) = \widehat{\omega}_G(-\chi_U, z) = \boldsymbol{\mathcal{E}}_{\Xi^{-\chi_U}_{\mathrm{Poi}}}(z)$$
$$= \inf\Big\{\frac{1}{2\pi}\int_0^{2\pi} -\chi_U(\varphi(e^{i\theta}))d\theta : \varphi \in \mathcal{O}(\overline{E}, G),\ \varphi(0) = z\Big\}$$
$$= -\sup\Big\{\frac{1}{2\pi}\Lambda(\{\xi \in \partial E : \varphi(\xi) \in U\}) : \varphi \in \mathcal{O}(\overline{E}, G),\ \varphi(0) = z\Big\}, \qquad z \in G.$$

A class of complex manifolds G for which the above result is true was presented in [Lár-Sig 1998b]. The case where G is an arbitrary complex manifold was proved in [Ros 2003], [Edi 2003].

Proof. By Remark 1.10.4, we may assume that $\boldsymbol{p} : G \to \mathbb{R}$ is continuous. Let $u_0 := \boldsymbol{\mathcal{E}}^{\boldsymbol{p}}_{\mathrm{Poi}}$. By Lemma 1.10.6 we only need to show that

$$u_0(\varphi(0)) \le \frac{1}{2\pi}\int_0^{2\pi} u_0(\varphi(e^{it}))dt, \qquad \varphi \in \mathcal{O}(\overline{E}, G).$$

Fix a $\varphi_0 \in \mathcal{O}(\overline{E}, G)$. It suffices to prove that for any $\varepsilon > 0$ there exists a $\widetilde{\varphi} \in \mathcal{O}(\overline{E}, G)$ such that $\widetilde{\varphi}(0) = \varphi_0(0)$ and

$$\Xi^{\boldsymbol{p}}_{\mathrm{Poi}}(\widetilde{\varphi}) \le \frac{1}{2\pi}\int_0^{2\pi} u_0(\varphi_0(e^{it}))\,dt + \varepsilon. \tag{1.10.16}$$

Fix an $\varepsilon > 0$. The proof will be divided into four steps (Lemmas 1.10.8–1.10.11).

LEMMA 1.10.8. *There exist $r > 1$ and $\Phi \in \mathcal{C}^\infty(U_r \times \partial E, G)$, where $U_r := \mathbb{B}(r) \subset \mathbb{C}$, such that*:

(i) $\Phi(\cdot, \xi) \in \mathcal{O}(\overline{E}, G)$, $\xi \in \partial E$,
(ii) $\Phi(0, \xi) = \varphi_0(\xi)$, $\xi \in \partial E$,
(iii)

$$\int_0^{2\pi} \Xi^{\boldsymbol{p}}_{\mathrm{Poi}}(\Phi(\cdot, e^{it}))\,dt \le \int_0^{2\pi} u_0(\varphi_0(e^{it}))\,dt + \varepsilon. \tag{1.10.17}$$

Proof. Since u_0 is upper semicontinuous (Lemma 1.10.6), there exists a $v \in \mathcal{C}(G, \mathbb{R})$ with $v \ge u_0$ such that

$$\int_0^{2\pi} v(\varphi_0(e^{it}))\,dt \le \int_0^{2\pi} u_0(\varphi_0(e^{it}))\,dt + \frac{\varepsilon}{2}.$$

For any $\xi_0 \in \partial E$ there exist $\varphi \in \mathcal{O}(\overline{E}, G)$, $0 < \delta < \operatorname{dist}(\varphi(\overline{E}), \partial G)$, an open arc $I \subset \partial E$, and $r > 1$ such that:

- $\xi_0 \in I$, $\varphi(0) = \varphi_0(\xi_0)$,
- $\Xi^{\boldsymbol{p}}_{\mathrm{Poi}}(\varphi + z - \varphi_0(\xi_0)) < v(z) + \varepsilon/4$, $z \in \mathbb{B}(\varphi_0(\xi_0), \delta)$,
- $\varphi_0(\xi) \in \mathbb{B}(\varphi_0(\xi_0), \delta)$, $\xi \in I$,
- $\Phi_0(U_r \times I) \Subset G$, where $\Phi_0(\lambda, \xi) := \varphi(\lambda) + \varphi_0(\xi) - \varphi_0(\xi_0)$.

By a compactness argument we find a finite covering $\partial E = \bigcup_{\nu=1}^{N_0} I_\nu$, $r > 1$, and functions $\Phi_\nu \in \mathcal{C}^\infty(U_r \times I_\nu, G)$, $\nu = 1, \dots, N_0$, such that:

- $\Phi_\nu(\cdot, \xi) \in \mathcal{O}(\overline{E}, G)$, $\xi \in I_\nu$,
- $\Phi_\nu(0, \xi) = \varphi_0(\xi)$, $\xi \in I_\nu$,
- $\Phi_\nu(U_r \times I_\nu) \Subset G$,
- $\Xi^{\boldsymbol{p}}_{\mathrm{Poi}}(\Phi_\nu(\cdot, \xi)) < v(\varphi_0(\xi)) + \varepsilon/4$, $\xi \in I_\nu$, $\nu = 1, \dots, N_0$.

Let K be the closure of the set $\varphi_0(\partial E) \cup \bigcup_{\nu=1}^{N_0} \Phi_\nu(U_r \times I_\nu)$ and let $C > 0$ be such that $C > \max\{v(z) : z \in K\}$. There exist disjoint closed arcs $J_\nu \subset I_\nu$, $\nu \in A \subset \{1, \dots, N_0\}$, such that

$$\Lambda\Big(\partial E \setminus \bigcup_{\nu \in A} J_\nu\Big) < \frac{\varepsilon}{8C}.$$

We may assume that $A = \{1, \dots, N\}$ for some $N \le N_0$. Fix open disjoint arcs K_ν with $J_\nu \subset K_\nu \subset I_\nu$, $\nu = 1, \dots, N$, and let $\varrho \in \mathcal{C}^\infty(\partial E, [0,1])$ be such that $\varrho = 1$ on $\bigcup_{\nu=1}^{N} J_\nu$ and $\operatorname{supp} \varrho \subset \bigcup_{\nu=1}^{N} K_\nu$. Now we define $\Phi : U_r \times \partial E \to G$ by the formula

$$\Phi(\lambda, \xi) := \begin{cases} \Phi_\nu(\varrho(\xi)\lambda, \xi), & (\lambda, \xi) \in U_r \times K_\nu, \\ \varphi_0(\xi), & (\lambda, \xi) \in U_r \times (\partial E \setminus \bigcup_{\nu=1}^{N} K_\nu). \end{cases}$$

It is clear that Φ is well defined, $\Phi \in \mathcal{C}^\infty(U_r \times \partial E, G)$, $\Phi(U_r \times \partial E) \subset K$, and Φ satisfies (i) and (ii). It remains to check (iii). Let $\widetilde{J}_\nu := \{t \in [0, 2\pi) : e^{it} \in J_\nu\}$, $\nu = 1, \dots, N$. We have

$$\begin{aligned} \int_0^{2\pi} \Xi^{\boldsymbol{p}}_{\mathrm{Poi}}(\Phi(\cdot, e^{it}))\, dt &\le \sum_{\nu=1}^{N} \int_{\widetilde{J}_\nu} \Xi^{\boldsymbol{p}}_{\mathrm{Poi}}(\Phi_\nu(\cdot, e^{it}))\, dt + \frac{\varepsilon}{8} \le \sum_{\nu=1}^{N} \int_{\widetilde{J}_\nu} v(\varphi_0(e^{it}))\, dt + \frac{3\varepsilon}{8} \\ &\le \int_0^{2\pi} v(\varphi_0(e^{it}))\, dt + \frac{\varepsilon}{2} \le \int_0^{2\pi} u_0(\varphi_0(e^{it}))\, dt + \varepsilon. \ \blacksquare \end{aligned}$$

Lemma 1.10.9. *There exists $1 < s < r$ such that for any $j \ge 1$ there exist an open annulus $A_j \supset \partial E$ and $\Phi_j \in \mathcal{O}(U_s \times A_j, G)$ with:*

(i) *$\Phi_j \to \Phi$ uniformly on $U_s \times \partial E$,*

(ii) *there exist $1 < s_j < s$ and $k_j \in \mathbb{N}$, $k_j \ge j$, such that the mapping $(\lambda, \xi) \mapsto \Phi_j(\lambda \xi^{k_j}, \xi)$ extends to a mapping $\Psi_j \in \mathcal{O}(U_{s_j} \times U_{s_j}, G)$,*

(iii) *$\Psi_j(0, \xi) = \varphi_0(\xi)$, $\xi \in U_{s_j}$.*

Proof. Let

$$\Phi_j(\lambda, \xi) := \varphi_0(\xi) + \sum_{k=-j}^{j} \bigg(\frac{1}{2\pi} \int_0^{2\pi} (\Phi(\lambda, e^{i\theta}) - \varphi_0(e^{i\theta})) e^{-ik\theta}\, d\theta\bigg) \xi^k, \qquad (\lambda, \xi) \in U_r \times (U_r)_*.$$

Observe that the second term is the jth partial sum of the Fourier series of the function $\xi \mapsto \Phi(\lambda, \xi) - \varphi_0(\xi)$; also, Φ_j is holomorphic and $\Phi_j(0, \xi) = \varphi_0(\xi)$, $\xi \in (U_r)_*$. Moreover, for any $1 < t < r$, $\Phi_j \to \Phi$ uniformly on $U_t \times \partial E$. Indeed, it follows directly from Fourier series theory that $\Phi_j(\lambda, \cdot) \to \Phi(\lambda, \cdot)$ uniformly on ∂E for any $\lambda \in U_r$. Thus we only need to show that the series

$$\sum_{k=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_0^{2\pi} (\Phi(\lambda, e^{i\theta}) - \varphi_0(e^{i\theta}))e^{-ik\theta}\, d\theta \right) \xi^k$$

converges uniformly on $U_t \times \partial E$. Using integration by parts, we obtain

$$\left| \left(\int_0^{2\pi} (\Phi(\lambda, e^{i\theta}) - \varphi_0(e^{i\theta}))e^{-ik\theta}\, d\theta \right) \xi^k \right| \le \frac{1}{k^2} \sup_{\lambda \in U_t,\, \theta \in [0,2\pi)} \left| \frac{\partial^2}{\partial \theta^2} (\Phi(\lambda, e^{i\theta}) - \varphi_0(e^{i\theta})) \right|,$$
$$k \in \mathbb{Z}_*,\ (\lambda, \xi) \in U_t \times \partial E,$$

which implies the required convergence.

Fix $1 < t < r$. It follows that $\Phi_j(U_t \times \partial E) \Subset G$ for $j \ge j_0$. Hence, one can find an open annulus $A_j \supset \partial E$ such that $\Phi_j(U_t \times A_j) \subset G$, $j \ge j_0$.

For any $\xi \in (U_r)_*$ the mapping $\Phi_j(\cdot, \xi) - \varphi_0(\xi)$ has a zero at $\lambda = 0$. For any $\lambda \in U_r$ the mapping $\Phi_j(\lambda, \cdot) - \varphi_0$ has a pole of order $\le j$ at $\xi = 0$. Consequently, for any $k \ge j$ the mapping $(\lambda, \xi) \mapsto \Phi_j(\lambda \xi^k, \xi)$ extends holomorphically to $\overline{E} \times \overline{E}$. It remains to check (ii). Recall that $\Phi_j(0, \cdot) = \varphi_0$. Hence there exists $\delta_j > 0$ such that $\Phi_j(U_{\delta_j} \times \overline{E}) \subset G$. Since $\Phi_j(U_t \times A_j) \subset G$, $j \ge j_0$, we can find $0 < \varrho_j < 1$ such that $\Phi_j(\overline{E} \times (\overline{E} \setminus U_{\varrho_j})) \subset G$, $j \ge j_0$. Now, let $k_j \ge j$ be so large that $\varrho_j^{k_j} < \delta_j$. Then $\Psi_j(\lambda, \xi) := \Phi_j(\lambda \xi^{k_j}, \xi) \in G$, $(\lambda, \xi) \in \overline{E} \times \overline{E}$, $j \ge j_0$. ∎

LEMMA 1.10.10. *There exist* $1 < s < r$ *and* $\Psi \in \mathcal{O}(U_s \times U_s, G)$ *such that*:

(i) $\Psi(0, \xi) = \varphi_0(\xi)$, $\xi \in U_s$,

(ii)

$$\int_0^{2\pi} \Xi^{\boldsymbol{p}}_{\mathrm{Poi}}(\Psi(\cdot, e^{it}))\, dt \le \int_0^{2\pi} \Xi^{\boldsymbol{p}}_{\mathrm{Poi}}(\Phi(\cdot, e^{it}))\, dt + \varepsilon. \tag{1.10.18}$$

Proof. Let Φ_j, Ψ_j be as in Lemma 1.10.9. Then, for $j \ge j(\varepsilon)$ we have

$$\int_0^{2\pi} \Xi^{\boldsymbol{p}}_{\mathrm{Poi}}(\Psi_j(\cdot, e^{it}))\, dt = \int_0^{2\pi} \left(\frac{1}{2\pi} \int_0^{2\pi} \boldsymbol{p}(\Phi_j(e^{i(\theta + k_j t)}, e^{it}))\, d\theta \right) dt = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \boldsymbol{p}(\Phi_j(e^{i\theta}, e^{it}))\, d\theta\, dt$$
$$\le \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \boldsymbol{p}(\Phi(e^{i\theta}, e^{it}))\, d\theta\, dt + \varepsilon = \int_0^{2\pi} \Xi^{\boldsymbol{p}}_{\mathrm{Poi}}(\Phi(\cdot, e^{it}))\, dt + \varepsilon. \ ∎$$

LEMMA 1.10.11. *There exists a* $\theta_0 \in \mathbb{R}$ *such that if we put*

$$\widetilde{\varphi}(\lambda) := \varphi_{\theta_0}(\lambda) := \Psi(e^{i\theta_0}\lambda, \lambda), \qquad \lambda \in U_s,$$

then

$$\Xi^{\boldsymbol{p}}_{\mathrm{Poi}}(\widetilde{\varphi}) \le \frac{1}{2\pi} \int_0^{2\pi} \Xi^{\boldsymbol{p}}_{\mathrm{Poi}}(\Psi(\cdot, e^{it}))\, dt. \tag{1.10.19}$$

Proof. We have

$$\int_0^{2\pi}\int_0^{2\pi} \boldsymbol{p}(\Psi(e^{i\theta}, e^{it}))\,d\theta\,dt = \int_0^{2\pi}\int_0^{2\pi} \boldsymbol{p}(\Psi(e^{i\theta}e^{it}, e^{it}))\,dt\,d\theta.$$

Consequently, there exists a $\theta_0 \in \mathbb{R}$ such that

$$\begin{aligned}\varXi^{\boldsymbol{p}}_{\mathrm{Poi}}(\varphi_{\theta_0}) &= \frac{1}{2\pi}\int_0^{2\pi} \boldsymbol{p}(\Psi(e^{i\theta_0}e^{it}, e^{it}))\,dt \\ &\le \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \boldsymbol{p}(\Psi(e^{i\theta}e^{it}, e^{it}))\,dt\,d\theta \\ &= \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \boldsymbol{p}(\Psi(e^{i\theta}, e^{it}))\,d\theta\,dt = \frac{1}{2\pi}\int_0^{2\pi} \varXi^{\boldsymbol{p}}_{\mathrm{Poi}}(\Psi(\cdot, e^{it}))\,dt. \ \blacksquare\end{aligned}$$

Now, using (1.10.19), (1.10.18), and (1.10.17) gives (1.10.16). ■

The following result is a direct corollary of the definition of the function $\omega_{U,G}$ and Theorem 1.10.7.

PROPOSITION 1.10.12. *Let $F : G \to D$ be a holomorphic covering, let $V \subset D$ be open, and let $U := F^{-1}(V)$. Then $\omega_{V,D} \circ F = \omega_{U,G}$.*

Proof. The inequality "$\le$" follows from Remark 1.9.2. The opposite inequality follows from Theorem 1.10.7 and Remark 1.10.3(c, d):

$$\omega_{U,G} = \boldsymbol{\mathcal{E}}_{\varXi^{-\chi_U}_{\mathrm{Poi}}} = \boldsymbol{\mathcal{E}}_{\varXi^{-\chi_V\circ F}_{\mathrm{Poi}}} = \boldsymbol{\mathcal{E}}_{\varXi^{-\chi_V}_{\mathrm{Poi}}\circ F} = \boldsymbol{\mathcal{E}}_{\varXi^{-\chi_V}_{\mathrm{Poi}}} \circ F = \omega_{V,D} \circ F. \ \blacksquare$$

1.10.2. Green, Lelong, and Lempert functionals. For any function $\boldsymbol{p} : G \to \mathbb{R}_+$ let

$$\mathcal{G}_{\boldsymbol{p}}(G) := \{u \in \mathcal{PSH}(G) : u \le 0,\ \forall_{a\in G}\ \exists_{C(u,a)\in\mathbb{R}}\ \forall_{z\in G} : u(z) \le \boldsymbol{p}(a)\log\|z-a\| + C(a)\}.$$

Observe that $\log g_G(\boldsymbol{p}, z) = \sup\{u(z) : u \in \mathcal{G}_{\boldsymbol{p}}(G)\}$, $z \in G$ (cf. Definition 1.5.1).

PROPOSITION 1.10.13. $\log g_G(\boldsymbol{p},\cdot) \le \boldsymbol{\mathcal{E}}^{\boldsymbol{p}}_{\mathrm{Lel}}$. *Consequently,*

- *for $\varXi \in \{\varXi^{\boldsymbol{p}}_{\mathrm{Gre}}, \varXi^{\boldsymbol{p}}_{\mathrm{Lel}}, \varXi^{\boldsymbol{p}}_{\mathrm{Lem}}\}$, if $\boldsymbol{\mathcal{E}}_{\varXi} \in \mathcal{PSH}(G)$, then $\boldsymbol{\mathcal{E}}_{\varXi} \in \mathcal{G}_{\boldsymbol{p}}(G)$ and*

$$\log g_G(\boldsymbol{p},\cdot) \equiv \boldsymbol{\mathcal{E}}_{\varXi},$$

- *if $\boldsymbol{\mathcal{E}}_{\varXi^{\widehat{k}_G(\boldsymbol{p},\cdot)}_{\mathrm{Poi}}} \in \mathcal{PSH}(G)$ (43), then $\boldsymbol{\mathcal{E}}_{\varXi^{\widehat{k}_G(\boldsymbol{p},\cdot)}_{\mathrm{Poi}}} \in \mathcal{G}_{\boldsymbol{p}}(G)$ and*

$$\boldsymbol{\mathcal{E}}_{\varXi^{\widehat{k}_G(\boldsymbol{p},\cdot)}_{\mathrm{Poi}}} \le \log g_G(\boldsymbol{p},\cdot) \le \boldsymbol{\mathcal{E}}^{\boldsymbol{p}}_{\mathrm{Lel}}.$$

Proof. Take $\varphi \in \mathcal{O}(\overline{E}, G)$, $u \in \mathcal{G}_{\boldsymbol{p}}(G)$, and a finite set $B \subset E_* \cap \varphi^{-1}(|\boldsymbol{p}|)$. We are going to prove that $u(\varphi(0)) \le \sum_{\lambda\in B}\boldsymbol{p}(\varphi(\lambda))\operatorname{ord}_\lambda(\varphi-\varphi(\lambda))\log|\lambda|$, which implies that $u(\varphi(0)) \le \varXi^{\boldsymbol{p}}_{\mathrm{Lel}}(\varphi)$ and, consequently, $\log g_G(\boldsymbol{p},\cdot) \le \boldsymbol{\mathcal{E}}^{\boldsymbol{p}}_{\mathrm{Lel}}$. Let

$$v(\xi) := u(\varphi(\xi)) - \sum_{\lambda\in B}\boldsymbol{p}(\varphi(\lambda))\operatorname{ord}_\lambda(\varphi-\varphi(\lambda))\log m_E(\lambda,\xi).$$

(43) Recall that $\widehat{k}_G(\boldsymbol{p},\cdot) = \log\widetilde{k}^*_G(\boldsymbol{p},\cdot)$.

Then $v \in \mathcal{SH}(U_r \setminus B)$ for some $r > 1$ and $v = u \circ \varphi \le 0$ on ∂E. Moreover, one can easily check that v is locally bounded from above in U_r. Hence v extends subharmonically to U_r and, by the maximum principle, $v \le 0$ on E. In particular, $v(0) \le 0$, which gives the required inequality. ∎

PROPOSITION 1.10.14. $\boldsymbol{\mathcal{E}}^{\boldsymbol{p}}_{\mathrm{Gre}} = \boldsymbol{\mathcal{E}}^{\boldsymbol{p}}_{\mathrm{Lel}}$.

Proof. We have to prove that

$$\begin{aligned} L(z) &:= \inf\Big\{\sum_{\lambda\in E_*} \boldsymbol{p}(\varphi(\lambda))\operatorname{ord}_\lambda(\varphi-\varphi(\lambda))\log|\lambda| : \varphi \in \mathcal{O}(\overline{E}, G),\ \varphi(0) = z\Big\} \\ &= \inf\Big\{\sum_{\lambda\in E_*} \boldsymbol{p}(\varphi(\lambda))\log|\lambda| : \varphi \in \mathcal{O}(\overline{E}, G),\ \varphi(0) = z\Big\} =: R(z), \qquad z \in G. \end{aligned}$$

The inequality "$L \le R$" is obvious. Fix a $z \in G$ and an arbitrary constant $C > L(z)$. We want to show that $C \ge R(z)$. Since $C > L(z)$, there exist $\varphi \in \mathcal{O}(\overline{E}, G)$, $\varphi(0) = z$, and a finite set $B \subset E_* \cap \varphi^{-1}(|\boldsymbol{p}|)$ such that

$$\sum_{\lambda\in B} \boldsymbol{p}(\varphi(\lambda))\operatorname{ord}_\lambda(\varphi-\varphi(\lambda))\log|\lambda| < C.$$

Write $B = \{b_1, \dots, b_N\}$, $a_j := \varphi(b_j)$, $r(j) := \operatorname{ord}_{b_j}(\varphi - a_j)$, $j = 1, \dots, N$. Consider the family of all systems c of pairwise different points $c_{j,k} \in E$, $j = 1, \dots, N$, $k = 1, \dots, r(j)$, such that $c_{j,1} \cdots c_{j,r(j)} = b_j^{r(j)}$, $j = 1, \dots, N$. Define polynomials

$$Q_{c,\mu,\nu}(\lambda) := \prod_{\substack{j=1,\dots,N \\ k=1,\dots,r(j) \\ (j,k)\neq(\mu,\nu)}} (\lambda - c_{j,k}), \qquad \nu = 1, \dots, r(\mu),$$

$$P_{c,\mu}(\lambda) := \sum_{\nu=1}^{r(\mu)} \frac{Q_{c,\mu,\nu}(\lambda)}{Q_{c,\mu,\nu}(c_{\mu,\nu})}, \qquad \mu = 1, \dots, N,\ \lambda \in \mathbb{C}.$$

Observe that

- $\deg P_{c,j} \le r(1) + \dots + r(N) - 1$,
- $P_{c,j}(c_{\mu,\nu}) = 0$ if $\mu \neq j$ and $P_{c,j}(c_{j,\nu}) = 1$,
- $P_{c,1} + \dots + P_{c,N} \equiv 1$.

Define

$$\varphi_c(\lambda) := \sum_{\mu=1}^{N} P_{c,\mu}(\lambda)\Big(\frac{\varphi(\lambda) - a_\mu}{(\lambda - b_\mu)^{r(\mu)}}(\lambda - c_{\mu,1})\cdots(\lambda - c_{\mu,r(\mu)}) + a_\mu\Big).$$

Notice that $\varphi_c \in \mathcal{O}(\overline{E}, \mathbb{C}^n)$, $\varphi_c(0) = \varphi(0) = z$, and $\varphi_c(c_{j,k}) = a_j$ for all $j = 1, \dots, N$, $k = 1, \dots, r(j)$. Moreover,

$$\sum_{j=1}^{N}\sum_{k=1}^{r(j)} \boldsymbol{p}(a_j)\log|c_{j,k}| = \sum_{j=1}^{N} \boldsymbol{p}(a_j)r(j)\log|b_j| < C.$$

It remains to observe that $\varphi_c(\overline{E}) \subset G$ provided that $c_{j,k} \approx b_j$, $j = 1, \dots, N$, $k = 1, \dots, r(j)$. ∎

THEOREM 1.10.15. *If* $|\boldsymbol{p}|$ *is finite, then* $\boldsymbol{\mathcal{E}}_{\varXi_{\mathrm{Poi}}^{\widehat{k}_G(\boldsymbol{p},\cdot)}} = \boldsymbol{\mathcal{E}}_{\mathrm{Lel}}^{\boldsymbol{p}}$. *Consequently, by Theorem* 1.10.7, $\boldsymbol{\mathcal{E}}_{\mathrm{Lel}}^{\boldsymbol{p}} \in \mathcal{PSH}(G)$ *and, therefore, by Propositions* 1.10.13 *and* 1.10.14,

$$\boldsymbol{\mathcal{E}}_{\varXi_{\mathrm{Poi}}^{\widehat{k}_G(\boldsymbol{p},\cdot)}} = \log g_G(\boldsymbol{p},\cdot) = \boldsymbol{\mathcal{E}}_{\mathrm{Lel}}^{\boldsymbol{p}} = \boldsymbol{\mathcal{E}}_{\mathrm{Gre}}^{\boldsymbol{p}}.$$

Moreover, by Proposition 1.6.2, *for arbitrary* $\boldsymbol{p} : G \to \mathbb{R}_+$ *we get the following* Poletsky formula:

$$\log g_G(\boldsymbol{p},\cdot) = \boldsymbol{\mathcal{E}}_{\mathrm{Lel}}^{\boldsymbol{p}} = \boldsymbol{\mathcal{E}}_{\mathrm{Gre}}^{\boldsymbol{p}}.$$

The Poletsky formula and the main ideas of the proof are due to E. A. Poletsky (cf. [Pol-Sha 1989], [Pol 1991], [Pol 1993]). The first complete proof was given by A. Edigarian in [Edi 1997b]. We follow A. Edigarian's exposition.

Proof. We may assume that $\boldsymbol{p} \not\equiv 0$. By Proposition 1.10.13 we only need to show that $\boldsymbol{\mathcal{E}}_{\mathrm{Lel}}^{\boldsymbol{p}} \le \boldsymbol{\mathcal{E}}_{\varXi_{\mathrm{Poi}}^{\widehat{k}_G(\boldsymbol{p},\cdot)}}$. Fix a $\varphi_0 \in \mathcal{O}(\overline{E}, G)$ and $\varepsilon > 0$. It suffices to find a $\widetilde{\varphi} \in \mathcal{O}(\overline{E}, G)$ such that $\widetilde{\varphi}(0) = \varphi_0(0)$ and

$$\varXi_{\mathrm{Lel}}^{\boldsymbol{p}}(\widetilde{\varphi}) \le \frac{1}{2\pi}\int_0^{2\pi} \widehat{k}_G(\boldsymbol{p}, \varphi_0(e^{it}))\,dt + \varepsilon.$$

The existence of φ will follow from a sequence of lemmas (Lemmas 1.10.16–1.10.20).

LEMMA 1.10.16. *There exist*:

- $1 < s < r$,
- $\varPhi \in \mathcal{C}^\infty(U_r \times \partial E, G)$,
- $N \in \mathbb{N}$,
- $a_1, \dots, a_N \in |\boldsymbol{p}|$,
- $\sigma_1, \dots, \sigma_N \in \mathcal{C}^\infty(\partial E, \mathbb{C}_*)$,
- *disjoint closed arcs* $J_1, \dots, J_N \subset \partial E$

such that:

(i) $\varPhi(\cdot, \xi) \in \mathcal{O}(\overline{E}, G)$, $\varPhi(0, \xi) = \varphi_0(\xi)$, $\xi \in \partial E$,
(ii) *if* $|\sigma_\nu(\xi)| < s$, *then* $|\sigma_\mu(\xi)| > s$, $\mu \neq \nu$, *and* $\varPhi(\sigma_\nu(\xi), \xi) = a_\nu$,
(iii) $|\sigma_\nu(\xi)| < 1$, $\xi \in J_\nu$, $\nu = 1, \dots, N$, $\varLambda(\partial E \setminus \bigcup_{\nu=1}^N J_\nu) < \varepsilon$,
(iv) $\sigma_\nu(\xi) \neq \sigma_\mu(\xi)$, $\xi \in \partial E$, $\nu \neq \mu$,
(v) $2\pi N \max_{\nu=1,\dots,N}\{\boldsymbol{p}(a_\nu) \max_{\partial E} \log|\sigma_\nu|\} < \varepsilon/2$,
(vi) $\sum_{\nu=1}^N \boldsymbol{p}(a_\nu) \int_0^{2\pi} \log|\sigma_\nu(e^{it})|\,dt \le \int_0^{2\pi} \widehat{k}_G(\boldsymbol{p}, \varphi_0(e^{it}))\,dt + \varepsilon$.

Proof. Let $u_0 := \widehat{k}_G(\boldsymbol{p}, \cdot)$. Since u_0 is upper semicontinuous, there exists a $v \in \mathcal{C}(G, \mathbb{R})$ with $v \ge u_0$ such that

$$\int_0^{2\pi} v(\varphi_0(e^{it}))\,dt \le \int_0^{2\pi} u_0(\varphi_0(e^{it}))\,dt + \frac{\varepsilon}{2}.$$

For any $\xi_0 \in \partial E$ there exist $\varphi \in \mathcal{O}(\overline{E}, G)$, $\lambda_0 \in E_*$, $\delta > 0$, an open arc $I \subset \partial E$, and $r > 1$ such that:

- $\xi_0 \in I$, $\varphi(0) = \varphi_0(\xi_0)$, $\varphi(\lambda_0) =: a \in |\boldsymbol{p}|$,
- $\boldsymbol{p}(a) \log|\lambda_0| < v(z) + \varepsilon/8$, $z \in \mathbb{B}(\varphi_0(\xi_0), \delta) \subset G$,

- $\varphi(\lambda) + (1 - \lambda/\lambda_0)(z - \varphi_0(\xi_0)) \in G$, $(\lambda, z) \in U_r \times \mathbb{B}(\varphi_0(\xi_0), \delta)$,
- $\varphi_0(\xi) \in \mathbb{B}(\varphi_0(\xi_0), \delta)$, $\xi \in I$,
- $\Phi_0(U_r \times I) \Subset G$, where $\Phi_0(\lambda, \xi) := \varphi(\lambda) + (1 - \lambda/\lambda_0)(\varphi_0(\xi) - \varphi_0(\xi_0))$.

By a compactness argument we find a covering $\partial E = \bigcup_{\nu=1}^{N_0} I_\nu$, $r > 1$, functions $\Phi_\nu \in \mathcal{C}^\infty(U_r \times I_\nu, G)$, $\nu = 1, \dots, N_0$, and points $\lambda_1, \dots, \lambda_{N_0} \in E_*$ such that:

- $\Phi_\nu(\cdot, \xi) \in \mathcal{O}(\overline{E}, G)$, $\xi \in I_\nu$,
- $\Phi_\nu(0, \xi) = \varphi_0(\xi)$, $\xi \in I_\nu$,
- $\Phi_\nu(\lambda_\nu, \xi) =: a_\nu \in |\boldsymbol{p}|$, $\xi \in I_\nu$,
- $\Phi_\nu(U_r \times I_\nu) \Subset G$,
- $\boldsymbol{p}(a_\nu) \log |\lambda_\nu| < v(\varphi_0(\xi)) + \varepsilon/8$, $\xi \in I_\nu$, $\nu = 1, \dots, N_0$.

Replacing Φ_ν by the function $(\lambda, \xi) \mapsto \Phi_\nu(e^{i\theta_\nu}\lambda, \xi)$ with suitable $\theta_\nu \approx 0$, we may assume that the points $\lambda_1, \dots, \lambda_{N_0}$ have different arguments.

Fix $1 < s < s_0 < r$ with $2\pi N_0 \max_{\nu=1,\dots,N_0} \boldsymbol{p}(a_\nu) \log s_0 < \varepsilon/8$. Let K be the closure of the set

$$\varphi_0(\partial E) \cup \bigcup_{\nu=1}^{N_0} \Phi_\nu(U_r \times I_\nu)$$

and let $C > 0$ be such that

$$C > 2\pi N_0 \max_{\nu=1,\dots,N_0} \boldsymbol{p}(a_\nu) \big|\log|\lambda_\nu|\big| + \max\{v(z) : z \in K\}.$$

There exist disjoint closed arcs $J_\nu \subset I_\nu$, $\nu \in A \subset \{1, \dots, N_0\}$, such that

$$\Lambda\Big(\partial E \setminus \bigcup_{\nu \in A} J_\nu\Big) < \frac{\varepsilon}{8C}.$$

We may assume that $A = \{1, \dots, N\}$ for some $N \le N_0$. Fix open disjoint arcs K_ν with $J_\nu \subset K_\nu \subset I_\nu$, $\nu = 1, \dots, N$, and let $\varrho \in \mathcal{C}^\infty(\partial E, [0,1])$ be such that $\varrho = 1$ on $\bigcup_{\nu=1}^N J_\nu$ and $\operatorname{supp} \varrho \subset \bigcup_{\nu=1}^N K_\nu$. We define $\Phi : U_r \times \partial E \to G$ by the formula

$$\Phi(\lambda, \xi) := \begin{cases} \Phi_\nu(\varrho(\xi)\lambda, \xi), & (\lambda, \xi) \in U_r \times K_\nu, \\ \varphi_0(\xi), & (\lambda, \xi) \in U_r \times (\partial E \setminus \bigcup_{\nu=1}^N K_\nu). \end{cases}$$

It is clear that Φ is well defined, $\Phi \in \mathcal{C}^\infty(U_r \times \partial E, G)$, and Φ satisfies (i).

Let $K_\nu = \{e^{i\theta} : \theta \in (\alpha_\nu, \beta_\nu)\}$, $J_\nu = \{e^{i\theta} : \theta \in [\gamma_\nu, \delta_\nu]\}$ with $\alpha_\nu < \gamma_\nu < \delta_\nu < \beta_\nu$. We may assume that ϱ increases on (α_ν, γ_ν) and decreases on (δ_ν, β_ν). Then the set $J'_\nu := \{\xi \in K_\nu : |\lambda_\nu|/\varrho(\xi) \le s\}$ is a closed arc with $J_\nu \subset J'_\nu \subset K_\nu$. Take a $\sigma_\nu \in \mathcal{C}^\infty(\partial E, \mathbb{R}_{>0}\lambda_\nu)$ with

- $\sigma_\nu(\xi) = \lambda_\nu/\varrho(\xi)$, $\xi \in J'_\nu$,
- $s < |\sigma_\nu(\xi)| < s_0$, $\xi \in K_\nu \setminus J'_\nu$,
- $|\sigma_\nu(\xi)| = s_0$, $\xi \in \partial E \setminus K_\nu$.

Then (ii)–(v) are satisfied. It remains to check (vi). Let $\widetilde{J}_\nu := \{\theta \in [0, 2\pi) : e^{i\theta} \in J_\nu\}$, $\nu = 1, \dots, N$. We have

$$\sum_{\nu=1}^{N}\boldsymbol{p}(a_\nu)\int_0^{2\pi}\log|\sigma_\nu(e^{it})|\,dt \le \sum_{\nu=1}^{N}\boldsymbol{p}(a_\nu)\int_{\widetilde{J}_\nu}\log|\lambda_\nu|\,dt+\frac{\varepsilon}{8}\le\sum_{\nu=1}^{N}\int_{\widetilde{J}_\nu}v(\varphi_0(e^{it}))\,dt+\frac{\varepsilon}{4}$$
$$\le\int_0^{2\pi}v(\varphi_0(e^{it}))\,dt+\frac{\varepsilon}{2}\le\int_0^{2\pi}\widehat{k}_G(\boldsymbol{p},\varphi_0(e^{it}))\,dt+\varepsilon.\ \blacksquare$$

Lemma 1.10.17. *There exists a* $j_0\in\mathbb{N}$ *such that for any* $j\ge j_0$ *there exist* $1<s_j<s$, $\Psi_j\in\mathcal{O}(U_{s_j}\times U_{s_j},G)$, *and* $\tau_{\nu,j}\in\mathcal{O}(U_{s_j}\setminus\overline{U}_{1/s_j})$, $\nu=1,\dots,N$, *such that*:

(i) $\Psi_j(0,\xi)=\varphi_0(\xi)$, $\xi\in U_{s_j}$,
(ii) $|\tau_{\nu,j}|\to|\sigma_\nu|$ *uniformly on* ∂E,
(iii) $\Psi_j(\tau_{\nu,j}(\xi),\xi)=a_\nu$, $\xi\in U_{s_j}\setminus\overline{U}_{1/s_j}$ *with* $|\tau_{\nu,j}(\xi)|<s_j$,
(iv) $|\tau_{\nu,j}(\xi)|<1$, $\xi\in J_\nu$, $\nu=1,\dots,N$.

Proof. Recall that for any $\xi\in\partial E$ the numbers $0,\sigma_1(\xi),\dots,\sigma_N(\xi)$ are pairwise different. Let $P:\mathbb{C}\times\partial E\to\mathbb{C}$ be defined by the formula

$$P(\lambda,\xi):=\varphi_0(\xi)\prod_{l=1}^{N}\frac{\lambda-\sigma_l(\xi)}{-\sigma_l(\xi)}+\sum_{\mu=1}^{N}\frac{\lambda a_\mu}{\sigma_\mu(\xi)}\prod_{\substack{l=1\\ l\ne\mu}}^{N}\frac{\lambda-\sigma_l(\xi)}{\sigma_\mu(\xi)-\sigma_l(\xi)};$$

observe that $P(\cdot,\xi)$ is the Lagrange interpolation polynomial with $P(0,\xi)=\varphi_0(\xi)$, $P(\sigma_\nu(\xi),\xi)=a_\nu$, $\nu=1,\dots,N$. We will prove that there exists a function $\Phi_0\in\mathcal{C}^\infty(U_s\times\partial U)$ such that

$$\Phi(\lambda,\xi)=P(\lambda,\xi)+(\lambda-\sigma_1(\xi))\cdots(\lambda-\sigma_N(\xi))\Phi_0(\lambda,\xi),\qquad(\lambda,\xi)\in U_s\times\partial E.$$

Indeed, the only problem is to check that Φ_0 is $\mathcal{C}^\infty$ near a point $(\sigma_\nu(\xi_0),\xi_0)$ with $|\sigma_\nu(\xi_0)|<s$. Then $|\sigma_\mu(\xi_0)|>s$ for $\mu\ne\nu$, and there exists a neighborhood V of ξ_0 such that $|\sigma_\nu(\xi)|<s$, $|\sigma_\mu(\xi)|>s$, $\mu\ne\nu$, for $\xi\in V$. Observe that

$$\Phi(\lambda,\xi)=a_\nu+(\lambda-\sigma_\nu(\xi))\widehat{\Phi}(\lambda,\xi),\quad P(\lambda,\xi)=a_\nu+(\lambda-\sigma_\nu(\xi))\widehat{P}(\lambda,\xi),\quad(\lambda,\xi)\in U_s\times V,$$

where $\widehat{\Phi}$ and $\widehat{P}$ are $\mathcal{C}^\infty$ mappings. Hence

$$\Phi_0(\lambda,\xi)=(\widehat{\Phi}(\lambda,\xi)-\widehat{P}(\lambda,\xi))\prod_{\mu\ne\nu}\frac{1}{\lambda-\sigma_\mu(\xi)},\qquad(\lambda,\xi)\in U_s\times V,$$

and, consequently, $\Phi_0\in\mathcal{C}^\infty(U_s\times V)$. Notice that $\Phi_0(0,\cdot)\equiv0$.

Let $\Phi_{0,j}$ and $\sigma_{\nu,j}$ be the jth partial sums of the Fourier series of Φ_0 and σ_ν, respectively, i.e.

$$\Phi_{0,j}(\lambda,\xi):=\sum_{k=-j}^{j}\Big(\frac{1}{2\pi}\int_0^{2\pi}\Phi_0(\lambda,e^{it})e^{-ikt}\,dt\Big)\xi^k,$$
$$\sigma_{\nu,j}(\xi):=\sum_{k=-j}^{j}\Big(\frac{1}{2\pi}\int_0^{2\pi}\sigma_\nu(e^{it})e^{-ikt}\,dt\Big)\xi^k$$

for $(\lambda,\xi)\in U_s\times\mathbb{C}_*$; cf. the proof of Lemma 1.10.9. One can easily show that $\Phi_{0,j}\to\Phi_0$ and $\sigma_{\nu,j}\to\sigma_\nu$ uniformly on $U_t\times\partial E$ for any $1<t<s$. Define

$$P_j(\lambda,\xi) := \varphi_0(\xi)\prod_{l=1}^{N}\frac{\lambda-\sigma_{l,j}(\xi)}{-\sigma_{l,j}(\xi)} + \sum_{\mu=1}^{N}\frac{\lambda a_\mu}{\sigma_{\mu,j}(\xi)}\prod_{l=1,\,l\neq\mu}^{N}\frac{\lambda-\sigma_{l,j}(\xi)}{\sigma_{\mu,j}(\xi)-\sigma_{l,j}(\xi)},$$
$$\Phi_j(\lambda,\xi) := P_j(\lambda,\xi) + (\lambda-\sigma_{1,j}(\xi))\cdots(\lambda-\sigma_{N,j}(\xi))\Phi_{0,j}(\lambda,\xi).$$

Then

- $\Phi_{0,j}\in\mathcal{O}(U_s\times\mathbb{C}_*)$,
- for any $\lambda\in U_s$ the function $\Phi_{0,j}(\lambda,\cdot)$ has a pole of order $\le j$ at $\xi=0$,
- for any $\xi\in\mathbb{C}_*$ the function $\Phi_{0,j}(\cdot,\xi)$ has a zero at $\lambda=0$,
- $\Phi_j\to\Phi$ uniformly on $U_t\times\partial E$, $1<t<s$,
- Φ_j is holomorphic on $U_t\times\partial E$, $1<t<s$, $j\gg 1$,
- for any $\lambda\in U_s$ the function $\Phi_j(\lambda,\cdot)$ has a pole of order $\le j$ at $\xi=0$, $j\gg 1$.

Suppose that $j\gg 1$ is such that $\sigma_{\nu,j}(\xi)\neq 0$, $\xi\in\partial E$. In particular, the set $Z_{\nu,j} := E_*\cap\sigma_{\nu,j}^{-1}(0)$ is finite. Observe that $\Phi_j\in\mathcal{O}(U_s\times(\overline{E}_*\setminus Z_j))$, where $Z_j := \bigcup_{\nu=1}^{N} Z_{\nu,j}$. Put $B_j := B_{1,j}\cdots B_{N,j}$, where $B_{\nu,j}$ denotes the Blaschke product for $Z_{\nu,j}$ with the zeros counted with multiplicities ([44]). For every $\xi\in\mathbb{C}_*\setminus Z_j$ with $|\sigma_{\nu,j}(\xi)|<s$, we get $\Phi_j(\sigma_{\nu,j}(\xi),\xi)=a_\nu$. For any $k\ge j$:

- the mapping $\Phi_{j,k}(\lambda,\xi) := \Phi_j(\lambda\xi^k B_j(\xi),\xi)$ is holomorphic on $\overline{E}\times\overline{E}$, and
- the mapping $\sigma_{\nu,j,k}(\xi) := \sigma_{\nu,j}(\xi)/(\xi^k B_j(\xi))$ is meromorphic in $\mathbb{C}_*$ and zero-free holomorphic in E_*.

Moreover, $\Phi_{j,k}(\sigma_{\nu,j,k}(\xi),\xi)=a_\nu$ for all $\xi\in(U_s)_*$ such that $|\sigma_{\nu,j,k}(\xi)\xi^k B_j(\xi)|<s$. Using the same method as in the proof of Lemma 1.10.9, we get the required result with

$$\Psi_j(\lambda,\xi) := \Phi_{j,k_j}(\lambda,\xi) = \Phi_j(\lambda\xi^{k_j}B_j(\xi),\xi),\qquad \tau_{\nu,j}(\xi) := \sigma_{\nu,j,k_j}(\xi) = \sigma_{\nu,j}(\xi)/(\xi^{k_j}B_j(\xi)),$$

where $k_j\ge j$ is sufficiently large (and $1<s_j<s$, $s_j\approx 1$). ■

Taking in Lemma 1.10.17 a $j\gg 1$ gives the following result.

LEMMA 1.10.18. *There exist* $1<t<s$, $\Psi\in\mathcal{O}(U_t\times U_t,G)$, $\tau_\nu\in\mathcal{O}(U_t\setminus\overline{U}_{1/t},\mathbb{C}_*)$, $\nu=1,\dots,N$, *such that*:

(i) $\Psi(0,\xi)=\varphi_0(\xi)$, $\xi\in U_t$,
(ii) $|\tau_\nu(\xi)|<1$, $\xi\in J_\nu$,
(iii) $\Psi(\tau_\nu(\xi),\xi)=a_\nu$, $\xi\in(U_t)_*$ *with* $|\tau_\nu(\xi)|<t$, $\nu=1,\dots,N$,
(iv) $2\pi N\max_{\nu=1,\dots,N}\{\boldsymbol{p}(a_\nu)\max_{\partial E}\log|\tau_\nu|\}<\varepsilon/2$,
(v) $\sum_{\nu=1}^{N}\boldsymbol{p}(a_\nu)\int_0^{2\pi}\log|\tau_\nu(e^{it})|\,dt\le\sum_{\nu=1}^{N}\boldsymbol{p}(a_\nu)\int_0^{2\pi}\log|\sigma_\nu(e^{it})|\,dt+\varepsilon$.

LEMMA 1.10.19. *There exist* $\eta_0\in\partial E$, k, $c>0$, *and* $0<\varrho<1$ *such that the functions*

$$f(\xi) := \Psi(\eta_0\xi^k,\xi),\qquad F(\lambda,\eta) := \eta\,\frac{\varrho\lambda+e^{-c/k}}{1+e^{-c/k}\varrho\lambda}$$

satisfy

$$\int_0^{2\pi}\Xi_{\mathrm{Lel}}^{\boldsymbol{p}}(f(F(\cdot,e^{it})))\,dt\le\sum_{\nu=1}^{N}\boldsymbol{p}(a_\nu)\int_0^{2\pi}\log|\tau_\nu(e^{it})|\,dt+\varepsilon.$$

([44]) That is, the function $\sigma_{\nu,j}/B_{\nu,j}$ extends to a zero-free holomorphic function on E_*.

Proof. Since $\tau_\nu(\xi) \neq 0$, $\xi \in U_t \setminus \overline{U}_{1/t}$, there exists $c \gg 1$ such that

$$\log\left|\frac{\eta e^{-c} - \tau_\nu(\xi)}{1 - \overline{\tau_\nu(\xi)}\eta e^{-c}}\right| < \log|\tau_\nu(\xi)| + \frac{\varepsilon}{2M}, \qquad \eta \in \overline{E},\ \xi \in \partial E,\ \nu = 1, \dots, N, \tag{1.10.20}$$

where $M := \sum_{\nu=1}^N \boldsymbol{p}(a_\nu)$. Let

$$\psi(\lambda) := \exp\left(c\,\frac{\lambda - 1}{\lambda + 1}\right), \qquad \lambda \in \mathbb{C} \setminus \{-1\};$$

observe that $\psi(E) = E_*$, $\psi(\partial E \setminus \{-1\}) = \partial E$. Define

$$\varphi_\nu(\lambda; \eta, \xi) := \frac{\eta\psi(\lambda) - \tau_\nu(\xi)}{1 - \overline{\tau_\nu(\xi)}\eta\psi(\lambda)}, \qquad (\lambda, \eta, \xi) \in (\mathbb{C} \setminus \{-1\}) \times \partial E \times J_\nu;$$

we have $|\varphi_\nu(\lambda; \eta, \xi)| = 1$, $(\lambda, \eta, \xi) \in (\partial E \setminus \{-1\}) \times \partial E \times J_\nu$. Moreover, $\varphi_\nu(t; \eta, \xi) \to -\tau_\nu(\xi)$ when $t \to -1^-$. Thus $\varphi_\nu(\cdot; \eta, \xi)$ is an inner function with non-zero radial limits. Consequently, by Proposition 1.12.2, $\varphi_\nu(\cdot; \eta, \xi)$ is a Blaschke product. Moreover, since $\psi'(\lambda) \neq 0$, the zeros of $\varphi_\nu(\cdot; \eta, \xi)$ are simple and, by the implicit mapping theorem, for any point $(\lambda_0, \eta_0, \xi_0)$ with $\varphi_\nu(\lambda_0; \eta_0, \xi_0) = 0$, there exists a holomorphic function $h = h_{\lambda_0, \eta_0, \xi_0}$ defined in a neighborhood V_0 of (η_0, ξ_0) such that $h(\eta_0, \xi_0) = \lambda_0$ and $\varphi_\nu(h(\eta, \xi); \eta, \xi) = 0$, $(\eta, \xi) \in V_0$. Observe that

$$h(\eta, \xi) = \frac{1}{2\pi i} \int\limits_{\partial\mathbb{B}(\lambda_0, r)} \frac{\lambda\eta\psi'(\lambda)}{\eta\psi(\lambda) - \tau_\nu(\xi)}\, d\lambda,$$

where $\mathbb{B}(\lambda_0, r)$ is so small that $\lambda = \lambda_0$ is the only zero of $\varphi_\nu(\cdot; \eta_0, \xi_0)$ in $\overline{\mathbb{B}}(\lambda_0, r)$. Let $(\lambda_{\nu,l})_{l=1}^\infty$ be the zeros of $\varphi_\nu(\cdot; \eta_0, \xi_0)$ in E_*. Since $\varphi_\nu(\cdot; \eta_0, \xi_0)$ is a Blaschke product, we get

$$|\varphi_\nu(0; \eta_0, \xi_0)| = \left|\frac{\eta e^{-c} - \tau_\nu(\xi)}{1 - \overline{\tau_\nu(\xi)}\eta e^{-c}}\right| = \prod_{l=1}^\infty |\lambda_{\nu,l}|.$$

Hence, using (1.10.20), we conclude that there exist $L \in \mathbb{N}$ and $\varrho > 1$ such that

$$\sum_{l=1}^L \log\frac{|\lambda_{\nu,l}|}{\varrho} < \log|\tau_\nu(\xi_0)| + \frac{\varepsilon}{2M}.$$

Consequently,

$$\sum_{l=1}^L \log\frac{|h_{\lambda_{\nu,l}, \eta_0, \xi_0}(\eta, \xi)|}{\varrho} < \log|\tau_\nu(\xi)| + \frac{\varepsilon}{2M}$$

for (η, ξ) in a neighborhood of (η_0, ξ_0).

Using a compactness argument we see that there exist $L \in \mathbb{N}$ and $\varrho > 1$ such that for any point $(\eta, \xi) \in \partial E \times J_\nu$ there exist $\lambda_{\nu,1}(\eta, \xi), \dots, \lambda_{\nu,L}(\eta, \xi)$ such that

$$\varphi_\nu(\lambda_{\nu,l}(\eta, \xi); \eta, \xi) = 0, \qquad l = 1, \dots, L,$$

and

$$\sum_{l=1}^L \log\frac{|\lambda_{\nu,l}(\eta, \xi)|}{\varrho} < \log|\tau_\nu(\xi)| + \frac{\varepsilon}{2M}.$$

Let
$$\psi_k(\lambda) := \frac{\lambda + e^{-c/k}}{1 + e^{-c/k}\lambda} = 1 + (1 - e^{-c/k})\,\frac{\lambda - 1}{1 + e^{-c/k}\lambda}, \qquad \lambda \in \mathbb{C} \setminus \{-e^{c/k}\}.$$
Observe that $\psi_k \to 1$ locally uniformly in E and
$$k \operatorname{Log} \psi_k(\lambda) \to c\,\frac{\lambda - 1}{\lambda + 1} \text{ (45)}$$
locally uniformly in E. Consequently, $\psi_k^k \to \psi$ locally uniformly in E.

Fix $1 < t_0 < 1/\varrho$ and let V_ν be a neighborhood of J_ν such that $|\tau_\nu(\xi)| < 1$, $\xi \in V_\nu$. Let $k_0 \in \mathbb{N}$ be such that $\xi\psi_k(\varrho\lambda) \in V_\nu$, $(\lambda, \xi) \in U_{t_0} \times J_\nu$, $k \geq k_0$. Hence, by (iii) of Lemma 1.10.18, we get
$$\varPsi(\tau_\nu(\xi\psi_k(\varrho\lambda)), \xi\psi_k(\varrho\lambda)) = a_\nu, \qquad (\lambda, \xi) \in U_{t_0} \times J_\nu,\ k \geq k_0.$$
Recall that
$$\eta\psi_k^k(\varrho\lambda) - \tau_\nu(\xi\psi_k(\varrho\lambda)) \to \eta\psi(\varrho\lambda) - \tau_\nu(\xi)$$
uniformly with respect to $(\lambda, \eta, \xi) \in U_{t_0} \times \partial E \times J_\nu$. Hence, by the Hurwitz theorem, for $k \gg 1$, there are zeros $\lambda_{\nu,l,k}(\eta, \xi)$ of the function $\lambda \mapsto \eta\psi_k^k(\varrho\lambda) - \tau_\nu(\xi\psi_k(\varrho\lambda))$ which are so close to $\lambda_{\nu,l}(\eta, \xi)$ that
$$\sum_{l=1}^{L} \log|\lambda_{\nu,l,k}(\eta, \xi)| < \log|\tau_\nu(\xi)| + \frac{\varepsilon}{2M}, \qquad (\eta, \xi) \in \partial E \times J_\nu.$$
Observe that
$$\varPsi(\eta\psi_k^k(\varrho\lambda_{\nu,l,k}(\eta, \xi)), \xi\psi_k(\varrho\lambda_{\nu,l,k}(\eta, \xi))) = a_\nu, \qquad (\eta, \xi) \in \partial E \times J_\nu,\ k \gg 1.$$
Hence
$$\varXi_{\mathrm{Lel}}^{\boldsymbol{p}}(\lambda \mapsto \varPsi(\eta\psi_k^k(\varrho\lambda), \xi\psi_k(\varrho\lambda))) < \sum_{\nu=1}^{N} \boldsymbol{p}(a_\nu)\log|\tau_\nu(\xi)| + \frac{\varepsilon}{2},$$
$$(\eta, \xi) \in Q := \bigcup_{\nu=1}^{N} (\partial E \times J_\nu).$$
Consider the diffeomorphism $H : (\partial E)^2 \to (\partial E)^2$ given by $H(\eta, \xi) := (\eta\xi^{-k}, \xi)$. Let $S := H(Q)$. Then $\varLambda(S) = \varLambda(Q) \geq 2\pi(2\pi - \varepsilon)$ (because the modulus of the Jacobian of H is equal to 1). Consequently, there exists an $\eta_0 \in \partial E$ such that $\varLambda(R) \geq 2\pi - \varepsilon$, where $R := \{\xi \in \partial E : (\eta_0, \xi) \in S\}$. We have
$$\varXi_{\mathrm{Lel}}^{\boldsymbol{p}}(\lambda \mapsto \varPsi(\eta_0(\xi\psi_k(\varrho\lambda))^k, \xi\psi_k(\varrho\lambda))) \leq \sum_{\nu=1}^{N} \boldsymbol{p}(a_\nu)\log|\tau_\nu(\xi)| + \frac{\varepsilon}{2}, \qquad \xi \in R.$$
Finally, by Lemma 1.10.18, we conclude that
$$\int_0^{2\pi} \varXi_{\mathrm{Lel}}^{\boldsymbol{p}}(\lambda \mapsto \varPsi(\eta_0(e^{it}\psi_k(\varrho\lambda))^k, e^{it}\psi_k(\varrho\lambda)))\,dt \leq \sum_{\nu=1}^{N} \boldsymbol{p}(a_\nu) \int_0^{2\pi} \log|\tau_\nu(e^{it})|\,dt + \varepsilon,$$
which implies directly the required result. ■

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(45) $\lim_{k\to+\infty} k \operatorname{Log}\psi_k(\lambda) = \lim_{k\to+\infty} k(1 - e^{-c/k})\,\frac{\lambda - 1}{1 + e^{-c/k}\lambda} = c\,\frac{\lambda - 1}{\lambda + 1}.$

LEMMA 1.10.20. *There exists a* $\theta_0 \in \mathbb{R}$ *such that the mapping*

$$\varphi(\xi) := f(F(e^{i\theta_0}\xi, \xi))$$

satisfies

$$\Xi^{\boldsymbol{p}}_{\mathrm{Lel}}(\varphi) \le \frac{1}{2\pi}\int_0^{2\pi} \Xi^{\boldsymbol{p}}_{\mathrm{Lel}}(f(F(\cdot, e^{it})))\, dt. \tag{1.10.21}$$

Proof. First we will prove that for any $\varphi \in \mathcal{O}(\overline{E}, G)$ we have

$$\Xi^{\boldsymbol{p}}_{\mathrm{Lel}}(\varphi) = \int_E (\log|\lambda|)\Delta v_\varphi(\lambda)\, d\Lambda_2(\lambda), \tag{1.10.22}$$

where

$$v_\varphi(\lambda) := \frac{1}{2\pi}\sum_{b\in B_\varphi} \boldsymbol{p}(\varphi(b))\,\mathrm{ord}_b(\varphi - \varphi(b))\log m_E(b,\lambda), \qquad \lambda \in U_r,$$

$$B_\varphi := \{b \in E_* : \boldsymbol{p}(\varphi(b)) > 0\} = E_* \cap \varphi^{-1}(|\boldsymbol{p}|)$$

(for some $r > 1$). Observe that B_φ is discrete and $v_\varphi \in \mathcal{SH}(U_r)$. To prove (1.10.22) we use the Riesz representation formula:

$$\begin{aligned}\int_E (\log|\lambda|)\Delta v_\varphi(\lambda) d\Lambda_2(\lambda) &= 2\pi v_\varphi(0) - \int_0^{2\pi} v_\varphi(e^{i\theta})\, d\theta \\ &= \sum_{b\in B_\varphi} \boldsymbol{p}(\varphi(b))\,\mathrm{ord}_b(\varphi - \varphi(b))\log|b| = \Xi^{\boldsymbol{p}}_{\mathrm{Lel}}(\varphi).\end{aligned}$$

Next we are going to show that for any function $h \in \mathcal{O}(\overline{E})$ with $h(\overline{E}) \subset \overline{E}$ we have

$$\Delta v_{\varphi\circ h} = \Delta(v_\varphi \circ h) \text{ in } E, \tag{1.10.23}$$

with $\Delta_{-\infty} := 0$. If $\varphi \equiv \mathrm{const}$ or $h \equiv \mathrm{const}$ or $h(E) \cap B_\varphi = \emptyset$, then (1.10.23) is obviously true. Assume that $\varphi \not\equiv \mathrm{const}$ and $h \not\equiv \mathrm{const}$ and $h(E) \cap B_\varphi \neq \emptyset$. It is clear that $v_{\varphi\circ h}$ and $v_\varphi \circ h$ are harmonic on $E \setminus h^{-1}(B_\varphi)$ and, consequently, $\Delta v_{\varphi\circ h} = \Delta(v_\varphi \circ h) = 0$ on $E \setminus h^{-1}(B_\varphi)$.

