## Contents

## Abstract

General position properties play a crucial role in geometric and infinite-dimensional topologies. Often such properties provide convenient tools for establishing various universality results. One of well-known general position properties is $\mathrm{DD}^n$, the property of disjoint $n$-cells. Each Polish $\mathrm{LC}^{n-1}$-space $X$ possessing $\mathrm{DD}^n$ contains a topological copy of each $n$-dimensional compact metric space. This fact implies, in particular, the classical Lefschetz–Menger–Nöbeling–Pontryagin–Tolstova embedding theorem which says that any $n$-dimensional compact metric space embeds into the $(2n+1)$-dimensional Euclidean space $\mathbb{R}^{2n+1}$. A parametric version of this result was recently proved by B. Pasynkov: any $n$-dimensional map $p : K \to M$ between metrizable compacta with $\dim M = m$ embeds into the projection $\mathrm{pr}_M : M \times \mathbb{R}^{2n+1+m} \to M$ in the sense that there is an embedding $e : K \to M \times \mathbb{R}^{2n+1+m}$ with $\mathrm{pr}_M \circ e = p$. This feature of $\mathbb{R}^{2n+1+m}$ can be derived from the fact that the space $\mathbb{R}^{2n+1+m}$ satisfies the general position property $m\text{-}\overline{\mathrm{DD}}^n = m\text{-}\overline{\mathrm{DD}}^{\{n,n\}}$, which is a particular case of the 3-parameter general position property $m\text{-}\overline{\mathrm{DD}}^{\{n,k\}}$ introduced and studied in this paper. We shall give convenient "arithmetic" tools for establishing the $m\text{-}\overline{\mathrm{DD}}^{\{n,k\}}$-property and on this base obtain simple proofs of some classical and recent results on (fiber) embeddings. In particular, the Pasynkov theorem mentioned above, as well as the results of P. Bowers and Y. Sternfeld on embedding into a product of dendrites, follow from our general approach. Moreover, the arithmetic of the $m\text{-}\overline{\mathrm{DD}}^{\{n,k\}}$-properties established in our paper generalizes some results of W. Mitchell, R. Daverman and D. Halverson.

The paper consists of two parts. In the first part we survey the principal results proved in this paper and discuss their applications and interplay with existing results in this area. The second part contains the proofs of the principal results announced in the first part.

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# Part I. SURVEY OF PRINCIPAL RESULTS

First, we fix some notation that will be often used in the subsequent text. Throughout the paper $m, n, k$ will stand for non-negative integers or $\infty$. We extend the arithmetic operations from $\omega = \{0, 1, 2, \dots\}$ onto $\omega \cup \{\infty\}$ letting $\infty = \infty + \infty = \infty + n = n + \infty = \infty - n$ for any $n \in \omega$. $\mathbb{I}$ denotes the unit interval $[0, 1]$ and $\mathbb{Q}$ the set of rational numbers on the real line $\mathbb{R}$. By a simplicial complex we shall always mean the geometric realization of an abstract simplicial complex equipped with the $CW$-topology. All topological spaces are assumed to be *Tychonoff* and all maps continuous.

A topological space $X$ is called *submetrizable* if it admits a continuous metric (equivalently, admits a bijective continuous map onto a metrizable space). A topological space $X$ is called *completely metrizable* if its topology is generated by a complete metric. A *Polish* space is a separable completely metrizable topological space.

By an ANR-space we mean a metrizable space $X$ which is a retract of every metrizable space $M$ containing $X$ as a closed subspace. It is well-known (see [12] or [41]) that a metrizable space $X$ is an ANR if and only if it is an ANE$[\infty]$ for the class of metrizable spaces. We recall that a space $X$ is called an ANE$[n]$ for a class $\mathcal{C}$ of spaces if every map $f : A \to X$ defined on a closed subset of a space $C \in \mathcal{C}$ with $\dim C \leq n$ can be extended to a continuous map $\bar{f} : U \to X$ defined on some neighborhood $U$ of $A$ in $X$.

Following [31], we define a subset $A$ of a space $X$ to be *relative* LC$^n$ in $X$ if given $x \in X$, $k < n + 2$, and a neighborhood $U$ of $x$ there is a neighborhood $V \subset U$ of $x$ such that each map $f : \partial\mathbb{I}^k \to A \cap V$ extends to a map $\bar{f} : \mathbb{I}^k \to U \cap A$. A space $X$ is an LC$^n$*-space* if it is relative LC$^n$ in $X$. According to [41, V.2.1], a metrizable space $X$ is LC$^n$ for a finite number $n$ if and only if $X$ is ANE$[n + 1]$ for the class of metrizable spaces.

By $\dim(X)$ we denote the covering dimension of a topological space $X$. For a map $f : X \to Y$ between topological spaces its *dimension* $\dim(f)$ is defined as $\dim(f) = \sup\{\dim(f^{-1}(y)) : y \in Y\}$. Maps $f$ with $\dim(f) = 0$ are called *light.*

Other (undefined here) notions can be found in the corresponding sections of Part II.

## 1. $m$-DD$^n$-property and fiber embeddings

We recall that a space $X$ has the DD$^n$P-property, the *disjoint $n$-disks property*, if any two maps $f, g : \mathbb{I}^n \to X$ from the $n$-dimensional cube $\mathbb{I}^n = [0, 1]^n$ can be approximated by maps with disjoint images. A parametric version of this property says that the same can be done for a continuous family $f_z, g_z : \mathbb{I}^n \to X$ of maps parameterized by points $z$

of some space $M$. More precisely, given a compact space $M$, we shall say that a space $X$ has the *$M$-parametric disjoint $n$-disk property* (briefly, the *$M$-$\mathrm{DD}^n$-property*) if any two maps $f, g : M \times \mathbb{I}^n \to X$ can be uniformly approximated by maps $f', g' : M \times \mathbb{I}^n \to X$ such that for any $z \in M$ the images $f'(\{z\} \times \mathbb{I}^n)$ and $g'(\{z\} \times \mathbb{I}^n)$ are disjoint.

We are mostly interested in the particular case of this property with $M = \mathbb{I}^m$ being the $m$-dimensional cube. In this case we write $m$-$\mathrm{DD}^n$ instead of $\mathbb{I}^m$-$\mathrm{DD}^n$. In the extremal cases when $m$ or $n$ is zero, the $m$-$\mathrm{DD}^n$-property turns out to be very familiar. Namely, the $0$-$\mathrm{DD}^n$-property is nothing other than the classical disjoint $n$-disks property, while the $m$-$\mathrm{DD}^0$-property is well-known to specialists in fixed point and coincidence theories: a space $X$ has the $m$-$\mathrm{DD}^0$-property iff any two maps $f, g : \mathbb{I}^m \to X$ can be approximated by maps with disjoint graphs!

It is well known (see [68] or [30]) that all one-to-one maps from a metrizable $n$-dimensional compactum $K$ into a completely metrizable $\mathrm{LC}^{n-1}$-space $X$ possessing the $\mathrm{DD}^n$-property form a dense $G_\delta$-set in the function space $C(K, X)$. Our first principal result is just a parametric version of this embedding theorem.

Theorem 1.1. *A completely metrizable* $\mathrm{LC}^{m+n}$*-space $X$ has the $m$-$\mathrm{DD}^n$-property if and only if for every perfect map $p : K \to M$ between finite-dimensional metrizable spaces with* $\dim M \leq m$ *and* $\dim(p) \leq n$ *the function space $C(K, X)$ contains a dense $G_\delta$-set of maps $f : K \to X$ that are injective on each fiber $p^{-1}(z)$, $z \in M$.*

The function space $C(K, X)$ appearing in this theorem is endowed with the *source limitation topology* whose neighborhood base at a given $f \in C(K, X)$ consists of the sets

$$B_\rho(f, \varepsilon) = \{g \in C(K, X) : \rho(g, f) < \varepsilon\},$$

where $\rho$ runs over all continuous pseudometrics on $X$ and $\varepsilon : K \to (0, \infty)$ runs over continuous positive functions on $K$. Here, the symbol $\rho(f, g) < \varepsilon$ means that $\rho(f(x), g(x)) < \varepsilon(x)$ for all $x \in K$. To the best of our knowledge, the notion of source limitation topology was introduced in the literature (see, for example, [44], [55], [50]) only for metrizable spaces $X$. In this case, for a fixed compatible metric $\rho$ on $X$, the sets $B_\rho(f, \varepsilon)$ with $\varepsilon \in C(K, (0, \infty)$ and $f \in C(K, X)$ form a base for a topology $\mathcal{T}_\rho$ on $C(K, X)$. If $K$ is paracompact, then the topology $\mathcal{T}_\rho$ does not depend on the metric $\rho$ [44]. Moreover, $\mathcal{T}_\rho$ has the Baire property provided $K$ is paracompact and $X$ is completely metrizable [55]. According to Lemma 18.4 below, $\mathcal{T}_\rho$ coincides with the topology obtained from our definition provided $K$ is paracompact and $X$ metrizable. Therefore, the source limitation topology on $C(K, X)$ also has the Baire property if $K$ is paracompact and $X$ is completely metrizable. We will use our more general definition (in terms of pseudometrics) and, unless stated otherwise, all function spaces will be considered with this topology.

