# Contents

## Abstract

We consider the motion of an incompressible magnetohydrodynamic (mhd) fluid in a domain bounded by a free surface. In the external domain there exists an electromagnetic field generated by some currents which keeps the mhd flow in the bounded domain. Then on the free surface transmission conditions for electromagnetic fields are imposed. In this paper we prove existence of local regular solutions by the method of successive approximations. The $L_2$ approach is used. This helps us to treat the transmission conditions.

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## 1. Introduction

In this paper we show local existence of solutions to a free boundary problem for the incompressible magnetohydrodynamics. The motion of an incompresible magnetohydrodynamic fluid is considered in a domain $\overset{1}{\Omega}_t \subset \mathbb{R}^3$ bounded by a free surface $S_t$ as being caused by an electromagnetic field in a domain $\overset{2}{\Omega}_t$ exterior to $\overset{1}{\Omega}_t$, generated by some currents located on a fixed external boundary $B$ of $\overset{2}{\Omega}_t$. In $\overset{1}{\Omega}_t$ the motion is described by

$$
\begin{aligned}
&v_t + v\cdot\nabla v - \operatorname{div}\mathbb{T}(v,p) - \operatorname{div}(\mu_1\mathbb{T}(\overset{1}{H})) = f && \text{in } \tilde{\Omega}_1^T,\\
&\operatorname{div} v = 0 && \text{in } \tilde{\Omega}_1^T,\\
&\mu_1(\overset{1}{H}_t + v\cdot\nabla\overset{1}{H} - \overset{1}{H}\cdot\nabla v) - \frac{1}{\sigma_1}\Delta\overset{1}{H} = 0 && \text{in } \tilde{\Omega}_1^T,\\
&\operatorname{div}\overset{1}{H} = 0 && \text{in } \tilde{\Omega}_1^T,
\end{aligned}
\tag{1.1}
$$

where $\tilde{\Omega}_1^T = \bigcup_{0\le t\le T}\overset{1}{\Omega}_t\times\{t\}$, $v = v(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t)) \in \mathbb{R}^3$ is the velocity of the fluid, $p = p(x,t) \in \mathbb{R}$ is the pressure, $x = (x_1, x_2, x_3)$ the global Cartesian coordinates, $\overset{i}{H} = \overset{i}{H}(x,t) = (\overset{i}{H}_1(x,t), \overset{i}{H}_2(x,t), \overset{i}{H}_3(x,t)) \in \mathbb{R}^3$, $i = 1, 2$, is the magnetic field, $f = f(x,t) = (f_1(x,t), f_2(x,t), f_3(x,t)) \in \mathbb{R}^3$ is the external force field, $\mu_1$ is the constant magnetic permeability coefficient and $\sigma_1$ the constant electric conductivity coefficient in $\overset{1}{\Omega}_t$. By $\mathbb{T}(v,p)$ we denote the stress tensor of the form

$$
\mathbb{T}(v,p) = \nu\mathbb{D}(v) - p\mathbb{I},
\tag{1.2}
$$

where $\nu$ is the constant viscosity coefficient, $\mathbb{I}$ is the unit matrix and

$$
\mathbb{D}(v) = \{v_{ix_j} + v_{jx_i}\}_{i,j=1,2,3},
\tag{1.3}
$$

is the dilatation tensor. Moreover, we denote by $\mathbb{T}(\overset{1}{H})$ the stress tensor of the magnetic field described by

$$
\mathbb{T}(\overset{1}{H}) = \left\{\overset{1}{H}_i\overset{1}{H}_j - \frac{\overset{1}{H}{}^2}{2}\delta_{ij}\right\}_{i,j=1,2,3}.
\tag{1.4}
$$

In $\overset{2}{\Omega}_t$ there is no fluid motion but there is a fluid under a constant pressure $p_0$ and the magnetic field is described by the system of equations

$$
\begin{aligned}
&\mu_2\overset{2}{H}{}'_t - \frac{1}{\sigma_2}\Delta\overset{2}{H}{}' = 0 && \text{in } \tilde{\Omega}_2^T,\\
&\operatorname{div}\overset{2}{H}{}' = 0 && \text{in } \tilde{\Omega}_2^T,
\end{aligned}
\tag{1.5}
$$

where $\sigma_2$, $\mu_2$ are the constant electric conductivity and magnetic permeability coefficients in $\overset{2}{\Omega}_t$ and $\tilde{\Omega}_2^T = \bigcup_{0\le t\le T} \overset{2}{\Omega}_t \times \{t\}$. We assume the following initial conditions hold:

$$\begin{aligned} &\overset{i}{\Omega}_t|_{t=0} = \overset{i}{\Omega}, \quad S_t|_{t=0} = S, \quad i = 1,2, \\ &v|_{t=0} = v_0, \quad \overset{1}{H}|_{t=0} = \overset{1}{H}_0 \quad && \text{in } \overset{1}{\Omega}, \\ &\overset{2}{H}'|_{t=0} = \overset{2}{H}'_0 && \text{in } \overset{2}{\Omega}. \end{aligned} \tag{1.6}$$

On $S_t$ we impose the following boundary conditions:

$$\begin{aligned} &\bar{n}\cdot\mathbb{T}(v,p) + \mu_1\bar{n}\cdot\mathbb{T}(\overset{1}{H}) = -p_0\bar{n} \quad && \text{on } S_t, \\ &v\cdot\bar{n} = -\frac{\varphi_t}{|\nabla\varphi|} && \text{on } S_t, \end{aligned} \tag{1.7}$$

where $\bar{n}$ is the unit outward vector normal to $\overset{1}{\Omega}_t$ and $\varphi(x,t) = 0$ describes $S_t$ at least locally. Since the fluid considered is incompressible we can set $p_0 = 0$. This also means that $p - p_0$ is $p$.

Finally, we assume that

$$\overset{2}{H}'|_B = H_*. \tag{1.8}$$

In view of the relation between the current and the magnetic field, for simplicity we assume that the magnetic field is given on $B$.

To derive transmission conditions on the free surface $S_t$ we recall the Maxwell equations in the form appropriate for mhd. In $\overset{1}{\Omega}_t$ the field equations take the form

$$\begin{aligned} &\mu_1\overset{1}{H}_t = -\operatorname{rot}\overset{1}{E}, \\ &\operatorname{rot}\overset{1}{H} = \sigma_1(\overset{1}{E} + \mu_1 v\times\overset{1}{H}), \\ &\operatorname{div}(\mu_1\overset{1}{H}) = 0, \end{aligned} \tag{1.9}$$

and in $\overset{2}{\Omega}_t$,

$$\mu_2\overset{2}{H}'_t = -\operatorname{rot}\overset{2}{E}, \qquad \operatorname{rot}\overset{2}{H}' = \sigma_2\overset{2}{E}, \tag{1.10}$$

where $\overset{1}{E}$, $\overset{2}{E}$ are electric fields. Eliminating the electric fields, equations (1.9) and (1.10) take the form

$$\mu_1\overset{1}{H}_t = -\frac{1}{\sigma_1}\operatorname{rot}\operatorname{rot}\overset{1}{H} + \mu_1\operatorname{rot}(v\times\overset{1}{H}), \tag{1.11}$$

$$\mu_2\overset{2}{H}'_t = -\frac{1}{\sigma_2}\operatorname{rot}\operatorname{rot}\overset{2}{H}'. \tag{1.12}$$

For the divergence free vectors $\overset{i}{H}$ we have

$$\operatorname{rot}\operatorname{rot}\overset{i}{H} = -\Delta\overset{i}{H}, \quad i = 1,2. \tag{1.13}$$

Hence, we obtain equations $(1.1)_{3,4}$ and (1.5).

To obtain energy type estimates we need homogenous boundary conditions on $B$. Therefore, we construct a divergence free extension of $H_*$ denoted by $H'_*$ such that it vanishes in a neighborhood of $S_t$. Then the function

$$\overset{2}{H} = \overset{2}{H}{}' - H'_* \tag{1.14}$$

is a solution to the problem

$$\begin{aligned}
&\mu_2 \overset{2}{H}_t + \frac{1}{\sigma_2}\operatorname{rot}\operatorname{rot}\overset{2}{H} = -\mu_2 H'_{*t} - \frac{1}{\sigma_2}\operatorname{rot}\operatorname{rot} H'_* \equiv \overset{2}{G},\\
&\operatorname{div}\overset{2}{H} = 0,\\
&\overset{2}{H}|_B = 0.
\end{aligned} \tag{1.15}$$

From [5, 3] we have the following transmission conditions on $S_t$:

$$\overset{1}{H}_\tau = \overset{2}{H}_\tau, \quad \mu_1 \overset{1}{H}_n = \mu_2 \overset{2}{H}_n, \quad \mu_1 = \mu_2, \tag{1.16}$$

$$\overset{1}{E}_\tau = \overset{2}{E}_\tau, \tag{1.17}$$

where the subscript $\tau$ means the tangent coordinates to $S_t$ and the subscript $n$ the normal ones.

In view of $(1.9)_2$ and $(1.10)_2$ condition (1.17) takes the form

$$\left(\frac{1}{\sigma_1}\operatorname{rot}\overset{1}{H} - \mu_1 v \times \overset{1}{H}\right)_\tau = \frac{1}{\sigma_2}(\operatorname{rot}\overset{2}{H})_\tau. \tag{1.18}$$

Since the electric currents are determined by the relations

$$j_1 = \sigma_1(\overset{1}{E} + \mu_1 v \times \overset{1}{H}), \qquad j_2 = \sigma_2 \overset{2}{E}, \tag{1.19}$$

continuity of the tangent components of the electric field for different conductivity coefficients means that the tangent components of currents are not continuous across $S_t$. Hence, there are different electric currents on different sides of $S_t$. Summarizing, we formulate our problem in compact form as

$$\begin{aligned}
&v_t + v\cdot\nabla v - \operatorname{div}\mathbb{T}(v,p) - \mu_1 \overset{1}{H}\cdot\nabla\overset{1}{H} + \frac{\mu_1}{2}\nabla\overset{1}{H}{}^2 = f && \text{in } \tilde\Omega_1^T,\\
&\operatorname{div} v = 0 && \text{in } \tilde\Omega_1^T,\\
&\bar n\cdot\mathbb{T}(v,p) + \mu_1\bar n\cdot\mathbb{T}(\overset{1}{H}) = 0 && \text{on } S_t,\\
&\mu_1(\overset{1}{H}_t + v\cdot\nabla\overset{1}{H} - \overset{1}{H}\cdot\nabla v) + \frac{1}{\sigma_1}\operatorname{rot}\operatorname{rot}\overset{1}{H} = 0 && \text{in } \tilde\Omega_1^T,\\
&\operatorname{div}\overset{1}{H} = 0 && \text{in } \tilde\Omega_1^T,\\
&\overset{1}{v}|_{t=0} = \overset{1}{v}(0), \quad \overset{1}{H}|_{t=0} = \overset{1}{H}(0), && \text{in } \overset{1}{\Omega}
\end{aligned} \tag{1.20}$$

and

$$\begin{aligned}
&\mu_2\overset{2}{H}_t + \frac{1}{\sigma_2}\operatorname{rot}\operatorname{rot}\overset{2}{H} = \overset{2}{G} && \text{in } \tilde\Omega_2^T,\\
&\operatorname{div}\overset{2}{H} = 0 && \text{in } \tilde\Omega_2^T,\\
&\overset{2}{H} = 0 && \text{on } B,\\
&\overset{2}{H}|_{t=0} = \overset{2}{H}(0) && \text{in } \overset{2}{\Omega}.
\end{aligned} \tag{1.21}$$

Moreover, on $S_t$ we have the transmission conditions (1.16), (1.18).

The equations of mhd (magnetohydrodynamics) describe mutual interaction between the magnetic field and the fluid. Therefore the motion of a highly conducting fluid under the electromagnetic field (the Lorentz force) and generation of electromagnetic field by the motion of currents are considered. To describe the above interaction in the mhd description we recall the Maxwell equations

$$\begin{aligned} &\operatorname{div} E = \varrho_e,\\ &\operatorname{div} H = 0,\\ &\operatorname{rot} E = -\mu H_t,\\ &\operatorname{rot} H = j + E_t, \end{aligned} \tag{1.22}$$

where $\mu$ is the magnetic permeability, $j$ is the current density, $\varrho_e$ is the charge density. Equations $(1.22)_{1,4}$ imply the charge conservation

$$\varrho_{et} + \operatorname{div} j = 0. \tag{1.23}$$

Moreover, the incompressible and divergence free fluid motion is described by the Navier–Stokes equations with the external electromagnetic force of the form

$$f_{em} = \varrho_e E + j \times H. \tag{1.24}$$

Finally, we have the Ohm law

$$j = \sigma(E + v \times H). \tag{1.25}$$

For simplicity, we assume that $B = H$ and $D = E$, where $B$ and $D$ are, respectively, the magnetic and electric inductions.

In the mhd framework the electric force $\varrho_e E$ is minute by comparison with the Lorentz force, so it is cancelled. Moreover, the displacement currents are negligible, so one sets $\varrho_{et} = 0$. This means that electric currents move in closed circuit. This also means that the concentration of charges does not influence the fluid motion because the charge density is negligible, unlike the current density $j$.

The above properties of conducting fluids imply the following electrodynamic equations used in mhd:

$$\begin{aligned} &\operatorname{rot} H = j, \quad \operatorname{div} j = 0, \quad \mu H_t = -\operatorname{rot} E, \quad \operatorname{div} H = 0,\\ &j = \sigma(E + v \times H), \quad f_{em} = \mu j \times H. \end{aligned} \tag{1.26}$$

Then

$$f_{em} = \mu \operatorname{rot} H \times H = -\mu\bigg(\frac{1}{2}\nabla H^2 - H \cdot \nabla H\bigg). \tag{1.27}$$

Finally, $H$ satisfies

$$H_t = -\frac{1}{\mu}\operatorname{rot}\bigg(\frac{1}{\sigma}\operatorname{rot} H - v \times H\bigg). \tag{1.28}$$

The above description implies problem (1.20) under some changes of constants. The aim of this paper is to prove

THEOREM 1.1. *Assume that* $v_0 \in H^2(\overset{1}{\Omega})$, $\overset{i}{H}_0 \in H^2(\overset{i}{\Omega})$, $i = 1,2$, $\partial_t^k v|_{t=0} \in L_2(\overset{1}{\Omega})$, $\partial_t^k \overset{i}{H}|_{t=0} \in L_2(\overset{i}{\Omega})$, $i = 1,2$, $k \le 2$, $\partial_t^i \bar{f} \in L_2(\overset{1}{\Omega}{}^T)$, $i \le 2$, $\overset{2}{\bar{G}}, \overset{2}{\bar{G}}_t \in L_2(\overset{2}{\Omega}{}^T)$, $S \in H^{5/2}$ *and*

$\int_{\overset{1}{\Omega}} p\,dx = 0$, *where* $\overset{i}{\Omega}{}^T = \overset{i}{\Omega} \times [0,T]$, $i = 1,2$. *Assume the transmission conditions* (1.16), (1.18) *hold on* $S_t$ *and* $\mu_1 = \mu_2$. *Then for* $T$ *sufficiently small there exists a solution to problem* (1.20), (1.21), (1.16), (1.18) *such that*

$$\partial_t^i \bar v \in L_\infty(0,T;H^{2-i}(\overset{1}{\Omega})) \cap L_2(0,T;H^{3-i}(\overset{1}{\Omega})), \qquad i \le 2,$$

$$\partial_t^k \bar{\overset{i}{H}} \in L_\infty(0,T;H^{2-k}(\overset{i}{\Omega})) \cap L_2(0,T;H^{3-k}(\overset{i}{\Omega})) \equiv X(\overset{i}{\Omega}{}^T), \quad k \le 2,\ i = 1,2,$$

*and*

$$\begin{aligned}(1.29)\qquad \|\bar v\|_{X(\overset{1}{\Omega}{}^T)} + \|\bar p\|^2_{L_2(0,T;\Gamma_0^2(\Omega))} + \sum_{i=1}^2 \|\bar{\overset{i}{H}}\|_{X(\overset{i}{\Omega}{}^T)} \\ \le c\Big[\sum_{k=0}^2 (\|\partial_t^k v|_{t=0}\|_{H^{2-k}(\Omega)} + \sum_{i=1}^2 \|\partial_t^k \overset{i}{H}|_{t=0}\|_{H^{2-k}(\overset{i}{\Omega})}) \\ + \|\bar f\|_{L_2(0,T;H^2(\overset{1}{\Omega}))} + \|\bar f_t\|_{L_2(\overset{1}{\Omega}{}^T)} + \|\bar f_{tt}\|_{L_2(\overset{1}{\Omega}{}^T)} \\ + \|\bar{\overset{2}{G}}\|_{L_2(0,T;H^2(\overset{2}{\Omega}))} + \|\bar{\overset{2}{G}}_t\|_{L_2(\overset{2}{\Omega}{}^T)} + \|\bar{\overset{2}{G}}_{tt}\|_{L_2(\overset{2}{\Omega}{}^T)}\Big] \equiv \bar D,\end{aligned}$$

*where* $\bar{\overset{2}{G}}$ *is given by* (1.15) *and* $\bar v$, $\bar H$, $\bar p$ *are expressed in lagrangian coordinates* (*see* (2.2)).

Theorem 1.1 follows from Lemmas 7.1 and 8.1. Moreover, for $\bar D$ small, the existence time $T$ can be chosen large (see Remark 7.2).

The paper is organized as follows. In Section 2 the notation is introduced. Moreover, the lagrangian coordinates are recalled and the Navier–Stokes equations are formulated in these coordinates. Finally, in Remark 2.4, the relations between the norms used in this paper, expressed in both the lagrangian and eulerian coordinates, are proved. In Section 3, by the Galerkin method, existence of solutions to the linearized problem (1.20)–(1.21) is proved. Then we prove local existence of solutions to problem (1.20)–(1.21) by the method of successive approximations in Sections 4–8. In Sections 4, 5 we derive appropriate estimates for velocity using the formulation of the Navier–Stokes equations in lagrangian coordinates. In Section 6 similar estimates are shown for the magnetic field. In this case we use eulerian coordinates because in these coordinates the transmission conditions are much simpler. For sequences constructed by the method of successive approximations, boundedness is shown in Section 7 and convergence is shown in Section 8.

## 2. Notation and auxiliary results

First we introduce the notation employed in this paper. We do not distinguish between norms of scalar and vector-valued functions. Let $\omega$ be a vector, $\omega = (\omega_1,\dots,\omega_n)$. Then

$$|\omega| = \Big(\sum_{i=1}^n |\omega_i|^2\Big)^{1/2}.$$

Let $L_p(\Omega) = \{u : \int_\Omega |u|^p\,dx < \infty\}$, $p \in [1,\infty]$. The space of functions with the norm

$$\|u\|_{V_2^0(\Omega^T)} = \|u\|_{L_\infty(0,T;L_2(\Omega))} + \|u\|_{L_2(0,T;H^1(\Omega))}$$

is denoted by $V_2^0(\Omega^T)$. We shall use the notation

$$\|u\|_{\Gamma_k^l(\Omega)} = \sum_{i\le l-k} \|\partial_t^i u\|_{H^{l-i}(\Omega)}, \quad l,k\in\mathbb{N},$$

where $H^l(\Omega) = \big\{u : \sum_{|\alpha|\le l} \|D_x^\alpha u\|_{L_2(\Omega)} < \infty\big\}$ and

$$\|u\|_{\Gamma_{k,r}^l(\Omega^T)} = \|u\|_{L_r(0,T;\Gamma_k^l(\Omega))}.$$

Let

$$\|u\|_{L_p^k(\Omega)} = \sum_{|\alpha|=k} \|D_x^\alpha u\|_{L_p(\Omega)}, \quad \|u\|_{L_2^1(\Omega^t)} = \|u_x\|_{L_2(\Omega^t)} + \|u_t\|_{L_2(\Omega^t)},$$

where $\alpha = (\alpha_1,\alpha_2,\alpha_3)$ and $|\alpha| = \alpha_1+\alpha_2+\alpha_3$, $\alpha_i\in\mathbb{N}_0\equiv\mathbb{N}\cup\{0\}$, $i=1,2,3$, $D_x^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$.

We denote by $c$ a generic constant which changes its value from formula to formula. Similarly we denote by $\varphi$ a generic function which is always positive and increasing.

To examine free boundary problems in hydrodynamics we use lagrangian coordinates which are the initial data to the following Cauchy problem:

$$\frac{dx}{dt} = v(x,t), \quad x|_{t=0} = \xi\in\overset{1}{\Omega}. \tag{2.1}$$

Therefore,

$$x_v(\xi,t) = \xi + \int_0^t \bar v(\xi,s)\,ds, \tag{2.2}$$

where $\bar v(\xi,t) = v(x_v(\xi,t),t)$. To define lagrangian coordinates in $\overset{2}{\Omega}_t$ we need

LEMMA 2.1 ([9]). *Let $X(\overset{1}{\Omega}_t)$ be some Sobolev space. Let $v\in X(\overset{1}{\Omega}_t)$ be divergence free. Then there exists an extension $v'$ of $v$ on $\overset{1}{\Omega}_t\cup\overset{2}{\Omega}_t$ such that $v'$ is divergence free, $v'|_{\overset{1}{\Omega}_t} = v$ and there exists a constant $c$ such that*

$$\|v'\|_{X(\overset{1}{\Omega}_t\cup\overset{2}{\Omega}_t)} \le c\|v\|_{X(\overset{1}{\Omega}_t)}. \tag{2.3}$$

In view of the definition of lagrangian coordinates we have

$$\begin{aligned}
&\overset{1}{\Omega}_t = \{x\in\mathbb{R}^3 : x = x_v(\xi,t), \xi\in\overset{1}{\Omega}\}, \qquad S_t = \{x\in\mathbb{R}^3 : x = x_v(\xi,t),\xi\in S\},\\
&\overset{1}{\Omega}_t\cup\overset{2}{\Omega}_t = \{x\in\mathbb{R}^3 : x = x_{v'}(\xi,t), \xi\in\overset{1}{\Omega}\cup\overset{2}{\Omega}\}.
\end{aligned}$$

To formulate our problem in lagrangian coordinates we need the notation

$$\begin{aligned}
&\nabla_{\bar v} = \frac{\partial\xi_k}{\partial x}\frac{\partial}{\partial\xi_k}, \qquad \mathbb{D}_{\bar v}\bar u = \nabla_{\bar v}\bar u + (\nabla_{\bar v}\bar u)^T,\\
&\mathbb{T}_{\bar v}(\bar u,\bar p) = \mathbb{D}_{\bar v}(\bar u) - \bar p\mathbb{I}, \quad \operatorname{div}_{\bar v}\bar v = \partial_{x_i}\xi_k\partial_{\xi_k}\bar v_i = \nabla_{\bar v}\cdot\bar v,
\end{aligned}\tag{2.4}$$

where summation over repeated indices is assumed, $\xi=\xi(x,t)$ is the inverse transformation to $x = x_{\bar v}(\xi,t)$. From [11, 10] we have

LEMMA 2.2. *Let $\Omega\subset\mathbb{R}^3$ be a given bounded domain. Let $v\in L_2(\Omega)$ be such that*

$$E_\Omega(v) = \int_\Omega (v_{jx_i} + v_{ix_j})^2\,dx. \tag{2.5}$$

*Then there exists a constant* $c$ *such that*

$$\|v\|^2_{H^1(\Omega)} \le c(E_\Omega(v) + \|v\|^2_{L_2(\Omega)}). \tag{2.6}$$

LEMMA 2.3. *Let* (2.2) *describe the relation between the eulerian* $x$ *and the lagrangian* $\xi$ *coordinates. Then*

$$|x_\xi - I| \le \Big|\int_0^t \bar{v}_\xi(\xi,s)\,ds\Big|, \tag{2.7}$$

$$|\xi_x| \le \exp\int_0^t |\bar{v}_\xi(\xi,s)|\,ds. \tag{2.8}$$

*Proof.* Expressing (2.2) in the form that

$$x = \xi + \int_0^t \bar{v}(\xi,s)\,ds, \tag{2.9}$$

we see that (2.7) is obvious. To show (2.8) we obtain from (2.9) the relation

$$\xi_x = I - \int_0^t \bar{v}_\xi(\xi,s)\xi_x\,ds,$$

so

$$\frac{d}{dt}\xi_x = -\bar{v}_\xi\xi_x. \tag{2.10}$$

From (2.10) it follows that

$$\frac{1}{2}\frac{d}{dt}\ln|\xi_x|^2 \le |\bar{v}_\xi|, \tag{2.11}$$

so (2.8) holds. ∎

To show higher space regularity of velocity we express equations (1.1) in lagrangian coordinates

$$\begin{aligned} &\bar{v}_t - \nabla_{\bar{v}}\cdot\mathbb{T}_{\bar{v}}(\bar{v},\bar{p}) = \bar{f} - \nabla_{\bar{v}}\bar{\S},\\ &\operatorname{div}_{\bar{v}}\bar{v} = 0,\\ &\bar{n}_{\bar{v}}\cdot\mathbb{T}_{\bar{v}}(\bar{v},\bar{p}) = -\mu_1\bar{n}_{\bar{v}}\cdot\bar{\S},\\ &\bar{v}|_{t=0} = v_0, \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} &\S = \S(H) = \{S_{ij}(H)\} = \big\{H_iH_j - \tfrac{1}{2}H^2\delta_{ij}\big\}_{i,j=1,2,3},\\ &\bar{\S} = \S(\bar{H}), \quad \bar{n}_{\bar{v}}(\xi,t) = \bar{n}(x(\xi,t),t). \end{aligned} \tag{2.13}$$

Continuing, we linearize (2.12) in the form

$$\begin{aligned} &\bar{v}_t - \nabla_\xi\cdot\mathbb{T}_\xi(\bar{v},\bar{p}) = -(\nabla^2_\xi - \nabla^2_{\bar{u}})\bar{v} + (\nabla_\xi - \nabla_{\bar{u}})\bar{p} + \bar{f} - \nabla_{\bar{u}}\bar{\S} \equiv F,\\ &\operatorname{div}_\xi\bar{v} = \operatorname{div}_\xi\bar{v} - \operatorname{div}_{\bar{u}}\bar{v} \equiv g,\\ &\bar{n}_\xi\cdot\mathbb{T}_\xi(\bar{v},\bar{p}) = \bar{n}_\xi\cdot\mathbb{T}_\xi(\bar{v},\bar{p}) - \bar{n}_{\bar{u}}\cdot\mathbb{T}_{\bar{u}}(\bar{v},\bar{p}) - \bar{n}_{\bar{u}}\cdot\bar{\S} \equiv h,\\ &\bar{v}|_{t=0} = v_0, \end{aligned} \tag{2.14}$$

where $\bar{n}_\xi$ is the unit outward vector normal to $S = \partial\Omega$, $T_\xi(\bar{v},\bar{p}) = \mathbb{D}_\xi(\bar{v}) - \bar{p}\mathbb{I}$ and $\bar{u}$ is a given function. Finally,

$$\mathbb{D}_\xi(\bar{v}) = \nabla_\xi \bar{v} + (\nabla_\xi \bar{v})^T.$$

To show higher regularity with respect to space variables we need local considerations. For this purpose we introduce a partition of unity $\{\overset{i}{\tilde{\Omega}}_k, \overset{i}{\zeta}_k\}$, $\overset{i}{\Omega} = \bigcup_k \overset{i}{\tilde{\Omega}}_k$, $\operatorname{supp} \overset{i}{\zeta}_k \subset \overset{i}{\tilde{\Omega}}_k$, $i = 1, 2$.

Let $\overset{i}{\tilde{\Omega}}$ be one of the $\overset{i}{\tilde{\Omega}}_k$ and $\overset{i}{\zeta}(\xi) = \overset{i}{\zeta}_k(\xi)$ the corresponding function. If $\overset{i}{\tilde{\Omega}}$ is an interior subdomain we introduce $\tilde{\omega}^i$ such that $\bar{\tilde{\omega}}^i \subset \overset{i}{\tilde{\Omega}}$ and $\overset{i}{\zeta}(\xi) = 1$ for $\xi \in \tilde{\omega}^i$, $i = 1, 2$. Otherwise we assume that $\overset{i}{\bar{\tilde{\Omega}}} \cap S \neq \emptyset$, $\bar{\tilde{\omega}}^i \cap S \neq \emptyset$, $\bar{\tilde{\omega}}^i \subset \overset{i}{\bar{\tilde{\Omega}}}$, $i = 1, 2$. In this case we introduce a local coordinate system $y = (y_1, y_2, y_3)$ with origin in the middle of $\tilde{S} = S \cap \overset{i}{\bar{\tilde{\Omega}}}$ obtained from the lagrangian coordinates by translation and rotation. In these local coordinates we can express $\tilde{S}$ by $y_3 = F(y_1, y_2)$ with the $y_3$ axis directed into $\overset{1}{\Omega}$. Then

$$(2.15)\qquad \begin{aligned} \overset{1}{\tilde{\Omega}} &= \{y : |y'| < \lambda,\ F(y') < y_3 < F(y') + \lambda,\ y' = (y_1, y_2)\}, \\ \overset{2}{\tilde{\Omega}} &= \{y : |y'| < \lambda,\ F(y') - \lambda < y_3 < F(y')\}. \end{aligned}$$

Having coordinates $y = (y_1, y_2, y_3)$ we introduce a new system of coordinates $z = z(y)$ by the relations

$$(2.16)\qquad z_i = y_i, \quad i = 1, 2, \qquad z_3 = y_3 - F(y').$$

In these coordinates $\tilde{S}$ is described by the equation $z_3 = 0$. Let

$$(2.17)\qquad z = \Phi(\xi) \equiv \Psi(y).$$

In view of the above transformation we have

$$(2.18)\qquad \overset{i}{\hat{\Omega}} = \Phi(\overset{i}{\Omega}), \quad i = 1, 2,$$

where by (2.15) we derive

$$(2.19)\qquad \overset{1}{\hat{\Omega}} = \{z : |z'| < \lambda,\ 0 < z_3 < \lambda\}, \qquad \overset{2}{\hat{\Omega}} = \{z : |z'| < \lambda,\ -\lambda < z_3 < 0\},$$

where $z' = (z_1, z_2)$.

Let $\bar{u} = \bar{u}(\xi, t)$ be any vector in lagrangian coordinates. Then

$$(2.20)\qquad \hat{u}(z,t) = \bar{u}(\Phi^{-1}(z), t), \qquad \tilde{u}(z,t) = \hat{u}(z,t)\hat{\zeta}(z).$$

For the interior subdomain $\overset{i}{\tilde{\Omega}}$ we have

$$(2.21)\qquad \tilde{u}(\xi,t) = \bar{u}(\xi,t)\zeta(\xi).$$

Moreover, we introduce

$$(2.22)\qquad \hat{\nabla}_j = \frac{\partial z_i}{\partial \xi_j} \nabla_{z_i}.$$

To derive the space regularity of weak solutions we restrict our considerations to neighborhoods of $S$, because proofs of estimates in interior subdomains are simpler. Necessary estimates in the whole domains $\overset{1}{\Omega}$ and $\overset{2}{\Omega}$ follow from the properties of the partition of unity.