Take $b \in B_\varphi$ and $c \in h^{-1}(b)$. Write $h(\lambda) = (\lambda - c)^m g(\lambda)$, where $g \in \mathcal{O}(\overline{E})$ and $g(c) \neq 0$. Then

$$v_\varphi \circ h(\lambda) = \frac{1}{2\pi}\boldsymbol{p}(\varphi(b))\,\mathrm{ord}_b(\varphi - \varphi(b)) m \log|\lambda - c| + u(\lambda),$$

where u is harmonic near c. Thus

$$\begin{aligned}\Delta(v_\varphi \circ h) &= \boldsymbol{p}(\varphi(b))\,\mathrm{ord}_b(\varphi - \varphi(b)) m\delta_c \\ &= \boldsymbol{p}(\varphi(h(c)))\,\mathrm{ord}_c(\varphi\circ h - \varphi\circ h(c))\delta_c = \Delta v_{\varphi\circ h}\end{aligned}$$

in a neighborhood of c. Applying (1.10.23) to $\varphi := f$ and $h := F(\cdot, \xi)$, we get

$$\begin{aligned}\Xi^{\boldsymbol{p}}_{\mathrm{Lel}}(f(F(\cdot,\xi))) &= \int_E (\log|\lambda|)\Delta v_{f\circ F(\cdot,\xi)}(\lambda)\, d\Lambda_2(\lambda) \\ &= \int_E (\log|\lambda|)\Delta_\lambda(v_f \circ F(\lambda,\xi))\, d\Lambda_2(\lambda).\end{aligned} \tag{1.10.24}$$

Now we need the following auxiliary result.

LEMMA 1.10.21. *Let $w \in \mathcal{PSH}(U_r \times U_r)$ $(r > 1)$ and let $w_\theta(\xi) := w(e^{i\theta}\xi, \xi)$. Then there exists a $\theta_0 \in \mathbb{R}$ such that*

$$\int_E (\log|\lambda|)\Delta w_{\theta_0}(\lambda)\,d\Lambda_2(\lambda) \le \frac{1}{2\pi}\int_0^{2\pi}\Big(\int_E (\log|\lambda|)\Delta_\lambda w(\lambda, e^{i\theta})\,d\Lambda_2(\lambda)\Big)\,d\theta.$$

Proof. The Riesz representation formula gives

$$w(0,0) = w_\theta(0) = \frac{1}{2\pi}\int_E (\log|\lambda|)\Delta w_\theta(\lambda)\,d\Lambda_2(\lambda) + \frac{1}{2\pi}\int_0^{2\pi} w(e^{i(\theta+t)}, e^{it})\,dt.$$

Hence

$$2\pi w(0,0) = \frac{1}{2\pi}\int_0^{2\pi}\Big(\int_E (\log|\lambda|)\Delta w_\theta(\lambda)\,d\Lambda_2(\lambda)\Big)\,d\theta + \frac{1}{2\pi}\int_0^{2\pi}\Big(\int_0^{2\pi} w(e^{i(\theta+t)}, e^{it})\,dt\Big)\,d\theta.$$

On the other hand, using the Riesz representation formula for the functions $w(0,\cdot)$ and $w(\cdot, e^{i\theta})$, we get

$$\begin{aligned}2\pi w(0,0) &= \int_E (\log|\lambda|)\Delta_\lambda w(0,\lambda)\,d\Lambda_2(\lambda) + \int_0^{2\pi} w(0, e^{i\theta})\,d\theta \le \int_0^{2\pi} w(0, e^{i\theta})\,d\theta\\ &= \int_0^{2\pi}\Big(\frac{1}{2\pi}\int_E (\log|\lambda|)\Delta_\lambda w(\lambda, e^{i\theta})\,d\Lambda_2(\lambda) + \frac{1}{2\pi}\int_0^{2\pi} w(e^{it}, e^{i\theta})\,dt\Big)\,d\theta.\end{aligned}$$

Consequently,

$$\int_0^{2\pi}\Big(\int_E (\log|\lambda|)\Delta w_\theta(\lambda)\,d\Lambda_2(\lambda)\Big)\,d\theta \le \int_0^{2\pi}\Big(\int_E (\log|\lambda|)\Delta_\lambda w(0,\lambda)\,d\Lambda_2(\lambda)\Big)\,d\theta,$$

which implies the required result. ∎

Applying Lemma 1.10.21 to (1.10.24) gives (1.10.21). ∎

This finishes the proof of Theorem 1.10.15. ∎

REMARK 1.10.22. (a) There is a counterpart of the Poletsky formula from Theorem 1.10.15 for the Azukawa pseudometric A_G (cf. §1.2). Recently, N. Nikolov and W. Zwonek [Nik-Zwo 2004a, Theorem 1] proved that for any domain $G \subset \mathbb{C}^n$ we have

$$A_D = \Gamma_G = \Gamma_G^1,$$

where

$$\begin{aligned}\Gamma_G(a;X) &:= \inf\{L_\varphi(a)/|t| : \varphi \in \mathcal{O}(E,G),\ \varphi(0) = a,\ \varphi^{(k)}(0) = k!tX,\ k := \operatorname{ord}_0(\varphi - a)\},\\ \Gamma_G^1(a;X) &:= \inf\{L_\varphi(a)/|t| : \varphi \in \mathcal{O}(E,G),\ \varphi(0) = a,\ \varphi'(0) = tX,\ \operatorname{ord}_0(\varphi - a) = 1\},\\ L_\varphi(a) &:= \prod_{\lambda\in\varphi^{-1}(a)\cap E_*} |\lambda|^{\operatorname{ord}_\lambda(\varphi - a)} = \exp(\Xi_{\mathrm{Lel}}^{\chi\{a\}}(\varphi)),\ ({}^{46}) \qquad (a,X) \in G \times \mathbb{C}^n.\end{aligned}$$

(b) Let $G \subset \mathbb{C}^n$ be a domain and let $a, z_0 \in G$, $z_0 \ne a$, $X_0 \in \mathbb{C}^n_*$. Following [Nik-Zwo 2004a], we say that a mapping $\varphi \in \mathcal{O}(E,G)$ is *g_G-extremal for (a, z_0)*

(46) $\prod_{\lambda\in\emptyset}\cdots := 1$. See Remark 1.10.3(f) for the definition of $\Xi_{\mathrm{Lel}}^{\chi\{a\}}(\varphi)$.

(resp. A_G*-extremal for* (a, X_0)) if $a, z_0 \in \varphi(E)$, $\varphi(0) = z_0$, and $\log g_G(a, z_0) = \Xi_{\mathrm{Lel}}^{\chi_{\{a\}}}(\varphi)$ (resp. $\varphi(0) = a$, $\varphi^{(k)}(0) = tk!X_0$, $k := \mathrm{ord}_0(\varphi - a)$, and $A_G(a; X_0) = L_\varphi(a)/|t|$) up to an automorphism of E (i.e. we are allowed to replace φ by $\varphi \circ h$, where $h \in \mathrm{Aut}(E)$). It follows from Proposition 3 in [Nik-Zwo 2004a] that for $\varphi \in \mathcal{O}(E, G)$, $\varphi \not\equiv \mathrm{const}$, $a \in \varphi(E)$, the following conditions are equivalent:

(i) φ is g_G-extremal for a pair (a, z_0) with $a \neq z_0 \in \varphi(E)$;

(ii) φ is g_G-extremal for any pair (a, z) with $a \neq z \in \varphi(E)$;

(iii) φ is A_G-extremal for any pair $(a, \varphi^{(k)}(\lambda))$ with $\lambda \in \varphi^{-1}(a)$, $k := \mathrm{ord}_\lambda(\varphi - a)$.

Moreover, if $G \subset \mathbb{C}$ is such that ∂G is not polar, then a mapping $\varphi \in \mathcal{O}(E, G)$, $a \in \varphi(E)$, $\varphi \not\equiv \mathrm{const}$, is g_G-extremal for some (a, z_0) $(a \neq z_0 \in \varphi(E))$ iff $\varphi = \pi \circ \psi$, where $\pi : E \to G$ is a universal covering, $\psi \in \mathcal{O}(E, E)$, and the function $h_\lambda \circ \psi = (\psi - \lambda)/(1 - \overline{\lambda}\psi)$ is a Blaschke product for any $\lambda \in \pi^{-1}(a)$.

1.11. Coman conjecture

Definition 1.11.1. Let G be a domain in $\mathbb{C}^n$ and let $\boldsymbol{p} : G \to \mathbb{R}_+$. Define the *Coman function*

$$\delta_G(\boldsymbol{p}, z) := \inf\Big\{ \prod_{a\in|\boldsymbol{p}|} |\mu_a|^{\boldsymbol{p}(a)} : (\mu_a)_{a\in|\boldsymbol{p}|} \subset E,$$
$$\exists_{\varphi\in\mathcal{O}(E,G)} : \varphi(0) = z,\ \varphi(\mu_a) = a,\ a \in |\boldsymbol{p}|\Big\}, \qquad z \in G;$$

we put $\delta_G(\boldsymbol{p}, z) := 1$ if the defining family is empty. We put $\delta_G(A, \cdot) := \delta_G(\chi_A, \cdot)$ $(A \subset G)$, $\delta_G(a, \cdot) := \delta_G(\{a\}, \cdot)$ $(a \in G)$.

Remark 1.11.2. (a) Directly from Proposition 1.10.13 it follows that $g_G(\boldsymbol{p}, \cdot) \leq \delta_G(\boldsymbol{p}, \cdot)$.

(b) Obviously, $\delta_G(a, \cdot) = \widetilde{k}_G^*(a, \cdot)$ $(a \in G)$.

(c) $\prod_{a\in|\boldsymbol{p}|}[m_E(a, \cdot)]^{\boldsymbol{p}(a)} = g_E(\boldsymbol{p}, \cdot) = \delta_E(\boldsymbol{p}, \cdot)$ (for any $\boldsymbol{p} : E \to \mathbb{R}_+$). Indeed, we only need to prove that $\delta_E(\boldsymbol{p}, \cdot) \leq \prod_{a\in|\boldsymbol{p}|}[m_E(a, \cdot)]^{\boldsymbol{p}(a)}$. Fix a $z_0 \in E$ and let $\varphi := h_{-z_0}$, where $h_a(z) := (z - a)/(1 - \overline{a}z)$ $(a, z \in E)$. Let $\mu_a := \varphi^{-1}(a)$, $a \in |\boldsymbol{p}|$. Then $\varphi(0) = z_0$, $\varphi(\mu_a) = a$, $a \in |\boldsymbol{p}|$, and $\prod_{a\in|\boldsymbol{p}|} |\mu_a|^{\boldsymbol{p}(a)} = \prod_{a\in|\boldsymbol{p}|}[m_E(\mu_a, 0)]^{\boldsymbol{p}(a)} = \prod_{a\in|\boldsymbol{p}|}[m_E(a, z_0)]^{\boldsymbol{p}(a)}$.

(d) Let $F : G \to D$ be a holomorphic mapping and let $\boldsymbol{q} : D \to \mathbb{R}_+$ be such that $\#F^{-1}(b) = 1$ for any $b \in |\boldsymbol{q}|$ (e.g. F is bijective). Then

$$\delta_D(\boldsymbol{q}, F(z)) \leq \delta_G(\boldsymbol{q} \circ F, z), \qquad z \in G.$$

Indeed,

$$\delta_D(\boldsymbol{q}, F(z)) = \inf\Big\{ \prod_{b\in|\boldsymbol{q}|} |\mu_b|^{\boldsymbol{q}(b)} : \exists_{\psi\in\mathcal{O}(E,D)} : \psi(0) = F(z),\ \psi(\mu_b) = b,\ b \in |\boldsymbol{q}|\Big\}$$
$$\leq \inf\Big\{ \prod_{a\in F^{-1}(|\boldsymbol{q}|)} |\mu_a|^{\boldsymbol{q}(F(a))} : \exists_{\varphi\in\mathcal{O}(E,G)} : \varphi(0) = z,\ \varphi(\mu_a) = a,\ a \in F^{-1}(|\boldsymbol{q}|)\Big\}.$$

The *Coman conjecture* says that $g_G(\boldsymbol{p}, \cdot) \equiv \delta_G(\boldsymbol{p}, \cdot)$ for any convex bounded domain G and function $\boldsymbol{p}$ with $\#|\boldsymbol{p}| < +\infty$ (cf. [Com 2000]). The conjecture was motivated by the Lempert theorem and Remark 1.11.2(b).

D. Coman proved that his conjecture is true in the case where $G = \mathbb{B}_2$ is the unit ball in $\mathbb{C}^2$, $|\boldsymbol{p}| = \{a_1, a_2\}$, and $\boldsymbol{p}(a_1) = \boldsymbol{p}(a_2)$ (cf. [Com 2000]).

EXAMPLE 1.11.3. The first counterexample was given by M. Carlehed and J. Wiegerinck in [Car-Wie 2003]: $G = E^2 \subset \mathbb{C}^2$, $|\boldsymbol{p}| = \{a_1, a_2\} \subset E \times \{0\}$, $\boldsymbol{p}(a_1) \neq \boldsymbol{p}(a_2)$.

Let $c_1, c_2, d \in E_*$, $c_1 \neq c_2$, $|c_1 c_2| < |d| < |c_1|$, $\boldsymbol{p}_{2,1} := 2\chi_{(c_1,0)} + \chi_{(c_2,0)}$. Then

$$g_{E^2}(\boldsymbol{p}_{2,1}, (0,d)) < \delta_{E^2}(\boldsymbol{p}_{2,1}, (0,d)).$$

Indeed, by Example 1.7.17,

$$g_{E^2}(\boldsymbol{p}_{2,1}, z) = [\max\{m_E(c_1, z_1), |z_2|\}][\max\{m_E(c_1, z_1) m_E(c_2, z_1), |z_2|\}].$$

Hence, by Example 1.7.2, if $\boldsymbol{p}_{1,1} := \chi_{(c_1,0)} + \chi_{(c_2,0)}$, then

$$\begin{aligned}
g_{E^2}(\boldsymbol{p}_{2,1}, z) &= [g_{E^2}((c_1,0), z)][g_{E^2}(\boldsymbol{p}_{1,1}, z)] \\
&\leq [\delta_{E^2}((c_1,0), z)][\delta_{E^2}(\boldsymbol{p}_{1,1}, z)] \\
&= [\inf\{|\lambda| : \exists_{\varphi \in \mathcal{O}(E,E^2)} : \varphi(0) = z,\ \varphi(\lambda) = (c_1, 0)\}] \\
&\qquad \times [\inf\{|\lambda_1 \lambda_2| : \exists_{\varphi \in \mathcal{O}(E,E^2)} : \varphi(0) = z,\ \varphi(\lambda_1) = (c_1,0),\ \varphi(\lambda_2) = (c_2,0)\}] \\
&\leq \inf\{|\lambda_1^2 \lambda_2| : \exists_{\varphi \in \mathcal{O}(E,E^2)} : \varphi(0) = z,\ \varphi(\lambda_1) = (c_1,0),\ \varphi(\lambda_2) = (c_2,0)\} \\
&= \delta_{E^2}(\boldsymbol{p}_{2,1}, z).
\end{aligned}$$

Suppose that $g_{E^2}(\boldsymbol{p}_{2,1}, (0,d)) = \delta_{E^2}(\boldsymbol{p}_{2,1}, (0,d))$. Then there exist $\varphi_\nu \in \mathcal{O}(E, E^2)$ and $\lambda_{\nu,1}, \lambda_{\nu,2} \in E_*$ such that $\varphi_\nu(0) = (0,d)$, $\varphi_\nu(\lambda_{\nu,1}) = (c_1, 0)$, $\varphi_\nu(\lambda_{\nu,2}) = (c_2, 0)$, and

$$|\lambda_{\nu,1}^2 \lambda_{\nu,2}| \to [\max\{m_E(c_1,0),\ |d|\}][\max\{m_E(c_1,0) m_E(c_2,0),\ |d|\}] = |c_1 d|.$$

Using a Montel argument we easily conclude that there are the following three situations:

(a) There exist $\psi_1, \psi_2 \in \mathcal{O}(E,E)$ and $\zeta_1, \zeta_2 \in E$ such that $\psi_1(0) = 0$, $\psi_2(0) = d$, $\psi_1(\zeta_1) = c_1$, $\psi_2(\zeta_1) = 0$, $\psi_1(\zeta_2) = c_2$, $\psi_2(\zeta_2) = 0$, and $|\zeta_1^2 \zeta_2| = |c_1 d|$. Then, by the Schwarz lemma,

$$|\psi_1(\lambda)| \leq |\lambda|, \qquad |\psi_2(\lambda)| \leq \left| \frac{\lambda - \zeta_1}{1 - \overline{\zeta}_1 \lambda} \cdot \frac{\lambda - \zeta_2}{1 - \overline{\zeta}_2 \lambda} \right|, \qquad \lambda \in E.$$

Hence $|c_1| \leq |\zeta_1|$, $|d| \leq |\zeta_1 \zeta_2|$ and, consequently, $|c_1| = |\zeta_1|$ and $|d| = |\zeta_1 \zeta_2|$. Thus $|\psi_1(\lambda)| \equiv |\lambda|$. It follows that $|c_2| = |\zeta_2|$ and $|d| = |\zeta_1 \zeta_2| = |c_1 c_2|$, a contradiction.

(b) There exist $\psi_1, \psi_2 \in \mathcal{O}(E,E)$ and $\zeta_1 \in E$ such that $\psi_1(0) = 0$, $\psi_2(0) = d$, $\psi_1(\zeta_1) = c_1$, $\psi_2(\zeta_1) = 0$, and $|\zeta_1^2| = |c_1 d|$. Then, by the Schwarz lemma,

$$|\psi_1(\lambda)| \leq |\lambda|, \qquad |\psi_2(\lambda)| \leq \left| \frac{\lambda - \zeta_1}{1 - \overline{\zeta}_1 \lambda} \right|, \qquad \lambda \in E.$$

Hence $|c_1| \leq |\zeta_1|$, $|d| \leq |\zeta_1|$ and, consequently, $|c_1| = |\zeta_1| = |d|$, a contradiction.

(c) There exist $\psi_1, \psi_2 \in \mathcal{O}(E,E)$ and $\zeta_2 \in E$ such that $\psi_1(0) = 0$, $\psi_2(0) = d$, $\psi_1(\zeta_2) = c_2$, $\psi_2(\zeta_2) = 0$, and $|\zeta_2| = |c_1 d|$. Then, by the Schwarz lemma,

$$|\psi_2(\lambda)| \leq \left| \frac{\lambda - \zeta_2}{1 - \overline{\zeta}_2 \lambda} \right|, \qquad \lambda \in E.$$

Hence $|c_1 d| = |\zeta_2| = |\psi_2(0)| \geq |d|$, a contradiction.

EXAMPLE 1.11.4. Recently, P. J. Thomas and N. V. Trao [Tho-Tra 2003] (see also [Die-Tra 2003]) found a counterexample with $G = E^2$, $\boldsymbol{p} = \chi_{B\times C}$, $\#B = \#C = 2$.

Let $a \in (0,1)$, $a^{3/2} < \gamma < a$, $A_\varepsilon := \{-a, a\} \times \{-\varepsilon, \varepsilon\}$, $\varepsilon \in (0,1)$. Then there exists a small $\varepsilon > 0$ such that $g_{E^2}(A_\varepsilon, (0,\gamma)) < \delta_{E^2}(A_\varepsilon, (0,\gamma))$. Indeed, first recall that

$$g_{E^2}(A_\varepsilon, (0,\gamma)) = \max\left\{a^2, \left|\frac{\gamma+\varepsilon}{1+\varepsilon\gamma}\frac{\gamma-\varepsilon}{1-\varepsilon\gamma}\right|\right\}$$

(cf. Example 1.7.2). Suppose that there exists a sequence $\varepsilon_k \searrow 0$ such that

$$g_{E^2}(A_{\varepsilon_k}, (0,\gamma)) = \delta_{E^2}(A_{\varepsilon_k}, (0,\gamma)), \qquad k = 1, 2, \ldots.$$

We may assume that $g_{E^2}(A_{\varepsilon_k}, (0,\gamma)) = a^2$, $k \in \mathbb{N}$. Let $\varphi_k : E \to E^2$ and $\xi^{(k)}_{\sigma,\tau} \in E$ $(\sigma, \tau \in \{-1, 1\})$ be such that

$$\varphi_k(0) = (0,\gamma), \qquad \varphi_k(\xi^{(k)}_{\sigma,\tau}) = (\sigma a, \tau\varepsilon_k), \qquad \prod_{\sigma,\tau\in\{-1,1\}} |\xi^{(k)}_{\sigma,\tau}| \to a^2.$$

By a Montel argument we may assume that $\varphi_k \to \varphi \in \mathcal{O}(E, E^2)$ locally uniformly in E and $\xi^{(k)}_{\sigma,\tau} \to \xi_{\sigma,\tau} \in \overline{E}$ with $\varphi(0) = (0,\gamma)$ and $\varphi(\xi_{\sigma,\tau}) = (\sigma a, 0)$ for $(\sigma,\tau) \in J$, where

$$J := \{(\sigma,\tau) \in \{-1,1\} : \xi_{\sigma,\tau} \in E\}.$$

Observe that

$$\prod_{(\sigma,\tau)\in J} |\xi_{\sigma,\tau}| = \prod_{(\sigma,\tau)\in\{-1,1\}} |\xi_{\sigma,\tau}| = a^2,$$

in particular, $J \neq \emptyset$.

It is clear that $\xi_{\sigma,\tau} \neq \xi_{\sigma',\tau'}$ for $(\sigma,\tau), (\sigma',\tau') \in J$ with $\sigma \neq \sigma'$. Put

$$T := \{\xi_{\sigma,\tau} : (\sigma,\tau) \in J\}.$$

Let $\varphi_k =: (f_k, g_k)$, $\varphi =: (f, g)$. We have $|g(z)| \le \prod_{\xi\in T} m_E(\xi, z)$, $z \in E$. In particular, $\gamma = |g(0)| \le \prod_{\xi\in T} |\xi|$. Consequently, if $\#T = \#J$, then $\gamma \le a^2 < a^{3/2}$, a contradiction. From now on assume that $\#T < \#J$. It suffices to consider the following four cases:

(a) $\#T = 1$, $\#J = 2$: $J = \{(-1,-1), (-1,1)\}$, $\xi_{-1,-1} = \xi_{-1,1} =: \xi_{-1}$.
(b) $\#T = 2$, $\#J = 3$: $J = \{(-1,-1), (-1,1), (1,-1)\}$, $\xi_{-1,1} = \xi_{-1,1} = \xi_{-1}$.
(c) $\#T = 3$, $\#J = 4$: $J = \{(-1,-1), (-1,1), (1,-1), (1,1)\}$, $\xi_{-1,-1} = \xi_{-1,1} =: \xi_{-1}$, $\xi_{1,-1} \neq \xi_{1,1}$.
(d) $\#T = 2$, $\#J = 4$: $J = \{(-1,-1), (-1,1), (1,-1), (1,1)\}$, $\xi_{-1,-1} = \xi_{-1,1} =: \xi_{-1}$, $\xi_{1,-1} = \xi_{1,1} =: \xi_1$.

Put $f_\sigma := h_{\sigma a} \circ f = (f - \sigma a)/(1 - \sigma a f)$.

If $(\sigma,\tau) \in J$, then $f_\sigma(\xi_{\sigma,\tau}) = 0$. Hence $|f_\sigma(z)| \le m_E(\xi_{\sigma,\tau}, z)$, $z \in E$. In particular, $a = |f_\sigma(0)| \le |\xi_{\sigma,\tau}|$.

If $(\sigma,-1), (\sigma,1) \in J$ and $\xi_{\sigma,-1} \neq \xi_{\sigma,1}$, then $|f_\sigma(z)| \le m_E(\xi_{\sigma,-1}, z) m_E(\xi_{\sigma,1}, z)$, $z \in E$. In particular, $a = |f_\sigma(0)| \le |\xi_{\sigma,-1}\xi_{\sigma,1}|$.

If $(\sigma,-1), (\sigma,1) \in J$ and $\xi_{\sigma,-1} = \xi_{\sigma,1} =: \xi_\sigma$, then $f'(\xi_\sigma) = 0$ (if $f'(\xi_\sigma) \neq 0$, then by the Hurwitz theorem, for large k, the equation $f_k(z) = \sigma a$ has exactly one solution in a neighborhood of ξ_σ, which is false since $f_k(\xi^{(k)}_{\sigma,\tau}) = \sigma a$, $\xi^{(k)}_{\sigma,-1} \neq \xi^{(k)}_{\sigma,1}$, and $\xi^{(k)}_{\sigma,\tau} \to \xi_\sigma$). We have $f_\sigma(\xi_\sigma) = f'_\sigma(\xi_\sigma) = 0$. Hence $|f_\sigma(z)| \le [m_E(\xi_\sigma, z)]^2$, $z \in E$. In particular, $a = |f_\sigma(0)| \le |\xi_\sigma|^2$.

Consequently:

In case (a) we get $a^2 = |\xi_{-1}|^2 \geq a$, which is a contradiction.

In case (b) we get $a^2 = |\xi_{-1}^2\xi_{1,-1}| \geq a \cdot a$. Hence $|\xi_{-1}^2| = |\xi_{1,-1}| = a$. Since $|g(z)| \leq m_E(\xi_{-1}, z)m_E(\xi_{1,-1}, z)$, $z \in E$, we have $\gamma = |g(0)| \leq |\xi_{-1}\xi_{1,-1}| = a^{1/2} \cdot a$, which is a contradiction.

In case (c) we get $a^2 = |\xi_{-1}^2\xi_{1,-1}\xi_{1,1}| \geq a \cdot a$. Hence $|\xi_{-1}^2| = |\xi_{1,-1}\xi_{1,1}| = a$. Since $|g(z)| \leq m_E(\xi_{-1}, z)m_E(\xi_{1,-1}, z)m_E(\xi_{1,1}, z)$, $z \in E$, we have $\gamma = |g(0)| \leq |\xi_{-1}\xi_{1,-1}\xi_{1,1}| = a^{1/2} \cdot a$, which is a contradiction.

In case (d) we get $a^2 = |\xi_{-1}^2\xi_1^2| \geq a \cdot a$. Hence $|\xi_{-1}^2| = |\xi_1^2| = a$ and, consequently, by the Schwarz lemma, $h_{\sigma a} \circ f = f_\sigma = \alpha_\sigma h_{\xi_\sigma}^2$, where $|\alpha_\sigma| = 1$, $\sigma \in \{-1, 1\}$, which implies that $f = h_a(\alpha_{-1}h_{\xi_{-1}}^2) = h_{-a}(\alpha_1 h_{\xi_1}^2)$. In particular, $-a = f(\xi_{-1}) = h_{-a}(\alpha_1 h_{\xi_1}^2(\xi_{-1}))$ and $a = f(\xi_1) = h_a(\alpha_{-1}h_{\xi_{-1}}^2(\xi_1)$. Then

$$\frac{2a}{1+a^2} = -\alpha_1 h_{\xi_1}^2(\xi_{-1}) = \alpha_{-1}h_{\xi_{-1}}^2(\xi_1).$$

Recall that $-\sigma a = f_\sigma(0) = \alpha_\sigma \xi_\sigma^2$. Hence

$$\frac{2}{1+a^2} = \frac{1}{\xi_1^2}\, h_{\xi_1}^2(\xi_{-1}) = \frac{1}{\xi_{-1}^2}\, h_{\xi_{-1}}^2(\xi_1).$$

Put $t := \xi_{-1}/\xi_1$. Note that $|t| = 1$ and $t \neq 1$. We have

$$\frac{2}{1+a^2} = \left(\frac{t-1}{1-at}\right)^2 = \left(\frac{1/t-1}{1-a/t}\right)^2,$$

which is a contradiction.

Example 1.11.5. Let D, A_t be as in Example 1.7.19. Taking $\varphi(\lambda) := (\lambda^2/4, \lambda/2)$, we easily see that $\delta_D(A_t, (0,0)) \leq 4t < t + \sqrt{t} = d_D^{\max}(A_t, (0,0))$, $0 < t \ll 1$.

$\boxed{?}$ We do not know whether $g_D(A_t, (0,0)) < \delta_D(A_t, (0,0))$ for small $t > 0$. $\boxed{?}$

1.12. Product property

1.12.1. Product property for relative extremal function

Theorem 1.12.1 ([Edi-Pol 1997], [Edi 2002]). *Let $G_j \subset \mathbb{C}^{n_j}$ be a domain, $A_j \subset G_j$, $j = 1, 2$. Assume that A_1, A_2 are open or A_1, A_2 are compact. Then*

$$\omega_{A_1\times A_2, G_1\times G_2}(z_1, z_2) = \max\{\omega_{A_1,G_1}(z_1),\, \omega_{A_2,G_2}(z_2)\}, \qquad (z_1, z_2) \in G_1 \times G_2.$$

Moreover, if G_1, G_2 are bounded, then for arbitrary subsets $A_1 \subset G_1$, $A_2 \subset G_2$ we have

$$\omega^*_{A_1\times A_2, G_1\times G_2}(z_1, z_2) = \max\{\omega^*_{A_1,G_1}(z_1),\, \omega^*_{A_2,G_2}(z_2)\}, \qquad (z_1, z_2) \in G_1 \times G_2.$$

We need a few auxiliary results.

Proposition 1.12.2 ([Nos 1960, Chapter III]). *Let $\varphi \in \mathcal{O}(E, E)$ be an inner function, $\varphi \not\equiv \text{const}$. Assume that φ is not a Blaschke product. Then there exists a $\zeta \in \partial E$ such that $\varphi^*(\zeta) = 0$* ([47]).

([47]) $\varphi^*(\zeta) := \lim_{r\to 1} \varphi(r\zeta)$.

PROPOSITION 1.12.3 ([Nos 1960, Chapter II]). *Let $\varphi \in \mathcal{H}^\infty(E)$ and let $A \subset \mathbb{C}$ be a compact polar set. Assume that there exists a set $I \subset \partial E$ of positive measure such that $\varphi^*(\zeta) \in A$, $\zeta \in I$. Then $\varphi \equiv \mathrm{const}$.*

LEMMA 1.12.4. *Let $A \subset E$ be a compact polar set and let $\pi : E \to E \setminus A$ be a universal covering. Then π is an inner function. Moreover, if $0 \notin A$, then π is a Blaschke product.*

Proof. Obviously $\pi^*(\zeta) \in A \cup \partial E$ for each $\zeta \in \partial E$ such that $\pi^*(\zeta)$ exists. Hence, by Proposition 1.12.3, we conclude that $\pi^*(\zeta) \in \partial E$ for almost all $\zeta \in \partial E$ and, consequently, π is an inner function. Now, if $0 \notin A$, then Proposition 1.12.2 implies that π is a Blaschke product. ∎

REMARK 1.12.5. Let B be a finite Blaschke product and let $\varphi \in \mathcal{O}(E, E)$. Then φ is an inner function iff $B \circ \varphi$ is inner.

LEMMA 1.12.6 (Löwner theorem, [Edi 2002]). *Let $\varphi \in \mathcal{O}(E, E)$ be an inner function such that $\varphi(0) = 0$. Then for any open set $I \subset \partial E$ we have $\Lambda((\varphi^*)^{-1}(I)) = \Lambda(I)$* ([48]).

Proof. We may assume that I is an arc. Put $J := (\varphi^*)^{-1}(I)$ (observe that J is measurable). Consider the following holomorphic functions:

$$\begin{aligned} u_I(z) &:= \frac{1}{2\pi}\int_0^{2\pi} P(z,\theta)\chi_I(e^{i\theta})\,d\theta, \\ u_J(z) &:= \frac{1}{2\pi}\int_0^{2\pi} P(z,\theta)\chi_J(e^{i\theta})\,d\theta, \qquad z \in E, \\ u &:= u_I \circ \varphi - u_J, \end{aligned}$$

where $P(z, \theta)$ denotes the Poisson kernel. Let A denote the set of all $\zeta \in \partial E$ such that:

- $u_J^*(\zeta)$ does not exist, or
- $u_J^*(\zeta)$ exists but $u_J^*(\zeta) \neq \chi_I(\zeta)$, or
- $\varphi^*(\zeta)$ does not exist, or
- $\varphi^*(\zeta)$ exists and $\varphi^*(\zeta) \in \partial I$ (here ∂I denotes the boundary of I in ∂E).

Note that A is of zero measure (use Proposition 1.12.3). Observe that $u^*(\zeta) = 0$ on $J \setminus A$. Moreover, $u^*(\zeta) \le 0$ on $(\partial E \setminus J) \setminus A$. Thus $u^* \le 0$ almost everywhere on ∂E and hence $u \le 0$. In particular, $u(0) = (1/2\pi)(\Lambda(I) - \Lambda(J)) \le 0$.

Applying the same argument to the arc $\partial E \setminus I$ shows that $\Lambda(\partial E \setminus I) \le \Lambda(\partial E \setminus J)$, which finishes the proof. ∎

LEMMA 1.12.7 ([Edi 2002]). *Let $(I_j)_{j=1}^k \subset \partial E$ be a family of disjoint open arcs, let $I := \bigcup_{j=1}^k I_j$, and let $\alpha := \Lambda(I)$. Then for every $\varepsilon > 0$ there exists a finite Blaschke product B such that:*

- $B(0) = 0$,
- $B'(z) \neq 0$ *for* $z \in B^{-1}(0)$,
- $B^{-1}(J_\varepsilon) \subset I$, *where* $J_\varepsilon = \{e^{i\theta} : 0 < \theta < \alpha - \varepsilon\}$.

([48]) Recall that Λ denotes the Lebesgue measure on ∂E.

Proof. We may assume that $\alpha < 2\pi$. Let $I_j = \{e^{i\theta} : \theta_{1,j} < \theta < \theta_{2,j}\}$, $j = 1, \dots, k$, $J_0 := \{e^{i\theta} : 0 < \theta < \alpha\}$. Define

$$B_0(z) = \frac{\prod_{j=1}^k (z - e^{i\theta_{2,j}}) - e^{i\alpha} \prod_{j=1}^k (z - e^{i\theta_{1,j}})}{\prod_{j=1}^k (z - e^{i\theta_{2,j}}) - \prod_{j=1}^k (z - e^{i\theta_{1,j}})}.$$

One can prove ([Edi 2002, the proof of Lemma 4.8]) that B_0 is a finite Blaschke product with $B_0(I) = J_0$, $B_0(\partial E \setminus I) \subset \partial E \setminus J_0$, and $B_0(\partial E \setminus \bar{I}) = \partial E \setminus \bar{J}_0$.

Suppose that

$$B_0(z) = e^{i\tau} \prod_{j=1}^N \left(\frac{z - a_j}{1 - \bar{a}_j z}\right)^{m_j}.$$

Take a closed arc $\widetilde{J}_0 \subset J_0$ such that $\Lambda(\widetilde{J}_0) \geq \alpha - \varepsilon$. Then for different points $a_{j,1}, \dots, a_{j,m_j}$, sufficiently close to a_j, such that $a_j \in \{a_{j,1}, \dots, a_{j,m_j}\}$, if

$$\widetilde{B}_0(z) = e^{i\tau} \prod_{j=1}^N \prod_{l=1}^{m_j} \left(\frac{z - a_{j,l}}{1 - \bar{a}_{j,l} z}\right),$$

then $\widetilde{B}_0(\partial E \setminus I) \subset \partial E \setminus \widetilde{J}_0$. Finally, we put $B(z) := \widetilde{B}(e^{i\theta} z)$ (with suitable θ). ∎

PROPOSITION 1.12.8 (cf. [Lev-Pol 1999]). *Let $G \subset \mathbb{C}^n$ be a domain and let $A \subset G$. Then*

$$\omega_{A,G} = \sup\{\omega_{U,G} : A \subset U \subset G,\ U \text{ open}\}.$$

In particular, if A is compact, then for any neighborhood basis $(U_j)_{j=1}^\infty$ of A with $G \supset U_{j+1} \subset U_j$, we have

$$\omega_{A,G} = \lim_{j\to\infty} \omega_{U_j,G}.$$

Proof. Let $u \in \mathcal{PSH}(G)$, $u \leq 0$, $u \leq -1$ on A. Fix $0 < \varepsilon < 1$ and define

$$U_\varepsilon := \{z \in G : u < -1 + \varepsilon\}.$$

Then $u/(1 - \varepsilon) \leq \omega_{U_\varepsilon,G}$. Consequently,

$$u \leq (1 - \varepsilon) \sup\{\omega_{U,X} : A \subset U,\ U \text{ open}\}.$$

Taking $\varepsilon \to 0$, we get the required result. ∎

PROPOSITION 1.12.9 (cf. [Bło 2000]). *Let $G \subset \mathbb{C}^n$ be a bounded domain and let $A \subset G$. Put $A_\varepsilon := \{z \in G : \omega^*_{A,G}(z) < -1 + \varepsilon\}$, $0 < \varepsilon < 1$. Then*

$$\frac{\omega^*_{A,G}}{1 - \varepsilon} \leq \omega_{A_\varepsilon,G} \leq \omega^*_{A,G}.$$

Consequently, $\omega_{A_\varepsilon,G} \nearrow \omega^*_{A,G}$ as $\varepsilon \searrow 0$.

Proof. Put $N := \{z \in G : \omega_{A,G}(z) < \omega^*_{A,G}(z)\}$ and $Q := A \setminus N$. It is well known (see e.g. Theorem 4.7.6 in [Kli 1991]) that N is pluripolar and $\omega^*_{Q,G} = \omega^*_{A,G}$ (cf. [Jar-Pfl 2000, Lemma 3.5.3]). We have $\omega^*_{Q,G} = \omega^*_{A,G} = \omega_{A,G} = -1$ on Q. Hence $Q \subset A_\varepsilon$ and $\omega^*_{A,G} = \omega^*_{Q,G} \geq \omega_{A_\varepsilon,\Omega}$. Put $u := \omega^*_{A,G}/(1 - \varepsilon)$. Note that $u \in \mathcal{PSH}(G)$, $u \leq 0$, and $u \leq -1$ on A_ε. Hence, $u \leq \omega_{A_\varepsilon,G}$. ∎

Proof of Theorem 1.12.1. For the proof of the inequality "$\geq$" it suffices to consider the projections $\mathrm{pr}_j : G_1 \times G_2 \to G_j$, $j = 1, 2$, and use Remark 1.6.1(e). We turn to the opposite inequality. First assume that A_1, A_2 are open. Put $u_1 = -\chi_{A_1}$ and $u_2 = -\chi_{A_2}$. Let $(z_1, z_2) \in G_1 \times G_2$ be fixed and let $\beta \in \mathbb{R}$ be such that

$$\max\{\omega_{A_1,G_1}(z_1), \omega_{A_2,G_2}(z_2)\} < \beta.$$

By Theorem 1.10.7 there are $\varphi_j \in \mathcal{O}(\overline{E}, G_j)$, $j = 1, 2$, such that $\varphi_1(0) = z_1$, $\varphi_2(0) = z_2$, and

$$\frac{1}{2\pi}\int_0^{2\pi} u_j(\varphi_j(e^{i\theta}))d\theta < \beta, \qquad j = 1, 2.$$

Note that $\varphi_1^{-1}(A_1) \cap \partial E$ is an open set in ∂E. So, we may choose a finite set of disjoint open arcs $I_1^1, \dots, I_m^1 \subset \varphi_1^{-1}(A_1) \cap \partial E$ such that $\Lambda(I^1) > -2\pi\beta$ where $I^1 = \bigcup_{j=1}^m I_j^1$. Similarly we choose $I_1^2, \dots, I_k^2$ with $I^2 = \bigcup_{j=1}^k I_j^2$. By Lemma 1.12.7 we may find Blaschke products B_1, B_2 and a closed arc $I \subset \partial E$ with $\Lambda(I) > -2\pi\beta$ such that $B_1^{-1}(I) \subset I^1$ and $B_2^{-1}(I) \subset I^2$.

Let A be the union of the sets of critical values of B_1 and B_2. Note that 0 is not in A. Let π be a holomorphic universal covering of $E \setminus A$ by E with $\pi(0) = 0$. Observe that π is inner (Lemma 1.12.4). If $\widetilde{I} = \pi^{-1}(I)$, then according to Lemma 1.12.6, $\Lambda(\widetilde{I}) = \Lambda(I)$. There are liftings $\psi_1, \psi_2 : E \to E$ of π such that $\pi = B_1 \circ \psi_1 = B_2 \circ \psi_2$ and $\psi_1(0) = \psi_2(0) = 0$. By Remark 1.12.5, ψ_1, ψ_2 are inner. Also non-tangential boundary values of ψ_1 and ψ_2 on $\widetilde{I}$ belong to I^1 and I^2, respectively. Put $\widetilde{\varphi}_1 = \varphi_1 \circ \psi_1$ and $\widetilde{\varphi}_2 = \varphi_2 \circ \psi_2$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} \max\{u_1(\widetilde{\varphi}_1(e^{i\theta})), u_2(\widetilde{\varphi}_2(e^{i\theta}))\}\, d\theta \leq -\frac{\Lambda(\widetilde{I})}{2\pi} < \beta.$$

By Fatou's theorem the same inequality holds if we replace $\widetilde{\varphi}_j(z)$, $j = 1, 2$, with $\widetilde{\varphi}_j(rz)$, where $r < 1$ is sufficiently close to 1. Hence, $\omega_{A_1 \times A_2, G_1 \times G_2}(z_1, z_2) < \beta$. Since β was arbitrary, we get the assertion.

The case where A_1, A_2 are compact follows from Proposition 1.12.8.

We turn to the second part of the theorem. First note that for any $(z_1, z_2) \in G_1 \times G_2$ we have

$$\begin{aligned}\max\{\omega_{A_1,G_1}(z_1),\, \omega_{A_2,G_2}(z_2)\} &\leq \omega_{A_1 \times A_2, G_1 \times G_2}(z_1, z_2) \\ &\leq -\omega^*_{A_1,G_1}(z_1)\omega^*_{A_2,G_2}(z_2).\end{aligned} \tag{1.12.25}$$

Indeed, we only need to prove the second inequality. Let $u \in \mathcal{PSH}(G_1 \times G_2)$, $u \leq 0$, $u \leq -1$ on $A_1 \times A_2$. Then

$$u(\cdot, z_2) \leq -\omega_{A_2,G_2}(z_2)\omega_{A_1,G_1}(\cdot), \quad z_2 \in A_2, \quad u(z_1, \cdot) \leq -\omega_{A_1,G_1}(z_1)\omega_{A_2,G_2}(\cdot), \quad z_1 \in A_1.$$

Take a $z_1 \in G_1$. If $\omega_{A_1,G_1}(z_1) = 0$, then $u(z_1, \cdot) \leq 0 = -\omega_{A_1,G_1}(z_1)\omega_{A_2,G_2}(\cdot)$. If $\omega_{A_1,G_1}(z_1) \neq 0$, then let $v := u(z_1, \cdot)/(-\omega_{A_1,G_1}(z_1))$. Then $v \in \mathcal{PSH}(G_2)$, $v \leq 0$, and $v \leq -1$ on A_2. Hence $v \leq \omega_{A_2,G_2}$.

Fix an $\varepsilon > 0$. Then by (1.12.25),

$$\omega_{A_1 \times A_2, G_1 \times G_2}(z_1, z_2) \leq -(1-\varepsilon)^2 \qquad \text{on } (A_1)_\varepsilon \times (A_2)_\varepsilon.$$

Hence

$$\omega^*_{A_1\times A_2,G_1\times G_2}(z_1,z_2) \le -(1-\varepsilon)^2 \qquad \text{on } (A_1)_\varepsilon \times (A_2)_\varepsilon.$$

It follows that on $G_1 \times G_2$,

$$(1-\varepsilon)^2\omega^*_{A_1\times A_2,G_1\times G_2} \le \omega^*_{(A_1)_\varepsilon\times(A_2)_\varepsilon,G_1\times G_2} \le \omega^*_{A_1\times A_2,G_1\times G_2}.$$

Thus, using the first part of the theorem and Proposition 1.12.9, we get

$$\begin{aligned}\omega^*_{A_1\times A_2,G_1\times G_2}(z_1,z_2) &= \lim_{\varepsilon\to 0}\omega_{(A_1)_\varepsilon\times(A_2)_\varepsilon,G_1\times G_2}(z_1,z_2)\\ &= \lim_{\varepsilon\to 0}\max\{\omega_{(A_1)_\varepsilon,G_1}(z_1),\,\omega_{(A_2)_\varepsilon,G_2}(z_2)\}\\ &= \max\{\omega^*_{A_1,G_1}(z_1),\,\omega^*_{A_2,G_2}(z_2)\}, \qquad (z_1,z_2)\in G_1\times G_2. \ \blacksquare\end{aligned}$$

REMARK 1.12.10. Using the analytic discs method F. Lárusson, P. Lassere, and R. Sigurdsson proved in [Lár-Las-Sig 1998a] the following result.

THEOREM. *Let $G \subset \mathbb{C}^n$ be a convex domain and let $A \subset G$ be an open or compact convex set. Then for any $\alpha \in [-1, 0)$ the level set $\{z \in G : \omega_{A,G}(z) < \alpha\}$ is convex.*

1.12.2. Product property for the generalized Green function. Proposition 1.6.2 and Theorems 1.10.15, 1.12.1 imply the following product property for the generalized Green function (cf. [Edi 2001]).

THEOREM 1.12.11. *For any domains $G_1 \subset \mathbb{C}^{n_1}$, $G_2 \subset G^{n_2}$ and for any sets $A_1 \subset G_1$, $A_2 \subset G_2$, the pluricomplex Green function with many poles has the* product property:

$$g_{G_1\times G_2}(A_1\times A_2,(z_1,z_2)) = \max\{g_{G_1}(A_1,z_1),\,g_{G_2}(A_2,z_2)\}, \qquad (z_1,z_2)\in G_1\times G_2.$$

In particular,

$$g_{G_1\times G_2}((a_1,a_2),(z_1,z_2)) = \max\{g_{G_1}(a_1,z_1),\,g_{G_2}(a_2,z_2)\}, \qquad (a_1,a_2),(z_1,z_2)\in G_1\times G_2.$$

A different proof, using Poletsky's methods, was given by A. Edigarian in [Edi 1999].

The case of the pluricomplex Green function with one pole has been solved in [Edi 1997a]; some particular cases have been solved previously (using different methods based on the Monge–Ampère operator):

- the case where both G_1 and G_2 are domains of holomorphy—in [J-P 1993, Theorem 9.8],
- the case where at least one of the domains G_1, G_2 is a domain of holomorphy—in [Jar-Pfl 1995b].

REMARK 1.12.12. One could try to generalize the above product property to arbitrary pole functions $\boldsymbol{p}_j : G_j \to \mathbb{R}_+$ with $\max_{G_j}\boldsymbol{p}_j = 1$, $j = 1, 2$. For instance, one could conjecture that

$$g_{G_1\times G_2}(\boldsymbol{p},(z_1,z_2)) = \max\{g_{G_1}(\boldsymbol{p}_1,z_1), g_{G_2}(\boldsymbol{p}_2,z_2)\}, \qquad (z_1,z_2)\in G_1\times G_2,$$

where $\boldsymbol{p}(a_1,a_2) := \min\{\boldsymbol{p}_1(a_1),\,\boldsymbol{p}_2(a_2)\}$. Unfortunately, such a formula is false. Take for instance $G_1 = G_2 = E$, $\boldsymbol{p}_1 := \chi_{\{0\}} + \frac{1}{2}\chi_{\{c\}}$, $\boldsymbol{p}_2 := \chi_{\{0\}}$, where $0 < c < 1$. Observe that $\boldsymbol{p} = \chi_{\{(0,0)\}} + \frac{1}{2}\chi_{\{(c,0)\}}$. Hence, by Example 1.7.17, we get

$$g_{E^2}(\boldsymbol{p},(z_1,z_2)) = (\max\{|z_1|,\,|z_2|\}\max\{|z_1|m(z_1,c),\,|z_2|\})^{1/2}.$$

In particular, if $z_1 = c$, $z_2 = c^2$, then

$$g_{E^2}(\boldsymbol{p}, (c, c^2)) = c^{3/2}.$$

On the other hand,

$$\max\{g_E(\boldsymbol{p}_1, c), g_E(\boldsymbol{p}_2, c^2)\} = c^2.$$

REMARK 1.12.13. Example 1.11.4 shows that, in general, the Coman function does not have the product property. Indeed, let B, C, and γ be as in the example. Then, by Remark 1.11.2(c), we have

$$\begin{aligned}\delta_{E^2}(B \times C, (0, \gamma)) &> g_{E^2}(B \times C, (0, \gamma)) \\ &= \max\{g_E(B, 0), g_E(C, \gamma)\} = \max\{\delta_E(B, 0), \delta_E(C, \gamma)\}.\end{aligned}$$

PROPOSITION 1.12.14 ([Die-Tra 2003]). *For any domains $G \subset \mathbb{C}^n$, $D \subset \mathbb{C}^m$, the following conditions are equivalent*:

(i) *for any finite set $A \subset G$, and for any point $b \in D$ we have*

$$\delta_{G\times D}(A \times \{b\}, (z, w)) = \max\{\delta_G(A, z),\ \delta_D(b, w)\}, \qquad (z, w) \in G \times D;$$

(ii) $\delta_D(b, w) = g_D(b, w)$, $b, w \in D$.

Proof. (i)⇒(ii). By Theorem 1.10.15 we have

$$g_D(b, w) = \inf_{N\in\mathbb{N}} \delta_D^{(N)}(b, w), \qquad w \in D,$$

where

$$\begin{aligned}\delta_D^{(N)}(b, w) := \inf\Big\{ \prod_{j=1}^{N} |\mu_j| : \mu_1, \dots, \mu_N \in E,\ \mu_j \neq \mu_k, \\ \exists_{\psi\in\mathcal{O}(E,D)} : \psi(\mu_j) = b,\ j = 1, \dots, N,\ \psi(0) = w\Big\}, \qquad w \in D.\end{aligned}$$

By Remark 1.11.2(a), it suffices to show that $\delta_D(b, \cdot) \leq \delta_D^{(N)}(b, \cdot)$ (for every N).

Fix $N \in \mathbb{N}$, $w_0 \in D$, and $\varepsilon > 0$. Let $\mu_1, \dots, \mu_N \in E$ and $\psi \in \mathcal{O}(E, D)$ be such that $\mu_j \neq \mu_k$, $\psi(\mu_j) = b$, $j = 1, \dots, N$, $\psi(0) = w_0$, and $\prod_{j=1}^N |\mu_j| \leq \delta_D^{(N)}(b, w_0) + \varepsilon$. Take an arbitrary $\varphi \in \mathcal{O}(E, G)$ such that $\varphi(\mu_j) \neq \varphi(\mu_k)$. Put $A := \{\varphi(\mu_1), \dots, \varphi(\mu_N)\}$, $z_0 := \varphi(0)$. Then

$$\begin{aligned}\delta_D(b, w_0) \leq \max\{\delta_G(A, z_0),\ \delta_D(b, w_0)\} = \delta_{G\times D}(A \times \{b\}, (z_0, w_0)) \\ \leq \prod_{j=1}^{N} |\mu_j| \leq \delta_D^{(N)}(b, w_0) + \varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0$ we conclude the proof.

(ii)⇒(i). Directly from the definition we get the inequality

$$\delta_G(A, z) \leq \delta_{G\times D}(A \times \{b\}, (z, w)), \qquad (z, w) \in G \times D.$$

Moreover, by Remark 1.11.2(a) and Theorem 1.12.11,

$$\begin{aligned}\delta_D(b, w) = g_D(b, w) \leq \max\{g_G(A, z),\ g_D(b, w)\} = g_{G\times D}(A \times \{b\}, (z, w)) \\ \leq \delta_{G\times D}(A \times \{b\}, (z, w)), \qquad (z, w) \in G \times D.\end{aligned}$$

Thus

$$\delta_{G\times D}(A\times\{b\},(z,w))\geq\max\{\delta_G(A,z),\delta_D(b,w)\},\qquad (z,w)\in G\times D.$$

Let $A=\{a_1,\dots,a_N\}$. Fix $(z_0,w_0)\in G\times D$ and $\varepsilon>0$. To prove the inequality

$$\delta_{G\times D}(A\times\{b\},(z_0,w_0))\leq\max\{\delta_G(A,z_0),\,\delta_D(b,w_0)\},$$

we may assume that $\max\{\delta_G(A,z_0),\,\delta_D(b,w_0)\}+\varepsilon<1$. Consider the following two cases:

(a) $\delta_D(b,w_0)\leq\delta_G(A,z_0)$. Then take $\mu_1,\dots,\mu_N\in E$ and $\varphi\in\mathcal{O}(E,G)$ such that $\varphi(0)=z_0$, $\varphi(\mu_j)=a_j$, $j=1,\dots,N$, and $\prod_{j=1}^N|\mu_j|<\delta_G(A,z_0)+\varepsilon$.

We may assume that $\mu_j\neq 0$, $j=1,\dots,N$. Indeed, suppose that $\mu_1\cdots\mu_{N-1}\neq 0$, and $\mu_N=0$. Then we may replace φ by $\widetilde{\varphi}:=\varphi\circ B$, where $B(z):=z(z-\varepsilon)/(1-\varepsilon z)$, $z\in E$. Observe that $B(E)=E$, so there exist $\widetilde{\mu}_1,\dots,\widetilde{\mu}_{N-1}\in E_*$ with $B(\widetilde{\mu}_j)=\mu_j$, $j=1,\dots,N-1$. Hence $\widetilde{\varphi}(\widetilde{\mu}_j)=a_j$, $j=1,\dots,N-1$, $\widetilde{\varphi}(\varepsilon)=\widetilde{\varphi}(0)=z_0$, and $|\mu_1\cdots\mu_{N-1}\varepsilon|<\varepsilon$.

We may also assume that $\delta_D(b,w_0)<\prod_{j=1}^N|\mu_j|$. Indeed, if $\delta_D(b,w_0)=\prod_{j=1}^N|\mu_j|$, then we may replace φ by $\widetilde{\varphi}(z):=\varphi(tz)$, $z\in E$, and μ_j by μ_j/t, $j=1,\dots,N$, with suitable $0<t<1$, $t\approx 1$.

Take $\eta\in E$ and $\psi\in\mathcal{O}(E,D)$ such that $\psi(0)=w_0$, $\psi(\eta)=b$, and $|\eta|<\prod_{j=1}^N|\mu_j|$. Define $\alpha:=\prod_{j=1}^N(-\mu_j)\in E$, $t:=-\eta/\alpha\in E$, $\widetilde{\psi}(z):=\psi(tz)$, $z\in E$, $B:=\prod_{j=1}^N h_{\mu_j}$, $\chi:E\to G\times D$, $\chi:=(\varphi,\widetilde{\psi}\circ h_\alpha\circ B)$.