In fact, finite-dimensionality of the spaces $K, M$ in Theorem 1.1 can be replaced by the $C$-space property. We recall that a topological space $X$ is defined to be a *$C$-space* if for any sequence $\{\mathcal{V}_n : n \in \omega\}$ of open covers of $X$ there exists a sequence $\{\mathcal{U}_n : n \in \omega\}$ of disjoint families of open sets in $X$ such that each $\mathcal{U}_n$ refines $\mathcal{V}_n$ and $\bigcup\{\mathcal{U}_n : n \in \omega\}$ is a cover of $X$. It is known that every finite-dimensional paracompact space (as well as every hereditarily paracompact countable-dimensional space) is a $C$-space and normal $C$-spaces are weakly infinite-dimensional (see [32, §6.3]).

Theorem 1.2. *A completely metrizable locally contractible space $X$ has the $m$-$\mathrm{DD}^n$-property if and only if for every perfect map $p : K \to M$ between metrizable $C$-spaces with $\dim M \le m$ and $\dim(p) \le n$ the function space $C(K, X)$ contains a dense $G_\delta$-set of maps $f : K \to X$ that are injective on each fiber $p^{-1}(z)$, $z \in M$.*

## 2. $\triangle$-dimension of maps

There is a natural temptation to remove the dimensional restrictions on the spaces $K, M$ from Theorems 1.1 and 1.2. This indeed can be done if we replace the usual dimension $\dim(p)$ of the map $p$ with the so-called $\triangle$-dimension $\dim_\triangle(p)$ (coinciding with $\dim(p)$ for perfect maps $p$ between finite-dimensional metrizable spaces.)

By definition, the *$\triangle$-dimension* $\dim_\triangle(p)$ of a map $p : X \to Y$ between Tychonoff spaces is equal to the smallest cardinal number $\tau$ for which there exists a map $g : X \to \mathbb{I}^\tau$ such that the diagonal product $f \triangle g : X \to Y \times \mathbb{I}^\tau$ has $\dim(f \triangle g) = 0$. The $\triangle$-dimension $\dim_\triangle(p)$ is a well-defined cardinal function not exceeding the weight $w(X)$ of $X$ (because we always can take $g$ to be an embedding in the Tychonoff cube $\mathbb{I}^{w(X)}$).

The following important result describing the interplay between the dimension and $\triangle$-dimension of perfect maps is actually a reformulation of results due to B. Pasynkov [58], M. Tuncali and V. Valov [71], and M. Levin [47] (see Section 19).

Proposition 2.1. *Let $f : X \to Y$ be a perfect map between paracompact spaces. Then*

(1) $\dim(f) \le \dim_\triangle(f)$;
(2) $\dim_\triangle(f) = 0$ *if and only if $f$ is a light map;*
(3) $\dim_\triangle(f) \le \omega$ *if $X$ is submetrizable;*
(4) $\dim_\triangle(f) = \dim(f)$ *if $X$ is submetrizable and $Y$ is a $C$-space;*
(5) $\dim_\triangle(f) \le \dim(f) + 1$ *if the spaces $X, Y$ are compact and metrizable.*

We know no example of a map $f : X \to Y$ between compacta with $\dim(f) < \dim_\triangle(f)$ (cf. [47]). The following theorem is a version of Theorem 1.1 with $\dim(p)$ replaced by $\dim_\triangle(p)$.

Theorem 2.2. *A completely metrizable ANR-space $X$ has the $m$-$\mathrm{DD}^n$-property if and only if for every perfect map $p : K \to M$ between submetrizable paracompact spaces with $\dim M \le m$ and $\dim_\triangle(p) \le n$ the function space $C(K, X)$ contains a dense $G_\delta$-set of maps $f : K \to X$ that are injective on each fiber $p^{-1}(z)$, $z \in M$.*

## 3. The $m$-$\overline{\mathrm{DD}}^n$-property and a general fiber embedding theorem

In fact, it is more convenient to work not with the $m$-$\mathrm{DD}^n$-property, but with its homotopical version defined as follows:

Definition 3.1. A space $X$ has the *$m$-$\overline{\mathrm{DD}}^n$-property* if for any open cover $\mathcal{U}$ of $X$ and any two maps $f, g : \mathbb{I}^m \times \mathbb{I}^n \to X$ there are maps $f', g' : \mathbb{I}^m \times \mathbb{I}^n \to X$ such that

- $f'$ is $\mathcal{U}$-homotopic to $f$;
- $g'$ is $\mathcal{U}$-homotopic to $g$;
- $f'(\{z\} \times \mathbb{I}^n) \cap g'(\{z\} \times \mathbb{I}^n) = \emptyset$ for all $z \in \mathbb{I}^m$.

We recall that two maps $f, g : K \to X$ are said to be *$\mathcal{U}$-homotopic* (briefly, $f \underset{\mathcal{U}}{\sim} g$), where $\mathcal{U}$ is a cover of $X$, if there is a homotopy $h : K \times [0,1] \to X$ such that for every $x \in K$ we have $h(x,0) = f(x)$, $h(x,1) = g(x)$ and $h(\{x\} \times [0,1])$ is contained in some $U \in \mathcal{U}$. It is clear that any $\mathcal{U}$-homotopic maps $f, g : K \to X$ are *$\mathcal{U}$-near* (i.e., for each point $z \in K$ the set $\{f(z), g(z)\}$ is contained in some $U \in \mathcal{U}$).

The notion of a $\mathcal{U}$-homotopy has a pseudometric counterpart. Given a continuous pseudometric $\rho$ on $X$ and a continuous map $\varepsilon : K \to (0,\infty)$ we shall say that two maps $f, g : K \to X$ are *$\varepsilon$-homotopic* if there is a homotopy $h : K \times [0,1] \to X$ such that $h(z,0) = f(z)$, $h(z,1) = g(z)$ and $\operatorname{diam}_\rho h(\{z\} \times [0,1]) < \varepsilon(z)$ for all $z \in K$. In this case $h$ is called an *$\varepsilon$-homotopy.*

The relation between the $m$-DD$^n$-property and its homotopical version is described in the next proposition.

PROPOSITION 3.2. *Each space $X$ with the $m$-$\overline{\mathrm{DD}}^n$-property has the $m$-DD$^n$-property. Conversely, each LC$^{n+m}$-space $X$ with the $m$-DD$^n$-property has the $m$-$\overline{\mathrm{DD}}^n$-property.*

Proposition 3.2 follows from the well-known property of LC$^n$-spaces which asserts that for any open cover $\mathcal{U}$ of an LC$^n$-space $X$ with $n < \infty$ there is another open cover $\mathcal{V}$ of $X$ such that two maps $f, g : \mathbb{I}^n \to X$ are $\mathcal{U}$-homotopic provided they are $\mathcal{V}$-near (see Lemma 21.2).

Thus, in the realm of LC$^{m+n}$-spaces both the $m$-$\overline{\mathrm{DD}}^n$-property and the $m$-DD$^n$-property are equivalent. The advantage of the $m$-$\overline{\mathrm{DD}}^n$-property is that it works for spaces without a nice local structure, while the $m$-DD$^n$-property is applicable only for LC$^k$-spaces with sufficiently large $k$. In particular, using the $m$-$\overline{\mathrm{DD}}^n$-property, we can establish the following general result implying Theorems 1.1, 1.2 and 2.2.