Next, we transform problem (2.14) to the local coordinates $z$ and localize it to $\operatorname{supp}\overset{1}{\zeta}$. Then problem (2.14) takes the form

$$
\begin{aligned}
&\tilde{v}_t - \nu \operatorname{div}_z \mathbb{D}_z(\tilde{v}) + \nabla_z \tilde{p} = \tilde{F} + (\nabla_z - \hat{\nabla})\tilde{p} + \hat{\nabla}\hat{\overset{1}{\zeta}}\hat{p} \\
&\qquad - \nu(\nabla_z^2 - \hat{\nabla}^2)\tilde{v} - \nu(2\hat{\nabla}\hat{\overset{1}{\zeta}}\hat{\nabla}\hat{v} + \hat{v}\hat{\nabla}^2\hat{\overset{1}{\zeta}}\hat{v}) \equiv F_0, \\
&\nabla_z \cdot \tilde{v} = \nabla_z \cdot \tilde{v} - \hat{\nabla} \cdot \tilde{v} + \hat{\nabla}\hat{\overset{1}{\zeta}}\hat{v} + \tilde{g} \equiv g_0, \\
&\bar{n}_z \cdot \mathbb{T}_z(\tilde{v}, \tilde{p}) = \bar{n}_z \cdot \mathbb{T}_z(\tilde{v}, \tilde{p}) - \hat{n} \cdot \hat{\mathbb{T}}(\tilde{v}, \tilde{p}) + \hat{n} \cdot \hat{B}(\hat{\overset{1}{\zeta}}) \cdot \hat{v} + \tilde{h} \equiv h_0, \\
&\tilde{v}|_{t=0} = \tilde{v}_0,
\end{aligned}
\tag{2.23}
$$

where $\hat{B}(\hat{\overset{1}{\zeta}}) = \{\hat{\nabla}_i\hat{\overset{1}{\zeta}}\delta_{jk} + \hat{\nabla}_j\hat{\overset{1}{\zeta}}\delta_{ik}\}_{i,j=1,2,3}$.

Finally, we formulate results describing relations between eulerian and lagrangian coordinates.

REMARK 2.4. Let $x = \xi + \int_0^t \bar{v}(\xi, t')\,dt' \equiv x(\xi, t)$ and $\xi = x - \int_0^t v(x, t')\,dt' \equiv \xi(x, t)$. Let $\alpha(\bar{v}) = t^{1/2}\|\bar{v}\|_{L_2(0,t;H^3(\Omega))}$. Let $\varphi$ be some increasing positive function. Let $\omega$ be any function. Then

$$
\omega(x, t) = \bar{\omega}(\xi(x, t), t), \quad \bar{\omega}(\xi, t) = \omega(x(\xi, t), t).
$$

First we have

$$
\begin{aligned}
&\int_\Omega \bar{\omega}_\xi^2\,d\xi = \int_\Omega \omega_x^2(x(\xi, t), t)x_\xi^2\,d\xi \le \|x_\xi\|_{L_\infty(\Omega)}^2 \int_{\Omega_t} \omega_x^2\,dx, \\
&\int_{\Omega_t} \omega_x^2\,dx = \int_{\Omega_t} \bar{\omega}_\xi^2(\xi(x, t), t)\xi_x^2\,dx \le \|\xi_x\|_{L_\infty(\Omega)}^2 \int_\Omega \bar{\omega}_\xi^2\,d\xi,
\end{aligned}
\tag{2.24}
$$

where

$$
x_\xi = \delta + \int_0^t \bar{v}_\xi(\xi, t')\,dt', \quad \text{so} \quad \|x_\xi\|_{L_\infty(\Omega)} \le c(1 + \alpha(\bar{v}))
$$

and

$$
\xi_x = \delta - \int_0^t \bar{v}_\xi(\xi, t')\xi_x\,dt', \quad \text{so} \quad \frac{d}{dt}\xi_x = -\bar{v}_\xi\xi_x,
$$

where $\delta$ is the unit matrix.

Then

$$
\xi_x^2 \le \exp\int_0^t |\bar{v}_\xi(\xi, t')|\,dt' \le \exp c\alpha(\bar{v}).
$$

Next we calculate

$$
\bar{\omega}_{\xi\xi} = \omega_{xx}x_\xi^2 + \omega_x x_{\xi\xi}.
$$

Hence

$$
\|\bar{\omega}_{\xi\xi}\|_{L_2(\Omega)}^2 \le c\|\omega\|_{H^2(\Omega_t)}^2\varphi(\alpha(\bar{v})),
\tag{2.25}
$$

because
$$\|x_{\xi\xi}\|_{L_4(\Omega)} \le c\int_0^t \|\bar v_{\xi\xi}(t')\|_{H^1(\Omega)}\,dt' \le c\alpha(\bar v).$$
Moreover
$$\omega_{xx} = \bar\omega_{\xi\xi}\xi_x^2 + \bar\omega_\xi\xi_{xx} \quad \text{and} \quad \frac{d}{dt}\xi_{xx} = -(\bar v_{\xi\xi}\xi_x^2 + \bar v_\xi\xi_{xx}).$$
Hence
$$\|\xi_{xx}\|_{L_4(\Omega)} \le \exp\Big(\int_0^t |\bar v_\xi|\,dt'\Big)\int_0^t \|\bar v_{\xi\xi}\|_{L_4(\Omega)}\,dt' \sup_t |\xi_x| \le \varphi(\alpha(\bar v)).$$
Therefore, we have
$$\|\omega_{xx}\|^2_{L_2(\Omega_t)} \le c\|\bar\omega\|^2_{H^2(\Omega)}\varphi(\alpha(\bar v)). \tag{2.26}$$
Next, we calculate
$$\bar\omega_{\xi\xi\xi} = \omega_{xxx}x_\xi^3 + 2\omega_{xx}x_\xi x_{\xi\xi} + \omega_x x_{\xi\xi\xi},$$
where $x_{\xi\xi\xi} = \int_0^t \bar v_{\xi\xi\xi}\,dt'$.

We have
$$\|\bar\omega_{\xi\xi\xi}\|_{L_2(\Omega)} \le \|\omega\|_{H^3(\Omega_t)}\varphi(\alpha(\bar v)) \tag{2.27}$$
and for the inverse transformation
$$\|\omega_{xxx}\|_{L_2(\Omega_t)} \le \|\bar\omega\|_{H^3(\Omega)}\varphi(\alpha(\bar v)). \tag{2.28}$$
Next,
$$\begin{aligned}\bar\omega_t(\xi,t) &= \omega_x(x(\xi,t),t)v(x(\xi,t),t) + \omega_t(x,t),\\ \omega_t(x,t) &= \bar\omega_\xi(\xi(x,t),t)\xi_t + \bar\omega_t(\xi,t),\\ \xi_t &= -\bar v(\xi,t).\end{aligned}$$
Then
$$\begin{aligned}\|\bar\omega_t\|_{L_2(\Omega)} &\le c\|\omega_x\|_{L_2(\Omega_t)}\|v\|_{H^2(\Omega_t)} + \|\omega_t\|_{L_2(\Omega_t)},\\ \|\omega_t\|_{L_2(\Omega_t)} &\le c\|\bar\omega_\xi\|_{L_2(\Omega)}\|\bar v\|_{H^2(\Omega)} + \|\bar\omega_t\|_{L_2(\Omega)}.\end{aligned} \tag{2.29}$$
Next, we calculate the second time derivatives:
$$\bar\omega_{tt} = \omega_{xx}v^2 + 2\omega_{xt}v + \omega_x(v_xv + v_t) + \omega_{tt}, \qquad \omega_{tt} = \bar\omega_{\xi\xi}\xi_t^2 + 2\bar\omega_{\xi t}\xi_t + \bar\omega_\xi\xi_{tt} + \bar\omega_{tt}.$$
Hence, we have
$$\begin{aligned}\|\bar\omega_{tt}\|_{L_2(\Omega)} &\le c\|\omega\|_{H^2(\Omega_t)}(\|v\|^2_{H^2(\Omega_t)} + \|v_t\|_{H^1(\Omega_t)})\\ &\quad + c\|\omega_{xt}\|_{L_2(\Omega_t)}\|v\|_{H^2(\Omega_t)} + \|\omega_{tt}\|_{L_2(\Omega_t)},\\ \|\omega_{tt}\|_{L_2(\Omega_t)} &\le c\|\bar\omega\|_{H^2(\Omega)}(\|\bar v\|^2_{H^2(\Omega)} + \|\bar v_t\|_{H^1(\Omega)})\\ &\quad + c\|\bar\omega_{\xi t}\|_{L_2(\Omega)}\|\bar v\|_{H^2(\Omega)} + \|\bar\omega_{tt}\|_{L_2(\Omega)}.\end{aligned} \tag{2.30}$$
Next we calculate the third derivatives:
$$\begin{aligned}\bar\omega_{tt\xi} &= \omega_{xxx}v^2x_\xi + 2\omega_{xx}vv_xx_\xi + 2\omega_{xxt}vx_\xi + \omega_{xt}v_xx_\xi\\ &\quad + \omega_{xx}(v_xv + v_t)x_\xi + \omega_x(v_xv + v_t)_xx_\xi + \omega_{ttx}x_\xi.\end{aligned}$$

Then

$$\|\bar{\omega}_{tt\xi}\|_{L_2(\Omega)} \le c[\|\omega\|_{H^3(\Omega_t)}(\|v\|^2_{H^2(\Omega_t)} + \|v_t\|_{H^1(\Omega_t)}) + \|\omega_t\|_{H^2(\Omega_t)}\|v\|_{H^2(\Omega_t)} + \|\omega_{ttx}\|_{L_2(\Omega_t)}]\varphi(\alpha(\bar{v})). \tag{2.31}$$

Similarly,

$$\omega_{ttx} = \bar{\omega}_{\xi\xi\xi}\xi_x\xi_t^2 + 2\bar{\omega}_{\xi\xi}\xi_{xt}\xi_t + \bar{\omega}_{\xi\xi t}\xi_x\xi_t + \bar{\omega}_{\xi t}\xi_{xt} + \bar{\omega}_{\xi\xi}\xi_x\xi_{tt} + \bar{\omega}_\xi\xi_{ttx} + \bar{\omega}_{tt\xi}\xi_x,$$

where $\xi_{tt} = -\bar{v}_t$, $\xi_{ttx} = -\bar{v}_{\xi t}\xi_x$. Hence

$$\|\omega_{ttx}\|_{L_2(\Omega_t)} \le c[\|\bar{\omega}\|_{H^3(\Omega)}(\|\bar{v}\|^2_{H^2(\Omega)} + \|\bar{v}_t\|_{H^1(\Omega)}) + \|\bar{\omega}_t\|_{H^2(\Omega)}\|\bar{v}\|_{H^2(\Omega)} + \|\bar{\omega}_{tt\xi}\|_{L_2(\Omega)}]\varphi(\alpha(\bar{v})). \tag{2.32}$$

Consider the relations

$$\begin{aligned}
\bar{\omega}_{t\xi} &= \omega_{xx}(x(\xi,t),t)v(x(\xi,t),t)x_\xi + \omega_x(x(\xi,t),t)v_x(x(\xi,t),t)x_\xi + \omega_{tx}(x(\xi,t),t)x_\xi,\\
\omega_{tx} &= \bar{\omega}_{\xi\xi}(\xi(x,t),t)\bar{v}(\xi(x,t),t)\xi_x + \bar{\omega}_\xi\bar{v}_\xi\xi_x + \bar{\omega}_{t\xi}\xi_x.
\end{aligned}$$

Then

$$\|\bar{\omega}_{t\xi}\|_{L_2(\Omega)} \le c(\|\omega_{xx}\|_{L_2(\Omega_t)}\|v\|_{H^2(\Omega_t)} + \|\omega_x\|_{H^1(\Omega_t)}\|v_x\|_{H^1(\Omega_t)} + \|\omega_{tx}\|_{L_2(\Omega_t)})\alpha(\bar{v}) \tag{2.33}$$

and

$$\|\omega_{tx}\|_{L_2(\Omega_t)} \le c(\|\bar{\omega}_{\xi\xi}\|_{L_2(\Omega)}\|\bar{v}\|_{H^2(\Omega)} + \|\bar{\omega}_\xi\|_{H^1(\Omega)}\|\bar{v}_\xi\|_{H^1(\Omega)} + \|\bar{\omega}_{t\xi}\|_{L_2(\Omega)})\varphi(\alpha(\bar{v})). \tag{2.34}$$

Finally, we calculate

$$\begin{aligned}
\bar{\omega}_{t\xi\xi} &= \omega_{xxx}x_\xi^2 v + \omega_{xx}x_{\xi\xi}v + 2\omega_{xx}x_\xi^2 v_x + \omega_x v_{xx}x_\xi^2 + \omega_x v_x x_{\xi\xi} + \omega_{txx}x_\xi^2 + \omega_{tx}x_{\xi\xi},\\
\omega_{txx} &= \bar{\omega}_{\xi\xi\xi}\xi_x^2\bar{v} + \bar{\omega}_{\xi\xi}\xi_{xx}\bar{v} + 2\omega_{\xi\xi}\xi_x^2\bar{v}_\xi + \bar{\omega}_\xi\bar{v}_{\xi\xi}\xi_x^2 + \bar{\omega}_\xi\bar{v}_\xi\xi_{xx} + \bar{\omega}_{t\xi\xi}\xi_x^2 + \bar{\omega}_{t\xi}\xi_{xx}.
\end{aligned}$$

Then we have

$$\|\bar{\omega}_{t\xi\xi}\|_{L_2(\Omega)} \le c(\|\omega\|_{H^3(\Omega_t)}\|v\|_{H^2(\Omega_t)} + \|\omega_{tx}\|_{H^1(\Omega_t)})\varphi(\alpha(\bar{v})), \tag{2.35}$$

$$\|\omega_{txx}\|_{L_2(\Omega_t)} \le c(\|\bar{\omega}\|_{H^3(\Omega)}\|\bar{v}\|_{H^2(\Omega)} + \|\bar{\omega}_{t\xi}\|_{H^1(\Omega)})\varphi(\alpha(\bar{v})). \tag{2.36}$$

Consider the problem

$$\begin{aligned}
&\operatorname{rot}\overset{i}{H} = \overset{i}{g} && \text{in } \overset{i}{\Omega}_t,\ i=1,2,\\
&\operatorname{div}(\mu_i\overset{i}{H}) = 0 && \text{in } \overset{i}{\Omega}_t,\ i=1,2,\\
&\overset{1}{H}_\tau = \overset{2}{H}_\tau,\ \mu_1\overset{1}{H}_n = \mu_2\overset{2}{H}_n && \text{on } S_t,\\
&\overset{2}{H}|_B = 0,
\end{aligned} \tag{2.37}$$

where $t$ is fixed. From [5, Theorem 6] we have

LEMMA 2.5. *Assume that* $\overset{i}{g} \in L_2(\overset{i}{\Omega}_t)$, $i = 1, 2$. *Then there exists a solution to problem* (2.37) *such that* $\overset{i}{H} \in \overset{1}{H}(\overset{i}{\Omega}_t)$, $i = 1, 2$, *and*

$$\sum_{i=1}^{2} \|\overset{i}{H}\|_{H^1(\overset{i}{\Omega}_t)} \le c \sum_{i=1}^{2} \|\overset{i}{g}\|_{L_2(\overset{i}{\Omega}_t)}. \tag{2.38}$$

## 3. Existence for the linearized problem

To prove existence of solutions to problem (1.16), (1.18) (1.20), (1.21) we formulate it in the form of integral identities. This approach implies a natural treatment of the transmission problem for the electromagnetic field. Moreover, the formulation restricts our considerations to the $L_2$-approach.

Now we present an integral formulation of problem (1.20)–(1.21). Multiplying $(1.20)_1$ by a divergence free function $\varphi$, integrating the result over $\overset{1}{\Omega}_t$ and using the boundary conditions $(1.20)_3$ we get

$$\int_{\overset{1}{\Omega}_t} (v_t + v \cdot \nabla v)\varphi \, dx + \int_{\overset{1}{\Omega}_t} \mathbb{D}(v) \cdot \mathbb{D}(\varphi) \, dx = \int_{\overset{1}{\Omega}_t} f \cdot \varphi \, dx - \mu_1 \int_{\overset{1}{\Omega}_t} \mathbb{T}(\overset{1}{H}) \cdot \mathbb{D}(\varphi) \, dx. \tag{3.1}$$

Multiplying $(1.20)_4$ by a function $\overset{1}{\varphi}$ and integrating over $\overset{1}{\Omega}_t$ we obtain

$$\mu_1 \int_{\overset{1}{\Omega}_t} (\overset{1}{H}_t + v \cdot \nabla \overset{1}{H}) \cdot \overset{1}{\varphi} \, dx - \mu_1 \int_{\overset{1}{\Omega}_t} \overset{1}{H} \cdot \nabla v \cdot \overset{1}{\varphi} \, dx - \frac{1}{\sigma_1} \int_{\overset{1}{\Omega}_t} \Delta \overset{1}{H} \cdot \overset{1}{\varphi} \, dx = 0. \tag{3.2}$$

Multiplying $(1.21)_1$ by a function $\overset{2}{\varphi}$ and integrating over $\overset{2}{\Omega}_t$ implies

$$\mu_2 \int_{\overset{2}{\Omega}_t} \overset{2}{H}_t \cdot \overset{2}{\varphi} \, dx - \frac{1}{\sigma_2} \int_{\overset{2}{\Omega}_t} \Delta \overset{2}{H} \cdot \overset{2}{\varphi} \, dx = \int_{\overset{2}{\Omega}_t} \overset{2}{G} \cdot \overset{2}{\varphi} \, dx. \tag{3.3}$$

To prove existence of solutions to integral identities (3.1), (3.2) and (3.3) we first transform them to lagrangian coordinates. Passing to lagrangian coordinates in (3.1) yields

$$\int_{\overset{1}{\Omega}} \bar{v}_t \cdot \bar{\varphi} \, d\xi + \int_{\overset{1}{\Omega}} \mathbb{D}_{\bar{v}}(\bar{v}) \cdot \mathbb{D}_{\bar{v}}(\bar{\varphi}) \, d\xi = \int_{\overset{1}{\Omega}} \bar{f} \cdot \bar{\varphi} \, d\xi - \mu_1 \int_{\overset{1}{\Omega}} \mathbb{T}(\overset{1}{\bar{H}}) \cdot \mathbb{D}_{\bar{v}}(\bar{\varphi}) \, d\xi \equiv \int_{\overset{1}{\Omega}} \overset{0}{\bar{K}}(\bar{v}) \cdot \bar{\varphi} \, d\xi. \tag{3.4}$$

To prove estimates and existence of solutions to integral identities (3.2)–(3.4) we transform identities (3.2) and (3.3) to the forms

$$\mu_1 \int_{\overset{1}{\Omega}_t} (\overset{1}{H}_t + v \cdot \nabla \overset{1}{H}) \cdot \overset{1}{\varphi} \, dx + \frac{1}{\sigma_1} \int_{\overset{1}{\Omega}_t} \operatorname{rot} \operatorname{rot} \overset{1}{H} \cdot \overset{1}{\varphi} \, dx = \mu_1 \int_{\overset{1}{\Omega}_t} \overset{1}{H} \cdot \nabla v \cdot \overset{1}{\varphi} \, dx \tag{3.5}$$

and

$$\text{(3.6)}\qquad \mu_2\int\limits_{\overset{2}{\Omega}_t}(\overset{2}{H}_t+v'\cdot\nabla\overset{2}{H})\cdot\overset{2}{\varphi}\,dx+\frac{1}{\sigma_2}\int\limits_{\overset{2}{\Omega}_t}\operatorname{rot}\operatorname{rot}\overset{2}{H}\cdot\overset{2}{\varphi}\,dx =\mu_2\int\limits_{\overset{2}{\Omega}_t}v'\cdot\nabla\overset{2}{H}\cdot\overset{2}{\varphi}\,dx+\int\limits_{\overset{2}{\Omega}_t}\overset{2}{G}\cdot\overset{2}{\varphi}\,dx.$$

Adding (3.5) and (3.6) and applying the transmission conditions (1.16), (1.18) we get

$$\begin{aligned}\frac{1}{\sigma_1}\int\limits_{\overset{1}{\Omega}_t}\operatorname{rot}\operatorname{rot}\overset{1}{H}\cdot\overset{1}{\varphi}\,dx+\frac{1}{\sigma_2}\int\limits_{\overset{2}{\Omega}_t}\operatorname{rot}\operatorname{rot}\overset{2}{H}\cdot\overset{2}{\varphi}\,dx&=\sum_{i=1}^2\frac{1}{\sigma_i}\int\limits_{\overset{i}{\Omega}_t}\operatorname{rot}\overset{i}{H}\cdot\operatorname{rot}\overset{i}{\varphi}\,dx\\&\quad+\frac{1}{\sigma_1}\int\limits_{S_t}\operatorname{rot}\overset{1}{H}\cdot\overset{1}{\bar n}\times\overset{1}{\varphi}\,dS_t+\frac{1}{\sigma_2}\int\limits_{S_t}\operatorname{rot}\overset{2}{H}\cdot\overset{2}{\bar n}\times\overset{2}{\varphi}\,dS_t\\&=\sum_{i=1}^2\frac{1}{\sigma_i}\int\limits_{\overset{i}{\Omega}_t}\operatorname{rot}\overset{i}{H}\cdot\operatorname{rot}\overset{i}{\varphi}\,dx-\mu_1\int\limits_{S_t}v\times\overset{1}{H}\cdot\overset{1}{\bar n}\times\varphi\,dS_t,\end{aligned}$$

where we have used the facts that $\overset{1}{\bar n}=-\overset{2}{\bar n}$, $\overset{i}{\bar n}$ is the unit normal vector to $S_t$ outward to $\overset{i}{\Omega}_t$ and $\overset{1}{\varphi}_\tau=\overset{2}{\varphi}_\tau=\varphi_\tau$ on $S_t$. Then we obtain

$$\begin{aligned}\text{(3.7)}\qquad &\sum_{i=1}^2\Bigg[\mu_i\int\limits_{\overset{i}{\Omega}_t}(\overset{i}{H}_t+\overset{i}{v}\cdot\nabla\overset{i}{H})\cdot\overset{i}{\varphi}\,dx+\frac{1}{\sigma_i}\int\limits_{\overset{i}{\Omega}_t}\operatorname{rot}\overset{i}{H}\cdot\operatorname{rot}\overset{i}{\varphi}\,dx\Bigg]\\&=\mu_1\int\limits_{\overset{1}{\Omega}_t}\overset{1}{H}\cdot\nabla\overset{1}{v}\cdot\overset{1}{\varphi}\,dx+\mu_2\int\limits_{\overset{2}{\Omega}_t}\overset{2}{v}\cdot\nabla\overset{2}{H}\cdot\overset{2}{\varphi}\,dx+\int\limits_{\overset{2}{\Omega}_t}\overset{2}{G}\cdot\overset{2}{\varphi}\,dx\\&\quad+\mu_1\int\limits_{S_t}\overset{1}{v}\times\overset{1}{H}\cdot\overset{1}{\bar n}\times\varphi\,dS_t\equiv\int\limits_{\overset{1}{\Omega}_t}\overset{1}{K}\cdot\overset{1}{\varphi}\,dx+\int\limits_{\overset{2}{\Omega}_t}\overset{2}{K}\cdot\overset{2}{\varphi}\,dx+\int\limits_{S_t}\overset{3}{K}\cdot\varphi\,dS_t,\end{aligned}$$

where we have used the notation

$$\text{(3.8)}\qquad \overset{1}{v}=v,\quad \overset{2}{v}=v'.$$

In (3.7) the quantities $\overset{i}{K}$, $i=1,2,3$, are treated as given.

To prove existence of solutions to (3.4) and (3.7) we introduce the lagrangian coordinates $\overset{1}{\xi}$, $\overset{2}{\xi}$ as the Cauchy data to the problems

$$\text{(3.9)}\qquad \frac{dx}{dt}=\overset{i}{v}(x,t),\quad x|_{t=0}=\overset{i}{\xi},\quad x\in\overset{i}{\Omega}_t,\quad i=1,2.$$

In this section we prove existence of solutions to linearized integral identities (3.4) and (3.7) by the Galerkin method. Existence of solutions to the nonlinear equations will be proved in Sections 4–8 by the method of successive approximations.

Let $\overset{1}{u}$ be given. Then the linearized (3.4) takes the form

$$\text{(3.10)}\qquad \int\limits_{\overset{1}{\Omega}}\bar v_t\cdot\bar\varphi\,d\overset{1}{\xi}+\int\limits_{\overset{1}{\Omega}}\mathbb{D}_{\overset{1}{\bar u}}(\bar v)\cdot\mathbb{D}_{\overset{1}{\bar u}}(\bar\varphi)\,d\overset{1}{\xi}=\int\limits_{\overset{1}{\Omega}}\overset{0}{\bar K}(\overset{1}{\bar u})\cdot\bar\varphi\,d\overset{1}{\xi}.$$

To linearize (3.7) we introduce the notation: $H=(\overset{1}{H},\overset{2}{H})$, $\psi=(\overset{1}{\varphi},\overset{2}{\varphi})$, $u=(\overset{1}{u},\overset{2}{u})$, $K=$

$(\overset{1}{K},\overset{2}{K})$, $\xi = (\overset{1}{\xi},\overset{2}{\xi})$, $\Omega = \overset{1}{\Omega} \cup \overset{2}{\Omega}$, $\mu = (\mu_1,\mu_2)$, $\sigma = (\sigma_1,\sigma_2)$. Then we express (3.7) in compact form as

$$\mu \int_\Omega \bar{H}_t \cdot \bar{\psi}\, d\xi + \frac{1}{\sigma} \int_\Omega \mathrm{rot}_{\bar{u}} \bar{H} \cdot \mathrm{rot}_{\bar{u}} \bar{\psi}\, d\xi = \int_\Omega \bar{K} \cdot \bar{\psi}\, d\xi + \int_S \overset{3}{\bar{K}} \cdot \bar{\varphi}\, dS. \tag{3.11}$$

Now we prove the existence of solutions to the integral identities (3.5)–(3.7).

LEMMA 3.1. *Assume that* $\overset{0}{\bar{K}} \in L_2(\overset{1}{\Omega}{}^T)$, $v(0) \in L_2(\overset{1}{\Omega})$, $H(0) \in L_2(\Omega)$, $\bar{K} \in L_2(\Omega^T)$, $\overset{3}{\bar{K}} \in L_2(S^T)$, $\overset{1}{\bar{u}} \in L_2(0,T,H^3(\overset{1}{\Omega}))$, $\bar{u} \in L_2(0,T;H^3(\overset{1}{\Omega}))$ *and $T$ is sufficiently small. Assume the transmission condition* (1.16), (1.18) *hold. Then there exist solutions to the integral identities* (3.10), (3.11) *and*

$$\|\bar{v}\|^2_{L_\infty(0,T;L_2(\overset{1}{\Omega}))} + \|\nabla \bar{v}\|^2_{L_2(\overset{1}{\Omega}{}^T)} \le c(\|\bar{v}(0)\|^2_{L_2(\overset{1}{\Omega})} + \|\overset{0}{\bar{K}}\|^2_{L_2(\overset{1}{\Omega}{}^T)}), \tag{3.12}$$

$$\|\bar{H}\|^2_{L_\infty(0,T;L_2(\Omega))} + \|\nabla \bar{H}\|^2_{L_2(\Omega^T)} \le c(\|\bar{H}(0)\|^2_{L_2(\Omega)} + \|\bar{K}\|^2_{L_2(\Omega^T)} + \|\overset{3}{\bar{K}}\|^2_{L_2(S^T)}). \tag{3.13}$$

*Proof.* To prove existence of solutions we use the Galerkin method. Therefore we are looking for the approximate solutions to system (3.10)–(3.11) in the form

$$\begin{aligned} &\bar{v}_n = \sum_{k=1}^n \alpha_{kn}(t)\varphi_k(\xi),\\ &\overset{1}{\bar{H}}_n = \sum_{k=1}^n \overset{1}{\beta}_{kn}(t)\overset{1}{\varphi}_k(\xi), \quad \overset{2}{\bar{H}}_n = \sum_{k=1}^n \overset{2}{\beta}_{kn}(t)\overset{2}{\varphi}_k(\xi), \quad \text{so}\\ &\bar{H}_n = \sum_{k=1}^n \beta_{kn}(t)\psi_k(\xi), \end{aligned} \tag{3.14}$$

where $\operatorname{supp} \overset{i}{\varphi}_k \subset \overset{i}{\Omega}$, $i = 1,2$. Inserting (3.14) in (3.10)–(3.11), respectively, and replacing $\bar{\varphi}$ by $\varphi_k$, $\overset{i}{\bar{\varphi}}$ by $\overset{i}{\varphi}_k$, $i = 1,2$, we obtain the system of ordinary differential equations:

$$\frac{d}{dt}\alpha_{kn} = a_{kl}\alpha_{ln} + d_k, \quad \mu_i \frac{d}{dt}\overset{i}{\beta}_{kn} = \overset{i}{a}_{kl}\overset{i}{\beta}_{ln} + \overset{i}{d}_k, \ i = 1,2, \tag{3.15}$$

where $k = 1,\dots,n$, and the summation convention over repeated indices is assumed. Moreover

$$\begin{aligned} &a_{kl} = \int_{\overset{1}{\Omega}} \mathbb{D}_{\overset{1}{u}}(\varphi_l) \cdot \mathbb{D}_{\overset{1}{u}}(\varphi_k)\, d\xi, \quad d_k = \int_{\overset{1}{\Omega}} \overset{0}{\bar{K}} \varphi_k\, d\overset{1}{\xi},\\ &\overset{i}{a}_{kl} = -\frac{1}{\sigma_i} \int_\Omega \mathrm{rot}_{\bar{u}} \psi_k \cdot \mathrm{rot}_{\bar{u}} \psi_l\, d\xi, \quad i = 1,2,\\ &\overset{i}{d}_k = \int_\Omega \bar{K} \cdot \psi_k\, d\xi + \int_S \overset{3}{\bar{K}} \cdot \overset{1}{\varphi}_k d\overset{1}{\xi}, \quad i = 1,2. \end{aligned} \tag{3.16}$$

Finally, we have the initial conditions

$$\alpha_{kn}|_{t=0} = \alpha_{kn}(0), \quad \overset{i}{\beta}_{kn}|_{t=0} = \overset{i}{\beta}_{kn}(0), \quad i = 1,2, \tag{3.17}$$

which follow from $(1.6)_{2,3}$.