We have $\chi(0)=(\varphi(0),\widetilde{\psi}(h_\alpha(B(0))))=(z_0,\widetilde{\psi}(h_\alpha(\alpha)))=(z_0,\widetilde{\psi}(0))=(z_0,\psi(0))=(z_0,w_0)$. Moreover, $\chi(\mu_j)=(\varphi(\mu_j),\widetilde{\psi}(h_\alpha(B(\mu_j))))=(a_j,\widetilde{\psi}(h_\alpha(0)))=(a_j,\widetilde{\psi}(-\alpha))=(a_j,\psi(\eta))=(a_j,b)$, $j=1,\dots,N$. Hence

$$\delta_{G\times D}(A\times\{b\},(z_0,w_0))\leq\prod_{j=1}^N|\mu_j|<\delta_G(A,z_0)+\varepsilon.$$

(b) $\delta_G(A,z_0)<\delta_D(b,w_0)$. Then take $\eta\in E$ and $\psi\in\mathcal{O}(E,D)$ such that $\psi(0)=w_0$, $\psi(\eta)=b$, and $|\eta|<\delta_D(b,w_0)+\varepsilon/2$. Take $\mu_1,\dots,\mu_N\in E$ and $\varphi\in\mathcal{O}(E,G)$ such that $\varphi(0)=z_0$, $\varphi(\mu_j)=a_j$, $j=1,\dots,N$, and $\prod_{j=1}^N|\mu_j|<\delta_D(b,w_0)+\varepsilon/2$. Using the same argument as in (a), we may assume that $\mu_j\neq 0$, $j=1,\dots,N$.

LEMMA 1.12.15. *Let $\mu_1,\dots,\mu_N\in E_*$, $\alpha:=\prod_{j=1}^N|\mu_j|$. Assume that $\alpha<\beta<\alpha^{1/(k+1)}$ with $k\in\mathbb{N}$. For $t\in[0,1]$, put $f_t(z):=z((z-t)/(1-tz))^k$, $z\in E$. Then there exist $t\in[0,1]$ and $\widetilde{\mu}_1,\dots,\widetilde{\mu}_N\in E$ such that $f_t(\widetilde{\mu}_j)=\mu_j$, $j=1,\dots,N$, and $\prod_{j=1}^N|\widetilde{\mu}_j|=\beta$.*

Proof (due to W. Zwonek [49]). Observe that $f_t(E)=E$. Hence, for any $j\in\{1,\dots,N\}$ there exists a $\widetilde{\mu}_j\in E$ such that $f_t(\widetilde{\mu}_j)=\mu_j$. Let $\Phi_j(t):=\min\{|\widetilde{\mu}_j|:f_t(\widetilde{\mu}_j)=\mu_j\}$, $\Phi:=\Phi_1\cdots\Phi_N$. We have $\Phi(0)=\alpha^{1/(k+1)}>\beta$, $\Phi(1)=\alpha<\beta$. We only need to show that each function Φ_j is continuous.

[49] The original proof in [Die-Tra 2003] contains an essential gap.

Fix a $j \in \{1, \dots, N\}$ and let $\Phi_j(t) = |\widetilde{\mu}_j(t)|$, $t \in [0,1]$. Suppose that $[0,1] \ni t_s \to t_0$ and $|\widetilde{\mu}_j(t_s)| \le m < \Phi_j(t_0)$. Then, without loss of generality, $\widetilde{\mu}_j(t_s) \to \widetilde{\mu}_j \in E$. Therefore, $f_{t_0}(\widetilde{\mu}_j) = \mu_j$, i.e. $\Phi_j(t_0) \le m$, a contradiction. So, Φ_j is lower semicontinuous.

On the other hand, by the Hurwitz theorem, for any $\varepsilon > 0$, the equation $f_{t_s}(z) = \mu_j$ must have a solution in the disc $\mathbb{B}(\widetilde{\mu}_j(t_0), \varepsilon)$, provided $s \gg 1$. Hence $\Phi_j(t_s) \le \Phi_j(t_0) + \varepsilon$, $s \gg 1$, and finally $\lim_{s\to+\infty} \Phi_j(t_s) \le \Phi_j(t_0)$. ■

Using Lemma 1.12.15 with $\beta := \delta_D(b, w_0) + \varepsilon/2$, we may modify φ and $\mu_1, \dots, \mu_N$ in such a way that $|\eta| < \prod_{j=1}^N |\mu_j| < \delta_D(b, w_0) + \varepsilon$. Now we continue as in (a). ■

REMARK 1.12.16. Lemma 1.12.15 was improved by N. Nikolov, namely:

Let $\mu_1, \dots, \mu_N \in E_*$, $\alpha := \prod_{j=1}^N |\mu_j|$, $\alpha < \beta < 1$. For $t \in [0,1]$, put $f_t(z) := z(z-t)/(1-tz)$, $z \in E$. Then there exist $t \in [0,1]$ and $\widetilde{\mu}_1, \dots, \widetilde{\mu}_N \in E$ such that $f_t(\widetilde{\mu}_j) = \mu_j$, $j = 1, \dots, N$, and $\prod_{j=1}^N |\widetilde{\mu}_j| = \beta$.

Indeed, the case where $\beta < \alpha^{1/2}$ reduces to the proof of Lemma 1.12.15 (with $k = 1$). If $\beta \ge \alpha^{1/2}$, then put $\Psi_j(t) := \max\{|\widetilde{\mu}_j| : f_t(\widetilde{\mu}_j) = \mu_j\}$, $\Psi := \Psi_1 \cdots \Psi_N$. Observe that $\Psi(0) = \alpha^{1/2}$. Similarly to Lemma 1.12.15 we prove that Ψ_j is continuous on $[0,1)$. Moreover, $\Psi_j(t) \to 1$ when $t \to 1$, $j = 1, \dots, N$.

REMARK 1.12.17. Recently, N. Nikolov and W. Zwonek have extended Theorem 1.12.14 in the following way:

THEOREM. *Let $D \subset \mathbb{C}^n$ and $G \subset \mathbb{C}^m$ be domains and let $z \in D$, $w, b \in G$, $A \subset D$. Then*

$$\max\{\delta_D(A,z), l_G^{\#A}(b,w)\} \le \delta_{D\times G}(A \times \{b\}, (z,w)) \le \max\{\delta_D(A,z), \delta_G(b,w)\},$$

where

$$l_G^N(b,w) := \inf\Big\{ \prod_{j=1}^N |\lambda_j| : (\lambda_j)_{j=1}^N \subset E,\ \exists_{\varphi\in\mathcal{O}(E,G)} : \varphi(0) = w,\ \varphi(\lambda_j) = b,\ \#\{k : \lambda_k = \lambda_j\} \le \operatorname{ord}_{\lambda_j}(\varphi - b),\ j = 1, \dots, N\Big\}, \quad N \in \mathbb{N},$$

$$l_G^\infty(b,w) := \inf\Big\{ \prod_{j=1}^\infty |\lambda_j| : (\lambda_j)_{j=1}^\infty \subset E,\ \exists_{\varphi\in\mathcal{O}(E,G)} : \varphi(0) = w,\ \varphi(\lambda_j) = b,\ \#\{k : \lambda_k = \lambda_j\} \le \operatorname{ord}_{\lambda_j}(\varphi - b),\ j = 1, 2, \dots\Big\}.$$

Moreover, for any $N \in \mathbb{N} \cup \{\infty\}$ the equality

$$\delta_{D\times G}(A \times \{b\}, (z,w)) = \max\{\delta_D(A,z),\ \delta_G(b,w)\}$$

holds for any $A \subset D$ with $\#A = N$ if and only if $\delta_G(b,w) = l_G^N(b,w)$.

They have also proved the following result.

THEOREM. *Let $A, B \subset E$, $\#A = \#B = 2$, and $z, w \in E$ be such that $\delta_E(A,z) = \delta_E(B,w)$. Then*

$$\delta_E(A,z) = \min\{\delta_{E^2}(C,(z,w)) : C \subset A \times B\}$$

if and only if there is an $h \in \operatorname{Aut}(E)$ *with* $h(z) = w$ *and* $h(A) = B$. *Consequently, if* $\zeta \in E \setminus A$, *then there exist uncountably many* $\xi \in E$ *for which*

$$\delta_E(A, \zeta) = \delta_E(B, \xi) < \min\{\delta_{E^2}(C, (\zeta, \xi)) : C \subset A \times B\} \leq \delta_{E^2}(A \times B, (\zeta, \xi))$$

and, therefore,

$$g_{E^2}(A \times B, (\zeta, \xi)) < \delta_{E^2}(A \times B, (\zeta, \xi))$$

(*cf. Example* 1.11.4).

1.12.3. Product property for $d_G^{\min}$ and $d_G^{\max}$. The case of the generalized Green function suggests that the product property might hold for other generalized holomorphically contractible families $(d_G)_G$, i.e.

$$\text{(P)} \qquad d_{G \times D}(A \times B, (z, w)) = \max\{d_G(A, z),\, d_D(B, w)\}, \qquad (z, w) \in G \times D,$$

for any domains $G \subset \mathbb{C}^n$, $D \subset \mathbb{C}^m$ and for any sets $\emptyset \neq A \subset G$, $\emptyset \neq B \subset D$. Notice that the inequality "$\geq$" follows from (H) applied to the projections $G \times D \to G$, $G \times D \to D$.

The definition applies to the standard holomorphically contractible families and means that

$$d_{G \times D}((a, b), (z, w)) = \max\{d_G(a, z),\, d_D(b, w)\}, \qquad (a, b), (z, w) \in G \times D.$$

Recall that the standard (non-generalized) families $(\widetilde{k}_G^*)_G$, $(c_G^*)_G$, $(g_G)_G$ have the product property; cf. [J-P 1993, Ch. 9], see also [Mey 1997] (for a proof of the product property for the Möbius functions based on functional analysis methods) and [Jar-Pfl 1999b] (for the case of complex spaces). Moreover, it is known that the higher order Möbius functions $(m_G^{(k)})_G$ with $k \geq 2$ fail the product property; cf. [J-P 1993, Ch. 9].

PROPOSITION 1.12.18. *The system* $(d_G^{\max})_G$ *has the product property.*

Proof. Fix $(z_0, w_0) \in G \times D$ and $\varepsilon > 0$. Let $(a, b) \in A \times B$ be such that $\widetilde{k}_G^*(a, z_0) \leq d_G^{\max}(A, z_0) + \varepsilon$, $\widetilde{k}_D^*(b, w_0) \leq d_G^{\max}(B, w_0) + \varepsilon$. Then using the product property for $(\widetilde{k}_G^*)_G$, we get

$$\begin{aligned} d_{G \times D}^{\max}(A \times B, (z_0, w_0)) &\leq \widetilde{k}_{G \times D}^*((a, b), (z_0, w_0)) = \max\{\widetilde{k}_G^*(a, z_0),\, \widetilde{k}_D^*(b, w_0)\} \\ &\leq \max\{d_G^{\max}(A, z_0),\, d_D^{\max}(B, w_0)\} + \varepsilon. \ \blacksquare \end{aligned}$$

$\boxed{?}$ We do not know whether the system $(d_G^{\min})_G$ has the product property. $\boxed{?}$ So far we were able to handle ([Jar-Jar-Pfl 2003]) only the case where $\#B = 1$: see Proposition 1.12.20 (cf. also [Die-Tra 2003]). Recall that $d_G^{\min}(A, \cdot) = m_G(A, \cdot)$ (Proposition 1.5.4).

PROPOSITION 1.12.19. *Assume that for any* $n \in \mathbb{N}$, *the system* $(m_G)_G$ *has the following special product property*:

$$(\mathrm{P}_0) \qquad |\Psi(z, w)| \leq (\max_{G \times D} |\Psi|) \max\{m_G(A, z),\, m_D(B, w)\}, \qquad (z, w) \in G \times D,$$

where $G, D \subset \mathbb{C}^n$ *are balls with respect to arbitrary* $\mathbb{C}$-*norms*, $A \subset D$, $B \subset G$ *are finite and non-empty*, $\Psi(z, w) := \sum_{j=1}^n z_j w_j$, *and* $\Psi|_{A \times B} = 0$. *Then the system* $(m_G)_G$ *has the product property* (P) *in full generality. Moreover, if* (P_0) *holds with* $\#B = 1$, *then* (P) *holds with* $\#B = 1$.

Proof (cf. [J-P 1993, the proof of Theorem 9.5]). Fix arbitrary domains $G \subset \mathbb{C}^n$, $D \subset \mathbb{C}^m$, non-empty sets $A \subset G$, $B \subset G$, and $(z_0, w_0) \in G \times D$. We have to prove that for any $F \in \mathcal{O}(G \times D, E)$ with $F|_{A \times B} = 0$ the following inequality is true:

$$|F(z_0, w_0)| \le \max\{m_G(A, z_0),\ m_D(B, w_0)\}.$$

By Remark 1.6.1(h), we may assume that A, B are finite.

Let $(G_\nu)_{\nu=1}^\infty$, $(D_\nu)_{\nu=1}^\infty$ be sequences of relatively compact subdomains of G and D, respectively, such that $A \cup \{z_0\} \subset G_\nu \nearrow G$, $B \cup \{w_0\} \subset D_\nu \nearrow D$. By Remark 1.6.1(h), it suffices to show that

$$|F(z_0, w_0)| \le \max\{m_{G_\nu}(A, z_0),\ m_{D_\nu}(B, w_0)\}, \qquad \nu \ge 1.$$

Fix a $\nu_0 \in \mathbb{N}$ and let $G' := G_{\nu_0}$, $D' := D_{\nu_0}$. It is well known that F may be approximated locally uniformly in $G \times D$ by functions of the form

$$F_s(z, w) = \sum_{\mu=1}^{N_s} f_{s,\mu}(z) g_{s,\mu}(w), \qquad (z, w) \in G \times D, \tag{1.12.26}$$

where $f_{s,\mu} \in \mathcal{O}(G)$, $g_{s,\mu} \in \mathcal{O}(D)$, $s \ge 1$, $\mu = 1, \dots, N_s$. Notice that $F_s \to 0$ uniformly on $A \times B$. Using the Lagrange interpolation formula, we find polynomials $P_s : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}$ such that $P_s|_{A\times B} = F_s|_{A \times B}$ and $P_s \to 0$ locally uniformly in $\mathbb{C}^n \times \mathbb{C}^m$. The functions $\widehat{F}_s := F_s - P_s$, $s \ge 1$, also have the form (1.12.26) and $\widehat{F}_s \to F$ locally uniformly in $G \times D$. Hence, without loss of generality, we may assume that $F_s|_{A \times B} = 0$, $s \ge 1$. Let $m_s := \max\{1, \|F_s\|_{G' \times D'}\}$ and $\widetilde{F}_s := F_s / m_s$, $s \ge 1$. Note that $m_s \to 1$ and, therefore, $\widetilde{F}_s \to F$ uniformly on $G' \times D'$. Consequently, we may assume that $F_s(G' \times D') \Subset E$, $s \ge 1$.

It is enough to prove that

$$|F_s(z_0, w_0)| \le \max\{m_{G'}(A, z_0), m_{D'}(B, w_0)\}, \qquad s \ge 1.$$

Fix an $s = s_0 \in \mathbb{N}$ and let $N := N_{s_0}$, $f_\mu := f_{s_0,\mu}$, $g_\mu := g_{s_0,\mu}$, $\mu = 1, \dots, N$. Let $f := (f_1, \dots, f_N) : G \to \mathbb{C}^N$ and $g := (g_1, \dots, g_N) : D \to \mathbb{C}^N$. Put

$$K := \{\xi = (\xi_1, \dots, \xi_N) \in \mathbb{C}^N : |\xi_\mu| \le \|f_\mu\|_{G'},\ \mu = 1, \dots, N,\ |\Psi(\xi, g(w))| \le 1,\ w \in D'\}.$$

It is clear that K is an absolutely convex compact subset of $\mathbb{C}^N$ with $f(G') \subset K$. Let

$$L := \{\eta = (\eta_1, \dots, \eta_N) \in \mathbb{C}^N : |\eta_\mu| \le \|g_\mu\|_{D'},\ \mu = 1, \dots, N,\ |\Psi(\xi, \eta)| \le 1,\ \xi \in K\}.$$

Then again L is an absolutely convex compact subset of $\mathbb{C}^N$, and, moreover, $g(D') \subset L$.

Let $(W_\sigma)_{\sigma=1}^\infty$ (resp. $(V_\sigma)_{\sigma=1}^\infty$) be a sequence of absolutely convex bounded domains in $\mathbb{C}^N$ such that $W_{\sigma+1} \Subset W_\sigma$ and $W_\sigma \searrow K$ (resp. $V_{\sigma+1} \Subset V_\sigma$ and $V_\sigma \searrow L$). Put $M_\sigma := \|\Psi\|_{W_\sigma \times V_\sigma}$, $\sigma \in \mathbb{N}$. By (P_0) and by the holomorphic contractibility applied to the mappings $f : G' \to W_\sigma$, $g : D' \to V_\sigma$, we have

$$\begin{aligned}
|F_{s_0}(z_0, w_0)| &= |\Psi(f(z_0), g(w_0))| \\
&\le M_\sigma \max\{m_{W_\sigma}(f(A), f(z_0)),\ m_{V_\sigma}(g(B), g(w_0))\} \\
&\le M_\sigma \max\{m_{G'}(f^{-1}(f(A)), z_0),\ m_{D'}(g^{-1}(g(B)), w_0)\} \\
&\le M_\sigma \max\{m_{G'}(A, z_0),\ m_{D'}(B, w_0)\}.
\end{aligned}$$

Letting $\sigma \to +\infty$ we get the required result. ∎

PROPOSITION 1.12.20. *The system* $(m_G)_G$ *has the product property* (P) *whenever* $\#B=1$, *i.e. for any domains* $G \subset \mathbb{C}^n$, $D \subset \mathbb{C}^m$, *for any set* $A \subset G$, *and for any point* $b \in D$ *we have*

$$m_{G\times D}(A \times \{b\}, (z,w)) = \max\{m_G(A,z),\, m_D(b,w)\}, \qquad (z,w) \in G \times D.$$

Proof. By Proposition 1.12.19, we only need to check (P) in the case where D is a bounded convex domain, A is finite, and $B = \{b\}$. Fix $(z_0, w_0) \in G \times D$. Let $\varphi : E \to D$ be a holomorphic mapping such that $\varphi(0) = b$ and $\varphi(m_D(b, w_0)) = w_0$ (cf. [J-P 1993, Ch. 8]). Consider the mapping $F : G \times E \to G \times D$, $F(z, \lambda) := (z, \varphi(\lambda))$. Then

$$m_{G\times D}(A \times \{b\}, (z_0, w_0)) \le m_{G\times E}(A \times \{0\}, (z_0, m_G(b, w_0))).$$

Consequently, it suffices to show that

$$m_{G\times E}(A \times \{0\}, (z_0, \lambda)) \le \max\{m_G(A, z_0),\, |\lambda|\}, \qquad \lambda \in E. \tag{1.12.27}$$

The case where $m_G(A, z_0) = 0$ is elementary: for an $f \in \mathcal{O}(G\times E, E)$ with $f|_{A\times\{0\}} = 0$ we have $f(z_0, 0) = 0$ and hence $|f(z_0, \lambda)| \le |\lambda|$, $\lambda \in E$ (by the Schwarz lemma).

Thus, we may assume that $r := m_G(A, z_0) > 0$. First observe that it suffices to prove (1.12.27) only on the circle $|\lambda| = r$. Indeed, if the inequality holds on that circle, then by the maximum principle for subharmonic functions (applied to the function $m_{G\times E}(A \times \{0\}, (z_0, \cdot)))$ it holds for all $|\lambda| \le r$. In the annulus $\{r < |\lambda| < 1\}$ we apply the maximum principle to the subharmonic function $\lambda \mapsto |\lambda|^{-1} m_{G\times E}(A \times \{0\}, (z_0, \lambda))$.

Now fix a $\lambda_0 \in E$ with $|\lambda_0| = r$. Let f be an extremal function for $m_G(A, z_0)$ with $f|_A = 0$ and $f(z_0) = \lambda_0$. Consider $F : G \to G \times E$, $F(z) := (z, f(z))$. Then

$$m_{G\times E}(A \times \{0\}, (z_0, \lambda_0)) \le m_G(A, z_0) = \max\{m_G(A, z_0),\, |\lambda_0|\},$$

which completes the proof. ∎

CHAPTER 2

Hyperbolicity and completeness

2.1. c^i-hyperbolicity versus c-hyperbolicity

Recall that a domain $G \subset \mathbb{C}^n$ is called c^i_G*-hyperbolic* (or briefly c^i*-hyperbolic*), respectively c_G*-hyperbolic* (briefly c*-hyperbolic*), if c^i_G, respectively c_G, is a true distance on G. In view of the inequality $c_G \le c^i_G$, if G is c_G-hyperbolic, then it is c^i_G-hyperbolic. If G is bounded, then c_G is a distance. In the general case, the following result due to J.-P. Vigué (cf. [Vig 1996]) gives a characterization of c^i_G-hyperbolicity.

THEOREM 2.1.1. *Let $G \subset \mathbb{C}^n$ be a domain. Then the following properties are equivalent*:

(i) *G is c^i_G-hyperbolic*;
(ii) *there is no non-constant $\mathcal{C}^1$-curve $\alpha : [0,1] \to G$ such that $\gamma_G(\alpha;\alpha') \equiv 0$*;
(iii) *for any point $a \in G$ there exists a neighborhood $U = U(a) \subset G$ such that $c_G(a,z) \neq 0$, $z \in U \setminus \{a\}$.*

Proof. (i)$\Rightarrow$(ii). Suppose the contrary, namely, that there exists a $\mathcal{C}^1$-curve $\alpha : [0,1] \to G$ such that

$$\gamma_G(\alpha;\alpha') \equiv 0, \qquad \alpha'(t_0) \neq 0 \ \text{ for a } t_0 \in [0,1].$$

Obviously, for any $0 \le t' < t'' \le 1$ we then have $c^i_G(\alpha(t'),\alpha(t'')) = 0$. Since $\alpha'(t_0) \neq 0$ there are two different points $\alpha(t')$, $\alpha(t'')$ showing that G is not c^i-hyperbolic. Contradiction.

(ii)$\Rightarrow$(iii). We proceed by assuming the contrary. So let $a \in G$ be a point such that there exists a sequence of points $(z^j)_{j\in\mathbb{N}} \subset G \setminus \{a\}$, $z^j \to a$, such that $c_G(a,z^j) = 0$, $j \in \mathbb{N}$. We have to find a $\mathcal{C}^1$-curve which does have the property stated in (ii).

Observe that $A := \{z \in G : c_G(a,z) = 0\} = \{z \in G : f(a) = f(z),\ f \in \mathcal{O}(G,E)\}$ is an analytic subset of G. In view of the existence of the points $z^j \in A \setminus \{a\}$ tending to a, the dimension of the analytic set A in a is at least 1. Therefore, there is a $\mathcal{C}^1$-curve $\alpha : [0,1] \to \operatorname{Reg} A$ such that $\alpha' \not\equiv 0$. On the other hand, since this curve lies in A, we have $\gamma_G(\alpha;\alpha') \equiv 0$, a contradiction.

(iii)$\Rightarrow$(i). Fix $a, b \in G$, $a \neq b$, and choose a neighborhood $U = U(a) \subset G$ according to (iii). Moreover, let $V = V(a) \Subset U$, $b \notin V$. Obviously, $0 < c_G(a,z)$, $z \in U \setminus \{a\}$. Applying the continuity of c_G there is a $C > 0$ such that $c_G(a,\cdot)|_{\partial V} \ge C$. Thus for any $\mathcal{C}^1$-curve $\alpha : [0,1] \to G$, $\alpha(0) = a$, $\alpha(1) = b$, there is a $t_0 \in (0,1)$ with $\alpha(t_0) \in \partial V$; therefore, $L_c(\alpha) \ge c_G(a,\alpha(t_0)) + c_G(\alpha(t_0),b) \ge C > 0$. Hence, $c^i_G(a,b) \ge C > 0$. ■

Moreover, there is the following general relation between γ_G-hyperbolicity and local c-hyperbolicity.

PROPOSITION 2.1.2. *Any domain $G \subset \mathbb{C}^n$ that is γ_G-hyperbolic* (*i.e.* $\gamma_G(z;X) > 0$, $z \in G$, $X \in \mathbb{C}^n \setminus \{0\}$) *is locally c_G-hyperbolic* (*i.e. for any $a \in G$ there exists a neighborhood $U = U(a) \subset G$ such that c_G is a distance on U*). *In particular, G is c^i-hyperbolic.*

Proof. Fix an $a \in G$ and suppose that $z^j, w^j \to a$, $z^j \neq w^j$, $c_G(z^j, w^j) = 0$, $j = 1, 2, \dots$. We may assume that $(z^j - w^j)/\|z^j - w^j\| \to X_0 \in \partial\mathbb{B}_n$. Then

$$\gamma_G(a; X_0) = \lim_{j\to\infty} \frac{c_G(z^j, w^j)}{\|z^j - w^j\|} = 0,$$

a contradiction (cf. §1.2). ∎

Observe that the result is true for any $\mathcal{C}^1$-pseudodistance (cf. §1.2.4).

REMARK 2.1.3. $\boxed{?}$ It seems to be unknown whether c-hyperbolicity implies γ-hyperbolicity. $\boxed{?}$

EXAMPLE 2.1.4. There is a domain $G \subset \mathbb{C}^3$ which is not c_G-hyperbolic and not γ_G-hyperbolic, but nevertheless c^i_G-hyperbolic (see [Vig 1996]). This G is constructed via an example of a 1-dimensional complex space and then applying the Remmert embedding theorem. We omit the details here.

REMARK 2.1.5. Notice that Example 2.1.4 is not explicitly given. So $\boxed{?}$ it is interesting to find an effective example of that type; moreover, the question whether such an example is possible in $\mathbb{C}^2$ is still open. $\boxed{?}$

2.2. Hyperbolicity for Reinhardt domains

Before we discuss the different notions of hyperbolicity in the case of pseudoconvex Reinhardt domains, we recall the effective formulas for the Kobayashi pseudodistance on elementary Reinhardt domains (cf. Theorem 1.3.1). Let

$$V_j := \{z \in \mathbb{C}^n : z_j = 0\}, \qquad j = 1, \dots, n.$$

Moreover, for a matrix $A = (A^j_k)_{j=1,\dots,n,\, k=1,\dots,n} \in \mathbb{Z}(n \times n)$, we denote by A^j its jth row. Put

$$\varPhi_A : \mathbb{C}^n_* \to \mathbb{C}^n_*, \qquad \varPhi(z) := (z^{A^1}, \dots, z^{A^n}).$$

THEOREM 2.2.1 ([Zwo 1999a]). *Let G be a pseudoconvex Reinhardt domain in $\mathbb{C}^n$. Then the following properties are equivalent*:

(i) *G is c_G-hyperbolic*;
(ii) *G is $\widetilde{k}_G$-hyperbolic*;
(iii) *G is Brody-hyperbolic* (*i.e.* $\mathcal{O}(\mathbb{C}, G) = \mathbb{C}$);
(iii′) $\log G$ (¹) *contains no affine lines, and either $V_j \cap G = \emptyset$ or $V_j \cap G$ is c-hyperbolic as a domain in $\mathbb{C}^{n-1}$, $j = 1, \dots, n$*;
(iv) *there exist* $A = (A^j_k)_{j=1,\dots,n,\, k=1,\dots,n} \in \mathbb{Z}(n \times n)$, $\operatorname{rank} A = n$, *and a vector $C = (C_1, \dots, C_n) \in \mathbb{R}^n$ such that*

(¹) $\log G := \{x \in \mathbb{R}^n : (e^{x_1}, \dots, e^{x_n}) \in G\}$.

- $G \subset G(A,C) := D_{A^1,C_1} \cap \cdots \cap D_{A^n,C_n}$,
- *either* $V_j \cap G = \emptyset$ *or* $V_j \cap G$ *is c-hyperbolic as a domain in* $\mathbb{C}^{n-1}$, $j = 1, \dots, n$;

(iv′) *there exist* $A \in \mathbb{Z}(n \times n)$, $|\det A| = 1$, *and a vector* $C \in \mathbb{R}^n$ *such that*

- $G \subset G(A,C)$ (cf. (iv)),
- *either* $V_j \cap G = \emptyset$ *or* $V_j \cap G$ *is c-hyperbolic as a domain in* $\mathbb{C}^{n-1}$, $j = 1, \dots, n$;

(v) *G is algebraically equivalent to a bounded domain (i.e. there is a matrix $A \in \mathbb{Z}(n \times n)$ such that Φ_A is defined on G and gives a biholomorphic mapping from G to the bounded domain $\Phi_A(G)$);*

(vi) *G is k_G-complete.*

In what follows, a domain of the type $G(A,C)$ (cf. (iv) in Theorem 2.2.1) will be briefly called a *quasi-elementary* Reinhardt domain.

To prove Theorem 2.2.1 we need the following lemmas.

LEMMA 2.2.2 ([Zwo 1999a]). *Let $G(A,C)$ be as in Theorem* 2.2.1. *Then*:

(a) *there is a matrix $\widetilde{A} \in \mathbb{Z}(n \times n)$, $|\det \widetilde{A}| = 1$, and a vector $\widetilde{C} \in \mathbb{R}^n$ such that $G(A,C) \subset G(\widetilde{A},\widetilde{C})$;*

(b) $c_{G(A,C)}(z,w) > 0$ *for any points* $z, w \in G(A,C) \cap \mathbb{C}^n_*$, $z \neq w$.

Proof. Fix a matrix A and a vector C as in Lemma 2.2.2.

STEP 1. To prove (a) it suffices to construct a sequence of quasi-elementary Reinhardt domains $G_0 := G(A,C) \subset \cdots \subset G_N$ such that $|\det G_j| < |\det G_{j-1}|$, where $\det G(A,C) := \det A$.

Assume that G_j has already been constructed. Let $G_j = G(B,D)$ with a matrix $B \in \mathbb{Z}(n \times n)$, $|\det B| \geq 1$, and a vector $D \in \mathbb{R}^n$. In the case when $|\det B| > 1$ we describe how to get G_{j+1}.

Put

$$\mathcal{S}(G_j) := \{\alpha \in \mathbb{Z}^n : z^\alpha \in \mathcal{H}^\infty(G_j)\}, \qquad \mathcal{B}(G_j) := \mathcal{S}(G_j) \setminus (\mathcal{S}(G_j) + \mathcal{S}(G_j)).$$

It is known (cf. [J-P 1993, Lemma 2.7.6]) that

$$\mathcal{S} := \mathcal{S}(G(B,D)) = \mathbb{Z}^n \cap (\mathbb{Q}_+ B^1 + \cdots + \mathbb{Q}_+ B^n),$$
$$\mathcal{B} := \mathcal{B}(G(B,D)) \subset \mathbb{Z}^n \cap (\mathbb{Q} \cap [0,1) B^1 + \cdots + \mathbb{Q} \cap [0,1) B^n) \cup \{B^1, \dots, B^n\}.$$

CLAIM. $\mathcal{B} \not\subset \{B^1, \dots, B^n\}$.

Assume the contrary, i.e. $\mathcal{B} \subset \{B^1, \dots, B^n\}$. Define

$$r(B) := \min\{r \in \mathbb{N} : \text{if } x \in \mathbb{Q}^n,\ xB \in \mathbb{Z}^n, \text{ then } rx \in \mathbb{Z}^n\}.$$

Observe that $B^{-1}B \in \mathbb{Z}(n \times n)$, i.e. all the rows of B^{-1} are special vectors in the definition of the number $r(B)$. So $r(B)B^{-1} \in \mathbb{Z}(n \times n)$, from which $r(B)^n = \det(r(B)B^{-1}B) = \det(r(B)B^{-1})\det(B)$ follows. Therefore, if $r(B) = 1$ then $|\det B| = 1$, which gives the contradiction.

So it remains to prove that $r(B) = 1$. Take an arbitrary $x \in \mathbb{Q}^n$ with $xB \in \mathbb{Z}^n$. We have to show that $x \in \mathbb{Z}^n$. In fact: we write $xB = uB + \nu B$, where $u = (u_1, \dots, u_n)$, $u_j := x_j - [x_j] \geq 0$ and $\nu = (\nu_1, \dots, \nu_n)$, $\nu_j := [x_j] \in \mathbb{Z}$, $j = 1, \dots, n$ (here $[x]$ denotes the largest integer smaller than or equal to x). Obviously, $uB \in \mathbb{Z}^n$. Applying the above description of $\mathcal{S}$, it follows that $uB = u_1B^1 + \dots + u_nB^n \in \mathcal{S}$ (recall that B^j is the jth row of B). By the assumption, uB is an entire linear combination of the vectors $B^1, \dots, B^n$; in particular (recall that the B^j's are linearly independent), $u \in \mathbb{Z}^n$. Hence, $x = u + \nu \in \mathbb{Z}^n$, i.e. $r(B) = 1$, as required.

Therefore, there is a $\beta \in \mathcal{B} \setminus \{B^1, \dots, B^n\}$ such that $\beta = t_1B^1 + \dots + t_nB^n$ with $t_j \in [0,1)$ and one of the t_j's is positive. We may assume that $t_1 > 0$. We denote by $\widetilde{B}$ the matrix whose rows $\widetilde{B}^j$ are given by $\widetilde{B}^1 := \beta$, $\widetilde{B}^j := B^j$, $j = 2, \dots, n$. Moreover, with $\widetilde{C}_1 := \sum_{j=1}^n t_jC_j$ and $\widetilde{C}_j := C_j$, $j = 2, \dots, n$, we put

$$G_{j+1} := G(\widetilde{B}, \widetilde{C}), \qquad \text{where } \widetilde{C} := (\widetilde{C}_1, \dots, \widetilde{C}_n).$$

Then $|\det \widetilde{B}| = t_1|\det B| < |\det B|$ and $G_j \subset G_{j+1}$. Hence, (a) is verified.

STEP 2. Recall that for a matrix $A \in \mathbb{Z}(n \times n)$ the mapping

$$\Phi_A : \mathbb{C}_*^n \to \mathbb{C}_*^n, \qquad \Phi_A(z) := (z^{A^1}, \dots, z^{A^n}),\ z \in \mathbb{C}_*^n,$$

is proper iff $\det A \neq 0$, and that in this case its multiplicity is given by $|\det A|$. In particular, the mapping $\Phi_{\widetilde{A}}$, $\widetilde{A}$ of (a), is a biholomorphic mapping from $\mathbb{C}_*^n$ to itself.

Now fix two different points $z, w \in G(A, C) \cap \mathbb{C}_*^n$. Then

$$c_{G(A,C)}(z,w) \geq c_{G(\widetilde{A},\widetilde{C})}(z,w) = c_{G(\widetilde{A},\widetilde{C}) \cap \mathbb{C}_*^n}(z,w) = c_{E^n}(\Psi(z), \Psi(w)) > 0,$$

where $\Psi(z) := (\Phi_1(z)/e^{\widetilde{C}_1}, \dots, \Phi_n(z)/e^{\widetilde{C}_n})$ with $\Phi_{\widetilde{A}} =: (\Phi_1, \dots, \Phi_n)$. ∎

LEMMA 2.2.3. *Let $\Omega \subset \mathbb{R}^n$ be a convex domain containing no straight lines. Then there are linearly independent vectors $A^1, \dots, A^n \in \mathbb{Z}^n$ and a $C \in \mathbb{R}^n$ such that*

$$\Omega \subset \{x \in \mathbb{R}^n : \langle x, A^j \rangle < C_j,\ j = 1, \dots, n\}.$$

Proof. See [Vla 1993]. ∎

Proof of Theorem 2.2.1. First, observe that the implications (i)⇒(ii)⇒(iii) are obvious and that (iv)⇒(iv′) is true due to Lemma 2.2.2.

The remaining proof uses induction on the dimension n. Obviously, the theorem is true in the case $n = 1$. Now, let $n \geq 2$.

(iii)⇒(iii′). The first condition is an obvious consequence of (iii). The second one follows from the induction process.

(iii′)⇒(iv). Note that the second condition in (iv) follows by applying the theorem in the case $n - 1$. From (iii) we see that $\log G$ does not contain straight lines. Therefore, we immediately get (iv) from Lemma 2.2.3.

(iv′)⇒(i). Take $z, w \in G$, $z \neq w$.

CASE 1. If both points belong to $\mathbb{C}_*^n$, then, in view of Lemma 2.2.2, we have

$$c_G(z,w) \geq c_{G(A,C)}(z,w) > 0.$$

CASE 2. Let $z \in \mathbb{C}_*^n$, $w \notin \mathbb{C}_*^n$. Without loss of generality we may assume that $w = (w_1, \dots, w_k, 0, \dots, 0)$ with $w_1 \cdots w_k \neq 0$. Then $k < n$ and $A_s^j \geq 0$, $j = 1, \dots, n$, $s = k+1, \dots, n$. Since $\operatorname{rank} A = n$ we find a $j \in \{1, \dots, n\}$ and an $r \in \{k+1, \dots, n\}$ such that $A^j_r > 0$. Thus $w^{A^j} = 0 \neq z^{A^j}$. Therefore,

$$c_G(z,w) \geq c_{G(A^j,C_j)}(z,w) \geq c_D(z^{A^j}, w^{A^j}) > 0, \qquad \text{where } D := e^{C_j}E.$$

CASE 3. Let $z, w \notin \mathbb{C}^n_*$. We may assume that $z_1 = 0$ and $z_2 \neq w_2$. Consequently, $\pi_{2,\dots,n}(G)$ ([2]) is c-hyperbolic and $\pi_{2,\dots,n}(z) \neq \pi_{2,\dots,n}(w)$. Therefore,

$$c_G(z,w) \geq c_{\pi_{2,\dots,n}(G)}(\pi_{2,\dots,n}(z), \pi_{2,\dots,n}(w)) > 0.$$

Hence G is c-hyperbolic.

(iv′)⇒(v). By (iv′) we know that there is a matrix $A \in \mathbb{Z}(n \times n)$, $|\det A| = 1$, and a vector $C \in \mathbb{R}^n$ with $G \subset G(A, C)$. Moreover, the mapping $\Phi_A : \mathbb{C}^n_* \to \mathbb{C}^n_*$, $\Phi_A(z) := (z^{A^1}, \dots, z^{A^n})$, is biholomorphic.

Therefore, if the domain G is contained in $\mathbb{C}^n_*$, then $\Phi_A : G \to \Phi_A(G)$ is a biholomorphic mapping and $\Phi_A(G)$ is bounded.

The remaining case is done by induction. Obviously, the case $n = 1$ is clear. So we may assume that $n \geq 2$ and, without loss of generality, that $V_n \cap G \neq \emptyset$.

CLAIM. It suffices to prove (v) under the additional assumption that

$$V_n \cap G \neq \emptyset \text{ and } \pi_j(G) \text{ is bounded}, \; j = 1\dots, n-1. \tag{2.2.1}$$

In fact, put $\widetilde{G} := G \cap V_n$. By assumption, $\widetilde{G}$ is a c-hyperbolic pseudoconvex Reinhardt domain in $\mathbb{C}^{n-1}$. By the induction hypothesis there exists a matrix $\widetilde{A} \in \mathbb{Z}((n-1) \times (n-1))$ such that $\Phi_{\widetilde{A}}$ is defined on $\widetilde{G}$, $\Phi_{\widetilde{A}}(\widetilde{G})$ is bounded, and $\Phi_{\widetilde{A}} : \widetilde{G} \to \Phi_{\widetilde{A}}(\widetilde{G})$ is biholomorphic. Put

$$B := \begin{bmatrix} \widetilde{A} & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{Z}(n \times n).$$

Then Φ_B satisfies condition (2.2.1), and so the claim has been verified.

For the remaining part of the proof of (v) we may now assume that (2.2.1) holds true. Without loss of generality assume further that

$$V_j \cap G \neq \emptyset, \quad j = 1, \dots, k, \qquad V_j \cap G = \emptyset, \quad j = k+1, \dots, n-1.$$

Put $\widetilde{G} := V_1 \cap \dots \cap V_k \cap G$. Then $\widetilde{G}$ is a (non-empty) c-hyperbolic pseudoconvex Reinhardt domain. Hence there is $\alpha = (0, \dots, 0, \alpha_{k+1}, \dots, \alpha_n) \in \mathcal{S}(\widetilde{G})$, $\alpha_n \neq 0$. The fact that $\widetilde{G} \cap V_n \neq \emptyset$ implies $\alpha_n > 0$. Moreover, by (2.2.1), it is clear that $e_j := (0, \dots, 0, 1, 0, \dots, 0) \in \mathcal{S}(\widetilde{G})$ (1 at the jth position), $j = k+1, \dots, n-1$. Thus

$$\widetilde{\alpha} := \frac{1}{\alpha_n}\alpha + \sum_{j=k+1}^{n-1} \left(\left[\frac{\alpha_j}{\alpha_n} \right] + 1 - \frac{\alpha_j}{\alpha_n} \right) e_j \in \mathcal{S}(\widetilde{G}) \subset \mathcal{S}(G).$$

Define

$$A := \begin{bmatrix} & & \mathbb{I}_{n-1} & & & & 0 \\ 0 & \dots & 0 & \widetilde{\alpha}_{k+1} & \dots & \widetilde{\alpha}_{n-1} & 1 \end{bmatrix}.$$

Then A has all the required properties. Hence condition (v) is proved.

(v)⇒(vi). By assumption we may assume that G is a bounded pseudoconvex Reinhardt domain. Fix a point $w \in G$. To verify that G is k-complete we only have to disprove the existence of a sequence $(z^j)_{j \in \mathbb{N}} \subset G$ such that $(k_G(w, z^j))_{j \in \mathbb{N}}$ is bounded, but $z^j \to z^0 \in \partial G$ as $j \to \infty$.

([2]) $\pi_{i_1,\dots,i_k}(z_1, \dots, z_n) := (z_{i_1}, \dots, z_{i_k})$.

CASE $z^0 \in \mathbb{C}^n_*$. We may assume that $z^0 = (1, \dots, 1)$. It is clear that there is an $\alpha \in \mathbb{R}^n$, $\alpha \neq 0$, such that $G \subset D_\alpha$, where D_α denotes the elementary Reinhardt domain for α. Moreover, we may assume that $\alpha_j \neq 0$, $j = 1, \dots, k$, and $\alpha_{k+1} = \dots \alpha_n = 0$, where $k \geq 1$. So we get

$$\begin{aligned}(2.2.2) \qquad k_G(w, z^j) &\geq k_{D_\alpha}(w, z^j) \\ &= \max\{k_{D_{\widetilde{\alpha}}}(\widetilde{w}, \widetilde{z}^j),\ k_{\mathbb{C}^{n-k}}((w_{k+1}, \dots, w_n), (z^j_{k+1}, \dots, z^j_n))\} = k_{D_{\widetilde{\alpha}}}(\widetilde{w}, \widetilde{z}^j),\end{aligned}$$

where $\widetilde{\alpha} := (\alpha_1, \dots, \alpha_k)$, $\widetilde{w} := (w_1, \dots, w_k)$, $\widetilde{z}^j := (z^j_1, \dots, z^j_k)$.

Observe that the sequence $(\widetilde{z}^j)_{j\in\mathbb{N}}$ converges to the boundary point $\widetilde{z}^0$ of $D_{\widetilde{\alpha}}$. Then, applying Theorem 1.3.1, we see that the sequence in (2.2.2) tends to infinity.

CASE $z^0 \notin \mathbb{C}^n_*$. Assume that $z^0_j \neq 0$ for $j = 1, \dots, k$ with a suitable k, $0 \leq k < n$, and $z^0_{k+1} = \dots = z^0_n = 0$. We have to discuss two subcases:

(a) There is an $s \in \{k+1, \dots, n\}$ such that $G \cap V_s = \emptyset$. Then

$$k_G(w, z^j) \geq k_{\pi_s(G)}(w_s, z^j_s).$$

Here, $\pi_s(G)$ is a plane Reinhardt domain not containing the origin, but $0 \in \partial\pi_s(G)$. Therefore, the right side tends to infinity.

(b) All the intersections $G \cap V_j$, $j = k+1, \dots, n$, are non-empty. Obviously, $k > 0$, otherwise $z^0 = 0 \in G$, a contradiction. Hence

$$k_G(w, z^j) \geq k_{\widetilde{G}}((w_1, \dots, w_k), (z^j_1, \dots, z^j_k)),$$

where $\widetilde{G} := \pi_{1,\dots,k}(G)$. Since $(z^0_1, \dots, z^0_k) \in \partial\widetilde{G}$ and $\widetilde{G}$ is a Reinhardt domain of the first case, the right side again tends to infinity.

Hence, the Kobayashi completeness of G has been verified.

It remains to mention that (vi) trivially implies (iv). ∎

REMARK 2.2.4. Observe that Theorem 2.2.1 shows that all notions of hyperbolicity coincide in the class of pseudoconvex Reinhardt domains. That is why we will often speak only of *hyperbolic pseudoconvex Reinhardt domains*. Moreover, in that class "hyperbolic" and "Kobayashi-complete" are the same notions.

REMARK 2.2.5. The pseudoconvex Reinhardt domain

$$D := \{z \in \mathbb{C}^3 : \max\{|z_1z_2|, |z_1z_3|, |z_2|, |z_3|\} < 1\}$$

is not k-hyperbolic since $\mathbb{C} \times \{0\} \times \{0\} \subset D$; in particular, D is not c-hyperbolic. Let $\widetilde{D} := D \setminus (\mathbb{C} \times \{0\} \times \{0\})$. Then $\widetilde{D}$ is c-hyperbolic (the functions z_1z_2, z_1z_3, z_2, and z_3 separate the points of $\widetilde{D}$). Observe that D is the envelope of holomorphy of $\widetilde{D}$, i.e. $D = \mathcal{H}(\widetilde{D})$. Hence, in general, c-hyperbolicity of a Reinhardt domain and its envelope of holomorphy may be different.

But in the two-dimensional case, there is the following positive result [Die-Hai 2003].

THEOREM 2.2.6. *Let $G \subset \mathbb{C}^2$ be a c-hyperbolic Reinhardt domain. Then its envelope of holomorphy $\mathcal{H}(G)$ is c-hyperbolic.*

Proof. Recall that the envelope of holomorphy $\mathcal{H}(D)$ of a Reinhardt domain $D \subset \mathbb{C}_*^n$ has the following properties:

- $\mathcal{H}(D) \subset \mathbb{C}_*^n$,
- $\log \mathcal{H}(D) = \operatorname{conv}(\log D)$.

Put $G_* := G \cap \mathbb{C}_*^2$. Then G_* is a Reinhardt domain. Assume that $\log \mathcal{H}(G_*)$ contains an affine line ℓ. Fix a point $x_0 \in \log G \setminus \ell$. Denote by ℓ' the line passing through x_0 which is parallel to ℓ. Then $\ell' \subset \log \mathcal{H}(G_*)$. Let

$$\ell' = \{(a_1 t + b_1, a_2 t + b_2) : t \in \mathbb{R}\},$$

where $a_1, a_2, b_1, b_2 \in \mathbb{R}$ and $a_1^2 + a_2^2 \neq 0$. Hence

$$A := \{(e^{a_1\lambda + b_1}, e^{a_2\lambda + b_2}) : \lambda \in \mathbb{C}\} \subset \mathcal{H}(G_*).$$

Using Liouville's theorem and the fact that G is c-hyperbolic, we get $A \cap G = \emptyset$ or $\ell' \cap \log G = \emptyset$, a contradiction.

Assume now that $\log \mathcal{H}(G)$ contains an affine line. As in the previous step, this leads to a non-trivial entire map $\varphi : \mathbb{C} \to \mathcal{H}(G) \cap \mathbb{C}_*^n$. Recall that $\mathcal{H}(G_*) = \mathcal{H}(G) \cap \mathbb{C}_*^n$ (see Theorem 2.5.9 in [Jar-Pfl 2000]). Hence, $\mathcal{H}(G_*)$ contains an affine line, a contradiction.

Without loss of generality, assume finally that $\mathcal{H}(G) \cap V_2 \neq \emptyset$. Denote this intersection by $G' \subset \mathbb{C}$. Suppose that G' is not c-hyperbolic. Then either $G' = \mathbb{C}$ or $G' = \mathbb{C}_*$. Therefore, either $A_1 := \mathbb{C} \times \{0\} \subset \mathcal{H}(G)$ or $A_2 := \mathbb{C}_* \times \{0\} \subset \mathcal{H}(G)$. In view of the c-hyperbolicity of G, we conclude that $A_1 \cap G = \emptyset$ or that $A_2 \cap G = \emptyset$. Therefore, $G \cap V_2 = \emptyset$, a contradiction. Thus Theorem 2.2.1 implies that $\mathcal{H}(G)$ is c-hyperbolic. ∎

We conclude this section with the following result which will be useful later.

Proposition 2.2.7 ([Zwo 2000a]). *Let $G \subset \mathbb{C}^n$ be a hyperbolic pseudoconvex Reinhardt domain. Then the following conditions are equivalent*:

(i) *G is algebraically equivalent to an unbounded Reinhardt domain*;

(ii) *G is algebraically equivalent to a bounded Reinhardt domain D for which there is a j_0, $1 \le j_0 \le n$, such that $\overline{D} \cap V_{j_0} \neq \emptyset$, but $D \cap V_{j_0} = \emptyset$.*

Proof. (i)⇒(ii). We may assume that G is an unbounded hyperbolic pseudoconvex Reinhardt domain. By Theorem 2.2.1, there are a bounded Reinhardt domain D and a biholomorphic mapping $\Phi_A : D \to G$ (here we use the notation from Theorem 2.2.1). Suppose that D satisfies the following condition:

$$\text{if } \overline{D} \cap V_j \neq \emptyset \text{ then } D \cap V_j \neq \emptyset,\ j = 1, \dots, n.$$

Without loss of generality, we may assume that there is a $k \in \{0, 1, \dots, n\}$ such that

$$\overline{D} \cap V_j \neq \emptyset, \quad j = 1, \dots, k, \qquad \overline{D} \cap V_j = \emptyset, \quad j = k+1, \dots, n. \tag{2.2.3}$$

Now, let $A = (A_j^r)_{r=1,\dots,n,\, j=1,\dots,n} \in \mathbb{Z}(n \times n)$. Then $A_j^r \ge 0$, $j = 1, \dots, k$, $r = 1, \dots, n$. Moreover, using (2.2.3) and the fact that D is bounded, we find a positive M such that

$$|z_j| \ge M, \qquad z \in D,\ k+1 \le j \le n.$$

Hence, $\sup\{|z^{A^r}| : z \in D\} < \infty$, $r = 1, \dots, n$, which implies that G is bounded, a contradiction.

(ii)$\Rightarrow$(i). Observe that the mapping

$$D \ni z \mapsto \left(z_1, \dots, z_{j_0-1}, \frac{1}{z_{j_0}}, z_{j_0+1}, \dots, z_n\right)$$

maps D biholomorphically onto an unbounded pseudoconvex Reinhardt domain $\widetilde{D}$. Thus D is algebraically equivalent to $\widetilde{D}$ and so is G. ■

2.3. Hyperbolicities for balanced domains

Recall that any balanced domain is k-hyperbolic if and only if it is bounded (cf. Theorem 7.1.2 in [Jar-Pfl 1999a]).

EXAMPLE 2.3.1 ([Azu 1983], see also [J-P 1993, Example 7.1.4]). Observe that there is an unbounded pseudoconvex balanced domain $G \subset \mathbb{C}^2$ that is Brody-hyperbolic. To be more concrete G is defined as

$$G := \{z \in \mathbb{C}^2 : h(z) < 1\},$$

where

$$h(z) := \begin{cases} |z_2| e^{\varphi(z_1/z_2)} & \text{if } z_2 \neq 0, \\ |z_1| & \text{if } z_2 = 0, \end{cases} \qquad \varphi(\lambda) := \max\left\{\log|\lambda|, \sum_{j=2}^{\infty} \frac{1}{k^2} \log\left|\lambda - \frac{1}{k}\right|\right\}, \quad \lambda \in \mathbb{C}.$$

Recently, S.-H. Park [Par 2003] has shown that G is almost $\widetilde{k}$-*hyperbolic*, i.e. $\widetilde{k}_G(z,w) > 0$, whenever $z_1 \neq w_1$ or ($z_1 = w_1 \neq 0$ and $z_2 \neq w_2$). $\boxed{?}$ It is still unclear whether $\widetilde{k}_G((0,z_2),(0,w_2)) > 0$ for $z_2 \neq w_2$. $\boxed{?}$

Nevertheless, there is the following result (cf. [Par 2003]).

PROPOSITION 2.3.2. *For any $n \geq 3$ there exists a pseudoconvex balanced domain $G \subset \mathbb{C}^n$ such that*

- *G is Brody-hyperbolic,*
- *G is not $\widetilde{k}_G$-hyperbolic.*

Proof. Obviously, it suffices to construct such an example G in $\mathbb{C}^3$. Then, in the general case, $G \times E^{n-3}$ will do the job in $\mathbb{C}^n$.

So let $n = 3$. Put $r_j := e^j$, $s_j := 1/(r_j^2 + r_j)$, $t_j := \sqrt{j}/s_j$, $\varepsilon_j := 2^{-j-1}$, and $\eta_j := t_j s_j$, $j \in \mathbb{N}$. Then

$$\sum_{j=1}^{\infty} \varepsilon_j = \frac{1}{2}, \qquad \sum_{j=1}^{\infty} \varepsilon_j \log\frac{1}{\eta_j} \geq \sum_{j=1}^{\infty} \varepsilon_j \log\frac{1}{t_j} > -\infty.$$

For $j \in \mathbb{N}$ define

$$Q_j(z) := z_1 z_2 - s_j(z_3 - z_2)(z_3 - 2z_2), \qquad z = (z_1, z_2, z_3) \in \mathbb{C}^3.$$

Put

$$G := \{z \in \mathbb{C}^3 : h(z) < 1\} \quad \text{with} \quad h(z) := \max\{|z_1|, |z_2|/2, h_0(z)\},$$

where

$$h_0(z) := \prod_{j=1}^{\infty} \left|\frac{Q_j(z)}{\eta_j}\right|^{\varepsilon_j} = \exp\left(\sum_{j=1}^{\infty} \varepsilon_j \log\frac{|Q_j(z)|}{\eta_j}\right).$$

We claim that G is a pseudoconvex balanced domain that is Brody-hyperbolic, but not $\widetilde{k}$-hyperbolic.

STEP 1. h is absolutely homogeneous and positive definite. Indeed, it suffices to discuss h_0. Fix $z \in \mathbb{C}^3$ and $\lambda \in \mathbb{C}$. Then

$$\sum_{j=1}^{\infty} \varepsilon_j \log \frac{|Q_j(\lambda z)|}{\eta_j} = \sum_{j=1}^{\infty} \varepsilon_j \log(|\lambda|^2) + \sum_{j=1}^{\infty} \varepsilon_j \log \frac{|Q_j(z)|}{\eta_j} = \log|\lambda| + \sum_{j=1}^{\infty} \varepsilon_j \log \frac{|Q_j(z)|}{\eta_j}.$$

Hence, $h_0(\lambda z) = |\lambda| h_0(z)$.

Assume now that $h(z) = 0$. Then $z_1 = z_2 = 0 = h_0(z)$, which implies that

$$-\infty = \sum_{j=1}^{\infty} \varepsilon_j \log \frac{|Q_j(0,0,z_3)|}{\eta_j} = \sum_{j=1}^{\infty} \varepsilon_j \log \frac{1}{t_j} + \frac{1}{2}\log(|z_3|^2),$$

from which we obtain $z_3 = 0$. Hence, h is positive definite.

STEP 2. $h_0 \in \mathcal{PSH}(\mathbb{C}^3)$ (in particular, G is pseudoconvex). Indeed, fix a positive R and let $z \in (RE)^3$. Then $|Q_j(z)| \le (1+6)R^2$. Recall that $\eta_j \to \infty$. Therefore, there is a j_R such that

$$|Q_j(z)|/\eta_j < 1, \qquad z \in (RE)^3,\ j \ge j_R.$$

So it follows that $h_0 \in \mathcal{PSH}((RE)^3)$ for arbitrary R. Hence, $h_0 \in \mathcal{PSH}(\mathbb{C}^3)$.

STEP 3. G is not $\widetilde{k}$-hyperbolic. Indeed, let

$$\varphi_j \in \mathcal{O}(\mathbb{C}, \mathbb{C}^3), \quad \varphi_j(\lambda) := (s_j\lambda(\lambda-1), 1, \lambda+1), \qquad j \in \mathbb{N}.$$

Observe that $Q_j \circ \varphi_j = 0$ on $\mathbb{C}$, $j \in \mathbb{N}$. Therefore, $\varphi_j(\lambda) \in G$ if $|\lambda| < r_j$. In particular,

$$\widetilde{k}_G((0,1,1),(0,1,2)) = \widetilde{k}_G(\varphi_j(0), \varphi_j(1)) \le k_E(0, 1/r_j) \xrightarrow[j\to\infty]{} 0,$$

meaning that G is not $\widetilde{k}$-hyperbolic.