THEOREM 3.3. *Let $p : K \to M$ be a perfect map defined on a paracompact submetrizable space $K$. If a subspace $X$ of a completely metrizable space $Y$ possesses the $m$-$\overline{\mathrm{DD}}^n$-property for $m = \dim M$ and $n = \dim_\triangle(p)$, then*

$$\mathcal{E}(p, Y) = \{f \in C(K, Y) : p \triangle f : K \to M \times Y \text{ is an embedding}\}$$

*is a $G_\delta$-set in $C(K, Y)$ whose closure $\overline{\mathcal{E}(p, Y)}$ contains all simplicially factorizable maps from $K$ to $X$. More precisely, for any continuous pseudometric $\rho$ on $Y$, a continuous function $\varepsilon : K \to (0, +\infty)$ and a simplicially factorizable map $f : K \to X$ there is a map $g \in \mathcal{E}(p, Y)$ and an $\varepsilon$-homotopy $h : K \times [0,1] \to Y$ connecting $f$ and $g$ so that $h(K \times [0,1)) \subset X$.*

A map $f : K \to X$ is called *simplicially factorizable* if there exist a simplicial complex $L$ and maps $\alpha : K \to L$ and $\beta : L \to X$ such that $f = \beta \circ \alpha$. It turns out that in many important cases simplicially factorizable maps form a dense set in the function space $C(K, X)$. To describe such cases, we need the notion of a Lefschetz ANE[$n$]-space that is a parameterized version of a space satisfying the Lefschetz condition (see [12, V.8]).

Let $\mathcal{U}$ be a cover of a space $X$ and $K$ be a simplicial complex. By a *partial $\mathcal{U}$-realization* of $K$ in $X$ we understand any continuous map $f : L \to X$ defined on a geometric subcomplex $L \subset K$ containing all vertices of $K$ and such that $\operatorname{diam} f(\sigma \cap L) < \mathcal{U}$ for every simplex $\sigma$ of $K$. If $L = K$, then the map $f$ is called *a full $\mathcal{U}$-realization* of $K$ in $X$.

A topological space $X$ is defined to be a *Lefschetz* ANE$[n]$ if for every open cover $\mathcal{U}$ of $X$ there is an open cover $\mathcal{V}$ of $X$ such that each partial $\mathcal{V}$-realization $f : L \to X$ of a simplicial complex $K$ with $\dim K \le n$ can be extended to a full $\mathcal{U}$-realization $\tilde{f} : K \to X$ of $K$.

Lefschetz ANE$[n]$-spaces are tightly connected with both ANR's and LC$^n$-spaces and have all basic properties of such spaces.

PROPOSITION 3.4. *Let $n$ be a non-negative integer or infinity.*

(1) *A metrizable space $X$ is a Lefschetz* ANE$[n]$ *if and only if $X$ is an* ANE$[n]$ *for the class of metrizable spaces.*
(2) *If $n$ is finite, then a regular* (*paracompact*) *space $X$ is a Lefschetz* ANE$[n]$ (*if and*) *only if $X$ is* LC$^{n-1}$.
(3) *Each convex subset $X$ of a* (*locally convex*) *linear topological space $L$ is a Lefschetz* ANE$[n]$ *for any finite $n$* (*is a Lefschetz* ANE$[\infty]$).
(4) *There exists a metrizable $\sigma$-compact linear topological space that fails to be a Lefschetz* ANE$[\infty]$.
(5) *A neighborhood retract of a Lefschetz* ANE$[n]$*-space is a Lefschetz* ANE$[n]$*-space.*
(6) *A functionally open subspace of a Lefschetz* ANE$[n]$ *is a Lefschetz* ANE$[n]$.
(7) *A topological space $X$ is a Lefschetz* ANE$[n]$ *if $X$ has a uniform open cover by Lefschetz* ANE$[n]$*-spaces.*
(8) *A metric space $(X, \rho)$ is a Lefschetz* ANE$[n]$ *if for every $\varepsilon > 0$ there is $\delta > 0$ such that each partial $\mathcal{B}_\rho(\delta)$-realization $f : L \to X$ of a simplicial complex $K$ with $\dim K \le n$ extends to a full $\mathcal{B}_\rho(\varepsilon)$-realization $\bar{f} : K \to X$ of $K$ in $X$.*
(9) *For each continuous pseudometric $\eta$ on a paracompact Lefschetz* ANE$[n]$*-space $X$ there is a continuous pseudometric $\rho \ge \eta$ such that for every $r \in (0, 1/2]$ each partial $\mathcal{D}_\rho(r/8)$-realization $f : L \to X$ of a simplicial complex $K$ with $\dim K \le n$ extends to a full $\mathcal{D}_\rho(r)$-realization $\bar{f} : K \to X$ of $K$ in $X$.*
(10) *Each map $f : X \to Y$ from a paracompact Lefschetz* ANE$[n]$*-space to a metrizable space $Y$ factorizes through a metrizable Lefschetz* ANE$[n]$*-space $Z$ in the sense that $f = g \circ h$ for some maps $h : X \to Z$ and $g : Z \to Y$.*

Here by $\mathcal{D}_\rho(\varepsilon)$ we denote the cover of a metric space $(X, \rho)$ by all open sets of diameter $< \varepsilon$. With the notion of Lefschetz ANE$[n]$-space at hand, we can return to simplicially factorizable maps.

PROPOSITION 3.5. *The simplicially factorizable maps from a paracompact space $K$ into a Tychonoff space $X$ form a dense set in the function space $C(K, X)$ if one of the following conditions is satisfied:*

(1) *$X$ is a Lefschetz* ANE$[k]$ *for $k = \dim K$;*
(2) *$K$ is a $C$-space and $X$ is a locally contractible paracompact space.*

Observe that Theorems 1.1, 1.2, and 2.2 follow immediately from Theorem 3.3 and Propositions 3.5, 3.4 and 2.1.

Combining Theorem 3.3 with Propositions 3.5(1), 3.4(2) and 2.1(4), we obtain another generalization of Theorem 1.1.

THEOREM 3.6. *Let $p : K \to M$ be a perfect map between finite-dimensional paracompact spaces with $K$ being submetrizable. If $X$ is a completely metrizable* $\mathrm{LC}^{k-1}$*-space possessing the $m$-$\overline{\mathrm{DD}}^n$-property, where $k = \dim K$, $m = \dim M$ and $n = \dim(p)$, then the function space $C(K, X)$ contains a dense $G_\delta$-set of maps injective on each fiber of $p$.*

## 4. Approximating perfect maps by perfect PL-maps

The proof of Theorem 3.3 heavily exploits the technique of approximation by PL-maps. By a *PL-map* (resp., a *simplicial map*) we understand a map $f : K \to M$ between simplicial complexes which maps each simplex $\sigma$ of $K$ into (resp., onto) some simplex $\tau$ of $M$ and $f$ is linear on $\sigma$.

THEOREM 4.1. *If $p : X \to Y$ is a perfect map between paracompact spaces, then for any open cover $\mathcal{U}$ of $X$ there exists an open cover $\mathcal{V}$ of $Y$ such that for any $\mathcal{V}$-map $\beta : Y \to M$ into a simplicial complex $M$ there are an $\mathcal{U}$-map $\alpha : X \to K$ into a simplicial complex $K$ and a perfect PL-map $f : K \to M$ with $f \circ \alpha = \beta \circ p$ and $\dim_\triangle(f) = \dim(f) \le \dim_\triangle(p)$.*

Since for each open cover $\mathcal{V}$ of a paracompact space $Y$ there is a $\mathcal{V}$-map $\beta : Y \to M$ into a simplicial complex with $\dim M \le \dim Y$, Theorem 4.1 implies the following approximation result.

COROLLARY 4.2. *If $p : X \to Y$ is a perfect map between paracompact spaces, then for any open covers $\mathcal{U}$ and $\mathcal{V}$ of $X$ and $Y$, respectively, there exist a $\mathcal{U}$-map $\alpha : X \to K$ into a simplicial complex $K$ with $\dim K \le \dim Y + \dim_\triangle(p)$, a $\mathcal{V}$-map $\beta : Y \to M$ to a simplicial complex $M$ with $\dim M \le \dim Y$, and a perfect PL-map $f : K \to M$ with $\dim(f) \le \dim_\triangle(p)$ making the following diagram commutative:*

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & K \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{\beta} & M \end{array}$$

For light maps $p : X \to Y$ between metrizable compacta this corollary was proved by A. Dranishnikov and V. Uspenskij in [27] and for arbitrary maps between metrizable compacta by Yu. Turygin [72].

Let us mention that Theorem 4.1 can be applied in different situations (see for example [8], [9], [76]).

## 5. $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-properties

Because of the presence of the $m$-$\overline{\mathrm{DD}}^n$-property in Theorems 3.3–3.6, it is important to have convenient methods for detecting that property. To establish such methods, we introduce the following three-parametrer version of the $m$-$\overline{\mathrm{DD}}^n$-property.