Existence of solutions to the ordinary differential equations (3.15) with initial conditions (3.17) is evident.

To prove the existence of solutions in $L_\infty(0,T;L_2(\overset{i}{\Omega}))\cap L_2(0,T;H^1(\overset{i}{\Omega}))$, only corresponding estimates will be shown. Multiply (3.15) by $\alpha_{kn}$, $\beta_{kn}$, respectively, next sum the results with respect to $k$ and add. Adding the expressions with respect to $k$, integrating with respect to time, letting $n\to\infty$ and using Lemma 2.5 we obtain (3.12), (3.13). ∎

Similarly we can prove

LEMMA 3.2. *Assume that* $\overset{0}{\bar K}\in H^2(0,T;L_2(\overset{1}{\Omega}))$, $\partial_t^k\bar v(0)\in L_2(\overset{1}{\Omega})$, $\partial_t^k\bar H(0)\in L_2(\Omega)$, $k=1,2$, $\bar K\in H^2(0,T;L_2(\Omega))$, $\overset{3}{\bar K}\in H^2(0,T;L_2(S))$, $\overset{1}{\bar u}\in L_2(0,T;H^3(\overset{1}{\Omega}))$, $\overset{1}{\bar u}_t\in L_2(0,T;H^2(\overset{1}{\Omega}))$, $\bar u\in L_2(0,T;H^3(\Omega))$, $\bar u_t\in L_2(0,T;H^2(\Omega))$, $\bar v_t\in L_2(0,T;H^2(\overset{1}{\Omega}))$, $\bar H_t\in L_2(0,T;H^2(\Omega))$ *and* $T$ *is sufficiently small. Assume the transmission conditions* (1.16), (1.18) *hold. Then a weak solution to problem* (3.10), (3.11) *satisfies*

$$
\begin{aligned}
(3.18)\qquad &\|\bar v\|^2_{W^2_\infty(0,T;L_2(\overset{1}{\Omega}))}+\|\bar v\|^2_{H^2(0,T;H^1(\overset{1}{\Omega}))}\\
&\le c\Big(\alpha(\overset{1}{\bar u})\Big(\|\bar v\|^2_{L_2(0,T;H^2(\overset{1}{\Omega}))}+\sqrt{T}\int_0^T\|\overset{1}{\bar u}_t\|^2_{H^1(\overset{1}{\Omega})}\,dt\,\|\bar v\|^2_{L_\infty(0,T;H^2(\overset{1}{\Omega}))}\\
&\quad+\|\bar v_t\|^2_{L_\infty(0,T;H^1(\overset{1}{\Omega}))}\Big)+\|\overset{0}{\bar K}\|^2_{H^2(0,T;L_2(\overset{1}{\Omega}))}+\sum_{k=0}^2\|\partial_t^k\bar v(0)\|^2_{L_2(\overset{1}{\Omega})}\Big),
\end{aligned}
$$

$$
\begin{aligned}
(3.19)\qquad &\|\bar H\|^2_{W^2_\infty(0,T;L_2(\Omega))}+\|\bar H\|^2_{H^2(0,T;H^1(\Omega))}\\
&\le c\Big(\alpha(\bar u)\Big(\|\bar H\|^2_{L_2(0,T;H^2(\Omega))}+\sqrt{T}\int_0^T\|\bar u_t\|^2_{H^1(\Omega)}\,dt\,\|\bar H\|^2_{L_\infty(0,T;H^2(\Omega))}\\
&\quad+\|\bar H_t\|^2_{L_\infty(0,T;H^1(\Omega))}\Big)+\|\bar K\|^2_{H^2(0,T;L_2(\Omega))}+\|\overset{3}{\bar K}\|^2_{H^2(0,T;L_2(S))}\\
&\quad+\sum_{k=0}^2\|\partial_t^k\bar H(0)\|^2_{L_2(\Omega)}\Big),
\end{aligned}
$$

*where* $\alpha(\overset{1}{\bar u})$ *is given by Remark* 2.4.

## 4. Space regularity of solutions to (2.14)

LEMMA 4.1. *Assume that* $F\in L_2(\overset{1}{\Omega}{}^t)$, $g\in\Gamma^1_{0,2}(\overset{1}{\Omega}{}^t)$, $h\in L_2(0,t;H^{1/2}(S))\cap L_\infty(0,t;L_2(S))$, $h_t\in L_2(S^t)$, $t\le T$, $v(0)\in H^1(\overset{1}{\Omega})$. *Moreover,* $\bar v_{\xi t}\in L_2(\overset{1}{\Omega}{}^t)$, $t\le T$. *Then a weak solution to problem* (2.14) *satisfies*

$$(4.1)\qquad \|\bar v(t)\|^2_{H^1(\overset{1}{\Omega})}+\|\bar v\|^2_{\Gamma^2_{1,2}(\overset{1}{\Omega}{}^t)}+\|\bar p\|^2_{L_2(0,t;H^1(\overset{1}{\Omega}))}$$
$$\le \varepsilon\|\bar v_{\xi t}\|^2_{L_2(\overset{1}{\Omega}{}^t)}+c(1/\varepsilon)(\|F\|^2_{L_2(\overset{1}{\Omega}{}^t)}+\|g\|^2_{\Gamma^1_{0,2}(\overset{1}{\Omega}{}^t)}$$
$$+\|h\|^2_{L_2(0,t;H^{1/2}(S))\cap L_\infty(0,t;L_2(S))}+\|h_t\|^2_{L_2(S^t)})+c\|v(0)\|^2_{H^1(\overset{1}{\Omega})},$$

*where $t\le T$, $\varepsilon\in(0,1)$.*

*Proof.* Multiplying (2.14) by $\bar v$, integrating the result over $\overset{1}{\Omega}$ and integrating by parts we obtain

$$(4.2)\qquad \frac12\frac{d}{dt}\int_{\overset{1}{\Omega}}|\bar v|^2\,d\xi+\nu\int_{\overset{1}{\Omega}}|\mathbb{D}(\bar v)|^2\,d\xi-\int_S\bar n\cdot\mathbb{T}(\bar v,\bar p)\cdot\bar v\,dS-\int_{\overset{1}{\Omega}}\bar pg\,d\xi=\int_{\overset{1}{\Omega}}F\bar v\,d\xi.$$

Applying the Korn inequality (2.6) to the second integral, the boundary condition $(2.14)_3$ and the Hölder and Young inequalities we obtain

$$(4.3)\qquad \frac12\frac{d}{dt}\|\bar v\|^2_{L_2(\overset{1}{\Omega})}+\nu\|\bar v\|^2_{H^1(\overset{1}{\Omega})}$$
$$\le c(\|\bar v\|^2_{L_2(\overset{1}{\Omega})}+\|h\|^2_{L_2(S)}+\|F\|^2_{L_2(\overset{1}{\Omega})})+\varepsilon\|\bar p\|^2_{L_2(\overset{1}{\Omega})}+c(1/\varepsilon)\|g\|^2_{L_2(\overset{1}{\Omega})},$$

where $\varepsilon\in(0,1)$, $c(1/\varepsilon)\sim\varepsilon^{-a}$, $a>0$.

Integrating (4.3) with respect to time and employing the Gronwall inequality yields

$$(4.4)\qquad \|\bar v(t)\|^2_{L_2(\overset{1}{\Omega})}+\nu\int_0^t\|\bar v(t')\|^2_{H^1(\overset{1}{\Omega})}\,dt'$$
$$\le c(t)\int_0^t(\|h(t')\|^2_{L_2(S)}+\|F(t')\|^2_{L_2(\overset{1}{\Omega})})\,dt'+\varepsilon\int_0^t\|\bar p(t')\|^2_{L_2(\overset{1}{\Omega})}\,dt'$$
$$+c(1/\varepsilon,t)\int_0^t\|g(t')\|^2_{L_2(\overset{1}{\Omega})}\,dt'+\|v(0)\|^2_{L_2(\overset{1}{\Omega})},$$

where $c(t)$ and $c(1/\varepsilon,t)$ are increasing functions of $t$.

Multiplying $(2.14)_1$ by $\bar v_t$ and integrating the result over $\overset{1}{\Omega}$ implies

$$(4.5)\qquad \int_{\overset{1}{\Omega}}\bar v_t^2\,d\xi+\frac{\nu}{2}\frac{d}{dt}\int_{\overset{1}{\Omega}}|\mathbb{D}(\bar v)|^2\,d\xi\le c\|F\|^2_{L_2(\overset{1}{\Omega})}+\int_S(h\bar v)_t\,dS$$
$$-\int_S h_t\bar v\,dS+\varepsilon\|\bar p\|^2_{L_2(\overset{1}{\Omega})}+c(1/\varepsilon)\|g_t\|^2_{L_2(\overset{1}{\Omega})}.$$

Integrating (4.5) with respect to time and applying the Hölder and the Young inequalities we get

$$(4.6)\qquad \|\bar v_t\|^2_{L_2(\overset{1}{\Omega}{}^t)}+\frac{\nu}{2}\|\bar v(t)\|^2_{H^1(\overset{1}{\Omega})}$$
$$\le\varepsilon(\|\bar v\|^2_{L_2(0,t;H^1(\overset{1}{\Omega}))}+\|\bar p\|^2_{L_2(\overset{1}{\Omega}{}^t)})+c(1/\varepsilon)(\|h_t\|^2_{L_2(S^t)}+\|g_t\|^2_{L_2(\overset{1}{\Omega}{}^t)})$$
$$+c(\|F\|^2_{L_2(\overset{1}{\Omega}{}^t)}+\|h(t)\|^2_{L_2(S)}+\|v(0)\|^2_{H^1(\overset{1}{\Omega})}+\|\bar v(t)\|^2_{L_2(\overset{1}{\Omega})}).$$

Adding (4.4) and (4.6) and assuming that $\varepsilon$ is sufficiently small we obtain

$$
\begin{aligned}
(4.7)\qquad \|\bar v(t)\|^2_{L_2(\overset{1}{\Omega})} + \|\bar v_t\|^2_{L_2(\overset{1}{\Omega}{}^t)} + \|\bar v(t)\|^2_{H^1(\overset{1}{\Omega})} + \|\bar v\|^2_{L_2(0,t;H^1(\overset{1}{\Omega}))} \\
\le \varepsilon\|\bar p\|^2_{L_2(\overset{1}{\Omega}{}^t)} + c(1/\varepsilon)(\|g\|^2_{L_2(\overset{1}{\Omega}{}^t)} + \|g_t\|^2_{L_2(\overset{1}{\Omega}{}^t)} + \|h_t\|^2_{L_2(S^t)}) \\
+ c(\|F\|^2_{L_2(\overset{1}{\Omega}{}^t)} + \|h\|^2_{L_\infty(0,t;L_2(S))} + \|v(0)\|^2_{H^1(\overset{1}{\Omega})}).
\end{aligned}
$$

To estimate pressure from the r.h.s. of (4.7) we introduce a function $\psi$ such that

$$
(4.8)\qquad \operatorname{div}\psi = \bar p, \qquad \psi|_S = 0.
$$

Then [4] implies the existence of such a function in $H^1(\overset{1}{\Omega})$ and the estimate

$$
(4.9)\qquad \|\psi\|_{H^1(\overset{1}{\Omega})} \le c\|\bar p\|_{L_2(\overset{1}{\Omega})}.
$$

Multiplying (2.14) by $\psi$, integrating the result over $\overset{1}{\Omega}{}^t$ and using (4.9) yields

$$
(4.10)\qquad \|\bar p\|_{L_2(\overset{1}{\Omega}{}^t)} \le c(\|\bar v_t\|_{L_2(\overset{1}{\Omega}{}^t)} + \|\nabla\bar v\|_{L_2(\overset{1}{\Omega}{}^t)} + \|F\|_{L_2(\overset{1}{\Omega}{}^t)}).
$$

In view of (4.7) and (4.10) and sufficiently small $\varepsilon$ we derive

$$
\begin{aligned}
(4.11)\qquad \|\bar v(t)\|^2_{H^1(\overset{1}{\Omega})} + \|\bar v_t\|^2_{L_2(\overset{1}{\Omega}{}^t)} + \|\bar v\|^2_{L_2(0,t;H^1(\overset{1}{\Omega}))} + \|\bar p\|^2_{L_2(\overset{1}{\Omega}{}^t)} \\
\le c(\|g\|^2_{L_2(\overset{1}{\Omega}{}^t)} + \|g_t\|^2_{L_2(\overset{1}{\Omega}{}^t)} + \|h_t\|^2_{L_2(S^t)} + \|F\|^2_{L_2(\overset{1}{\Omega}{}^t)} \\
+ \|h\|^2_{L_\infty(0,t;L_2(S))} + \|v(0)\|^2_{H^1(\overset{1}{\Omega})}).
\end{aligned}
$$

To estimate higher derivatives we use the local coordinates. Therefore we shall restrict our considerations to examine problem (2.23) only because estimates in interior subdomains are simpler. Finally, the estimate in the whole $\overset{1}{\Omega}$ follows from the properties of the partition of unity. Therefore, we consider problem (2.23) in the half-space $z_3 > 0$. In reality we consider it in $\overset{1}{\hat\Omega}$ because all functions in (2.23) have compact supports in $\overset{1}{\hat\Omega}$.

Differentiating $(2.23)_1$ with respect to $z'$, multiplying the result by $\tilde v_{z'}$ and integrating over $\overset{1}{\hat\Omega}$ we get

$$
(4.12)\qquad \frac12\frac{d}{dt}\|\tilde v_{z'}\|^2_{L_2(\overset{1}{\hat\Omega})} - \nu\int_S h_{0z'}\tilde v_{z'}\,dS + \frac{\nu}{2}\int_{\overset{1}{\hat\Omega}} |\mathbb{D}_z(\tilde v_{z'})|^2\,dz - \int_{\overset{1}{\hat\Omega}} \tilde p_{z'}g_{0z'}\,dz = \int_{\overset{1}{\hat\Omega}} F_0\tilde v_{z'}dz.
$$

By the Korn inequality and some imbedding we obtain

$$
\begin{aligned}
(4.13)\qquad \frac{d}{dt}\|\tilde v_{z'}\|^2_{L_2(\overset{1}{\hat\Omega})} + \nu\|\tilde v_{z'}\|^2_{H^1(\overset{1}{\hat\Omega})} \le c(\|\tilde v_{z'}\|^2_{L_2(\overset{1}{\hat\Omega})} + \|\partial^{1/2}_{z'}h_0\|^2_{L_2(S)} + \|F_0\|^2_{L_2(\overset{1}{\hat\Omega})}) \\
+ \varepsilon\|\tilde p_{z'}\|^2_{L_2(\overset{1}{\hat\Omega})} + c(1/\varepsilon)\|g_{0z'}\|^2_{L_2(\overset{1}{\hat\Omega})}.
\end{aligned}
$$

Let $\psi$ be a solution to the problem

$$
(4.14)\qquad
\begin{aligned}
&\operatorname{div}\psi = \tilde p_{z'} && \text{in } \overset{1}{\hat\Omega},\\
&\psi = 0 && \text{on } \partial\overset{1}{\hat\Omega}.
\end{aligned}
$$

Then by [4] there exists a solution to (4.14) such that $\psi \in H^1(\overset{1}{\hat{\Omega}})$ and

$$\|\psi\|_{H^1(\overset{1}{\hat{\Omega}})} \le c\|\tilde{p}_{z'}\|_{L_2(\overset{1}{\hat{\Omega}})}. \tag{4.15}$$

Multiplying $(2.23)_1$ by $\psi$, integrating the result over $\overset{1}{\hat{\Omega}}$ and using (4.15) we get

$$\|\tilde{p}_{z'}\|^2_{L_2(\overset{1}{\hat{\Omega}})} \le c(\|\tilde{v}_t\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\tilde{v}_{z'}\|^2_{H^1(\overset{1}{\hat{\Omega}})} + \|F_0\|^2_{L_2(\overset{1}{\hat{\Omega}})}). \tag{4.16}$$

Employing (4.16) in (4.13) and assuming that $\varepsilon$ is sufficiently small we obtain

$$\begin{aligned} &\frac{d}{dt}\|\tilde{v}_{z'}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\tilde{v}_{z'}\|^2_{H^1(\overset{1}{\hat{\Omega}})} + \|\tilde{p}_{z'}\|^2_{L_2(\overset{1}{\hat{\Omega}})} \\ &\quad \le c(\|\tilde{v}_t\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\tilde{v}_{z'}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\partial^{1/2}_{z'} h_0\|^2_{L_2(S)} + \|F_0\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|g_{0z'}\|^2_{L_2(\overset{1}{\hat{\Omega}})}). \end{aligned} \tag{4.17}$$

Let us express $(2.23)_1$ in coordinates

$$\tilde{v}_{it} - \nu\Delta\tilde{v}_i + \nabla_i\tilde{p} = \nu\nabla_i g_0 + F_{0i}, \quad i = 1,2,3. \tag{4.18}$$

From (4.18) for $i = 1,2$, we calculate

$$\begin{aligned} &\|\tilde{v}_{iz_3z_3}\|^2_{L_2(\overset{1}{\hat{\Omega}})} \\ &\quad \le c(\|\tilde{v}_t\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\tilde{v}_{iz'z'}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\tilde{p}_{z'}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|g_{0z'}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|F_0\|^2_{L_2(\overset{1}{\hat{\Omega}})}). \end{aligned} \tag{4.19}$$

Differentiating $(2.23)_2$ with respect to $z_3$ yields

$$\|\tilde{v}_{3z_3z_3}\|^2_{L_2(\overset{1}{\hat{\Omega}})} \le c(\|\tilde{g}_{0z_3}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\tilde{v}_{z'z_3}\|^2_{L_2(\overset{1}{\hat{\Omega}})}). \tag{4.20}$$

Finally from (4.18) for $i = 3$ we have

$$\begin{aligned} \|\tilde{p}_{z_3}\|^2_{L_2(\overset{1}{\hat{\Omega}})} &\le c(\|\tilde{v}_{3z_3z_3}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\tilde{v}_{3,z'z'}\|^2_{L_2(\overset{1}{\hat{\Omega}})} \\ &\quad + \|\tilde{v}_t\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|F_0\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|g_{0z_3}\|^2_{L_2(\overset{1}{\hat{\Omega}})}). \end{aligned} \tag{4.21}$$

Using the inequality

$$\frac{d}{dt}\|\tilde{v}_{z_3}\|^2_{L_2(\overset{1}{\hat{\Omega}})} \le \varepsilon\|\tilde{v}_{z_3t}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + c(1/\varepsilon)\|\tilde{v}_{z_3}\|^2_{L_2(\overset{1}{\hat{\Omega}})}$$

in (4.17) yields

$$\begin{aligned} &\frac{d}{dt}\|\tilde{v}_z\|_{L_2(\overset{1}{\hat{\Omega}})} + \|\tilde{v}_{z'}\|^2_{H^1(\overset{1}{\hat{\Omega}})} + \|\tilde{p}_{z'}\|^2_{L_2(\overset{1}{\hat{\Omega}})} \\ &\qquad \le \varepsilon\|\tilde{v}_{z_3t}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + c(1/\varepsilon)\|\tilde{v}_{z_3}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + c(\|\tilde{v}_t\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\tilde{v}_{z'}\|^2_{L_2(\overset{1}{\hat{\Omega}})}) \\ &\qquad\quad + c(\|F_0\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|g_{0z'}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\partial^{1/2}_{z'} h_0\|^2_{L_2(S)}). \end{aligned} \tag{4.22}$$

Adding (4.19)–(4.22) and integrating the result with respect to time we obtain

$$
\begin{aligned}
(4.23)\qquad & \|\tilde v_z(t)\|^2_{L_2(\overset{1}{\hat\Omega})} + \|\tilde v_z\|^2_{L_2(0,t;H^1(\overset{1}{\hat\Omega}))} + \|\tilde p_z\|^2_{L_2(\overset{1}{\hat\Omega}{}^t)} \\
&\le \varepsilon\|\tilde v_{zt}\|^2_{L_2(\overset{1}{\hat\Omega}{}^t)} + c(1/\varepsilon)\|\tilde v_z\|^2_{L_2(\overset{1}{\hat\Omega}{}^t)} \\
&\quad + c(\|\tilde v_t\|^2_{L_2(\overset{1}{\hat\Omega}{}^t)} + \|\tilde v_z\|^2_{L_2(\overset{1}{\hat\Omega}{}^t)}) \\
&\quad + c(\|F_0\|^2_{L_2(\overset{1}{\hat\Omega}{}^t)} + \|g_0\|^2_{L_2(0,t;H^1(\overset{1}{\hat\Omega}))} + \|h_0\|^2_{L_2(0,t;H^{1/2}(S))}) + c\|\tilde v(0)\|^2_{H^1(\overset{1}{\hat\Omega})}.
\end{aligned}
$$

From (2.23) and the properties of the partition of unity we have

$$
\begin{aligned}
(4.24)\qquad & \|F_0\|_{L_2(\overset{1}{\hat\Omega}{}^t)} \le c\lambda\|\tilde v_{zz}\|_{L_2(\overset{1}{\hat\Omega}{}^t)} + c\lambda\|\tilde p_z\|_{L_2(\overset{1}{\hat\Omega}{}^t)} \\
&\qquad + \frac{c}{\lambda}(\|\hat p\|_{L_2(\overset{1}{\hat\Omega}{}^t)} + \|\hat v\|_{L_2(0,t;H^1(\overset{1}{\hat\Omega}))}) + c\|\tilde F\|_{L_2(\overset{1}{\hat\Omega}{}^t)},
\end{aligned}
$$

$$
(4.25)\qquad \|g_0\|_{L_2(0,t;H^1(\overset{1}{\hat\Omega}))} \le c\lambda\|\tilde v_{zz}\|_{L_2(\overset{1}{\hat\Omega}{}^t)} + \frac{c}{\lambda}\|\hat v\|_{L_2(0,t;H^1(\overset{1}{\hat\Omega}))} + \|\tilde g\|_{L_2(0,t;H^1(\overset{1}{\hat\Omega}))},
$$

$$
\begin{aligned}
(4.26)\qquad & \|h_0\|_{L_2(0,t;H^{1/2}(S))} \le c\lambda\|\tilde v_{zz}\|_{L_2(\overset{1}{\hat\Omega}{}^t)} + c\lambda\|\tilde p\|_{L_2(0,t;H^1(\overset{1}{\hat\Omega}))} \\
&\qquad + \frac{c}{\lambda}\|\hat v\|_{L_2(0,t;H^1(\overset{1}{\hat\Omega}))} + c\|\tilde h\|_{L_2(0,t;H^{1/2}(S))}.
\end{aligned}
$$

Employing (4.24)–(4.26) in (4.23), passing to variables $\xi$, summing over all subdomains of the partition of unity and using that $\lambda$ is sufficiently small we have

$$
\begin{aligned}
(4.27)\qquad & \|\bar v(t)\|^2_{H^1(\overset{1}{\Omega})} + \|\bar v\|^2_{L_2(0,t;H^2(\overset{1}{\Omega}))} + \|\bar p\|^2_{L_2(0,t;H^1(\overset{1}{\Omega}))} \\
&\le c(1/\varepsilon)(\|\bar v\|^2_{L_2(0,t;H^1(\overset{1}{\Omega}))} + \|\bar p\|^2_{L_2(\overset{1}{\Omega}{}^t)} + \|\bar v_t\|^2_{L_2(\overset{1}{\Omega}{}^t)}) + \varepsilon\|\bar v_{\xi t}\|^2_{L_2(\overset{1}{\Omega}{}^t)} \\
&\quad + c(\|F\|^2_{L_2(\overset{1}{\Omega}{}^t)} + \|g\|^2_{L_2(0,t;H^1(\overset{1}{\Omega}))} + \|h\|^2_{L_2(0,t;H^{1/2}(S))}) + c\|v(0)\|^2_{H^1(\overset{1}{\Omega})}.
\end{aligned}
$$

Finally, inequalities (4.11) and (4.27) imply

$$
\begin{aligned}
(4.28)\qquad & \|\bar v(t)\|^2_{H^1(\overset{1}{\Omega})} + \|\bar v_t\|^2_{L_2(\overset{1}{\Omega}{}^t)} + \|\bar v\|^2_{L_2(0,t;H^1(\overset{1}{\Omega}))} + \|\bar p\|^2_{L_2(0,t;H^1(\overset{1}{\Omega}))} \\
&\le \varepsilon\|\bar v_{\xi t}\|^2_{L_2(\overset{1}{\Omega}{}^t)} \\
&\quad + c(1/\varepsilon)(\|F\|^2_{L_2(\overset{1}{\Omega}{}^t)} + \|g\|^2_{L_2(0,t;H^1(\overset{1}{\Omega}))} + \|g_t\|^2_{L_2(\overset{1}{\Omega}{}^t)} \\
&\quad + \|h\|^2_{L_2(0,t;H^{1/2}(S))} + \|h\|^2_{L_\infty(0,t;L_2(S))} + \|h_t\|^2_{L_2(S^t)}) \\
&\quad + c\|v(0)\|^2_{H^1(\overset{1}{\Omega})}, \quad t \le T.
\end{aligned}
$$

Inequality (4.28) gives (4.1). ■

Next, we need estimates for the third derivatives.

Lemma 4.2. *Assume that* $F \in L_2(0,T;\Gamma_0^1(\overset{1}{\Omega}))$, $g \in \Gamma_{1,2}^2(\overset{1}{\Omega}{}^T)$, $h \in L_2(0,T;H^{3/2}(S))$, $h_t \in L_2(0,T;H^{1/2}(S))$, $v_0 \in H^2(\overset{1}{\Omega})$, $\bar v \in \Gamma_{0,2}^2(\overset{1}{\Omega}{}^T)$, $\bar v_{tt} \in L_2(0,T;H^1(\overset{1}{\Omega}))$, $\bar p \in \Gamma_{0,2}^1(\overset{1}{\Omega}{}^T)$.

*Then a weak solution to problem* (2.14) *satisfies*

$$
\begin{aligned}
(4.29)\qquad & \|\bar v_{\xi\xi}(t)\|^2_{L_2(\overset{1}{\Omega})} + \|\bar v_{\xi t}(t)\|^2_{L_2(\overset{1}{\Omega})} + \nu\|\bar v_{\xi\xi}\|^2_{L_2^1(\overset{1}{\Omega}{}^t)} + \|\bar p_\xi\|^2_{L_2^1(\overset{1}{\Omega}{}^t)} \\
&\le \varepsilon\|\bar v_{\xi tt}\|^2_{L_2(\overset{1}{\Omega}{}^t)} + c(\|\bar v\|^2_{\Gamma^2_{0,2}(\overset{1}{\Omega}{}^t)} + \|\bar p\|^2_{\Gamma^1_{0,2}(\overset{1}{\Omega}{}^t)}) \\
&\quad + c(\|F\|^2_{L_2(0,t;H^1(\overset{1}{\Omega}))} + \|F_t\|^2_{L_2(\overset{1}{\Omega}{}^t)} + \|g\|^2_{\Gamma^2_{1,2}(\overset{1}{\Omega}{}^t)} + \|h\|^2_{L_2(0,t;H^{3/2}(S))} + \|h_t\|^2_{L_2(0,t;H^{1/2}(S))}) \\
&\quad + c(\|v_{\xi\xi}(0)\|^2_{L_2(\overset{1}{\Omega})} + \|v_{\xi t}(0)\|^2_{L_2(\overset{1}{\Omega})}),
\end{aligned}
$$

*where* $t \le T$.

*Proof.* Differentiating $(2.23)_1$ twice with respect to $z'$, multiplying the result by $\tilde v_{z'z'}$ and integrating over $\overset{1}{\hat\Omega}$ we get

$$
(4.30)\qquad \frac{1}{2}\frac{d}{dt}\int_{\overset{1}{\hat\Omega}} \tilde v^2_{z'z'}\,dz + \nu\int_{\overset{1}{\hat\Omega}} |\mathbb{D}(\tilde v_{z'z'})|^2\,dz = \int_S h_{0z'z'}\tilde v_{z'z'}\,dS - \int_{\overset{1}{\hat\Omega}} \tilde p_{z'z'} g_{0z'z'}\,dz + \int_{\overset{1}{\hat\Omega}} F_{0z'z'}\tilde v_{z'z'}\,dz.
$$

Applying the Korn inequality and performing some integration by parts yields

$$
\begin{aligned}
(4.31)\qquad & \frac{d}{dt}\|\tilde v_{z'z'}\|^2_{L_2(\overset{1}{\hat\Omega})} + \nu\|\tilde v_{z'z'}\|^2_{H^1(\overset{1}{\hat\Omega})} \\
&\le c(\|\tilde v_{z'z'}\|^2_{L_2(\overset{1}{\hat\Omega})} + \varepsilon\|\tilde p_{z'z'}\|^2_{L_2(\overset{1}{\hat\Omega})} + \|F_{0z'}\|^2_{L_2(\overset{1}{\hat\Omega})} + \|\partial_{z'}^{1/2} h_{0z'}\|^2_{L_2(S)} + c(1/\varepsilon)\|g_{0z'z'}\|^2_{L_2(\overset{1}{\hat\Omega})}),
\end{aligned}
$$

where $\varepsilon \in (0,1)$.

Let us introduce a function $\varphi_1$ such that

$$
(4.32)\qquad
\begin{aligned}
&\operatorname{div}\varphi_1 = \tilde p_{z'z'} && \text{in } \overset{1}{\hat\Omega}, \\
&\varphi_1 = 0 && \text{on } \partial\overset{1}{\hat\Omega}.
\end{aligned}
$$

Assuming that $\tilde p_{z'z'} \in L_2(\overset{1}{\hat\Omega})$, [4] implies existence of solutions to (4.32) such that $\varphi_1 \in H^1(\overset{1}{\hat\Omega})$ and

$$
(4.33)\qquad \|\varphi_1\|_{H^1(\overset{1}{\hat\Omega})} \le c\|\tilde p_{z'z'}\|_{L_2(\overset{1}{\hat\Omega})},
$$

where $\tilde p_{z'z'}$ means that only one derivative is chosen from the derivatives $\tilde p_{z_iz_j}$, $i,j = 1,2$.