STEP 4. G is Brody-hyperbolic. Indeed, let $f = (f_1, f_2, f_3) \in \mathcal{O}(\mathbb{C}, G)$. In view of the form of G, f_j is bounded and so $f_j \equiv: a_j$, $j = 1, 2$. Suppose that f_3 is not constant. Then, by Picard's theorem, we have $\mathbb{C} \setminus \{w\} \subset f_3(\mathbb{C})$ for a suitable $w \in \mathbb{C}$. Hence, $h(a_1, a_2, \cdot) < 1$ on $\mathbb{C} \setminus \{w\}$. Using Liouville's theorem for subharmonic functions, we conclude that $h_0(a_1, a_2, \cdot) \equiv \text{const}$. Note that $h_0(a_1, a_2, \lambda) = 0$ if $Q_j(a_1, a_2, \lambda) = 0$ for at least one j. Therefore, $h_0(a_1, a_2, \cdot) \equiv 0$.

To get a contradiction we discuss different cases of a_1, a_2.

CASE $a_2 = 0$. Then $Q_j(a_1, 0, \lambda) = -s_j\lambda^2$, $j \in \mathbb{N}$. Therefore,

$$\log h_0(a_1, 0, 1) = \sum_{j=1}^{\infty} \varepsilon_j \log \frac{|Q_j(a_1,0,1)|}{\eta_j} = \sum_{j=1}^{\infty} \varepsilon_j \log \frac{1}{t_j} > -\infty,$$

a contradiction.

CASE $a_2 \ne 0$, $a_1 = 0$. Then $Q_j(0, a_2, 0) = -2s_j a_2^2$, $j \in \mathbb{N}$. Therefore,

$$\log h_0(0, a_2, 0) = \sum_{j=1}^{\infty} \varepsilon_j \log \frac{1}{t_j} + \log(2|a_2|^2) \sum_{j=1}^{\infty} \varepsilon_j > -\infty,$$

a contradiction.

CASE $a_1a_2 \neq 0$. Then

$$\log h_0(a_1, a_2, a_2) = \sum_{j=1}^{\infty} \varepsilon_j \log \frac{|a_1a_2|}{\eta_j} > -\infty,$$

a contradiction.

Hence, G is Brody-hyperbolic. ■

REMARK 2.3.3. $\boxed{?}$ It remains an open question whether such an example exists in $\mathbb{C}^2$. $\boxed{?}$

2.4. Hyperbolicities for Hartogs type domains

Let $G \subset \mathbb{C}^n$ be an arbitrary domain. A domain $D = D(G) \subset G \times \mathbb{C}^m$ is called a *Hartogs domain over G with m-dimensional balanced fibers* if for any $z \in G$ the fiber $D_z := \{w \in \mathbb{C}^m : (z, w) \in D\}$ is a non-empty balanced domain in $\mathbb{C}^m$. Recall that for such a D there exists exactly one upper semicontinuous function $H : G \times \mathbb{C}^m \to [0, \infty)$, $H(z, \lambda w) = |\lambda| H(z, w)$, $z \in G$, $w \in \mathbb{C}^m$, $\lambda \in \mathbb{C}$, such that

$$D_H = D = \{(z, w) \in G \times \mathbb{C}^m : H(z, w) < 1\}.$$

Conversely, any such H leads to a Hartogs domain over G with m-dimensional balanced fibers.

Recall that $D = D_H$ is pseudoconvex if and only if G is pseudoconvex and $\log H \in \mathcal{PSH}(G \times \mathbb{C}^m)$.

Then we have the following hyperbolicity criterion (cf. [DDT-Tho 1998], see also [DDT-PVD 2000]).

THEOREM 2.4.1. *Let $D = D_H \subset G \times \mathbb{C}^m$ be a Hartogs domain over $G \subset \mathbb{C}^n$ with m-dimensional balanced fibers. If D is k-hyperbolic, then G is k-hyperbolic and, for any compact set $K \subset G$, the function $\log H$ is bounded from below on $K \times \partial\mathbb{B}_m$.*

Proof. If D is k-hyperbolic then $k_G(z', z'') \geq k_D((z', 0), (z'', 0)) > 0$ for all $z', z'' \in G$, $z' \neq z''$. Hence G is k-hyperbolic.

Assume now that there are two sequences $(z^j)_{j\in\mathbb{N}} \subset G$ with $\lim z^j =: z^0 \in G$ and $(w^j)_{j\in\mathbb{N}} \subset \partial\mathbb{B}_m$ with $\lim w^j =: w^0 \in \partial\mathbb{B}_m$ such that $\lim_{j\to\infty} H(z^j, w^j) = 0$. We may assume that $(z^j, w^j) \in D$, $j \in \mathbb{N}$. Then $\varphi_j \in \mathcal{O}(\mathbb{C}, G \times \mathbb{C}^m)$, $\varphi_j(\lambda) := (z^j, \lambda w^j)$, maps R_jE into D for a suitable sequence $(R_j)_{j\in\mathbb{N}}$ with $R_j \to \infty$ as $j \to \infty$. Therefore,

$$k_D((z^j, 0), (z^j, w^j)) = k_D(\varphi_j(0), \varphi_j(1)) \leq k_E(0, 1/R_j) \to 0;$$

hence, $k_D((z^0, 0), (z^0, w^0)) = 0$, a contradiction. ■

REMARK 2.4.2. $\boxed{?}$ It seems unknown whether the converse of Theorem 2.4.1 also holds. $\boxed{?}$ Nevertheless, the following special case is true (cf. [DDT-Tho 1998]).

PROPOSITION 2.4.3. *Let $G := E$ and $u : E \to [-\infty, \infty)$ be upper semicontinuous. Put $H(z, w) := |w|e^{u(z)}$ and $D := D_H$. Assume that u is locally bounded from below. Then D is k-hyperbolic.*

Proof. By Theorem 7.2.2 in [J-P 1993], it suffices to show that the Kobayashi–Royden pseudometric is locally positive definite, i.e. for any point $p_0 = (z_0, w_0) \in D$ there exist a neighborhood $U = U(p_0) \subset D$ and a positive number C such that $\varkappa_D(p; X) \geq C\|X\|$, $p \in U$, $X \in \mathbb{C}^2$.

First, observe that

$$g(r) := -\inf\{u(\lambda) : \lambda \in E,\ |\lambda| \leq r\} < \infty, \qquad r \in (0,1).$$

Now, let $s \in (0,1)$ and fix $(z_0, w_0) \in D$, $|z_0| < s$, and $X \in \mathbb{C}^2 \setminus \{0\}$. Let $f \in \mathcal{O}(E, D)$ with $f(0) = (z_0, w_0)$ and $\alpha f'(0) = X$ for $\alpha \in \mathbb{C}_*$. By the Schwarz lemma, we see that $|f'(0)| \leq 1 - |z_0|^2 \leq 1$.

Put $r_0 := (1+2s)/(2+s)$. Applying the Schwarz lemma, it follows that, if $|f_1(\lambda)| \geq r_0$, then

$$|\lambda| \geq \left|\frac{z_0 - f_1(\lambda)}{1 - \overline{z}_0 f_1(\lambda)}\right| \geq \frac{|f_1(\lambda)| - |z_0|}{1 - |f_1(\lambda)|\,|z_0|} \geq \frac{r_0 - |z_0|}{1 - r_0|z_0|} \geq \frac{1}{2}.$$

Put $\Omega := \{\lambda \in E : |f_1(\lambda)| < r_0\}$. Then $\sup_\Omega |f_2| \leq e^{g(r_0)}$ and $\mathbb{B}_1(0, 1/2) \subset \Omega$. Thus, $|f_2'(0)| \leq 2e^{g(r_0)}$. Then

$$|\alpha| \geq \max\left\{|X_1|, \frac{|X_2|}{2e^{g(r_0)}}\right\} \geq \frac{1}{\sqrt{2}} \min\left\{1, \frac{1}{2e^{g(r_0)}}\right\}\|X\|.$$

Since f was arbitrarily chosen we get

$$\varkappa_D((z,w); X) \geq \frac{1}{\sqrt{2}} \min\left\{1, \frac{1}{2e^{g(r_0)}}\right\}\|X\|, \qquad (z,w) \in D,\ |z| < s.$$

Hence, D is k-hyperbolic. ∎

REMARK 2.4.4. In Remark 2.2.5 we mentioned that, if a Reinhardt domain in $\mathbb{C}^2$ is c-hyperbolic, then so is its envelope of holomorphy. In the class of Hartogs domains and the case of k-hyperbolicity, such a conclusion is false even in dimension 2 (cf. [Die-Hai 2003]).

Let $u : [0,1) \to (-\infty, 0)$ be a continuous function satisfying $\lim_{t \nearrow 1} \varphi(t) = -\infty$. Put $u(z_1) := \varphi(|z_1|)$. Then the domain

$$D := \{z \in E \times \mathbb{C} : |z_2| < e^{-u(z_1)}\}$$

is k-hyperbolic (see Proposition 2.4.3). Recall that

$$\mathcal{H}(D) = \{z \in E \times \mathbb{C} : |z_2| < e^{-\widehat{u}(z_1)}\},$$

where $\widehat{u}$ is the largest subharmonic minorant of u. By the maximum principle for subharmonic functions, it is clear that $\widehat{u} \equiv -\infty$. Therefore, $\mathcal{H}(D) = E \times \mathbb{C}$, which is not k-hyperbolic.

REMARK 2.4.5. So far, we discussed hyperbolicity. We close this part by a remark on the opposite situation. There is the following result due to E. Fornæss and N. Sibony [For-Sib 1981]: Let $D \subset \mathbb{C}^2$ be a domain which can be monotonically exhausted by domains D_j, where each D_j is biholomorphically equivalent to $\mathbb{B}_2$. If $\varkappa_D \not\equiv 0$, then D is biholomorphically equivalent either to $\mathbb{B}_2$ or to $E \times \mathbb{C}$. Observe that $\mathbb{B}_2(0, j) \nearrow \mathbb{C}^2$, $\varkappa_{\mathbb{C}^2} \equiv 0$, but, obviously, $\mathbb{C}^2$ is biholomorphic neither to $\mathbb{B}_2$ nor to $E \times \mathbb{C}$.

It turns out that there is a domain $D \subset \mathbb{C}^n$, $n \geq 2$, $D_j \nearrow D$, each D_j biholomorphically equivalent to $\mathbb{B}_n$, such that

- $\varkappa_D \equiv 0$,
- $\exists_{u\in\mathcal{PSH}(\mathbb{C}^n)} : D = \{z \in \mathbb{C}^n : u(z) < 0\}$ and $u|_D \not\equiv$ const; in particular, D is not biholomorphic to $\mathbb{C}^n$.

Domains of that type are called *short* $\mathbb{C}^n$*'s* (see [For 2004]). An example is obtained in the following way: Let $d \in \mathbb{N}$, $d \geq 2$, and $\eta > 0$. Denote by $\mathrm{Aut}_{d,\eta}$ the set of all polynomial automorphisms Φ of $\mathbb{C}^n$ of the form

$$\Phi(z) = \Phi(z_1, \dots, z_n) = (z_1^d + P_1(z), P_2(z), \dots, P_n(z)),$$

where $\deg P_j \leq d-1$, $j = 1, \dots, n$, and where each coefficient of the polynomials P_j has modulus at most η. Choosing sufficiently good sequences $a_j \searrow 0$, $a_j \in (0,1)$, and $F_j \in \mathrm{Aut}_{d,\eta_j}$ where $\eta_j := a_j^{d^j}$, $j \in \mathbb{N}$, we define

$$D = \{z \in \mathbb{C}^n : \lim_{k\to\infty} F_k \circ \dots \circ F_1(z) = 0\}.$$

2.5. c-completeness for Reinhardt domains

In this chapter Carathéodory completeness for Reinhardt domains will be discussed. Recall that a domain $G \subset \mathbb{C}^n$ is called c_G*-complete* (briefly, *c-complete*) (respectively, c_G-finitely compact (briefly, *c-finitely compact*)) if c_G is a distance and if any c_G-Cauchy sequence converges to a point in G (in the standard topology) (respectively, if c_G is a distance and if any c_G-ball with a finite radius is a relatively compact subset of G). Moreover, recall that any c_G-complete domain G is pseudoconvex.

THEOREM 2.5.1. *Let $G \subset \mathbb{C}^n$ be a pseudoconvex Reinhardt domain. Then the following conditions are equivalent*:

(i) *G is c_G-finitely compact*;
(ii) *G is c_G-complete*;
(iii) *there is no sequence $(z_\nu)_{\nu\in\mathbb{N}} \subset G$ with $\sum_{\nu=1}^{\infty} g_G(z_\nu, z_{\nu+1}) < \infty$*;
(iv) *G is bounded and satisfies the following* Fu condition:

$$\text{if } \overline{G} \cap V_j \neq \emptyset, \text{ then } G \cap V_j \neq \emptyset, \tag{2.5.4}$$

where $V_j := \{z \in \mathbb{C}^n : z_j = 0\}$.

This result is due W. Zwonek ([Zwo 2000a], see also [Zwo 2000b]); earlier partial results can be found in [Pfl 1984] (see also [J-P 1993]) and [Fu 1994].

For the proof of Theorem 2.5.1 we shall need the following three lemmas.

LEMMA 2.5.2 ([Zwo 2000a]). *Let $G \subset \mathbb{C}^n_*$ be a pseudoconvex Reinhardt domain. Then $k_G = \widetilde{k}_G$. In particular, the Lempert function $\widetilde{k}_G$ is continuous on $G \times G$.*

Proof. Observe that $T := \log G$ is a convex domain in $\mathbb{R}^n$ and that the mapping

$$T + i\mathbb{R}^n \ni z \overset{\Phi}{\mapsto} (e^{z_1}, \dots, e^{z_n}) \in G$$

is a holomorphic covering. Therefore, for $z, w \in G$ we have

$$\begin{aligned}\widetilde{k}_G(z,w) &= \inf\{\widetilde{k}_{T+i\mathbb{R}^n}(\widetilde{z}, \widetilde{w}) : \widetilde{z}, \widetilde{w} \in T + i\mathbb{R}^n \text{ with } \Phi(\widetilde{z}) = z,\ \Phi(\widetilde{w}) = w\} \\ &= \inf\{k_{T+i\mathbb{R}^n}(\widetilde{z}, \widetilde{w}) : \widetilde{z}, \widetilde{w} \in T + i\mathbb{R}^n \text{ with } \Phi(\widetilde{z}) = z,\ \Phi(\widetilde{w}) = w\} = k_G(z,w).\end{aligned}$$

Here we have used the theorem of Lempert. ∎

LEMMA 2.5.3. *Let $\Omega \subset \mathbb{R}^n$ be a unbounded convex domain which is contained in $\mathbf{X}_{j=1}^n(-\infty, R)$ for a certain number R. Then, for any point $a \in \Omega$, there exist a vector $v \in \mathbb{R}_-^n \setminus \{0\}$ and a neighborhood $V = V(a) \subset \Omega$ such that $V + \mathbb{R}_+ v \subset \Omega$.*

Proof. Take without loos of generality the point $a = 0$. Then the continuity of the Minkowski function h of Ω and the assumptions on Ω lead to a vector v on the unit sphere with $h(v) = 0$. Obviously, $v \in \mathbb{R}_-^n \setminus \{0\}$ and $\mathbb{R}_+ v \subset \Omega$. Finally, using the convexity of Ω, we see that for any open ball $V \subset \Omega$ with center a the following inclusion holds: $V + \mathbb{R}_+ v \subset \Omega$. ∎

LEMMA 2.5.4 ([Hay-Ken 1976]). *Let $H := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\}$, $b < 0$, and $M < 0$. Moreover, let $u \in \mathcal{SH}(H)$, $u < 0$, and $u(\lambda) \le M$ for all λ with $\operatorname{Re}\lambda = b$. Then $u \le M$ on $\{\lambda \in \mathbb{C}\ \operatorname{Re}\lambda \le b\}$.*

Now we are in a position to proceed with the proof of the above theorem.

Proof of Theorem 2.5.1. Observe that the implications (i)⇒(ii)⇒(iii) are obvious. Moreover, (iv)⇒(i) follows along the same lines as Theorem 7.4.6 in [J-P 1993].

Therefore, we need to prove (iii)⇒(iv) only. Suppose that this implication is false. Then, by Proposition 2.2.7, we may assume that G is bounded and does not satisfy the Fu condition (2.5.4). Moreover, without loss of generality we only have to deal with the following situation:

$$\begin{aligned}&\overline{G} \cap V_j \neq \emptyset \text{ but } G \cap V_j = \emptyset, \quad j = 1, \dots, k, \quad 1 \le k \le n,\\ &\overline{G} \cap V_j = \emptyset, \quad j = k+1, \dots, n,\end{aligned}$$

In fact, if $G \cap V_j \neq \emptyset$, then one can pass to the intersection of G with those coordinate axes.

Hence $G \subset \mathbb{C}_*^n$. We may also assume that $(1, \dots, 1) \in G$. Observe that $\log G$ is convex, bounded in all positive directions, unbounded in the first k negative directions, and bounded in the remaining negative directions. Thus, in view of Lemma 2.5.3 we find a small ball $V = V(0) \subset \log G$ with center 0 and a vector $v \in \mathbb{R}_-^n \setminus \{0\}$ such that $V + \mathbb{R}_+ v \subset \log G$. It is clear that $v_j = 0$, $j = k+1, \dots, n$. Without loss of generality, we may assume that $v_j < 0$, $j = 1, \dots, l$, where $l \le k$, $v_1 = -1$, and $v_{l+1} = \dots = v_n = 0$. Hence,

$$(e^{x_1}e^{-t}, e^{x_2}e^{tv_2}, \dots, e^{x_n}e^{tv_n}) \in G, \quad t > 0,\ x \in V.$$

Then, with $\alpha := -v$, we have an $\varepsilon > 0$ such that

$$(e^{\lambda}, \mu_2 e^{\lambda\alpha_2}, \dots, \mu_l e^{\lambda\alpha_l}, 1, \dots, 1) \in G, \quad \lambda \in H,\ e^{-\varepsilon} < |\mu_j| < e^{\varepsilon},\ j = 2, \dots, l.$$

Put

$$\begin{aligned}&A := \{(\mu_2, \dots, \mu_l) \in \mathbb{C}^{l-1} : e^{-\varepsilon} < |\mu_j| < e^{\varepsilon},\ j = 2, \dots, l\},\\ &H_R := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < R\},\ R \ge 0,\\ &\Phi : \mathbb{C} \times A \to \mathbb{C}^l, \quad \Phi(\lambda, \mu) := (e^{\lambda}, \mu_2 e^{\lambda\alpha_2}, \dots, \mu_l e^{\lambda\alpha_l}).\end{aligned}$$

It is clear that $\Phi(H_R \times A) =: D_R \subset \mathbb{C}_*^l$, $R \geq 0$, is a pseudoconvex Reinhardt domain with $D_R \nearrow D_\infty := \Phi(\mathbb{C} \times A) \subset \mathbb{C}_*^l$ as $R \to \infty$. Therefore,

$$\widetilde{k}_{D_\infty}(\Phi(-1, 1\dots, 1), \Phi(\lambda, 1, \dots, 1)) \leq \widetilde{k}_{\mathbb{C}}(-1, \lambda) = 0, \qquad \lambda \in \mathbb{C}.$$

By virtue of Lemma 2.5.3, $\widetilde{k}_{D_\infty}(\Phi(-1, 1, \dots, 1), z) = 0$ for all $z \in D_\infty \cap M$, where $M := \overline{\Phi(\mathbb{C} \times \{(1, \dots, 1)\})}$.

Observe that $D_0 \times \{(1, \dots, 1)\} \subset G$, but $(0, \dots, 0, 1, \dots, 1) \notin G$. Now choose positive numbers a_j, $j \in \mathbb{N}$, with $\sum_{j=1}^\infty a_j < \infty$. It suffices to find points $z^j \in D_0$, $j \in \mathbb{N}$, with $\lim_{j\to\infty} z_1^j = 0$ such that

$$g_G((z^j, 1, \dots, 1), (z^{j+1}, 1, \dots, 1)) \leq g_{D_0}(z^j, z^{j+1}) \leq a_j, \qquad j \in \mathbb{N}.$$

Applying the fact that $\widetilde{k}_{D_R}$ is continuous on $G_R \times G_R$, the theorem of Dini, and the equality

$$\lim_{R\to\infty} \widetilde{k}_{D_R}(\Phi(-1, 1, \dots, 1), z) = \widetilde{k}_{D_\infty}(\Phi(-1, 1, \dots, 1), z) = 0,$$
$$z \in D_\infty \cap \overline{\Phi(\mathbb{C} \times \{(1, \dots, 1)\})},\ e^{-2} < |z_1| < e^{-1},$$

we conclude that this convergence is uniform. Hence we have a sequence $(R_j)_{j\in\mathbb{N}}$ with $\lim_{j\to\infty} R_j = \infty$ such that

$$\widetilde{k}^*_{D_{R_j}}(\Phi(-1, 1, \dots, 1), \Phi(\lambda, 1, \dots, 1)) < a_j, \qquad -2 \leq \operatorname{Re}\lambda \leq -1.$$

Observe that the mapping $\psi_R : D_0 \to D_R$, $\psi(z) := (e^R z_1, z_2 e^{\alpha_2 R}, \dots, z_l e^{\alpha_l R})$, is biholomorphic. Therefore,

$$\widetilde{k}^*_{D_0}(\Phi(-1 - R_j, 1\dots, 1), \Phi(\lambda, 1, \dots, 1)) < a_j, \qquad -2 - R_j \leq \operatorname{Re}\lambda \leq -1 - R_j.$$

Define

$$u_j(\lambda) := \log g_{D_0}(\Phi(-1 - R_j, 1, \dots, 1), \Phi(\lambda, 1, \dots, 1)), \qquad \lambda \in H_0.$$

Observe that $u_j \in \mathcal{SH}(H_0)$. By Lemma 2.5.4 it follows that $u_j(\lambda) < \log a_j$ whenever $\operatorname{Re}\lambda \leq -1 - R_j$. Therefore, we may take $z^j := \Phi(-1 - R_j, 1, \dots, 1)$ as the desired point sequence. ■

REMARK 2.5.5. Obviously, any c_G-finitely compact domain G is c_G-complete. We point out that the converse (due to N. Sibony and M. A. Selby) is also known for domains in the plane (see [J-P 1993, Theorem 7.4.7]). $\boxed{?}$ Whether the two notions of c-completeness coincide for all bounded domains is still unknown. $\boxed{?}$ We only mention that there is a one-dimensional complex space X that is c_X-complete but not c_X-finitely compact (see [Jar-Pfl-Vig 1993]).

REMARK 2.5.6. Let $G \subset \mathbb{C}^2$ be a bounded pseudoconvex Reinhardt domain, $a \in G$, and $z^0 \in \partial G \cap \mathbb{C}_*^2$. Then $c_G(a, z) \to \infty$ as $z \to z^0$ (see [Zwo 2000b]). So the part of ∂G not lying on a coordinate axis is c_G-infinitely far away from any point of G.

We point out that this phenomenon does not occur in higher dimensions.

EXAMPLE 2.5.7 (cf. [Zwo 2000b]). Let $\alpha > 0$ be an irrational number. Put

$$G := \{z \in \mathbb{C}^3 : |z_1|\,|z_2|^\alpha |z_3|^{\alpha+1} < 1,\ |z_2|\,|z_3| < 1,\ |z_3| < 1\}.$$

Then G is a pseudoconvex Reinhardt domain. Fixing points $z^0 \in G \cap \mathbb{C}^3_*$ and $w \in \mathbb{C}^3_*$ with $|w_1|\,|w_2|^{\alpha}|z_3|^{\alpha+1} = 1$, $|w_2|^{-1}|w_3|^2 < 1$, and $|w_3| < 1$, we get

$$\limsup_{G \ni z \to w} c_G(z^0, z) < \infty.$$

Moreover, the biholomorphic map

$$\Phi : G \cap \mathbb{C}^3_* \to \mathbb{C}^3_*, \qquad \Phi(z) := (z_1 z_2^{[\alpha]+1} z_3^{[\alpha]+3}, z_2 z_3^2, z_3),$$

has as its image a bounded pseudoconvex Reinhardt G^* domain contained in

$$\{z \in \mathbb{C}^3_* : |z_2| < 1,\ |z_3| < 1,\ |z_1|\,|z_2|^{\alpha-[\alpha]-1}|z_3|^{[\alpha]-\alpha} < 1\}.$$

In the class of Reinhardt domains we have the following characterization of hyperconvexity (cf. [Zwo 2000a] and [Car-Ceg-Wik 1999]).

THEOREM 2.5.8. *Let $G \subset \mathbb{C}^n$ be a pseudoconvex Reinhardt domain. Then the following conditions are equivalent*:

(i) *G is hyperconvex*;
(ii) *G is bounded and satisfies the Fu condition.*

Proof. The implication (ii)⇒(i) follows directly from Theorem 2.5.1(i). To prove the converse, suppose that G does not satisfy the conditions in (ii). According to Proposition 2.2.7, we may assume that G is bounded and does not satisfy the Fu condition. Hence, without loss of generality, we may assume that $G = D_0$ (compare the proof of Theorem 2.5.1), i.e.

$$G := \{(\zeta, \mu_2\zeta^{\alpha_2}, \dots, \mu_n\zeta^{\alpha_n}) \in \mathbb{C}^n : \zeta \in E_*,\ \mu_j \in \mathbb{C},\ e^{-\varepsilon_0} < |\mu_j| < e^{\varepsilon_0},\ j = 2, \dots, n\},$$

where $\varepsilon_0 > 0$, $\alpha_j > 0$, $j = 2, \dots, n$.

Let $u \in \mathcal{PSH}(G) \cap \mathcal{C}(G)$, $u < 0$, be such that $\{z \in G : u(z) < -\varepsilon\} \Subset G$ for any $\varepsilon > 0$. Define

$$v(z) := \sup\{u(z_1 e^{i\theta_1}, \dots, z_n e^{i\theta_n}) : \theta_j \in \mathbb{R}\}.$$

Obviously, v is an exhausting function of G with $v(z) = v(|z_1|, \dots, |z_n|)$. Therefore, the function

$$E_* \ni \lambda \mapsto v(|\lambda|, |\lambda|^{\alpha_2}, \dots, |\lambda|^{\alpha_n})$$

is subharmonic and bounded from above by 0. Hence it can be continued as a function $v^* \in \mathcal{SH}(E)$. Then, because of the hyperconvexity of G, it follows that $v^*(0) = 0$ implying that $v = 0$, a contradiction. ∎

Recall that the Carathéodory distance is not inner. So, in general, we have $c_G \le c^i_G$, $c_G \ne c^i_G$ (cf. §1.2.1). Moreover, it is known that $c^i_G = \int\gamma_G = \int\gamma^{(1)}_G \le \int\gamma^{(k)}_G$, $k \in \mathbb{N}$. Thus,

$$c_G\text{-complete} \Rightarrow c^i_G\text{-complete} \Rightarrow \textstyle\int\gamma^{(k)}_G\text{-complete},\ k \in \mathbb{N}.$$

(A domain G is called *δ_G-complete* (briefly, *δ-complete*) if δ_G is a distance and any δ_G-Cauchy sequence in G converges to a point in G, $\delta \in \{c^i,\ \int\gamma^{(k)},\ k \in \mathbb{N}\}$. See [J-P 1993] for more details.) In fact the following holds (see [Zwo 2001b], [Zap 2003]).

THEOREM 2.5.9. *Let $G \subset \mathbb{C}^n$ be a bounded pseudoconvex Reinhardt domain. Then the following properties are equivalent*:

(i) *G is c_G-complete;*
(ii) *G is c^i_G-complete;*
(iii) *G is $\int\gamma_G^{(k)}$-complete, $k \in \mathbb{N}$;*
(iv) *there is a $k \in \mathbb{N}$ such that G is $\int\gamma_G^{(k)}$-complete.*

In order to be able to prove Theorem 2.5.9, we first recall a fact on multi-dimensional Vandermonde determinants (for example, see [Sic 1962]), namely:

Let $X_s := (s,\dots,s) \in \mathbb{C}^n$ ([3]), $s \in \mathbb{N}$, and $N_k := \#\{\alpha \in \mathbb{Z}^n_+ : |\alpha| \le k\}$, $k \in \mathbb{N}$. Then

$$\det((X_s^\alpha)_{1\le s\le N_k,\,|\alpha|\le k}) \neq 0. \tag{2.5.5}$$

Using this fact we get the following

LEMMA 2.5.10. *Let $P(z) = \sum_{1\le|\beta|\le k} b_\beta z^\beta$, $z \in \mathbb{C}^n$, be a polynomial in $\mathbb{C}^n$. Then there are numbers $(N_j)_{1\le j\le k} \subset \mathbb{N}$, $(c_{j,s})_{1\le j\le k,\,1\le s\le N_j} \subset \mathbb{C}$, and vectors $(X_{j,s})_{1\le j\le k,\,1\le s\le N_j} \subset \mathbb{C}^n$ such that*

$$P(z) = \sum_{j=1}^{k}\sum_{s=1}^{N_j} c_{j,s} \sum_{|\beta|=j} \frac{j!}{\beta!}\, p_\beta(z) X^\beta_{j,s}, \qquad z \in \mathbb{C}^n,$$

where

$$p_\beta(z) := \prod_{j=1}^{n} p_{\beta,j}(z), \quad p_{\beta,j}(z) := z_j(z_j-1)\cdots(z_j-\beta_j+1), \qquad z=(z_1,\dots,z_n)\in\mathbb{C}^n.$$

Proof. The proof is by induction on $k \in \mathbb{N}$. The case $k=1$ is obvious. So we may assume that the assertion holds for a $k \in \mathbb{N}$. Now take a polynomial $P(z) = \sum_{1\le|\beta|\le k+1} b_\beta z^\beta$, $z \in \mathbb{C}^n$, and write

$$P(z) = \sum_{1\le|\beta|\le k} b_\beta z^\beta + \sum_{|\beta|=k+1} b_\beta(z^\beta - p_\beta(z)) + \sum_{|\beta|=k+1} b_\beta p_\beta(z), \qquad z \in \mathbb{C}^n.$$

Observe that the first two terms are of degree less than or equal to k. The third may be written as

$$\sum_{|\beta|=k+1} b_\beta p_\beta(z) = \sum_{|\beta|=k+1} \frac{(k+1)!}{\beta!}\, p_\beta(z)\, \frac{\beta! b_\beta}{(k+1)!}.$$

Using (2.5.5), we find $(c_s)_{1\le s\le N_{k+1}} \subset \mathbb{C}$ and $(X_s)_{1\le s\le N_{k+1}} \subset \mathbb{C}^n$ such that

$$\frac{\beta! b_\beta}{(k+1)!} = \sum_{s=1}^{N_{k+1}} c_s X_s^\beta, \qquad |\beta| = k+1.$$

Hence, Lemma 2.5.10 has been proved. ∎

Proof of Theorem 2.5.9. It remains to prove (iv)⇒(i). We may assume that $n \ge 2$. Suppose that G does not satisfy the Fu condition, but is $\int\gamma_G^{(k)}$-complete for a suitable k. According to the proof of Theorem 2.5.1, we may assume that $G \subset \mathbb{C}^n_*$ and that

$$\log G = \{0\} \times (\log\delta, -\log\delta)^{n-1} + \mathbb{R}_{>0} v,$$

where $\delta \in (0,1)$, $v \in (-\infty,0)^n$, and $v_1 = -1$. Put $\gamma := -v$.

([3]) Notice that here X_s is a vector and not the sth coordinate of a vector.

Observe that a monomial z^α ($\alpha \in \mathbb{Z}^n$) is bounded on G if and only if $\langle \alpha, \gamma \rangle \geq 0$. Put

$$\chi : (0,1) \to G, \qquad \chi(t) := (t^{\gamma_1}, \dots, t^{\gamma_n}).$$

For a fixed $t \in (0,1)$, we are going to estimate $\gamma_G^{(k)}(\chi(t); \chi'(t))$.

Fix an $f \in \mathcal{O}(G, E)$, $\operatorname{ord}_{\chi(t)} f \geq k$. Then, using Laurent expansion, we get

$$f(z) = \sum a_\alpha z^\alpha, \quad \text{where} \quad a_\alpha = \frac{1}{(2\pi i)^n} \int\limits_{|\zeta_1| = r_1, \dots, |\zeta_n| = r_n} \frac{f(\zeta)\, d\zeta_1 \dots d\zeta_n}{\zeta^{\alpha + \mathbf{1}}}$$

is independent of $r = (r_1, \dots, r_n) \in G \cap \mathbb{R}^n_{>0}$. Note that

$$|a_\alpha| \leq \frac{1}{r^\alpha} \quad \text{for any } r \in G. \tag{2.5.6}$$

From (2.5.6) it follows that $a_\alpha = 0$ if $\langle \alpha, \gamma \rangle < 0$. Therefore,

$$f(z) = \sum_{\alpha \in \mathbb{Z}^n : \langle \alpha, \gamma \rangle \geq 0} a_\alpha z^\alpha, \qquad z \in G.$$

Taking $r_1 < 1$ in (2.5.6) arbitrarily large and r_j arbitrarily close to δ (or to δ^{-1}), $j = 2, \dots, n$, we obtain

$$|a_\alpha| \leq \delta^{|\alpha_2| + \dots + |\alpha_n|}.$$

Taking derivatives we have

$$\frac{1}{s!} f^{(s)}(\chi(t))(X_1 t^{\gamma_1 - k/s}, \dots, X_n t^{\gamma_n - k/s}) = \sum_{\langle \alpha, \gamma \rangle \geq 0} \Big(a_\alpha \sum_{|\beta| = s} \frac{1}{\beta!} p_\beta(\alpha) X^\beta t^{\langle \alpha, \gamma \rangle - k} \Big),$$
$$s \in \mathbb{N},\ X = (X_1, \dots, X_n) \in \mathbb{C}^n.$$

Since $\operatorname{ord}_{\chi(t)} f \geq k$, it follows that

$$\sum_{\langle \alpha, \gamma \rangle \geq 0} a_\alpha t^{\langle \alpha, \gamma \rangle - k} \sum_{|\beta| = s} \frac{1}{\beta!} p_\beta(\alpha) X^\beta = 0, \qquad 0 \leq s < k,\ X \in \mathbb{C}^n. \tag{2.5.7}$$

Moreover,

$$\begin{aligned} \frac{1}{k!} f^{(k)}(\chi(t))(\chi'(t)) = {} & \frac{1}{k!} \sum_{\langle \alpha, \gamma \rangle \geq 0} a_\alpha \langle \alpha, \gamma \rangle^k t^{\langle \alpha, \gamma \rangle - k} \\ & + \frac{1}{k!} \sum_{\langle \alpha, \gamma \rangle \geq 0} a_\alpha t^{\langle \alpha, \gamma \rangle - k} \sum_{|\beta| = k} \frac{k!}{\beta!} (p_\beta(\alpha) - \alpha^\beta) \gamma^\beta. \end{aligned} \tag{2.5.8}$$

Applying Lemma 2.5.10 and (2.5.7) shows that the second term in (2.5.8) vanishes. Hence, we have

$$\frac{1}{k!} f^{(k)}(\chi(t))(\chi'(t)) = \frac{1}{k!} \sum_{\langle \alpha, \gamma \rangle \geq 0} a_\alpha \langle \alpha, \gamma \rangle^k t^{\langle \alpha, \gamma \rangle - k}.$$

In view of the above estimate, we obtain

$$\gamma_G^{(k)}(\chi(t)); \chi'(t)) \leq \frac{1}{\sqrt[k]{k!}} \sum_{\alpha \in \mathbb{Z}^n : \langle \alpha, \gamma \rangle > 0} \delta^{(|\alpha_2| + \dots + |\alpha_n|)/k} \langle \alpha, \gamma \rangle t^{(\langle \alpha, \gamma \rangle / k) - 1}, \qquad t \in (0,1).$$

It remains to show that $L_{\gamma_G^{(k)}}(\chi|_{(0,1/2]})$ is finite, which would give the desired contradiction. We have the following estimate:

$$\begin{aligned}
L_{\gamma_G^{(k)}}(\chi|_{(0,1/2]}) &\le \int_0^{1/2} \gamma_G^{(k)}(\chi(t);\chi'(t))\,dt \\
&\le \frac{1}{\sqrt[k]{k!}} \sum_{\langle\alpha,\gamma\rangle\ge 0} \delta^{(|\alpha_2|+\dots+|\alpha_n|)/k} \int_0^{1/2} \langle\alpha,\gamma\rangle t^{(\langle\alpha,\gamma\rangle/k)-1}dt \\
&= \frac{1}{\sqrt[k]{k!}} \sum_{\langle\alpha,\gamma\rangle> 0} \delta^{(|\alpha_2|+\dots+|\alpha_n|)/k} \frac{k}{2^{\langle\alpha,\gamma\rangle/k}} \\
&= \frac{k}{\sqrt[k]{k!}} \sum_{\alpha_2,\dots,\alpha_n\in\mathbb{Z}} \delta^{(|\alpha_2|+\dots+|\alpha_n|)/k} \sum_{\alpha_1\in\mathbb{Z}:\alpha_1>-\langle\alpha',\gamma'\rangle} \frac{1}{2^{\langle\alpha,\gamma\rangle/k}} \\
&\le \frac{k}{\sqrt[k]{k!}} \sum_{\alpha_2,\dots,\alpha_n\in\mathbb{Z}} \delta^{(|\alpha_2|+\dots+|\alpha_n|)/k} \frac{1}{2^{(\langle\alpha',\gamma'\rangle+[-\langle\alpha',\gamma'\rangle])/k}},
\end{aligned}$$

where $\alpha' := (\alpha_2,\dots,\alpha_n)$, $\gamma' := (\gamma_2,\dots,\gamma_n)$. Obviously, the last number is finite, which finishes the proof. ∎

REMARK 2.5.11. Observe that in the case $\gamma\in\mathbb{Q}^n$ the above proof may be essentially simplified. Namely, the punctured unit disc can then be embedded into G. So the non-$\int\gamma_G^{(k)}$-completeness of G follows immediately from that of E_*.

REMARK 2.5.12. Let $G\subset\mathbb{C}^n$ be an arbitrary domain and $A\subset G$ be finite. In generalization of the notion of c_G-finite compactness we say that G is *$m_G(A,\cdot)$-finitely compact* if for any $R>0$ the set $\{z\in G : m_G(A,z)<R\}$ is relatively compact in G. Obviously, any $m_G(A,\cdot)$-finitely compact domain is c_G-finitely compact. $\boxed{?}$ Is there a geometrical characterization for $m_G(A,\cdot)$-finite compactness in the class of all pseudoconvex Reinhardt domains as in Theorem 2.5.1 $\boxed{?}$

2.6. c-completeness for complete circular domains

Let $G\subset\mathbb{C}^n$ be a bounded pseudoconvex balanced (:= complete circular) domain. Then there is an $h=h_G\in\mathcal{PSH}(\mathbb{C}^n)$ with $h(\lambda z)=|\lambda|h(z)$ ($\lambda\in\mathbb{C}$, $z\in\mathbb{C}^n$) such that

$$G=G_h=\{z\in\mathbb{C}^n : h(z)<1\}$$

and, since G is pseudoconvex, $\log h\in\mathcal{PSH}(\mathbb{C}^n)$ (see Chapter 1).

It is known (due to work of T. Barth) that h is continuous whenever G is k_G-complete. In dimensions larger than 2 the converse statement becomes false; in fact there is a counterexample due to Jarnicki–Pflug ([J-P 1993, Theorem 7.5.7]). In particular, this example is not c-complete.

$\boxed{?}$ It is still open which conditions on h may imply that $G=G_h$ is c-complete. Moreover, in dimension 2, so far it is not known whether the continuity of h implies the Kobayashi completeness or even the Carathéodory completeness. $\boxed{?}$

2.7. $\int\gamma_G^{(k)}$-completeness for Zalcman domains

First, we introduce the class of domains we wish to study. Let $(a_j)_{j\in\mathbb{N}}$ and $(r_j)_{j\in\mathbb{N}}$ be sequences of positive real numbers such that:

- $2r_j < a_j$, $j \in \mathbb{N}$,
- $a_j \searrow 0$ as $j \to \infty$,
- $\overline{\mathbb{B}}(a_j, r_j) \subset E$, $\overline{\mathbb{B}}(a_j, r_j) \cap \overline{\mathbb{B}}(a_k, r_k) = \emptyset$, $j \neq k$.

Then $G := E_* \setminus \bigcup_{j=1}^{\infty} \overline{\mathbb{B}}(a_j, r_j)$ is called a *Zalcman type domain.*

The main result here is the following one due to P. Zapałowski (see [Zap 2002] and [Zap 2004]).

THEOREM 2.7.1. *For any $k \in \mathbb{N}$ there exists a Zalcman type domain G which is $\int\gamma_G^{(l)}$-complete, but not $\int\gamma_G^{(m)}$-complete, whenever $m \le k < l$.*

REMARK 2.7.2. $\boxed{?}$ It seems to be an open problem whether for different $k, l \in \mathbb{N}$, $k < l$, there exists a Zalcman type domain G, which is $\int\gamma_G^{(k)}$-complete, but not $\int\gamma_G^{(l)}$-complete. $\boxed{?}$ Note that for $l = sk$, $s \in \mathbb{N}$, this is impossible because $\int\gamma_G^{(k)} \le \int\gamma_G^{(sk)}$.

Before giving the proof of Theorem 2.7.1 we mention the following sufficient condition for a Zalcman type domain not to be $\int\gamma^{(k)}$-complete.

PROPOSITION 2.7.3. *Let $G \subset \mathbb{C}$ be a Zalcman type domain (as above) and let $k \in \mathbb{N}$, $\alpha \in (0,1)$, and $c > 0$. Assume that*

$$\gamma_G^{(k)}(t;1) \le c|t|^{-\alpha}, \qquad t \in (-1,0). \tag{2.7.9}$$

Then G is not $\int\gamma_G^{(l)}$-complete for any $l \ge k$.

Proof. Fix $l \in \mathbb{N}$, $l \ge k$, and a point $t \in (-1,0)$. Take an $f \in \mathcal{O}(G,E)$, $f(t) = f'(t) = \dots = f^{(l-1)}(t) = 0$, with $(\gamma_G^{(l)}(t;1))^l = (l!)^{-1}|f^{(l)}(t)|$. We define

$$g(z) = \begin{cases} f(z)/(z-t)^{l-k} & \text{if } z \neq t, \\ 0, & \text{if } z = t. \end{cases}$$

Then g is holomorphic with

$$g^{(m)}(t) = 0, \quad m = 0, \dots, k-1, \qquad g^{(k)}(t) = \frac{k!}{l!}\, f^{(l)}(t).$$

Moreover, by the maximum principle, we have

$$\|g\|_G \le \operatorname{dist}(t, \partial G)^{-(l-k)}.$$

Therefore, $h := g \operatorname{dist}(t, \partial G)^{l-k} \in \mathcal{O}(G,E)$ and we obtain

$$\begin{aligned}(\gamma_G^{(k)}(t;1))^k &\ge \frac{1}{k!}\,|h^{(k)}(t)| = \frac{\operatorname{dist}(t,\partial G)^{l-k}}{l!}\,|f^{(l)}(t)| \\ &= \operatorname{dist}(t,\partial G)^{(l-k)}(\gamma_G^{(l)}(t;1))^l.\end{aligned} \tag{2.7.10}$$

Finally, the assumed inequality (2.7.9) implies the following estimate:

$$\gamma_G^{(l)}(t;1) \le \frac{c^{k/l}|t|^{-\alpha k/l}}{|t|^{(l-k)/l}} = c'|t|^{-\alpha'},$$

where $c' := c^{k/l}$ and $\alpha' := (\alpha k + (l-k))/l < 1$. Then integrating along the segment $(-1/2, 0)$ shows that G is not $\int \gamma_G^{(l)}$-complete. ∎

Consequently, to find examples as claimed in Theorem 2.7.1 we should try to deal with situations where the boundary behavior of $\gamma_G^{(k)}$ is of the type

$$\gamma_G^{(k)}(\cdot; 1) \le c\,\mathrm{dist}(\cdot, \partial G)^{-1} |\log \mathrm{dist}(\cdot, \partial G)|^{-\alpha}$$

with some $\alpha > 1$, $c > 0$.

LEMMA 2.7.4. *Let $G \subset \mathbb{C}$ be a Zalcman type domain and $k \in \mathbb{N}$. Then there exists a $C > 0$ such that*

$$|f^{(k)}(z)| \le C\Big(1 + \sum_{j=1}^{\infty} \frac{r_j}{(a_j - z)^{k+1}}\Big), \qquad z \in (-1/2, 0),\ f \in \mathcal{O}(G, E).$$

Proof. Choose numbers $\widetilde{a}_j \in (0, a_j)$ and $\widetilde{r}_j \in (\widetilde{a}_j, 1)$ such that

$$\overline{\mathbb{B}}(a_s, r_s) \subset \mathbb{B}(\widetilde{a}_j, \widetilde{r}_j), \quad s > j, \qquad \overline{\mathbb{B}}(\widetilde{a}_j, \widetilde{r}_j) \cap \overline{\mathbb{B}}(a_j, r_j) = \emptyset.$$

Put

$$G_j := E \setminus \Big(\overline{\mathbb{B}}(\widetilde{a}_j, \widetilde{r}_j) \cup \bigcup_{s=1}^{j} \overline{\mathbb{B}}(a_s, r_s)\Big).$$

Then G_j is a $(j+2)$-connected domain with $G_j \subset G$, $j \in \mathbb{N}$. Then, for a sufficiently small positive ε_j (we may assume that $\varepsilon_j \to 0$ as $j \to \infty$), we have

$$G_{j,\varepsilon_j} := (1-\varepsilon_j)E \setminus \Big(\overline{\mathbb{B}}(\widetilde{a}_j, \widetilde{r}_j + \varepsilon_j) \cup \bigcup_{s=1}^{j} \overline{\mathbb{B}}(a_s, r_s + \varepsilon_j)\Big) \Subset G_j.$$

By the Cauchy integral formula, we see that

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int\limits_{|\zeta| = 1 - \varepsilon_j} \frac{f(\zeta)}{(\zeta - z)^{k+1}}\, d\zeta - \frac{k!}{2\pi i} \int\limits_{|\zeta - \widetilde{a}_j| = \widetilde{r}_j + \varepsilon_j} \frac{f(\zeta)}{(\zeta - z)^{k+1}}\, d\zeta$$
$$- \sum_{s=1}^{j} \frac{k!}{2\pi i} \int\limits_{|\zeta - a_s| = r_s + \varepsilon_j} \frac{f(\zeta)}{(\zeta - z)^{k+1}}\, d\zeta, \qquad z \in G_{j,\varepsilon_j},\ f \in \mathcal{O}(G, E).$$

Let $z \in (-1/2, 0)$. Then $z \in (-1/2, \widetilde{a}_j - \widetilde{r}_j - \varepsilon_j - \sqrt[2(k+1)]{\widetilde{r}_j + \varepsilon_j})$ and $z < -\varepsilon_j$ for all sufficiently large j. Hence we obtain

$$|f^{(k)}(z)| \le \frac{k!}{2\pi} \int_0^{2\pi} \frac{1 - \varepsilon_j}{|(1-\varepsilon_j)e^{it} - z|^{k+1}}\, dt + \frac{k!}{2\pi} \int_0^{2\pi} \frac{\widetilde{r}_j + \varepsilon_j}{|(\widetilde{r}_j + \varepsilon_j)e^{it} + \widetilde{a}_j - z|^{k+1}}\, dt$$
$$+ \sum_{s=1}^{j} \frac{k!}{2\pi} \int_0^{2\pi} \frac{r_s + \varepsilon_j}{|(r_s + \varepsilon_j)e^{it} + a_s - z|^{k+1}}\, dt$$
$$\le k!\Big(\frac{1 - \varepsilon_j}{(1/2 - \varepsilon_j)^{k+1}} + \frac{\widetilde{r}_j + \varepsilon_j}{(\sqrt[2(k+1)]{\widetilde{r}_j + \varepsilon_j})^{k+1}} + \sum_{s=1}^{j} \frac{r_s + \varepsilon_j}{(1/2(a_s - z - \varepsilon_j))^{k+1}}\Big)$$

Observe that here the assumption $2r_j < a_j$, $j \in \mathbb{N}$, is used to estimate the third term. Since $\varepsilon_j \to 0$, we finally arrive at the following inequality:

$$|f^{(k)}(z)| \le k!\Big(2^{k+1} + \sqrt{\widetilde{r}_j} + 2^{k+1}\sum_{s=1}^{j}\frac{r_s}{(a_s - z)^{k+1}}\Big).$$

Recall that $\widetilde{r}_j \to 0$ as $j \to \infty$. Therefore, letting $j \to \infty$, we obtain

$$|f^{(k)}(z)| \le k!2^{k+1}\Big(1 + \sum_{s=1}^{\infty}\frac{r_s}{(a_s - z)^{k+1}}\Big). \ \blacksquare$$

LEMMA 2.7.5. *For every $k \in \mathbb{N}$ there are a $\widetilde{k} \in \mathbb{N}$ and a Zalcman type domain G such that*

(a) $\limsup\limits_{(-1,0)\ni z\to 0}(\int\gamma_G^{(m)})(-1/2^{\widetilde{k}-1}, z) < \infty$, $1 \le m \le k$,

(b) $\lim\limits_{G\ni z\to 0}(\int\gamma_G^{(l)})(w, z) = \infty$, $w \in G$, $k < l$.

Observe that Lemma 2.7.5 immediately implies Theorem 2.7.1:

Proof of Theorem 2.7.1. Fix a $k \in \mathbb{N}$ and take the Zalcman type domain from Lemma 2.7.5. Let $m \in \mathbb{N}$, $1 \le m \le k$. As a direct consequence of (a) and the fact that the $\int\gamma_G^{(m)}$-completeness is equivalent to the $\int\gamma_G^{(m)}$-finite compactness we see that G is not $\int\gamma_G^{(m)}$-complete.

It remains to see that G is $\int\gamma_G^{(l)}$-complete for $l > k$. So let us fix such an l, a point $w \in G$, and a boundary point $z^0 \in \partial G$. We have to show that $\lim_{z\to z^0}(\int\gamma_G^{(l)})(w, z) = \infty$.

CASE 1. If $z^0 = 0$, then using (b) we are done.

CASE 2. If $|z^0| = 1$, it follows that

$$\lim_{z\to z^0}(\textstyle\int\gamma_G^{(l)})(w, z) \ge \lim\limits_{z\to z^0}(\int\gamma_E^{(l)})(w, z) = \lim\limits_{z\to z^0} c_E(w, z) = \infty.$$

CASE 3. If $z^0 \in \partial\mathbb{B}(a_j, r_j)$ for some j, then

$$\begin{aligned}\lim_{z\to z^0}(\textstyle\int\gamma_G^{(l)})(w, z) &\ge \lim\limits_{z\to z^0} c_G(w, z) \ge \lim\limits_{z\to z^0} c_{\mathbb{C}\setminus\overline{\mathbb{B}}(a_j,r_j)}(w, z)\\ &= \lim_{z\to z^0} c_{E_*}\Big(\frac{r_j}{w - a_j}, \frac{r_j}{z - a_j}\Big) = \lim_{z\to z^0} c_E\Big(\frac{r_j}{w - a_j}, \frac{r_j}{z - a_j}\Big) = \infty,\end{aligned}$$

since $|r_j/(z - a_j)| \to 1$ as $z \to z^0$. $\blacksquare$

What remains is:

Proof of Lemma 2.7.5. Let $k \in \mathbb{N}$, $a_j := 2^{-j}$, and $r_{k,j} := 2^{-j}j^{-k-1}$, $j \in \mathbb{N}$. Since

$$\lim_{s\to\infty}\Big(\frac{s}{s-1}\Big)^2\frac{1}{\sqrt[k]{2}} = \frac{1}{\sqrt[k]{2}} < 1,$$

we may choose a $\widetilde{k} \in \mathbb{N}$ such that $(s/(s-1))^2/\sqrt[k]{2} < 1$, $s \ge \widetilde{k}$. Put

$$G_k := E_* \setminus \bigcup_{j\ge\widetilde{k}}\overline{\mathbb{B}}(a_j, r_{k,j}).$$

Obviously, G_k is a Zalcman type domain.

To prove (a) it suffices to verify the following inequality:

$$\exists_{c=c(k)>0} : \gamma^{(m)}_{G_k}(z;1) \le \frac{c}{-z(-\log(-z))^{(k+1)/m}}, \qquad z \in [-1/2^{\widetilde{k}-1}, 0),\ m \le k. \tag{2.7.11}$$

In fact, let $z \in [-1/2^{\widetilde{k}-1}, 0)$. Then there exist a unique $N \in \mathbb{N}$, $N \ge \widetilde{k}$, and a $b \in (1,2]$ such that $z = -b/2^N$. Therefore,

$$\begin{aligned}\sum_{j=\widetilde{k}}^{N} \frac{r_{k,j}}{(a_j - z)^{m+1}} &\le \sum_{j=\widetilde{k}}^{N} \frac{r_{k,j}}{a_j^{m+1}} = \sum_{j=\widetilde{k}}^{N} \frac{2^{jm}}{j^{k+1}} \\ &\le \frac{2^{Nm}}{N^{k+1}} \sum_{j=0}^{\infty} \delta^j \le \frac{1}{1-\delta} \frac{2^{Nm}}{N^{k+1}},\end{aligned} \tag{2.7.12}$$

$$\begin{aligned}\sum_{j=N}^{\infty} \frac{r_{k,j}}{(a_j - z)^{m+1}} &\le \sum_{j=N}^{\infty} \frac{r_{k,j}}{(-z)^{m+1}} = \sum_{j=N}^{\infty} \frac{2^{N(m+1)}}{2^j j^{k+1} b^{m+1}} \\ &\le \frac{2^{N(m+1)}}{2^N N^{k+1}} \sum_{j=0}^{\infty} \frac{1}{2^j} \le \frac{2^{Nm+1}}{N^{k+1}}.\end{aligned} \tag{2.7.13}$$

(The second inequality in (2.7.12) follows easily from the observation that there is a positive $\delta < 1$ such that

$$\frac{2^{(s-1)m}}{(s-1)^{k+1}} \le \delta \frac{2^{sm}}{s^{k+1}},$$

for $s \ge \widetilde{k}$, $m \le k$.) We put $\widehat{c} := 1/(1-\delta)$. Using (2.7.12) and (2.7.13) we get

$$\sum_{j=\widetilde{k}}^{\infty} \frac{r_{k,j}}{(a_j - z)^{m+1}} \le \frac{(\widehat{c}+2) 2^k (\log 2)^{k+1} 2^{Nm}}{b^m (\log(2^m/b))^{k+1}} =: \frac{C_1}{(-z)^m (-\log(-z))^{k+1}}.$$

In view of Lemma 2.7.4, we obtain

$$|f^{(m)}(z)| \le C\left(1 + \frac{C_1}{(-z)^m(-\log(-z))^{k+1}}\right) \le \frac{2CC_1}{(-z)^m(-\log(-z))^{k+1}}, \qquad f \in \mathcal{O}(G,E),$$

which finally proves (2.7.11). (Note that we may take $C = k! 2^{k+1} \ge m! 2^{m+1}$, $m \le k$; thus the constant $C = C(k)$ from Lemma 2.7.4 works for all m, $m \le k$.)

To prove (b) we claim that

$$\forall_{l>k}\ \exists_{c=c(k,l)>0} : \gamma^{(l)}_{G_k}(z;1) \ge \frac{c}{|z| \log(1/|z|)}, \qquad |z| < \frac{1}{2^{\widetilde{k}-2}},\ z \in G_k. \tag{2.7.14}$$

Assume for a while that (2.7.14) is correct. Fix an $l \in \mathbb{N}$, $l > k$, and a $w \in G_k$. Take a $z \in G_k$, $|z| < 1/2^{\widetilde{k}-2}$, and a $\mathcal{C}^1$-curve $\alpha : [0,1] \to G_k$ connecting z with w. Then we have

$$\begin{aligned}\int_0^1 \gamma^{(l)}_{G_k}(\alpha(t);\alpha'(t))\,dt &\ge c\int_0^{t_\alpha} \frac{|\alpha'(t)|\,dt}{|\alpha(t)|\log(1/|\alpha(t)|)} \ge c\int_0^{t_\alpha} \frac{\frac{d}{dt}|\alpha(t)|\,dt}{|\alpha(t)|\log(1/|\alpha(t)|)} \\ &\ge c\int_0^{t_\alpha} \frac{d}{dt}(-\log\log(1/|\alpha(t)|))\,dt = c\left(\log\log\frac{1}{|z|} - \log\log 2^{\widetilde{k}-2}\right),\end{aligned}$$

where $t_\alpha := \sup\{t \in [0,1] : |\alpha(\tau)| < 1/2^{\widetilde{k}-2},\ 0 \le \tau \le t\}$. Since the curve α was an arbitrary one connecting z and w in G_k, it follows that

$$\int\gamma_{G_k}^{(l)}(w,z) \ge c(\log\log(1/|z|) - \log\log 2^{\widetilde{k}-2}) \underset{z\to 0}{\longrightarrow} \infty,$$

Hence, (b) is verified.