Definition 5.1. A space $X$ is defined to have the *$m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property* if for any open cover $\mathcal{U}$ of $X$ and two maps $f : \mathbb{I}^m \times \mathbb{I}^n \to X$, $g : \mathbb{I}^m \times \mathbb{I}^k \to X$ there exist maps $f' : \mathbb{I}^m \times \mathbb{I}^n \to X$, $g' : \mathbb{I}^m \times \mathbb{I}^k \to X$ such that $f' \underset{\mathcal{U}}{\sim} f$, $g' \underset{\mathcal{U}}{\sim} g$, and $f'(\{z\}\times\mathbb{I}^n)\cap g'(\{z\}\times\mathbb{I}^k) = \emptyset$ for all $z \in \mathbb{I}^m$.

By $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$ we shall denote the class of all spaces with the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property. It is clear that the $m$-$\overline{\mathrm{DD}}^{n}$-property coincides with the $m$-$\overline{\mathrm{DD}}^{\{n,n\}}$-property. If some of the numbers $m, n, k$ are infinite, the detection of the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property can be reduced to the detection of $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$ with finite $m, n, k$.

Proposition 5.2. *A Tychonoff space $X$ has the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property if and only if it has the $a$-$\overline{\mathrm{DD}}^{\{b,c\}}$-property for all $a < m + 1$, $b < n + 1$, $c < k + 1$.*

The proof of Theorem 3.3 is based on the following simplicial characterization of the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property.

Theorem 5.3. *A submetrizable space $X$ has the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property if and only if for any*

- *simplicial maps $p_N : N \to M$, $p_K : K \to M$ between finite simplicial complexes with $\dim M \le m$, $\dim(p_N) \le n$, $\dim(p_K) \le k$,*
- *open cover $\mathcal{U}$ of $X$, and*
- *maps $f : N \to X$, $g : K \to X$,*

*there exist maps $f' : N \to X$, $g' : K \to X$ such that $f' \underset{\mathcal{U}}{\sim} f$, $g' \underset{\mathcal{U}}{\sim} g$ and, for every $z \in M$ we have $f'(p_N^{-1}(z)) \cap g'(p_K^{-1}(z)) = \emptyset$.*

Using the above simplicial characterization, we can establish the local nature of the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property.

Proposition 5.4. *Let $m, n, k$ be non-negative integers or $\infty$.*

(1) *If a space $X$ has the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property, then so does each open subspace of $X$.*
(2) *A paracompact submetrizable space $X$ has the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property if and only if it admits a cover by open subspaces with that property.*

The $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property is also preserved by taking homotopically $n$-dense subspaces. We define a subset $A$ of a topological space $X$ to be *homotopically $n$-dense* in $X$ if the following conditions are satisfied:

- for every map $f : \mathbb{I}^n \to X$ and an open cover $\mathcal{U}$ of $X$ there is a map $f' : \mathbb{I}^n \to A$ that is $\mathcal{U}$-homotopic to $f$;
- for every open cover $\mathcal{U}$ of $X$ there is an open cover $\mathcal{V}$ of $X$ such that if two maps $f, g : \mathbb{I}^n \to A$ are $\mathcal{V}$-homotopic in $X$, then they are $\mathcal{U}$-homotopic in $A$.

By Toruńczyk's Theorem 2.8 in [67], each dense relative $\mathrm{LC}^n$-subset $X$ of a metrizable space $\tilde{X}$ is homotopically $n$-dense in $\tilde{X}$. The following useful proposition follows immediately from the definitions and the above mentioned theorem of Toruńczyk.

Proposition 5.5. *A homotopically $\max\{m+n, m+k\}$-dense subspace $X$ of a topological space $\tilde{X}$ has the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property if and only if $\tilde{X}$ does. Consequently, a dense relative $\mathrm{LC}^{m+\max\{n,k\}}$-set $X$ in a space $\tilde{X}$ has the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property if and only if $\tilde{X}$ does.*

This fact will be often applied in combination with Proposition 2.8 from [22] asserting that each metrizable $\mathrm{LC}^n$-space $X$ embeds into a completely metrizable $\mathrm{LC}^n$-space $\tilde{X}$ as a dense relative $\mathrm{LC}^n$-set. This enables us to apply Baire category arguments for establishing the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-properties in arbitrary (not necessarily complete) metric spaces.

Next, we elaborate tools for detecting the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-properties. Recall that a space $X$ *has no free arcs* if $X$ contains no open subset homeomorphic to a non-empty connected subset of the real line. In particular, a space without free arcs has no isolated points.

Proposition 5.6.

(1) *A topological space $X$ has the $0$-$\overline{\mathrm{DD}}^{\{0,0\}}$-property if and only if each path-connected component of $X$ is non-degenerate.*
(2) *An $\mathrm{LC}^0$-space $X$ has the $0$-$\overline{\mathrm{DD}}^{\{0,0\}}$-property if and only if $X$ has no isolated point.*
(3) *A metrizable $\mathrm{LC}^1$-space $X$ has the $0$-$\overline{\mathrm{DD}}^{\{0,1\}}$-property if and only if $X$ has the $1$-$\overline{\mathrm{DD}}^{\{0,0\}}$-property if and only if $X$ has no free arc.*
(4) *Any metrizable $\mathrm{LC}^n$-space $X$ with the $0$-$\overline{\mathrm{DD}}^{\{0,n\}}$-property and with $\dim X \le n$ has the $0$-$\overline{\mathrm{DD}}^{\{0,\infty\}}$-property.*
(5) *A Polish $\mathrm{ANE}[\max\{n,k\}+1]$-space $X$ has the $0$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property if and only if there are two disjoint dense $\sigma$-compact subsets $A, B$ of $X$ such that $A$ is relative $\mathrm{LC}^{n-1}$ and $B$ is relative $\mathrm{LC}^{k-1}$ in $X$.*

Items (3) and (4) of Proposition 5.6 imply that each one-dimensional $\mathrm{LC}^1$-space without free arcs has the $0$-$\overline{\mathrm{DD}}^{\{0,\infty\}}$-property. In particular, each dendrite with a dense set of end-points has that property.

The last item of Proposition 5.6 is a particular case of a more general characterization of the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property in terms of mapping absorption properties.

Let $M, X$ be topological spaces. We shall say that a subset $A \subset M \times X$ has the *absorption property for $n$-dimensional maps in $M$* (briefly, $M$-$\mathrm{MAP}^n$) if for any $n$-dimensional map $p : K \to M$ with $K$ being a finite-dimensional compact space, a closed subset $C \subset K$, a map $f : K \to X$, and an open cover $\mathcal{U}$ of $X$ there is a map $f' : K \to X$ such that $f'$ is $\mathcal{U}$-homotopic to $f$, $f'|C = f|C$ and $(p \triangle f')(K \setminus C) \subset A$. If $M = \mathbb{I}^m$, then we write $m$-$\mathrm{MAP}^n$ instead of $\mathbb{I}^m$-$\mathrm{MAP}^n$.

Theorem 5.7. *Let $m, n, k$ be non-negative integers or infinity and $d = 1+m+\max\{n,k\}$. A (Polish $\mathrm{ANE}[d]$-)space $X$ has the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property if (and only if) for any separable polyhedron $M$ with $\dim M \le m$ there are two disjoint ($\sigma$-compact) sets $E, F \subset M \times X$ such that $E$ has $M$-$\mathrm{MAP}^n$ and $F$ has $M$-$\mathrm{MAP}^k$.*

Let us observe that the existence of such disjoint sets $E, F$ is not obvious even for a dendrite with a dense set of end-points. Such a dendrite $D$ has the $1$-$\overline{\mathrm{DD}}^{\{0,0\}}$-property and thus the product $\mathbb{I} \times D$ contains two disjoint $\sigma$-compact subsets with $1$-$\mathrm{MAP}^0$.

## 6. A selection theorem for $Z_n$-set-valued functions

Many results on $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-properties are based on a selection theorem for $Z_n$-valued functions, discussed in this subsection.

A subset $A$ of a topological space $X$ is called a (*homotopical*) $Z_n$*-set* in $X$ if $A$ is closed in $X$ and for any an open cover $\mathcal{U}$ of $X$ and a map $f : \mathbb{I}^n \to X$ there is a map $g : \mathbb{I}^n \to X$ such that $g(\mathbb{I}^n) \cap A = \emptyset$ and $g$ is $\mathcal{U}$-near ($\mathcal{U}$-homotopic) to $f$. Each homotopical $Z_n$-set in a topological space $X$ is a $Z_n$-set in $X$. The converse is true if $X$ is an LC$^n$-space (see Theorem 10.1).