Differentiating $(2.23)_1$ twice with respect to $z'$, multiplying the result by $\varphi_1$, integrating over $\overset{1}{\hat\Omega}$, next integrating by parts and using (4.32) and (4.33) we obtain

$$
(4.34)\qquad \|\tilde p_{z'z'}\|^2_{L_2(\overset{1}{\hat\Omega})} \le c(\|\tilde v_{tz'}\|^2_{L_2(\overset{1}{\hat\Omega})} + \|\nabla\tilde v_{z'z'}\|^2_{L_2(\overset{1}{\hat\Omega})} + \|F_{0z'}\|^2_{L_2(\overset{1}{\hat\Omega})}).
$$

Using (4.34) in (4.31) and assuming that $\varepsilon$ is sufficiently small we obtain

$$\frac{d}{dt}\|\tilde{v}_{z'z'}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \nu\|\tilde{v}_{z'z'}\|^2_{H^1(\overset{1}{\hat{\Omega}})} + \|\tilde{p}_{z'z'}\|^2_{L_2(\overset{1}{\hat{\Omega}})} \le c(\|\tilde{v}_{tz'}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\tilde{v}_{z'z'}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|F_{0z'}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|g_{0z'z'}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\partial^{1/2}_{z'}h_{0z'}\|^2_{L_2(S)}). \tag{4.35}$$

Consider (4.18) for $i = 1,2$. Differentiating it with respect to $z'$ and integrating over $\overset{1}{\hat{\Omega}}$ we get the inequality

$$\|\tilde{v}_{iz_3z_3z'}\|^2_{L_2(\overset{1}{\hat{\Omega}})} \le c(\|\tilde{v}_{iz'z'z'}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\tilde{p}_{z'z'}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\tilde{g}_{0z'z'}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|F_{0z'}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\tilde{v}_{itz'}\|^2_{L_2(\overset{1}{\hat{\Omega}})}), \tag{4.36}$$

where $i = 1,2$.

Differentiating $(2.23)_2$ with respect to $z'$ and $z_3$, integrating the result over $\overset{1}{\hat{\Omega}}$ we derive

$$\|\tilde{v}_{3z_3z_3z'}\|^2_{L_2(\overset{1}{\hat{\Omega}})} \le c\sum_{i=1}^{2}\|\tilde{v}_{iz'z'z_3}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + c\|g_{0z_3z'}\|^2_{L_2(\overset{1}{\hat{\Omega}})}. \tag{4.37}$$

Differentiating the third component of (4.18) with respect to $z'$ and integrating the result over $\overset{1}{\hat{\Omega}}$ yields

$$\|\tilde{p}_{z_3z'}\|^2_{L_2(\overset{1}{\hat{\Omega}})} \le c(\|\tilde{v}_{3zzz'}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\tilde{v}_{tz'}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|g_{0z'}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|F_{0z'}\|^2_{L_2(\overset{1}{\hat{\Omega}})}). \tag{4.38}$$

Adding appropriately (4.35)–(4.38) gives

$$\frac{d}{dt}\|\tilde{v}_{z'z'}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \nu\|\tilde{v}_{z'zz}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\tilde{p}_{z'z}\|^2_{L_2(\overset{1}{\hat{\Omega}})} \le c(\|\tilde{v}_{tz'}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\tilde{v}_{z'z'}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|F_{0z}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|g_{0z'}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|g_{0zz'}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\partial^{1/2}_{z'}h_{0z'}\|^2_{L_2(S)}). \tag{4.39}$$

Differentiating the first two components of $(2.23)_1$ with respect to $z_3$ and integrating the result over $\overset{1}{\hat{\Omega}}$ implies

$$\|\tilde{v}_{iz_3z_3z_3}\|^2_{L_2(\overset{1}{\hat{\Omega}})} \le c(\|\tilde{v}_{iz'z'z_3}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\tilde{p}_{z_3z_3}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|F_{0z_3}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|g_{0z'z_3}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\tilde{v}_{tz_3}\|^2_{L_2(\overset{1}{\hat{\Omega}})}), \tag{4.40}$$

where $i = 1,2$.

Differentiating $(2.23)_2$ twice with respect to $z_3$ and integrating the result over $\overset{1}{\hat{\Omega}}$ we get

$$\|\tilde{v}_{3z_3z_3z_3}\|^2_{L_2(\overset{1}{\hat{\Omega}})} \le c\Big(\sum_{i=1}^{2}\|\tilde{v}_{iz'z_3z_3}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|g_{0z_3z_3}\|^2_{L_2(\overset{1}{\hat{\Omega}})}\Big). \tag{4.41}$$

Finally, differentiating the third component of (4.18) with respect to $z_3$ and integrating over $\overset{1}{\hat{\Omega}}$ we have

$$\|\tilde{p}_{z_3z_3}\|^2_{L_2(\overset{1}{\hat{\Omega}})} \le c(\|\tilde{v}_{3zzz_3}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\tilde{v}_{tz}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|F_{0z}\|^2_{L_2(\overset{1}{\hat{\Omega}})}). \tag{4.42}$$

Adding (4.39)–(4.42) yields

$$\begin{aligned} &\frac{d}{dt}\|\tilde{v}_{z'z'}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \nu\|\tilde{v}_{zzz}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\tilde{p}_{zz}\|^2_{L_2(\overset{1}{\hat{\Omega}})} \\ &\quad\le c(\|\tilde{v}_{tz}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\tilde{v}_{zz}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|F_{0z}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|g_{0z}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|g_{0zz}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\partial_{z'}^{1/2}h_{0z'}\|^2_{L_2(S)}). \end{aligned} \tag{4.43}$$

Employing the inequality

$$\frac{d}{dt}\|\tilde{v}_{zz}\|^2_{L_2(\overset{1}{\hat{\Omega}})} \le \varepsilon\|\tilde{v}_{zzt}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + c(1/\varepsilon)\|\tilde{v}_{zz}\|^2_{L_2(\overset{1}{\hat{\Omega}})} \tag{4.44}$$

in (4.43) gives

$$\begin{aligned} &\frac{d}{dt}\|\tilde{v}_{zz}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \nu\|\tilde{v}_{zzz}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\tilde{p}_{zz}\|^2_{L_2(\overset{1}{\hat{\Omega}})} \\ &\qquad\le \varepsilon\|\tilde{v}_{zzt}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + c(\|\tilde{v}_{tz}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + c(1/\varepsilon)\|\tilde{v}_{zz}\|^2_{L_2(\overset{1}{\hat{\Omega}})} \\ &\qquad\quad + \|F_{0z}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|g_{0z}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|g_{0zz}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\partial_{z'}^{1/2}h_{0z'}\|^2_{L_2(S)}), \end{aligned} \tag{4.45}$$

where $\varepsilon \in (0,1)$.

Now we have to find an estimate for the first term on the r.h.s. of (4.45). Adding (4.19)–(4.22) we obtain

$$\begin{aligned} &\frac{d}{dt}\|\tilde{v}_{z}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\tilde{v}_{zz}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\tilde{p}_{z}\|^2_{L_2(\overset{1}{\hat{\Omega}})} \\ &\qquad\le \varepsilon\|\tilde{v}_{zt}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + c(1/\varepsilon)(\|\tilde{v}_{z}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\tilde{v}_{t}\|^2_{L_2(\overset{1}{\hat{\Omega}})}) \\ &\qquad\quad + c(\|F_0\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|g_{0z}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\partial_{z'}^{1/2}h_0\|^2_{L_2(S)}). \end{aligned} \tag{4.46}$$

Differentiating (2.23) with respect to time and repeating the considerations leading to (4.46) we have

$$\begin{aligned} &\frac{d}{dt}\|\tilde{v}_{tz}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \nu\|\tilde{v}_{tzz}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\tilde{p}_{tz}\|^2_{L_2(\overset{1}{\hat{\Omega}})} \\ &\qquad\le \varepsilon\|\tilde{v}_{ztt}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + c(1/\varepsilon)(\|\tilde{v}_{zt}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\tilde{v}_{tt}\|^2_{L_2(\overset{1}{\hat{\Omega}})}) \\ &\qquad\quad + c(\|F_{0t}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|g_{0zt}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\partial_{z'}^{1/2}h_{0t}\|^2_{L_2(S)}). \end{aligned} \tag{4.47}$$

Adding (4.45) and (4.47) and assuming that $\varepsilon$ in (4.45) is sufficiently small we derive

$$\begin{aligned} &\frac{d}{dt}(\|\tilde{v}_{zz}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\tilde{v}_{tz}\|^2_{L_2(\overset{1}{\hat{\Omega}})}) \\ &\qquad + \nu(\|\tilde{v}_{zzz}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\tilde{v}_{tzz}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\tilde{p}_{zz}\|^2_{L_2(\overset{1}{\hat{\Omega}})} + \|\tilde{p}_{tz}\|^2_{L_2(\overset{1}{\hat{\Omega}})}) \end{aligned} \tag{4.48}$$

$$
\begin{aligned}
&\le \varepsilon\|\tilde v_{ztt}\|^2_{L_2(\overset{1}{\hat\Omega})}+c(1/\varepsilon)(\|\tilde v_{zt}\|^2_{L_2(\overset{1}{\hat\Omega})}+\|\tilde v_{tt}\|^2_{L_2(\overset{1}{\hat\Omega})}+\|\tilde v_{zz}\|^2_{L_2(\overset{1}{\hat\Omega})})\\
&\quad+c(\|F_{0z}\|^2_{L_2(\overset{1}{\hat\Omega})}+\|F_{0t}\|^2_{L_2(\overset{1}{\hat\Omega})}+\|g_{0zz}\|^2_{L_2(\overset{1}{\hat\Omega})}+\|g_{0tz}\|^2_{L_2(\overset{1}{\hat\Omega})}\\
&\quad+\|\partial_{z'}^{1/2}h_{0z'}\|^2_{L_2(S)}+\|\partial_{z'}^{1/2}h_{0t}\|^2_{L_2(S)}).
\end{aligned}
$$

Using the form of $F_0$, $g_0$, $h_0$ from the r.h.s. of (2.23) we estimate the last term in parentheses on the r.h.s. of (4.48). Then we have

$$
\begin{aligned}
\|F_{0z}\|_{L_2(\overset{1}{\hat\Omega})}&\le c\lambda(\|\tilde v_{zzz}\|_{L_2(\overset{1}{\hat\Omega})}+\|\tilde p_{zz}\|_{L_2(\overset{1}{\hat\Omega})})\\
&\quad+c(1/\lambda)(\|\hat v\|_{H^2(\overset{1}{\hat\Omega})}+\|\hat p\|_{H^1(\overset{1}{\hat\Omega})})+c\|\tilde F_z\|_{L_2(\overset{1}{\hat\Omega})},
\end{aligned}
\tag{4.49}
$$

$$
\begin{aligned}
\|F_{0t}\|_{L_2(\overset{1}{\hat\Omega})}&\le c\lambda(\|\tilde v_{zzt}\|_{L_2(\overset{1}{\hat\Omega})}+\|\tilde p_{zt}\|_{L_2(\overset{1}{\hat\Omega})})\\
&\quad+c(1/\lambda)(\|\hat v_t\|_{H^1(\overset{1}{\hat\Omega})}+\|\hat p_t\|_{L_2(\overset{1}{\hat\Omega})})+c\|\tilde F_t\|_{L_2(\overset{1}{\hat\Omega})},
\end{aligned}
\tag{4.50}
$$

$$
\|g_{0zz}\|_{L_2(\overset{1}{\hat\Omega})}\le c\lambda\|\tilde v_{zzz}\|_{L_2(\overset{1}{\hat\Omega})}+c(1/\lambda)\|\hat v\|_{H^2(\overset{1}{\hat\Omega})}c\|\tilde g_{zz}\|_{L_2(\overset{1}{\hat\Omega})},
\tag{4.51}
$$

$$
\|g_{0tz}\|_{L_2(\overset{1}{\hat\Omega})}\le c\lambda\|\tilde v_{tzz}\|_{L_2(\overset{1}{\hat\Omega})}+c(1/\lambda)\|\tilde v_t\|_{H^1(\overset{1}{\hat\Omega})}+c\|\tilde g_{tz}\|_{L_2(\overset{1}{\hat\Omega})},
\tag{4.52}
$$

$$
\begin{aligned}
\|\partial_{z'}^{1/2}h_{0z'}\|_{L_2(S)}&\le c\lambda(\|\tilde v_{zz}\|_{H^1(\overset{1}{\hat\Omega})}+\|\tilde p_z\|_{H^1(\overset{1}{\hat\Omega})})\\
&\quad+c(1/\lambda)(\|\hat v\|_{H^2(\overset{1}{\hat\Omega})}+\|\hat p\|_{H^1(\overset{1}{\hat\Omega})})+c\|\partial_{z'}^{1/2}\tilde h_{z'}\|_{L_2(S)},
\end{aligned}
\tag{4.53}
$$

$$
\begin{aligned}
\|\partial_{z'}^{1/2}h_{0t}\|_{L_2(S)}&\le c\lambda(\|\tilde v_{zt}\|_{H^1(\overset{1}{\hat\Omega})}+\|\tilde p_t\|_{H^1(\overset{1}{\hat\Omega})})\\
&\quad+c(1/\lambda)(\|\hat v_t\|_{L_2(S)}+\|\tilde p_t\|_{L_2(S)})+c\|\partial_{z'}^{1/2}\tilde h_t\|_{L_2(S)}.
\end{aligned}
\tag{4.54}
$$

Employing (4.49)–(4.54) in (4.48) and using that $\lambda$ is sufficiently small we have

$$
\begin{aligned}
&\frac{d}{dt}(\|\tilde v_{zz}\|^2_{L_2(\overset{1}{\hat\Omega})}+\|\tilde v_{zt}\|^2_{L_2(\overset{1}{\hat\Omega})})\\
&\qquad+\nu(\|\tilde v_{zzz}\|^2_{L_2(\overset{1}{\hat\Omega})}+\|\tilde v_{zzt}\|_{L_2(\overset{1}{\hat\Omega})}+\|\tilde p_{zz}\|^2_{L_2(\overset{1}{\hat\Omega})}+\|\tilde p_{zt}\|^2_{L_2(\overset{1}{\hat\Omega})})\\
&\quad\le \varepsilon\|\tilde v_{ztt}\|^2_{L_2(\overset{1}{\hat\Omega})}+c(1/\varepsilon)(\|\tilde v_{zz}\|^2_{L_2(\overset{1}{\hat\Omega})}+\|\tilde v_{zt}\|^2_{L_2(\overset{1}{\hat\Omega})}+\|\tilde v_{tt}\|^2_{L_2(\overset{1}{\hat\Omega})})\\
&\qquad+c(\|\hat v\|^2_{H^2(\overset{1}{\hat\Omega})}+\|\hat p\|^2_{H^1(\overset{1}{\hat\Omega})}+\|\hat v_t\|^2_{H^1(\overset{1}{\hat\Omega})}+\|\hat p_t\|^2_{L_2(\overset{1}{\hat\Omega})})\\
&\qquad+c(\|\tilde F\|^2_{L_2(\overset{1}{\hat\Omega})}+\|\tilde F_t\|^2_{L_2(\overset{1}{\hat\Omega})}+\|\tilde g_{zz}\|^2_{L_2(\overset{1}{\hat\Omega})}+\|\tilde g_{zt}\|^2_{L_2(\overset{1}{\hat\Omega})}\\
&\qquad+\|\partial_{z'}^{1/2}\tilde h_{z'}\|^2_{L_2(S)}+\|\partial_{z'}^{1/2}\tilde h_t\|^2_{L_2(S)}),
\end{aligned}
\tag{4.55}
$$

where we have used that

$$
\|\tilde p_t\|^2_{L_2(S)}\le\varepsilon\|\tilde p_{zt}\|^2_{L_2(\overset{1}{\hat\Omega})}+c(1/\varepsilon)\|\tilde p_t\|^2_{L_2(\overset{1}{\hat\Omega})}
$$

and $\varepsilon\in(0,1)$ is sufficiently small and $\|\tilde p_t\|_{L_2(\overset{1}{\hat\Omega})}\le\|\hat p_t\|_{L_2(\overset{1}{\hat\Omega})}$.

Integrating (4.55) with respect to time, passing to variables $\xi$ and summing over all neighborhoods of the partition of unity we obtain

$$
\begin{aligned}
(4.56)\qquad & \|\bar v_{\xi\xi}(t)\|^2_{L_2(\overset{1}{\Omega})}+\|\bar v_{\xi t}(t)\|^2_{L_2(\overset{1}{\Omega})}\\
&\quad+\nu(\|\bar v_{\xi\xi\xi}\|^2_{L_2(\overset{1}{\Omega}{}^t)}+\|\bar v_{\xi\xi t}\|^2_{L_2(\overset{1}{\Omega}{}^t)}+\|\tilde p_{\xi\xi}\|^2_{L_2(\overset{1}{\Omega}{}^t)}+\|\tilde p_{\xi t}\|^2_{L_2(\overset{1}{\Omega}{}^t)})\\
&\le\varepsilon\|\bar v_{\xi tt}\|^2_{L_2(\overset{1}{\Omega}{}^t)}+c(\|\bar v\|^2_{L_2(0,t;H^2(\overset{1}{\Omega}))}+\|\bar v_t\|^2_{L_2(0,t;H^1(\overset{1}{\Omega}))}+\|\bar v_{tt}\|^2_{L_2(\overset{1}{\Omega}{}^t)})\\
&\quad+c(\|\bar p\|^2_{L_2(0,t;H^1(\overset{1}{\Omega}))}+\|\bar p_t\|_{L_2(\overset{1}{\Omega}{}^t)})\\
&\quad+c(\|F\|_{L_2(\overset{1}{\Omega}{}^t)}+\|F_t\|^2_{L_2(\overset{1}{\Omega}{}^t)}+\|g\|^2_{L_2(0,t;H^2(\overset{1}{\Omega}))}+\|g_t\|^2_{L_2(0,t;H^1(\overset{1}{\Omega}))}\\
&\quad+\|h\|^2_{L_2(0,t;H^{3/2}(S))}+\|h_t\|^2_{L_2(0,t;H^{1/2}(S))}+\|v_{\xi\xi}(0)\|^2_{L_2(\overset{1}{\Omega})}+\|v_{\xi t}(0)\|^2_{L_2(\overset{1}{\Omega})}).
\end{aligned}
$$

Hence we obtain (4.29). ∎

Finally, we formulate the lemma which summarizes all estimates concerning velocity and pressure.

LEMMA 4.3. *Assume that the following quantities are finite:*

$$
\begin{aligned}
A_1^2(t)&=\|F\|^2_{L_2(0,t;\Gamma^1_0(\overset{1}{\Omega}))}+\|g\|^2_{\Gamma^2_{1,2}(\overset{1}{\Omega}{}^t)}+\|h\|^2_{L_2(0,t;\Gamma^{3/2}_0(S))}\\
&\quad+\|h_t\|^2_{L_2(0,t;H^{1/2}(S))}+\|h\|^2_{L_\infty(0,t;L_2(S))}+\|v(0)\|^2_{\Gamma^2_0(\overset{1}{\Omega})},\\
A_2^2(t)&=\sum_{i=1}^2\langle\partial_t^iF_1,\partial_t^i\bar v\rangle_{L_2(\overset{1}{\Omega}{}^t)}+\sum_{i=0}^2\|\bar f\|^2_{H^i(0,t;L_2(\overset{1}{\Omega}))},
\end{aligned}
$$

*where $t\le T$ and*

$$
\langle F_1,\bar v\rangle_{L_2(\overset{1}{\Omega}{}^t)}=\mu_1\int_{\overset{1}{\Omega}{}^t}\mathbb{T}(\overset{1}{\bar H})\cdot\mathbb{D}_{\overset{1}{\bar u}}(\bar v)\,d\xi\,dt'.
$$

*Then*

$$
(4.57)\qquad \|\bar v(t)\|^2_{\Gamma^2_0(\overset{1}{\Omega})}+\|\bar v\|^2_{L_2(0,t;\Gamma^3_1(\overset{1}{\Omega}))}+\|\bar p\|^2_{L_2(0,t;\Gamma^2_1(\overset{1}{\Omega}))}\le c(A_1^2(t)+A_2^2(t)),\quad t\le T.
$$

*Proof.* Consider problem (2.14). For sufficiently small $\varepsilon$ inequality (4.1) implies

$$
\begin{aligned}
(4.58)\qquad & \|\bar v(t)\|^2_{H^1(\overset{1}{\Omega})}+\|\bar v\|^2_{\Gamma^2_{1,2}(\overset{1}{\Omega}{}^t)}+\|\bar p\|^2_{L_2(0,t;H^1(\overset{1}{\Omega}))}\\
&\le c(\|F\|^2_{L_2(\overset{1}{\Omega}{}^t)}+\|g\|^2_{\Gamma^1_{0,2}(\overset{1}{\Omega}{}^t)}+\|h\|^2_{L_2(0,t;H^{1/2}(S))}\\
&\quad+\|h\|^2_{L_\infty(0,t;L_2(S))}+\|h_t\|^2_{L_2(S^t)}+\|v(0)\|^2_{H^1(\overset{1}{\Omega})}),
\end{aligned}
$$

where $t\le T$ and $F$, $g$, $h$ are defined in (2.14).

Inequality (4.29) can be expressed in the form

$$
\begin{aligned}
(4.59)\qquad & \|\bar v_{\xi\xi}(t)\|^2_{L_2(\overset{1}{\Omega})}+\|\bar v_{\xi t}(t)\|^2_{L_2(\overset{1}{\Omega})}+\|\bar v_{\xi\xi}\|^2_{L^1_2(\overset{1}{\Omega}{}^t)}+\|\bar p_\xi\|^2_{L^1_2(\overset{1}{\Omega}{}^t)}\\
&\le\varepsilon\|\bar v_{\xi tt}\|^2_{L_2(\overset{1}{\Omega}{}^t)}+c(\|\bar v_{tt}\|^2_{L^1_2(\overset{1}{\Omega}{}^t)}+\|\bar v\|^2_{\Gamma^2_{1,2}(\overset{1}{\Omega}{}^t)}+\|\bar p_t\|^2_{L_2(\overset{1}{\Omega}{}^t)}+\|\bar p\|^2_{L_2(0,t;H^1(\overset{1}{\Omega}))})
\end{aligned}
$$

$$+\, c(\|F\|^2_{L_2(0,t;H^1(\overset{1}{\Omega}))} + \|F_t\|^2_{L_2(\overset{1}{\Omega}{}^t)} + \|g\|^2_{\Gamma^2_{1,2}(\overset{1}{\Omega}{}^t)}$$
$$+\, \|h\|^2_{L_2(0,t;H^{3/2}(S))} + \|h_t\|^2_{L_2(0,t;H^{1/2}(S))}$$
$$+\, \|v_{\xi\xi}(0)\|^2_{L_2(\overset{1}{\Omega})} + \|v_{\xi t}(0)\|^2_{L_2(\overset{1}{\Omega})}).$$

Adding (4.58) and (4.59) yields

$$\begin{aligned}\text{(4.60)}\quad & \|\bar v(t)\|^2_{H^2(\overset{1}{\Omega})} + \|\bar v_{\xi t}\|^2_{L_2(\overset{1}{\Omega})} + \|\bar v\|^2_{L_2(0,t;H^3(\overset{1}{\Omega}))}\\ &\quad + \|\bar v_t\|^2_{L_2(0,t;H^2(\overset{1}{\Omega}))} + \|\bar p\|^2_{L_2(0,t;H^2(\overset{1}{\Omega}))} + \|\bar p_{\xi t}\|^2_{L_2(\overset{1}{\Omega}{}^t)}\\ &\quad \le \varepsilon\|\bar v_{\xi tt}\|^2_{L_2(\overset{1}{\Omega}{}^t)} + c(\|\bar v_{tt}\|^2_{L_2(\overset{1}{\Omega}{}^t)} + \|\bar p_t\|^2_{L_2(\overset{1}{\Omega}{}^t)})\\ &\quad + cA_1^2(t), \quad t \le T.\end{aligned}$$

Employing (3.18) in (4.60) and assuming that $T$ is sufficiently small we get

$$\begin{aligned}\text{(4.61)}\quad & \|\bar v(t)\|^2_{\Gamma^2_0(\overset{1}{\Omega})} + \|\bar v\|^2_{L_2(0,t;\Gamma^3_1(\overset{1}{\Omega}))} + \|\bar p\|^2_{L_2(0,t;\Gamma^2_1(\overset{1}{\Omega}))}\\ &\quad \le c\|\bar p_t\|^2_{L_2(\overset{1}{\Omega}{}^t)} + c(A_1^2(t) + A_2^2(t)) + \sum_{k=0}^2 \|\partial_t^k v(0)\|^2_{L_2(\overset{1}{\Omega})}, \quad t \le T.\end{aligned}$$

To estimate the first term on the r.h.s. of (4.61) we use $(2.14)_1$ in the form

$$\text{(4.62)}\qquad \bar v_t - \operatorname{div}_\xi \mathbb{D}_\xi(\bar v) + \nabla_\xi \bar p = F.$$

Differentiating (4.62) with respect to $t$ gives

$$\text{(4.63)}\qquad \bar v_{tt} - \operatorname{div}_\xi \mathbb{D}_\xi(\bar v_t) + \nabla_\xi \bar p_t = F_t.$$

Let $\varphi$ be a function such that

$$\text{(4.64)}\qquad \operatorname{div}_\xi \varphi = \bar p_t, \qquad \varphi|_S = 0.$$

Heving $\bar p_t \in L_2(\overset{1}{\Omega})$, [4] implies existence of solutions to (4.64) in $H^1(\overset{1}{\Omega})$ and the estimate

$$\text{(4.65)}\qquad \|\varphi\|_{H^1(\overset{1}{\Omega})} \le c\|\bar p_t\|_{L_2(\overset{1}{\Omega})}.$$

We multiply (4.63) by $\varphi$, integrate the result over $\overset{1}{\Omega}$, integrate by parts using $(4.64)_2$ and next apply $(4.64)_1$. Finally, employing (4.65) we get

$$\text{(4.66)}\qquad \|\bar p_t\|_{L_2(\overset{1}{\Omega}{}^t)} \le c(\|\bar v_t\|_{L_2(0,t;\Gamma^1_0(\overset{1}{\Omega}))} + \|F_t\|_{L_2(\overset{1}{\Omega}{}^t)}).$$

Exploiting (3.18) in (4.66) and assuming that $T$ is sufficiently small, we get

$$\begin{aligned}\text{(4.67)}\quad & \|\bar v_{tt}(t)\|^2_{L_2(\overset{1}{\Omega})} + \|\bar v_t\|^2_{L_2(0,t;\Gamma^1_0(\overset{1}{\Omega}))} + \|\bar p_t\|^2_{L_2(\overset{1}{\Omega}{}^t)}\\ &\quad \le c\sum_{i=1}^2 (\langle \partial_t^i \bar F_1, \partial_t^i \bar v\rangle_{L_2(\overset{1}{\Omega}{}^t)} + \|\partial_t^i \bar f\|^2_{L_2(\overset{1}{\Omega}{}^t)} + \|\partial_t^i v(0)\|^2_{L_2(\overset{1}{\Omega})}) + A_1^2(t).\end{aligned}$$

From (4.61) and (4.67) we obtain (4.57). ■

## 5. Method of successive approximations. Estimates for velocity

We prove the existence of solutions to problem (1.1) by the method of successive approximations. We restrict ourselves to proving estimates only because convergence is then easy to show. In this section we derive estimates for velocity and pressure. For given $v_n$, $\overset{1}{\bar{H}}_n$ we calculate $v_{n+1}$, $p_{n+1}$ from the following problem:

$$
(5.1)\quad
\begin{aligned}
& v_{n+1\,t} + v_n\cdot\nabla v_{n+1} - \operatorname{div}\mathbb{T}(v_{n+1},p_{n+1}) = -\mu_1\operatorname{div}\mathbb{T}(\overset{1}{H}_n)+f \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{in } \textstyle\bigcup_{0\le t\le T}\overset{1}{\Omega}_{nt}\times\{t\}=\tilde{\Omega}^T_{1n},\\
& \operatorname{div} v_{n+1}=0 \qquad\qquad\qquad\qquad\qquad\qquad\text{in } \tilde{\Omega}^T_{1n},\\
& \bar{n}\cdot\mathbb{T}(v_{n+1},p_{n+1}) = -\mu_1\bar{n}\cdot\mathbb{T}(\overset{1}{H}_n) \qquad\text{on } \textstyle\bigcup_{0\le t\le T} S_{nt}\times\{t\}=\tilde{S}^T_n,\\
& v_{n+1}|_{t=0}=v(0) \qquad\qquad\qquad\qquad\qquad\quad\text{in } \overset{1}{\Omega},
\end{aligned}
$$

where $\overset{1}{\Omega}_{nt}=\{x\in\mathbb{R}^3 : x=\xi+\int_0^t\bar{v}_n(\xi,t')\,dt',\ \xi\in\overset{1}{\Omega}\}$, $S_{nt}=\{x\in\mathbb{R}^3 : x=\xi+\int_0^t\bar{v}_n(\xi,t')\,dt',\ \xi\in S\}$.

To show convergence of the sequence we have to use lagrangian coordinates.