What remains is the proof of (2.7.14). Fix a $z \in G_k \cap \mathbb{B}(0, 1/2^{\widetilde{k}-2})$. Then we have to find an $f \in \mathcal{O}(G_k, E)$ satisfying the following conditions:

- $f(z) = f'(z) = \dots f^{(l-1)}(z) = 0$,
- $|f^{(l)}(z)| \ge c/(|z|\log(1/|z|))^l$, where c is independent of z.

Again we write z as $z = be^{i\theta}/2^N$ with $N \in \mathbb{N}$, $b \in (1,2]$, and $\theta \in [0,2\pi)$. Observe that $N \ge \widetilde{k} - 1$. Put

$$f(\lambda) := \sum_{j=0}^{l-1} \alpha_{b,\theta,j}(2^{-N-j-1} - \lambda)^{-1} + 2^{N+1}\beta_{b,\theta}, \qquad \lambda \in G_k, \tag{2.7.15}$$

where $\alpha_{b,\theta,0} := 1$ and $\alpha_{b,\theta,1}, \dots, \alpha_{b,\theta,l-1}, \beta_{b,\theta} \in \mathbb{C}$ depend only on b and θ and are such that (obviously $f \in \mathcal{O}(G_k)$) $f(z) = \dots = f^{(l-1)}(z) = 0$.

We proceed under the assumption that we have already chosen f as in (2.7.15). Then

$$\begin{aligned}|f^{(l)}(z)| &= \left|\sum_{j=0}^{l-1} \alpha_{b,\theta,j}\, l!\left(\frac{1}{2^{N+j+1}} - \frac{be^{i\theta}}{2^N}\right)^{-l-1}\right| \\ &\ge \left|\sum_{j=0}^{l-1} \alpha_{b,\theta,j}\left(\frac{2^{N+j+1}}{1 - 2^{j+1}be^{i\theta}}\right)^{l+1}\right| = 2^{(N+1)(l+1)}|B_{l,b,\theta}|,\end{aligned}$$

where

$$B_{l,b,\theta} := \sum_{j=0}^{l-1} \alpha_{b,\theta,j}\left(\frac{2^j}{1 - 2^{j+1}be^{i\theta}}\right)^{l+1}.$$

Moreover, assume that $|B_{l,b,\theta}| \ge B_l > 0$, where B_l is independent of b and θ. Then

$$\begin{aligned}\|f\|_{G_k} &\le \sum_{j=0}^{l-1} \frac{|\alpha_{b,\theta,j}|}{r_{k,N+j+1}} + 2^{N+1}|\beta_{b,\theta}| \le \alpha\,\frac{l+1}{r_{k,N+l}} \\ &= \alpha(l+1)2^{N+l}(N+l)^{k+1} \le c2^N(N-1)^{k+1} \le c2^N(N-1)^l,\end{aligned}$$

where $\alpha := \max\{|\alpha_{b,\theta,j}|, |\beta_{b,\theta}|\}$ and c depends only on k and l. Put $g := f/\|f\|_{G_k} \in \mathcal{O}(G_k, E)$. Then

$$|g^{(l)}(z)| \ge \frac{2^{l+1}B_l 2^{Nl}}{c(N-1)^l} \ge \widetilde{c}_1\left(\frac{2^N\log 2}{b(\log 2^N - \log b)}\right)^l \ge \frac{\widetilde{c}}{(|z|\log(1/|z|))^l},$$

where $\widetilde{c}_1$ and $\widetilde{c}$ are constants that only depend on k.

In order to finish the proof of Lemma 2.7.5 we need the following lemma.

LEMMA 2.7.6. *For an $l \in \mathbb{N}$ there are positive numbers α and B_l such that for every $z = be^{i\theta}/2^N$, where $b \in [1,2)$, $\theta \in [0,2\pi]$, and $N \ge \widetilde{k} - 1$, there exist complex numbers $\alpha_{b,\theta,j}$, $j = 1, \dots, l-1$, and $\beta_{b,\theta}$ such that*

- $\max\{|\alpha_{b,\theta,j}|,\ |\beta_{b,\theta}|\colon\ j=1,\dots,l-1,\ b$ *and* θ *as above*$\}| \leq \alpha$,
- $\min\{|B_{l,b,\theta}| : b \in [1,2],\ \theta \in [0,2\pi]\} \geq B_l$,
- $f(z) = f'(z) = \cdots = f^{(l-1)}(z) = 0$ (*for* f *see* (2.7.15)).

Proof. Let f be a function as in (2.7.15) with unknown numbers $\alpha_{b,\theta,j}$. Then the condition $f'(z) = \cdots = f^{(l-1)}(z) = 0$ gives the following system of $l-1$ equations:

$$\sum_{j=0}^{l-1} s!\left(\frac{2^{N+j+1}}{1-2^{j+1}be^{i\theta}}\right)^{s+1} \alpha_{b,\theta,j} = 0, \qquad s = 1,\dots,l-1,$$

which is equivalent to

$$\sum_{j=1}^{l-1} s!\left(\frac{2^{j}}{1-2^{j+1}be^{i\theta}}\right)^{s+1} \alpha_{b,\theta,j} = -\left(\frac{1}{1-2be^{i\theta}}\right)^{s+1}, \qquad s = 1,\dots,l-1. \tag{2.7.16}$$

To simplify further discussions we put

$$A_{b,\theta,j} := \frac{2^j}{1-2^{j+1}be^{i\theta}}, \qquad j = 0,\dots,l-1.$$

Observe that $|A_{b,\theta,j}| \in [1/8,1]$ and that $A_{b,\theta,\mu} \neq A_{b,\theta,\nu}$ for $\mu \neq \nu$. Now we can rewrite (2.7.16) in the form

$$\sum_{j=1}^{l-1} A_{b,\theta,j}^{s+1}\alpha_{b,\theta,j} = -A_{b,\theta,0}^{s+1}, \qquad s = 1,\dots,l-1.$$

We conclude that

$$|\det[A_{b,\theta,j}^{s+1}]_{j,s=1,\dots,l-1}| = \left|\prod_{j=1}^{l-1} A_{b,\theta,j}\right|^2 \prod_{1\leq\mu<\nu\leq k} |A_{b,\theta,\mu} - A_{b,\theta,\nu}| \geq \varepsilon > 0,$$

where ε is independent of b and θ. Hence the claimed choice of the $\alpha_{b,\theta,j}$, $j = 1,\dots,l-1$, is always possible. Next, we put

$$\beta_{b,\theta} := -\sum_{j=0}^{l-1} A_{b,\theta,j}\alpha_{b,\theta,j}.$$

For an upper estimate, observe that

$$|\beta_{b,\theta}| \leq \sum_{j=0}^{l-1} |A_{b,\theta,j}\alpha_{b,\theta,j}| \leq l\max\{|\alpha_{b,\theta,j}| : j = 0,\dots,l-1\}.$$

Therefore it suffices to estimate the $\alpha_{b,\theta,j}$'s. Recall that $|A_{b,\theta,j}|^{s+1} \in [2^{-3l},1], j = 0,\dots,l-1$, $s = 1,\dots,l-1$, $b \in [1,2]$, and $\theta \in [0,2\pi]$. Applying Cramer's formula and the continuity of the det-function, we see there is a number $\widetilde{\alpha} > 0$ such that all the $|\alpha_{b,\theta,j}| \leq \widetilde{\alpha}$.

Finally, the lower estimate remains. Since $|B_{l,b,\theta}|$ is continuous with respect to (b,θ) it suffices to show that $B_{l,b,\theta} \neq 0$, or equivalently,

$$\sum_{j=1}^{l-1} A_{b,\theta,j}^{l+1}\alpha_{b,\theta,j} \neq -A_{b,\theta,0}^{l+1}.$$

Suppose that this is false. Then the $\alpha_{b,\theta,j}$'s satisfy the following l equations:

$$\sum_{j=1}^{l-1} A_{b,\theta,j}^{s+1}\alpha_{b,\theta,j} = -A_{b,\theta,0}^{s+1}, \qquad s = 1, \dots, l,$$

implying that $A_{b,\theta,0}/A_{b,\theta,j} = 1$, $j = 1, \dots, l-1$. But this is impossible. Thus also the lower estimate has been proved. ∎

This finishes the proof of Lemma 2.7.5. ∎

2.8. Kobayashi completeness and smoothly bounded pseudoconvex domains

It is well known that there is a bounded pseudoconvex domain G (due to N. Sibony) with a $\mathcal{C}^\infty$-boundary except of one point that is not k_G-complete (see [J-P 1993, Theorem 7.5.9]). $\boxed{?}$ On the other hand, for a smoothly bounded pseudoconvex domain G it is still an open question whether it is k_G-complete. $\boxed{?}$

Let $D \subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain and let $z_0 \in \partial D$. Then there is a neighborhood $U = U(z_0)$ and a function $r \in \mathcal{C}^\infty(U, \mathbb{R})$ such that

$$D \cap U = \{z \in U : r(z) < 0\}$$

and $\operatorname{grad} r(z) \neq 0$, $z \in U$. Moreover, let $\mathcal{V}(z_0)$ be the set of all germs of non-constant holomorphic mappings $\psi : \mathbb{C}_0 \to \mathbb{C}^n$ with $\psi(0) = z_0$. According to d'Angelo, the domain D is said to be of *finite type at* z_0 if

$$\tau(D, z_0) := \sup\left\{\frac{\operatorname{ord}_0(r \circ \psi)}{\operatorname{ord}_0 \psi} : \psi \in \mathcal{V}(z_0)\right\} < \infty.$$

The domain D is said to be of *finite type* if D is of finite type at all of its boundary points.

The following result is due to [Bed-For 1978] (see also [For-Sib 1989], [For-McN 1994]).

THEOREM 2.8.1. *Let $D \subset \mathbb{C}^2$ be a bounded pseudoconvex domain. Assume that D is of finite type. Then any boundary point $a \in \partial D$ is a peak point with respect to $\mathcal{C}(\overline{D}) \cap \mathcal{O}(D)$, i.e. there exists an $f \in \mathcal{C}(\overline{D}) \cap \mathcal{O}(D)$ such that $f(a) = 1$ and $|f(z)| < 1$, $z \in \overline{D} \setminus \{a\}$.*

In particular, we have

COROLLARY 2.8.2. *Any bounded pseudoconvex domain with a smooth boundary, which is of finite type, is c-complete.*

Observe that to conclude that a domain is c-complete, a weaker condition is already sufficient; namely we have (see [J-P 1993]):

Let $D \subset \mathbb{C}^n$ be a c-hyperbolic domain. Then the following two conditions are equivalent:

(i) *D is c-finitely compact*;

(ii) *for any $z_0 \in D$ and for any sequence $(z_j)_j \subset D$ without accumulation points in D, there is an $f \in \mathcal{O}(D, E)$ with $f(z_0) = 0$ and* $\sup\{|f(z_j)| : j \in \mathbb{N}\} = 1$.

$\boxed{?}$ It is an open problem whether all bounded pseudoconvex domains of finite type are k-complete or even c-complete. $\boxed{?}$

2.9. Kobayashi completeness and unbounded domains

Let $D \subset \mathbb{C}^n$ be an arbitrary domain and let $a \in \partial D$. The point a is called a *local holomorphic peak point* of D if there is a neighborhood $U = U(a)$ such that a is a peak point with respect to $\mathcal{C}(\overline{U \cap D}) \cap \mathcal{O}(U \cap D)$. When D is unbounded, we say that D has a *local holomorphic peak point at infinity* if there is an $r > 0$ and an $f \in \mathcal{C}(\overline{D} \setminus \mathbb{B}(0,r), E) \cap \mathcal{O}(D \setminus \overline{\mathbb{B}}(0,r), E)$ such that $\lim_{z\to\infty} f(z) = 1$.

Recall that a bounded domain is locally k-complete iff it is k-complete (see [J-P 1993, Theorem 7.5.5]). For an unbounded domain we have the following result (see [Gau 1999]).

THEOREM 2.9.1. *Let $D \subset \mathbb{C}^n$ be an unbounded domain. Assume that D has a local holomorphic peak point at any point of $\partial D \cup \{\infty\}$. Then D is k-complete.*

EXAMPLE 2.9.2. Put

$$D := \{z \in \mathbb{C}^2 : u(z) := |z_1|^2(1 + |z_2|^2) < 1\}.$$

Obviously, $\{0\} \times \mathbb{C} \subset D$. Thus, D is not k-hyperbolic. On the other hand, since u is strongly psh, any $a \in \partial D$ is a local holomorphic peak point. So this example shows that the condition at infinity in Theorem 2.9.1 is in some sense necessary.

The proof of Theorem 2.9.1 is based on the following lemma.

LEMMA 2.9.3. *Let $D \subset \mathbb{C}^n$ be an arbitrary domain and let $a \in \mathbb{C}^n \cup \{\infty\}$ be a boundary point of D. Assume that a is a local holomorphic peak point of D. Then for any neighborhood $U = U(a)$ there exists a neighborhood $V = V(a) \subset U$ such that for any $\varphi \in \mathcal{O}(E, D)$, $\varphi(0) \in V$, one has $\varphi(\lambda) \in U$, $|\lambda| < 1/2$.*

Proof. We give the proof only for $a = \infty$ (the finite case is similar). Without loss of generality, let $U = U(\infty) := \mathbb{C}^n \setminus \overline{\mathbb{B}}(0, \varrho)$. By assumption, there is an $r > 0$ and an $f \in \mathcal{C}(\overline{D} \setminus \mathbb{B}(0,r), E) \cap \mathcal{O}(D \setminus \overline{\mathbb{B}}(0,r), E)$ such that $\lim_{\overline{D}\ni z\to\infty} f(z) = 1$. We may assume that $r = \varrho$. Put $u(z) := \log|f(z)|$, $z \in \overline{D} \setminus \mathbb{B}(0,r)$. Then

$$u \in \mathcal{C}(\overline{D} \setminus \mathbb{B}(0,r), [-\infty, 0)) \cap \mathcal{PSH}(D \setminus \overline{\mathbb{B}}(0,r)), \qquad \lim_{\overline{D}\ni z\to\infty} u(z) = 0.$$

Fix numbers r', r'', $r < r' < r''$, such that

$$\begin{aligned} &\sup\{u(z) : z \in \overline{D} \cap \partial\mathbb{B}(0,r')\} =: c' < 0, \qquad f(z) \neq 0,\ \|z\| \geq r', \\ &\inf\{u(z) : z \in \overline{D} \cap \partial\mathbb{B}(0,r'')\} =: c'' > c'. \end{aligned}$$

We define $\widehat{u} : \overline{D} \to (-\infty, 0)$ by

$$\widehat{u}(z) := \begin{cases} u(z), & \text{if } \|z\| \geq r'', \\ \max\{u(z), (c' + c'')/2\}, & \text{if } r' < \|z\| < r'', \\ (c' + c'')/2, & \text{if } \|z\| \leq r. \end{cases}$$

Obviously, $\widehat{u}$ is a *global negative psh peak function at* ∞, i.e. $\widehat{u}$ is a negative continuous function on $\overline{D}$, psh on D, such that $\lim_{\overline{D}\ni z\to\infty} \widehat{u}(z) = 0$.

Fix a $\psi \in \mathcal{C}(\overline{E}, D) \cap \mathcal{O}(E, D)$. Observe that $\widehat{u} \circ \psi \in \mathcal{C}(\overline{E}) \cap \mathcal{SH}(E)$. Let $\alpha < 0$. Put

$$E(\psi, \alpha) := \{\theta \in [0, 2\pi] : \widehat{u} \circ \psi(e^{i\theta}) \geq 2\alpha\}.$$

Assume that $\alpha \le \widehat{u} \circ \psi(0)$. Then

$$\begin{aligned}\alpha \le \widehat{u} \circ \psi(0) &\le \frac{1}{2\pi} \int_0^{2\pi} \widehat{u} \circ \psi(e^{i\theta})\, d\theta \\ &\le \frac{1}{2\pi} \int_{[0,2\pi]\setminus E(\psi,\alpha)} \widehat{u} \circ \psi(e^{i\theta})\, d\theta \le \frac{\alpha}{\pi}\,(2\pi - \Lambda_1(E(\psi,\alpha))).\end{aligned}$$

Hence,

$$\Lambda_1(E(\psi,\alpha)) \ge \pi. \tag{2.9.17}$$

Put $v(z) := \log(|f(z) - 1|/2)$, $z \in \overline{D}$. Then $v \in \mathcal{C}(\overline{D} \setminus \mathbb{B}(0,r)) \cap \mathcal{PSH}(D \setminus \overline{\mathbb{B}}(0,r))$. Choose an $\varepsilon > 0$ such that

$$\begin{aligned}\sup\{(u+\varepsilon v)(z) : z \in \overline{D} \cap \partial\mathbb{B}(0,r')\} &=: c_1' < 0,\\ \inf\{(u+\varepsilon v)(z) : z \in \overline{D} \cap \partial\mathbb{B}(0,r'')\} &=: c_1'' > c_1'.\end{aligned}$$

Define $\widehat{v} : \overline{D} \to (-\infty, 0)$ by

$$\widehat{v}(z) := \begin{cases} (u+\varepsilon v)(z), & \text{if } \|z\| \ge r'', \\ \max\{(u+\varepsilon v)(z), (c_1' + c_1'')/2\}, & \text{if } r' < \|z\| < r'', \\ (c_1' + c_1'')/2, & \text{if } \|z\| \le r'. \end{cases}$$

Then $\widehat{v} \in \mathcal{C}(\overline{D}) \cap \mathcal{PSH}(D)$, $\widehat{v} < 0$, and $\lim_{\overline{D} \ni z \to \infty} \widehat{v}(z) = -\infty$. (Such a function is sometimes called a *global psh antipeak function at* ∞.)

Let ψ be as above. Applying the Poisson integral representation, we get

$$\widehat{v} \circ \psi(\lambda) \le \frac{1}{2\pi} \int_0^{2\pi} \widehat{v} \circ \psi(e^{i\theta})\, \frac{1 - |\lambda|^2}{|e^{i\theta} - \lambda|^2}\, d\theta \le \frac{1}{6\pi} \int_0^{2\pi} \widehat{v} \circ \psi(e^{i\theta})\, d\theta, \qquad |\lambda| \le \frac{1}{2}. \tag{2.9.18}$$

Now, choose $L > 0$ such that

$$U' := \{z \in \overline{D} : \widehat{v}(z) < -L/6\} \subset \mathbb{C}^n \setminus \overline{\mathbb{B}}(0,r).$$

Then there is an $\alpha_0 > 0$ such that

$$V := \{z \in \overline{D} : \widehat{u}(z) > -\alpha_0\} \subset \{z \in \overline{D} : \widehat{u}(z) \ge -2\alpha_0\} \subset \{z \in \overline{D} : \widehat{v}(z) < -L\}.$$

Now let $\varphi \in \mathcal{O}(E, D)$ be such that $\varphi(0) \in V$. Obviously, we may also assume that $\varphi \in \mathcal{C}(\overline{E}, D)$. Applying (2.9.17) and (2.9.18) we have, for $\lambda \in E$, $|\lambda| \le 1/2$,

$$\widehat{v} \circ \varphi(\lambda) \le \frac{1}{6\pi} \int_0^{2\pi} \widehat{v} \circ \varphi(e^{i\theta})\, d\theta \le \frac{1}{6\pi} \int_{E(\varphi,\alpha_0)} \widehat{v} \circ \varphi(e^{i\theta})\, d\theta \le -\frac{L\Lambda_1(E(\varphi,\alpha_0))}{6\pi} \le -\frac{L}{6},$$

i.e. $\varphi(\lambda) \in U'$. ■

COROLLARY 2.9.4. *Let $D \subset \mathbb{C}^n$ and a be as in Lemma 2.9.3. Let $U = U(a)$ be any neighborhood of a. Then there exists a neighborhood $V = V(a) \subset U$ such that for any connected component V' of $D \cap V$ the following inequality is true:*

$$2\varkappa_D(z; X) \ge \varkappa_{V'}(z; X), \qquad z \in V', X \in \mathbb{C}^n.$$

REMARK 2.9.5. Observe that in Lemma 2.9.3 only the existence of a local psh peak function and a local psh antipeak function was needed. Other localization results for unbounded domains may be found in [Nik 2002].

Proof of Theorem 2.9.1. STEP 1. We prove that D is k-hyperbolic if D has a local holomorphic peak point at infinity. Assume this is not the case. Then there exist $z_0 \in D$, $(z_j)_j \subset D$ with $z_j \to z_0$, and $X_j \in \mathbb{C}^n$ with $\|X_j\| = 1$ such that $\varkappa_D(z_j; X_j) < 1/j$, $j \in \mathbb{N}$ (see Theorem 7.2.2 in [J-P 1993]). Therefore, we find functions $\varphi_j \in \mathcal{O}(E, D)$ such that $\varphi_j(0) = z_j$ and $\|\varphi_j'(0)\| > j$, $j \in \mathbb{N}$. By the Cauchy inequalities, we may further assume that there is a sequence $(\lambda_j)_j \in \frac{1}{2}E$, $\lambda_j \to 0$, such that $\|\varphi_j(\lambda_j)\| \to \infty$.

Put $\widetilde{\varphi}_j := \varphi_j \circ (-h_{\lambda_j})$. Then $\widetilde{\varphi}_j \in \mathcal{O}(E, D)$ with $\widetilde{\varphi}_j(\lambda_j) = z_j$ and $\|\widetilde{\varphi}_j(0)\| \to \infty$. Put $R := 2\|z_0\| + 1$. Then there is an $R' > R$ for which Lemma 2.9.3 can be used. Since $\|\widetilde{\varphi}_j(0)\| > R'$ for large j, we deduce that $\|\widetilde{\varphi}_j(\lambda_j)\| > R$ for these j, which contradicts the fact that $z_j \to z_0$.

STEP 2. Here we prove that D is k-complete. Assume the contrary. Then there are a point $z_0 \in D$ and a sequence $(z_j)_j \subset D$ such that $A := \sup\{k_D(z_0, z_j) : j \in \mathbb{N}\} < \infty$ and either $z_j \to z^* \in \partial D$ or $z_j \to \infty$. Again we discuss only the second case. The first is similar.

Let $f \in \mathcal{C}(\overline{D} \setminus \mathbb{B}(0, r), E) \cap \mathcal{O}(D \setminus \overline{\mathbb{B}}(0, r), E)$ be the local holomorphic peak function at infinity. Choose $\mathcal{C}^1$-curves $\alpha_j : [0, 1] \to D$ with

$$\alpha_j(0) = z_0, \qquad \alpha_j(1) = z_j, \qquad \int_0^1 \varkappa_D(\alpha_j(t); \alpha_j'(t))\, dt < A + 1.$$

According to Corollary 2.9.4, we find an $R > \max\{\|z_0\|, r\}$ such that for any connected component U of $D \cap (\mathbb{C}^n \setminus \overline{\mathbb{B}}(0, R))$ the following is true: $2\varkappa_D(z; X) \geq \varkappa_U(z; X)$, $z \in U$, $X \in \mathbb{C}^n$. Moreover, observe that $|f| \leq C < 1$ on $\overline{D} \cap \partial\mathbb{B}(0, R)$.

We may assume that all $\|z_j\| > R$. Fix an j, put $t_j := \sup\{t \in [0, 1] : \|\alpha_j(t)\| \leq R\}$, and let U_j denote that connected component of $(\mathbb{C}^n \setminus \overline{\mathbb{B}}(0, R)) \cap D$ containing $\alpha_j((t_j, 1])$. Then

$$\begin{aligned} 2\int_0^1 \varkappa_D(\alpha_j(t); \alpha_j'(t))\, dt &\geq \int_{t_j}^1 \varkappa_{U_j}(\alpha_j(t); \alpha_j'(t))\, dt \geq \int_{t_j}^1 \varkappa_E(f \circ \alpha_j(t); (f \circ \alpha_j)'(t))\, dt \\ &\geq \min\{k_E(f(z_j), \lambda) : |\lambda| \leq C\} \xrightarrow[j \to \infty]{} \infty, \end{aligned}$$

a contradiction. ■

REMARK 2.9.6. The domain (see [Par 2003])

$$D := \{(z, w) \in \mathbb{C}^3 \times \mathbb{C} : |z_1 z_2 z_3| < 1,\ 0 < |w| < e^{-\max\{|z_j| : j = 1,2,3\}}\}$$

is k-complete, but there is no local psh peak function at infinity; in particular, there is no local holomorphic peak function at infinity.

Indeed, D is a pseudoconvex Reinhardt domain which is Brody-hyperbolic. Hence, it is k-complete (see Theorem 2.2.1). Assume now that there exists a local psh peak function at ∞. Hence there is an $R > 1$ and a $\varphi \in \mathcal{C}(\overline{D} \setminus \mathbb{B}(0, R)) \cap \mathcal{PSH}(D \setminus \overline{\mathbb{B}}(0, R))$,

$\varphi < 0$, such that $\lim_{\overline{D} \ni (z,w) \to \infty} \varphi(z,w) = 0$. Fix an $a \in \mathbb{C}$ with $|a| = 2R$ and define

$$D_a := \{z \in \mathbb{C}^2 : 2R|z_1 z_2| < 1\}, \qquad u_a(z) := \max\{|z_1|, |z_2|, |a|\}, \quad z \in D_a.$$

Moreover, put

$$\Omega := \{(z, \lambda) \in D_a \times \mathbb{C} : |\lambda| < e^{-u_a(z)}\}.$$

Finally, define $\varphi_a : \Omega \to [-\infty, \infty)$ as $\varphi_a(z,\lambda) := \varphi(z, a, \lambda)$. Then $\varphi_a \in \mathcal{PSH}(\Omega)$ and $\varphi_a(\cdot, 0) \in \mathcal{PSH}(D_a)$. By the Liouville theorem, there is a $\psi \in \mathcal{SH}((1/2R)E)$ such that $\varphi_a(z, 0) = \psi(z_1 z_2)$, $z \in D_a$. So, applying the Oka theorem, we get

$$\psi(0) = \limsup_{\mathbb{R} \ni t \to \infty} \psi\left(\frac{1}{2Rt}\right) = \limsup_{\mathbb{R} \ni t \to \infty} \varphi_a\left(t, \frac{1}{2Rt^2}, 0\right) =: C \le 0.$$

In the case when $C = 0$, the maximum principle would imply that $\psi \equiv 0$ on $(1/2R)E$ and thus $\varphi(\cdot, a, 0) \equiv 0$ on D_a, which contradicts the assumption $\varphi < 0$. Hence, $C < 0$.

Now choose a $t_0 > 2|a|$ such that for all $t > t_0$,

$$\lim_{0 \ne \lambda \to 0} \varphi\left(t, \frac{1}{2Rt^2}, a, \lambda\right) = \varphi_a\left(t, \frac{1}{2Rt^2}, 0\right) < \frac{3}{4} C.$$

So, for $t > t_0$, there is an $\varepsilon_t > t$ such that

$$\varphi\left(t, \frac{1}{2Rt^2}, a, \lambda\right) < \frac{1}{2} C, \qquad 0 < |\lambda| < e^{-\varepsilon_t}.$$

Now observe that $\max\{t, 1/2Rt^2, |a|\} = t$, $t > t_0$, and $\varepsilon_t \to \infty$ if $t \to \infty$. Therefore, we may choose a sequence $((t_j, 1/2Rt_j^2, a, \lambda_j))_j \subset D \cap (\mathbb{R}^2 \times \{a\} \times \mathbb{C})$ such that $t_0 < t_j \to \infty$ and $|\lambda_j| < e^{-\varepsilon_j}$. Hence, we have

$$0 = \lim_{j \to \infty} \varphi\left(t_j, \frac{1}{2Rt_j^2}, a, \lambda_j\right) \le \frac{C}{4},$$

a contradiction.

The above example shows that the conditions in Theorem 2.9.1 are too strong. Observe that any finite boundary point z_0 is obviously a local psh antipeak point (take simply $\log(\|z - z_0\|/R)$ for a large R).

$\boxed{?}$ Does Theorem 2.9.1 remain true if one only assumes that any boundary point admits a local psh peak and antipeak function $\boxed{?}$

In this context observe that there exists a smoothly bounded pseudoconvex domain $D \subset \mathbb{C}^3$ such that each boundary point of D is a global psh peak point, but some boundary point is not a local holomorphic peak point (see [Yu 1997]).

EXAMPLE 2.9.7. Theorem 2.9.1 has been used in [Gau 1999] to prove the following results.

(a) Let P be a real-valued subharmonic polynomial on $\mathbb{C}$ without harmonic terms. Then

$$D := \{(z, w) \in \mathbb{C} \times \mathbb{C} : \operatorname{Re} w + P(z) < 0\}$$

is k-complete.

(b) Let P be a real-valued convex polynomial on $\mathbb{C}^n$, $P(0) = \operatorname{grad} P(0) = 0$, without harmonic terms, such that the set $\{z \in \mathbb{C}^n : P(z) = 0\}$ does not contain a non-trivial analytic set. Then

$$D := \{(z, w) \in \mathbb{C}^n \times \mathbb{C} : \operatorname{Re} w + P(z) < 0\}$$

is a convex k-complete domain ([4]).

([4]) See also Theorem 7.1.8 in [J-P 1993] for the following characterization of k-complete convex domains: *A convex domain G is k-complete iff G contains no complex lines iff G is biholomorphic to a bounded convex domain.*

CHAPTER 3

Bergman metric

3.1. The Bergman kernel

In this chapter we will discuss a metric on domains which is invariant under biholomorphic mappings, namely the Bergman metric. To do so we have to first recall the Bergman kernel function and the Bergman kernel.

Let $G \subset \mathbb{C}^n$ be a domain. We denote by $L_{\mathrm{h}}^2(G)$ the Hilbert space of all square integrable functions on G which are holomorphic; it is a closed subspace of $L^2(G)$. The key tool in this chapter is the following extension theorem due to T. Ohsawa and K. Takegoshi [Ohs-Tak 1987].

THEOREM 3.1.1. *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ and H an affine subspace of $\mathbb{C}^n$. Then there is a positive constant C, which depends only on the diameter of D and on n, such that for any $f \in L_{\mathrm{h}}^2(D \cap H)$ there is an $F \in L_{\mathrm{h}}^2(D)$ such that $F|_{D\cap H} = f$ and $\|F\|_{L_{\mathrm{h}}^2(D)} \leq C\|f\|_{L_{\mathrm{h}}^2(D\cap H)}$.*

Moreover, we recall the following one-dimensional result (see [Lin 1977], [Che 2000]) which will be used subsequently.

THEOREM 3.1.2. *Let $D \subset \mathbb{C}$ be a bounded domain, $z_0 \in \partial D$, and $f \in L_{\mathrm{h}}^2(D)$. Then for any $\varepsilon > 0$ there exist a neighborhood $U = U(z_0)$ and a function $g \in L_{\mathrm{h}}^2(D \cup U)$ such that $\|f - g\|_{L_{\mathrm{h}}^2(D)} \leq \varepsilon$. In particular, the subspace of all functions in $L_{\mathrm{h}}^2(D)$, bounded near z_0, is dense in $L_{\mathrm{h}}^2(D)$.*

In [Che 2000], complete Kaehler metrics were used to solve a corresponding $\bar{\partial}$-problem in order to find g. Here we give a proof which is based on Berndtsson's solution of a $\bar{\partial}$-problem (see [Pfl 2000]).

Proof. We may assume that $z_0 = 0 \in \partial D$ and that $\overline{D} \subset E$. Fix $f \in L_{\mathrm{h}}^2(D)$ and a sufficiently small $\varepsilon \in (0, 1/2)$. Put $\psi(z) := -\log(\log(1/|z|))$, $z \neq 0$. Observe that $\psi \in \mathcal{C}^\infty(\mathbb{C}_*) \cap \mathcal{SH}(\mathbb{C}_*)$ and

$$\left|\frac{\partial\psi}{\partial z}\right|^2 = \frac{\partial^2\psi}{\partial z\partial\overline{z}} = (\log|z|^2)^{-2}|z|^{-2} > 0.$$

Moreover, let $\chi \in \mathcal{C}^\infty(\mathbb{R}, [0, 1])$,

$$\chi(t) := \begin{cases} 1 & \text{if } t \leq 1 - \log 2, \\ 0 & \text{if } t > 1, \end{cases}$$

be such that $|\chi'| \leq 3$.

Finally, we define $\varrho_\varepsilon(z) := \chi(-\psi(z) - \log(\log(1/\varepsilon)) + 1)$, $z \in \mathbb{C}_*$. Observe that $\varrho_\varepsilon(z) = 0$ if $0 < |z| < \varepsilon$, and $\varrho_\varepsilon(z) = 1$ if $|z| > \sqrt{\varepsilon}$. Then $\alpha := \bar\partial(\varrho_\varepsilon f)$ is a $\mathcal{C}^\infty$ $\bar\partial$-closed $(0,1)$-form on $D_\varepsilon := D \cup \mathbb{B}(0,\varepsilon)$. Now we wish to apply the following theorem of Berndtsson.

THEOREM ([Ber 1996]). *Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain. Let $\varphi, \psi \in \mathcal{PSH}(\Omega)$, ψ strongly psh, be such that for any $X \in \mathbb{C}^n$,*

$$\sum_{j,k=1}^n \frac{\partial^2\psi}{\partial z_j \partial \overline{z}_k}(z) X_j \overline{X}_k \geq \left| \sum_{j=1}^n \frac{\partial \psi}{\partial z_j}(z) X_j \right|^2$$

on Ω. Let $\delta \in (0,1)$ and $\alpha = \sum_{j=1}^n \alpha_j d\overline{z}_j$ be a $\bar\partial$-closed $(0,1)$-form. Then there exists a solution $u \in L^2_{\mathrm{loc}}(\Omega)$ of $\bar\partial u = \alpha$ such that

$$\int_\Omega |u|^2 e^{-\varphi+\delta\psi}\, d\Lambda_{2n}(z) \leq \frac{4}{\delta(1-\delta)^2} \int_\Omega \sum_{j,k=1}^n \psi^{j,k} \alpha_j \overline{\alpha}_k e^{-\varphi+\delta\psi}\, d\Lambda_{2n}(z),$$

where (ψ^{jk}) denotes the inverse matrix of $(\partial^2\psi/\partial z_j \partial \overline{z}_k)$.

Take $\varphi := (1/2)\psi$ and $\delta := 1/2$. Then there exists a function u_ε, $\bar\partial u_\varepsilon = \alpha$ on $D_\varepsilon \setminus \{0\}$, such that

$$\begin{aligned}\int_{D_\varepsilon \setminus \{0\}} |u|^2 e^{-\varphi+\delta\psi}\, d\Lambda_2(z) &\leq \frac{4}{\delta(1-\delta)^2} \int_{D_\varepsilon \setminus \{0\}} \frac{|\alpha|^2}{|\partial^2\psi/\partial z \partial \overline{z}|} e^{-\varphi+\delta\psi}\, d\Lambda_2(z) \\ &= 16 \int_{z \in D,\, \varepsilon \leq |z| \leq \sqrt{\varepsilon}} |\chi'|^2 |f|^2\, d\Lambda_2(z).\end{aligned}$$

Then the function $f_\varepsilon := u_\varepsilon - \varrho_\varepsilon f$ belongs to $L^2_{\mathrm{h}}(D_\varepsilon \setminus \{0\})$ and

$$\|f - f_\varepsilon\|_{L^2_{\mathrm{h}}(D)} \leq \|(1-\varrho_\varepsilon) f\|_{L^2_{\mathrm{h}}(D)} + 160 \|f\|_{L^2_{\mathrm{h}}(D \cap \mathbb{B}(0,\sqrt{\varepsilon}))} \leq C \|f\|_{L^2_{\mathrm{h}}(D \cap \mathbb{B}(0,\sqrt{\varepsilon}))} \underset{\varepsilon \to 0}{\longrightarrow} 0,$$

where C is a general positive constant. It remains to note that $f_\varepsilon \in \mathcal{O}(D_\varepsilon)$ (use Laurent series), which finishes the proof. ∎

We note that under suitable assumptions this result can be generalized to higher dimensions (see [Blo 2002]).

Observe that the point evaluation functional $L^2_{\mathrm{h}}(G) \ni f \mapsto f(w)$ $(w \in G)$ is continuous. Therefore, there is a uniquely defined function $K_G(\cdot, w) \in L^2_{\mathrm{h}}(G)$ such that

$$f(w) = \int_G f(z) \overline{K_G(z,w)}\, d\Lambda(z), \qquad f \in L^2_{\mathrm{h}}(g), w \in G.$$

The function K_G is the *Bergman kernel function for G*. Recall that K_G can be given with the help of a complete orthonormal system $(\varphi_j)_{j \in N} \subset L^2_{\mathrm{h}}$, where $N \subset \mathbb{N}$; namely

$$K_G(z,w) = \sum_{j \in N} \varphi_j(z) \overline{\varphi_j(w)}, \qquad z, w \in G.$$

REMARK 3.1.3. Recall that there are domains $G_k \subset \mathbb{C}^2$ for which $\dim L^2_{\mathrm{h}}(G_k) = k$ [Wig 1984]. $\boxed{?}$ It is unknown whether $\dim L^2_{\mathrm{h}}(G) = \infty$ if $G \subset \mathbb{C}^n$, $n > 1$, is a pseudoconvex domain with $L^2_{\mathrm{h}}(G) \neq \{0\}$. $\boxed{?}$

The function K_G is holomorphic in z and antiholomorphic in w; moreover, we have $K_G(z,w) = \overline{K_G(w,z)}$, $z,w \in G$. If $\Phi : G \to D$ is a biholomorphic mapping between the domains D and G, then

$$K_D(\Phi(z), \Phi(w)) \det \Phi'(z) \overline{\det \Phi'(w)} = K_G(z,w), \qquad z,w \in G.$$

Moreover, there is a transformation law even for proper holomorphic mappings due to S. Bell (see [J-P 1993, Theorem 6.1.8]).

THEOREM 3.1.4. *Let $F : G \to D$ be a proper holomorphic mapping of order m between the bounded domains $G, D \subset \mathbb{C}^n$. Let $u := \det F'$ and denote by $\Phi_1, \dots, \Phi_m$ the local inverses of F defined on $D' := D \setminus \{F(z) : z \in G,\ u(z) = 0\}$. Put $U_k := \det \Phi_k'$. Then*

$$\sum_{k=1}^{m} K_G(z, \Phi_k(w))\overline{U_k(w)} = u(z) K_D(F(z), w), \qquad z \in G,\ w \in D'.$$

The function $k_G(z) := K_G(z,z)$ (1), $z \in G$, is called *the Bergman kernel of G*. If $L^2_{\mathrm{h}}(G) \neq \{0\}$, then k_G is also given as

$$k_G(z) = \sup\left\{ \frac{|f(z)|^2}{\|f\|^2_{L^2_{\mathrm{h}}}} : f \in L^2_{\mathrm{h}}(G) \setminus \{0\} \right\}.$$

Observe that $k_D|_G \le k_G$ whenever $G \subset D$.

For the Bergman kernel there is the following localization result (see [J-P 1993, Theorem 6.3.5]).

THEOREM 3.1.5. *Let $D_j \subset \mathbb{C}^n$, $j = 1, 2$, be two bounded pseudoconvex domains and let $z_0 \in \partial D_1$. Assume that there is a neighborhood $U = U(z_0)$ of z_0 such that $D_1 \cap U = D_2 \cap U$. Then there exist positive numbers m, M and a neighborhood $V = V(z_0)$ such that*

$$m k_{D_1}(z) \le k_{D_2}(z) \le M k_{D_1}(z), \qquad z \in V \cap D_1.$$

In general, it is not easy to find explicit formulas for the Bergman kernel function. In most of the known examples the formulas are obtained using an explicit complete orthonormal system $(\varphi_j)_j \in L^2_{\mathrm{h}}(G)$.

EXAMPLE 3.1.6 (see Examples 6.1.5 and 6.1.6 in [J-P 1993]). (a) For the Euclidean ball $\mathbb{B}_n$ we have

$$K_{\mathbb{B}_n}(z,w) = \frac{n!}{\pi^n}(1 - \langle z, w\rangle)^{-(n+1)}, \qquad z,w \in \mathbb{B}_n.$$

(b) Let E^n be the n-dimensional polydisc. Then

$$K_{E^n}(z,w) = \frac{1}{\pi^n}\prod_{j=1}^{n}(1 - z_j\overline{w}_j)^{-2}, \qquad z,w \in E^n.$$

(c) Put $D_p := \{z \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2/p} < 1\}$, $p > 0$. Then

$$K_{D_p}(z,w) = \frac{1}{\pi^2}(1 - z_1\overline{w}_1)^{p-2}\,\frac{(p+1)(1 - z_1\overline{w}_1)^p + (p-1)z_2\overline{w}_2}{((1 - z_1\overline{w}_1)^p - z_2\overline{w}_2)^3}, \qquad z,w \in D_p.$$

Observe (by a simple calculation) that K_{D_2} has no zeros on $D_2 \times D_2$.

---

(1) Observe that the symbol $k_D(\cdot)$ is a function on D, while the Kobayashi pseudodistance $k_D(\,\cdot\,,\,\cdot\,)$ is defined on $D \times D$; we hope no confusion arises.

(d) Recently, using Theorem 3.1.4, the following formula for the Bergman kernel function of $\mathbb{G}_n$ has been found in [Edi-Zwo 2004] (see Remark 1.4.17 for a definition of $\mathbb{G}_n$):

$$K_{\mathbb{G}_n}(\pi_n(\lambda), \pi_n(\mu)) = K_{\mathbb{G}_n}(z, w) = \frac{\det[(1 - \lambda_j \overline{\mu}_k)^{-2}]_{1 \le j,k \le n}}{\pi^n \det \pi_n'(\lambda) \overline{\det \pi_n'(\mu)}} = \frac{F_n(z, \overline{w})}{\pi^n \prod_{j,k=1}^n (1 - \lambda_j \overline{\mu}_k)^2},$$
$$\lambda, \mu \in E^n \setminus \{\zeta \in E^n : \det \pi_n'(\zeta) = 0\} = E^n \setminus \{\zeta \in E^n : \zeta_j = \zeta_k \text{ for some } j \neq k\},$$

where $\pi_n : \mathbb{C}^n \to \mathbb{C}^n$,

$$\pi_n(\lambda_1, \dots, \lambda_n) := \Big(\sum_{1 \le j_1 < \cdots < j_k \le n} \lambda_{j_1} \cdots \lambda_{j_k} \Big)_{k=1,\dots,n}$$

($\mathbb{G}_n = \pi_n(E^n)$). In particular,

$$K_{\mathbb{G}_2}(\pi_2(\lambda), \pi_2(\mu)) = K_{\mathbb{G}_2}(z, w) = \frac{F_2(z, \overline{w})}{\pi^2 \prod_{j,k=1}^2 (1 - \lambda_j \overline{\mu}_k)^2}, \qquad \lambda, \mu \in E^2,$$

where

$$F_2(z, w) := 2 - z_1 w_1 + 2 z_2 w_2,$$

and

$$K_{\mathbb{G}_3}(\pi_3(\lambda), \pi_3(\mu)) = K_{\mathbb{G}_3}(z, w) = \frac{F_3(z, \overline{w})}{\pi^3 \prod_{j,k=1}^3 (1 - \lambda_j \overline{\mu}_k)^2}, \qquad \lambda, \mu \in E^3,$$

where

$$F_3(z, w) := 6 - 4z_1w_1 - 2z_2w_2 + 2z_1^2w_2 + 2z_2w_1^2 - 3z_1z_2w_3 - 3z_3w_1w_2 + 15z_3w_3 \\ - z_1z_2w_1w_2 - 2z_1z_3w_1w_3 + 2z_1z_3w_2^2 + 2z_2^2w_1w_3 - 4z_2z_3w_2w_3 + 6z_3^2w_3^2.$$

It is easily seen that $K_{\mathbb{G}_2}$ has no zeros on $\mathbb{G}_2 \times \mathbb{G}_2$: use simply the description of $\mathrm{Aut}(\mathbb{G}_2)$ (Theorem 1.4.14) to reduce the discussion to the case $\mu_2 = 0$. $\boxed{?}$ What remains as an open question is whether $K_{\mathbb{G}_n}$ with $n \ge 3$ has zeros. $\boxed{?}$

EXAMPLE 3.1.7 ([D'Ang 1994]). Generalizing Example 3.1.6(c), let

$$D := \{\zeta = (z, w) \in \mathbb{C}^n \times \mathbb{C}^m : \|z\|^2 + \|w\|^{2p} < 1\}$$

with $p \in (0, \infty)$. Then we have the following formula for its Bergman kernel:

$$k_D(\zeta) = \sum_{j=0}^{n+1} c_k \frac{(1 - \|z\|^2)^{-n-1+k/p}}{((1 - \|z\|^2)^{1/p} - \|w\|^2)^{m+k}}, \qquad \zeta = (z, w) \in D,$$

where the constants c_k depend on k, n, m, and p. Even more, they can be explicitly calculated.

Other examples and methods to proceed may be found in [Boa-Fu-Str 1999] (see also [Boa 2000]).

3.1.1. Deflation. Fix a bounded domain $D \subset \mathbb{C}^n$ which is given as

$$D = \{z \in U : \varphi(z) < 1\},$$

where $\varphi \in \mathcal{C}(U, [0,\infty))$ for a suitable open neighborhood U of $\overline{D}$. Put

$$G_1 := \{(z,\zeta) \in D \times \mathbb{C}^1 : \varphi(z) + |\zeta|^{2/(p+q)} < 1\},$$
$$G_2 := \{(z,\zeta) \in D \times \mathbb{C}^2 : \varphi(z) + |\zeta_1|^{2/p} + |\zeta_2|^{2/q} < 1\},$$

where p, q are positive real numbers. Then we have the following *deflation identity*:

$$\text{(D)} \quad \pi K_{G_1}((z,0),(w,0)) = \frac{\pi^2 \Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+1)} K_{G_2}((z,0,0),(w,0,0)), \qquad z,w \in D.$$

In fact, the identity (D) holds because both sides represent the unique reproducing kernel for the Hilbert space $L^2_{\mathrm{h}}(D, \pi(1-\varphi)^{p+q})$ (2). To be more precise, fix an $h \in L^2_{\mathrm{h}}(D)$. Then h can also be thought to belong to $L^2_{\mathrm{h}}(G_j)$, $j = 1,2$. Therefore, by the reproducing property of the Bergman kernel function, we see that

$$h(z) = \int_{G_1} h(w) K_{G_1}((z,0),(w,\zeta))\, d\Lambda(w,\zeta).$$

Observe that the fiber over a point $w \in D$ is a disc of radius $(1-\varphi(w))^{(p+q)/2}$. Therefore, applying the mean value property for harmonic functions leads to

$$h(z) = \int_D h(w)(1-\varphi(w))^{p+q} \pi K_{G_1}(z,0),(w,0))\, d\Lambda(w).$$

Hence, $K_{G_1}(\cdot,0),(\cdot,0))$ is the reproducing kernel for $L^2_{\mathrm{h}}(D, \pi(1-\varphi)^{p+q})$.

A similar reasoning leads to the same conclusion for the right side of (D), which finally proves the deflation identity.

EXAMPLE 3.1.8. Let for instance $G := \{z \in \mathbb{C}^2 : |z_1| + |z_2|^{1/2} < 1\}$. Suppose we knew the formula for $K_G(z,w)$ (see Example 3.2.1). Now, let $D = E$ and $p = q = 2$. Applying the deflation method from above, for $G^* := \{z \in \mathbb{C}^3 : |z_1| + |z_2| + |z_3| < 1\}$ we get

$$\pi K_G((z,0),(w,0)) = \frac{\pi^2}{3!} K_{G^*}((z,0,0),(w,0,0)), \qquad z,w \in E.$$

Observe that if $K_G((z,0),(w,0)) = 0$ for certain points $z,w \in E$, then we also have $K_{G^*}((z,0,0),(w,0,0)) = 0$.

Most of the domains for which an explicit formula for their Bergman kernel is known are Reinhardt domains. Here we describe the Bergman kernel function of a domain that is not biholomorphically equivalent to a Reinhardt domain. Define

$$N(z) := \sqrt{\|z\|^2 + |z \bullet z|}, \qquad z \in \mathbb{C}^n,$$

where $z \bullet w := \langle z, \overline{w}\rangle = z_1 w_1 + \cdots + z_n w_n$. Recall that $N/\sqrt{2}$ is the smallest $\mathbb{C}$-norm dominated by $\|\cdot\|$ which coincides with the Euclidean norm on $\mathbb{R}^n$ (cf. [Hah-Pfl 1988]). The N-ball $\mathbb{M}_n$ is called *the minimal ball*, i.e.

$$\mathbb{M} := \mathbb{M}_n := \{z \in \mathbb{C}^n : N(z) < 1\}.$$

Observe that $\mathbb{M}_n$ can be thought of as a model for domains with non-smooth boundary.

(2) Recall that, if ψ is a non-negative measurable function, then $L^2_{\mathrm{h}}(D,\psi) := \{f \in \mathcal{O}(D) : \int_D |f|^2 \psi \, d\Lambda_{2n} < \infty\}$.

For $n = 2$, a formula for its Bergman kernel function can be found in [J-P 1993]. The general case is contained in [Oel-Pfl-You 1997] and [Men-You 1999].

THEOREM 3.1.9. *The Bergman kernel function of* $\mathbb{M}$ *is given by the formula*

$$K_{\mathbb{M}}(z,w) = \frac{1}{n(n+1)\Lambda_{2n}(\mathbb{M})}\,\frac{\sum_{j=0}^{[n/2]}\binom{n+1}{2j+1}X^{n-1-2j}Y^j(2nX-(n-2j)(X^2-Y))}{(X^2-Y)^{n+1}},$$

where $z, w \in \mathbb{M}$, $X = X(z,w) := 1 - \langle z, w\rangle$, *and* $Y = Y(z,w) := (z \bullet z)\overline{(w \bullet w)}$.

Proof. The main ideas of the proof are:

1) to establish a formula for the Bergman kernel function of the "domain"

$$\mathfrak{K} := \{z \in \mathbb{C}^{n+1} \setminus \{0\} : \|z\| < 1,\ z \bullet z = 0\},$$

2) to use the proper mapping $\pi : \mathfrak{K} \to \mathbb{M} \setminus \{0\}$, $\pi(\widetilde{z}, z_{n+1}) := \widetilde{z}$, to get a formula for the Bergman kernel function of $\mathbb{M}$.

Now, we present the proof in more detail. First, observe that the following n-form on $\mathbb{C}_*^{n+1}$:

$$\widetilde{\alpha}(z) := \sum_{j=1}^{n+1} \frac{(-1)^{j+1}}{z_j}\, dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_{n+1}$$

induces by restriction an $\mathrm{SO}(n+1)$-invariant holomorphic n-form α on the complex manifold $\mathfrak{H} := \{z \in \mathbb{C}^{n+1} \setminus \{0\} : z \bullet z = 0\}$. Put

$$\omega(z)(V_1, \dots, V_{2n-1}) := \alpha(z) \wedge \overline{\alpha(z)}(z, V_1, \dots, V_{2n-1}), \qquad z \in \mathfrak{K},$$

where $(V_1, \dots, V_{2n-1}) \in T_z(\partial\mathfrak{K})$. Observe that ω is a volume form on $T_z(\partial\mathfrak{K})$. Since $\alpha \wedge \overline{\alpha}$ is $\mathrm{SO}(n+1)$-invariant, so is ω. Hence the measure on $\partial\mathfrak{K}$ induced by ω is proportional to the unique $\mathrm{O}(n+1,\mathbb{R})$-invariant measure μ on $\partial\mathfrak{K}$ with $\mu(\partial\mathfrak{K}) = 1$. Put $\omega(\partial\mathfrak{K}) := \int_{\mathfrak{K}} \omega$.

Exploiting the definition of the form α, the following statement may be proved: For any $\mathcal{C}^\infty$-function f on $\mathfrak{H}$ we have

$$\int_{\mathfrak{H}} f(z)\alpha(z) \wedge \overline{\alpha(z)} = \omega(\partial\mathfrak{K}) \int_0^\infty t^{2n-3} \int_{\partial\mathfrak{K}} f(t\zeta)\, d\mu(\zeta)\, dt, \tag{3.1.1}$$

provided the integrals make sense.

Moreover, using spherical harmonics, one obtains the following result:

$$\int_{\partial\mathfrak{K}} (z \bullet w)^k (\xi \bullet \overline{w})^k\, d\mu(w) = \frac{(z \bullet \xi)^k}{N(k,n)}, \qquad z \in \mathfrak{K},\ \xi \in \mathbb{C}^{n+1}, \tag{3.1.2}$$

where

$$N(k,n) := \frac{(2k+n-1)(k+n-1)!}{k!(n-1)!}.$$

Next, let $f \in \mathcal{O}(\mathfrak{K})$ be a homogeneous polynomial of degree k. Fix $z \in \mathfrak{K}$. Then

$$f(z) = C(k,n) \int_{\mathfrak{K}} \langle z, w\rangle^k f(w)\alpha(w) \wedge \overline{\alpha(w)}, \tag{3.1.3}$$

where

$$C(k,n) := \frac{2(2k+n-1)(n+k-1)!}{\omega(\partial\mathfrak{K})(n-1)!k!}.$$

In fact, using (3.1.1) we have

$$\int_{\mathfrak{K}} \langle z,w\rangle^k f(w)\alpha(w)\wedge\overline{\alpha(w)} = \omega(\partial\mathfrak{K})\int_0^1 t^{2n-3+2k}dt \int_{\partial\mathfrak{K}} \langle z,w\rangle^k f(w)\,d\mu(w).$$

Recall that f is a linear combination of a finite number of polynomials of the form $z \mapsto (z\bullet\xi)^k$, $\xi \in S^n$. So, it remains to apply (3.1.2) to get the claim (3.1.3).

In order to be able to continue, we prove the following

LEMMA 3.1.10. *Let $f \in \mathcal{O}(\mathfrak{K})$. Then there are homogeneous polynomials f_k of degree k, $k \in \mathbb{N}_0$, such that*

$$f(z) = \sum_{k=0}^{\infty} f_k(z), \qquad z \in \mathfrak{K},$$

and the convergence is uniform on compact subsets of $\mathfrak{K}$.

Proof. Observe that $A := \{0\}\cup\mathfrak{K}$ is an analytic subset of $\mathbb{B} = \mathbb{B}_{n+1}$. It is clear that 0 is the only singularity of A; it is a normal singularity for $n \geq 2$. Hence A is a normal complex space and the function f extends holomorphically to a function $\widetilde{f} \in \mathcal{O}(A)$. Applying Cartan's Theorem B we find an $\widehat{f} \in \mathcal{O}(\mathbb{B})$ for which $\widehat{f}|_A = \widetilde{f}$. Therefore there are homogeneous polynomials f_k of degree k such that $\widehat{f}(z) = \sum_{k=0}^{\infty} f_k(z)$, $z \in \mathbb{B}$, and the convergence is locally uniform. ∎

Denote by $L^2(\mathfrak{K})$ the space of all measurable functions on $\mathfrak{K}$ satisfying

$$\|f\|_{L^2(\mathfrak{K})} := \left(\int_{\mathfrak{K}} |f(z)|^2 \frac{\alpha(z)\wedge\overline{\alpha(z)}}{(-1)^{n(n+1)/2}(2i)^n}\right)^{1/2} < \infty,$$

and let $L^2_{\mathrm{h}}(\mathfrak{K}) := L^2(\mathfrak{K})\cap\mathcal{O}(\mathfrak{K})$. Then we have the following formula for the Bergman kernel function of the space $L^2_{\mathrm{h}}(\mathfrak{K})$:

LEMMA 3.1.11. *The Bergman kernel function is given by*

$$K_{\mathfrak{K}}(z,w) = \frac{2(-1)^{n(n+1)/2}(2i)^n}{\omega(\partial\mathfrak{K})}\left(\frac{(n-1)}{(1-\langle z,w\rangle)^{n+1}} + \frac{2n\langle z,w\rangle)}{(1-\langle z,w\rangle)^{n+1}}\right), \qquad z,w \in \mathfrak{K}.$$

Proof. Fix $z \in \mathfrak{K}$. Then, applying Lemma 3.1.10 and (3.1.3), we obtain

$$f(z) = \sum_{k=0}^{\infty} f_k(z) = \sum_{k=0}^{\infty} C(k,n)\int_{\mathfrak{K}} \langle z,w\rangle^k f_k(w)\alpha(w)\wedge\overline{\alpha(w)} \tag{3.1.4}$$

$$= \int_{\mathfrak{K}} K_{\mathfrak{K}}(z,w)f(w)\frac{\alpha(w)\wedge\overline{\alpha(w)}}{(-1)^{n(n+1)/2}(2i)^n}.$$

Exploiting the last formula leads to the statement in Lemma 3.1.11. ∎

To summarize, we have finished the first step of the proof of Theorem 3.1.9. Now we continue with the second one.