A set-valued function $\Phi : X \multimap Y$ is defined to be *compactly semicontinuous* if for every compact subset $K \subset Y$ the preimage $\Phi^{-1}(K) = \{x \in X : \Phi(x) \cap K \neq \emptyset\}$ is closed in $X$.

Theorem 6.1. *Let $\Phi : X \multimap Y$ be a compactly semicontinuous set-valued function from a paracompact $C$-space $X$ into a topological space $Y$, assigning to each point $x \in X$ a homotopical $Z_n$-set $\Phi(x)$, where $n = \dim X \leq \infty$. If $X$ is a retract of an open subset of a locally convex linear topological space, then for any map $f : X \to Y$ and any continuous pseudometric $\rho$ on $Y$ there is a map $f' : X \to Y$ such that $f'(x) \notin \Phi(x)$ for all $x \in X$ and $f'$ is 1-homotopic to $f$ with respect to $\rho$.*

In particular, this theorem is true for stratifiable ANR's $X$ (which are neighborhood retracts of stratifiable locally convex spaces, see [62]). Theorem 6.1 can be seen as a generalization of Selection Theorem 1.4 of Uspenskij [75].

## 7. Homotopical $Z_n$-sets and $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-properties

It turns out that homotopical $Z_n$-sets are tightly connected with the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-properties. A point $x$ of a topological space $X$ is called a (*homotopical*) $Z_n$*-point* if the singleton $\{x\}$ is a (homotopical) $Z_n$-set in $X$. By $\mathcal{Z}_n(X)$ we shall denote the set of all homotopical $Z_n$-points of a space $X$.

Let

- $\mathcal{Z}_n$ be the class of Tychonoff spaces $X$ with $\mathcal{Z}_n(X) = X$;
- $\overline{\mathcal{Z}}_n$ be the class of Tychonoff spaces $X$ with $\overline{\mathcal{Z}_n(X)} = X$;
- $\Delta\mathcal{Z}_n$ be the class of Tychonoff spaces $X$ whose diagonal $\Delta_X$ is a homotopical $Z_n$-set in $X^2$.

For example, $\mathbb{R}^{n+1} \in \mathcal{Z}_n \cap \Delta\mathcal{Z}_n$.

Besides the classes of spaces related to $Z$-sets, we also need some other (more familiar) classes of topological spaces:

- $\mathsf{Br}$, the class of metrizable separable Baire spaces;
- $\Pi^0_2$, the class of Polish spaces;
- LC$^n$, the class of all LC$^n$-spaces.

Theorem 7.1. *Let $m, n, k$ be non-negative integers or infinity.*

(1) *A space $X$ has the $n$-$\overline{\mathrm{DD}}^{\{0,0\}}$-property if and only if the diagonal of $X^2$ is a homotopical $Z_n$-set in $X^2$. This can be written as*

$$\boxed{\Delta\mathcal{Z}_n = n\text{-}\overline{\mathrm{DD}}^{\{0,0\}}}$$

(2) *An* $\mathrm{LC}^0$*-space* $X$ *has the* $0\text{-}\overline{\mathrm{DD}}^{\{0,n\}}$*-property provided the set* $\mathcal{Z}_n(X)$ *is dense in* $X$*. This can be written as*

$$\boxed{\mathrm{LC}^0 \cap \overline{\mathcal{Z}}_n \subset 0\text{-}\overline{\mathrm{DD}}^{\{0,n\}}}$$

(3) *If a metrizable separable Baire* ($\mathrm{LC}^n$*-*)*space* $X$ *has the* $0\text{-}\overline{\mathrm{DD}}^{\{0,n\}}$*-property then the set of* (*homotopical*) $Z_n$*-points is a dense* $G_\delta$*-set in* $X$*:*

$$\boxed{\mathsf{Br} \cap \mathrm{LC}^n \cap 0\text{-}\overline{\mathrm{DD}}^{\{0,n\}} \subset \overline{\mathcal{Z}}_n}$$

(4) *If each point of a space* $X$ *is a homotopical* $Z_{m+k}$*-point, then* $X$ *has the* $m\text{-}\overline{\mathrm{DD}}^{\{0,k\}}$*-property:*

$$\boxed{\mathcal{Z}_{m+k} \subset m\text{-}\overline{\mathrm{DD}}^{\{0,k\}}}$$

(5) *If a topological space* $X$ *has either the* $n\text{-}\overline{\mathrm{DD}}^{\{n,0\}}$*- or the* $0\text{-}\overline{\mathrm{DD}}^{\{n,n\}}$*-property, then each point of* $X$ *is a homotopical* $Z_n$*-point:*

$$\boxed{0\text{-}\overline{\mathrm{DD}}^{\{n,n\}} \cup n\text{-}\overline{\mathrm{DD}}^{\{n,0\}} \subset \mathcal{Z}_n}$$

(6) *If a Tychonoff space* $X$ *has the* $2\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$*-property, then each point of* $X$ *is a homotopical* $Z_1$*-point:*

$$\boxed{2\text{-}\overline{\mathrm{DD}}^{\{0,0\}} \subset \mathcal{Z}_1}$$

## 8. Arithmetic of $m\text{-}\overline{\mathrm{DD}}^{\{n,k\}}$-properties

In this subsection we study the behavior of the $m\text{-}\overline{\mathrm{DD}}^{\{n,k\}}$-properties under arithmetic operations. The combination of the results from this subsection and Propositions 5.4–5.6 provides convenient tools for detecting the $m\text{-}\overline{\mathrm{DD}}^{\{n,k\}}$-properties of more complex spaces (like products or manifolds).

For a better visual presentation of our subsequent results, let us introduce the following operations on subclasses $\mathcal{A}, \mathcal{B} \subset \mathit{Top}$ of the class $\mathit{Top}$ of topological spaces:

$$\mathcal{A} \times \mathcal{B} = \{A \times B : A \in \mathcal{A},\ B \in \mathcal{B}\},$$
$$\frac{\mathcal{A}}{\mathcal{B}} = \{X \in \mathit{Top} : \exists B \in \mathcal{B} \text{ with } X \times B \in \mathcal{A}\},$$
$$\mathcal{A}^k = \{A^k : A \in \mathcal{A}\} \quad \text{and} \quad \sqrt[k]{\mathcal{A}} = \{A \in \mathit{Top} : A^k \in \mathcal{A}\}.$$

A space $X$ will be identified with the one-element class $\{X\}$. So $X \times \mathcal{A}$ and $\frac{\mathcal{A}}{X}$ mean $\{X\} \times \mathcal{A}$ and $\frac{\mathcal{A}}{\{X\}}$.

We recall that $m\text{-}\overline{\mathrm{DD}}^{\{n,k\}}$ stands for the class of all spaces possessing the $m\text{-}\overline{\mathrm{DD}}^{\{n,k\}}$-property and $\mathrm{LC}^n$ is the class of $\mathrm{LC}^n$-spaces.

THEOREM 8.1 (Multiplication formulas). *Let* $X, Y$ *be metrizable spaces and* $k_1, k_2, k,\ n_1, n_2,\ n,\ m_1, m_2, m$ *be non-negative integers or infinity.*

(1) (First multiplication formula) *If $X$ has the $m$-$\overline{\mathrm{DD}}^{\{n,k_1\}}$-property and $Y$ has the $m$-$\overline{\mathrm{DD}}^{\{n,k_2\}}$-property, then the product $X \times Y$ has the $m$-$\overline{\mathrm{DD}}^{\{n,k_1+k_2+1\}}$-property. This can be written as*

$$\boxed{m\text{-}\overline{\mathrm{DD}}^{\{n,k_1\}} \times m\text{-}\overline{\mathrm{DD}}^{\{n,k_2\}} \subset m\text{-}\overline{\mathrm{DD}}^{\{n,k_1+k_2+1\}}}$$

(2) (Second multiplication formula) *If $X$ has the $m$-$\overline{\mathrm{DD}}^{\{n_1,k_1\}}$-property and $Y$ has the properties $m$-$\overline{\mathrm{DD}}^{\{n,k_2\}}$ and $m$-$\overline{\mathrm{DD}}^{\{n_2,k\}}$ for $n = n_1 + n_2 + 1$ and $k = k_1 + k_2 + 1$, then the product $X \times Y$ has the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property. This can be written as*

$$\boxed{m\text{-}\overline{\mathrm{DD}}^{\{n_1,k_1\}} \times \big(m\text{-}\overline{\mathrm{DD}}^{\{n,k_2\}} \cap m\text{-}\overline{\mathrm{DD}}^{\{n_2,k\}}\big) \subset m\text{-}\overline{\mathrm{DD}}^{\{n,k\}}}$$