Using the notation introduced in Section 2 we express problem (5.1) in lagrangian coordinates in the form

$$
(5.2)\quad
\begin{aligned}
& \bar{v}_{n+1\,t} - \operatorname{div}_{\bar{v}_n}\mathbb{T}_{\bar{v}_n}(\bar{v}_{n+1},\bar{p}_{n+1}) = -\mu_1\operatorname{div}_{\bar{v}_n}\mathbb{T}(\overset{1}{\bar{H}}_n)+\bar{f},\\
& \operatorname{div}_{\bar{v}_n}\bar{v}_{n+1}=0,\\
& \bar{n}_{\bar{v}_n}\cdot\mathbb{T}_{\bar{v}_n}(\bar{v}_{n+1},\bar{p}_{n+1}) = -\mu_1\bar{n}_{\bar{v}_n}\mathbb{T}(\overset{1}{\bar{H}}_n),\\
& \bar{v}_{n+1}|_{t=0}=v(0).
\end{aligned}
$$

To apply Lemma 4.3 we express (5.2) in the form

$$
(5.3)\quad
\begin{aligned}
& \bar{v}_{n+1\,t} - \operatorname{div}_\xi\mathbb{T}_\xi(\bar{v}_{n+1},\bar{p}_{n+1})\\
&\quad = -(\operatorname{div}_\xi\mathbb{T}_\xi(\bar{v}_{n+1},\bar{p}_{n+1}),\ -\operatorname{div}_{\bar{v}_n}\mathbb{T}_{\bar{v}_n}(\bar{v}_{n+1},\bar{p}_{n+1})) - \mu_1\operatorname{div}_{\bar{v}_n}\mathbb{T}(\overset{1}{\bar{H}}_n)+\bar{f},\\
& \operatorname{div}_\xi\bar{v}_{n+1} = \operatorname{div}_\xi\bar{v}_{n+1} - \operatorname{div}_{\bar{v}_n}\bar{v}_{n+1},\\
& \bar{n}_\xi\cdot\mathbb{T}_\xi(\bar{v}_{n+1},\bar{p}_{n+1}) = \bar{n}_\xi\cdot\mathbb{T}_\xi(\bar{v}_{n+1},\bar{p}_{n+1}) - \bar{n}_{\bar{v}_n}\mathbb{T}_{\bar{v}_n}(\bar{v}_{n+1},\bar{p}_{n+1}) - \mu_1\bar{n}_{\bar{v}_n}\mathbb{T}(\overset{1}{\bar{H}}_n),\\
& \bar{v}_{n+1}|_{t=0}=v(0).
\end{aligned}
$$

Comparing (5.3) with (2.14) we have

$$
(5.4)\quad
\begin{aligned}
& F = -(\operatorname{div}_\xi\mathbb{T}_\xi(\bar{v}_{n+1},\bar{p}_{n+1}) - \operatorname{div}_{\bar{v}_n}\mathbb{T}_{\bar{v}_n}(\bar{v}_{n+1},\bar{p}_{n+1})) - \mu_1\operatorname{div}_{\bar{v}_n}\mathbb{T}(\overset{1}{\bar{H}}_n)+\bar{f},\\
& g = \operatorname{div}_\xi\bar{v}_{n+1} - \operatorname{div}_{\bar{v}_n}\bar{v}_{n+1},\\
& h = \bar{n}_\xi\cdot\mathbb{T}_\xi(\bar{v}_{n+1},\bar{p}_{n+1}) - \bar{n}_{\bar{v}_n}\cdot\mathbb{T}_{\bar{v}_n}(\bar{v}_{n+1},\bar{p}_{n+1}) - \mu_1\bar{n}_{\bar{v}_n}\mathbb{T}(\overset{1}{\bar{H}}_n),\\
& \langle F_1,\bar{v}_{n+1}\rangle_{L_2(\overset{1}{\Omega}{}^t)} = \mu_1\int_{\overset{1}{\Omega}{}^t}\mathbb{T}(\overset{1}{\bar{H}}_n)\cdot\mathbb{D}_{\bar{v}_n}(\bar{v}_{n+1})\,d\xi\,dt'.
\end{aligned}
$$

Let us introduce the quantities

$$
(5.5)\quad
\begin{aligned}
& \bar{X}_1(u,t) = \|\bar{u}\|_{L_\infty(0,t;\Gamma_0^2(\overset{1}{\Omega}))} + \|\bar{u}\|_{L_2(0,t;\Gamma_1^3(\overset{1}{\Omega}))}, \qquad \bar{P}_1(u,t) = \|\bar{u}\|_{L_2(0,t;\Gamma_1^2(\overset{1}{\Omega}))},\\
& \bar{D}_1 = \|\bar{f}\|_{L_2(0,t;\Gamma_0^1(\overset{1}{\Omega}))} + \|\bar{f}_{tt}\|_{L_2(\overset{1}{\Omega}{}^t)} + \|v(0)\|_{\Gamma_0^2(\overset{1}{\Omega})}.
\end{aligned}
$$

Lemma 5.1. *Assume that* $\bar D_1 < \infty$, $\bar X_1(\bar v_n) < \infty$, $\bar X_1(\bar{\overset{1}{H}}_n) < \infty$. *Then for sufficiently small* $t$ *there exists a positive increasing function* $\varphi$ *such that*

$$\bar X_1(\bar v_{n+1}, t) + \bar P_1(\bar p_{n+1}, t) \le \varphi(t^a \bar X_1(\bar v_n, t))\bar D_1 + \varphi(t^a \bar X_1(\bar{\overset{1}{H}}_n, t))\bar D_1 + c\bar D_1, \tag{5.6}$$

*where* $a > 0$.

*Proof.* We take some extension of $v(0)$ and $\overset{1}{H}(0)$ as a zero approximation. Then $p(0)$ is a solution to the problem

$$\begin{aligned} &\Delta p(0) = \operatorname{div}(\overset{1}{H}(0)\cdot\nabla\overset{1}{H}(0) - \overset{1}{H}_k(0)\nabla\overset{1}{H}_k(0)) - \operatorname{div}(v(0)\cdot\nabla v(0)) + f(0),\\ &p(0)|_S = \bar n\cdot\mathbb{D}(v(0)) + \bar n\cdot(\overset{1}{H}(0)\cdot\nabla\overset{1}{H}(0) - \overset{1}{H}_k(0)\cdot\nabla\overset{1}{H}_k(0)).\end{aligned} \tag{5.7}$$

Since $v(0), \overset{1}{H}(0) \in H^2(\Omega)$ we have $p(0) \in H^1(\Omega)$.

To prove the lemma we use Lemma 4.3. In view of (5.4) we use assumptions of Lemma 4.3. Consider the expression

$$\begin{aligned} \|F\|^2_{L_2(0,t;H^1(\overset{1}{\Omega}))} &\le \|\nabla_\xi\mathbb{D}_\xi(\bar v_{n+1}) - \nabla_{\bar v_n}\mathbb{D}_{\bar v_n}(\bar v_{n+1})\|_{L_2(0,t;H^1(\overset{1}{\Omega}))}\\ &\quad + \|\nabla_\xi\bar p_{n+1} - \nabla_{\bar v_n}\bar p_{n+1}\|^2_{L_2(0,t;H^1(\overset{1}{\Omega}))} + \|\operatorname{div}_{\bar v_n}\mathbb{T}(\bar{\overset{1}{H}}_n)\|_{L_2(0,t;H^1(\overset{1}{\Omega}))}\\ &\quad + \|\bar f\|^2_{L_2(0,t;H^1(\overset{1}{\Omega}))} \equiv \sum_{i=1}^4 I_i. \end{aligned}$$

Hence,

$$\begin{aligned} I_1 &\le \|(1-\xi_x^2(\bar v_n))\nabla_\xi^2\bar v_{n+1}\|^2_{L_2(0,t;H^1(\overset{1}{\Omega}))} + \|\xi_x(\bar v_n)\xi_{xx}(\bar v_n)\nabla_\xi\bar v_{n+1}\|^2_{L_2(0,t;H^1(\overset{1}{\Omega}))}\\ &\le \alpha(\bar v_n)\varphi(\bar v_n)\|\bar v_{n+1}\|^2_{L_2(0,t;H^3(\overset{1}{\Omega}))}\\ &\quad + \varphi(\alpha(\bar v_n))\|\xi_{xx}\bar v_{n+1\,\xi\xi}\|^2_{L_2(\overset{1}{\Omega}{}^t)} + \varphi(\alpha(\bar v_n))\|\xi_{xxx}\bar v_{n+1\,\xi}\|^2_{L_2(\overset{1}{\Omega}{}^t)} \equiv I_1^1, \end{aligned}$$

where $\alpha(\bar v_n) = t^{1/2}\|\bar v_n\|_{L_2(0,t;H^3(\overset{1}{\Omega}))}$.

Using

$$|\xi_x| \le \varphi(\alpha(\bar v_n)), \qquad |\xi_{xx}| \le \varphi(\alpha(\bar v_n))\Big|\int_0^t \bar v_{n\xi\xi}\,dt'\Big|, \qquad |\xi_{xxx}| \le \varphi(\alpha(\bar v_n))\Big|\int_0^t \bar v_{n\xi\xi\xi}\,dt'\Big|,$$

we obtain

$$I_1^1 \le \alpha(\bar v_n)\varphi(\alpha(\bar v_n))\|\bar v_{n+1}\|^2_{L_2(0,t;H^3(\overset{1}{\Omega}))}.$$

Hence for $\alpha(\bar v_n)$ sufficiently small, so for $t$ sufficiently small because $\|\bar v_n\|_{L_2(0,t;H^3(\overset{1}{\Omega}))}$ is bounded by a given constant, in view of inductive assumption, we see that $I_1^1$ is absorbed by the second term on the l.h.s. of (4.57).

Similarly we have

$$\begin{aligned} \|\operatorname{div}_{\bar v_n}\mathbb{T}(\bar{\overset{1}{H}}_n)\|_{L_2(0,t;H^1(\overset{1}{\Omega}))} &\le \varphi(\alpha(\bar v_n))(\|\bar{\overset{1}{H}}_n\bar{\overset{1}{H}}_{n\xi\xi}\|_{L_2(\overset{1}{\Omega}{}^t)} + \|\bar{\overset{1}{H}}{}^2_{n\xi}\|_{L_2(\overset{1}{\Omega}{}^t)})\\ &\quad + \varphi(\alpha(\bar v_n))\|\,|\xi_{xx}|\,|\overset{1}{H}_n|\,|\overset{1}{H}_{n\xi}|\,\|_{L_2(\overset{1}{\Omega}{}^t)}. \end{aligned}$$

Consider the integral

$$\Big(\int_0^t \|\bar{\overset{1}{H}}_n \bar{\overset{1}{H}}_{n\xi\xi}\|^2_{L_2(\overset{1}{\Omega})}\,dt'\Big)^{1/2} \le \Big(\int_0^t \|\bar{\overset{1}{H}}_n\|^2_{L_4(\overset{1}{\Omega})} \|\bar{\overset{1}{H}}_{n\xi\xi}\|^2_{L_4(\overset{1}{\Omega})}\,dt'\Big)^{1/2}$$

$$\le \sup_t (\varepsilon_1 \|\bar{\overset{1}{H}}_n\|_{H^1(\overset{1}{\Omega})} + c(1/\varepsilon_1)\|\bar{\overset{1}{H}}_n\|_{L_2(\overset{1}{\Omega})}) \Big[\int_0^t (\varepsilon_2 \|\bar{\overset{1}{H}}_{n\xi\xi}\|^2_{H^1(\overset{1}{\Omega})} + c(1/\varepsilon_2)\|\bar{\overset{1}{H}}_n\|^2_{L_2(\overset{1}{\Omega})})\,dt'\Big]^{1/2}$$

$$\le \sup_t \Big(\varepsilon_1 \|\bar{\overset{1}{H}}_n\|_{H^1(\overset{1}{\Omega})} + c(1/\varepsilon_1)\Big\|\int_0^t \bar{\overset{1}{H}}_{nt'}\,dt' + \overset{1}{H}(0)\Big\|_{L_2(\overset{1}{\Omega})}\Big)$$

$$\cdot \Big[\varepsilon_2 \int_0^t \|\bar{\overset{1}{H}}_{n\xi\xi}\|^2_{H^1(\overset{1}{\Omega})}\,dt' + c(1/\varepsilon_2)t\Big\|\int_0^t \bar{\overset{1}{H}}_{nt'}\,dt' + \overset{1}{H}(0)\Big\|^2_{L_2(\overset{1}{\Omega})}\Big]^{1/2}$$

$$\le \sup_t \Big[\varepsilon_1 \Big\|\int_0^t \bar{\overset{1}{H}}_{nt'}\,dt' + \overset{1}{H}(0)\Big\|_{H^1(\overset{1}{\Omega})} + c(1/\varepsilon_1)(t^{1/2}\|\bar{\overset{1}{H}}_{nt}\|_{L_2(\overset{1}{\Omega}{}^t)} + \|\overset{1}{H}(0)\|_{L_2(\overset{1}{\Omega})})\Big]$$

$$\cdot [\varepsilon_2 \|\bar{\overset{1}{H}}_n\|_{L_2(0,t;H^3(\overset{1}{\Omega}))} + c(1/\varepsilon_2)t(t^{1/2}\|\bar{\overset{1}{H}}_{nt}\|_{L_2(\overset{1}{\Omega}{}^t)} + \|\overset{1}{H}(0)\|_{L_2(\overset{1}{\Omega})})]$$

$$\le ct^a(\bar{X}_1^2(\bar{\overset{1}{H}}_n) + \|\overset{1}{H}(0)\|^2_{H^1(\overset{1}{\Omega})}).$$

Similar considerations can be applied to the other terms in $I_3$. Finally, we have

$$I_3 \le \varphi(\alpha(\bar{v}_n))t^a(\bar{X}_1^2(\bar{\overset{1}{H}}_n, t) + \|\overset{1}{H}(0)\|^2_{H^1(\overset{1}{\Omega})}).$$

Applying similar considerations to the other terms on the r.h.s. of (5.3) we obtain (5.6). ∎

## 6. Method of successive approximations. Estimates for magnetic fields

The functions $\overset{i}{H}_{n+1}$, $i = 1, 2$, are solutions to the problems

$$\text{(6.1)}\qquad \begin{aligned} &\mu_1(\overset{1}{H}_{n+1\,t} + \overset{1}{v}_n \cdot \nabla \overset{1}{H}_{n+1} - \overset{1}{H}_n \cdot \nabla \overset{1}{v}_n) - \frac{1}{\sigma_1}\Delta \overset{1}{H}_{n+1} = 0 && \text{in } \tilde{\Omega}^T_{1n},\\ &\operatorname{div} \overset{1}{H}_{n+1} = 0 && \text{in } \tilde{\Omega}^T_{1n},\\ &\overset{1}{H}_{n+1}|_{t=0} = \overset{1}{H}(0) && \text{in } \overset{1}{\Omega}, \end{aligned}$$

and

$$\text{(6.2)}\qquad \begin{aligned} &\mu_2(\overset{2}{H}_{n+1\,t} + \overset{2}{v}_n \cdot \nabla \overset{2}{H}_{n+1}) - \frac{1}{\sigma_2}\Delta \overset{2}{H}_{n+1} = \mu_2 \overset{2}{v}_n \cdot \nabla \overset{2}{H}_n + \overset{2}{G} \\ & && \text{in } \textstyle\bigcup_{0\le t\le T} \overset{2}{\Omega}_{nt} \times \{t\} = \tilde{\Omega}^T_{2n},\\ &\operatorname{div} \overset{2}{H}_{n+1} = 0 && \text{in } \tilde{\Omega}^T_{2n},\\ &\overset{2}{H}_{n+1}|_{t=0} = \overset{2}{H}(0) && \text{in } \overset{2}{\Omega}, \end{aligned}$$

where $v_n|_{\overset{i}{\Omega}_{nt}} = \overset{i}{v}_n$, $\overset{i}{\Omega}_{nt} = \big\{x \in \mathbb{R}^3 : x = \xi + \int_0^t \overset{i}{v}_n(\xi, t')\,dt',\ \xi \in \overset{i}{\Omega}\big\}$, $i = 1, 2$.

Assume the following transmission conditions hold:

$$
\begin{aligned}
&[\bar\tau\cdot H_{n+1}]=0,\ [\mu\bar n\cdot H_{n+1}]=0,\\
&\Big(\frac{1}{\sigma_1}\operatorname{rot}\overset{1}{H}_{n+1}-\mu_1 v_n\times\overset{1}{H}_{n+1}\Big)_\tau=\frac{1}{\sigma_2}(\operatorname{rot}\overset{2}{H}_{n+1})_\tau\quad\text{on } \tilde S_n^T.
\end{aligned}\tag{6.3}
$$

Moreover, we have

$$
[v_n]=0,\quad [\bar\tau\cdot H_n]=0,\quad [\mu\bar n\cdot H_n]=0,\quad \text{on } \tilde S_n^T,\tag{6.4}
$$

where $[\mu\bar n\cdot H]=\mu_1\bar n\cdot\overset{1}{H}-\mu_2\bar n\cdot\overset{2}{H}$ and $\mu_1=\mu_2$.

In this section we perform calculations in eulerian coordinates because the transmission conditions in these coordinates are clearer. In virtue of Remark 2.4 the derived estimates in eulerian coordinates will be easily transformed to lagrangian coordinates.

Let us introduce the quantities

$$
X_i(u,n)=\|u\|_{V_2^0(\overset{i}{\Omega}{}_{nt}^t)}+\|u_t\|_{V_2^0(\overset{i}{\Omega}{}_{nt}^t)}+\|u\|_{L_\infty(0,t;\Gamma_0^2(\overset{i}{\Omega}_{nt}))}+\|u\|_{H^{3,1}(\overset{i}{\Omega}{}_{nt}^t)},\qquad i=1,2,
$$

where

$$
\|u\|_{H^{3,1}(\Omega_t^t)}=\|u\|_{H^{2,1}(\Omega_t\times(0,t))}+\|u\|_{L_2(0,t;H^3(\Omega_t))}.
$$

Moreover

$$
\begin{aligned}
D_2={}&\|f\|_{L_2(0,t;H^1(\overset{1}{\Omega}_t))}+\|f_t\|_{L_2(\overset{1}{\Omega}{}_t^t)}+\|\overset{2}{G}\|_{H^1(\overset{2}{\Omega}{}_t^t)}+\|\overset{2}{G}_t\|_{L_2(\overset{2}{\Omega}{}_t^t)}\\
&+\sum_{i=1}^2(\|\overset{i}{v}(0)\|_{\Gamma_0^1(\overset{i}{\Omega})}+\|\overset{i}{H}(0)\|_{\Gamma_0^1(\overset{i}{\Omega})}),
\end{aligned}
$$

where we use the notation

$$
\|u\|_{L_2(\Omega_t^t)}=\|u\|_{L_2(0,t;L_2(\Omega_t))},\qquad X_0(n)=\sum_{i=1}^2(X_i(\overset{i}{v}_n,n)+X_i(\overset{i}{H}_n,n)).
$$

LEMMA 6.1. *Assume that $X_0(n)<\infty$, $D_2<\infty$. Then*

$$
\sum_{i=1}^2(\|\overset{i}{H}_{n+1}\|_{V_2^0(\overset{i}{\Omega}_{nt})}+\|\overset{i}{H}_{n+1\,t}\|_{V_2^0(\overset{i}{\Omega}_{nt})})\le t^a\varphi(t^aX_0(n))(1+D_2^2)+cD_2^2.\tag{6.5}
$$

*Proof.* The subscript $n$ in $\overset{i}{\Omega}_{nt}$ will be omitted for simplicity. Multiply (6.1) by $\overset{1}{H}_{n+1}$ and integrate the result over $\overset{1}{\Omega}_t$, next multiply (6.2) by $\overset{2}{H}_{n+1}$ and integrate over $\overset{2}{\Omega}_t$. Adding the results, using the transmission conditions and Lemma 2.5 yields

$$
\begin{aligned}
&\sum_{i=1}^2\Big(\mu_i\int_{\overset{i}{\Omega}_t}\overset{i}{H}{}_{n+1}^2\,dx+\frac{1}{\sigma_i}\int_0^t\|\overset{i}{H}_{n+1}\|^2_{H^1(\overset{i}{\Omega}_{t'})}\,dt'\Big)\\
&\quad\le c\Big|\int_0^t\int_{\overset{1}{\Omega}_{t'}}\overset{1}{H}_n\cdot\nabla\overset{1}{v}_n\cdot\overset{1}{H}_{n+1}\,dx\,dt'\Big|\\
&\qquad+c\Big|\int_0^t\int_{\overset{2}{\Omega}_{t'}}\overset{2}{v}_n\cdot\nabla\overset{2}{H}_n\cdot\overset{2}{H}_{n+1}\,dx\,dt'\Big|+c\Big|\int_0^t\int_{\overset{2}{\Omega}_{t'}}\overset{2}{G}\overset{2}{H}_{n+1}\,dx\,dt'\Big|\\
&\qquad+\mu_1\Big|\int_0^t\int_{S_{t'}}v_n\times\overset{1}{H}_{n+1}\cdot\bar n\times\overset{1}{H}_{n+1}S_{t'}\,dt'\Big|+\sum_{i=1}^2\mu_i\|\overset{i}{H}(0)\|^2_{L_2(\overset{i}{\Omega})}.
\end{aligned}\tag{6.6}
$$

The first term on the r.h.s. of (6.6) is estimated by

$$\varepsilon_1\int_0^t\|\overset{1}{H}_{n+1}\|^2_{L_6(\overset{1}{\Omega}_{t'})}\,dt'+c(1/\varepsilon_1)\sup_t\|\overset{1}{v}_n\|^2_{H^1(\overset{1}{\Omega}_t)}\int_0^t\|\overset{1}{H}_n\|^2_{L_3(\overset{1}{\Omega}_{t'})}\,dt',$$

where

$$\begin{aligned}\int_0^t\|\overset{1}{H}_n\|^2_{L_3(\overset{1}{\Omega}_{t'})}\,dt'&\le\varepsilon_2\int_0^t\|\overset{1}{H}_n\|^2_{H^1(\overset{1}{\Omega}_{t'})}\,dt'+c(1/\varepsilon_2)\int_0^t\|\overset{1}{H}_n\|^2_{L_2(\overset{1}{\Omega}_{t'})}\,dt'\\&\le\varepsilon_2\|\overset{1}{H}_n\|^2_{L_2(0,t;H^1(\overset{1}{\Omega}_t))}+c(1/\varepsilon_2)t\Big(\varphi(\alpha(\overset{1}{v}_n))t\int_0^t(\|\overset{1}{H}_{nt'}\|^2_{L_2(\overset{1}{\Omega}_{t'})}\\&\quad+\|\overset{1}{H}_{nx}\|^2_{L_2(\overset{1}{\Omega}_{t'})}\|\overset{1}{v}_n\|^2_{H^2(\overset{1}{\Omega}_{t'})})\,dt'+\|\overset{1}{H}(0)\|^2_{L_2(\overset{1}{\Omega})}\Big).\end{aligned}$$

Summarizing, the first term on the r.h.s. of (6.6) is bounded by

$$\varepsilon_1\|\overset{1}{H}_{n+1}\|^2_{L_2(0,t;H^1(\overset{1}{\Omega}_t))}+c(1/\varepsilon_1)t^a\varphi(t^aX_1(\overset{1}{v}_n,n))[X_1(\overset{1}{H}_n,n)+\|\overset{1}{H}(0)\|_{L_2(\overset{1}{\Omega})}].$$

The second term on the r.h.s. of (6.6) is estimated by

$$\varepsilon_2\int_0^t\|\overset{2}{H}_{n+1}\|^2_{L_6(\overset{2}{\Omega}_{t'})}\,dt'+c(1/\varepsilon_2)\sup_t\|\overset{2}{H}_n\|^2_{H^1(\overset{2}{\Omega}_t)}\int_0^t\|\overset{2}{v}_n\|^2_{L_3(\overset{2}{\Omega}_{t'})}\,dt',$$

where

$$\begin{aligned}&\int_0^t\|\overset{2}{v}_n\|^2_{L_3(\overset{2}{\Omega}_{t'})}\,dt'\le\varepsilon_3\int_0^t\|\overset{2}{v}_n\|^2_{H^1(\overset{2}{\Omega}_{t'})}\,dt'\\&\quad+c(1/\varepsilon_3)t\varphi(\alpha(\overset{2}{v}_n))\Big(t\int_0^t(\|\overset{2}{v}_{nt'}\|^2_{L_2(\overset{2}{\Omega}_{t'})}+\|\overset{2}{v}_{nx}\|^2_{L_2(\overset{2}{\Omega}_{t'})}\|\overset{2}{v}_n\|^2_{H^2(\overset{2}{\Omega}_{t'})})\,dt'+\|\overset{2}{v}(0)\|^2_{L_2(\overset{2}{\Omega})}\Big).\end{aligned}$$

Hence the second term on the r.h.s. of (6.6) is bounded by

$$\varepsilon_2\|\overset{2}{H}_{n+1}\|^2_{L_2(0,t;H^1(\overset{2}{\Omega}_t))}+c(1/\varepsilon_2)t^a\varphi(t^aX_2(\overset{2}{v}_n,n))[X_2(\overset{2}{v}_n,n)+\|\overset{2}{v}(0)\|^2_{L_2(\overset{2}{\Omega})}].$$

Finally, the third term on the r.h.s. of (6.6) is bounded by

$$\varepsilon_3\|\overset{2}{H}_{n+1}\|^2_{L_2(\overset{2}{\Omega}^t_t)}+c(1/\varepsilon_3)\|\overset{2}{G}\|^2_{L_2(\overset{2}{\Omega}^t_t)}$$

and the fourth term by

$$\int_0^t[\varepsilon_4\|\overset{1}{H}_{n+1}\|^2_{H^1(\overset{1}{\Omega}_{t'})}+c(1/\varepsilon_4)\|\overset{1}{H}_{n+1}\|^2_{L_2(\overset{1}{\Omega}_{t'})}]\,dt'\varphi(t^aX_1(\overset{1}{v}_n,n)).$$

Employing the above estimates in (6.6), using the fact that $\varepsilon_1$–$\varepsilon_4$ are sufficiently small and the Gronwall inequality we get

$$\text{(6.7)}\qquad \sum_{i=1}^{2}(\sigma_i\|\overset{i}{H}_{n+1}\|^2_{L_2(\overset{i}{\Omega}_t)}+\|\overset{i}{H}_{n+1}\|^2_{L_2(0,t;H^1(\overset{i}{\Omega}_t))})$$
$$\le ct^a\varphi(t^aX_1(\overset{1}{v}_n,n))[X_1(\overset{1}{H}_n,n)+\|\overset{1}{H}(0)\|^2_{L_2(\overset{1}{\Omega})}]$$
$$+ct^a\varphi(t^aX_2(\overset{2}{v}_n,n))X_2(\overset{2}{H}_n,n)[X_2(\overset{2}{v}_n,n)+\|\overset{2}{v}(0)\|^2_{L_2(\overset{2}{\Omega})}]+c\|\overset{2}{G}\|^2_{L_2(\overset{2}{\Omega}{}^t_t)}.$$

Differentiate (6.1) with respect to $t$, multiply by $\overset{1}{H}_{n+1\,t}$ and integrate over $\overset{1}{\Omega}_t$. Next, differentiate (6.2) with respect to $t$, multiply by $\overset{2}{H}_{n+1\,t}$ and integrate over $\overset{2}{\Omega}_t$. Adding the results and using the transmission conditions we get

$$\text{(6.8)}\qquad \sum_{i=1}^{2}\Big(\mu_i\int\limits_{\overset{i}{\Omega}_t}\overset{i}{H}{}^2_{n+1\,t}\,dx+\int\limits_0^t\|\overset{i}{H}_{n+1\,t'}\|^2_{H^1(\overset{i}{\Omega}_{t'})}\,dt'\Big)$$
$$=\sum_{i=1}^{2}\mu_i\int\limits_0^t\int\limits_{\overset{i}{\Omega}_{t'}}\overset{i}{v}_{nt'}\cdot\nabla\overset{i}{H}_{n+1}\cdot\overset{i}{H}_{n+1\,t'}\,dx\,dt'$$
$$+\mu_1\int\limits_0^t\int\limits_{\overset{1}{\Omega}_{t'}}(\overset{1}{H}_n\cdot\nabla\overset{1}{v}_n)_{t'}\cdot\overset{1}{H}_{n+1\,t'}\,dx\,dt'$$
$$+\mu_2\int\limits_0^t\int\limits_{\overset{2}{\Omega}_{t'}}(\overset{2}{v}_n\cdot\nabla\overset{2}{H}_n)_{t'}\cdot\overset{2}{H}_{n+1\,t'}\,dx\,dt'+\int\limits_0^t\int\limits_{\overset{2}{\Omega}_{t'}}\overset{2}{G}_{t'}\cdot\overset{2}{H}_{n+1\,t'}\,dx\,dt'$$
$$+\mu_1\int\limits_0^t\int\limits_{S_{t'}}(v_n\times\overset{1}{H}_{n+1})_{t'}\cdot\bar n\times\overset{1}{H}_{n+1\,t'}\,dS_{t'}\,dt'+\sum_{i=1}^{2}\mu_i\|\overset{i}{H}_t(0)\|^2_{L_2(\overset{i}{\Omega})}.$$

Now we examine the particular terms from the r.h.s. of (6.8). The first integral is bounded by

$$\varepsilon\sum_{i=1}^{2}\int\limits_0^t\|\overset{i}{H}_{n+1\,t'}\|^2_{L_6(\overset{i}{\Omega}_{t'})}\,dt'+c(1/\varepsilon)\sum_{i=1}^{2}\int\limits_0^t\|\overset{i}{v}_{nt'}\|^2_{L_3(\overset{i}{\Omega}_{t'})}\|\nabla\overset{i}{H}_{n+1}\|^2_{L_2(\overset{i}{\Omega}_{t'})}\,dt',$$

where the second integral is estimated by

$$\sum_{i=1}^{2}\sup_t\|\nabla\overset{i}{H}_{n+1}\|^2_{L_2(\overset{i}{\Omega}_t)}\int\limits_0^t\|\overset{i}{v}_{nt'}\|^2_{L_3(\overset{i}{\Omega}_{t'})}\,dt'$$
$$\le\sum_{i=1}^{2}\sup_t\|\nabla\overset{i}{H}_{n+1}\|^2_{L_2(\overset{i}{\Omega}_t)}\int\limits_0^t(\varepsilon\|\overset{i}{v}_{nt'x}\|^2_{L_2(\overset{i}{\Omega}_{t'})}+c(1/\varepsilon)\|\overset{i}{v}_{nt'}\|^2_{L_2(\overset{i}{\Omega}_{t'})})\,dt'$$
$$\le\sum_{i=1}^{2}\sup_t\|\nabla\overset{i}{H}_{n+1}\|^2_{L_2(\overset{i}{\Omega}_t)}[\varepsilon\|\overset{i}{v}_{ntx}\|^2_{L_2(0,t;L_2(\overset{i}{\Omega}_t))}+c(1/\varepsilon)t\sup_t\|\overset{i}{v}_{nt}\|^2_{L_2(\overset{i}{\Omega}_t)}]$$
$$\le\sum_{i=1}^{2}\sup_t\|\nabla\overset{i}{H}_{n+1}\|^2_{L_2(\overset{i}{\Omega}_t)}t^aX_i^2(\overset{i}{v}_n,n).$$

The second integral on the r.h.s. of (6.8) is bounded by

$$\Big|\int_0^t\int_{\overset{1}{\Omega}_t}\overset{1}{H}_{nt'}\cdot\nabla\overset{1}{v}_n\cdot\overset{1}{H}_{n+1\,t'}\,dx\,dt'\Big|+\Big|\int_0^t\int_{\overset{1}{\Omega}_t}\overset{1}{H}_n\cdot\nabla\overset{1}{v}_{nt'}\cdot\overset{1}{H}_{n+1\,t'}\,dx\,dt'\Big|\equiv I_1+I_2.$$