Let $\pi : \mathfrak{K} \to \mathbb{M} \setminus \{0\}$, $\pi(\widetilde{z}, z_{n+1}) := \widetilde{z}$, $z = (\widetilde{z}, z_{n+1}) \in \mathfrak{K}$. Then π is a proper map of degree 2. Its branching locus is called W; let $V := \pi(W)$. Denote the local inverses of π by φ and ψ. They are given for $z \in \mathbb{M} \setminus V$ by

$$\varphi(z) = (z, i\sqrt{z \bullet z}), \qquad \psi(z) = (z, -i\sqrt{z \bullet z}).$$

Then a calculation leads to the following description of the pull-back of α under φ and ψ on $\mathbb{M} \setminus V$:

$$\begin{aligned} \varphi^*(\alpha) &= \frac{n+1}{i\sqrt{z \bullet z}} (-1)^n dz_1 \wedge \cdots \wedge dz_n, \\ \psi^*(\alpha) &= \frac{n+1}{-i\sqrt{z \bullet z}} (-1)^n dz_1 \wedge \cdots \wedge dz_n. \end{aligned} \tag{3.1.5}$$

Let $P_{\mathfrak{K}}$ denote the Bergman projection on $\mathfrak{K}$ and $P_{\mathbb{M}}$ the Bergman projection on $\mathbb{M}$. Then we have the following relation.

LEMMA 3.1.12. *Let $f \in L^2(\mathbb{M})$. Then*

$$P_{\mathfrak{K}}(\chi \cdot f \circ \pi)(z) = z_{n+1} P_{\mathbb{M}}(f)(\pi(z)), \qquad z \in \mathfrak{K},$$

where $\chi(z) := z_{n+1}$, $z \in \mathfrak{K}$.

Now applying Lemma 3.1.12, we find a way to express the Bergman kernel function of $\mathbb{M}$ in terms of the Bergman kernel function for $\mathfrak{K}$.

LEMMA 3.1.13. *Let φ and ψ be the local inverses as above. Then*

$$z_{n+1} K_{\mathbb{M}}(\pi(z), w) = (n+1)^2 \left(\frac{K_{\mathfrak{K}}(z, \varphi(w))}{\overline{\varphi_{n+1}(w)}} + \frac{K_{\mathfrak{K}}(z, \psi(w))}{\overline{\psi_{n+1}(w)}} \right), \qquad z \in \mathfrak{K},\ w \in \mathbb{M} \setminus V.$$

Proof. Fix a $w \in \mathbb{M} \setminus V$ and choose an $r > 0$ such that $w + rE^n \Subset \mathbb{M} \setminus V$. In view of Remark 6.1.4 in [J-P 1993], we may find a $\mathcal{C}^\infty$-function $u : \mathbb{C}^n \to [0, \infty)$, $\operatorname{supp} u \subset w + rE^n$, such that

$$f(w) = \int_{\mathbb{M}} f(z) u(z)\, d\Lambda_{2n}(z), \qquad f \in \mathcal{O}(\mathbb{M}).$$

Therefore,

$$K_{\mathbb{M}}(\cdot, w) = P_{\mathbb{M}}(u).$$

From Lemma 3.1.12 it follows that

$$\begin{aligned} z_{n+1} K_{\mathbb{M}}(\pi(z), w) &= z_{n+1} P_{\mathbb{M}}(u)(\pi(z)) = P_{\mathfrak{K}}(\chi \cdot u \circ \pi)(z) \\ &= \int_{\mathfrak{K}} \zeta_{n+1} u \circ \pi(\zeta) K_{\mathfrak{K}}(z, \zeta)\, \frac{\alpha(\zeta) \wedge \overline{\alpha(\zeta)}}{(-1)^{n(n+1)/2} (2i)^n} \\ &= (n+1)^2 \int_{\mathbb{M} \setminus V} u(\eta) \left(\frac{K_{\mathfrak{K}}(z, \varphi(\eta))}{\overline{\varphi_{n+1}(\eta)}} + \frac{K_{\mathfrak{K}}(z, \psi(\eta))}{\overline{\psi_{n+1}(\eta)}} \right) d\Lambda_{2n}(\eta) \\ &= (n+1)^2 \left(\frac{K_{\mathfrak{K}}(z, \varphi(w))}{\overline{\varphi_{n+1}(\eta)}} + \frac{K_{\mathfrak{K}}(z, \psi(w))}{\overline{\psi_{n+1}(\eta)}} \right), \qquad z \in \mathfrak{K}. \end{aligned}$$

Hence the lemma is proved. ∎

Now we are in a position to finish the proof of Theorem 3.1.9. According to Lemma 3.1.11, we have $K_{\mathfrak{R}}(z,w) = Ch(z \bullet w)$, where

$$C = \frac{2(2i)^n(-1)^{n(n+1)/2}}{\omega(\partial\mathfrak{R})}, \qquad h(t) := \frac{2n}{(1-t)^{n+1}} - \frac{n+1}{(1-t)^n}, \quad t \in \mathbb{C}.$$

Hence,

$$K_{\mathfrak{R}}(\varphi(z),\varphi(w)) = Ch(x), \qquad K_{\mathfrak{R}}(\varphi(z),\psi(w)) = Ch(y), \tag{3.1.6}$$

where

$$x := \langle z,w\rangle + t, \quad y := \langle z,w\rangle - t, \qquad t := \varphi_{n+1}(z)\overline{\varphi_{n+1}(w)}.$$

In view of Lemma 3.1.13, we get

$$K_{\mathbb{M}}(z,w) = C(n+1)^2\left(\frac{h(x)-h(y)}{t}\right).$$

Using the abbreviation $r := 1 - \langle z,w\rangle$, we can rewrite the expression in brackets as

$$Q := \frac{h(x)-h(y)}{t} = 2n\,\frac{(r+t)^{n+1}-(r-t)^{n+1}}{t(r^2-t^2)^{n+1}} - (n+1)\,\frac{(r+t)^n-(r-t)^n}{t(r^2-t^2)^n}.$$

Then

$$Q = \frac{2n}{(r^2-t^2)^{n+1}}\,2\sum_{k=0}^{[n/2]}\binom{n+1}{2k+1}r^{n-2k}t^{2k} - \frac{n+1}{(r^2-t^2)^n}\,2\sum_{k=0}^{[(n-1)/2]}\binom{n}{2k+1}r^{n-2k-1}t^{2k}.$$

Since

$$\binom{n}{2k+1} = \frac{n-2k}{2k+1}\binom{n+1}{2k+1},$$

we proceed with our calculations to get

$$Q = \frac{2}{(r^2-t^2)^{n+1}}\sum_{k=0}^{[n/2]}\binom{n+1}{2k+1}r^{n-1-2k}t^{2k}(2nr-(n-2k)(r^2-t^2)),$$

which immediately leads to the formula in Theorem 3.1.9. ∎

EXAMPLE 3.1.14 (see Example 6.1.9 in [J-P 1993]). By Theorem 3.1.9, the biholomorphic mapping

$$\mathbb{M}_2 \ni (z_1,z_2) \mapsto \frac{1}{\sqrt{2}}(z_1+iz_2, z_1-iz_2) \in G_2 := \{z \in \mathbb{C}^2 : |z_1|+|z_2|<1\}$$

leads to the following formula for the Bergman kernel function of the domain G_2:

$$K_{G_2}(z,w) = \frac{2}{\pi^2}\,\frac{3(1-\langle z,w\rangle)^2(1+\langle z,w\rangle)+4z_1z_2\overline{w}_1\overline{w}_2(5-3\langle z,w\rangle)}{((1-\langle z,w\rangle)^2-4z_1z_2\overline{w}_1\overline{w}_2)^3}, \qquad z,w \in G_2.$$

Observe that this formula may also be derived from the one in Example 3.1.6(c) using the proper holomorphic mapping $D_2 \ni z \mapsto (z_1^2, z_2) \in G_2$ and Bell's transformation.

Fix points $z,w \in G_2$ and write, for abbreviation, $\xi_j := z_j\overline{w}_j$. Then $\sqrt{|\xi_1|}+\sqrt{|\xi_2|}<1$, and so $4|\xi_1\xi_2| < (1-|\xi_1|-|\xi_2|)^2$. Therefore, the numerator in the formula above admits the following estimate:

$$\begin{aligned}3(1-\langle z,w\rangle)^2(1+\langle z,w\rangle)+4z_1z_2\overline{w}_1\overline{w}_2(5-3\langle z,w\rangle) &= 3(1-\xi_1-\xi_2)(1-(\xi_1-\xi_2)^2)+8\xi_1\xi_2\\ &\geq 3(1-|\xi_1|-|\xi_2|)(1-|\xi_1-\xi_2|^2)-2(1-|\xi_1|-|\xi_2|)^2\\ &\geq 3(1-|\xi_1|-|\xi_2|)^2(1+|\xi_1-\xi_2|)-2(1-|\xi_1|-|\xi_2|)^2>(1-|\xi_1|-|\xi_1|)^2>0.\end{aligned}$$

Hence, the Bergman kernel function K_{G_2} has no zeros on $G_2\times G_2$.

REMARK 3.1.15. In [You 2002] an explicit formula for the Bergman kernel function is given even for a more general domain Ω, which could be thought of as some interpolation between the minimal balls and the Euclidean balls. Here, we only describe Ω. Fix $d\in\mathbb{N}$ and two d-tuples $m=(m_1,\dots,m_d)\in\mathbb{N}^d$ and $n=(n_1,\dots,n_d)\in\mathbb{N}^d$. Moreover, let $a=(a_1,\dots,a_d)\in[1,\infty)^d$. Then the domain $\Omega=\Omega_{d,m,n,a}$ is given as

$$\Omega:=\Big\{Z=(Z(1),\dots,Z(d))\in\mathbb{C}(m_1\times n_1)\times\cdots\times\mathbb{C}(m_d\times n_d):\sum_{j=1}^{d}\|Z(j)\|_*^{2a_j}<1\Big\},$$

where $\mathbb{C}(p\times q)$ denotes the space of all $(p\times q)$-matrices with complex entries, and where

$$\|M\|_*:=\Big(\sum_{j=1}^{p}\Big(\sum_{k=1}^{q}|z_{jk}|^2+\Big|\sum_{k=1}^{q}z_{jk}^2\Big|\Big)\Big)^{1/2},\qquad M=(z_{jk})_{j=1,\dots,p,\,k=1,\dots,q}\in\mathbb{C}(p\times q).$$

Observe that for $d=1=a=m$ and $n_1=n$ the domain $\Omega_{d,m,n,a}$ is just the minimal ball $\mathbb{M}\subset\mathbb{C}^n$.

3.2. The Lu Qi-Keng problem

For quite a while it was a question (posed by Lu Qi-Keng [LQK 1966]) whether the Bergman kernel function of a simply connected domain $G\subset\mathbb{C}^n$, $n\geq 2$, has no zeros. Such a domain is called a *Lu Qi-Keng domain.* A first example of a simply connected domain of holomorphy which is not a Lu Qi-Keng domain was given by H. P. Boas [Boa 1986] (see also [Skw 1980]). In fact, it turned out that the domains of holomorphy which are not Lu Qi-Keng form a nowhere dense set in a suitable topology. For a more detailed discussion of this topic see [Boa 1996] (also [Boa 2000]).

EXAMPLE 3.2.1. Let $D=D_p=\{z\in\mathbb{C}^2:|z_1|^2+|z_2|^{2/p}<1\}$, p a positive integer, be the third example of 3.1.6. Then there is the proper holomorphic mapping

$$F:D_p\to G_p:=\{z\in\mathbb{C}^2:|z_1|+|z_2|^{2/p}<1\},\qquad F(z_1,z_2):=(z_1^2,z_2).$$

Using Bell's transformation law (see Theorem 3.1.4), we obtain

$$K_{G_p}((z_1^2,0),(w_1,0))2z_1=(K_D((z_1,0),(\sqrt{w_1},0))-K_D((z_1,0),(-\sqrt{w_1},0)))\,\frac{1}{2\sqrt{w_1}},$$

whenever $z_1\in E$, $w_1\in E\setminus\{0\}$.

Now, applying Example 3.1.6(c), it follows that

$$K_{G_p}((z_1^2,0),(w_1^2,0))2z_1=\frac{p+1}{2\overline{w}_1\pi^2}\,((1-z_1\overline{w}_1)^{-p-2}-(1+z_1\overline{w}_1)^{-p-2}).$$

Then, if $z_1\neq 0$, the kernel function $K_{G_p}((z_1^2,0)(w_1^2,0))$ has a zero iff $(1+x)^{p+2}=(1-x)^{p+2}$, where $x:=z_1\overline{w}_1$. Observe that $\lambda\mapsto(1+\lambda)/(1-\lambda)$ maps E biholomorphically to the right half-plane. Hence, $((1+\lambda)/(1-\lambda))^{p+2}=1$ has a non-zero solution iff $p>2$.

We point out that also $K_{G_p}((0,z_2),(0,w_2))$ has zeros.

EXAMPLE 3.2.2. Next, we study domains of the following type:

$$\Omega_{n,m} := \Big\{(z,w) \in \mathbb{C}^n \times \mathbb{C}^m : \sum_{j=1}^{n} |z_j| + \sum_{k=1}^{m} |w_k|^2 < 1\Big\},$$

where $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$.

First, let $n = 1$ and $m \in \mathbb{N}_0$. Then, using Bell's transformation for the proper holomorphic mapping $F : \mathbb{B}_k \to \Omega_{1,m}$, $F(z) := (z_1^2, z_2, \dots, z_k)$, where $k := m+1$, we get

$$\begin{aligned} &K_{\Omega_{1,m}}((z_1^2, z_2, \dots, z_k), (w_1^2, w_2, \dots, w_k)) \\ &= \frac{k!}{\pi^k 4 z_1 \overline{w}_1}\left(\frac{1}{(1-\langle z,w\rangle)^{k+1}} - \frac{1}{(1+z_1\overline{w}_1 - \langle \widetilde{z}, \widetilde{w}\rangle)^{k+1}}\right), \qquad z,w \in \mathbb{B}_k,\ z_1 w_1 \neq 0, \end{aligned}$$

where $\widetilde{z} := (z_2, \dots, z_k)$ and $\widetilde{w} := (w_2, \dots, w_k)$. In the case $m+2 > 4$, a similar reasoning gives $z_1, w_1 \in E_*$ such that $K_{\Omega_{1,m}}((z_1^2, 0, \dots, 0), (w_1^2, 0, \dots, 0)) = 0$. If $m+2 \le 4$, an easy calculation shows that $K_{\Omega_{1,m}}$ has no zeros on $\Omega_{1,m} \times \Omega_{1,m}$. Hence, the Bergman kernel function of $\Omega_{1,m}$ has a zero iff $m+2 > 4$.

Finally, using the above result for $n = 1$, induction over n, and the deflation method, we are led to the following result:

The Bergman kernel function of $\Omega_{n,m}$ has zeros iff $2n+m > 4$ ([3]). In particular, the convex domain $\Omega_{n,0}$, $n \ge 3$, is not Lu Qi-Keng.

$\boxed{?}$ So far it is not known whether there is a convex domain in $\mathbb{C}^2$ which is not a Lu Qi-Keng domain. $\boxed{?}$

Let $n, k \in \mathbb{N}$, $m \in \mathbb{N}_0$, and $a \in (0,1]$. Put

$$N_{a,k}(z) := \Big(\sum_{\varepsilon_1,\dots,\varepsilon_{n+1} \in \{+1,-1\}} \alpha^{2k}_{\varepsilon_1,\dots,\varepsilon_{n+1}}(z) + a^{2k}\alpha^{2k}(z)\Big)^{1/2k}, \qquad z \in \mathbb{C}^n \times \mathbb{C}^m,$$

where $\alpha_{\varepsilon_1,\dots,\varepsilon_{n+1}}(z) := \sum_{j=1}^{n} \varepsilon_j |z_j| + \varepsilon_{n+1} \sum_{j=1}^{m} |z_{n+j}|^2$ and $\alpha(z) := \sum_{j=1}^{n+m} |z_j|^2$. Moreover, put

$$\Omega_{a,k,n,m} := \{z \in \mathbb{C}^{n+m} : N_{a,k}(z) < 1\}.$$

The following result is due to Nguyên Viêt Anh [Viê 2000].

THEOREM 3.2.3. *The domain $\Omega_{a,k,n,m}$ is strongly convex, algebraic* ([4]), *complete Reinhardt. Moreover, if $2n - m > 4$, then there is a positive integer $M = M(a,n,m)$ such that for all $k \ge M$ the domain $\Omega_{a,k,n,m}$ is not a Lu Qi-Keng domain. In particular, for $m = 0$ there are strongly convex algebraic complete Reinhardt domains in $\mathbb{C}^n$, $n \ge 3$, which are not Lu Qi-Keng.*

$\boxed{?}$ What are effective values for the number $M(a,n,m)$ $\boxed{?}$

To prove Theorem 3.2.3, we need the following lemma.

([3]) Recall Example 3.1.14.

([4]) Here "algebraic" means that the domain is the sublevel set of a real polynomial.

LEMMA 3.2.4. *Suppose $f_j : \mathbb{R}^q \to \mathbb{R}_+$ is a convex function, $j = 1, \dots, p$. Then for $k \in \mathbb{N}$ the function*

$$\varrho(z) := \sum_{\varepsilon_1,\dots,\varepsilon_p \in \{-1,+1\}} (\varepsilon_1 f_1(z) + \dots + \varepsilon_p f_p(z))^{2k}, \qquad z \in \mathbb{R}^q,$$

is also convex.

Proof. Fix $z, w \in \mathbb{R}^q$. Then

$$\frac{\varrho(z)+\varrho(w)}{2} \geq \sum_{\varepsilon_1,\dots,\varepsilon_p \in \{-1,+1\}} \left(\varepsilon_1 \frac{f_1(z)+f_1(w)}{2} + \dots + \varepsilon_p \frac{f_p(z)+f_p(w)}{2}\right)^{2k}.$$

Now recall the following formula:

$$\sum_{\varepsilon_1,\dots,\varepsilon_p \in \{-1,+1\}} (\varepsilon_1 b_1 + \dots + \varepsilon_p b_p)^{2k} = 2^p \sum_{k_1+\dots+k_p=k} \frac{(2k)! b_1^{2k_1} \cdots b_p^{2k_p}}{(2k_1)! \cdots (2k_p)!}.$$

Plugging it into the first expression we get

$$\frac{\varrho(z)+\varrho(w)}{2} \geq \sum_{k_1+\dots+k_p=k} \frac{2^p (2k)!}{(2k_1)! \cdots (2k_p)!} \left(\frac{f_1(z)+f_1(w)}{2}\right)^{2k_1} \cdots \left(\frac{f_p(z)+f_p(w)}{2}\right)^{2k_p}.$$

In view of the positivity and convexity of the functions f_j the last inequality gives $(\varrho(z) + \varrho(w))/2 \geq \varrho((z+w)/2)$, i.e. ϱ is a convex function. ∎

Proof of Theorem 3.2.3. Put

$$\varrho(z) := \sum_{\varepsilon_1,\dots,\varepsilon_{n+1} \in \{-1,+1\}} \alpha^{2k}_{\varepsilon_1,\dots,\varepsilon_{n+1}}(z) + a^{2k}\alpha^{2k}(z) - 1.$$

Then ϱ is the defining function of the domain $\Omega = \Omega_{a,k,n,m}$. Using the above expansion, we see that ϱ is a polynomial with positive coefficients in $|z_1|^2, \dots, |z_n|^2$ and $\sum_{j=1}^m |z_{n+j}|^2$. Hence, Ω is an algebraic complete Reinhardt domain with a smooth boundary. Moreover, by Lemma 3.2.4, Ω is strongly convex.

Observe that $\Omega \subset \Omega_{n,m}$, where $\Omega_{n,m}$ is the domain from Example 3.2.2, and that $N_{a,k} \leq N_{a,l}$ when $l \leq k$. Moreover,

$$\lim_{k\to\infty} N_{a,k}(z) = \sum_{j=1}^n |z_j| + \sum_{k=1}^m |z_{n+k}|^2, \qquad z \in \Omega_{n,m}.$$

It remains to apply Ramadanov's theorem (see [J-P 1993, Theorem 6.1.15]), Example 3.2.2, and the Hurwitz theorem. ∎

EXAMPLE 3.2.5. We also mention that the minimal ball $\mathbb{M} \subset \mathbb{C}^n$, $n \geq 4$, is not Lu Qi-Keng [Pfl-You 1998]. This is proved by exploiting the explicit formula given in Theorem 3.1.9. In fact, let first $n \geq 5$. Put

$$f : \mathbb{R} \to \mathbb{R},\ f(t) := -(n+1)\arctan\frac{2t}{1-t^2} + 2\pi - \arctan\frac{2(n^2-1)t}{(n-1)^2-(n+1)^2t^2}.$$

Observe that $f(0) = 2\pi$ and $f(1/2) < 0$ (here we need the condition $n \geq 5$); so $f(t_0) = 0$ for a certain $t_0 \in (0, 1/2)$. Therefore

$$\left(\frac{1-it_0}{1+it_0}\right)^{n+1} = \frac{n-1+it_0(n+1)}{n-1-it_0(n+1)}.$$

Put $z_0 := \sqrt{it_0}(1,0,\dots,0), w_0 := \sqrt{-it_0}(0,1,0,\dots,0) \in \mathbb{C}^n$. A simple calculation gives $N(z_0) = N(w_0) = t_0 < 1/2$; thus, $z_0, w_0 \in \mathbb{M}$. Then, in view of Theorem 3.1.9,

$$K_{\mathbb{M}}(z_0,w_0) = \frac{1}{n(n+1)\Lambda_{2n}(\mathbb{M})}\,\frac{\sum_{j=0}^{[n/2]}\binom{n+1}{2j+1}(it_0)^{2j}(n+2j+(n-2j)(it_0)^2)}{(1-(it_0)^2)^{n+1}}.$$

Computing the binomial expression leads to

$$K_{\mathbb{M}}(z_0,w_0) = \frac{(n-1+(n+1)it_0)(1+it_0)^{n+1} - (n-1-(n+1)it_0)(1-it_0)^{n+1}}{n(n+1)\Lambda_{2n}(\mathbb{M})2it_0(1-(it_0)^2)^{n+1}} = 0.$$

Let now $n=4$. Consider the function

$$g:\mathbb{R}\to\mathbb{R}, \qquad g(s) := -28s^4 + 50s^3 - 10s^2 - 15s + 5.$$

Then $g(0) = 5$ and $g(2/5) < 0$. Therefore, there exists an $s_0 \in (0,2/5)$ with $g(s_0) = 0$. Put

$$z_0 := \frac{\sqrt{s_0(i-1)}}{2}(i+\sqrt{i}, -i+\sqrt{i}, 0, 0), \qquad w_0 := \frac{\sqrt{s_0(1-i)}}{2}(i-\sqrt{i}, -i-\sqrt{i}, 0, 0).$$

Then $N(z_0) = N(w_0) < 1/2$, i.e. $z_0, w_0 \in \mathbb{M}$. By a little calculation we deduce from the formula in Theorem 3.1.9 that

$$K_{\mathbb{M}}(z_0,w_0) = \frac{g(s_0)}{5\Lambda_{2n}(\mathbb{M})((1-s_0)^2+s_0^2)^5} = 0.$$

Hence, the Bergman kernel function vanishes at (z_0, w_0).

$\boxed{?}$ It is an open question whether the three-dimensional minimal ball is a Lu Qi-Keng domain. $\boxed{?}$ For further open problems see also [Boa 2000]. Other examples of domains that are not Lu Qi-Keng may be found in [Die-Her 1999], [Eng 2000], and [Che 2002].

We close this section by discussing consequences of the following result.

THEOREM 3.2.6 ([Eng 1997], [Eng 2000], [Che 2002]). *Let $D \subset \mathbb{C}^n$ be a bounded pseudoconvex domain, $\varphi > 0$ a positive function on D, $-\log\varphi \in \mathcal{PSH}(D)$, such that $1/\varphi \in L^\infty_{\mathrm{loc}}(D)$ fails to have a sesqui-holomorphic extension near a point $z^0 \in D$ (i.e. there is no function $f: V\times V \to \mathbb{C}$, $V \subset D$ a neighborhood of z^0, such that f is holomorphic in the first coordinates and antiholomorphic in the latter, $f(z,z) = 1/\varphi(z)$ for all $z \in V$). Let $U = U(z^0) \subset D$ be a neighborhood. Then there is an $m_U \in \mathbb{N}$ such that the Bergman kernel function $K_{\Omega_m}((\cdot,0),(\cdot,0))$ of*

$$\Omega_m := \{(z,w) \in D\times\mathbb{C}^m : \|w\|^2 < \varphi(z)\}$$

has a zero in $U \times U$, $m \ge m_U$.

Proof. The proof is based on an extension theorem for L^2_{h}-functions (see [Ohs 2001]) and a description of the Bergman kernel with weights due to E. Ligocka (see [Lig 1989]).

Applying the first result we are led ([5]) to the following formula:

$$\lim_{k\to\infty} K_{\Phi^k}(z,z)^{1/k} = \frac{1}{\Phi(z)}, \qquad z \in D, \tag{3.2.7}$$

where K_{Φ^k} denotes the reproducing kernel function of the Hilbert space $L^2_{\mathrm{h}}(D,\Phi^k)$. (Observe that in the case $\Phi \equiv 1$ this is just the classical Bergman kernel function.)

([5]) We omit that proof.

Then there is an m_U such that K_{Φ^m} has zeros on $U \times U$, $m \geq m_U$. Otherwise, we may assume that U is simply connected and that all the functions K_{Φ^k} have no zeros on $U \times U$. Next we choose a sesqui-holomorphic branch of $K_{\Phi^k}^{1/k}$ on $U \times U$. Since the function $1/\Phi$ is locally bounded, using (3.2.7) we see that the sequence $(K_{\Phi^k}(z,z)^{1/k})$ is locally bounded on U. Therefore, applying $|K_{\Phi^k}(z,w)|^2 \leq K_{\Phi^k}(z,z)K_{\Phi^k}(w,w)$, $z, w \in U$, shows that (K_{Φ^k}) is locally bounded on $U \times U$. Hence, it converges locally uniformly to a sesqui-holomorphic function L on $U \times U$ with $L(z,z) = 1/\Phi(z)$, $z \in U$, a contradiction.

It remains to recall Ligocka's formula:

$$K_{\Omega_m}((z,t),(w,s)) = \sum_{j=0}^{\infty} \frac{(j+m)!}{j!\pi^m} K_{\Phi^{m+j}}(z,w)\langle t,s\rangle^j, \qquad (z,t),(w,s) \in \Omega_m.$$

Thus we have

$$K_{\Omega_m}((z,0),(w,0)) = \frac{m!}{\pi^m} K_{\Phi^m}(z,w), \qquad z, w \in D.$$

Therefore, in view of the above claim, it follows that there is an m_U such that for any $m \geq m_U$ the function $K_{\Omega_m}((\cdot,0),(\cdot,0))$ has zeros on $U \times U$. ∎

We should mention that the original formulation in [Che 2002] is much stronger than the one given here. Applying Theorem 3.2.6 for certain complex ellipsoids we obtain the following consequences.

COROLLARY 3.2.7 ([Che 2002]). (a) *For any $k \geq 1$, not an even integer, there exists an $m = m(k) \in \mathbb{N}$ such that*

$$\Omega := \Omega_k := \{(z,w) \in E \times \mathbb{C}^m : |z|^k + \|w\|^2 < 1\}$$

is not Lu Qi-Keng.

(b) *For any $k \in \mathbb{N}$ there exists a natural number $m = m(k)$ such that, if*

$$\Omega := \Omega_k := \{(z,w) \in E \times \mathbb{C}^m : |z|^{(2k+1)/2} + \|w\|^2 < 1\},$$
$$(z^0, w^0) := (0,0,\dots,-1) \in \partial\Omega,$$

then Ω is convex with a $\mathcal{C}^k$-boundary and there are sequences

$$((z_j', w_j'))_j, ((z_j'', w_j''))_j \subset \Omega, \qquad \lim_{j\to\infty}(z_j', w_j') = \lim_{j\to\infty}(z_j'', w_j'') = (z^0, w^0),$$

such that $K_\Omega((z_j', w_j'), (z_j'', w_j'')) = 0$, $j \in \mathbb{N}$. In particular, the set $\{(z,w) \in \Omega \times \Omega : K_\Omega(z,w) = 0\}$ accumulates at $((z^0,w^0),(z^0,w^0))$.

Proof. (a) Take $D = E$ and $\varphi(z) := 1 - |z|^k$, $z \in E$. Then $-\log\varphi \in \mathcal{SH}(E)$ and $1/\varphi$ is not real-analytic at 0. So it cannot be extended to a sesqui-holomorphic function near 0. Hence, by Theorem 3.2.6, there is a neighborhood $U = U(0)$ and an $m = m(k) \in \mathbb{N}$ such that $K_{\Omega_k}((\,\cdot\,,0),(\,\cdot\,,0))$ has at least one zero in $U \times U$.

(b) Fix a k. In view of part (a), there is an $m = m(k)$ such that K_Ω has a zero at a point $((z',w'),(z'',w'')) \in \Omega \times \Omega$. Put

$$D := \Big\{\zeta \in \mathbb{C}^{m+1} : |\zeta_1|^{(2k+1)/2} + \sum_{j=2}^{m} |\zeta_j|^2 + \operatorname{Re}\zeta_{m+1} < 0\Big\}.$$

Observe that

$$\Phi(\zeta) := \left(\frac{4^{2/(2k+1)}}{(\zeta_{m+1}-1)^{4/(2k+1)}}, \frac{2\zeta_2}{\zeta_{m+1}-1}, \dots, \frac{2\zeta_m}{\zeta_{m+1}-1}, \frac{\zeta_{m+1}+1}{\zeta_{m+1}-1}\right)$$

defines a biholomorphic map from D to Ω. Moreover, for any positive ε,

$$F_\varepsilon(\zeta) := (\varepsilon^{2/(2k+1)}\zeta_1, \sqrt{\varepsilon}\,\zeta_2, \dots, \sqrt{\varepsilon}\,\zeta_m, \varepsilon\zeta_{m+1})$$

is a biholomorphic mapping from D to D. Therefore,

$$K_\Omega(\Phi \circ F_\varepsilon \circ \Phi^{-1}(z', w'), \Phi \circ F_\varepsilon \circ \Phi^{-1}(z'', w'')) = 0, \qquad \varepsilon > 0.$$

It remains to mention that $\lim_{\varepsilon\to 0} \Phi \circ F_\varepsilon \circ \Phi^{-1}(z', w') = \lim_{\varepsilon\to 0} \Phi \circ F_\varepsilon \circ \Phi^{-1}(z'', w'') = (z^0, w^0)$. ∎

$\boxed{?}$ It would be interesting to find in the situation of Corollary 3.2.7 concrete numbers $m = m(k)$. $\boxed{?}$

So far, we saw that some of the domains $\mathbb{E}_p \subset \mathbb{C}^n$ are not Lu Qi-Keng, some of them are. $\boxed{?}$ Describe all the vectors $p = (p_1, \dots, p_n)$ for which the Bergman kernel function of $\mathbb{E}_p$ is zero-free. $\boxed{?}$

REMARK 3.2.8. Recall that in the situation of Corollary 3.2.7(b) we have

$$\lim_{z\to z^0} k_\Omega(z) = \lim_{z\to z^0} K_\Omega(z, z) = \infty$$

(apply Theorem 6.1.17 of [J-P 1993]). Therefore, Corollary 3.2.7(b) shows that K_Ω does not continuously extend as a map $\overline{\Omega} \times \overline{\Omega} \to \overline{\mathbb{C}}$. $\boxed{?}$ It is unknown whether this negative phenomenon also occurs for $\mathcal{C}^\infty$-smooth convex domains. $\boxed{?}$

In addition to Remark 3.2.8 we recall that the Bergman kernel function K_D, $D \subset \mathbb{C}^n$ a smooth bounded strictly pseudoconvex domain, can be smoothly extended to $\overline{D} \times \overline{D} \setminus \nabla(\partial D)$, where $\nabla(\partial D) := \{(z, z) : z \in \partial D\}$ (see [Ker 1972]). This result was generalized by Bell and Boas (see [Bel 1986], [Boa 1987]) to the following statements:

(a) Let $D \subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain. Let $\Gamma_1, \Gamma_2 \subset \partial D$ be two open disjoint subsets of the boundary consisting of points of finite type (in the sense of D'Angelo). Then K_D extends smoothly to $(D \cup \Gamma_1) \times (D \cup \Gamma_2)$.

(b) Let D be as in (a) and assume that D satisfies condition (R) ([6]). If Γ_1, Γ_2 are disjoint open subsets of ∂D and Γ_1 consists of points of finite type, then K_D extends smoothly to $(D \cup \Gamma_1) \times (D \cup \Gamma_2)$.

There was the question whether a similar extension phenomenon might be possible for any smoothly bounded pseudoconvex domain. That this is not true was shown by So-Chin Chen [Chen 1996].

THEOREM 3.2.9. *Let $D \subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain*, $n \geq 2$. *Suppose that its boundary contains a non-trivial complex variety V. Then K_D cannot be continuously extended to $\overline{D} \times \overline{D} \setminus \nabla(\partial D)$.*

Proof. Take a regular point $z^0 \in V$ and denote by $\mathfrak{n}$ the outward unit normal at z^0. Then the smoothness assumption gives an $\varepsilon_0 > 0$ such that

$$w - \varepsilon\mathfrak{n} \in D, \qquad \varepsilon \in (0, \varepsilon_0),\ w \in \partial D \cap \mathbb{B}(z^0, \varepsilon_0).$$

([6]) A bounded domain is said to satisfy *condition* (R) if the Bergman projection $L^2(D) \to L^2_{\mathrm{h}}(D)$ sends $\mathcal{C}^\infty(\overline{D}) \cap L^2(D)$ to $\mathcal{C}^\infty(\overline{D}) \cap \mathcal{O}(D)$.

Moreover, we choose a holomorphic disc in V, i.e. a holomorphic embedding $\varphi : E \to V$, with $\varphi(0) = z^0$ and $\varphi(E) \subset V \cap \mathbb{B}(z^0, \varepsilon_0)$.

Now assume that $K_D \in \mathcal{C}(\overline{D} \times \overline{D} \setminus \nabla(\partial D))$. Then

$$\sup_{|\lambda|=1/2} |K_D(z^0, \varphi(\lambda))| < \infty.$$

Applying Theorem 6.1.17 of [J-P 1993] and the maximum principle leads to

$$\begin{aligned}\sup_{|\lambda|=1/2} |K_D(z^0, \varphi(\lambda))| &= \lim_{\varepsilon\to 0} \sup_{|\lambda|=1/2} |K_D(z^0 - \varepsilon\mathfrak{n}, \varphi(\lambda) - \varepsilon\mathfrak{n})| \\ &\geq \lim_{\varepsilon\to 0} K_D(z^0 - \varepsilon\mathfrak{n}, z^0 - \varepsilon\mathfrak{n}) = \infty,\end{aligned}$$

a contradiction. ∎

EXAMPLE 3.2.10 ([Chen 1996]). Fix a smooth real-valued function $r : \mathbb{R} \to \mathbb{R}$ with the following properties:

(i) $r(t) = 0$ if $t \leq 0$,
(ii) $r(t) > 1$ if $t > 1$,
(iii) $r''(t) \geq 100r'(t)$ for all t,
(iv) $r''(t) > 0$ if $t > 0$,
(v) $r'(t) > 100$, if $r(t) > 1/2$.

For $s > 1$ put

$$\Omega := \Omega_s := \{z \in \mathbb{C}^2 : \varrho(z) < 0\}, \quad \text{where} \quad \varrho(z) := \varrho_s(z) := |z_1|^2 - 1 + r(|z_2|^2 - s^2).$$

Then Ω_s is a smoothly bounded pseudoconvex domain in $\mathbb{C}^2$, it is convex and satisfies condition (R), and it is strictly pseudoconvex everywhere except on the set

$$\{z \in \mathbb{C}^2 : |z_1| = 1, \, 0 \leq |z_2| \leq s\} \subset \partial\Omega.$$

Obviously, this set contains non-trivial analytic varieties. So Ω is an example of a domain treated in Theorem 3.2.9.

3.3. Bergman exhaustiveness

In the study of the Bergman kernel it is important to know its boundary behavior. We define

DEFINITION 3.3.1. Let $D \subset \mathbb{C}^n$ be a domain and $z^0 \in \partial D$. We say that D is *Bergman exhaustive at* z^0 (for short, *b-exhaustive at* z^0) if $\lim_{D\ni z\to z^0} k_D(z) = \infty$. Moreover, if D is b-exhaustive at any of its boundary points, then D is called *b-exhaustive.*

Obviously, any b-exhaustive domain is pseudoconvex. There are a lot of general results giving sufficient conditions for a pseudoconvex domain to be b-exhaustive at a boundary point. Besides Theorem 6.1.17 in [J-P 1993] the most general is the following one that relates b-exhaustiveness to the boundary behavior of certain level sets of the Green function. For an arbitrary domain $D \subset \mathbb{C}^n$ and a point $z \in D$ we define

$$A_z := A_z(D) := \{w \in D : \log g_D(z, w) \leq -1\}.$$

Then:

THEOREM 3.3.2. *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ and $z_0 \in \partial D$. Assume that*

$$\lim_{z \to z_0} \Lambda_{2n}(A_z(D)) = 0.$$

Then D is b-exhaustive at z_0.

Theorem 3.3.2 is a simple consequence of the following result ([Che 1999], [Her 1999]).

THEOREM 3.3.3. *For any $n \in \mathbb{N}$ there exists a positive number C_n such that for every bounded pseudoconvex domain $D \subset \mathbb{C}^n$,*

$$\frac{|f(z)|^2}{k_D(z)} \le C_n \int_{A_z} |f(w)|^2 \, d\Lambda_{2n}(w), \qquad f \in L^2_{\mathrm{h}}(D),\ z \in D.$$

Proof. Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$, $z_0 \in D$, and fix an $f \in L^2_{\mathrm{h}}(D)$, $f \not\equiv 0$. Put

$$D_t := \{z \in D : \operatorname{dist}(z, \partial D) > t\}, \qquad 0 < t < 1 \text{ sufficiently small.}$$

Moreover, let $\psi_1 \in \mathcal{C}^\infty(\mathbb{C}^n, \mathbb{R})$ be a non-negative polyradial symmetric function with $\int_{\mathbb{C}^n} \psi(z) \, d\Lambda_{2n}(z) = 1$ and $\operatorname{supp} \psi_1 \subset \mathbb{B}_n(0,1)$; put $\psi_t(z) := t^{-2n}\psi_1(z/t)$, $z \in \mathbb{C}^n$, $t > 0$. On D_t we define

$$\varphi_t(z) := 2nV_t(z) + \exp(V_t(z)) + t\|z\|^2, \qquad \varphi(z) := 2n \log g_D(z_0, \cdot) + g_D(z_0, \cdot),$$

where $V_t := \log g_D(z_0, \cdot) * \psi_t$. Finally, we choose a $\chi \in \mathcal{C}^\infty(\mathbb{R}, [0,1])$ with $\chi(t) = 1$ if $t \le -2$, $\chi(t) = 0$ if $t \ge -1$, and $|\chi'| \le 2$.

We define the following $\bar\partial$-closed $(0,1)$-form α_t on D_t:

$$\alpha_t := \bar\partial(\chi \circ V_t \cdot f) = \chi'(V_t) f \bar\partial V_t.$$

Observe that α_t is a smooth form whose support is contained in the set $\{-2 \le V_t \le -1\}$. Moreover, $\varphi_t \ge -4n$ on $\operatorname{supp} \alpha_t$. Therefore, $\int_{D_t} |\alpha_t|^2 e^{-\varphi_t} \, d\Lambda_{2n} < \infty$.

For the Levi form of φ_t we have the following estimate:

$$\mathcal{L}\varphi_t(z; X) \ge e^{V_t(z)} |V_t'(z)X|^2 \ge e^{-2}|V_t'(z)X|^2, \qquad z \in \operatorname{supp} \alpha_t,\ X \in \mathbb{C}^n.$$

Let Q denote the inverse matrix of the coefficient matrix of $\mathcal{L}$. Then, if $z \in \operatorname{supp} \alpha_t$, we have

$$\sum_{j,k=1}^n Q_{j,k}(z) \alpha_{tj}(z) \overline{\alpha_{tk}(z)} \exp(-\varphi_t(z)) \le e^2 |\chi'(V_t(z))|^2 |f(z)|^2 e^{-\varphi_t(z)} \le 4e^{4n+2}|f(z)|^2.$$

Therefore, by Lemma 4.4.1 in [Hör 1979], there exists a solution $u_t \in \mathcal{C}^\infty(D_t)$ of the equation $\bar\partial u_t = \alpha_t$ with the estimate

$$\int_{D_t} |u_t|^2 e^{-\varphi_t} \, d\Lambda_{2n} \le 4e^{4n+2} \int_{\operatorname{supp} \alpha_t} |f|^2 \, d\Lambda_{2n}.$$

Put

$$v_t := \begin{cases} u_t e^{-\varphi_t/2} & \text{on } D_t, \\ 0 & \text{on } D \setminus D_t. \end{cases}$$

Then the family $(v_t)_t$ belongs to $L^2(D)$ and satisfies the following uniform estimate:

$$\int_D |v_t|^2 \, d\Lambda_{2n} \le 4e^{4n+2} \int_{A_{z_0}} |f|^2 \, d\Lambda_{2n}$$

(observe that $\operatorname{supp} \alpha_t \subset \{-2 \le V_t \le -1\} \subset A_{z_0}$).

By the Alaoglu–Bourbaki theorem, we may find a function $v \in L^2(D)$ satisfying

$$\int_D |v|^2 \, d\Lambda_{2n} \le 4e^{4n+2} \int_{A_{z_0}} |f|^2 \, d\Lambda_{2n}.$$

Put $u := ve^{\varphi/2}$. Then

$$\int_D |u|^2 \, d\Lambda_{2n} \le e \int |v|^2 \, d\Lambda_{2n} \le 4e^{4n+3} \int_{A_{z_0}} |f|^2 \, d\Lambda_{2n}. \tag{3.3.8}$$

Using distributional derivatives, we find an $\widetilde{f} \in \mathcal{O}(D)$ such that

$$\widetilde{f} = \chi \circ \log g_D(z_0, \cdot) - u$$

almost everywhere on D. Moreover, take a neighborhood $U \subset D$ of z_0 such that $\log g_D(z_0, \cdot) \cdot f \le -3$ on U. Then $f - \widetilde{f} = u$ almost everywhere on U. By (3.3.8) it follows that

$$\int_U |f - \widetilde{f}|^2 e^{-\varphi} \, d\Lambda_{2n} < \infty.$$

Observe that e^{φ} is not locally integrable near z_0; hence $\widetilde{f}(z_0) = f(z_0)$.

Summarizing, we have found an $\widetilde{f} \in L^2_{\mathrm{h}}(D)$ with $f(z_0) = \widetilde{f}(z_0)$ and

$$\|\widetilde{f}\|_{L^2_{\mathrm{h}}(D)} \le (1 + 4e^{4n+3}) \int_{A_{z_0}} |f|^2 \, d\Lambda_{2n}.$$

Consequently,

$$\frac{|f(z_0)|^2}{k_D(z_0)} \le \|\widetilde{f}\|^2_{L^2_{\mathrm{h}}(D)} \le (1 + 4e^{4n+3}) \int_{A_{z_0}} |f|^2 \, d\Lambda_{2n},$$

which finishes the proof. ∎

Proof of Theorem 3.3.2. By Theorem 3.3.3 we know that there is a constant $C_n > 0$ such that

$$\frac{1}{k_D(z)} \le C_n \int_{A_z} d\Lambda_{2n}(w) \le C_n \Lambda_{2n}(A_z(D)) \to 0$$

Therefore, $k_D(z) \to \infty$ as $z \to z_0$. ∎

Moreover, combining Theorem 3.3.2 and a result due to Błocki we have the following (see also [Ohs 1993]):

THEOREM 3.3.4. *For a bounded hyperconvex domain $D \subset \mathbb{C}^n$ (i.e. there is a negative $u \in \mathcal{PSH}(D)$ such that the sublevel sets $\{z \in D : u(z) < -\varepsilon\}$, $\varepsilon > 0$, are relatively compact in D), $\Lambda_{2n}(A_z(D)) \to 0$ as $z \to \partial D$. In particular, any hyperconvex domain is b-exhaustive.*

Proof. According to [Bło 1996], there is a function $u \in \mathcal{C}(\overline{D}) \cap \mathcal{PSH}(D)$ with

$$u|_{\partial D} = 0 \quad \text{and} \quad (dd^c u)^n \ge \Lambda_{2n}.$$

Applying results from [Bło 1993], for a point $z_0 \in \partial D$ we get

$$\int_D (-\log g_D(z,w))^n \, d\Lambda_{2n}(w) \le \lim_{k\to\infty} \int_D (-\max\{\log g_D(z,\cdot), -k\})^n (dd^c u)^n$$
$$\le n!\|u\|^{n-1}_{L^\infty(D)} |u(z)| \xrightarrow[z\to z_0]{} 0,$$

where the last inequality is due to Demailly (see [Dem 1987]).

Finally, in view of Theorem 3.3.3, we get

$$\frac{1}{k_D(z)} \le C_n \int_{A_z(D)} d\Lambda_{2n}(w) \le C_n \int_D (-\log g_D(z,w))^n \, d\Lambda_{2n}(w) \xrightarrow[z\to z_0]{} 0.$$

Since z_0 is arbitrary, it follows that $k_D(z) \to \infty$ as $z \to \partial D$, i.e. D is b-exhaustive. ■

EXAMPLE 3.3.5. (1) There is a large class of bounded pseudoconvex domains which are hyperconvex, namely:

THEOREM 3.3.6 ([Ker-Ros 1981], [Dem 1987]). *Any bounded pseudoconvex domain $D \subset \mathbb{C}^n$ with Lipschitz boundary is hyperconvex. In particular, if D has $\mathcal{C}^1$-boundary, then it is hyperconvex.*

(2) Hyperconvexity is even a local property:

THEOREM 3.3.7 ([Ker-Ros 1981]). *Suppose that D is a bounded domain in $\mathbb{C}^n$ such that every $z_0 \in \partial D$ has a neighborhood $U = U(z_0)$ for which $D \cap U$ is hyperconvex. Then D itself is hyperconvex.*

(3) Put $D := \{z \in \mathbb{C}^2 : |z_1| < |z_2| < 1\}$. Then D is b-exhaustive but not hyperconvex. (For other examples of this type see also Theorems 3.3.8 and 3.3.9 and Example 3.3.23.) For D even more is true: there is a sequence $(z_k)_k \subset D$ tending to 0 such that $\Lambda_{2n}(A_{z_k}(D))_k \not\to 0$.

For Reinhardt domains in $\mathbb{C}^2$ we have (see [Zwo 2001a]) the following general result for the pole boundary behavior of the Green function:

THEOREM 3.3.8. *Let $D \subset \mathbb{C}^2$ be a bounded pseudoconvex Reinhardt domain such that $D \cap (\mathbb{C}_* \times \{0\}) = E_* \times \{0\}$. Moreover, suppose that for a $z^0 \in D$,*

$$\{v \in \mathbb{R}^2 : (\log|z_1^0|, \log|z_2^0|) + \mathbb{R}_+ v \subset \log D\} = \mathbb{R}_+(0,-1).$$

Then

$$g_D(z,w) \xrightarrow[D\ni z\to 0]{} 0, \quad w \in D \cap \mathbb{C}_*, \quad \textit{and therefore}, \quad \Lambda_{2n}(A_z(D)) \xrightarrow[D\ni z\to 0]{} 0.$$

In particular, D is b-exhaustive at the origin but not hyperconvex.

Proof. We may assume that $D = \{z \in E^2 : |z_2| < \varrho(|z_1|)\}$, where $\varrho : [0,1) \to [0,1]$, $\varrho(r) = 0$ iff $r = 0$ and

$$\forall_{A>0}\ \exists_{B\in\mathbb{R}} : \log\varrho(e^t) \le At + B, \quad t \in -\mathbb{R}_+.$$

Take a $z \in D$ close to 0. Then for $w \in D$ with $|w_1| = 2|z_1|$ we have

$$g_D(z,w) \ge g_E(z_1,w_1) \ge \frac{|z_1 - w_1|}{|1 - z_1\overline{w}_1|} \ge \frac{|z_1|}{2}.$$

Now we claim that

$$\log g_D(z,w) \ge \log\frac{|z_1|}{2}\,\frac{\log|w_2|}{\log\varrho(2|z_1|)}, \quad w \in D,\ |w_1| \ge 2|z_1|. \tag{3.3.9}$$

If $|w_1| = 2|z_1|$ then (3.3.9) is true since the second factor is larger than or equal to 1. Moreover, by Remark 2.5.6, $c_D(z,w) \to \infty$ whenever $w \to w^*$, $w^* \in \partial D \cap \mathbb{C}^2_*$. In particular, $g_D(z,w) \to 1$ as $w \to w^*$. Therefore, the maximality of the Green function implies inequality (3.3.9).

It remains to show that the right side of (3.3.9) tends to 0 if z tends to 0. In fact, the right side can be written with $t = \log|z_1| < 0$ and $v(t) := \log \varrho(e^t)$ as

$$\frac{t-\log 2}{v(t+\log 2)} \log|w_2| = \frac{t-\log 2}{t+\log 2} \log|w_2| \, \frac{t+\log 2}{v(t+\log 2)} =: f(t).$$

According to our assumption, for any $A > 0$ we have $\liminf_{t\to-\infty} v(t)/t \geq A$. In particular, since A is arbitrary, $\lim_{t\to-\infty} v(t)/t = \infty$. Therefore, $\lim_{t\to-\infty} f(t) = 0$, which finishes the proof. ∎

A Reinhardt domain satisfying the conditions of Theorem 3.3.8 is given, for example, by $D := \{z \in E_* \times E : |z_2| < e^{-1/|z_1|}\}$. Hence, D is b-exhaustive but not hyperconvex.

For circular domains we have the following result:

THEOREM 3.3.9 ([Jar-Pfl-Zwo 2000]). *Any bounded pseudoconvex balanced domain is b-exhaustive.*

Proof. Let $D = D_h = \{z \in \mathbb{C}^n : h(z) < 1\}$ be a bounded pseudoconvex balanced domain. Fix a boundary point z_0 and let M be an arbitrary positive number. Put $H := \mathbb{C}z_0$. Then, by the theorem of Ohsawa (see Theorem 3.1.1), we have $k_{D\cap H}(z) \leq C k_D(z)$, $z \in D \cap H$, where C is a suitable positive number. Since $D \cap H$ is a plane disc, there is an $s \in (0,1)$ such that $M < k_{D\cap H}(sz_0)$. Using the continuity of k_D leads to an open neighborhood $U = U(z_0) \subset D \setminus \{0\}$ such that $k_D(z) > M$, $z \in U$.

Now fix a $z \in U$ and define $u_z : (1/h(z))E \to \mathbb{R}$, $u_z(\lambda) := k_D(\lambda z)$. This function is subharmonic and radial, so $u|_{[0,1/h(z))}$ is increasing. Therefore, $M < u_z(1) \leq u_z(\lambda) = k_D(\lambda z)$, $1 \leq |\lambda| < 1/h(z)$. Obviously,

$$V = V_{z_0,M} := \{\lambda z : z \in U,\ \lambda \in \mathbb{C},\ |\lambda| > 1\}$$

is an open neighborhood of z_0. Since M is arbitrary, we have

$$\liminf_{D\ni z\to z_0} k_D(z) = \infty,$$

proving the theorem. ∎

In the case of a bounded pseudoconvex balanced domain with a continuous Minkowski function, Theorem 3.3.9 was proved in [Jar-Pfl 1989] (see Theorem 7.6.7 in [J-P 1993]).

Observe that any bounded hyperconvex balanced domain is taut and therefore its Minkowski function h is continuous. Obviously, there are a lot of bounded balanced pseudoconvex domains with a non-continuous Minkowski function. Moreover, we mention that there exists a bounded pseudoconvex balanced domain D which is not fat (i.e. $\overline{\operatorname{int} \overline{D}} \neq D$); see Example 3.1.12 in [J-P 1993].

$\boxed{?}$ Describe all bounded pseudoconvex circular domains D (i.e. $\forall_{z\in D,\, \theta\in\mathbb{R}} : e^{i\theta}z \in D$) which are b-exhaustive. $\boxed{?}$

EXAMPLE 3.3.10. Let $D \subset \mathbb{C}^n$ be a bounded domain, and $H : D \times \mathbb{C}^m \to [0,\infty)$ such that $\log H \in \mathcal{PSH}(D \times \mathbb{C}^m)$, $H(z,\lambda w) = |\lambda| H(z,w)$, $(z,w) \in D \times \mathbb{C}^m$ and $\lambda \in \mathbb{C}$. Put

$$G_D := \{(z,w) \in D \times \mathbb{C}^m : H(z,w) < 1\}.$$

Notice that G_D is a Hartogs domain with m-dimensional fibers. Assume that G_D is bounded and pseudoconvex. Then we have the following result:

THEOREM 3.3.11. *Let G_D be bounded pseudoconvex as above and let $(z_0, w_0) \in \partial G_D$. Assume that one of the following conditions is satisfied:*

(a) $z_0 \in D$,

(b) $z_0 \in \partial D$ *and* $\lim_{D \ni z \to z_0} k_D(z) = \infty$,

(c) *there is a neighborhood* $U = U((z_0, w_0))$ *such that*

$$U \cap G_D \subset \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^m : \|w\| < \|z - z_0\|^\delta\}$$

for some $\delta > 0$.

Then $\lim_{G_D \ni (z,w) \to (z_0, w_0)} k_{G_D}((z, w)) = \infty$. *In particular, if D is b-exhaustive, then so is G_D.*

For a proof see [Jar-Pfl-Zwo 2000].

EXAMPLE 3.3.12. The following example shows that Theorem 3.3.11 is far from being optimal. Fix sequences $(a_j)_{j \in \mathbb{N}} \subset (0, 1)$ and $(n_j)_{j \in \mathbb{N}} \subset \mathbb{N}$ with $\lim_{j \to \infty} a_j = 0$ and $n_j \geq j$. Put

$$E_k := E \setminus \{a_j : j = 1, \dots, k\}, \qquad u_k(\lambda) := \sum_{j=1}^{k} \left(\frac{a_j}{2|\lambda - a_j|} \right)^{n_j}.$$

Observe that $u_k(0) < 0$. Define $E_\infty := E \setminus (\{0\} \cup \{a_j : j \in \mathbb{N}\})$. Then the sequence $(u_k)_k$ is locally bounded from above on E_∞ and globally bounded from below; moreover, it is an increasing sequence of subharmonic functions. It turns out that $u := \lim_{k \to \infty} u_k \in \mathcal{SH}(E_\infty)$ and $\lim_{(-1,0) \ni x \to 0} u(x) \leq 0$. Finally, we define the following bounded pseudoconvex Hartogs domain with one-dimensional fibers:

$$G_{E_\infty} := \{(z, w) \in E_\infty \times \mathbb{C} : |w| < e^{-u(z)}\}.$$

Obviously, the point $(0, 0) \in \partial G_{E_\infty}$ does not satisfy any of the conditions in Theorem 3.3.11. Nevertheless, a correct choice of the n_j's may show that G_{E_∞} satisfies the cone condition of the Theorem in [J-P 1993] at $(0, 0)$. Therefore,

$$k_{G_{E_\infty}}((z, w)) \to \infty \qquad \text{as } G_{E_\infty} \ni (z, w) \to (0, 0).$$

The discussion of the other boundary points with the help of Theorem 3.3.11 and Theorem 6.1.17 of [J-P 1993] even proves that G_{E_∞} is b-exhaustive. $\boxed{?}$ Try to give a complete description of those bounded pseudoconvex Hartogs domains with m-dimensional fibers that are b-exhaustive. $\boxed{?}$

In the complex plane there is even a full characterization of bounded domains which are b-exhaustive in terms of potential theory (see [Zwo 2002]). To be able to present this result we recall a few facts from the classical plane potential theory.

3.3.1. A short course in plane potential theory. (See [Ran 1995].) Let $K \subset \mathbb{C}$ be compact and $\mathcal{P}(K) := \{\mu : \mu \text{ a probability measure of } K\}$. For $\mu \in \mathcal{P}(K)$,

$$p_\mu(\lambda) := \int_K \log|\lambda - \zeta| \, d\mu(\zeta), \qquad \lambda \in \mathbb{C},$$

is the *logarithmic potential* of μ. Recall that $p_\mu \in \mathcal{SH}(\mathbb{C})$ and that $p_\mu|_{\mathbb{C}\setminus K}$ is a harmonic function. To any such a μ one associates its *energy*

$$I(\mu) := \int_K p_\mu(\lambda)\,d\mu(\lambda) = \int_K\int_K \log|\lambda-\zeta|\,d\mu(\lambda)\,d\mu(\zeta).$$

A probability Borel measure $\nu \in \mathcal{P}(K)$ is called the *equilibrium measure* of K if $I(\nu) = \sup_{\mu\in\mathcal{P}(K)} I(\mu)$. It is known that the equilibrium measure exists and is unique if K is not a polar set; then we write ν_K. Moreover, the *logarithmic capacity* of any set $M \subset \mathbb{C}$ is given by

$$\operatorname{cap}(M) := \exp(\sup\{I(\mu) : K \subset M \text{ compact}, \ \mu \in \mathcal{P}(K)\}).$$

If $M = K$ is compact and not polar then $\operatorname{cap}(K) = e^{I(\nu_K)}$. Moreover, if M is any Borel set then: M is polar iff $\operatorname{cap}(M) = 0$.