(3) (Multiplication by a cell) *If $X$ has the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property, then for any $d < m+1$ the product $\mathbb{I}^d \times X$ has the $(m-d)$-$\overline{\mathrm{DD}}^{\{d+n,d+k\}}$-property. This can be written as*

$$\boxed{\mathbb{I}^d \times m\text{-}\overline{\mathrm{DD}}^{\{n,k\}} \subset (m-d)\text{-}\overline{\mathrm{DD}}^{\{d+n,d+k\}}}$$

Remark 8.2. Let us mention that, since $\mathbb{R} \in 0$-$\overline{\mathrm{DD}}^{\{0,0\}}$, the second multiplication formula implies the following result of W. Mitchell [54, Theorem 4.3(3)] (see also R. Daverman [16, Proposition 2.8]): If $X$ is a compact metric ANR-space with $X \in 0$-$\overline{\mathrm{DD}}^{\{p,p+1\}}$, then $X \times \mathbb{R} \in 0$-$\overline{\mathrm{DD}}^{\{p+1,p+1\}}$. Moreover, by Theorem 8.1(2), for any metrizable space $X \in m$-$\overline{\mathrm{DD}}^{\{n,k\}} \cap m$-$\overline{\mathrm{DD}}^{\{n+p+1,k-1\}}$ we get $X \times \mathbb{R}^{m+p+1} \in m$-$\overline{\mathrm{DD}}^{\{n+p+1,k\}}$. The particular case of this result when $m = 0$ and $p = 1$ was proved by W. Mitchell in [54, Theorem 4.3(2)]. Similarly, we can see that Theorem 8.1(3) generalizes the following result of D. Halverson [39]: If $X$ is a locally compact *ANR* with $X \in 1$-$\overline{\mathrm{DD}}^{\{1,1\}}$, then $X \times \mathbb{R} \in 0$-$\overline{\mathrm{DD}}^{\{2,2\}}$.

We can express the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property via $0$-$\overline{\mathrm{DD}}^{\{n',k'\}}$-properties for sufficiently large $n', k'$.

Theorem 8.3 (Base enlargement formulas). *Let $X$ be a metrizable space and $n$, $k$, $m$, $m_1$, $m_2$ be non-negative integers or infinity.*

(1) *If $X$ has the $0$-$\overline{\mathrm{DD}}^{\{n+m_1,k+m_2\}}$-, $m_1$-$\overline{\mathrm{DD}}^{\{n,k+m-m_1\}}$-, and $m_2$-$\overline{\mathrm{DD}}^{\{n+m-m_2,k\}}$-properties simultaneously with $m = m_1 + m_2 + 1$, then $X$ has the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property. This can be written as*

$$\boxed{0\text{-}\overline{\mathrm{DD}}^{\{n+m_1,k+m_2\}} \cap m_1\text{-}\overline{\mathrm{DD}}^{\{n,k+m-m_1\}} \cap m_2\text{-}\overline{\mathrm{DD}}^{\{n+m-m_2,k\}} \subset m\text{-}\overline{\mathrm{DD}}^{\{n,k\}}}$$

(2) *If $X \in 0$-$\overline{\mathrm{DD}}^{\{n,k+m+1\}} \cap m$-$\overline{\mathrm{DD}}^{\{n+1,k\}}$, then $X$ has the $(m+1)$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property. This can be written as*

$$\boxed{0\text{-}\overline{\mathrm{DD}}^{\{n,k+m+1\}} \cap m\text{-}\overline{\mathrm{DD}}^{\{n+1,k\}} \subset (m+1)\text{-}\overline{\mathrm{DD}}^{\{n,k\}}}$$

(3) *$X$ has the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property if $X$ has the $0$-$\overline{\mathrm{DD}}^{\{n+i,k+j\}}$-property for all $i, j \in \omega$ with $i + j < m + 1$. This can be written as*

$$\boxed{\bigcap_{i+j<m+1} 0\text{-}\overline{\mathrm{DD}}^{\{n+i,k+j\}} \subset m\text{-}\overline{\mathrm{DD}}^{\{n,k\}}}$$

The second base enlargement formula implies that if $X$ is a metrizable space with $X \in 0\text{-}\overline{\mathrm{DD}}^{\{1,2\}}$, then $X \in 1\text{-}\overline{\mathrm{DD}}^{\{1,1\}}$. This result was established by D. Halverson [39] in the particular case when $X$ is a separable locally compact ANR.

## 9. $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-properties of products

In this subsection we apply the arithmetic formulas from the previous subsection to establish the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-properties of products.

THEOREM 9.1. *Let $m, n, k, d, l$ be non-negative integers and $L, D$ be metrizable spaces such that $L$ has the $0$-$\overline{\mathrm{DD}}^{\{0,0\}}$-property and $D$ has the $0$-$\overline{\mathrm{DD}}^{\{0,d+l\}}$-property. If $m+n+k < 2d+l$, then the product $D^d \times L^l$ has the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property. This can be written as*

$$\boxed{\big(0\text{-}\overline{\mathrm{DD}}^{\{0,0\}}\big)^l \times \big(0\text{-}\overline{\mathrm{DD}}^{\{0,d+l\}}\big)^d \subset \bigcap_{m+n+k<2d+l} m\text{-}\overline{\mathrm{DD}}^{\{n,k\}}}$$

Combining Theorem 9.1 with Theorem 3.3, Proposition 3.2 and Proposition 5.6 we obtain

THEOREM 9.2. *Let $l, d$ be non-negative integers or infinity and $L$, $D$ be completely metrizable locally path-connected spaces such that $L$ has no isolated points and $D$ is $1$-dimensional without free arcs. Then the product $D^d \times L^l$ has the $m$-$\overline{\mathrm{DD}}^n$-property for all $m, n \in \omega$ with $m + 2n < l + 2d$. Consequently, if $p : K \to M$ is a perfect map between paracompact submetrizable spaces with $\dim M + 2\dim_\triangle(p) < l + 2d$, then any simplicially factorizable map $f : K \to D^d \times L^l$ can be approximated by maps which are injective on each fiber of $p$.*

The case $m = 0$ from Theorem 9.2 yields

COROLLARY 9.3. *Let $L, D$ be completely metrizable ANR's such that $L$ has no isolated points and $D$ is $1$-dimensional without free arcs. Then the product $D^d \times L^l$ has the $\mathrm{DD}^n\mathrm{P}$ for all $n < d + l/2$. Consequently, for any compact space $X$ with $\dim X < d + l/2$ the set of all embeddings is a dense $G_\delta$ in the function space $C(X, D^d \times L^l)$.*

REMARK 9.4. Corollary 9.3 generalizes many (if not all) results on embeddings into products. Indeed, letting $L = \mathbb{R}$ be the real line and $D$ be a dendrite with a dense set of end-points we obtain the following well known results:

- the case $d = 0$ and $l = 2n+1$ is the Lefschetz–Menger–Nöbeling–Pontrjagin embedding theorem that $\mathbb{R}^{2n+1}$ has $\mathrm{DD}^n\mathrm{P}$;
- the case $d = n + 1$ and $l = 0$ is the embedding theorem of P. Bowers [13] that $D^{n+1}$ has $\mathrm{DD}^n\mathrm{P}$;
- the case $d = n$ and $l = 1$ is the embedding theorem of Y. Sternfeld [66] that $D^n \times \mathbb{I}$ has $\mathrm{DD}^n\mathrm{P}$;

Also, for $d = 0$ and $m = 0$ Theorem 9.2 is close to the embedding theorem of T. Banakh and Kh. Trushchak [7] while for $l = 0$ and $m = 0$ it is close to the one of T. Banakh, R. Cauty, Kh. Trushchak and L. Zdomskyy [5].

Remark 9.5. Letting $d = 0$ and $L = \mathbb{R}$ in Theorem 9.2, we obtain Pasynkov's result [57] asserting that for a map $p : X \to Y$ between finite-dimensional metrizable compacta the function space $C(X, \mathbb{R}^{\dim Y + 2\dim(p)+1})$ contains a dense $G_\delta$-set of maps that are injective on each fiber of the map $p$.