First, we examine

$$\begin{aligned}
I_1&\le\varepsilon\|\overset{1}{H}_{n+1\,t}\|^2_{L_2(0,t;L_6(\overset{1}{\Omega}_t))}\\
&\quad+c(1/\varepsilon)\sup_t\|\overset{1}{H}_{nt}\|^2_{L_2(\overset{1}{\Omega}_t)}\int_0^t(\varepsilon_1\|\nabla^2\overset{1}{v}_n\|^2_{L_2(\overset{1}{\Omega}_{t'})}+c(1/\varepsilon_1)\|\overset{1}{v}_n\|^2_{L_2(\overset{1}{\Omega}_{t'})})\,dt'\\
&\le\varepsilon\|\overset{1}{H}_{n+1\,t}\|^2_{L_2(0,t;H^1(\overset{1}{\Omega}_t))}\\
&\quad+c(1/\varepsilon)\sup_t\|\overset{1}{H}_{nt}\|^2_{L_2(\overset{1}{\Omega}_t)}\Big[\varepsilon_1\|\nabla^2\overset{1}{v}_n\|^2_{L_2(0,t;L_2(\overset{1}{\Omega}_t))}+c(1/\varepsilon_1)t\varphi(\alpha(\overset{1}{v}_n))\\
&\quad\cdot\Big(t\int_0^t(\|\overset{1}{v}_{nt'}\|^2_{L_2(\overset{1}{\Omega}_{t'})}+\|\overset{1}{v}_{nx}\|^2_{L_2(\overset{1}{\Omega}_{t'})}\|\overset{1}{v}_n\|^2_{H^2(\overset{1}{\Omega}_{t'})})\,dt'+\|\overset{1}{v}(0)\|^2_{L_2(\overset{1}{\Omega})}\Big)\Big]\\
&\le\varepsilon\|\overset{1}{H}_{n+1\,t}\|^2_{L_2(0,t;H^1(\overset{1}{\Omega}_t))}+c(1/\varepsilon)\varphi(t^aX_1(\overset{1}{v}_n,n))t^aX_1^2(\overset{1}{v}_n,n)X_1^2(\overset{1}{H}_n,n).
\end{aligned}$$

Next, we have

$$\begin{aligned}
I_2&=\Big|\int_0^t\int_{\overset{1}{\Omega}}\Big(\int_0^{t'}(\overset{1}{H}_{nt''}+\overset{1}{H}_{nx}\overset{1}{v}_n)\,dt''+\overset{1}{H}_n(0)\Big)\cdot\nabla\overset{1}{v}_{nt'}\cdot\overset{1}{H}_{n+1\,t'}\,d\xi\,dt'\Big|\\
&\le\Big|\int_0^t\int_{\overset{1}{\Omega}}\int_0^{t'}(\overset{1}{H}_{nt''}+\overset{1}{H}_{nx}\overset{1}{v}_n)\,dt''\cdot\nabla\overset{1}{v}_{nt'}\overset{1}{H}_{n+1\,t'}\,d\xi\,dt'\Big|\\
&\quad+\Big|\int_0^t\int_{\overset{1}{\Omega}}\overset{1}{H}(0)\cdot\nabla\overset{1}{v}_{nt'}\cdot\overset{1}{H}_{n+1\,t'}\,d\xi\,dt'\Big|=I_2^1+I_2^2,
\end{aligned}$$

where

$$\begin{aligned}
I_2^1&\le\varepsilon\int_0^t\|\overset{1}{H}_{n+1\,t'}\|^2_{L_6(\overset{1}{\Omega}_{t'})}\,dt'+c(1/\varepsilon)\|\nabla\overset{1}{v}_{nt}\|^2_{L_2(\overset{1}{\Omega}^t_t)}t\varphi(\alpha(\overset{1}{v}_n))\\
&\quad\cdot\int_0^t(\|\overset{1}{H}_{nt'}\|^2_{L_3(\overset{1}{\Omega}_{t'})}+\|\overset{1}{H}_{nx}\|^2_{L_2(\overset{1}{\Omega}_{t'})}\|\overset{1}{v}_n\|^2_{H^2(\Omega_{t'})})\,dt'\\
&\le\varepsilon\int_0^t\|\overset{1}{H}_{n+1\,t}\|^2_{H^1(\overset{1}{\Omega}_{t'})}\,dt'\\
&\quad+c(1/\varepsilon)t\varphi(\alpha(\overset{1}{v}_n))X_1^2(\overset{1}{v}_n,n)\int_0^t(\|\overset{1}{H}_{nt'}\|^2_{H^1(\overset{1}{\Omega}_{t'})}+\|\overset{1}{H}_{nx}\|^2_{L_2(\overset{1}{\Omega}_{t'})}\|\overset{1}{v}_n\|^2_{H^2(\overset{1}{\Omega}_{t'})})\,dt',
\end{aligned}$$

and

$$
\begin{aligned}
I_2^2 &\le \|\overset{1}{H}_{n+1\,t}\|_{L_{10/3}(\overset{1}{\Omega}{}_t^t)} \Big(\int_0^t \int_{\overset{1}{\Omega}_{t'}} |\overset{1}{H}(0)\nabla\overset{1}{v}_{nt'}|^{10/7}\,dx\,dt'\Big)^{7/10} \\
&\le \|\overset{1}{H}_{n+1\,t}\|_{L_{10/3}(\overset{1}{\Omega}{}_t^t)} \Big(\int_0^t \|\overset{1}{H}(0)\|^{10/7}_{L_6(\overset{1}{\Omega}_{t'})} \|\nabla\overset{1}{v}_{nt'}\|^{10/7}_{L_{15/8}(\overset{1}{\Omega}_{t'})}\,dt'\Big)^{7/10} \\
&\le \|\overset{1}{H}_{n+1\,t}\|_{L_{10/3}(\overset{1}{\Omega}{}_t^t)} \|\overset{1}{H}(0)\|_{H^1(\overset{1}{\Omega})} \Big(\int_0^t \|\nabla\overset{1}{v}_{nt'}\|^{10/7}_{L_{15/8}(\overset{1}{\Omega}_{t'})}\,dt'\Big)^{7/10} \\
&\le \|\overset{1}{H}_{n+1\,t}\|_{L_{10/3}(\overset{1}{\Omega}{}_t^t)} \|\overset{1}{H}(0)\|_{H^1(\overset{1}{\Omega})} t^a \|\nabla\overset{1}{v}_{nt}\|_{L_2(\overset{1}{\Omega}{}_t^t)} \\
&\le \varepsilon \|\overset{1}{H}_{n+1\,t}\|^2_{V_2^0(\overset{1}{\Omega}{}_t^t)} + c(1/\varepsilon) t^a X_1^2(\overset{1}{v}_n, n) \|\overset{1}{H}(0)\|^2_{H^1(\overset{1}{\Omega})}.
\end{aligned}
$$

Finally, we examine the third term on the r.h.s. of (6.8). We express it in the form

$$
\int_0^t \int_{\overset{2}{\Omega}_{t'}} \overset{2}{v}_{nt'} \cdot \nabla\overset{2}{H}_n \cdot \overset{2}{H}_{n+1\,t'}\,dx\,dt' + \int_0^t \int_{\overset{2}{\Omega}_{t'}} \overset{2}{v}_n \cdot \nabla\overset{2}{H}_{nt'} \cdot \overset{2}{H}_{n+1\,t'}\,dx\,dt' \equiv J_1 + J_2.
$$

First, examine

$$
|J_1| \le \varepsilon \int_0^t \|\overset{2}{H}_{n+1\,t'}\|^2_{L_6(\overset{2}{\Omega}_{t'})}\,dt' + c(1/\varepsilon) \sup_t \|\overset{2}{v}_{nt}\|^2_{L_2(\overset{2}{\Omega}_t)} \int_0^t \|\nabla\overset{2}{H}_n\|^2_{L_3(\overset{2}{\Omega}_{t'})}\,dt'.
$$

To find an estimate for $J_1$ we examine

$$
\int_0^t \|\nabla\overset{2}{H}_n\|^2_{L_3(\overset{2}{\Omega}_{t'})}\,dt' \le \varepsilon_1 \int_0^t \|\nabla^2\overset{2}{H}_n\|^2_{L_2(\overset{2}{\Omega}_{t'})}\,dt' + c(1/\varepsilon_1) \int_0^t \|\overset{2}{H}_n\|^2_{L_2(\overset{2}{\Omega}_{t'})}\,dt',
$$

where

$$
\begin{aligned}
&\int_0^t \|\overset{2}{H}_n\|^2_{L_2(\overset{2}{\Omega}_{t'})}\,dt' \\
&\le t^a \varphi(\alpha(\overset{2}{v}_n)) \Big( t \int_0^t (\|\overset{2}{H}_{nt'}\|^2_{L_2(\overset{2}{\Omega}_{t'})} + \|\overset{2}{H}_{nx}\|^2_{L_2(\overset{2}{\Omega}_{t'})} \|\overset{2}{v}_n\|^2_{H^2(\overset{2}{\Omega}_{t'})})\,dt' + \|\overset{2}{H}(0)\|^2_{L_2(\overset{2}{\Omega})}\Big).
\end{aligned}
$$

Summarizing,

$$
\begin{aligned}
|J_1| &\le \varepsilon \int_0^t \|\overset{2}{H}_{n+1\,t'}\|^2_{H^1(\overset{2}{\Omega}_{t'})}\,dt' \\
&\quad + c(1/\varepsilon) t^a X_2^2(\overset{2}{v}_n, n) \varphi(t^a X_2(\overset{2}{v}_n, n)) (t X_2^2(\overset{2}{H}_n, n) + \|\overset{2}{H}(0)\|^2_{L_2(\overset{2}{\Omega})}).
\end{aligned}
$$

Next, we examine

$$
|J_2| \le \Big| \int_0^t \int_{\overset{2}{\Omega}} \Big( \int_0^{t'} (\overset{2}{v}_{nt''} + \overset{2}{v}_{nx}\overset{2}{v}_n)\,dt'' + \overset{2}{v}(0)\Big) \cdot \nabla\overset{2}{H}_{nt'} \cdot \overset{2}{H}_{n+1\,t'}\,d\xi\,dt' \Big|
$$

$$\leq \Big|\int_0^t\int_{\overset{2}{\Omega}}\int_0^{t'}(\overset{2}{v}_{nt''}+\overset{2}{v}_{nx}\overset{2}{v}_n)\,dt''\cdot\nabla\overset{2}{H}_{nt'}\cdot\overset{2}{H}_{n+1\,t'}\,d\xi\,dt'\Big|$$
$$+\Big|\int_0^t\int_{\overset{2}{\Omega}}\overset{2}{v}(0)\cdot\nabla\overset{2}{H}_{nt'}\cdot\overset{2}{H}_{n+1\,t'}\,d\xi\,dt'\Big|$$
$$\equiv J_2^1+J_2^2,$$

where

$$J_2^1\leq\varepsilon\int_0^t\|\overset{2}{H}_{n+1\,t'}\|^2_{L_6(\overset{2}{\Omega}_{t'})}\,dt'+c(1/\varepsilon)\|\nabla\overset{2}{H}_{nt}\|^2_{L_2(\Omega_t^t)}t\varphi(\alpha(\overset{2}{v}_n))$$
$$\cdot\Big(\|\overset{2}{v}_{nt}\|^2_{L_2(0,t;H^1(\overset{2}{\Omega}_t))}+\int_0^t\|\overset{2}{v}_{nx}\|^2_{L_2(\overset{2}{\Omega}_{t'})}\|\overset{2}{v}_n\|^2_{H^2(\overset{2}{\Omega}_{t'})}\,dt'\Big)$$

and

$$J_2^2\leq\|\overset{2}{H}_{n+1\,t}\|_{L_{10/3}(\overset{2}{\Omega}_t^t)}\|\overset{2}{v}(0)\nabla\overset{2}{H}_{nt}\|_{L_{10/7}(\overset{2}{\Omega}_t^t)}$$
$$\leq\varepsilon\|\overset{2}{H}_{n+1\,t}\|^2_{V_2^0(\overset{2}{\Omega}_t^t)}+c(1/\varepsilon)t^aX_2^2(\overset{2}{H}_n,n)\|\overset{2}{v}(0)\|^2_{H^1(\overset{2}{\Omega})},$$

where similar estimations to those used in the bound of $I_2^2$ are employed.

The last but one term on the r.h.s. of (6.8) is estimated by

$$\int_0^t[\varepsilon\|\overset{1}{H}_{n+1\,t'}\|^2_{H^1(\overset{1}{\Omega}_{t'})}+c(1/\varepsilon)\|\overset{1}{H}_{n+1\,t'}\|^2_{\Gamma_0^1(\overset{1}{\Omega}_{t'})}\varphi(t^aX_1(\overset{1}{v}_n,n))]\,dt'.$$

Employing the above estimates in (6.8) and using the Gronwall lemma yields

$$\begin{aligned}(6.9)\qquad \sum_{i=1}^2\|\overset{i}{H}_{n+1\,t}\|^2_{V_2^0(\overset{i}{\Omega}_t^t)}&\leq\varepsilon\sum_{i=1}^2\|\overset{i}{H}_{n+1\,t}\|^2_{V_2^0(\overset{i}{\Omega}_t^t)}\\
&+c(1/\varepsilon)\sum_{i=1}^2\sup_t\|\nabla\overset{i}{H}_{n+1}\|^2_{L_2(\overset{i}{\Omega}_t)}t^aX_i^2(\overset{i}{v}_n,n)\\
&+c(1/\varepsilon)\varphi(t^aX_1(\overset{1}{v}_n,n))t^aX_1(\overset{1}{v}_n,n)(X_1^2(\overset{1}{H}_n,n)+\|\overset{1}{H}(0)\|^2_{H^1(\overset{1}{\Omega})})\\
&+c(1/\varepsilon)t^a\varphi(t^aX_2(\overset{2}{v}_n,n))X_2^2(\overset{2}{H}_n,n)(X_2^2(\overset{2}{v}_n,n)+\|\overset{2}{v}(0)\|^2_{H^1(\overset{2}{\Omega})})\\
&+c(1/\varepsilon)t^a\varphi(t^aX_2(\overset{2}{v}_n,n))X_2^2(\overset{2}{v}_n,n)\|\overset{2}{H}(0)\|^2_{H^1(\overset{2}{\Omega})}\\
&+\sum_{i=1}^2\|\overset{i}{H}_t(0)\|^2_{L_2(\overset{i}{\Omega})}+c\|\overset{2}{G}\|^2_{L_2(\overset{2}{\Omega}_t^t)}+c\|\overset{2}{G}_t\|^2_{L_2(\overset{2}{\Omega}_t^t)}.\end{aligned}$$

From (6.9), for sufficiently small $\varepsilon$, we obtain

$$\begin{aligned}(6.10)\qquad \sum_{i=1}^2\|\overset{i}{H}_{n+1\,t}\|^2_{V_2^0(\overset{i}{\Omega}_t^t)}&\leq c\sum_{i=1}^2\sup_t\|\nabla\overset{i}{H}_{n+1}\|^2_{L_2(\overset{i}{\Omega}_t)}t^aX_i^2(\overset{i}{v}_n,n)\\
&+c\varphi(t^aX_1(\overset{1}{v}_n,n),t^aX_1(\overset{1}{H}_n,n),t^aX_2(\overset{2}{v}_n,n),t^aX_2(\overset{2}{H}_n,n))\end{aligned}$$

$$\cdot\big[1+(\|\overset{2}{v}(0)\|^2_{H^1(\overset{2}{\Omega})}+\|\overset{1}{H}(0)\|^2_{H^1(\overset{1}{\Omega})}+\|\overset{2}{H}(0)\|^2_{H^1(\overset{2}{\Omega})})t^a\big]$$

$$+\sum_{i=1}^2 c\|\overset{i}{H}_t(0)\|^2_{L_2(\overset{i}{\Omega})}+c\|\overset{2}{G}\|^2_{L_2(\overset{2}{\Omega}{}^t)}+c\|\overset{2}{G}_t\|^2_{L_2(\overset{2}{\Omega}{}^t)}.$$

From (6.7) and (6.10) we obtain (6.5). ∎

We need local considerations to obtain estimates for space derivatives. For this purpose we make

Remark 6.2. Let us introduce a partition of unity $\overset{i}{\zeta}_k(x)$, $i=1,2$, $k\in\mathcal{M}\cup\mathcal{N}$, where $\operatorname{supp}\overset{i}{\zeta}_k$, $k\in\mathcal{M}$, is an interior subdomain of $\overset{i}{\Omega}_t$ and $\operatorname{supp}\overset{i}{\zeta}_k$, $k\in\mathcal{N}$, is a boundary subdomain. Then $\overset{i}{\zeta}_k$, $k\in\mathcal{N}$, are equal to 1 in some neighborhood of $S_t$. Moreover, $\overset{1}{\zeta}_k(x)=\overset{2}{\zeta}_k(x)$, $x\in S_t$, $k\in\mathcal{N}$.

Let $\xi_k\in S_t$, $k\in\mathcal{N}$, be in $\operatorname{supp}\overset{i}{\zeta}_k\cap S_t$, $i=1,2$. In the subdomain $\operatorname{supp}\overset{i}{\zeta}_k$, $k\in\mathcal{N}$, we introduce local curvilinear coordinates $\tau_1(x)$, $\tau_2(x)$, $n(x)$ such that for $x\in S_t$, $\tau_1(x)$, $\tau_2(x)$ are tangent coordinates to $S_t$ and $n(x)$ is the normal coordinate. With these coordinates there is connected a system of orthonormal vectors such that $\bar\tau_1(x)$, $\bar\tau_2(x)$, $x\in S_t$, are vectors tangent to $S_t$ and $\bar n(x)$, $x\in S_t$, is the unit outward normal vector to $S_t$. Moreover we assume that $\nabla\tau_\alpha=\bar\tau_\alpha$, $\nabla n=\bar n$, $|\bar\tau_\alpha|=1$, $|\bar n|=1$, $\alpha=1,2$, and $\bar\tau_1$, $\bar\tau_2$, $\bar n$ are orthogonal.

To localize problems (6.1) and (6.2) we introduce the notation $\overset{i,k}{H}=\overset{i}{H}\overset{i}{\zeta}_k$, $i=1,2$, $k\in\mathcal{M}\cup\mathcal{N}$.

To obtain an estimate for second space derivatives for solutions to (6.1) and (6.2) we restrict our considerations to a neighbourhood of $S_t$. For this purpose we take a partition of unity such that $\overset{1}{\zeta}_k$ and $\overset{2}{\zeta}_k$ are equal to one near $S_t$ and they vanish at some positive distance from $S_t$. Moreover, in a neighborhood of $S_t$ there exists a curvilinear system of orthonormal vectors $\bar\tau_1(x)$, $\bar\tau_2(x)$, $\bar n(x)$ for $x$ from this neighborhood.

Multiplying (6.1) by $\overset{1}{\zeta}_k$ we obtain

$$\begin{aligned}&\mu_1(\overset{1,k}{H}_{n+1\,t}+\overset{1}{v}_n\cdot\nabla\overset{1,k}{H}_{n+1}-\overset{1,k}{H}_n\cdot\nabla\overset{1}{v}_n)+\frac{1}{\sigma_1}\operatorname{rot}\operatorname{rot}\overset{1,k}{H}_{n+1}\\ &\quad=\mu_1\overset{1}{H}_{n+1}\overset{1}{\zeta}_{kt}+\mu_1\overset{1}{H}_{n+1}\overset{1}{v}_n\cdot\nabla\overset{1}{\zeta}_k+\frac{1}{\sigma_1}\operatorname{rot}(\nabla\overset{1}{\zeta}_k\times\overset{1}{H}_{n+1})+\frac{1}{\sigma_1}\nabla\overset{1}{\zeta}_k\times\operatorname{rot}\overset{1}{H}_{n+1},\\ &\overset{1,k}{H}_{n+1}|_{t=0}=\overset{1,k}{H}(0).\end{aligned}\tag{6.11}$$

Similarly, multiplying (6.2) by $\overset{2}{\zeta}_k$ yields

$$\begin{aligned}&\mu_2(\overset{2,k}{H}_{n+1\,t}+\overset{2}{v}_n\cdot\nabla\overset{2,k}{H}_{n+1})+\frac{1}{\sigma_2}\operatorname{rot}\operatorname{rot}\overset{2,k}{H}_{n+1}\\ &\qquad=\mu_2\overset{2}{H}_{n+1}\overset{2}{\zeta}_{kt}+\mu_2\overset{2}{v}_n\cdot\nabla\overset{2}{\zeta}_k\overset{2}{H}_{n+1}+\frac{1}{\sigma_2}\operatorname{rot}(\nabla\overset{2}{\zeta}_k\times\overset{2}{H}_{n+1})\\ &\qquad\quad+\frac{1}{\sigma_2}\nabla\overset{2}{\zeta}_k\times\operatorname{rot}\overset{2}{H}_{n+1}+\mu_2\overset{2}{v}_n\cdot\nabla\overset{2}{H}_n\overset{2}{\zeta}_k+\overset{k}{G}.\end{aligned}\tag{6.12}$$

LEMMA 6.3. *Let $X_0(n) < \infty$, $D_2 < \infty$. Then*

$$\sum_{i=1}^{2} \|\overset{i}{H}_{n+1\,x}\|^2_{V_2^0(\overset{i}{\Omega}{}^t_t)} \le t^a\varphi(t^a X_0(n))[t^a X_0^2(n) + D_2^2] + cD_2^2. \tag{6.13}$$

*Proof.* We need

$$\begin{aligned}
&\bar\tau \cdot \nabla \overset{i,k}{H}_t = \overset{i,k}{H}_{\tau t} - \bar\tau_t \cdot \nabla \overset{i,k}{H},\\
&\bar\tau \cdot \nabla(v \cdot \nabla \overset{i,k}{H}) = v_\tau \cdot \nabla \overset{i,k}{H} + v \cdot \nabla \overset{i,k}{H}_\tau - v \cdot \nabla(\tau \cdot \nabla)\overset{i,k}{H},\\
&\bar\tau \cdot \nabla\Delta \overset{i,k}{H} = \nabla(\bar\tau \cdot \nabla\nabla \overset{i,k}{H}) - \nabla(\bar\tau \cdot \nabla)\nabla \overset{i,k}{H} = \Delta \overset{i,k}{H}_\tau - \Delta(\bar\tau \cdot \nabla)\overset{i,k}{H} - \nabla(\bar\tau \cdot \nabla)\nabla \overset{i,k}{H},\\
&\bar\tau \cdot \nabla(\nabla \operatorname{div} \overset{i,k}{H}) = \nabla(\bar\tau \cdot \nabla \overset{i,k}{H}) - \nabla(\bar\tau \cdot \nabla)\operatorname{div} \overset{i,k}{H}\\
&\qquad = \nabla \operatorname{div} \overset{i,k}{H}_\tau - \nabla((\nabla_j \bar\tau \cdot \nabla)\overset{i,k}{H}_j) - \nabla(\bar\tau \cdot \nabla)\operatorname{div} \overset{i,k}{H}.
\end{aligned} \tag{6.14}$$

Applying the tangent derivative to (6.11), multiplying the result by $\overset{1,k}{H}_{n+1\,\tau}$ and integrating over $\overset{1}{\Omega}_k = \overset{1}{\Omega}_t \cap \operatorname{supp} \overset{1}{\zeta}_k$ and using (6.14) we get

$$\begin{aligned}
&\mu_1 \int_{\overset{1}{\Omega}_k} (\overset{1,k}{H}_{n+1\,\tau t} + \overset{1}{v}_n \cdot \nabla \overset{1,k}{H}_{n+1\,\tau}) \cdot \overset{1,k}{H}_{n+1\,\tau}\,dx + \frac{1}{\sigma_1}\int_{\overset{1}{\Omega}_k} \operatorname{rot}\operatorname{rot} \overset{1,k}{H}_{n+1\,\tau} \cdot \overset{1,k}{H}_{n+1\,\tau}\,dx\\
&\quad = \mu_1 \int_{\overset{1}{\Omega}_k} (\bar\tau_t \cdot \nabla \overset{1,k}{H}_{n+1} - \bar\tau \cdot \nabla(\overset{1}{v}_n) \cdot \nabla \overset{1,k}{H}_{n+1} + \overset{1}{v}_n \cdot \nabla(\bar\tau \cdot \nabla)\overset{1,k}{H}_{n+1}) \cdot \overset{1,k}{H}_{n+1\,\tau}\,dx\\
&\qquad + \mu_1 \int_{\overset{1}{\Omega}_k} \partial_\tau(\overset{1}{H}_{n+1}\overset{1}{\zeta}_{kt} + \overset{1}{H}_n \cdot \nabla \overset{1}{v}_n \overset{1}{\zeta}_k + \overset{1}{H}_{n+1}\overset{1}{v}_n \cdot \nabla \overset{1}{\zeta}_k) \cdot \overset{1,k}{H}_{n+1\,\tau}\,dx\\
&\qquad + \frac{1}{\sigma_1}\int_{\overset{1}{\Omega}_k} \partial_\tau(\operatorname{rot}(\nabla \overset{1}{\zeta}_k \times \overset{1}{H}_{n+1}) + \nabla \overset{1}{\zeta}_k \times \operatorname{rot} \overset{1}{H}_{n+1}) \cdot \overset{1,k}{H}_{n+1\,\tau}\,dx\\
&\qquad - \frac{1}{\sigma_1}\int_{\overset{1}{\Omega}_k} [(\operatorname{rot}\operatorname{rot} \overset{1,k}{H}_{n+1})_\tau - \operatorname{rot}\operatorname{rot} \overset{1,k}{H}_{n+1\,\tau}] \cdot \overset{1,k}{H}_{n+1\,\tau}\,dx.
\end{aligned} \tag{6.15}$$

Similarly we obtain

$$\begin{aligned}
&\mu_2 \int_{\overset{2}{\Omega}_k} (\overset{2,k}{H}_{n+1\,\tau t} + \overset{2}{v}_n \cdot \nabla \overset{2,k}{H}_{n+1\,\tau}) \cdot \overset{2,k}{H}_{n+1}\,dx + \frac{1}{\sigma_2}\int_{\overset{2}{\Omega}_k} \operatorname{rot}\operatorname{rot} \overset{2,k}{H}_{n+1\,\tau} \cdot \overset{2,k}{H}_{n+1\,\tau}\,dx\\
&= \mu_2 \int_{\overset{2}{\Omega}_k} (\bar\tau_t \cdot \nabla \overset{2,k}{H}_{n+1} - \bar\tau \cdot \nabla(\overset{2}{v}_n) \cdot \nabla \overset{2,k}{H}_{n+1} + \overset{2}{v}_n \cdot \nabla(\bar\tau \cdot \nabla)\overset{2,k}{H}_{n+1}) \cdot \overset{2,k}{H}_{n+1\,\tau}\,dx\\
&\quad + \mu_2 \int_{\overset{2}{\Omega}_k} \partial_\tau(\overset{2}{H}_{n+1}\overset{2}{\zeta}_{kt} + \overset{2}{H}_{n+1} \cdot \nabla \overset{2}{v}_n \overset{2}{\zeta}_k + \overset{2}{H}_{n+1}\overset{2}{v}_n \cdot \nabla \overset{2}{\zeta}_k + \overset{2}{v}_n \cdot \nabla \overset{2}{H}_n \overset{2}{\zeta}_k) \cdot \overset{2,k}{H}_{n+1\,\tau}\,dx\\
&\quad + \frac{1}{\sigma_2}\int_{\overset{2}{\Omega}_k} \partial_\tau(\operatorname{rot}(\nabla \overset{2}{\zeta}_k \times \overset{2}{H}_{n+1}) + \nabla \overset{2}{\zeta}_k \times \operatorname{rot} \overset{2}{H}_{n+1}) \cdot \overset{2,k}{H}_{n+1\,\tau}\,dx\\
&\quad + \int_{\overset{2}{\Omega}_k} \overset{k}{G}_\tau \cdot \overset{2,k}{H}_{n+1\,\tau}\,dx - \frac{1}{\sigma_2}\int_{\overset{2}{\Omega}_k} [(\operatorname{rot}\operatorname{rot} \overset{2,k}{H}_{n+1})_\tau - \operatorname{rot}\operatorname{rot} \overset{2,k}{H}_{n+1\,\tau}] \cdot \overset{2,k}{H}_{n+1\,\tau}\,dx,
\end{aligned} \tag{6.16}$$

where $\overset{2}{\Omega}_k = \overset{2}{\Omega}_t \cap \operatorname{supp} \overset{2}{\zeta}_k$.

Adding (6.15) and (6.16) and integrating with respect to time we obtain

$$\tag{6.17} \sum_{i=1}^{2}\frac{\mu_i}{2}\int_{\overset{i}{\Omega}_k}|\overset{i,k}{H}_{n+1\,\tau}|^2\,dx+\sum_{i=1}^{2}\frac{1}{\sigma_i}\int_0^t\int_{\overset{i}{\Omega}_k}\operatorname{rot}\operatorname{rot}\overset{i,k}{H}_{n+1\,\tau}\cdot\overset{i,k}{H}_{n+1\,\tau}\,dx\,dt'$$

$$= \sum_{i=1}^{2}\mu_i\int_0^t\int_{\overset{i}{\Omega}_k}(\bar\tau_{t'}\cdot\nabla\overset{i,k}{H}_{n+1}-\bar\tau\cdot\nabla(\overset{i}{v}_n)\cdot\nabla\overset{i,k}{H}_{n+1}+\overset{i}{v}_n\cdot\nabla(\bar\tau\cdot\nabla)\overset{i,k}{H}_{n+1})\cdot\overset{i,k}{H}_{n+1\,\tau}\,dx\,dt'$$

$$+\sum_{i=1}^{2}\mu_i\int_0^t\int_{\overset{i}{\Omega}_k}\partial_\tau(\overset{i}{H}_{n+1}\overset{i}{\zeta}_{kt}+\overset{i}{H}_{n+1}\cdot\nabla\overset{i}{v}_n\overset{i}{\zeta}_k+\overset{i}{H}_{n+1}\overset{i}{v}_n\cdot\nabla\overset{i}{\zeta}_k)\cdot\overset{i,k}{H}_{n+1\,\tau}\,dx\,dt'$$

$$+\mu_2\int_0^t\int_{\overset{2}{\Omega}_k}\partial_\tau(\overset{2}{v}_n\cdot\nabla\overset{2}{H}_n\overset{2}{\zeta}_k)\cdot\overset{2,k}{H}_{n+1\,\tau}\,dx\,dt'$$

$$+\sum_{i=1}^{2}\frac{1}{\sigma_i}\int_0^t\int_{\overset{i}{\Omega}_k}\partial_\tau(\operatorname{rot}(\nabla\overset{i}{\zeta}_k\times\overset{i}{H}_{n+1})+\nabla\overset{i}{\zeta}_k\times\operatorname{rot}\overset{i}{H}_{n+1})\cdot\overset{i,k}{H}_{n+1\,\tau}\,dx\,dt'$$

$$+\int_0^t\int_{\overset{2}{\Omega}_k}\overset{k}{G}_\tau\cdot\overset{2,k}{H}_{n+1\,\tau}\,dx\,dt'+\sum_{i=1}^{2}\frac{\mu_i}{2}\|\overset{i}{H}_\tau(0)\|^2_{L_2(\overset{i}{\Omega}_k)}$$

$$-\sum_{i=1}^{2}\frac{1}{\sigma_i}\int_{\overset{i}{\Omega}{}^t_k}[(\operatorname{rot}\operatorname{rot}\overset{i,k}{H}_{n+1})_\tau-\operatorname{rot}\operatorname{rot}\overset{i,k}{H}_{n+1\,\tau}]\cdot\overset{i,k}{H}_{n+1\,\tau}\,dx\,dt.$$

Now, we estimate the terms from the r.h.s. of (6.17). The first term is estimated by

$$\sum_{i=1}^{2}\Big(\varepsilon_1\int_0^t\|\overset{i,k}{H}_{n+1\,\tau}\|^2_{H^1(\overset{i}{\Omega}_k)}\,dt'$$
$$+c(1/\varepsilon_1)\sup_t\|\overset{i}{v}_n\|^2_{H^2(\overset{i}{\Omega}_k)}t\Big(\int_0^t\|\overset{i,k}{H}_{n+1\,t'}\|^2_{H^1(\overset{i}{\Omega}_k)}\,dt'+\|\overset{i}{H}(0)\|^2_{H^1(\overset{i}{\Omega}_k)}\Big)\Big),$$

and the second by

$$\sum_{i=1}^{2}\Big(\varepsilon_2\int_0^t\|\overset{i,k}{H}_{n+1\,\tau}\|^2_{H^1(\overset{i}{\Omega}_k)}\,dt'$$
$$+c(1/\varepsilon_2)\sup_t\|\overset{i}{v}_n\|^2_{H^2(\overset{i}{\Omega}_k)}t\Big(\int_0^t\|\overset{i,k}{H}_{n+1\,t'}\|^2_{H^1(\overset{i}{\Omega}_k)}\,dt'+\|\overset{i}{H}(0)\|^2_{H^1(\overset{i}{\Omega}_k)}\Big)\Big).$$

In view of (6.14) the last term on the r.h.s. of (6.17) is bounded by

$$\sum_{i=1}^{2}\varepsilon_3\int_0^t\|\overset{i,k}{H}_{n+1\,\tau}\|_{H^1(\overset{i}{\Omega}_k)}\,dt'+c(1/\varepsilon_3)t\int_0^t\|\overset{i}{v}_n\|^2_{H^3(\overset{i}{\Omega}_k)}\,dt\cdot\int_0^t\|\overset{i,k}{H}_{n+1}\|^2_{H^2(\overset{i}{\Omega}_k)}\,dt'.$$

The other terms in (6.17) can be estimated in a similar way.