For further applications we collect a few well known properties of the logarithmic capacity:

(1) If $M_1 \subset M_2$ then $\operatorname{cap}(M_1) \le \operatorname{cap}(M_2)$.
(2) If $M_1 \subset M_2 \subset M_3 \subset \cdots$ are Borel sets then $\operatorname{cap}(\bigcup_{j=1}^\infty M_j) = \lim_{j\to\infty}\operatorname{cap}(M_j)$.
(3) If $K_1 \supset K_2 \supset K_3 \supset \cdots$ are compact sets, then $\operatorname{cap} K_k \to \operatorname{cap}(\bigcap_{k=1}^\infty K_k)$ as $k \to \infty$;
(4) If $M = \bigcup_{j=1}^N M_j$, M_j Borel sets with $\operatorname{diam} M \le d$, $N \in \mathbb{N}\cup\{\infty\}$, then
$$\frac{1}{\log d - \log\operatorname{cap} M} \le \sum_{j=1}^N \frac{1}{\log d - \log\operatorname{cap} M_j};$$
(4′) If $M = \bigcup_{j=1}^N M_j$, M_j Borel sets with $\operatorname{dist}(M_j, M_k) \ge d > 0$, $k \ne j$, $N \in \mathbb{N}\cup\{\infty\}$, then
$$\frac{1}{\log^+(d/\operatorname{cap} M)} \ge \sum_{j=1}^N \frac{1}{\log^+(d/\operatorname{cap} M_j)}.$$
(5) THEOREM OF FROSTMAN. *Let $K \subset \mathbb{C}$ be a non-polar compact subset and ν_K its equilibrium measure. Then $p_{\nu_K} \ge \log\operatorname{cap} K$ on $\mathbb{C}$ and $p_{\nu_K} = \log\operatorname{cap} K$ on $K\setminus F$, $F \subset \partial K$ a suitable polar F_σ-set. Moreover, $p_{\nu_K}(z) = \log\operatorname{cap} K$ for $z \in \partial K$, whenever z is regular for the Dirichlet problem for the unbounded component of $\mathbb{C}\setminus K$.*
(6) $\operatorname{cap}\mathbb{B}(z,r) = \operatorname{cap}(\partial\mathbb{B}(z,r)) = r$ and $\operatorname{cap} K = \operatorname{cap}(\partial K) \le \operatorname{diam} K$ for any compact set $K \subset \mathbb{C}$.

For a compact set in the complex plane we introduce its Cauchy transform.

DEFINITION 3.3.13. Let $K \subset \mathbb{C}$ be compact. The function $f_K : \mathbb{C}\setminus K \to \mathbb{C}$ defined by

$$f_K(z) := \begin{cases} \displaystyle\int_K \frac{d\nu_K(\zeta)}{z-\zeta} & \text{if } K \text{ is not polar,} \\ 0 & \text{if } K \text{ is polar,} \end{cases}$$

is called the *Cauchy transform* of K. (Recall that ν_K is the equilibrium measure of K.)

Obviously, $f_K \in \mathcal{O}(\mathbb{C}\setminus K)$ and $f_K|_D \in L^2_{\mathrm{h}}(D)$ for any bounded domain $D \subset \mathbb{C}\setminus K$.

LEMMA 3.3.14 ([Zwo 2002]). *For a $\varrho \in (0,1/2)$ there exist positive numbers C_1, C_2 such that for any pair of disjoint compact sets $K, L \subset \varrho E$ and any domain $D \subset \varrho E\setminus(K\cup L)$*

the following inequalities hold:

$$|\langle f_K, f_L\rangle_{L^2_{\mathrm{h}}(D)}| \le C_2 - C_1 \log \operatorname{dist}(K, L), \tag{3.3.10}$$

$$\|f_K\|^2_{L^2_{\mathrm{h}}(D)} \le C_2 - C_1 \log(\operatorname{cap} K). \tag{3.3.11}$$

Proof. Obviously, both inequalities are true for any constants C_j when K or L is a polar set. So we may assume that neither set is polar.

Applying the Fubini theorem, we get the inequality

$$\begin{aligned} |\langle f_K, f_L\rangle_{L^2_{\mathrm{h}}(D)}| &= \left| \int_D \int_K \frac{d\nu_K(\zeta)}{z-\zeta} \int_L \frac{d\nu_L(\eta)}{\overline{z}-\overline{\eta}}\, d\Lambda_2(z)\right| \\ &\le \int_K \int_L \int_{\varrho E} \frac{1}{|z-\zeta|\,|z-\eta|}\, d\Lambda_2(z)\, d\nu_L(\eta)\, d\nu_K(\zeta). \end{aligned}$$

Now we discuss the inner integral.

Take $\zeta, \eta \in \varrho E$, $\zeta \neq \eta$. Then

$$\begin{aligned} \int_{\varrho E} \frac{d\Lambda_2(z)}{|z-\zeta|\,|z-\eta|} &\le \int_E \frac{d\Lambda_2(z)}{|z|\,|z-(\zeta-\eta)|} = \int_{(1/|\zeta-\eta|)E} \frac{d\Lambda_2(z)}{|z|\,|z-1|} \\ &= \int_{(1/2\varrho)E} \frac{d\Lambda_2(z)}{|z|\,|z-1|} + \int_{(1/|\zeta-\eta|)E\setminus(1/2\varrho)E} \frac{d\Lambda_2(z)}{|z|\,|z-1|}. \end{aligned}$$

Observe that the first term on the right hand side is finite and independent of η and ζ. For the second summand we proceed as follows:

$$\begin{aligned} \int_{(1/|\zeta-\eta|)E\setminus(1/2\varrho)E} \frac{d\Lambda_2(z)}{|z|\,|z-1|} &= \int_{1/2\varrho}^{(1/|\zeta-\eta|)} \int_0^{2\pi} \frac{dr\, d\theta}{|1-re^{i\theta}|} \\ &= \int_{1/2\varrho}^{1/|\zeta-\eta|} \int_0^{2\pi} \left| 1 + \frac{e^{i\theta}}{r} + \frac{e^{2i\theta}}{r^2(1-e^{i\theta}/r)}\right| \frac{dr\, d\theta}{r} \\ &\le \int_{1/2\varrho}^{1/|\zeta-\eta|} \frac{C_1\, dr}{r} \le -C_1 \log|\zeta-\eta|, \end{aligned}$$

where C_1 is independent of ζ, η. Consequently,

$$\int_{\varrho E} \frac{d\Lambda_2(z)}{|z-\zeta|\,|z-\eta|} \le C_2 - C_1 \log|\zeta-\eta|, \qquad \zeta, \eta \in \varrho E,\ \zeta \neq \eta,$$

where C_1, C_2 are positive constants.

Coming back to the beginning, we obtain

$$|\langle f_K, f_L\rangle_{L^2_{\mathrm{h}}(D)}| \le C_2 - C_1 \int_K \int_L \log|\zeta-\eta|\, d\nu_K\, d\nu_L,$$

which ends the proof. ∎

The main notion here will be the following potential-theoretic function.

DEFINITION 3.3.15. Let $D \subset \mathbb{C}$ be a bounded domain. Define $\alpha_D : \overline{D} \to (-\infty, \infty]$ by

$$\alpha_D(z) := \int_0^{1/2} \frac{dr}{-r^3 \log(\operatorname{cap}(\mathbb{B}(z,r) \setminus D))} = \int_0^{1/2} \frac{dr}{-r^3 \log(\operatorname{cap}(\overline{\mathbb{B}}(z,r) \setminus D))}.$$

REMARK 3.3.16. We denote by $A_k(z)$ the annulus with center z and radii $1/2^{k+1}, 1/2^k$, i.e.

$$A_k(z) := \{w \in \mathbb{C} : 1/2^{k+1} \le |w-z| \le 1/2^k\}.$$

Then for a bounded domain $D \subset \mathbb{C}$ there is an alternative description of α_D:

$$\frac{1}{8} \sum_{k=2}^{\infty} \frac{2^{2k}}{-\log \operatorname{cap}(A_k(z) \setminus D)} \le \alpha_D(z) \le 8 \sum_{k=1}^{\infty} \frac{2^{2k}}{-\log \operatorname{cap}(A_k(z) \setminus D)}, \qquad z \in \overline{D}.$$

To get the lower estimate one only has to use the monotonicity of cap, whereas the upper estimate is based on property (4) of cap.

Moreover, α_D is semicontinuous from below on $\overline{D}$ and continuous on D; here use properties (4) and (6) of cap and Fatou's lemma, respectively the Lebesgue theorem.

REMARK 3.3.17. For a point $z_0 = x_0 + iy_0 \in \mathbb{C}$ we define the annuli with respect to the maximum norm, i.e. $\widetilde{A}_k(z_0) := \{z = x+iy \in \mathbb{C} : 1/2^{k+1} \le \max\{|x-x_0|, |y-y_0|\} \le 1/2^k\}$, where $k \in \mathbb{N}$. Moreover, let $\widetilde{\mathbb{B}}(a,r) := \{z = x+iy \in \mathbb{C} : \max\{|x - \operatorname{Re} a|, |y - \operatorname{Im} a|\} < r\}$, where $a \in \mathbb{C}$ and $r > 0$. Then we may define a notion similar to α_D, namely

$$\widetilde{\alpha}_D(z) := \int_0^{1/2} \frac{dr}{-r^3 \log \operatorname{cap}(\overline{\widetilde{\mathbb{B}}}(z,r) \setminus D)}, \qquad z \in D.$$

We only note that α_D and $\widetilde{\alpha}_D$ are comparable and that for the new functions inequalities like the ones in Remark 3.3.16 hold.

It turns out that, in general, the function α_D is not continuous on $\overline{D}$ (see the next Example 3.3.18).

EXAMPLE 3.3.18 ([Zwo 2002]). Now fix $n \in \mathbb{N}$ and put

$$M_n := \widetilde{A}_n(0) \cap \left\{ \frac{j}{2^n 2^{n^3}} + i\, \frac{k}{2^n 2^{n^3}} : |j|, |k| = 0, \dots, 2^{n^3} - 1 \right\}.$$

Then M_n has $l_n := (2^{1+n^3} - 1)^2 - (2^{n^3} - 1)^2$ elements. We denote them by $z_{n,k}$, $k = 1, \dots, l_n$. We define the following plane domain:

$$D := \widetilde{\mathbb{B}}(0, 1/4) \setminus \Big(\bigcup_{n=2}^{\infty} \bigcup_{k=1}^{l_n} \overline{\widetilde{\mathbb{B}}}(z_{n,k}, r_n) \cup \{0\} \Big).$$

Here the radii $r_n > 0$ are chosen such that $-\log(\operatorname{cap}(\overline{\widetilde{\mathbb{B}}}(0, r_n)) = n^2 2^{2n(1+n^2)}$, $n \ge 2$. Observe that the distance between two different $z_{n,k}$'s is equal to $d_n := 1/2^n 2^{n^3}$. Therefore,

$$\begin{aligned} d_n - 2r_n &\ge d_n - 2 \operatorname{cap} \overline{\widetilde{\mathbb{B}}}(0, r_n) \ge d_n - 2e^{-n^2 2^{2n+2n^3}} \\ &\ge d_n - 2\frac{1}{n^2 2^{2n+2n^3}} = \frac{1}{2^n 2^{n^3}} \left(1 - \frac{2}{n^2 2^{n+n^3}}\right) =: b_n \ge \frac{1}{2^{n+1} 2^{n^3}} > 0. \end{aligned}$$

Hence two different "balls" have a distance at least b_n.

In the next step we estimate $\widetilde{\alpha}_D(0)$:

$$\widetilde{\alpha}_D(0) \le C_1 \sum_{n=1}^{\infty} \frac{2^{2n}}{-\log \operatorname{cap}(\widetilde{A}_n(0) \setminus D)}$$
$$\le C_1 \sum_{n=1}^{\infty} \sum_{k=1}^{l_n} \frac{2^{2n}}{-\log \operatorname{cap}(\overline{\overline{\mathbb{B}}}(z_{n,k}, r_n))} = C_1 \sum_{n=1}^{\infty} \frac{2^{2n} 2^{2+2n^3}}{n^2 2^{2n} 2^{2n^3}} = 4C_1 \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Therefore, $\alpha_D(0) < \infty$.

To see that α_D is not continuous at 0 take an arbitrary point $z \in \widetilde{A}_n(0)$. Then

$$\widetilde{\alpha}_D(z) \ge C_2 \sum_{j=1}^{n^3-1} \frac{2^{2(n+j)}}{-\log \operatorname{cap}(\widetilde{A}_{n+j}(z) \setminus D)}$$
$$= C_2 \sum_{j=1}^{n^3-1} \frac{2^{2(n+j)}}{\log \dfrac{1}{2^{1+n+n^3} \operatorname{cap}(\widetilde{A}_{n+j}(z) \setminus D)} + \log 2^{1+n+n^3}}.$$

To continue with the estimate we note that there exists a $C_3 > 0$ such that

$$\#\{k = 1, \dots, l_n : \overline{\overline{\mathbb{B}}}(z_{n,k}, r_n) \subset \widetilde{A}_{n+j}(z)\} \ge C_3 2^{2(n^3-j)}, \qquad j = 1, \dots, n^3 - 1.$$

Moreover, applying properties (4) and (4′) of cap we obtain

$$\frac{1}{-\log \operatorname{cap}(\widetilde{A}_{n+j}(z) \setminus D)} \le \sum_{k=1}^{l_n} \frac{1}{-\log \operatorname{cap}(\overline{\overline{\mathbb{B}}}(z_{n,k}, r_n))} \le \frac{2^{2+2n^3}}{n^2 2^{2n+2n^3}} = \frac{4}{n^2 2^{2n}}$$

and

$$\frac{1}{\log^+ \dfrac{1}{2^{n+1} 2^{n^3} \operatorname{cap}(\widetilde{A}_{n+j} \setminus D)}} \ge \frac{C_3 2^{2(n^3-j)}}{\log^+ \dfrac{1}{2^{n+1} 2^{n^3} \operatorname{cap}(\overline{\overline{\mathbb{B}}}(0, r_n))}}, \qquad j = 1 \dots, n^3 - 1.$$

Now, observing that $2^{2(1+n+n^3)} \max\{\operatorname{cap}(\overline{\overline{\mathbb{B}}}(0, r_n)), \operatorname{cap}(\widetilde{A}_{n+j}(z) \setminus D)\} < 1$ if $n \ge n_0$ for a suitable $n_0 \in \mathbb{N}$, we deduce for $n \ge n_0$ that

$$\widetilde{\alpha}_D(z) \ge C_2 \sum_{j=1}^{n^3-1} \frac{2^{2(n+j)}}{2 \log \dfrac{1}{2^{1+n+n^3} \operatorname{cap}(\widetilde{A}_{n+j}(z) \setminus D)}}$$
$$\ge \frac{C_2 C_3}{2} \sum_{j=1}^{n^3-1} \frac{2^{2(n+j)} 2^{2(n^3-j)}}{n^2 2^{2n^2+2n^3}} = C_4 \frac{n^3 - 1}{n^2} \xrightarrow[n \to \infty]{} \infty.$$

Hence $\lim_{D \ni z \to 0} \alpha_D(z) = \lim_{D \ni z \to 0} \widetilde{\alpha}_D(z) = \infty$.

Finally, we formulate the main result.

THEOREM 3.3.19 ([Zwo 2002]). *Let $D \subset \mathbb{C}$ be a bounded domain, $z_0 \in \partial D$. Then the following properties are equivalent*:

(i) *D is b-exhaustive at z_0 (i.e. $\lim_{D \ni z \to z_0} k_D(z) = \infty$)*;
(ii) *$\lim_{D \ni z \to z_0} \alpha_D(z) = \infty$.*

Proof. For the whole proof we may assume that $D \subset \frac{1}{2}E$ and $z_0 = 0 \in \partial D$.

(ii)⇒(i). Assume that the statement in (i) is not true. Then there exists a sequence $(z_k)_{k\in\mathbb{N}} \subset D \cap \mathbb{B}(0,1/8)$ with $\lim_{k\to\infty} z_k = 0$ and $\sup_{k\in\mathbb{N}} k_D(z_k) =: M < \infty$. Put $K_n^k := A_n(z_k) \setminus D$, $n \geq 2$, $k \in \mathbb{N}$. Since $z_k \in D$ there is an $N_k \in \mathbb{N}$ such that $K_n^k = \emptyset$ for all $k > N_k$. Observe that necessarily $N_k \to \infty$ as $k \to \infty$. By assumption (see Remark 3.3.16) we know that

$$\frac{1}{8}\alpha_D(z_k) \leq S_k := \sum_{n=2}^{N_k} \frac{2^{2n}}{-\log \operatorname{cap} K_n^k} \xrightarrow[k\to\infty]{} \infty.$$

Put

$$K_{n,j}^k := K_n^k \cap \{z_k + re^{i\theta} : r > 0, -\pi/3 + (j-1)2\pi/3 \leq \theta \leq \pi/3 + (j-1)2\pi/3\}, \quad j = 1,2,3.$$

By property (4) of cap, we have

$$\frac{1}{-\log \operatorname{cap} K_n^k} \leq \sum_{j=1}^{3} \frac{1}{-\log \operatorname{cap} K_{n,j}^k}.$$

Choose $j(n,k)$ such that $\operatorname{cap} K_{n,j}^k \leq \operatorname{cap} K_{n,j(n,k)}^k$ and put $\widetilde{K}_n^k := K_{n,j(n,k)}^k$. Then

$$\frac{1}{3}S_k \leq \sum_{n=2}^{N_k} \frac{2^{2n}}{-\log \operatorname{cap} \widetilde{K}_n^k} \xrightarrow[k\to\infty]{} \infty.$$

Define

$$f_{n,k}(z) := \begin{cases} \displaystyle\int_{\widetilde{K}_n^k} \frac{d\nu_{n,k}(\zeta)}{z - e^{i\theta_{n,k}}\zeta} & \text{if } \operatorname{cap} K_n^k \neq 0, \\ 0 & \text{if } \operatorname{cap} K_n^k = 0, \end{cases} \qquad z \in \mathbb{C} \setminus \widetilde{K}_n^k,$$

where $\nu_{n,k} := \nu_{\widetilde{K}_n^k}$ and $\theta_{n,k}$ such that $\arg(z_k - e^{i\theta_{n,k}}\zeta) \in [-\pi/3, \pi/3]$ for all $\zeta \in \widetilde{K}_n^k$. Then

$$|f_{n,k}(z_k)| \geq \operatorname{Re}\left(\int_{\widetilde{K}_n^k} \frac{d\nu_{n,k}(\zeta)}{z_k - e^{i\theta_{n,k}}\zeta}\right) \geq \widetilde{C}_3 \int_{\widetilde{K}_n^k} \frac{d\nu_{n,k}(\zeta)}{|z_k - \zeta|} \geq C_3 2^n,$$

where $\widetilde{C}_3, C_3$ are positive constants.

There are two cases to be discussed. First, assume that there are a subsequence of (z_k), denoted again by (z_k), and a sequence $(n_k)_k \subset \mathbb{N}$ with $n_k \leq N_k$, $k \in \mathbb{N}$, such that

$$\lim_{k\to\infty} \frac{2^{2n_k}}{-\log \operatorname{cap} \widetilde{K}_{n_k}^k} = \infty. \tag{3.3.12}$$

Put $f_k := f_{n_k,k}$. Then, by Lemma 3.3.14, we have

$$\|f_k\|^2_{L_h^2(D)} \leq C_2 - C_1 \log \operatorname{cap} \widetilde{K}_{n_k}^k.$$

Therefore, taking (3.3.12) into account, it follows that $\lim_{k\to\infty} k_D(z_k) = \infty$, a contradiction.

Now suppose that there is a positive constant C_4 such that

$$\frac{2^{2n}}{-\log \operatorname{cap} \widetilde{K}_n^k} \leq C_4, \qquad k \in \mathbb{N},\ n = 2,3,\dots,N_k.$$

Put $c_{k,n} := \operatorname{cap} \widetilde{K}_n^k$, $f_{k,n} := f_{\widetilde{K}_n^k}$. We are going to choose complex numbers $a_{k,n}$ with $a_{k,n} f_{k,n}(z_k) \geq 0$ such that if

$$f_k := \sum_{n=2}^{N_k} a_{k,n} f_{k,n},$$

then $(|f_k(z_k)|)_k$ is unbounded whereas $(\|f_k\|_{L^2_{\mathrm{h}}(D)})_k$ remains bounded by a positive constant C. In that situation we have

$$M \geq k_D(z_k) \geq \frac{|f_k(z_k)|^2}{\|f_k\|^2_{L^2_{\mathrm{h}}(D)}} \geq \frac{1}{C}\,|f_k(z_k)|^2,$$

a contradiction.

First observe the following inequalities (see Lemma 3.3.14):

$$|2\operatorname{Re}\langle f_{k,m}, f_{k,n}\rangle_{L^2_{\mathrm{h}}(D)}| \leq \|f_{k,m}\|^2_{L^2_{\mathrm{h}}(D)} + \|f_{k,n}\|^2_{L^2_{\mathrm{h}}(D)} \leq 2C_2 - C_1 \log(c_{k,m} c_{k,n}),$$

when $|n-m| \leq 1$, and

$$\begin{aligned} |2\operatorname{Re}\langle f_{k,m}, f_{k,n}\rangle_{L^2_{\mathrm{h}}(D)}| &\leq 2C_2 + 2C_1 \max\left|\log\left(\frac{1}{2^{m+1}} - \frac{1}{2^{m+2}}\right)\right|, \left|\log\left(\frac{1}{2^{n+1}} - \frac{1}{2^{n+2}}\right)\right| \\ &\leq 2C_2 + C_5 mn, \end{aligned}$$

when $|n-m| \geq 2$. Put $a_{k,n} := 0$ if $\operatorname{cap} \widetilde{K}_{k,n} = 0$. Then

$$\begin{aligned} \|f_k\|^2_{L^2_{\mathrm{h}}(D)} &\leq C_6 \sum_{n=1}^{N_k} |a_{k,n}|^2(-\log c_{k,n}) + C_6 \sum_{n,m=2,\, |n-m|\geq 2}^{N_k} |a_{k,n}|\,|a_{k,m}| nm \\ &\leq C_7\Big(\sum_{n=2}^{N_k} |a_{k,n}|^2(-\log c_{k,n}) + \Big(\sum_{n=2}^{N_k} n|a_{k,n}|\Big)^2\Big). \end{aligned}$$

Let $|a_{k,n}| := -(2^n/\log c_{k,n}) b_{k,n}$, where the numbers $b_{k,n} \geq 0$ will be fixed later. Then

$$|f_k(z_k)| = \sum_{n=2}^{N_k} a_{k,n} f_{k,n}(z_k) \geq C_3 \sum_{n=2}^{N_k} \frac{2^n}{-\log c_{k,n}}\, b_{k,n} 2^n.$$

So we have to look for numbers $b_{k,n}$ such that $|f_k(z_k)| \to \infty$ as $k \to \infty$, but

$$\|f_k\|^2_{L^2_{\mathrm{h}}(D)} \leq C_7\Big(\sum_{n=2}^{N_k} \frac{2^{2n}}{-\log c_{k,n}}\,(b_{k,n})^2 + \Big(\sum_{n=2}^{N_k} \frac{n}{2^n}\frac{2^{2n}}{-\log c_{k,n}}\, b_{k,n}\Big)^2\Big) \tag{3.3.13}$$

remains bounded.

Put $\nu_{k,n} := -2^{2n}/\log c_{k,n}$, $k \in \mathbb{N}$. Recall that $S_k = \sum_{n=2}^{N_k} \nu_{k,n} \to \infty$ as $k \to \infty$, $\nu_{k,n} \leq C_4$, $k \in \mathbb{N}$, and $N_k \to \infty$ as $k \to \infty$. So we may find sequences $(n_{k,j})_{j=0}^{q_k}$, where $n_{k,0} = 1$, $n_{k,q_k} = N_k$, and $q_k \to \infty$ as $k \to \infty$, such that

$$\nu_{k,n_{k,j}+1} + \cdots + \nu_{k,n_{k,j+1}} > 1, \quad \frac{l}{2^l} < \frac{1}{j+1}, \quad j = 0, \ldots, q_k - 1,\ l > n_{k,j+1}.$$

Now we take

$$b_{k,n_{k,j}+1} = \cdots = b_{k,n_{j+1}} := \frac{1}{(j+1)(\nu_{k,n_j+1} + \cdots + \nu_{k,n_{k,j+1}})}, \quad j = 0, \ldots, q_k - 1.$$

We finally see that $|f_k(z_k)| \to \infty$ as $k \to \infty$ and that $(\|f_k\|_{L^2_{\mathrm{h}}(D)})_k$ remains bounded (compare (3.3.13)). So this part of the proof is complete.

(i)$\Rightarrow$(ii). Suppose that there is a sequence $(z_k)_k \subset D$, $z_k \to 0$ as $k \to \infty$, such that, for a suitable positive number M, $\alpha_D(z_k) \le M$ for all k. Then

$$\sum_{n=2}^{\infty} \frac{2^{2n}}{-\log \operatorname{cap}(A_n(z_k) \setminus D)} \le 8M.$$

In particular, if $c_{k,n} := \operatorname{cap}(A_n(z_k) \setminus D)$ then $\log c_{k,n} \le -2^{2n}/8M$, $k, n \in \mathbb{N}$, $n \ge 2$, and therefore we may find an $n_0 \in \mathbb{N}$ such that $\log c_{k,n} + 1 < -(n+1)\log 2 - 1$, $n > n_0$, $k \in \mathbb{N}$. Let $z \in A_n(z_k)$, $1 \le n < n_0$. Then

$$\frac{1}{2} + \frac{1}{2^{n_0+1}} \ge |z - z_k| + |z_k| \ge |z| \ge |z - z_k| - |z_k| \ge \frac{1}{2^{n_0}} - \frac{1}{2^{n_0+1}} = \frac{1}{2^{n_0+1}},$$

when $|z_k| < 1/2^{n_0+1}$, i.e. for any $k \ge k_0$, k_0 suitably chosen. Choose a domain $D' \supset D$ with $D' \cap \mathbb{B}(0, 1/2^{2n_0}) = D' \cap \mathbb{B}(0, 1/2^{2n_0})$ such that $A_n(z_k) \setminus D' = \emptyset$, $1 \le n < n_0$, $k \ge k_0$. Applying the localization result (cf. Theorem 3.1.5) for the Bergman kernel, we still know that $\lim_{k\to\infty} k_{D'}(z_k) = \infty$.

Now, fix a $k \ge k_0$. Recall that there is an $n_1 > 2n_0$ such that $\mathbb{B}(z_k, 1/2^{n_1}) \subset D'$. We exhaust D' by a sequence of domains $D'_j \Subset D'$ with real-analytic boundaries such that $\sum_{n=2}^{\infty} -2^{2n}/\log \operatorname{cap}(A_n(z_k) \setminus D'_j) < 8M$, $\partial(A_2(z_k) \setminus D'_j) = \partial\mathbb{B}(z_k, 1/4)$, $\widetilde{K}_n := A_N(z_K) \setminus D'_j$ is either empty or non-polar, and any boundary point of $\widetilde{K}_n$, if $\widetilde{K}_n \ne \emptyset$, is a regular point with respect to the unbounded component of its complement. So Frostman's theorem (see Theorem 3.3.4 in [Ran 1995]) together with the continuity principle for logarithmic potentials (see Theorem 3.1.3 in [Ran 1995]) implies that the logarithmic potential $p_n := p_{\mu_{\widetilde{K}_n}}$ if $n \ge 3$ and $\widetilde{K}_n \ne \emptyset$, is continuous on $\mathbb{C}$. For an $n \ge 3$ such that $\widetilde{K}_n = \emptyset$ put $p_n := -\infty$.

For $n \ge 3$ choose $\chi_n \in \mathcal{C}^\infty(\mathbb{R}, [0,1])$ such that $\chi_n = 0$ if $\widetilde{K}_n = \emptyset$, and

$$\chi_n(t) := \begin{cases} 1 & \text{if } t \le \log \operatorname{cap} \widetilde{K}_n + 1/2, \\ 0 & \text{if } t \ge -(n+1)\log 2 - 1/2, \end{cases}$$

and $|\chi_n'(t)| \le -2/M_1 \log \operatorname{cap} \widetilde{K}_n$ where M_1 is a suitable positive number. For $n \ge 3$ define $f_n := f_{\widetilde{K}_n}$ and $\varphi_n := \chi_n \circ p_n$. Note that if $\widetilde{K}_n \ne \emptyset$ then $p_n(z) \ge -(n+1)\log 2$, $z \notin A_{n-1}(z_k) \cup A_n(z_k) \cup A_{n+1}(z_k)$, and that $p_n \in \mathcal{C}^\infty(\mathbb{C} \setminus \widetilde{K}_n)$. So φ_n is a smooth function with support in $A_{n-1}(z_k) \cup A_n(z_k) \cup A_{n+1}(z_k)$ such that $\varphi_n|_{\widetilde{K}_n} = 1$ and

$$\frac{\partial p_n}{\partial \overline{z}}(z) = \frac{1}{2}\overline{f}_n(z), \qquad z \notin \widetilde{K}_n.$$

For $n = 2$ we put $p_2(z) := \log|z|$ and take a $\chi_2 \in \mathcal{C}^\infty(\mathbb{R}, [0,1])$ such that

$$\chi_2(t) = \begin{cases} 0 & \text{if } t \le -\log 8 \text{ or } t \ge -\log 2, \\ 1 & \text{if } t \text{ is near } -\log 4, \end{cases}$$

and $|\chi_2'| \le 2/\log 4$. Again, let $\varphi_2 := \chi_2 \circ p_2$ and put $f_2 := 1$. Then

$$\left|\frac{\partial \varphi_n}{\partial \overline{z}}(z)\right| \le \frac{|f_n(z)|}{-M_2 \log \operatorname{cap} \widetilde{K}_n}, \qquad z \in \frac{1}{2}E \setminus \widetilde{K}_n,\ n \ge 2.$$

Finally, we define

$$\varphi := \sup\{\varphi_n : n \ge 2\}.$$

Note that the supremum is taken over at most three functions. φ is a Lipschitz function satisfying $\varphi|_{\partial D} = 1$ and $\varphi = 0$ in a neighborhood of z_k.

Now let $f \in L^2_{\mathrm{h}}(D')$. Then the Cauchy formula and the Green formula lead to

$$|f(z_k)| = \frac{1}{2\pi}\Bigg|\int\limits_{\partial D_j} \frac{f(\lambda)\,d\lambda}{\lambda - z_k}\Bigg| = \frac{1}{2\pi}\Bigg|\int\limits_{\partial D_j} \frac{(f\varphi)(\lambda)\,d\lambda}{\lambda - z_k}\Bigg| = \frac{1}{\pi}\Bigg|\int\limits_{D_j} \frac{f(\lambda)}{\lambda}\,\frac{\partial\varphi}{\partial\overline{\lambda}}\,d\Lambda_2(\lambda)\Bigg|.$$

Applying various versions of the Schwarz inequalities and Lemma 3.3.14 finally gives

$$\begin{aligned}
|f(z_k)| &\le \sum_{n=2}^{\infty} 2^n \int\limits_{A_n(z_k)\setminus\widetilde{K}_n} |f(\lambda)|\bigg(\frac{|f_{n-1}(\lambda)|}{-\log\operatorname{cap}\widetilde{K}_{n-1}} \\
&\qquad + \frac{|f(\lambda)|}{-\log\operatorname{cap}\widetilde{K}_n} + \frac{|f_{n+1}(\lambda)|}{-\log\operatorname{cap}\widetilde{K}_{n+1}}\bigg)\,d\Lambda_2(\lambda) \\
&\le M_4 \sum_{n=2}^{\infty} \|f\|_{L^2_{\mathrm{h}}(A_n(z_k)\setminus\widetilde{K}_n)}\bigg(\frac{\|f_{n-1}\|^2_{D'}}{(-\log\operatorname{cap}\widetilde{K}_{n-1})^2} \\
&\qquad + \frac{\|f_n\|^2_{D'}}{(-\log\operatorname{cap}K_n)^2} + \frac{\|f_{n+1}\|^2_{D'}}{(-\log\operatorname{cap}\widetilde{K}_{n+1})^2}\bigg)^{1/2} \\
&\le M_5\bigg(\sum_{n=2}^{\infty} \|f\|^2_{L^2_{\mathrm{h}}(A_n(z_k)\setminus\widetilde{K}_n)}\bigg)^{1/2}\bigg(\sum_{n=2}^{\infty} \frac{2^{2n}}{-\log\operatorname{cap}\widetilde{K}_n}\bigg)^{1/2} \le \sqrt{\widetilde{M}}\,\|f\|_{D'},
\end{aligned}$$

where the constant on the right side is independent of k. This estimate is true for all sufficiently large k. Therefore, $(k_{D'}(z_k))_k$ is bounded, a contradiction. ∎

REMARK 3.3.20. There are similar considerations to those in Theorem 3.3.19 for the so-called point evaluation. To be more precise, let $z_0 \in \partial D$, where $D \subset \mathbb{C}$ is a bounded domain. Recall that $V := \{f \in L^2_{\mathrm{h}}(D) : f \text{ is holomorphic in } D \cup \{z_0\}\}$ is dense in $L^2_{\mathrm{h}}(D)$ (cf. Theorem 3.1.2). Therefore, we may define the evaluation functional on V, i.e. $\Phi_{z_0} : V \to \mathbb{C}$, $\Phi_{z_0}(f) := f(z_0)$. The point z_0 is called a *bounded evaluation point* for $L^2_{\mathrm{h}}(D)$ if Φ_{z_0} extends to a continuous functional on $L^2_{\mathrm{h}}(D)$. There is the following description of such points [Hed 1972]:

THEOREM. *Let D and z_0 be as above. Then $\alpha_D(z_0) = \infty$ iff z_0 is not a bounded evaluation point for $L^2_{\mathrm{h}}(D)$.*

Observe that if z_0 is not a bounded evaluation point then D is b-exhaustive at z_0. Nevertheless, the converse statement is false (see Example 3.3.18).

REMARK 3.3.21. For a bounded domain $D \subset \mathbb{C}$ there are analogous notions like the Bergman kernel taking derivatives into account, namely the *nth Bergman kernel*

$$k_D^{(n)}(z) := \sup\{|f^{(n)}(z)|^2 : f \in L^2_{\mathrm{h}}(D) \setminus \{0\},\ \|f\|_{L^2_{\mathrm{h}}(D)} = 1\}, \qquad n \in \mathbb{N}_0,\ z \in D.$$

Observe that $k_D = k_D^{(0)}$. Moreover, one has the following potential-theoretic function:

$$\alpha_D^{(n)}(z) := \int\limits_0^{1/2} \frac{dr}{r^{2n+3}(-\log\operatorname{cap}(\mathbb{B}(z,r) \setminus D))}, \qquad z \in \overline{D},\ n \in \mathbb{N}_0.$$

Observe that $\alpha_D = \alpha_D^{(0)}$. There is the following relation between these notions (see [Pfl-Zwo 2003]):

THEOREM. *Let $n \in \mathbb{N}_0$ and $d > 1$. Then there is a $C > 0$ such that*

- *for any domain $D \subset \mathbb{C}$ with* $\operatorname{diam} D < d$,
$$C\alpha_D^{(n)}(z) \le k_D^{(n)}(z), \qquad z \in D;$$
- *for any domain $D \subset \mathbb{C}$ with* $1/d < \operatorname{diam} D < d$,
$$k_D^{(n)}(z) \le C \max\{1, \alpha_D^{(n)}(z)(\log \alpha_D^{(n)}(z))^2\}, \qquad z \in D.$$

$\boxed{?}$ Let $D \subset \mathbb{C}$ be a domain and $z_0 \in \partial D$. Is it true that $\lim_{D\ni z\to z_0} k_D^{(n)}(z) = \infty$ implies that $\lim_{D\ni z \to z_0} \alpha_D^{(n)}(z) = \infty$ $\boxed{?}$

With the help of the above theorem there is a complete description of those Zalcman domains which are b-exhaustive at each boundary point.

COROLLARY 3.3.22 ([Juc 2004]). *Let*
$$D := E \setminus \Big(\bigcup_{k=1}^{\infty} \overline{\mathbb{B}}(x_k, r_k) \cup \{0\}\Big)$$
be a Zalcman domain (7), *where* $x_k > x_{k+1} > 0$, $\lim_{k\to\infty} x_k = 0$, $r_k > 0$ *with* $\overline{\mathbb{B}}(x_k, r_k) \subset E$, $\overline{\mathbb{B}}(x_k, r_k) \cap \overline{\mathbb{B}}(x_j, r_j) = \emptyset$, $k, j \ge 1$, $k \ne j$. *Assume that*
$$\exists_{\Theta_1\in(0,1)}\ \exists_{\Theta_2\in(\Theta_1,1)}: \quad \Theta_1 \le x_{k+1}/x_k \le \Theta_2, \quad k \in \mathbb{N}.$$
Then D is b-exhaustive iff D is b-exhaustive at 0 *iff* $\sum_{k=1}^{\infty}(-1/x_k^2 \log r_k) = \infty$ *iff* $\alpha_D(0) = \infty$.

Observe that special cases were treated also in [Ohs 1993] and [Che 1999]. Moreover, we mention that the domains D in Corollary 3.3.22 are fat domains, but not all of them are b-exhaustive (for another example see [Jar-Pfl-Zwo 2000]).

Proof. First, observe that for every boundary point z_0 except the origin we have
$$\lim_{D\ni z\to z_0} k_D(z) = \infty$$
(use Theorem 6.1.17 of [J-P 1993]).

Obviously, $\overline{\mathbb{B}}(x_{k+1} - r_{k+1}/2, r_{k+1}) \subset \overline{\mathbb{B}}(0,\delta) \setminus D$, $\delta \in (x_{k+1}, x_k)$. Then
$$\begin{aligned}\alpha_D(0) &\ge \sum_{k=1}^{\infty} \int_{x_{k+1}}^{x_k} \frac{dr}{-r^3 \log \operatorname{cap}(\overline{\mathbb{B}}(0,r)\setminus D)} \ge \sum_{k=k_0}^{\infty} \int_{x_{k+1}}^{x_k} \frac{dr}{-r^3 \log (r_{k+1}/2)} \\ &\ge \sum_{k=k_0}^{\infty} (x_k - x_{k+1}) \frac{-1}{x_k^3 \log (r_{k+1}/2)} \ge C \sum_{k=k_0}^{\infty} \frac{-1}{x_{k+1}^2 \log r_{k+1}},\end{aligned}$$
where C is a constant. Observe that for the last inequality the assumption on the centers x_k was used.

Now, the divergence of the series in the corollary implies that $\alpha_D(0) = \infty$. In view of the lower semicontinuity of the function α_D it follows that $\lim_{D\ni z\to 0} \alpha_D(z) = \infty$.

---

(7) Observe that we use here a slightly more general notion than the one of a Zalcman type domain in Section 2.7.

On the other hand we have

$$\alpha_D(0) = \Big(\int_{x_1}^{1/2} + \sum_{k=1}^{\infty}\int_{x_{k+1}}^{x_k}\Big)\frac{dr}{-r^3\log\operatorname{cap}(\overline{\mathbb{B}}((0,r)\setminus D)}$$

$$\le C_1 + \sum_{k=1}^{\infty}\frac{x_k - x_{k+1}}{x_{k+1}^3}\sum_{j=k}^{\infty}\frac{-1}{\log r_j} \le C_1 + C_2\sum_{j=1}^{\infty}\frac{-1}{\log r_j}\sum_{k=1}^{j}\frac{1}{x_k^2}$$

$$\le C_1 + C_2\sum_{j=1}^{\infty}\frac{-1}{\log r_j}\sum_{k=1}^{j}\frac{\Theta_2^{2(j-k)}}{x_j^2} \le C_1 + C_3\sum_{j=1}^{\infty}\frac{-1}{x_j^2\log r_j},$$

where $C_1 \ge 0$ and $C_2, C_3 > 0$ are suitable numbers. Observe that the last three inequalities follow from the assumptions on the centers x_k.

If the series in the corollary does converge, then $\alpha_D(0) < \infty$. Moreover, directly from the definition we see that α_D restricted to the interval $(-1/4, 0]$ is increasing. Hence $\limsup_{0>x\to 0}\alpha_D(x) \le \alpha_D(0) < \infty$. So, the corollary is proved. ■

EXAMPLE 3.3.23. We discuss the particular case of a Zalcman domain, namely $x_k := (1/2)^k$ and $r_k := (1/2)^{kN(k)}$, where $N_k \in \mathbb{N}$, $k \ge 2$. Then we have

$$D \text{ is } b\text{-exhaustive} \quad \text{iff} \quad \sum_{k=2}^{\infty}\frac{2^{2k}}{kN(k)\log 2} = \infty.$$

On the other hand, following Ohsawa [Ohs 1993] we have

$$D \text{ is hyperconvex} \quad \text{iff} \quad \sum_{k=2}^{\infty} 1/N(k) = \infty.$$

So we see that there are plenty of Zalcman domains which are not hyperconvex but, nevertheless, they are b-exhaustive.

3.4. L^2_{h}-domains of holomorphy

The boundary behavior of the Bergman kernel may be used to give a complete description of L^2_{h}-domains of holomorphy ([8]). The precise result is the following.

THEOREM 3.4.1 ([Pfl-Zwo 2002]). *For a bounded domain $D \subset \mathbb{C}^n$ the following conditions are equivalent*:

(i) *D is an L^2_{h}-domain of holomorphy*;
(ii) $\limsup_{D\ni z\to z_0} k_D(z) = \infty$ *for every boundary point $z_0 \in \partial D$.*

REMARK 3.4.2. There is also the following more geometric condition which is equivalent to (i) of Theorem 3.4.1:

(iii) *for any boundary point $z_0 \in \partial D$ and for any open neighborhood $U = U(z_0)$ the set $U \setminus D$ is not pluripolar* ([9]).

([8]) Recall that a domain $D \subset \mathbb{C}^n$ is an L^2_{h}-*domain of holomorphy* if for any pair of open sets $U_1, U_2 \subset \mathbb{C}^n$ with $\emptyset \ne U_1 \subset D \cap U_2 \ne U_2$, U_2 connected, there is an $f \in L^2_{\mathrm{h}}(G)$ such that $f|_{U_1} \ne F|_{U_1}$ for any $F \in \mathcal{O}(U_2)$.

([9]) Recall that a set $P \subset \mathbb{C}^n$ is called *pluripolar* if there is a $u \in \mathcal{PSH}(\mathbb{C}^n)$, $u \not\equiv -\infty$, such that $P \subset u^{-1}(-\infty)$.

Proof of Theorem 3.4.1. The case $n = 1$ may be found in [Con 1995]. So we will always assume that $n \geq 2$.

(ii)$\Rightarrow$(i). Suppose that G is not an L^2_{h}-domain of holomorphy. Then there are concentric polydiscs $P \Subset \widetilde{P}$ satisfying $P \subset D$, $\partial P \cap \partial D \neq \emptyset$, and $\widetilde{P} \not\subset D$ such that for any function $g \in L^2_{\mathrm{h}}(D)$ there exists a $\widehat{g} \in \mathcal{H}^\infty(\widetilde{P})$ with $\widehat{g}|_P = g|_P$.

Let a be the center of P and let L be an arbitrary complex line through a. Then $L \cap \widetilde{P} \setminus D =: K$ is a polar set (in L).

Indeed, suppose that K is not polar. Fix a compact non-polar subset $K' \subset K$. Then, according to Theorem 9.5 in [Con 1995], there is a non-trivial function $f \in L^2_{\mathrm{h}}(L \setminus K')$ which has no holomorphic extension to L. Since $K' \cap D = \emptyset$, Theorem 3.1.1 guarantees the existence of a function $F \in L^2_{\mathrm{h}}(D)$ with $F|_{L\cap D} = f|_{L \cap D}$. Hence, we find $\widehat{F} \in \mathcal{O}(\widetilde{P})$ such that $\widehat{F}|_P = F|_P$. In particular, $\widehat{F}|_{L\cap\widetilde{P}}$ extends f to the whole of L, a contradiction.

So $L \cap \widetilde{P} \cap D$ is connected ([10]). Since L is arbitrary, $D \cap \widetilde{P}$ is connected. Therefore, for any function $g \in L^2_{\mathrm{h}}(D)$ there exists a unique holomorphic extension $\widehat{g} \in \mathcal{H}^\infty(\widetilde{P})$ with $\widehat{g}|_{D\cap\widetilde{P}} = g|_{D\cap\widetilde{P}}$. Consider the linear space

$$A := \{(g, \widehat{g}) : g \in L^2_{\mathrm{h}}(D)\} \subset L^2_{\mathrm{h}}(D) \times \mathcal{H}^\infty(\widetilde{P})$$

equipped with the norm $\|(g,\widehat{g})\| := \|g\|_{L^2_{\mathrm{h}}(D)} + \|\widehat{g}\|_{\mathcal{H}^\infty(\widetilde{P})}$. Then A is a Banach space.

Observe that $A \ni (g, \widehat{g}) \mapsto g \in L^2_{\mathrm{h}}(D)$ is a one-to-one, surjective, continuous, linear mapping. Hence, in view of the Banach open mapping theorem, its inverse map is also continuous, i.e. there is a $C > 0$ such that

$$\|(g, \widehat{g})\| \leq C\|g\|_{L^2_{\mathrm{h}}(D)}, \qquad g \in L^2_{\mathrm{h}}(D).$$

In particular, $\|\widehat{g}\|_{\mathcal{H}^\infty(\widetilde{P})} \leq C\|g\|_{L^2_{\mathrm{h}}(D)}$. So we are led to the following estimate:

$$\sup\{k_D(z) : z \in D \cap \widetilde{P}\} = \sup\left\{\frac{|g(z)|^2}{\|g\|^2_{L^2_{\mathrm{h}}(D)}} : z \in D \cap \widetilde{P},\ 0 \not\equiv g \in L^2_{\mathrm{h}}(D)\right\} \leq C^2.$$

In particular, $\limsup_{z\to w} k_D(z) \leq C^2$ for a point $w \in \partial P \cap \partial D \neq \emptyset$ (recall that such a point exists), a contradiction.

Before we are able to start the proof of (i)$\Rightarrow$(ii) we need some auxiliary results.

LEMMA 3.4.3. *Let $G \subset \mathbb{C}$ be a bounded domain and let $a \in \partial G$. Assume that $\limsup_{G\ni z\to a} k_G(z) < \infty$. Then there is a neighborhood $U = U(a)$ such that $U \setminus G$ is polar.*

Proof. Suppose that Lemma 3.4.3 is not true. First we claim that for any $r > 0$ the intersection $\mathbb{B}_1(a, r) \cap \partial G$ is not polar. Otherwise, there is an $r_0 > 0$ such that $\mathbb{B}_1(a, r_0) \cap \partial G$ is polar. Observe that $\mathbb{B}_1(a, r_0/4) \setminus G$ is not polar. Therefore, there exists a $b_0 \in \mathbb{B}_1(a, r_0/4) \setminus \overline{G}$. Choose a point $b \in \mathbb{B}_1(a, r_0/4) \cap G$. Since $\mathbb{B}_1(a, r_0) \cap \partial G$ is polar, there exists an $s \in (0, r_0/2)$ such that $\partial\mathbb{B}_1(b_0, s) \cap \partial G = \emptyset$, $\partial\mathbb{B}_1(b_0, s) \cap G \neq \emptyset$, and $\partial\mathbb{B}_1(b_0, s) \subset \mathbb{B}_1(a, r_0)$. Hence, $\partial\mathbb{B}_1(b_0, s) \subset G$. Therefore, for any $z \in \partial\mathbb{B}_1(b_0, s)$, one has $[b_0, z] \cap \partial G \neq \emptyset$. Then, by Theorem 5.1.7 in [Arm-Gar 2001], $\partial\mathbb{B}_1(b_0, s)$ is a polar set, a contradiction.

([10]) Recall that for a plane domain G and a relatively closed polar subset $M \subset G$ the open set $G \setminus M$ is connected.

Hence, there is a sequence $(a_j)_j \subset \partial G$, $a_j \to a$, where all the points a_j are regular boundary points of G. Suppose for a moment that we knew that $\lim_{G\ni z\to b} k_G(z) = \infty$ for a regular boundary point b of G. Then $\limsup_{G\ni z\to a} k_G(z) = \infty$, which obviously contradicts the assumption of Lemma 3.4.3.

It remains to prove the following claim: Let $b \in \partial G$ be a regular point. Then $k_G(z) \to \infty$ when $G \ni z \to b$.

Here we will use the following relation of the Bergman kernel and the Azukawa metric given in [Ohs 1995]: there is a $C > 0$ such that

$$\sqrt{k_G(z)} \ge C A_G(z;1), \qquad z \in G.$$

Define $G_p := \{z \in G : \log g_G(p,z) < -1\}$, $p \in G$, and $r(p) := \operatorname{diam} G_p$. Then, using [Zwo 2000c], we get

$$A_G(p,1) = eA_{G_p}(p,1) \ge eA_{\mathbb{B}_1(p,r(p))}(p,1) = \frac{e}{r(p)} \to \infty \qquad \text{as } p \to b.$$

So it remains to show that $r(p) \to 0$ when $p \to b$. Suppose this is not true. Then we find an $\varepsilon > 0$ and sequences $G \ni p_j \to b$ and $G \ni z_j \to z^* \in \overline{G}$ such that $|p_j - z_j| \ge \varepsilon$ and $\log g_G(p_j,z_j) < -1$, $j \in \mathbb{N}$. Choose a small disc V around z^* with $b \notin \overline{V}$. Then we have $g_G(p_j,z_j) \ge g_{\widetilde{G}}(p_j,z_j)$, where $\widetilde{G} := G \cup V$. Now, observe that $g_{\widetilde{G}}(p_j,\cdot) = g_{\widetilde{G}}(\cdot,p_j)$ and that $\log g_{\widetilde{G}}(p_j,\cdot) \to 0$ pointwise. Since $\log g_{\widetilde{G}}(p_j,\cdot)$ are harmonic functions, the Vitali theorem implies that $\log g_{\widetilde{G}}(p_j,\cdot)$ tends uniformly to 0 on some small neighborhood of z^*, contradicting the fact that $\log g_G(p_j,z_j) < -1$ for all j. ∎

LEMMA 3.4.4. *Let $D \subset \mathbb{C}^n$, $n \ge 2$, be a domain and let $0 < r < t$. For any $z' \in \mathbb{C}^{n-1}$ define*

$$D_{z'} := \{z_n \in tE : (z',z_n) \in D\} =: tE \setminus K(z').$$

Assume that $K(0')$ is polar and that there is a neighborhood V of $0'$ such that for almost all $z' \in V$ the set $K(z')$ is also polar. Then there is a neighborhood $V' \subset V$ of $0'$ such that for any $f \in L^2_{\mathrm{h}}(D)$ there exists an $\widehat{f} \in \mathcal{O}(V' \times rE)$ with $f = \widehat{f}$ on $D \cap (V' \times rE)$.

Proof. Since $K(0')$ is a polar set, there is an s with $r < s < t$ such that $K(0') \cap \partial\mathbb{B}_1(0,s) = \emptyset$. Therefore, we find a neighborhood $V' = V'(0') \subset V$ such that $K(z') \cap \partial\mathbb{B}_1(0,s) = \emptyset$, $z' \in V'$. Then we may define

$$\widehat{f}(z',z_n) := \frac{1}{2\pi} \int\limits_{\partial\mathbb{B}_1(0,s)} \frac{f(z',\lambda)}{\lambda - z_n}\, d\lambda, \qquad (z',z_n) \in V' \times \mathbb{B}_1(0,s).$$

Obviously, $\widehat{f} \in \mathcal{O}(V' \times \mathbb{B}_1(0,s))$.

On the other hand, using the fact that $f \in L^2_{\mathrm{h}}(D)$, the Fubini theorem and the assumptions made in Lemma 3.4.4 we deduce that for almost all $z' \in V'$ the function $f(z',\cdot) \in L^2_{\mathrm{h}}(\mathbb{B}_1(0,t) \setminus K(z'))$ and $K(z')$ is polar. Hence, $f(z',\cdot)$ extends to a holomorphic function on $\mathbb{B}_1(0,t)$ for almost all $z' \in V'$ ([11]). Applying the Cauchy integral formula, we obtain $f(z',z_n) = \widehat{f}(z',z_n)$, $(z',z_n) \in V' \times \mathbb{B}_1(0,s)$, for almost all $z' \in V'$. Since this set is dense in $(V' \times \mathbb{B}_1(0,s)) \cap D$, we have reached the claim of Lemma 3.4.4. ∎

Now we are able to complete the proof of Theorem 3.4.1.

([11]) Recall that a relatively closed polar subset of a plane domain is a removable set of singularities for square-integrable holomorphic functions.

(i)$\Rightarrow$(ii). Fix a boundary point $w \in \partial D$. First we discuss the case when $w \notin \operatorname{int}(\overline{D})$. Then there is a sequence $(z_j)_j \subset \mathbb{C}^n$ such that $z_j \to w$ and $z_j \notin \overline{D}$, $j \in \mathbb{N}$. By r_j we denote the largest radius such that $B_j := \mathbb{B}_n(z_j, r_j)$ does not intersect $\overline{D}$. Select $w_j \in \partial B_j \cap \partial D$. Then $w_j \to w$. Observe that the domain D satisfies the general outer cone condition at w_j (see Theorem 6.1.17 in [J-P 1993]). Therefore, $\lim_{D\ni z\to w_j} k_D(z) = \infty$. Hence, (ii) follows.

From now on we assume that $w \in \operatorname{int}(\overline{D})$. Suppose that (ii) is not true for w. Then there are a polydisc $P \Subset \overline{D}$ with center at w and a constant $C > 0$ such that

$$k_D(z) \le C, \qquad z \in D \cap P.$$

Now, let L be a complex line through P. Then $(L \cap P) \setminus D$ is a polar set (in L) or it is empty. Indeed, otherwise we apply Lemma 3.4.3. Therefore,

$$\sup\{k_{D\cap L}(z) : z \in L \cap P \cap D\} = \infty.$$

Then, by Theorem 3.1.1, $\sup\{k_D(z) : z \in L \cap D \cap P\} = \infty$, a contradiction.

Observe that there is a complex line L^* passing through w and $P \cap D$. We may assume that $w = 0$ and, after a linear change of coordinates, that $P = E^n$ and $L^* = \{(0,\dots,0)\} \times \mathbb{C}$. So the assumptions of Lemma 3.4.4 are satisfied with respect to some neighborhood $V \subset E^{n-1}$ of $0' \in \mathbb{C}^{n-1}$. Therefore, there is a neighborhood $V' = V'(0') \subset V$ such that for any $f \in L^2_{\mathrm{h}}(D)$ there is an $\widehat{f} \in \mathcal{O}(V' \times \mathbb{B}_1(0, 1/2))$ with $f = \widehat{f}$ on $D \cap (V' \times \mathbb{B}_1(0, 1/2))$, contradicting the assumption in (i). ∎

REMARK 3.4.5. In [Irg 2003], the following generalization of Theorem 3.4.1 may be found:

THEOREM. *Let (X, π) be a Riemann domain over $\mathbb{C}^n$ such that $\pi(X)$ is bounded. Let $(\widehat{X}, \widehat{\pi})$ be the envelope of holomorphy and $(\widetilde{X}, \widetilde{\pi})$ the $L^2_{\mathrm{h}}(X)$-envelope of holomorphy of (X, π). Then $(\widehat{X}, \widehat{\pi})$ embeds into $(\widetilde{X}, \widetilde{\pi})$ and the difference of these two sets is a pluripolar subset of $\widetilde{X}$.*

Theorems 3.3.9 and 3.4.1 may be used to get the following result.