Therefore, Theorem 9.2 can be considered as a generalization of [57]. However, Theorem 9.2 does not cover another generalization of Pasynkov's result due to H. Toruńczyk [70]: *If $p : X \to Y$ is a map between compacta, then the space $C(X, \mathbb{R}^{\dim X + \dim(p)+1})$ contains a dense $G_\delta$-set of maps that are injective on each fiber of the map $p$.*

Taking into account that the Euclidean space $\mathbb{R}^d$ has the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-properties for all $m, n, k$ with $m + n + k < d$, we may ask whether this theorem of H. Toruńczyk is true in the following more general form.

Problem 9.6. *Let $p : K \to M$ be a map between finite-dimensional compact metrizable spaces and $X$ be a Polish AR-space possessing the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property for all $m, n, k$ with $m + n + k \le \dim K + \dim(p)$. Does $p$ embed into the projection $\mathrm{pr} : M \times X \to M$?*

Let us also note that the above mentioned result of H. Toruńczyk would follow from our Theorem 1.1 if the following problem had an affirmative answer.

Problem 9.7. *Let $f : X \to Y$ be a $k$-dimensional map between finite-dimensional metrizable compacta. Is it true that there is a map $g : Y \to Z$ to a compact space $Z$ with $\dim Z \le \dim X - k$ such that the map $g \circ f$ is still $k$-dimensional?*

## 10. A short survey on homological $Z_n$-sets

The most exciting results on $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-properties (like multiplication and $k$-root formulas) are obtained by using homological $Z_n$-sets. In this subsection we survey some basic facts about such sets, and refer the interested reader to [4] where all these results are established. We use singular homology with coefficients in an Abelian group $G$. If $G = \mathbb{Z}$, we write $H_k(X)$ instead of $H_k(X; \mathbb{Z})$. By $\widetilde{H}_*(X; G)$ we denote the singular homology of $X$ reduced in dimension zero.

It can be shown that a closed subset $A$ of a topological space $X$ is a homotopical $Z_n$-set in $X$ if and only if for every open set $U \subset X$ the inclusion $U \setminus A \to U$ is a weak homotopy equivalence, which means that the relative homotopy groups $\pi_k(U, U \setminus A)$ vanish for all $k < n + 1$. Replacing the relative homotopy groups by relative homology groups, we obtain the notion of a homological $Z_n$-set.

A closed subset $A$ of a space $X$ is defined to be

- a *$G$-homological $Z_n$-set* in $X$ for a coefficient group $G$ if $H_k(U, U \setminus A; G) = 0$ for all open sets $U \subset X$ and all $k < n + 1$;
- an *$\exists G$-homological $Z_n$-set* if $A$ is a $G$-homological $Z_n$-set in $X$ for some coefficient group $G$;
- a *homological $Z_n$-set* if $A$ is a $\mathbb{Z}$-homological $Z_n$-set in $X$ (equivalently, if $A$ is a $G$-homological $Z_n$-set for every coefficient group $G$).

In [19] homological $Z_\infty$-sets are referred to as closed sets of infinite codimension. On the other hand, the term "homological $Z_n$-set" has been used in [42], [4], [3], and [6].

The following theorem whose proof can be found in [4, Theorems 3.2-3.3] describes the interplay between various sorts of $Z_n$-sets.

THEOREM 10.1. *Let $X$ be a topological space.*

(1) *Each homotopical $Z_n$-set in $X$ is both a $Z_n$-set and a homological $Z_n$-set.*
(2) *Each $Z_n$-set in an $\mathrm{LC}^n$-space is a homotopical $Z_n$-set.*
(3) *A set is a homotopical $Z_0$-set in $X$ iff it is a $\exists G$-homological $Z_0$-set.*
(4) *Each $\exists G$-homological $Z_1$-set in $X$ is a $Z_1$-set.*
(5) *If $X$ is an $\mathrm{LC}^1$-space, then a homotopical $Z_2$-set in $X$ is a homotopical $Z_n$-set if and only if it is a homological $Z_n$-set.*

The last item of this theorem has fundamental importance since it allows application of powerful tools of algebraic topology for studying homotopical $Z_n$-sets and related $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-properties. The study of $G$-homological $Z_n$-sets for an arbitrary group $G$ can be reduced to considering Bockstein groups. Denote by $\Pi$ the set of all prime numbers and consider the following groups:

- $\mathbb{Q}$, the group of rational numbers;
- $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$, the cyclic group of a prime order $p \in \Pi$;
- $\mathbb{Q}_p = \{z \in \mathbb{C} : \exists k \in \mathbb{N}\ z^{p^k} = 1\}$, the quasicyclic $p$-group;
- $R_p = \{m/n : m \in \mathbb{Z}$ and $n \in \mathbb{N}$ is not divisible by $p\}$.

The Bockstein family $\sigma(G)$ of a group $G$ is a subfamily of $\{\mathbb{Q}, \mathbb{Z}_p, \mathbb{Q}_p, R_p : p \in \Pi\}$ such that:

- $\mathbb{Q} \in \sigma(G)$ iff $G/\mathrm{Tor}(G) \neq 0$ is divisible;
- $\mathbb{Z}_p \in \sigma(G)$ iff the $p$-torsion part $p$-$\mathrm{Tor}(G)$ is not divisible by $p$;
- $\mathbb{Q}_p \in \sigma(G)$ iff $p$-$\mathrm{Tor}(G) \neq 0$ is divisible by $p$;
- $R_p \in \sigma(G)$ iff the group $G/p$-$\mathrm{Tor}(G)$ is not divisible by $p$.

Here

$$\mathrm{Tor}(G) = \{x \in G : \exists n \in \mathbb{N}\ n \cdot x = 0\} \quad \text{and} \quad p\text{-}\mathrm{Tor}(G) = \{x \in G : \exists k \in \mathbb{Z}\ p^k \cdot x = 0\}$$

are the torsion and $p$-torsion parts of $G$. In particular, $\sigma(\mathbb{Z}) = \{R_p : p \in \Pi\}$.

THEOREM 10.2. *Let $A$ be a closed subset of a space $X$, $G$ be a coefficient group, and $p$ be a prime number.*

(1) *$A$ is a $G$-homological $Z_n$-set in $X$ if and only if $A$ is an $H$-homological $Z_n$-set in $X$ for all groups $H \in \sigma(G)$.*
(2) *If $A$ is a $R_p$-homological $Z_n$-set in $X$, then $A$ is a $\mathbb{Q}$-homological and $\mathbb{Z}_p$-homological $Z_n$-set in $X$.*
(3) *If $A$ is a $\mathbb{Z}_p$-homological $Z_n$-set in $X$, then $A$ is a $\mathbb{Q}_p$-homological $Z_n$-set in $X$.*
(4) *If $A$ is a $\mathbb{Q}_p$-homological $Z_{n+1}$-set in $X$, then $A$ is a $\mathbb{Z}_p$-homological $Z_n$-set in $X$.*
(5) *$A$ is a $R_p$-homological $Z_n$-set in $X$ provided $A$ is a $\mathbb{Q}$-homological $Z_n$-set in $X$ and a $\mathbb{Q}_p$-homological $Z_{n+1}$-set in $X$.*

By analogy with multiplication formulas for the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-properties, there are multiplication formulas for homotopical and homological $Z_n$-sets (see [4, Theorem 6.1]).

Theorem 10.3. *Let $A \subset X$, $B \subset Y$ be closed subsets in Tychonoff spaces $X, Y$.*

(1) *If $A$ is a homotopical $Z_n$-set in $X$ and $B$ is a homotopical $Z_m$-set in $X$ then $A \times B$ is a homotopical $Z_{n+m+1}$-set in $X \times Y$.*
(2) *If $A$ is a homological $Z_n$-set in $X$ and $B$ is a homological $Z_m$-set in $X$ then $A \times B$ is a homological $Z_{n+m+1}$-set in $X \times Y$.*

Surprisingly, the multiplication formulas for homological $Z_n$-sets can be reversed:

Theorem 10.4. *Let $n, m \in \omega \cup \{\infty\}$, $k \in \omega$, and $A \subset X$, $B \subset Y$ be closed subsets of Tychonoff spaces $X$ and $Y$. Let $\mathfrak{D} = \{\mathbb{Q}, \mathbb{Q}_p : p \in \Pi\}$ and for every group $G \in \mathfrak{D}$ let $B_G \subset Y$ be a closed subset which fails to be a $G$-homological $Z_m$-set in $Y$.*

(1) *$A$ is a homological $Z_n$-set in $X$ if and only if $A^k$ is a homological $Z_{kn+k-1}$-set in $X^k$.*
(2) *If $A \times B$ is an $\mathbb{F}$-homological $Z_{n+m}$-set in $X \times Y$ for some field $\mathbb{F}$, then either $A$ is an $\mathbb{F}$-homological $Z_n$-set in $X$ or $B$ is an $\mathbb{F}$-homological $Z_m$-set in $Y$.*
(3) *If $A \times B$ is a homological $Z_{n+m}$-set in $X \times Y$, then either $A$ is a homological $Z_n$-set in $X$ or $B$ is an $\exists G$-homological $Z_m$-set in $Y$.*
(4) *$A$ is a homological $Z_n$-set in $X$ provided $A \times B_G$ is a homological $Z_{n+m}$-set in $X \times Y$ for every group $G \in \mathfrak{D}$.*

Theorem 10.4 is the principal tool in the proof of the $k$-root and multiplication formulas for the classes $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$. We first discuss these formulas for the classes $\mathcal{Z}_n$, $\overline{\mathcal{Z}}_n$, and $\mathcal{Z}_n^{\mathbb{Z}}$ because they are tightly connected with the classes $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$.