Next we examine the last two terms on the l.h.s. of (6.17). Integrating by parts they are equal to

$$\sum_{i=1}^{2}\bigg[\frac{1}{\sigma_i}\int_0^t\int_{\overset{i}{\Omega}_k}|\operatorname{rot}\overset{i,k}{H}_\tau|^2\,dx\,dt'-\frac{1}{\sigma_i}\int_0^t\int_{S_k}\operatorname{rot}\overset{i,k}{H}_{n+1\,\tau}\cdot\overset{i,k}{H}_{n+1\,\tau}\times\overset{i}{\bar n}\,dS_{t'}\,dt'\bigg]. \tag{6.18}$$

By applying transmission conditions the sum of the last two terms becomes

$$\int_0^t\int_{S_k}\partial_\tau(\overset{1}{v}_n\times\overset{1,k}{H}_{n+1})\cdot\bar\tau_\mu\bar\tau_\mu\cdot\bar n\times\overset{1,k}{H}_{n+1\,\tau}\,dS_k\,dt'\equiv I_0, \tag{6.19}$$

where $S_k=S_t\cap\operatorname{supp}\overset{1}{\zeta}_k$. Hence

$$\begin{aligned}|I_0|&\le\varepsilon\|\overset{1,k}{H}_{n+1\,\tau}\|^2_{L_2(0,t;H^1(\overset{1}{\Omega}_k))}\\&\quad+c(1/\varepsilon)\int_0^t(\|\overset{1}{v}_n\|^2_{L_3(S_k)}\|\overset{1,k}{H}_{n+1\,\tau}\|^2_{L_3(S_k)}+\|\overset{1}{v}_{n\tau}\|^2_{L_3(S_k)}\|\overset{1,k}{H}\|^2_{L_3(S_k)})\,dt'\\&\le\varepsilon\|\overset{1,k}{H}_{n+1\,\tau}\|^2_{L_2(0,t;H^1(\overset{1}{\Omega}_k))}\\&\quad+c(1/\varepsilon)\sup_t\|\overset{1}{v}_{n\tau}\|^2_{H^1(\overset{1}{\Omega}_k)}\bigg[\varepsilon_1\int_0^t\|\overset{1,k}{H}_{n+1\,\tau}\|^2_{H^1(\overset{1}{\Omega}_k)}\,dt'+c(1/\varepsilon_1)t\sup_t\|\overset{1,k}{H}_{n+1\,\tau}\|^2_{L_2(\overset{1}{\Omega}_k)}\bigg].\end{aligned}$$

Using $-\Delta+\nabla\operatorname{div}=\operatorname{rot}\operatorname{rot}$ we express problems (6.11) and (6.12) in the form

$$\begin{aligned}&\mu_1(\overset{1,k}{H}_{n+1\,t}+\overset{1}{v}_n\cdot\nabla\overset{1,k}{H}_{n+1}-\overset{1,k}{H}_n\cdot\nabla\overset{1}{v}_n)-\frac{1}{\sigma_1}\Delta\overset{1,k}{H}_{n+1}\\&\quad=\mu_1\overset{1}{H}_{n+1}\overset{1}{\zeta}_{kt}+\mu_1\overset{1}{v}_n\cdot\nabla\overset{1}{\zeta}_k\overset{1}{H}_{n+1}-\frac{1}{\sigma_1}\nabla\overset{1}{H}_{n+1}\nabla\overset{1}{\zeta}_k-\frac{1}{\sigma_1}\overset{1}{H}_{n+1}\Delta\overset{1}{\zeta}_k,\\&\overset{1,k}{H}_{n+1}|_{t=0}=\overset{1}{H}(0)\end{aligned} \tag{6.20}$$

and

$$\begin{aligned}&\mu_2(\overset{2,k}{H}_{n+1\,t}+\overset{2}{v}_n\cdot\nabla\overset{2,k}{H}_{n+1})-\frac{1}{\sigma_2}\Delta\overset{2,k}{H}_{n+1}=\overset{k}{G}+\mu_2\overset{2}{H}_{n+1}\overset{2}{\zeta}_{kt}\\&\quad+\mu_2\overset{2}{v}_n\cdot\nabla\overset{2}{\zeta}_k\overset{2}{H}_{n+1}+\mu_2\overset{2}{v}_n\cdot\nabla\overset{2}{H}_n\overset{2}{\zeta}_k-\frac{1}{\sigma_2}\nabla\overset{2}{\zeta}_k\nabla\overset{2}{H}_{n+1}-\frac{1}{\sigma_2}\overset{2}{H}_{n+1}\Delta\overset{2}{\zeta}_k,\\&\overset{2,k}{H}|_B=0,\quad\overset{2,k}{H}|_{t=0}=\overset{2}{H}(0).\end{aligned} \tag{6.21}$$

Using curvilinear coordinates we have

$$\Delta u=u_{nn}+u_{\tau_\alpha\tau_\alpha}+\Delta\tau_\alpha u_{\tau_\alpha}+\Delta n u_n,$$

where the summation convention over $\alpha=1,2$ is assumed. Then (6.20) implies

$$\begin{aligned}\overset{1,k}{H}_{n+1\,nn}=&-\overset{1,k}{H}_{n+1\,\tau_\alpha\tau_\alpha}-\Delta\tau_\alpha\overset{1,k}{H}_{n+1\,\tau_\alpha}-\Delta n\overset{1,k}{H}_{n+1\,n}\\&+\sigma_1\mu_1(\overset{1,k}{H}_{n+1\,t}+\overset{1}{v}_n\cdot\nabla\overset{1,k}{H}_{n+1}-\overset{1,k}{H}_n\cdot\nabla\overset{1}{v}_n)\\&-\sigma_1\mu_1(\overset{1}{H}_{n+1}\overset{1}{\zeta}_{kt}+\overset{1}{v}_n\cdot\nabla\overset{1}{\zeta}_k\overset{1}{H}_{n+1})+\nabla\overset{1}{H}_{n+1}\nabla\overset{1}{\zeta}_k+\overset{1}{H}_{n+1}\Delta\overset{1}{\zeta}_k\end{aligned} \tag{6.22}$$

and (6.21) gives

$$
\begin{aligned}
\overset{2,k}{H}_{n+1\,nn} = {} & -\overset{2,k}{H}_{n+1\,\tau_\alpha\tau_\alpha} - \Delta\tau_\alpha \overset{2,k}{H}_{n+1\,\tau_\alpha} - \Delta n \overset{2,k}{H}_{n+1\,n} \\
& + \sigma_2\mu_2(\overset{2,k}{H}_{n+1\,t} + \overset{2}{v}_n \cdot \nabla \overset{2,k}{H}_{n+1}) - \sigma_2 \overset{k}{G} \\
& - \sigma_2\mu_2(\overset{2}{v}_n \cdot \nabla \overset{2}{\zeta}_k \overset{2}{H}_{n+1} + \overset{2}{v}_n \cdot \nabla \overset{2}{H}_n \overset{2}{\zeta}_k) \\
& + \nabla \overset{2}{\zeta}_k \nabla \overset{2}{H}_{n+1} + \overset{2}{H}_{n+1} \Delta \overset{2}{\zeta}_k.
\end{aligned} \tag{6.23}
$$

We estimate (6.22), (6.23) in a similar way to (6.39). Now using the expression

$$
\frac{1}{2}\mu_i \frac{d}{dt} \int_{\overset{i}{\Omega}_t} \overset{i}{H}{}^2_{n+1\,x}\,dx = \mu_i \int_{\overset{i}{\Omega}_t} (\overset{i}{H}_{n+1\,x} \overset{i}{H}_{n+1\,xt} + \overset{i}{v}_n \cdot \nabla \overset{i}{H}_{n+1\,x} \cdot \overset{i}{H}_{n+1\,x})\,dx, \quad i = 1, 2,
$$

and summing up over all neighborhoods of the partition of unity, taking $t$, $\varepsilon_i$ sufficiently small and using (6.5) we derive (6.13). ∎

Let

$$
\begin{aligned}
X^2(\overset{1}{v}_n, \overset{2}{v}_n, \overset{1}{H}_n, \overset{2}{H}_n, n, t) = {} & \sum_{i=1}^{2} (\|\overset{i}{v}_n\|^2_{L_\infty(0,t;\Gamma^2_0(\overset{i}{\Omega}_t))} + \|\overset{i}{v}_n\|^2_{L_2(0,t;\Gamma^3_1(\overset{i}{\Omega}_t))} \\
& + \|\overset{i}{H}_n\|^2_{L_\infty(0,t;\Gamma^2_0(\overset{i}{\Omega}_t))} + \|\overset{i}{H}_n\|^2_{L_2(0,t;\Gamma^3_1(\overset{i}{\Omega}_t))}),
\end{aligned} \tag{6.24}
$$

$$
D^2 = \sum_{i=1}^{2} (\|\overset{i}{v}(0)\|^2_{\Gamma^2_0(\overset{i}{\Omega})} + \|\overset{i}{H}(0)\|^2_{\Gamma^2_0(\overset{i}{\Omega})}) + \|f\|^2_{L_2(0,t;\Gamma^2_0(\overset{1}{\Omega}_t))} + \|\overset{2}{G}\|^2_{L_2(0,t;\Gamma^2_0(\overset{2}{\Omega}_t))}.
$$

Lemma 6.4. *Let $X(\overset{1}{v}_n, \overset{2}{v}_n, \overset{1}{H}_n, \overset{2}{H}_n, n, t) < \infty$, $D < \infty$. Then*

$$
\begin{aligned}
\sum_{i=1}^{2} (\|\overset{i}{H}_{n+1\,tt}\|^2_{V^0_2(\overset{i}{\Omega}{}^t_t)} + \|\overset{i}{H}_{n+1\,xt}\|^2_{V^0_2(\overset{i}{\Omega}{}^t_t)} + \|\overset{i}{H}_{n+1\,xx}\|^2_{V^0_2(\overset{i}{\Omega}{}^t_t)}) \\
\le t^a \varphi(t^a X(\overset{1}{v}_n, \overset{2}{v}_n, \overset{1}{H}_n, \overset{2}{H}_n, n, t)) + \varphi(D).
\end{aligned} \tag{6.25}
$$

*Proof.* Differentiate $(6.1)_1$ and $(6.2)_1$ with the operator $-\Delta$ replaced by rot rot twice with respect to time, multiply by $\overset{1}{H}_{n+1\,tt}$ and $\overset{2}{H}_{n+1\,tt}$, respectively, integrate over $\overset{1}{\Omega}_t$ and $\overset{2}{\Omega}_t$, and with respect to time. Adding we get

$$
\begin{aligned}
& \sum_{i=1}^{2} \Big(\mu_i \int_{\overset{i}{\Omega}_t} \overset{i}{H}{}^2_{n+1\,tt}\,dx + \frac{1}{\sigma_i} \int_0^t \int_{\overset{i}{\Omega}_{t'}} |\mathrm{rot}\, \overset{i}{H}_{n+1\,t't'}|^2\,dx\,dt'\Big) + I_0 \\
& \le \Big|\mu_1 \int_{\overset{1}{\Omega}{}^t_t} (\overset{1}{v}_{nt't'} \cdot \nabla \overset{1}{H}_{n+1} + 2\overset{1}{v}_{nt'} \cdot \nabla \overset{1}{H}_{n+1\,t'}) \cdot \overset{1}{H}_{n+1\,t't'}\,dx\,dt'\Big| \\
& \quad + \Big|\mu_1 \int_{\overset{1}{\Omega}{}^t_t} (\overset{1}{H}_{nt't'} \cdot \nabla \overset{1}{v}_n + 2\overset{1}{H}_{nt'} \cdot \nabla \overset{1}{v}_{nt'} + \overset{1}{H}_n \cdot \nabla \overset{1}{v}_{nt't'}) \cdot \overset{1}{H}_{n+1\,t't'}\,dx\,dt'\Big| \\
& \quad + \Big|\mu_2 \int_{\overset{2}{\Omega}{}^t_t} (\overset{2}{v}_{nt't'} \cdot \nabla \overset{2}{H}_{n+1} + \overset{2}{v}_{nt'} \cdot \nabla \overset{2}{H}_{n+1\,t'}) \cdot \overset{2}{H}_{n+1\,t't'}\,dx\,dt'\Big|
\end{aligned} \tag{6.26}
$$

$$+\Big|\mu_2\int\limits_{\overset{2}{\Omega}{}^t_t}(\overset{2}{v}_{nt't'}\cdot\nabla\overset{2}{H}_n+\overset{2}{v}_{nt'}\cdot\nabla\overset{2}{H}_{nt'}+\overset{2}{v}_n\cdot\nabla\overset{2}{H}_{nt't'})\cdot\overset{2}{H}_{n+1\,t't'}\,dx\,dt'\Big|$$

$$+\Big|\int\limits_{\overset{2}{\Omega}{}^t_t}\overset{2}{G}_{t't'}\cdot\overset{2}{H}_{n+1\,t't'}\,dx\,dt'\Big|+\sum_{i=1}^2\sigma_i\|\overset{i}{H}_{tt}(0)\|^2_{L_2(\overset{i}{\Omega})}\equiv\sum_{i=1}^6 I_i,$$

where

$$I_0=\frac{1}{\sigma_1}\int\limits_0^t\int\limits_{S_{t'}}\operatorname{rot}\overset{1}{H}_{n+1\,t't'}\cdot\overset{1}{\bar n}\times\overset{1}{H}_{n+1\,t't'}\,dS_{t'}+\frac{1}{\sigma_2}\int\limits_{S_{t'}}\operatorname{rot}\overset{2}{H}_{n+1\,t't'}\cdot\overset{2}{\bar n}\times\overset{2}{H}_{n+1\,t't'}\,dS_{t'}.$$

Since $\overset{1}{\bar n}=-\overset{2}{\bar n}\equiv\bar n$ we express $I_0$ in the form

$$\begin{aligned}I_0=&\frac{1}{\sigma_1}\int\limits_0^t\int\limits_{S_{t'}}\operatorname{rot}\overset{1}{H}_{n+1\,t't'}\cdot\bar\tau_\mu\bar\tau_\mu\cdot\bar n\times\overset{1}{H}_{n+1\,t't'}\,dS_{t'}\,dt'\\&-\frac{1}{\sigma_2}\int\limits_0^t\int\limits_{S_{t'}}\operatorname{rot}\overset{2}{H}_{n+1\,t't'}\cdot\bar\tau_\mu\bar\tau_\mu\cdot\bar n\times\overset{2}{H}_{n+1\,t't'}\,dS_{t'}\,dt',\end{aligned}$$

where $\bar\tau_\mu$, $\mu=1,2$, is a tangent vector to $S_t$, $\bar\tau_1$, $\bar\tau_2$ are linearly independent and the summation convention over repeated indices is assumed.

Continuing,

$$\begin{aligned}I_0=&\bigg[\frac{1}{\sigma_1}\int\limits_0^t\int\limits_{S_{t'}}(\operatorname{rot}\overset{1}{H}_{n+1}\cdot\bar\tau_\mu)_{t't'}\bar\tau_\mu\times\bar n\cdot\overset{1}{H}_{n+1\,t't'}\,dS_{t'}\,dt'\\&-\frac{1}{\sigma_2}\int\limits_0^t\int\limits_{S_{t'}}(\operatorname{rot}\overset{2}{H}_{n+1}\cdot\bar\tau_\mu)_{t't'}\bar\tau_\mu\times\bar n\cdot\overset{2}{H}_{n+1\,t't'}\,dS_{t'}\,dt'\bigg]\\&-\frac{1}{\sigma_1}\int\limits_0^t\int\limits_{S_{t'}}(\operatorname{rot}\overset{1}{H}_{n+1}\cdot\bar\tau_{\mu t't'}+2\operatorname{rot}\overset{1}{H}_{n+1\,t'}\cdot\bar\tau_{\mu t'})\bar\tau_\mu\times\bar n\cdot\overset{1}{H}_{n+1\,t't'}\,dS_{t'}\,dt'\\&+\frac{1}{\sigma_2}\int\limits_0^t\int\limits_{S_{t'}}(\operatorname{rot}\overset{2}{H}_{n+1}\cdot\bar\tau_{\mu t't'}+2\operatorname{rot}\overset{2}{H}_{n+1\,t'}\cdot\bar\tau_{\mu t'})\cdot\bar\tau_\mu\times\bar n\cdot\overset{2}{H}_{n+1\,t't'}\,dS_{t'}\,dt'\\&\equiv I_0^1+I_0^2+I_0^3,\end{aligned}$$

where

$$I_0^1=\int\limits_0^t\int\limits_{S_{t'}}(v_n\times\overset{1}{H}_{n+1}\cdot\bar\tau_\mu)_{t't'}\bar\tau_\mu\times\bar n\cdot\overset{1}{H}_{n+1\,t't'}\,dS_{t'}\,dt'.\tag{6.27}$$

Hence

$$\begin{aligned}|I_0^1|\le{}&\varepsilon\|\overset{1}{H}_{n+1\,tt}\|^2_{L_2(0,t;H^1(\overset{1}{\Omega}_t))}\\&+c(1/\varepsilon)\int\limits_0^t(\|\overset{1}{H}_{n+1\,t't'}\|^2_{L_3(S_{t'})}\|v_n\|^2_{L_3(S_{t'})}\\&+\|\overset{1}{H}_{n+1\,t'}\|^2_{L_3(S_{t'})}\|v_{nt'}\|^2_{L_3(S_{t'})}+\|\overset{1}{H}_{n+1}\|^2_{L_3(S_{t'})}\|v_{nt't'}\|^2_{L_3(S_{t'})})\,dt',\end{aligned}$$

where

$$\int_0^t \|\overset{1}{H}_{n+1\,t't'}\|^2_{L_3(S_{t'})}\|v_n\|^2_{L_3(S_{t'})}\,dt'$$
$$\le \sup_{t'}\|v_n\|^2_{H^1(\overset{1}{\Omega}_{t'})}\cdot\int_0^t[\varepsilon_1\|\overset{1}{H}_{n+1\,t't'}\|^2_{H^1(\overset{1}{\Omega}_{t'})}+c(1/\varepsilon_1)\|\overset{1}{H}_{n+1\,t't'}\|^2_{L_2(\overset{1}{\Omega}_{t'})}]\,dt'$$
$$\le c\sup_{t'}\|v_n\|^2_{H^1(\overset{1}{\Omega}_{t'})}\varepsilon_1\int_0^t\Big[\|\overset{1}{H}_{n+1\,t't'}\|^2_{H^1(\overset{1}{\Omega}_{t'})}+c(1/\varepsilon_1)t'\sup_{t'}\|\overset{1}{H}_{n+1\,t't'}\|^2_{L_2(\overset{1}{\Omega}_{t'})}\Big]\,dt'.$$

Let us consider $I_0^2$. Then we have

$$|I_0^2|\le c\int_0^t\int_{S_{t'}}(|\overset{1}{H}_{n+1\,x}|\,|\overset{1}{v}_{nt'}|+|\overset{1}{H}_{n+1\,xt'}|\,|\overset{1}{v}_n|)|\overset{1}{H}_{n+1\,t't'}|\,dS_{t'}\,dt'$$
$$\le\varepsilon\int_0^t\|\overset{1}{H}_{n+1\,t't'}\|^2_{H^1(\overset{1}{\Omega}_{t'})}dt'$$
$$+c(1/\varepsilon)\int_0^t(\|\overset{1}{H}_{n+1\,x}\|^2_{L_3(S_{t'})}\|\overset{1}{v}_{nt'}\|^2_{L_3(S_{t'})}+\|\overset{1}{H}_{n+1\,xt'}\|^2_{L_3(S_{t'})}\|\overset{1}{v}_n\|^2_{L_3(S_{t'})})\,dt',$$

where the second integral is bounded by

$$c(1/\varepsilon)\Big(\sup_{t'}\|\overset{1}{v}_{nt'}\|^2_{H^1(\overset{1}{\Omega}_{t'})}+\sup_{t'}\|\overset{1}{v}_n\|^2_{H^1(\overset{1}{\Omega}_{t'})}\Big)\Big[\int_0^t(\varepsilon_1\|\overset{1}{H}_{n+1}\|^2_{H^2(\overset{1}{\Omega}_{t'})}+\varepsilon_1\|\overset{1}{H}_{n+1\,t'}\|^2_{H^2(\overset{1}{\Omega}_{t'})})\,dt'$$
$$+c(1/\varepsilon_1)t\sup_t(\|\overset{1}{H}_{n+1}\|^2_{L_2(\overset{1}{\Omega}_t)}+\|\overset{1}{H}_{n+1\,t}\|^2_{L_2(\overset{1}{\Omega}_t)})\Big].$$

The other terms in $I_0$ can be estimated in a similar way.

To obtain an estimate for the r.h.s. of (6.26) we restrict ourselves to $I_1$ only, because the other integrals can be treated in a similar way. We have

$$I_1\le\varepsilon_1\int_0^t\|\overset{1}{H}_{n+1\,t't'}\|^2_{L_6(\overset{1}{\Omega}_{t'})}dt'+c(1/\varepsilon_1)\int_0^t\|\overset{1}{v}_{nt't'}\|^2_{L_3(\overset{1}{\Omega}_{t'})}\|\nabla\overset{1}{H}_{n+1}\|^2_{L_2(\overset{1}{\Omega}_{t'})}dt'$$
$$+c(1/\varepsilon_1)\int_0^t\|\overset{1}{v}_{nt'}\|^2_{L_2(\overset{1}{\Omega}_{t'})}\|\overset{1}{H}_{n+1\,xt'}\|^2_{L_3(\overset{1}{\Omega}_{t'})}dt',$$

where the second integral in bounded by

$$\Big(\varepsilon_2\int_0^t\|\overset{1}{v}_{nxt't'}\|^2_{L_2(\overset{1}{\Omega}_{t'})}dt'+c(1/\varepsilon_2)\sup_t\|\overset{1}{v}_{ntt}\|^2_{L_2(\overset{1}{\Omega}_t)}\Big)\cdot\sup_t\|\overset{1}{H}_{n+1}\|^2_{H^1(\overset{1}{\Omega}_t)}.$$

Continuing, assuming that $t$ is sufficiently small and applying Lemma 6.1 we get

$$\sum_{i=1}^2\|\overset{i}{H}_{n+1\,tt}\|^2_{V_2^0(\overset{i}{\Omega}{}_t^t)}\le t^a\varphi(t^aX(\overset{1}{v}_n,\overset{2}{v}_n,\overset{1}{H}_n,\overset{2}{H}_n,n,t))+\varphi(D).\ \blacksquare$$

LEMMA 6.5. *Let* $X(\overset{1}{v}_n,\overset{2}{v}_n,\overset{1}{H}_n,\overset{2}{H}_n,n,t)<\infty$, $D<\infty$. *Then*

$$\sum_{i=1}^{2}(\|\overset{i}{H}_{n+1\,xx}\|^2_{L_2(\overset{i}{\Omega}_t)}+\|\overset{i}{H}_{n+1}\|^2_{L_2(0,t;H^3(\overset{i}{\Omega}_t))})\le t^\alpha\varphi(t^\alpha X(\overset{1}{v}_n,\overset{2}{v}_n,\overset{1}{H}_n,\overset{2}{H}_n,n,t))+\varphi(D). \tag{6.28}$$

*Proof.* As in the proof of Lemma 6.3, we restrict ourselves to neighborhoods of $S_t$. First we differentiate (6.11) twice with respect to $\tau$. To obtain an energy estimate, we need

$$\begin{aligned}
&\bar\tau\cdot\nabla(\bar\tau\cdot\nabla\overset{l,k}{H}_t)=\overset{l,k}{H}_{\tau\tau t}-\bar\tau_t\cdot\nabla(\bar\tau)\cdot\nabla\overset{l,k}{H}-\bar\tau\cdot\nabla(\bar\tau_t\cdot\nabla\overset{l,k}{H})+\bar\tau\cdot\nabla(\bar\tau_t)\cdot\nabla\overset{l,k}{H},\\
&\bar\tau\cdot\nabla(\bar\tau\cdot\nabla(v\cdot\overset{l,k}{H}))=\bar\tau\cdot\nabla(\bar\tau\cdot\nabla(v)\cdot\nabla\overset{l,k}{H})+\bar\tau\cdot\nabla(v_j\bar\tau\cdot\nabla\nabla_j\overset{l,k}{H})\\
&\quad=\partial_\tau(v_{j\tau}\cdot\nabla_j\overset{l,k}{H})+\bar\tau\cdot\nabla(v_j)\bar\tau\cdot\nabla\nabla_j\overset{l,k}{H}+v_j\bar\tau\cdot\nabla(\bar\tau\cdot\nabla\nabla_j\overset{l,k}{H})\\
&\quad=v\cdot\nabla\overset{l,k}{H}_{\tau\tau}+\partial_\tau(v_{j\tau}\cdot\nabla\overset{l,k}{H})+\bar\tau\cdot\nabla(v_j)\bar\tau\nabla\nabla_j\overset{l,k}{H}-v_j\nabla_j(\tau_l\tau_i)\nabla_l\nabla_i\overset{l,k}{H},\\
&\bar\tau\cdot\nabla(\bar\tau\cdot\nabla)\Delta\overset{l,k}{H}=\nabla_i(\bar\tau\cdot\nabla(\bar\tau\cdot\nabla)\nabla_i\overset{l,k}{H})-\nabla_i(\bar\tau\cdot\nabla(\bar\tau\cdot\nabla))\nabla_i\overset{l,k}{H}\\
&\quad=\Delta(\bar\tau\cdot\nabla(\bar\tau\cdot\nabla)\overset{l,k}{H})-\nabla_i(\nabla_i(\bar\tau\cdot\nabla(\bar\tau\cdot\nabla))\overset{l,k}{H})\\
&\qquad-\nabla_i(\bar\tau\cdot\nabla(\bar\tau\cdot\nabla))\nabla_i\overset{l,k}{H}=\Delta\overset{l,k}{H}_{\tau\tau}+\overset{l,k}{A},\\
&\bar\tau\cdot\nabla(\bar\tau\cdot\nabla)\nabla\operatorname{div}\overset{l,k}{H}=\nabla_i(\bar\tau\cdot\nabla(\bar\tau\cdot\nabla)\nabla\overset{l,k}{H}_i)-\nabla_i(\bar\tau\cdot\nabla(\bar\tau\cdot\nabla))\nabla\overset{l,k}{H}_i\\
&\quad=\nabla\operatorname{div}\overset{l,k}{H}_{\tau\tau}-\nabla(\nabla_i\bar\tau\cdot\nabla(\bar\tau\cdot\nabla))\overset{l,k}{H}_i-\nabla_i(\bar\tau\cdot\nabla(\bar\tau\cdot\nabla))\nabla\overset{l,k}{H}_i\\
&\quad\equiv\nabla\operatorname{div}\overset{l,k}{H}_{\tau\tau}+\overset{l,k}{B},\quad l=1,2,
\end{aligned}\tag{6.29}$$

where the summation convention over repeated indices from 1 to 3 is assumed.