COROLLARY 3.4.6 ([Jar-Pfl 1996]). *Any bounded balanced domain of holomorphy is an L^2_{h}-domain of holomorphy.*

$\boxed{?}$ It is an open problem to characterize those unbounded domains of holomorphy that are L^2_{h}-domains of holomorphy. Even more, so far there is no description of such unbounded domains that carry a non-trivial L^2_{h}-function. $\boxed{?}$

3.5. Bergman completeness

Let $G \subset \mathbb{C}^n$ be a domain such that $k_G(z) > 0$, $z \in G$ ([12]). Then the Bergman kernel k_G is a logarithmically psh function on G. So

$$\beta_G(z; X) := \Big(\sum_{j,k=1}^{n} \frac{\partial^2}{\partial z_j \partial \overline{z}_k} \log k_G(z) X_j \overline{X}_k \Big)^{1/2}, \qquad z \in G,\ X \in \mathbb{C}^n,$$

([12]) Observe that this condition holds if for any $z \in G$ there exists an $f \in L^2_{\mathrm{h}}(G)$ such that $f(z) \neq 0$.

gives a hermitian pseudometric on G. It is the *Bergman pseudometric.* Recall that there is another description of the Bergman pseudometric. Let $G \subset \mathbb{C}^n$ be as above. We define

$$M_G(z;X) = \sup\{|f'(z)X| : f \in L^2_{\mathrm{h}}(G),\ \|f\|_{L^2_{\mathrm{h}}(G)} = 1,\ f(z) = 0\}, \quad z \in G,\ X \in \mathbb{C}^n.$$

Then

$$\beta_G(z;X) = \frac{M_G(z;X)}{\sqrt{k_G(z)}}. \tag{3.5.14}$$

Observe that β_G is a metric if

$$\forall_{z\in G}\ \forall_{X\in\mathbb{C}^n,\, X\neq 0}\ \exists_{g,f\in L^2_{\mathrm{h}}(g)} : \quad g(z) \neq 0 \text{ and } f(z) = 0,\ f'(z)X \neq 0. \tag{3.5.15}$$

The *Bergman pseudodistance* on G is given by

$$b_G(z,w) := \inf\Big\{\int_0^1 \beta_G(\gamma(t);\gamma'(t))dt : \gamma \in \mathcal{C}^1([0,1],G) : \gamma(0) = z,\ \gamma(1) = w\Big\}, \qquad z,w \in G.$$

Under the condition (3.5.15), the function b_G is in fact a distance.

One of the main questions here is to decide which domain in $\mathbb{C}^n$ is b_G-complete.

DEFINITION 3.5.1. A domain $G \subset \mathbb{C}^n$ satisfying $k_G(z) > 0$ for all $z \in G$ is called *Bergman-complete* (for short *b-complete* or *b_G-complete*) if b_G is a distance and if for any b_G-Cauchy sequence $(z_j)_j \subset G$ there is a point $a \in G$ such that $\lim_{j\to\infty} z_j = a$.

Obviously, for any bounded domain $D \subset \mathbb{C}^n$, β_D is a metric and b_D is a distance on D. For a not necessarily bounded domain D, we have the following sufficient criterion (see [Che-Zha 2002]).

THEOREM 3.5.2. *Let $D \subset \mathbb{C}^n$ be a pseudoconvex domain (not necessarily bounded). Assume that for any point $w \in D$ there is an $r > 0$ such that*

$$A_w(D;r) := \{z \in D : \log g_D(w,z) < -r\} \subset\subset D. \ (^{13})$$

Then β_D is a metric on D and b_D is a distance.

Proof. The functions satisfying condition (3.5.15) are found by solving a $\bar\partial$-problem. For more details, the reader may consult [Che-Zha 2002]. ∎

From the above theorem we immediately get the following one-dimensional result (see [Che-Zha 2002]):

COROLLARY 3.5.3. *Any hyperbolic Riemann surface has a Bergman metric and distance. In particular, any plane domain $D \subset \mathbb{C}$ such that $\mathbb{C} \setminus D$ is not a polar set has a Bergman distance.*

REMARK 3.5.4. Moreover, any domain $D \subset \mathbb{C}^n$ which carries either a bounded continuous strictly psh function or a negative function $u \in \mathcal{PSH}(D)$ such that $\{z\in D : u(z) < -r\} \Subset D$, $r > 0$, satisfies (3.5.15), i.e. D admits a Bergman distance. For more details see [Che-Zha 2002].

Moreover, the following result due to N. Nikolov (private communication) is also a consequence of Theorem 3.5.2.

(^{13}) Observe that this condition is always true for a bounded domain.

COROLLARY 3.5.5. *Let $D \subset \mathbb{C}^n$ be an unbounded domain. Assume that there are $R > 0$ and $\psi \in \mathcal{PSH}(D \setminus \overline{\mathbb{B}(R)})$ such that:*

- $\psi < 0$ *on* $D \setminus \overline{\mathbb{B}(R)}$,
- $\lim_{z\to\infty} \psi(z) = 0$,
- $\limsup_{z\to a} \psi(z) < 0$, $a \in (\partial D) \setminus \overline{\mathbb{B}(R)}$.

Then D has the Bergman metric.

Proof. Fix a $z_0 \in D$ and choose positive numbers $R_3 > R_2 > R_1 > R$ such that $\|z_0\| < R_1$ and

$$2 \inf_{D\setminus\mathbb{B}(R_2)} \psi \geq \sup_{D\cap\partial\mathbb{B}(R_1)} \psi =: c < 0.$$

Moreover, put

$$d := \inf_{w\in D\cap\partial\mathbb{B}(R_1)} \log g_{\mathbb{B}(R_3)}(z_0, w) > -\infty, \qquad u(w) := 2\psi(w)(d/c) - d, \qquad w \in D \setminus \overline{\mathbb{B}(R)}.$$

Observe that

$$\begin{aligned} u(w) &\leq d \leq \log g_{\mathbb{B}(R_3)}(z_0, w), \quad && w \in D \cap \partial\mathbb{B}(R_1), \\ u(w) &\geq 0 \geq \log g_{\mathbb{B}(R_3)}(z_0, w), \quad && w \in D \cap \partial\mathbb{B}(R_2). \end{aligned}$$

Hence, the function

$$v(w) := \begin{cases} \log g_{\mathbb{B}(R_3)}(z_0, w), & w \in D \cap \mathbb{B}(R_1), \\ \max\{\log g_{\mathbb{B}(R_3)}(z_0, w),\, u(w)\}, & w \in D \cap (\mathbb{B}(R_2) \setminus \mathbb{B}(R_1)), \\ u(w), & w \in D \setminus \mathbb{B}(R_2), \end{cases}$$

is psh on D with a logarithmic pole at z_0. Therefore, $v + d \leq \log g_D(z_0, \cdot)$ on D. Since $v = u \geq 0$ on $D \setminus \mathbb{B}(R_2)$, we have $\log_D(z_0, \cdot) \geq d$ on $D \setminus \mathbb{B}(R_2)$. And if $w \in D \cap \mathbb{B}(R_2)$, then $\log g_D(z_0, w) \geq \log g_{\mathbb{B}(R_3)}(z_0, w) + d \geq \log g_{\mathbb{B}(z_0, R_3+\|z_0\|)}(z_0, w) + d$.

Let $\mathbb{B}(z_0, s) \Subset D \cap \mathbb{B}(R_2)$. Then there is a $d_1 < 0$ such that $\log g_D(z_0, w) \geq d + d_1$, $w \in D \cap \mathbb{B}(R_2) \setminus \mathbb{B}(z_0, s)$. Therefore, $A_{z_0}(D; d + d_1) \Subset D$ and, since z_0 was arbitrarily chosen, Theorem 3.5.2 finishes the proof. ■

It is an old result due to Bremermann that a bounded b-complete domain in $\mathbb{C}^n$ is pseudoconvex. On the other hand, the following sufficient conditions for a bounded domain to be b-complete are mainly due to Kobayashi.

THEOREM 3.5.6. *Let $D \subset \mathbb{C}^n$ be a bounded pseudoconvex domain.*

(a) *Assume that*

$$\limsup_{z\to\partial D} \frac{|f(z)|}{\sqrt{k_D(z)}} < \|f\|^2_{L^2_{\mathrm{h}}(D)}, \qquad f \in L^2_{\mathrm{h}}(D) \setminus \{0\}.$$

Then D is b-complete.

(a′) *Let $H \subset L^2_{\mathrm{h}}(D)$ be a dense subspace. Moreover, assume that for any sequence $(z_j)_j \in D$, $z_j \to z_0 \in \partial D$, and any $g \in H$, there is a subsequence $(z_{j_k})_k$ such that*

$$\lim_{k\to\infty} \frac{|g(z_{j_k})|}{\sqrt{k_D(z_{j_k})}} = 0.$$

Then D is b-complete.

(b) *For all $z, w \in D$ we have*

$$b_D(z,w) \geq \arccos \frac{|K_D(z,w)|}{\sqrt{k_D(z)}\sqrt{k_D(w)}}.$$

The statements (a) and (b) are due to Z. Błocki (see [Bło 2002], [Bło 2003]). Observe that (b) explains the connection between the Bergman distance and the Skwarczyński distance (for more details see [J-P 1993]).

Proof. (a) Suppose D is not b-complete. Then, by the proof of Lemma 7.6.4 in [J-P 1993], we may find an $f \in L^2_{\mathrm{h}}(D)$, $\|f\|_{L^2_{\mathrm{h}}(D)} = 1$, and real numbers θ_j such that

$$\frac{e^{i\theta_j} K_D(\cdot, z_j)}{\sqrt{k_D(z_j)}} \xrightarrow[j\to\infty]{} f \qquad \text{in } L^2_{\mathrm{h}}(D).$$

Therefore, taking the scalar product with f, we get $|f(z_j)|/\sqrt{k_D(z_j)} \to \|f\|^2$, a contradiction.

(a′) Suppose again that D is not b-complete and choose f and θ_j as above. Moreover, take a $g \in H$ with $\|g - f\|_{L^2_{\mathrm{h}}(D)} < 1/2$. Then, because of our assumption, there is a subsequence $(z_{j_k})_k$ such that

$$1 \xleftarrow[k\to\infty]{} \frac{|f(z_{j_k})|}{\sqrt{k_D(z_{j_k})}} \leq \|f-g\|_{L^2_{\mathrm{h}}(D)} + \frac{|g(z_{j_k})|}{\sqrt{k_D(z_{j_k})}} \xrightarrow[k\to\infty]{} \|f-g\|_{L^2_{\mathrm{h}}(D)} < \frac{1}{2},$$

a contradiction. ∎

REMARK 3.5.7. Most of the results on b-completeness will be based on Theorem 3.5.6. In order to verify b-completeness one could also try to find good quantitative estimates for the Bergman distance or the Bergman metric near the boundary. For example, there are the following two positive results.

THEOREM ([Die-Ohs 1995]). *Let $D \subset \mathbb{C}^n$ be a $\mathcal{C}^2$-smooth bounded pseudoconvex domain and let $z_0 \in D$. Then there exist positive constants C_1 and C_2 such that*

$$\text{(3.5.16)} \qquad b_D(z_0, z) \geq C_1 \log|\log(C_2 \operatorname{dist}(z, \partial D))| - 1, \qquad z \in D.$$

THEOREM ([Bło 2002]). *Let D be as above and let $z_0 \in D$. Then there is a positive constant C such that*

$$\text{(3.5.17)} \qquad b_D(z_0, z) \geq C \frac{\log(1/\operatorname{dist}(z,\partial D))}{\log\log(1/\operatorname{dist}(z,\partial D))}, \qquad z \in D,\ z \textit{ sufficiently close to } \partial D.$$

In fact, both estimates (3.5.16) and (3.5.17) remain true in a more general situation, namely, for bounded pseudoconvex domains, not necessarily smooth, which admit a good bounded psh exhaustion function.

On the other hand, the following example shows that there are certain obstacles, even for smooth domains, to good boundary behavior of the Bergman metric.

THEOREM ([Die-Her 2000]). *Let $a \in (0,1)$. Then there exists a bounded pseudoconvex domain $D \subset \mathbb{C}^2$ given as*

$$D = \{z \in \mathbb{C}^2 : r(z) < 0\}$$

with a smooth boundary, $0 \in \partial D$, where the defining function r is of the form $r(z) = \operatorname{Re} z_1 + b|z_1|^2 + \varrho(z_2)$ for suitable $\varrho \in \mathcal{PSH}(\mathbb{C})$, $\varrho(0) = 0$, and $b > 0$ such that there are *no positive constant C and no neighborhood $U = U(0) \subset \mathbb{C}^2$ such that*

$$\beta_D(z;X) \geq C \frac{\left|\sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(z) X_j\right|}{|r(z)|(\log(1/|r(z)|))^{1/(1+2a)}}, \qquad z \in D \cap U.$$

As a consequence of Theorem 3.5.6 we get (see [Bło-Pfl 1998], [Her 1999])

THEOREM 3.5.8. *Let $D \subset \mathbb{C}^n$ be a bounded pseudoconvex domain. Assume that*

$$\lim_{D \ni z \to \partial D} \Lambda_{2n}(A_z(D)) = 0.$$

Then D is b-complete. In particular, any hyperconvex bounded domain is b-complete.

Proof. Fix an $f \in L^2_{\mathrm{h}}(D) \setminus \{0\}$. Using Theorem 3.3.3 we have

$$\frac{|f(z)|^2}{k_D(z)} \leq C_n \int\limits_{A_z(D)} |f(w)|^2 d\Lambda_{2n}(w), \qquad z \in D.$$

Then the assumption and Theorem 3.5.6(a′) immediately give the proof.

Finally, it suffices to recall that a hyperconvex domain satisfies the condition on the level sets of the Green function. ∎

REMARK 3.5.9. In [Che 2004], a similar result is announced even for arbitrary domains, namely:

Let $D \subset \mathbb{C}^n$ be a (not necessarily bounded) domain. Assume that there is a strictly psh function $u : D \to [-1,0)$ such that all sublevel sets $\{z \in D : u(z) < c\}$, $c \in (-1,0)$, are relatively compact subsets of D. Then D is b-complete.

For weaker results see also [Che-Zha 2002].

A direct consequence of Theorem 3.5.6 is the following sufficient criterion for b-completeness.

COROLLARY 3.5.10. *Let $D \subset \mathbb{C}^n$ be a bounded b-exhaustive domain. Assume that there is a dense subspace $H \subset L^2_{\mathrm{h}}(D)$ such that*

$$\text{any } f \in H \text{ is bounded near } z_0, \qquad z_0 \in \partial D.$$

Then D is b-complete.

Applying this result together with Theorem 3.1.2 leads to the following sufficient criterion for a plane domain to be b-complete (see [Che 2000]).

COROLLARY 3.5.11. *Any bounded b-exhaustive domain $D \subset \mathbb{C}$ is b-complete.*

REMARK 3.5.12. For a bounded pseudoconvex domain, a localization result for the Bergman metric is well known ([J-P 1993]; for a sharper version see also [Her 2003]) This implies that a bounded pseudoconvex domain in $\mathbb{C}^n$ is b-complete iff D is *locally b-complete*, i.e. for any $a \in \partial D$ there is an open neighborhood $U = U(a)$ such that any connected component V of $D \cap U$ is b-complete.

There is an analogous result, due to N. Nikolov [Nik 2003], in the plane case for the unbounded situation, namely:

THEOREM 3.5.13. *Let $D \subset \mathbb{C}$ be a domain such that $\mathbb{C} \setminus D$ is not a polar set. Assume that D is locally b-complete* ([14]). *Then D is b-complete.*

The proof of Theorem 3.5.13 is based on the following lemma.

LEMMA 3.5.14. *Let $D \subset \mathbb{C}$ be a domain such that $\mathbb{C} \setminus D$ is not polar. Moreover, let $a \in \partial D$ and $U = U(a)$ be an open neighborhood of a. Then there exists a neighborhood $V = V(a) \subset U$ and a constant $C > 0$ such that*

$$C\beta_{\widehat{U}}(z;1) \le \beta_D(z;1), \qquad z \in V \cap D,$$

where $\widehat{U}$ denotes the connected component of $D \cap U$ with $z \in \widehat{U}$.

Proof. Since $\mathbb{C} \setminus D$ is not polar, there is an $r_0 > 0$ such that $\mathbb{C} \setminus (D \cup \mathbb{B}(a, r_0))$ is not polar. Hence, $\log g_{D\cup\mathbb{B}(a,r_0)}$ is harmonic on $(D \cup \mathbb{B}(a, r_0)) \setminus \{a\}$. Fix an $r_1 \in (0, r_0)$ and define $D_1 := D \cup \mathbb{B}(a, r_1)$. Applying the fact that $g_{D_1}(a, z) \ge g_{D\cup\mathbb{B}(a,r_0)}(a, z)$, $z \in D_1$, we have

$$\inf\{\log g_{D_1}(a, z) - |z - a|^2 : z \in \partial\mathbb{B}(a, r_1) \cap D\} =: m > -\infty.$$

Put

$$u(z) := \begin{cases} \max\{|z-a|^2 + m,\ \log g_{D_1}(a,z)\} & \text{if } z \in D \cap \mathbb{B}(a, r_1), \\ \log g_{D_1}(a,z) & \text{if } z \in D \setminus \mathbb{B}(a, r_1). \end{cases}$$

Observe that $|z - a|^2 + m \le g_{D_1}(a, z)$, $z \in D \cap \mathbb{B}(a, r_1)$. Therefore, $0 \ge u \in \mathcal{SH}(D_1)$ and $u(z) = |z - a|^2 + m$, $z \in \mathbb{B}(a, r_2)$, for a sufficiently small $r_2 < r_1$.

Choose numbers $0 < r_4 < r_3 < r_2$ and a $\mathcal{C}^\infty$ cut-off function χ such that $\chi \equiv 1$ on $\mathbb{B}(a, r_4)$ and $\chi \equiv 0$ outside $\mathbb{B}(a, r_3)$. Fix a point $z_0 \in D \cap \mathbb{B}(a, r_4)$ and let $\widetilde{U}$ be the connected component of $D \cap U$ with $z_0 \in \widetilde{U}$. Take an $f \in L^2_{\mathrm{h}}(\widetilde{U})$ with $f(z_0) = 0$. Put

$$\alpha(z) := \begin{cases} \bar\partial(\chi f)(z) & \text{if } z \in \widetilde{U}, \\ 0 & \text{if } z \in D \setminus \widetilde{U}. \end{cases}$$

Then α is a $\bar\partial$-closed $\mathcal{C}^\infty_{(0,1)}$-form on D satisfying the following inequality:

$$\int_D |\alpha(z)|^2 e^{-6\log g_D(z_0,z) - u(z)}\, d\varLambda_2(z) \le \widetilde{C} \int_{\widetilde{U}} |f(z)|^2\, d\varLambda_2(z) < \infty,$$

where $\widetilde{C} > 0$ is independent of f and z_0. Observe that the subharmonic weight function is strictly subharmonic near the support of α. Therefore, using Hörmander's L^2-theory ([15]), we get a function $h \in \mathcal{C}^\infty(D)$ with $\bar\partial h = \alpha$ on D such that

$$\|h\|^2_{L^2(D)} \le \int_D |h(z)|^2 e^{-6\log g_D(z_0,z) - u(z)}\, d\varLambda_2(z) \le C'\|f\|^2_{L^2_{\mathrm{h}}(\widetilde{U})},$$

([14]) Observe that here the point ∞ is counted as a boundary point of D; so we also assume that there is a compact set $K \subset \mathbb{C}$ such that any (non-empty) connected component of $D \setminus K$ is b-complete.

([15]) Here we use the following form of Hörmander's result:

THEOREM. *Let $D \subset \mathbb{C}^n$ be a pseudoconvex domain, $\varphi \in \mathcal{PSH}(D)$, and $\alpha \in \mathcal{C}^\infty_{(0,1)}(D)$. Assume that $\bar\partial\alpha = 0$ and that on an open set $U \subset D$ with* $\operatorname{supp}\alpha \subset U$, *the function φ can be written as $\varphi = \psi + \chi$, $\psi, \chi \in \mathcal{PSH}(U)$, such that $\mathcal{L}\psi(z; X) \ge C\|X\|^2$ for $z \in U$, $X \in \mathbb{C}^n$. Then there exists an $h \in \mathcal{C}^\infty(D)$ with $\bar\partial h = \alpha$, such that $\int_D |h|^2 e^{-\varphi} d\varLambda_{2n} \le C' \int_D |\alpha|^2 e^{-\varphi}\, d\varLambda_{2n}$, where $C' > 0$ depends only on C.*

where C' is a positive number which is independent of f and z_0. Moreover, since the second integral is finite, it follows that $h(z_0) = h'(z_0) = 0$. Hence, the function

$$\widehat{f}(z) := \begin{cases} (\chi f)(z) - h(z) & \text{if } z \in \widetilde{U}, \\ -h(z) & \text{if } z \in D \setminus \widetilde{U}, \end{cases}$$

is holomorphic on D and satisfies $\widehat{f}(z_0) = 0$, $\widehat{f}'(z_0) = f'(z_0)$, and $\|\widehat{f}\|_{L^2_{\mathrm{h}}(D)} \le C\|f\|_{L^2_{\mathrm{h}}(\widetilde{U})}$, with $C > 0$ independent of f and z_0. Therefore, in view of (3.5.14), we get $\widetilde{C}\beta_D(z_0;1) \ge \beta_{\widetilde{U}}(z_0;1)$. Since z_0 was arbitrary, the lemma is proved. ∎

Proof of Theorem 3.5.13. First of all, let us mention that, by Corollary 3.5.3, D has a Bergman metric.

Suppose now that D is not b-complete. Then there is a b-Cauchy sequence $(z_j)_j \subset D$ with $z_j \to a \in \partial D$ or $z_j \to \infty$. The second case can be reduced to the first one by using the biholomorphic transformation $z \mapsto 1/(z-c)$, where $c \notin D$. So we only have to deal with the first case.

By the assumption, there is an open neighborhood $U = U(a)$ such that any connected component of $U \cap D$ is b-complete. Fix a positive r_1 such that $\mathbb{B}(a, r_1) \Subset U$. Applying Lemma 3.5.14, we may find positive numbers $r_2 < r_1$ and C such that $C\beta_D(z;1) \le \min\{\beta_{\widehat{U}}(z;1), \beta_{\widetilde{U}}(z;1)\}$, $z \in D \cap \mathbb{B}(a, r_2)$, where $\widehat{U}$ and $\widetilde{U}$ denote the connected components of $D \cap U$ and $D \cap \mathbb{B}(a, r_1)$, respectively, with $z \in \widehat{U} \cap \widetilde{U}$. Choose an $r_3 \in (0, r_2)$. Put

$$\begin{aligned} d := \inf\{b_{\widetilde{U}}(z,w) : z \in \partial\mathbb{B}(a, r_3) \cap D,\ w \in \partial\mathbb{B}(a, r_2) \cap D, \\ z, w \in \widetilde{U},\ \widetilde{U} \text{ a connected component of } D \cap \mathbb{B}(a, r_1)\}. \end{aligned}$$

In view of the inequality $c \le b$, it follows that $d > 0$. So we may take an index $k_0 \in \mathbb{N}$ such that $b_D(z_k, z_l) < Cd/2$ and $z_k \in \mathbb{B}(a, r_3)$, $k, l \ge k_0$.

Fix such k, l with $z_k \ne z_l$. Then there is a $\mathcal{C}^1$-curve $\alpha_{k,l} : [0,1] \to D$ such that

$$2b_D(z_k, z_l) > \int_0^1 \beta_D(\alpha_{k,l}(t); \alpha'_{k,l}(t))\, dt.$$

Suppose this curve does not lie in $\mathbb{B}(a, r_1)$. Then there are numbers $0 < s_1 < s_2 < 1$ such that $\alpha_{k,l}(s_1) \in \partial\mathbb{B}(a, r_3)$, $\alpha_{k,l}(s_2) \in \partial\mathbb{B}(a, r_2)$, and $\alpha_{k,l}([s_1, s_2]) \subset \mathbb{B}(a, r_1)$. Hence,

$$2b_D(z_k, z_l) > \int_{s_1}^{s_s} \beta_D(\alpha_{k,l}(t); \alpha'_{k,l}(t))\, dt \ge Cb_{\widetilde{U}}(z_k, z_l) \ge dc,$$

where $\widetilde{U}$ is the connected component of $D \cap \mathbb{B}(a, r_1)$ containing this part of the curve, a contradiction.

Hence, we deduce for $k, l \ge k_0$ that $Cb_{\widetilde{U}_{k,l}}(z_k, z_l) \le b_D(z_k, z_l)$, where $\widetilde{U}_{k,l}$ denotes the connected component of $U \cap D$ which contains the curve $\alpha_{k,l}$. Hence, $(z_j)_j$ is even a $b_{\widetilde{U}_{k,l}}$-Cauchy sequence, a contradiction. ∎

REMARK 3.5.15. Let $D \subset \mathbb{C}$ be an unbounded b-complete domain. The following converse to Theorem 3.5.13 is due to N. Nikolov (private communication): For any open disc $U \subset \mathbb{C}$, any connected component of $U \cap D$ (resp. $D \setminus \overline{U}$) is also b-complete.

Moreover, we have the following general result for balanced domains:

THEOREM 3.5.16. *Let $D \subset \mathbb{C}^n$ be a bounded pseudoconvex balanced domain. Then D is b-complete.*

Proof. Recall that any $f \in \mathcal{O}(D)$ can be written as a series $\sum_{k=1}^{\infty} Q_k$, where Q_k are homogeneous polynomials, and the convergence is L^2_{h}-convergence. Therefore, the bounded holomorphic functions on D are dense in $L^2_{\mathrm{h}}(D)$. Then Theorem 3.3.9 and Corollary 3.5.10 finish the proof. ■

$\boxed{?}$ Characterize the b-complete bounded circular pseudoconvex domains. $\boxed{?}$

REMARK 3.5.17. There are also sufficient conditions for Hartogs domains with m-dimensional fibers to be b-complete (see [Jar-Pfl-Zwo 2000]).

THEOREM. *Let $D \subset \mathbb{C}^n$ be a domain and let G_D be a bounded pseudoconvex Hartogs domain with m-dimensional balanced fibers.*

(a) *Assume that D is b-exhaustive, that $\mathcal{H}^\infty(D)$ is dense in $L^2_{\mathrm{h}}(D)$, and that there is an $\varepsilon > 0$ such that $D \times P(0, \varepsilon) \subset G_D$. Then G_D is b-complete.*

(b) *Assume that D is c^i-complete. Then G_D is b-complete.*

For further results on b-complete Hartogs domains see also [Che 2001b]. $\boxed{?}$ Is there a complete characterization of such domains which are b-complete $\boxed{?}$

Moreover, the following result may be found in [Che 2001a].

THEOREM. *Let $u \in \mathcal{PSH}(\mathbb{C}^n)$, $u \not\equiv -\infty$, and $h \in \mathcal{O}(\mathbb{C}^m)$, $h \not\equiv 0$, be such that $u \in \mathcal{C}(\mathbb{C}^n \setminus u^{-1}(-\infty))$. Let $r > 0$ and assume that*

$$\Omega := \{(z', z'') \in \mathbb{B}_n(r) \times \mathbb{B}_m(r) \subset \mathbb{C}^n \times \mathbb{C}^m : u(z') + e^{1/|h(z'')|} < 1\}$$

is a domain. Then Ω is b-complete. If, in addition, there is a point $(z_0', z_0'') \in \mathbb{B}_n(r) \times \mathbb{B}_m(r)$ with $u(z_0') = -\infty$, $h(z_0'') = 0$, then Ω is not hyperconvex.

Observe that the boundary behavior of the level sets of the Green function implies both b-exhaustiveness and b-completeness. We already saw that there exist b-exhaustive domains which are not b-complete. It was a long-standing question whether any b-complete domain was automatically b-exhaustive. The first counterexample was given by W. Zwonek [Zwo 2001a] (see also [Zwo 2002]). The following Theorem 3.5.18 (see [Juc 2004]) gives even a large variety of domains that are b-complete but not b-exhaustive.

THEOREM 3.5.18. *Let $D \subset \mathbb{C}$ be a Zalcman domain as in Corollary 3.3.22. Then*

$$D \text{ is } b\text{-complete} \quad \text{iff} \quad \sum_{k=1}^{\infty} \frac{1}{x_k\sqrt{-\log r_k}} = \infty.$$

Proof. The proof "$\Rightarrow$" is similar to the one of Theorem 3.3.19, so it is omitted here. Proof of "$\Leftarrow$": Suppose that D is not b-complete. Then D is not b-exhaustive. Therefore, in view of Corollary 3.3.22, we have

$$\sum_{k=1}^{\infty} \frac{1}{x_k\sqrt{-\log r_k}} = \infty, \qquad \lim_{j\to\infty} \frac{1}{-x_j^2 \log r_j} = 0. \tag{3.5.18}$$

Moreover, there is a b_D-Cauchy sequence $(z_k)_k \subset D$ with $\lim_{k\to\infty} z_k = 0$. We may even assume that $b_D(z_k, z_{k+1}) < 1/2^k$. So there exist $\mathcal{C}^1$-curves $\gamma_k : [0,1] \to D$ such that $L_{\beta_D}(\gamma_k) < 1/2^k$. Gluing all these curves together we obtain a piecewise-$\mathcal{C}^1$-curve $\gamma : [0,1) \to D$ of finite β_D-length.

We claim that the Bergman kernel remains bounded along γ. In fact, if not then there is a sequence $(w_k)_k \subset \gamma([0,1))$ such that

$$\lim_{k\to\infty} k_D(w_k) = \infty, \qquad \lim_{k\to\infty} w_k = 0.$$

Obviously, $(w_k)_k$ is again a b_D-Cauchy sequence. As in the proof of Theorem 3.5.6, there exist an $f \in L^2_{\mathrm{h}}(D)$ and a subsequence $(w_{k_j})_j$ such that

$$\lim_{j\to\infty} \frac{|f(w_{k_j}|^2}{k_D(w_{k_j})} = 1.$$

Applying Theorem 3.1.2, we find a $g \in L^2_{\mathrm{h}}(D)$, locally bounded near 0, such that $\|f-g\|_{L^2_{\mathrm{h}}(D)} < 1/2$. Therefore,

$$0 \xleftarrow[j\to\infty]{} \frac{|g(w_{k_j})|}{\sqrt{k_D(w_{k_j})}} \geq \frac{|f(w_{k_j})|}{\sqrt{k_D(w_{k_j})}} - \|f-g\|_{L^2_{\mathrm{h}}(D)} \geq \frac{|f(w_{k_j})|}{\sqrt{k_D(w_{k_j})}} - \frac{1}{2} \xrightarrow[j\to\infty]{} \frac{1}{2},$$

a contradiction. Hence, there is a positive C such that $k_D(\gamma(t)) \leq C$, $t \in [0,1)$.

To be able to continue we need the following lemma.

LEMMA 3.5.19. *Let D be a domain as above satisfying* (3.5.18) *and let $\gamma : [0,1) \to D$ be a piecewise-$\mathcal{C}^1$-curve with* $\lim_{t\to1}\gamma(t) = 0$. *Then*

$$\lim_{\tau\to1} \int_0^\tau \sqrt{M_D(\gamma(t);\gamma'(t))}\,dt = \infty.$$

Proof. We may assume that $|\gamma(0)| > x_1$ and that $x_1\sqrt{-\log r_1} < x_j\sqrt{-\log r_j}$, $j \geq j_0$, for a suitable j_0 (use (3.5.18)). Now, fix an $N \in \mathbb{N}$, $N \geq j_0$, and let $z_N \in D$ be an arbitrary point with $x_{N+2} \leq |z_N| \leq x_{N+1}$. We define

$$f := f_{\overline{\mathbb{B}}(x_1,r_1)} - \frac{x_N - z_N}{x_1 - z_N} f_{\overline{\mathbb{B}}(x_N,r_N)},$$

where f_K denotes the Cauchy transform of K. Or more explicitly, we have

$$f(z) = \frac{1}{x_1 - z} - \frac{x_N - z_N}{x_1 - z_N}\frac{1}{x_N - z}, \qquad z \in D.$$

Therefore, we see that

$$f(z_N) = 0, \qquad f'(z_N) = \frac{x_N - x_1}{(x_1 - z_N)^2(x_N - z_N)}.$$

What remains is to estimate the $L^2_{\mathrm{h}}(D)$-norm of the function f. Applying the relation between x_n and x_{n+1}, we get

$$\|f\|_{L^2_{\mathrm{h}}(D)} \leq \|f_{\overline{\mathbb{B}}(x_1,r_1)}\|_{L^2_{\mathrm{h}}(D)} + \frac{|x_N - z_N|}{|x_1 - z_N|}\|f_{\overline{\mathbb{B}}(x_N,r_N)}\|_{L^2_{\mathrm{h}}(D)} \leq C_2 \frac{|x_N - z_N|}{|x_1 - z_N|}\sqrt{-\log r_N},$$

where C_1, C_2 are positive constants, independent of N and z_N.

Therefore, if $x_{N+2} \leq |z| \leq x_{N+1}$ then

$$\sqrt{M_D(z;X)} \geq |X| \frac{|x_1 - x_N|}{C_2|x_1 - \gamma(t)|\,|x_n - \gamma(t)|^2\sqrt{-\log r_N}} \geq |X| \frac{C_3}{x_N^2\sqrt{-\log r_N}},$$

where $C_3 > 0$ is a constant (use again the condition $x_{k+1} \le \Theta_2 x_k$ for all k). Finally, we obtain

$$\lim_{\tau\to 1}\int_0^\tau \sqrt{M_D(\gamma(t);\gamma'(t))}\,dt \ge \sum_{N=j_0}^{\infty} C_3\frac{x_{n+1}-x_{N+2}}{x_N^2\sqrt{-\log r_N}} \ge \sum_{N=j_0}^{\infty} C_3\,\frac{\Theta_1 x_N - \Theta_2^2 x_N}{x_N^2\sqrt{-\log r_N}}$$
$$\ge C_4\sum_{N=j_0}^{\infty}\frac{1}{x_N\sqrt{-\log r_N}} = \infty,$$

where $C_4 > 0$. Hence, the proof of this lemma is complete. ■

Now, applying Lemma 3.5.19 leads to the following contradiction:

$$\infty > \lim_{\tau\to 1}\int_0^\tau \beta_D(\gamma(t);\gamma'(t))\,dt \ge \frac{1}{C}\int_0^\tau \sqrt{M_D(\gamma(t);\gamma'(t))}\,dt = \infty. \ \blacksquare$$

The boundary behavior of the Bergman metric on a Zalcman domain is partially described in the following result, whose proof is based on methods of the proof of Theorem 3.5.18.

THEOREM 3.5.20 ([Juc 2004]). *Let D be a domain as in Theorem* 3.5.18.

(a) *If* $\displaystyle\sum_{k=1}^{\infty}\frac{1}{x_k^2\sqrt{-\log r_k}} < \infty$, *then* $\displaystyle\limsup_{(-1,0)\ni t\to 0}\beta_D(t;1) < \infty$.

(b) *If* $\displaystyle\limsup_{(-1,0)\ni t\to 0}\beta_D(t;1) < \infty$, *then* $\displaystyle\limsup_{k\to\infty}\frac{1}{x_k^2\sqrt{-\log r_k}} < \infty$.

$\boxed{?}$ It seems to be open how to characterize those Zalcman domains that are β-*exhaustive*, i.e. $\lim_{z\to\partial D}\beta_D(z;1) = \infty$. $\boxed{?}$

The b-completeness means heuristically that boundary points are infinitely far away from inner points. So one might think that for a b-complete domain the Bergman metric β_D becomes infinite at the boundary. The following example shows that this is not true.

EXAMPLE 3.5.21. There exists a b-complete bounded domain D in the plane which is not β_D-exhaustive, i.e. there is a boundary sequence $(w_k)_k \subset D$ such that $(\beta_D(w_k;1))_{k\in\mathbb{N}}$ is bounded (cf. [Pfl-Zwo 2003]). To be more precise, put

$$x_n := \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}},\quad z_{n,j} := \exp\left(i\frac{2\pi j}{2^{4n}}\right),\qquad n\in\mathbb{N},\ j=0,\dots,2^{4n}-1.$$

Moreover, let $r_n := \exp(-C_1 2^{9n})$, $n\in\mathbb{N}$, where $C_1 > 0$ is chosen such that

- the discs $\mathbb{B}(z_{n,j},r_n)\subset\mathbb{C}$, $n\in\mathbb{N}$, $j=0,\dots 2^{4n}-1$, are pairwise disjoint,
- $\overline{\mathbb{B}}(z_{n,j},r_n)\subset A_n(0)$, $n\in\mathbb{N}$, $j=0,\dots,2^{4n}-1$.

Then there is a sequence $(n_k)_k\subset\mathbb{N}$ such that

$$D := E\setminus\Big(\bigcup_{k=1}^{\infty}\bigcup_{j=0}^{2^{4n_k}-1}\overline{\mathbb{B}}(zn_k, j, r_n)\Big)$$

is a domain with the desired properties.

3.5.1. Reinhardt domains and b-completeness. Within the class of pseudoconvex Reinhardt domains there is a complete geometric characterization of b-complete domains (see [Zwo 1999b], [Zwo 2000b]).

Let $D \subset \mathbb{C}^n$ be a pseudoconvex Reinhardt domain. Then $\Omega := \Omega_D := \log D$ is a convex domain in $\mathbb{R}^n$. Fix a point $a \in \Omega$. Put

$$\mathfrak{C}(\Omega, a) := \{v \in \mathbb{R}^n : a + \mathbb{R}_+ v \subset \Omega\}.$$

It is easy to see that $\mathfrak{C}(\Omega, a)$ is a closed convex cone with vertex at 0, i.e. $tx \in \mathfrak{C}(\Omega, a)$ for all $x \in \mathfrak{C}(\Omega, a)$ and $t \in \mathbb{R}_+$. Moreover, this cone is independent of the point a, i.e. $\mathfrak{C}(\Omega, a) = \mathfrak{C}(\Omega, b)$, $b \in \Omega$. So we will write briefly $\mathfrak{C}(\Omega) := \mathfrak{C}(\Omega, a)$. Observe that $\mathfrak{C}(\Omega) = \{0\}$ iff $\Omega \subset\subset \mathbb{R}^n$.

We now define

$$\widetilde{\mathfrak{C}}(D) := \{v \in \mathfrak{C}(\Omega_D) : \overline{\exp(a + \mathbb{R}_+ v)} \subset D\}, \qquad \mathfrak{C}'(D) := \mathfrak{C}(\Omega_D) \setminus \widetilde{\mathfrak{C}}(D).$$

Observe that $\widetilde{\mathfrak{C}}(D)$ and $\mathfrak{C}'(D)$ are independent of the point a.

With the help of these geometric notions, there is the following complete description of those bounded Reinhardt domains which are Bergman complete:

THEOREM 3.5.22 ([Zwo 1999b]). *Let $D \subset \mathbb{C}^n$ be a bounded pseudoconvex Reinhardt domain. Then the following conditions are equivalent*:

(i) *D is b-complete*;
(ii) $\mathfrak{C}'(D) \cap \mathbb{Q}^n = \emptyset$.

EXAMPLE 3.5.23. Put

$$D_1 := \{z \in \mathbb{C}^2 : |z_1|^2/2 < |z_2| < 2|z_1|^2,\ |z_1| < 2\}.$$

Obviously, D_1 is a bounded pseudoconvex domain which contains the point $(1, 1)$. Then it turns out that $\mathfrak{C}'(D_1) = \mathbb{R}_{>0}(-1, -2)$, so $\mathfrak{C}'(D_1)$ contains the rational vector $(-1, -2)$.

Using the map

$$\Phi : \mathbb{C}_*^2 \to \mathbb{C}_*^2, \qquad \Phi(z) := (z_1^3 z_2^{-1}, z_1^{-1} z_2), \qquad z = (z_1, z_2),$$

we see that D_1 is biholomorphic to

$$\widetilde{D}_1 := \{z \in \mathbb{C}_*^2 : 1/2 < |z_2| < 2,\ |z_1 z_2| < 2\}.$$

It may be directly seen that $\widetilde{D}_1$, and therefore also D_1, is not b-complete.

On the other hand, let

$$D_2 := \left\{z \in \mathbb{C}^2 : \tfrac{1}{2}|z_1|^{\sqrt{2}} < |z_2| < 2|z_1|^{\sqrt{2}},\ |z_1| < 2\right\}.$$

Again, D_2 is a bounded pseudoconvex Reinhardt domain; now a simple calculation gives $\mathfrak{C}'(D_2) = \mathbb{R}_{>0}(-1, -\sqrt{2})$, i.e. $\mathfrak{C}'(D_2)$ does not contain any rational vector. Hence Theorem 3.5.22 tells us D_2 is b-complete. Recall that D_2 is not hyperconvex.

The next example can be found in [Her 1999]. Let

$$D := \{z \in \mathbb{C}^2 : |z_2|^2 < \exp(-1/|z_1|^2),\ |z_1| < 1\}.$$

Again, D is a bounded pseudoconvex Reinhardt domain. Here we have $\mathfrak{C}(D) = \widetilde{\mathfrak{C}}(D) = \{0\} \times \mathbb{R}_-$ and $\mathfrak{C}'(D) = \emptyset$. So Theorem 3.5.22 entails that D is b-complete (in [Her 1999], a direct proof of this fact is given). Again, observe that D is not hyperconvex.

For the proof of Theorem 3.5.22 we need the following lemma.

LEMMA 3.5.24. *Let $C \subset \mathbb{R}^n$ be a convex closed cone with $C \cap \mathbb{Q}^n = \{0\}$. Assume that C contains no straight lines. Then for any positive δ and any vector $v \in C \setminus \{0\}$ there is a $\beta \in \mathbb{Z}^n$ such that*

$$\langle \beta, v \rangle > 0, \qquad \langle \beta, w \rangle < \delta, \quad w \in C,\ \|w\| = 1.$$

Since this lemma is based on geometric number theory, we omit its proof. For more details, we refer to [Zwo 1999b].

Proof of Theorem 3.5.22. First we are going to verify (i)$\Rightarrow$(ii). Suppose that (ii) does not hold, i.e. there is a non-trivial vector $v \in \mathfrak{C}'(D) \cap \mathbb{Q}^n$. We may assume that $0 \in \log D$, $v = (v_1, \dots, v_n) \in \mathbb{Z}^n_-$, and that $v_1, \dots, v_n$ are relatively prime. It suffices to see that the Bergman length L_{β_D} of the curve $(0,1) \overset{\gamma}{\mapsto} (t^{-v_1}, \dots, t^{-v_n}) \in D$ is finite.

In fact, put $\varphi(\lambda) := (\lambda^{-v_1}, \dots, \lambda^{-v_n})$, $\lambda \in E_*$. Then $\varphi \in \mathcal{O}(E_*, D)$. Now let $u(\lambda) := k_D(\varphi(\lambda))$, $\lambda \in E_*$. To continue we need a part of the following lemma (see [Zwo 2000b]).

LEMMA 3.5.25. *Let $D \subset \mathbb{C}^n$ be a pseudoconvex Reinhardt domain, $\alpha \in \mathbb{Z}^n$, and $p \in (0, \infty)$. Then:*

- *the monomial z^α belongs to $L^p_{\mathrm{h}}(D)$ iff $\langle (p/2)\alpha + \mathbf{1}, v \rangle < 0$ for any $v \in \mathfrak{C}(D) \setminus \{0\}$;*
- *if $\langle \alpha, v \rangle < 0$ for any $v \in \mathfrak{C}(D) \setminus \{0\}$, then $z^\alpha \in \mathcal{H}^\infty(D)$;*
- *if $z^\alpha \in \mathcal{H}^\infty(D)$, then $\langle \alpha, v \rangle \le 0$ for any $v \in \mathfrak{C}(D)$.*

From Lemma 3.5.25 ($p = 2$) it follows that

$$u(\lambda) = \sum_{\alpha \in \mathbb{Z}^n :\, \langle \alpha + \mathbf{1}, v \rangle < 0} a_\alpha |\lambda|^{-2\langle \alpha, v \rangle} = \sum_{j=j_0}^{\infty} b_j |\lambda|^{2j},$$

where $b_{j_0} \neq 0$. Therefore,

$$\beta_D(\varphi(\lambda); \varphi'(\lambda)) = \frac{\partial^2 \log u(\lambda)}{\partial\lambda\partial\overline{\lambda}} = \frac{\partial^2}{\partial\lambda\partial\overline{\lambda}}\Big(\log \sum_{j=j_0}^{\infty} b_j |\lambda|^{2j-2j_0}\Big). \ (^{16})$$

Obviously, the last expression remains bounded along $(0,1)$, which gives the desired contradiction.

Now we turn to the proof of (ii)$\Rightarrow$(i). We start with the following observation. Put $\mathfrak{E} := \operatorname{span}\{z^\alpha : z^\alpha \in L^2_{\mathrm{h}}(D)\}$. Then $\mathfrak{E}$ is a dense subspace of $L^2_{\mathrm{h}}(D)$. In view of Theorem 3.5.6, we only have to show that for any point $z^0 \in \partial D$ and for any sequence $(z^j)_j \subset D$ with $\lim_{j\to\infty} z^j = z_0$ we can find a subsequence $(z^{j_k})_k$ such that

$$\text{(3.5.19)} \qquad \frac{|f(z^{j_k})|}{\sqrt{k_D(z^{j_k})}} \xrightarrow[k\to\infty]{} 0, \qquad f \in \mathfrak{E}.$$

First, we discuss the case when z^0 has the following property: if $z^0_j = 0$ then $V_j \cap D \neq \emptyset$, $j = 1, \dots, n$, where $V_j := \{z \in \mathbb{C}^n : z_j = 0\}$.

Fix an $\alpha \in \mathbb{Z}^n$ such that $z^\alpha \in L^2_{\mathrm{h}}(D)$. Then $\alpha_j \ge 0$ for all j with $z^0_j = 0$. So it suffices to verify that $k_D \to \infty$ as $z \to z^0$. Without loss of generality, we may assume that $z^0_j = 0$, $j = 1, \dots, s$, and $z^0_j \neq 0$, $j = s+1, \dots, n$. Obviously, $s < n$. Then there is an $R > 0$ such that $D \subset \mathbb{B}_s(0,R) \times \widetilde{\pi}_s(D)$, where $\widetilde{\pi}_s := \pi_{s+1,\dots,n}$ denotes the projection of $\mathbb{C}^n$ onto

(16) Observe here that $\log|\lambda|^{2j_0}$ is harmonic on E_*.

$\mathbb{C}^{n-s}$ if $s \geq 1$ or the identity if $s = 0$. Then $\widetilde{\pi}_s(D)$ is a bounded pseudoconvex Reinhardt domain with $\widetilde{\pi}_s(z^0) \in \partial\widetilde{\pi}_s(D)$, where all coordinates of $\widetilde{\pi}_s(z^0)$ are different from zero. Hence, $\widetilde{\pi}_s(D)$ satisfies the general outer cone condition at $\widetilde{\pi}_s(z^0)$. By Theorem 6.1.17 in [J-P 1993], it follows that $\lim_{z''\to\widetilde{\pi}_s(z^0)} k_{\widetilde{\pi}_s(D)}(z'') = \infty$. Using the monotonicity and the product formula of the Bergman kernel, we finally get

$$k_D(z) \geq k_{\mathbb{B}_s(0,R)}(z')k_{\widetilde{\pi}_s(D)}(z'') \xrightarrow[D\ni z=(z',z'')\to z^0]{} \infty.$$

In the remaining part of the proof we assume that there is at least one j such that $z_j^0 = 0$, but $V_j \cap D = \emptyset$. We may also assume that $D \cap V_j \neq \emptyset$, $j = 1,\dots,k$, $D \cap V_j = \emptyset$, $j = k+1,\dots,n$ (17), $z_{k+1}^0 = 0$, and $\mathbf{1} \in D$.

Let $v \in (\mathbb{Q}^n \cap \mathfrak{C}(D)) \setminus \{0\}$. Then, by assumption, we know that $v \in \widetilde{\mathfrak{C}}(D)$. Therefore, $\lim_{t\to\infty} \exp(tv) = w \in D$. So, if $v_j < 0$ then $w_j = 0$, and if $v_j = 0$ then $w_j = 1$. In particular, if there is a $v \in \mathfrak{C}(D) \cap \mathbb{Q}^n$, $v_j < 0$, then $j \leq k$.

Observe that $\mathbb{R}_-^k \times \{0\}^{n-k} \subset \mathfrak{C}(D)$. Now, we claim that $v \notin \mathbb{R}^k \times \mathbb{Q}^{n-k}$ for any $v \in \mathfrak{C}(D) \setminus (\mathbb{R}^k \times \{0\}^{n-k})$. In fact, suppose that $v \in \mathbb{R}^k \times \mathbb{Q}^{n-k}$. So $v_j < 0$ for a some $j > k$. So we may choose a suitable vector $w \in \mathbb{R}_-^k \times \{0\}^{n-k} \subset \mathfrak{C}(D)$ such that $\widetilde{v} := v + w \in \mathfrak{C}(D) \cap \mathbb{Q}^n$ and $\widetilde{v}_j < 0$. Hence, $j \leq k$, a contradiction.

Put $\pi : \mathbb{R}^n \to \mathbb{R}^n$, $\pi(x) := (0,\dots,0,x_{k+1},\dots,x_n)$, where $x = (x_1,\dots,x_n)$. Then $\pi(\mathfrak{C}(D))$ is a closed convex cone in $\{0\}^k \times \mathbb{R}_-^k$. In view of the above property, we conclude that

$$\pi(\mathfrak{C}(D)) \cap (\{0\}^k \times \mathbb{Q}^{n-k}) = \{0\}.$$

Recall that $z_{k+1}^0 = 0$. Now, let $z^j \in D \cap \mathbb{C}_*^n$ be a sequence tending to z^0 (18). Put $x^j := (\log|z_1^j|,\dots,\log|z_n^j|) \in \mathbb{R}^n$. Obviously, $\|x^j\| \to \infty$. Moreover, without loss of generality, we may assume that the sequence $(x^j/\|x^j\|)_j$ converges to a vector $\widetilde{v} \in \mathfrak{C}(D)$.

Fix an $\alpha \in \mathbb{Z}^n$ such that $z^\alpha \in L_{\mathrm{h}}^2(D)$. Then, using Lemma 3.5.25, we conclude that

$$\inf\{-\langle \alpha + \mathbf{1}, w\rangle : w \in \mathfrak{C}(D),\ \|w\| = 1\} =: \delta_0 > 0.$$

Two cases have to be discussed.

CASE 1: $\widetilde{v}_j < 0$ for some $j > k$. Applying Lemma 3.5.24 for $C = \pi(\mathfrak{C}(D))$, $v = \pi(\widetilde{v})$, and δ_0, we get the existence of a $\beta \in \{0\}^k \times \mathbb{Z}^{n-k}$ such that

$$\langle \beta, \widetilde{v}\rangle = \langle \beta, \pi(\widetilde{v})\rangle > 0, \quad \langle \beta, w\rangle = \|\pi(w)\| \left\langle \beta \frac{\pi(w)}{\|\pi(w)\|} \right\rangle < \delta, \quad w \in \mathfrak{C}(D),\ \pi(w) \neq 0.$$

Observe that $\langle \beta, w\rangle = 0$ if $\pi(w) = 0$. Then $z^{\alpha+\beta} \in L_{\mathrm{h}}^2(D)$ (use Lemma 3.5.25) and

$$\frac{|(z^j)^\alpha|}{\sqrt{k_D(z^j)}} \leq \|z^{\alpha+\beta}\|_{L_{\mathrm{h}}^2(D)} \frac{|(z^j)^\alpha|}{|(z^j)^{\alpha+\beta}|} = \|z^{\alpha+\beta}\|_{L_{\mathrm{h}}^2(D)} |(z^j)^{-\beta}| \xrightarrow[j\to\infty]{} 0.$$

Hence, the assumption of Theorem 3.5.6 is satisfied.

CASE 2: $\widetilde{v}_{k+1} = \dots = \widetilde{v}_n = 0$. Recall that $\|\pi(x^j)\| \to \infty$. So we may assume that

$$\frac{\pi(x^j)}{\|\pi(x^j)\|} \to \widetilde{w} = (0,\dots,0,\widetilde{w}_{k+1},\dots,\widetilde{w}_n).$$

(17) Then necessarily $k < n$.

(18) Observe that it suffices to prove (3.5.19) for sequences in $\mathbb{C}_*^n$.

If $\widetilde{w} \in \pi(\mathfrak{C}(D))$, then, by Lemma 3.5.24, there is a $\beta \in \{0\}^k \times \mathbb{Z}^{n-k}$ such that $\langle \beta, \widetilde{w} \rangle > 0$ and $\langle \beta, w \rangle < \delta_0$, $w \in \mathfrak{C}(D) \setminus \{0\}$.

If $\widetilde{w} \notin \pi(\mathfrak{C}(D))$, let $\widetilde{C}$ be the smallest convex closed cone containing $\pi(\mathfrak{C}(D))$ and $-\widetilde{w}$. Then $\widetilde{C} \subset \{0\}^k \times \mathbb{R}^{n-k}$ and $\widetilde{w} \notin \widetilde{C}$. Therefore,

$$\{\widetilde{\beta} \in \{0\}^k \times \mathbb{R}^{n-k} : \langle \widetilde{\beta}, u \rangle < 0,\ u \in \widetilde{C} \setminus \{0\}\}$$

is a non-empty convex open cone (see [Vla 1993, §25]). So it contains a $\beta \in \{0\}^k \times \mathbb{Z}^{n-k}$. Thus, $\langle \beta, -\widetilde{w} \rangle < 0$ and $\langle \beta, w \rangle = \|\pi(w)\| \langle \beta, \pi(w)/\|\pi(w)\| \rangle < 0 < \delta_0$, $w \in \mathfrak{C}(D)$, $\pi(w) \neq 0$.

Now we are able to complete the proof as in case 1 using the β we just constructed. Namely, we conclude that $z^{\alpha+\beta} \in L^2_{\mathrm{h}}(D)$ and

$$\frac{|(z^j)^\alpha|}{\sqrt{k_D(z^j)}} \leq \|z^{\alpha+\beta}\|_{L^2_{\mathrm{h}}(D)} |(z^j)^{-\beta}| = \|z^{\alpha+\beta}\|_{L^2_{\mathrm{h}}(D)} \Big| \prod_{\nu=k+1}^{n} |(z^j_\nu)^{-\beta_\nu}| \xrightarrow[j \to \infty]{} 0.$$

Hence, Theorem 3.5.6 may be applied. ∎

Finally, we will prove the part of Lemma 3.5.25 used during the proof of Theorem 3.5.22.

Proof of Lemma 3.5.25. We restrict ourselves to proving only the following statement (the other ones in Lemma 3.5.25 may be taken as an exercise!):

(†) if D is as in Theorem 3.5.22 (in particular, D is bounded) and if $\langle \alpha + \mathbf{1}, v \rangle < 0$, $v \in \mathfrak{C}(D) \setminus \{0\}$, then $z^\alpha \in L^2_{\mathrm{h}}(D)$.

Assume that $\mathbf{1} \in D$. If $\mathfrak{C}(D) = \{0\}$, then (†) is obvious. So let us assume that $\mathfrak{C}(D) \neq \{0\}$. Then there is a $\delta_0 < 0$ such that $\langle \alpha + \mathbf{1}, v \rangle < \delta_0$, $v \in \mathfrak{C}(D)$, $\|v\| = 1$.

We claim that for any $\varepsilon > 0$ there is a cone T such that $\log D \setminus T$ is bounded and $\|w - v\| < \varepsilon$, $v \in T$, $w \in \mathfrak{C}(D)$, $\|v\| = \|w\| = 1$.

Indeed, fix an $\varepsilon > 0$ and let h be the Minkowski function of the convex set $\log D$. Observe that h is continuous and $h^{-1}(0) = \mathfrak{C}(D)$. Therefore, there is a $\delta > 0$ such that

$$\{w \in \mathbb{R}^n : h(w) \leq \delta,\ \|w\| = 1\} \subset \{w \in \mathbb{R}^n : \|w\| = 1,\ \exists_{v \in \mathfrak{C}(D)} : \|v\| = 1,\ \|v - w\| < \varepsilon\}.$$

Let T be the smallest cone containing $\{w \in \mathbb{R}^n : h(w) \leq \delta,\ \|w\| = 1\}$. Then $\log D \setminus T$ is bounded. Otherwise there would exist an unbounded sequence $x^j \in \log D \setminus T$ such that $h(x^j) < 1$; then $h(x^j/\|x^j\|) < 1/\|x^j\|$, i.e. $x^j \in T$ for large j, a contradiction.

Now observe that $\langle \alpha + \mathbf{1}, v \rangle \leq (\delta_0/2)\|v\|$, $v \in T$, and

$$\int_D |z^\alpha|^2 d\Lambda_{2n}(z) < \infty \quad \text{iff} \quad \int_{\log D} e^{2\langle \alpha+\mathbf{1}, x \rangle}\, d\Lambda_n(x) < \infty \quad \text{iff} \quad \int_T e^{2\langle \alpha+\mathbf{1}, x \rangle}\, d\Lambda_n(x) < \infty.$$

So it remains to estimate the last integral. We get

$$\int_T e^{2\langle \alpha+\mathbf{1}, x \rangle}\, d\Lambda_n(x) \leq \int_T e^{\delta_0 \|x\|}\, d\Lambda_n(x) < \int_{\mathbb{R}^n} e^{\delta_0 \|x\|}\, d\Lambda_n(x) < \infty.$$

Hence, the monomial z^α is in $L^2_{\mathrm{h}}(D)$. ∎

REMARK 3.5.26. $\boxed{?}$ Up to our knowledge, so far there is no complete description of b-complete unbounded Reinhardt domains. $\boxed{?}$

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