In view of (6.29) we obtain from (6.11) the equation

$$\begin{aligned}
&\mu_1(\overset{1,k}{H}_{n+1\,\tau\tau t}+v_n\cdot\nabla\overset{1,k}{H}_{n+1\,\tau\tau})-\frac{1}{\sigma_1}\operatorname{rot}\operatorname{rot}\overset{1,k}{H}_{n+1\,\tau\tau}\\
&\quad=\bar\tau_t\cdot\nabla(\bar\tau)\cdot\nabla\overset{1,k}{H}_{n+1}+\bar\tau\cdot\nabla(\bar\tau_t\cdot\nabla\overset{1,k}{H}_{n+1})-\bar\tau\cdot\nabla(\bar\tau_t)\cdot\nabla\overset{1,k}{H}_{n+1}\\
&\qquad-\partial_\tau(v_{nj\tau}\cdot\nabla_j\overset{1,k}{H}_{n+1})-\bar\tau\cdot\nabla(v_{nj})\bar\tau\cdot\nabla\nabla_j\overset{1,k}{H}_{n+1}\\
&\qquad+v_{nj}\nabla_j(\tau_l\tau_i)\nabla_l\nabla_i\overset{1,k}{H}_{n+1}-\overset{1,k}{A}-\overset{1,k}{B}\\
&\qquad+\partial_\tau^2\bigg(\mu_1\overset{1}{H}_{n+1}\overset{1}{\zeta}_{kt}+\mu_1v_n\cdot\nabla\overset{1}{\zeta}_k\overset{1}{H}_n+\mu_1\overset{1,k}{H}_n\cdot\nabla v_n\\
&\qquad+\frac{1}{\sigma_1}\operatorname{rot}(\nabla\overset{1}{\zeta}_k\times\overset{1}{H}_{n+1})+\frac{1}{\sigma_1}\nabla\overset{1}{\zeta}_k\times\operatorname{rot}\overset{1}{H}_{n+1}\bigg).
\end{aligned}\tag{6.30}$$

Multiplying (6.30) by $\overset{1,k}{H}_{n+1\,\tau\tau}$ and integrating over $\overset{1}{\Omega}_k$ we get

$$\begin{aligned}
&\frac{d}{dt}\int_{\overset{1}{\Omega}_k}|\overset{1,k}{H}_{n+1\,\tau\tau}|^2\,dx+\frac{1}{\sigma_1}\int_{\overset{1}{\Omega}_k}\operatorname{rot}\operatorname{rot}\overset{1,k}{H}_{n+1\,\tau\tau}\cdot\overset{1,k}{H}_{n+1\,\tau\tau}\,dx\\
&\quad=\int_{\overset{1}{\Omega}_k}(\bar\tau_t\cdot\nabla(\bar\tau)\cdot\nabla\overset{1,k}{H}_{n+1}-\bar\tau\cdot\nabla(\bar\tau_t)\cdot\nabla\overset{1,k}{H}_{n+1}
\end{aligned}\tag{6.31}$$

$$
\begin{aligned}
&-\bar\tau\cdot\nabla(v_{jn})\bar\tau\cdot\nabla\nabla_j\overset{1,k}{H}_{n+1}+v_{nj}\nabla_j(\tau_l\tau_i)\nabla_l\nabla_i\overset{1,k}{H}_{n+1})\cdot\overset{1,k}{H}_{n+1\,\tau\tau}\,dx\\
&+\int\limits_{\overset{1}{\Omega}_k}\partial_\tau(\bar\tau_t\cdot\nabla\overset{1,k}{H}_{n+1}-v_{nj\tau}\nabla_j\overset{1,k}{H}_{n+1})\cdot\overset{1,k}{H}_{n+1\,\tau\tau}\,dx\\
&-\int\limits_{\overset{1}{\Omega}_k}(\overset{1,k}{A}+\overset{1,k}{B})\cdot\overset{1,k}{H}_{n+1\,\tau\tau}\,dx+\int\limits_{\overset{1}{\Omega}_k}\partial_\tau^2\Big[(\mu_1\overset{1}{H}_n\overset{1}{\zeta}_{kt}+\mu_1v_n\cdot\nabla\overset{1}{\zeta}_k\overset{1}{H}_n+\mu_1\overset{1,k}{H}_n\cdot\nabla v_n\\
&+\frac{1}{\sigma_1}(\operatorname{rot}(\nabla\overset{1}{\zeta}_k\times\overset{1}{H}_{n+1})+\nabla\overset{1}{\zeta}_k\times\operatorname{rot}\overset{1}{H}_{n+1}))\Big]\cdot\overset{1,k}{H}_{n+1\,\tau\tau}\,dx.
\end{aligned}
$$

Now we estimate the terms from the r.h.s. of (6.31). The first term is estimated by

$$\varepsilon_1\|\overset{1,k}{H}_{n+1\,\tau\tau}\|^2_{L_6(\overset{1}{\Omega}_k)}+c(1/\varepsilon_1)\|v_n\|^2_{H^2(\overset{1}{\Omega}_k)}\|\overset{1,k}{\tilde H}_{n+1}\|^2_{H^2(\overset{1}{\Omega}_k)}.$$

Integrating by parts in the second term we can bound it by

$$\varepsilon_2\|\overset{1,k}{H}_{n+1\,\tau\tau\tau}\|^2_{L_2(\overset{1}{\Omega}_k)}+c(1/\varepsilon_2)\|v_n\|^2_{H^2(\overset{1}{\Omega}_k)}\|\overset{1,k}{H}_{n+1}\|^2_{H^2(\overset{1}{\Omega}_k)}.$$

We estimate the third term by

$$\varepsilon_3\|\overset{1,k}{H}_{n+1\,\tau\tau}\|^2_{L_2(\overset{1}{\Omega}_k)}+c(1/\varepsilon_3)\|\overset{1,k}{H}_{n+1}\|^2_{H^2(\overset{1}{\Omega}_k)},$$

and the last one by

$$\varepsilon_4\|\overset{1,k}{H}_{n+1\,\tau\tau\tau}\|^2_{L_2(\overset{1}{\Omega}_k)}+c(1/\varepsilon_4)(\|v_n\|^2_{H^1(\overset{1}{\Omega}_k)}\|\overset{1}{H}_n\|^2_{H^2(\overset{1}{\Omega}_k)}+\|\overset{1}{H}_{n+1}\|^2_{H^2(\overset{1}{\Omega}_k)}).$$

Employing the above estimates in (6.31) yields

$$
\begin{aligned}
(6.32)\qquad &\frac{d}{dt}\|\overset{1,k}{\tilde H}_{n+1\,\tau\tau}\|^2_{L_2(\overset{1}{\Omega}_k)}+\frac{1}{\sigma_1}\int\limits_{\overset{1}{\Omega}_k}\operatorname{rot}\operatorname{rot}\overset{1,k}{H}_{\tau\tau}\cdot\overset{1,k}{H}_{\tau\tau}\,dx\\
&\le\varepsilon\|\overset{1,k}{H}_{n+1\,\tau\tau}\|^2_{H^1(\overset{1}{\Omega}_k)}+c(1/\varepsilon)(\|v_n\|^2_{H^2(\overset{1}{\Omega}_k)}(\|\overset{1}{H}_{n+1}\|^2_{H^2(\overset{1}{\Omega}_k)}\\
&\quad+\|\overset{1}{H}_n\|^2_{H^2(\overset{1}{\Omega}_k)})+\|\overset{1}{H}_n\|^2_{H^2(\overset{1}{\Omega}_k)}+\|\overset{1}{H}_{n+1}\|^2_{H^2(\overset{1}{\Omega}_k)}).
\end{aligned}
$$

Expressions (6.29) hold also for $\overset{2,k}{H}$, where $v$ is replaced by $v'$. Then from (6.12) we obtain

$$
\begin{aligned}
(6.33)\qquad &\mu_2(\overset{2,k}{H}_{n+1\,\tau\tau t}+v'\cdot\nabla\overset{2,k}{H}_{n+1\,\tau\tau})+\frac{1}{\sigma_2}\operatorname{rot}\operatorname{rot}\overset{2,k}{H}_{n+1\,\tau\tau}\\
&=\mu_2(\bar\tau_t\cdot\nabla(\bar\tau)\cdot\nabla\overset{2,k}{H}_{n+1}+\bar\tau\cdot\nabla(\bar\tau_t\cdot\nabla\overset{2,k}{H}_{n+1})-\bar\tau\cdot\nabla(\bar\tau_t)\cdot\nabla\overset{2,k}{H}_{n+1})\\
&\quad-\mu_2\partial_\tau(v'_{nj\tau}\cdot\nabla_j\overset{2,k}{H}_{n+1})-\mu_2\bar\tau\cdot\nabla v'_{nj}\bar\tau\cdot\nabla\nabla_j\overset{2,k}{H}_{n+1}\\
&\quad+\mu_2v'_{nj}(\tau_l\tau_i)\nabla_l\nabla_i\overset{2,k}{H}_{n+1}-(\overset{2,k}{A}+\overset{k,2}{B})\\
&\quad+\partial_\tau^2\Big(\overset{k}{G}+\mu_2\overset{2}{H}_n\overset{2}{\zeta}_{kt}+\mu_2v'\cdot\nabla\overset{2}{\zeta}_k\overset{2}{H}_{n+1}+\mu_2v'\cdot\nabla\overset{2}{H}_n\overset{2}{\zeta}_k\\
&\quad+\frac{1}{\sigma_2}\operatorname{rot}(\nabla\overset{2}{\zeta}_k\times\overset{2}{H}_{n+1})+\frac{1}{\sigma_2}\nabla\overset{2}{\zeta}_k\times\operatorname{rot}\overset{2}{H}_{n+1}\Big).
\end{aligned}
$$

Multiplying (6.33) by $\overset{2,k}{H}_{n+1\,\tau\tau}$ and integrating over $\overset{2}{\Omega}_k$ we obtain

$$
\begin{aligned}
(6.34)\qquad &\frac{d}{dt}\int_{\overset{2}{\Omega}_k} |\overset{2,k}{H}_{n+1\,\tau\tau}|^2\,dx + \frac{1}{\sigma_2}\int_{\overset{2}{\Omega}_k} \operatorname{rot}\operatorname{rot}\overset{2,k}{H}_{n+1\,\tau\tau}\cdot\overset{2,k}{H}_{n+1\,\tau\tau}\,dx\\
&= \int_{\overset{2}{\Omega}_k} \mu_2(\bar\tau_t\cdot\nabla(\bar\tau)\cdot\nabla\overset{2,k}{H}_{n+1} - \bar\tau\cdot\nabla(\bar\tau_t)\cdot\nabla\overset{2,k}{H}_{n+1}\\
&\quad - \bar\tau\cdot\nabla(v'_{nj})\bar\tau\cdot\nabla\nabla_j\overset{2,k}{H}_{n+1} + v'_{nj}\nabla_j(\tau_l\tau_i)\nabla_l\nabla_i\overset{2,k}{H}_{n+1})\cdot\overset{2,k}{H}_{n+1\,\tau\tau}\,dx\\
&\quad + \int_{\overset{2}{\Omega}_k} \mu_2\partial_\tau(\bar\tau_t\cdot\nabla\overset{2,k}{H}_{n+1} - v'_{nj\tau}\nabla_j\overset{2,k}{H}_{n+1})\cdot\overset{2,k}{H}_{n+1\,\tau\tau}\,dx\\
&\quad - \int_{\overset{2}{\Omega}_k} (\overset{2,k}{A} + \overset{2,k}{B})\overset{2,k}{H}_{n+1\,\tau\tau}\,dx\\
&\quad + \int_{\overset{2}{\Omega}_k} \partial_\tau^2(\overset{k}{G} + \mu_2(\overset{2}{H}_{n+1}\overset{2}{\zeta}_{kt} + v'\cdot\nabla\overset{2}{\zeta}_k\overset{2}{H}_{n+1} + v'_n\cdot\nabla\overset{2}{H}_n\overset{2}{\zeta}_k)\\
&\quad + \frac{1}{\sigma_2}(\operatorname{rot}(\nabla\overset{2}{\zeta}_k\times\overset{2}{H}_{n+1}) + \nabla\overset{2}{\zeta}_k\times\operatorname{rot}\overset{2}{H}_{n+1}))\cdot\overset{2,k}{H}_{\tau\tau}\,dx.
\end{aligned}
$$

Now, we estimate the terms from the r.h.s. of (6.34). We estimate the first term by

$$\varepsilon_1\|\overset{2,k}{H}_{n+1\,\tau\tau}\|^2_{L_6(\overset{2}{\Omega}_k)} + c(1/\varepsilon_1)\|v'_n\|^2_{H^2(\overset{2}{\Omega}_k)}\|\overset{2,k}{H}_{n+1}\|^2_{H^2(\overset{2}{\Omega}_k)}.$$

Integrating by parts in the second term we bound it by

$$\varepsilon_2\|\overset{2,k}{H}_{n+1\,\tau\tau\tau}\|^2_{L_2(\overset{2}{\Omega}_k)} + c(1/\varepsilon_2)\|v'_n\|^2_{H^2(\overset{2}{\Omega}_k)}\|\overset{2,k}{H}_{n+1}\|_{H^2(\overset{2}{\Omega}_k)}.$$

The third term is estimated by

$$\varepsilon_3\|\overset{2,k}{H}_{n+1\,\tau\tau}\|^2_{L_2(\overset{2}{\Omega}_k)} + c(1/\varepsilon_3)\|\overset{2,k}{H}_{n+1}\|^2_{H^2(\overset{2}{\Omega}_k)}.$$

Integrating by parts in the last term we bound it by

$$\varepsilon_4\|\overset{2,k}{H}_{n+1\,\tau\tau\tau}\|^2_{L_2(\overset{2}{\Omega}_k)} + c(1/\varepsilon_4)(\|\overset{k}{G}_\tau\|^2_{L_2(\overset{2}{\Omega}_k)} + \|v'_n\|^2_{H^2(\overset{2}{\Omega}_k)}\|\overset{2}{H}_n\|^2_{H^2(\overset{2}{\Omega}_k)} + \|\overset{2}{H}_{n+1}\|^2_{H^2(\overset{2}{\Omega}_k)}).$$

Employing the above estimates in (6.34) yields

$$
\begin{aligned}
(6.35)\qquad &\frac{d}{dt}\|\overset{2,k}{H}_{n+1\,\tau\tau}\|^2_{L_2(\overset{2}{\Omega}_k)} + \frac{1}{\sigma_2}\int_{\overset{2}{\Omega}_k} \operatorname{rot}\operatorname{rot}\overset{2,k}{H}_{n+1\,\tau\tau}\cdot\overset{2,k}{H}_{n+1\,\tau\tau}\,dx\\
&\le \varepsilon\|\overset{2,k}{H}_{n+1\,\tau\tau}\|^2_{H^1(\overset{2}{\Omega}_k)} + c(1/\varepsilon)(\|\overset{k}{G}_\tau\|^2_{L_2(\overset{2}{\Omega}_k)}\\
&\quad + \|v'_n\|^2_{H^2(\overset{2}{\Omega}_k)}(\|\overset{2}{H}_n\|^2_{H^2(\overset{2}{\Omega}_k)} + \|\overset{2}{H}_{n+1}\|_{H^2(\overset{2}{\Omega}_k)} + \|\overset{2}{H}_{n+1}\|^2_{H^2(\overset{2}{\Omega}_k)})).
\end{aligned}
$$

Adding (6.32) and (6.35) implies

$$
\begin{aligned}
(6.36)\quad & \frac{d}{dt}\|\overset{1,k}{H}_{n+1\,\tau\tau}\|^2_{L_2(\overset{1}{\Omega}_k)} + \frac{d}{dt}\|\overset{2,k}{H}_{n+1\,\tau\tau}\|^2_{L_2(\overset{2}{\Omega}_k)} \\
&+ \frac{1}{\sigma_1}\int\limits_{\overset{1}{\Omega}_k} \operatorname{rot}\operatorname{rot}\overset{1,k}{H}_{n+1\,\tau\tau}\cdot\overset{1,k}{H}_{n+1\,\tau\tau}\,dx + \frac{1}{\sigma_2}\int\limits_{\overset{2}{\Omega}_k} \operatorname{rot}\operatorname{rot}\overset{2,k}{H}_{n+1\,\tau\tau}\cdot\overset{2,k}{H}_{n+1\,\tau\tau}\,dx \\
&\le \varepsilon\sum_{i=1}^2 \|\overset{i,k}{H}_{n+1\,\tau\tau}\|^2_{H^1(\overset{i}{\Omega}_k)} \\
&+ c(1/\varepsilon)\sum_{i=1}^2 (\|\overset{i}{v}_n\|^2_{H^2(\overset{i}{\Omega}_k)}(\|\overset{i}{H}_n\|^2_{H^2(\overset{i}{\Omega}_k)} + \|\overset{i}{H}_{n+1}\|^2_{H^2(\overset{i}{\Omega}_k)}) \\
&+ \|\overset{i}{H}_{n+1}\|^2_{H^2(\overset{i}{\Omega}_i)} + \|\overset{i}{H}_{n+1}\|^2_{H^2(\overset{i}{\Omega}_k)}) + c(1/\varepsilon)\|\overset{k}{G}_\tau\|^2_{L_2(\overset{2}{\Omega}_k)}.
\end{aligned}
$$

Next we examine the last two terms on the l.h.s. of (6.36). Integrating by parts they are equal to

$$
\begin{aligned}
(6.37)\quad & \left[\frac{1}{\sigma_1}\int\limits_{\overset{1}{\Omega}_k} |\operatorname{rot}\overset{1,k}{H}_{n+1\,\tau\tau}|^2\,dx + \frac{1}{\sigma_2}\int\limits_{\overset{2}{\Omega}_k} |\operatorname{rot}\overset{2,k}{H}_{n+1\,\tau\tau}|^2\,dx\right] \\
&+ \int\limits_{S_k}\left[\frac{1}{\sigma_1}\operatorname{rot}\overset{1,k}{H}_{n+1\,\tau\tau}\cdot\overset{1,k}{H}_{n+1\,\tau\tau}\times\overset{1}{\bar n} + \frac{1}{\sigma_2}\operatorname{rot}\overset{2,k}{H}_{n+1\,\tau\tau}\cdot\overset{2,k}{H}_{n+1\,\tau\tau}\times\overset{2}{\bar n}\right]dS_k \\
&\equiv I_1 + I_2.
\end{aligned}
$$

$I_2$ can be estimated as in (6.27), where $t$ is replaced by $\tau$.

From the above inequalities we obtain

$$
\begin{aligned}
(6.38)\quad & \sum_{i=1}^2\left(\frac{d}{dt}\|\overset{i,k}{H}_{n+1\,\tau\tau}\|^2_{L_2(\overset{i}{\Omega}_k)} + \|\overset{i,k}{H}_{n+1\,\tau\tau}\|^2_{H^1(\overset{i}{\Omega}_k)}\right) \\
&\le \sum_{i=1}^2[\varepsilon\|\overset{i,k}{H}_{n+1}\|^2_{H^3(\overset{i}{\Omega}_k)} + c(1/\varepsilon)\|\overset{i}{v}_n\|^2_{H^2(\overset{i}{\Omega}_k)}(\varepsilon_1(\|\overset{i}{H}_n\|^2_{H^3(\overset{i}{\Omega}_k)} \\
&+ \|\overset{i}{H}_{n+1}\|^2_{H^3(\overset{i}{\Omega}_k)} + \|\overset{i,k}{H}_{n+1}\|^2_{H^3(\overset{i}{\Omega}_k)}) \\
&+ c(1/\varepsilon_1)(\|\overset{i}{H}_n\|^2_{L_2(\overset{i}{\Omega}_k)} + \|\overset{i}{H}_{n+1}\|^2_{L_2(\overset{i}{\Omega}_k)} + \|\overset{i,k}{H}_{n+1}\|^2_{L_2(\overset{i}{\Omega}_k)}) \\
&+ \|\overset{i}{v}_{n\tau\tau}\|^2_{H^1(\overset{i}{\Omega}_k)}\|\overset{i,k}{H}_{n+1}\|^2_{L_3(S_k)} + \|\overset{i}{H}_n\|^2_{H^2(\overset{i}{\Omega}_k)} + \|\overset{i}{H}_{n+1}\|^2_{H^2(\overset{i}{\Omega}_k)})] + c\|\overset{k}{G}_{\tau\tau}\|^2_{L_2(\overset{i}{\Omega}_k)},
\end{aligned}
$$

where $v = \overset{1}{v}$, $v' = \overset{2}{v}$.

From (6.20) and (6.21) we have

$$
\begin{aligned}
(6.39)\quad & \sum_{i=1}^2\|\overset{i,k}{H}_{n+1\,nn}\|^2_{H^1(\overset{i}{\Omega}_k)} \le c\sum_{i=1}^2(\|\overset{i,k}{H}_{n+1\,\tau\tau}\|^2_{H^1(\overset{i}{\Omega}_k)} + \|\overset{i,k}{H}_{n+1\,t}\|^2_{H^1(\overset{i}{\Omega}_k)} \\
&+ \|\overset{i}{H}_n\|^2_{H^2(\overset{i}{\Omega}_k)} + \|\overset{i}{H}_{n+1}\|^2_{H^2(\overset{i}{\Omega}_k)} \\
&+ \|\overset{i}{v}_n\|^2_{H^2(\overset{i}{\Omega}_k)}(\|\overset{i}{H}_{n+1}\|^2_{H^2(\overset{i}{\Omega}_k)} + \|\overset{i}{H}_n\|^2_{H^2(\overset{i}{\Omega}_k)})) + c\|\overset{k}{G}\|^2_{H^1(\overset{2}{\Omega}_k)}.
\end{aligned}
$$

Then we calculate the derivatives with respect to time of the second derivatives with respect to space variables:

$$
\begin{aligned}
(6.40)\qquad J' &= \frac{d}{dt}\sum_{i=1}^{2}\frac{\sigma_i}{2}\|\overset{i}{H}_{xx}\|^2_{L_2(\overset{1}{\Omega}_t)} = \sum_{i=1}^{2}\sigma_i\frac{d}{dt}\int_{\overset{i}{\Omega}_t}\overset{i}{H}{}^2_{xx}\,dx \\
&= \sum_{i=1}^{2}\frac{\sigma_i}{2}\frac{d}{dt}\int_{\overset{i}{\Omega}_t}(\overset{i}{H}{}^2_{xx})(x(\xi,t),t)\,d\xi \\
&= \sum_{i=1}^{2}\frac{\sigma_i}{2}\int_{\overset{i}{\Omega}_t}[(\overset{i}{H}{}^2_{xx})_x(x(\xi,t),t)\overset{i}{v} + (\overset{i}{H}{}^2_{xx})_t]\,d\xi \\
&= \sum_{i=1}^{2}\sigma_i\int_{\overset{i}{\Omega}_t}(\overset{i}{H}_{xx}\overset{i}{H}_{xxx}\overset{i}{v} + \overset{i}{H}_{xx}\overset{i}{H}_{xxt})\,dx \\
&\le \varepsilon\sum_{i=1}^{2}(\|\overset{i}{H}_{xxx}\|^2_{L_2(\overset{i}{\Omega}_t)} + \|\overset{i}{H}_{xxt}\|^2_{L_2(\overset{i}{\Omega}_t)}) \\
&\quad + c(1/\varepsilon)\sum_{i=1}^{2}(\|\overset{i}{v}\|^2_{L_\infty(\overset{i}{\Omega}_t)}\|\overset{i}{H}_{xx}\|^2_{L_2(\overset{i}{\Omega}_t)} + \|\overset{i}{H}_{xx}\|^2_{L_2(\overset{i}{\Omega}_t)}).
\end{aligned}
$$

Repeating the considerations leading to (6.38) for interior subdomains we obtain

$$
\begin{aligned}
(6.41)\qquad &\sum_{i=1}^{2}\bigg(\frac{d}{dt}\|\overset{i,k}{H}_{n+1\,xx}\|^2_{L_2(\overset{i}{\Omega}_k)} + \|\overset{i,k}{H}_{n+1}\|^2_{H^3(\overset{i}{\Omega}_k)}\bigg) \\
&\le c\sum_{i=1}^{2}[\|\overset{i}{v}_n\|^2_{H^2(\overset{i}{\Omega}_k)}(\|\overset{i}{H}_n\|^2_{H^2(\overset{i}{\Omega}_k)} + \|\overset{i}{H}_{n+1}\|^2_{H^2(\overset{i}{\Omega}_k)}) \\
&\quad + \|\overset{i}{H}_n\|_{H^2(\overset{i}{\Omega}_k)} + \|\overset{i}{H}_{n+1}\|^2_{H^2(\overset{i}{\Omega}_k)}] + c\|\overset{2}{G}_{xx}\|^2_{L_2(\overset{2}{\Omega}_k)}.
\end{aligned}
$$

Integrating (6.41) with respect to time we obtain (6.28). ■

Lemma 6.6. *Let the assumptions of Lemma* 6.5 *be satisfied. Then*

$$
\begin{aligned}
(6.42)\qquad &\sum_{i=1}^{2}(\|\overset{i}{H}_{n+1\,xt}\|^2_{L_2(\overset{i}{\Omega})} + \|\overset{i}{H}_{n+1\,xt}\|^2_{L_2(0,T;H^1(\overset{i}{\Omega}))}) \\
&\le ct^\alpha\varphi(t^\alpha X(\overset{1}{v}_n,\overset{2}{v}_n,\overset{1}{H}_n,\overset{2}{H}_n,n,t)) + \varphi(D).
\end{aligned}
$$

The proof is similar to the proof of Lemma 6.5.

## 7. Boundedness

From Sections 5 and 6 we have

Lemma 7.1. *Let* (6.24) *for lagrangian coordinates be introduced. Let* $\bar{D} < \infty$. *Then there exists a constant* $M$ *such that for* $t$ *sufficiently small we have*

$$
(7.1)\qquad \bar{X}(\overset{1}{\bar{v}}_n,\overset{2}{\bar{v}}_n,\overset{1}{\bar{H}}_n,\overset{2}{\bar{H}}_n,n,t) \le M.
$$

*Proof.* From Sections 5, 6 and Remark 2.4 there exists an increasing positive function $\varphi$ such that

$$\bar X(\overset{1}{\bar v}_{n+1}\overset{2}{\bar v}_{n+1},\overset{1}{\bar H}_{n+1},\overset{2}{\bar H}_{n+1},n+1,t)\le t^a\varphi(t^a\bar X(\overset{1}{\bar v}_n,\overset{2}{\bar v}_n,\overset{1}{\bar H}_n,\overset{2}{\bar H}_n,n,t),\bar D)+\varphi(\bar D),$$

where $a>0$. Let $M$ be such that $t^a\varphi(0,\bar D)+\varphi(\bar D)\le\frac12 M$. Then there exists $t$ so small that

$$t^a\varphi(t^aM,\bar D)+\varphi(\bar D)\le M. \tag{7.2}$$

This implies (7.1). ∎

Remark 7.2. Since $M$ can be calculated in terms of $\bar D$, inequality (7.2) for $\bar D$ small implies (7.1) for large time.

## 8. Convergence

Let us introduce the quantities

$$V_n=v_n-v_{n-1},\quad P_n=p_n-p_{n-1},\quad \overset{i}{\mathcal H}_n=\overset{i}{H}_n-\overset{i}{H}_{n-1},\quad i=1,2. \tag{8.1}$$

The functions, after passing to lagrangian coordinates, are solutions to the problems

$$\begin{aligned}
&\bar V_{n+1\,t}-\operatorname{div}_{\bar v_n}\mathbb T_{\bar v_n}(\bar V_{n+1},\bar P_{n+1})=-(\nabla_{\bar v_n}\mathbb T_{\bar v_n}-\nabla_{\bar v_{n-1}}\mathbb T_{\bar v_{n-1}})(\bar v_n,\bar p_n)\\
&\quad+\overset{1}{\bar{\mathcal H}}_n\cdot\nabla_{\bar v_n}\overset{1}{\bar H}_n+\overset{1}{\bar H}_{n-1}\cdot\nabla_{\bar v_n}\overset{1}{\bar{\mathcal H}}_n+\overset{1}{\bar H}_{n-1}\cdot(\nabla_{\bar v_n}-\nabla_{\bar v_{n-1}})\overset{1}{\bar H}_{n-1}\\
&\quad+\overset{1}{\bar{\mathcal H}}_n\nabla_{\bar v_n}\overset{1}{\bar H}_n+\overset{1}{\bar H}_{n-1}\nabla_{\bar v_n}\overset{1}{\bar{\mathcal H}}_n+\overset{1}{\bar H}_{n-1}\nabla_{\bar v_n}\overset{1}{\bar H}_{n-1},\\
&\operatorname{div}_{\bar v_n}\bar V_{n+1}=-(\operatorname{div}_{\bar v_n}-\operatorname{div}_{\bar v_{n-1}})\bar v_n,\\
&\bar n_{\bar v_n}\cdot\mathbb T_{\bar v_n}(\bar V_{n+1},\bar P_{n+1})=-(\bar n_{\bar v_n}\mathbb T_{\bar v_n}-\bar n_{\bar v_{n-1}}\mathbb T_{\bar v_{n-1}})(\bar v_n,\bar p_n)\\
&\quad+\overset{1}{\bar H}_{n+1}\bar n_{\bar v_n}\cdot\nabla\overset{1}{\bar H}_{n+1}-\overset{1}{\bar H}_n\bar n_{\bar v_{n-1}}\cdot\nabla\overset{1}{\bar H}_n\\
&\quad+\tfrac12(\bar n_{\bar v_n}\cdot\nabla\overset{1}{\bar H}{}^2_{n+1}-\bar n_{\bar v_{n-1}}\cdot\nabla\overset{1}{\bar H}{}^2_n),\\
&\sigma_1\overset{1}{\bar{\mathcal H}}_{n+1\,t}-\nabla^2_{\overset{1}{\bar v}_n}\overset{1}{\bar{\mathcal H}}_{n+1}=\nabla^2_{\bar v_n}\overset{1}{\bar{\mathcal H}}_n-(\nabla^2_{\overset{1}{\bar v}_n}-\nabla^2_{\overset{1}{\bar v}_{n-1}})\bar H_n\\
&\quad+\sigma_1(\overset{1}{\bar H}_n\nabla_{\overset{1}{\bar v}_n}\overset{1}{\bar v}_n-\overset{1}{\bar H}_{n-1}\nabla_{\overset{1}{\bar v}_{n-1}}\overset{1}{\bar v}_{n-1}),\\
&\sigma_2\overset{2}{\bar{\mathcal H}}_{n+1\,t}-\nabla^2_{\overset{2}{\bar v}_n}\overset{2}{\bar{\mathcal H}}_{n+1}=-\nabla^2_{\overset{2}{\bar v}_n}\overset{2}{\bar{\mathcal H}}_n-(\nabla^2_{\overset{2}{\bar v}_n}-\nabla^2_{\overset{2}{\bar v}_{n-1}})\overset{2}{\bar H}_n\\
&\quad+\sigma_2(\overset{2}{\bar v}_n\cdot\nabla_{\overset{2}{\bar v}_n}\overset{2}{\bar H}_n-\overset{2}{\bar v}_{n-1}\cdot\nabla_{\overset{2}{\bar v}_{n-1}}\overset{2}{\bar H}_{n-1}).
\end{aligned}\tag{8.2}$$

Let

$$\bar Y_n(t)=\|\bar V_n\|_{V_2^0(\overset{1}{\Omega}{}^t)}+\|\bar V_{n\xi}\|_{V_2^0(\overset{1}{\Omega}{}^t)}+\sum_{i=1}^2(\|\overset{i}{\bar{\mathcal H}}_n\|_{V_2^0(\overset{i}{\Omega}{}^t)}+\|\overset{i}{\bar{\mathcal H}}_{n\xi}\|_{V_2^0(\overset{i}{\Omega}{}^t)}). \tag{8.3}$$

Lemma 8.1. *Let the assumptions of Lemma* 7.1 *hold. Then*

$$\bar Y_{n+1}(t)\le t^a\varphi(M,\bar D)\bar Y_n(t),\qquad a>0. \tag{8.4}$$

The proof can be conducted similarly to Sections 4–6.

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