Let us start with some definitions. A point $x$ of a space $X$ is defined to be a *homological $Z_n$-point* if the singleton $\{x\}$ is a homological $Z_n$-set in $X$. By analogy, we define $G$-homological and $\exists G$-homological $Z_n$-points.

Let $\mathcal{Z}_n^G(X)$ denote the set of all $G$-homological $Z_n$-points in the space $X$ and let $\overline{\mathcal{Z}}_n^G$ (resp., $\mathcal{Z}_n^G$) be the class of Tychonoff spaces $X$ such that the set $\mathcal{Z}_n^G(X)$ is dense in (resp., coincides with) $X$. We also recall that $\overline{\mathcal{Z}}_n$ (resp., $\mathcal{Z}_n$) stands for the class of Tychonoff spaces $X$ such that the set $\mathcal{Z}_n(X)$ of homotopical $Z_n$-points of $X$ is dense in (resp., coincides with) $X$. Using these notations, Theorem 10.1 can be written in the following form.

Theorem 10.5. *Let $n \in \omega \cup \{\infty\}$ and $G$ be a non-trivial Abelian group. Then:*

(1) $\mathcal{Z}_n \subset \mathcal{Z}_n^{\mathbb{Z}} \subset \mathcal{Z}_n^G$;
(2) $\mathcal{Z}_0 = \mathcal{Z}_0^{\mathbb{Z}} = \mathcal{Z}_0^G$;
(3) $\mathrm{LC}^1 \cap \mathcal{Z}_1^G \subset \mathcal{Z}_1$;
(4) $\mathrm{LC}^1 \cap \mathcal{Z}_2 \cap \mathcal{Z}_n^{\mathbb{Z}} \subset \mathcal{Z}_n$;
(5) $\mathrm{LC}^1 \cap \mathsf{Br} \cap \overline{\mathcal{Z}}_2 \cap \overline{\mathcal{Z}}_n^{\mathbb{Z}} \subset \overline{\mathcal{Z}}_n$.

The last item of Theorem 10.5 follows from the fact that each of the sets $\mathcal{Z}_n(X)$ and $\mathcal{Z}_n^{\mathbb{Z}}(X)$ is $G_\delta$ in $X$ provided $X$ is a separable metrizable $\mathrm{LC}^n$-space [4, Theorem 9.2].

In its turn, Theorem 10.3 implies multiplication formulas for the classes $\mathcal{Z}_n$, $\overline{\mathcal{Z}}_n$, and $\mathcal{Z}_n^{\mathbb{Z}}$:

Theorem 10.6 (Multiplication formulas). *Let $n, m \in \omega \cup \{\infty\}$. Then:*

(1) $\mathcal{Z}_n \times \mathcal{Z}_m \subset \mathcal{Z}_{m+n+1}$;
(2) $\mathcal{Z}_n^{\mathbb{Z}} \times \mathcal{Z}_m^{\mathbb{Z}} \subset \mathcal{Z}_{n+m+1}^{\mathbb{Z}}$;
(3) $\overline{\mathcal{Z}}_n \times \overline{\mathcal{Z}}_m \subset \overline{\mathcal{Z}}_{m+n+1}$;
(4) $\overline{\mathcal{Z}}_n^{\mathbb{Z}} \times \overline{\mathcal{Z}}_m^{\mathbb{Z}} \subset \overline{\mathcal{Z}}_{n+m+1}^{\mathbb{Z}}$.

The multiplication formulas can be reversed, which yields division and $k$-root formulas for the classes $\mathcal{Z}_n^{\mathbb{Z}}$ (we recall that, for a class $\mathcal{A}$, we put $\sqrt[k]{\mathcal{A}} = \{X : X^k \in \mathcal{A}\}$).

Theorem 10.7 ($k$-root formulas). *Let $n \in \omega \cup \{\infty\}$ and $k \in \mathbb{N}$.*

(1) *A space $X$ belongs to the class $\mathcal{Z}_n^{\mathbb{Z}}$ if and only if $X^k$ belongs to $\mathcal{Z}_{kn+k-1}^{\mathbb{Z}}$:*

$$\boxed{\mathcal{Z}_n^{\mathbb{Z}} = \sqrt[k]{\mathcal{Z}_{kn+k-1}^{\mathbb{Z}}}}$$

(2) *A metrizable separable Baire $\mathrm{LC}^{kn+k-1}$-space $X$ belongs to the class $\overline{\mathcal{Z}}_n^{\mathbb{Z}}$ if and only if $X^k$ belongs to $\overline{\mathcal{Z}}_{kn+k-1}^{\mathbb{Z}}$:*

$$\boxed{\overline{\mathcal{Z}}_n^{\mathbb{Z}} \supset \sqrt[k]{\overline{\mathcal{Z}}_{kn+k-1}^{\mathbb{Z}}} \cap \mathrm{LC}^{nk+k-1} \cap \mathsf{Br}}$$

To state the division formula for the classes $\mathcal{Z}_n^{\mathbb{Z}}$ and $\overline{\mathcal{Z}}_n^{\mathbb{Z}}$ we need some more notations (which will be used for the classes $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$ as well). Consider the following classes of topological spaces:

- $\bigcup_G \mathcal{Z}_n^G = \bigcup\{\mathcal{Z}_n^G : G$ is a non-trivial Abelian group$\}$;
- $\bigcup_G \overline{\mathcal{Z}}_n^G = \bigcup\{\overline{\mathcal{Z}}_n^G : G$ is a non-trivial Abelian group$\}$;
- $\exists_\circ \bigcup_G \overline{\mathcal{Z}}_n^G$, the class of spaces containing a non-empty open subspace $U \in \bigcup_G \overline{\mathcal{Z}}_n^G$.

For example, the space $\mathbb{R}^n$ belongs to none of these classes.

Now, we can state the division formulas for the classes $\mathcal{Z}_n^{\mathbb{Z}}$ and $\overline{\mathcal{Z}}_n^{\mathbb{Z}}$ (recall that if $\mathcal{A}$ and $\mathcal{B}$ are two classes, then $\frac{\mathcal{A}}{\mathcal{B}}$ stands for the class $\{X \in \mathit{Top} : \exists B \in \mathcal{B}$ with $X \times B \in \mathcal{A}\}$).

Theorem 10.8 (Division formulas). *Let $n \in \omega \cup \{\infty\}$ and $k \in \omega$.*

(1) *A space $X$ belongs to the class $\mathcal{Z}_n^{\mathbb{Z}}$ if and only if $X \times Y \in \mathcal{Z}_{n+k}^{\mathbb{Z}}$ for some space $Y \notin \bigcup_G \mathcal{Z}_k^G$. This can be written as*

$$\boxed{\frac{\mathcal{Z}_{n+k}^{\mathbb{Z}}}{\mathit{Top} \setminus \bigcup_G \mathcal{Z}_k^G} = \mathcal{Z}_n^{\mathbb{Z}}}$$

(2) *A metrizable separable Baire $\mathrm{LC}^n$-space $X$ belongs to the class $\overline{\mathcal{Z}}_n^{\mathbb{Z}}$ if and only if $X \times Y \in \overline{\mathcal{Z}}_{n+k}^{\mathbb{Z}}$ for some space $Y \notin \exists_\circ \bigcup_G \overline{\mathcal{Z}}_n^G$. This can be written as*

$$\boxed{\mathsf{Br} \cap \mathrm{LC}^n \cap \frac{\overline{\mathcal{Z}}_{n+k}^{\mathbb{Z}}}{\mathit{Top} \setminus \exists_\circ \bigcup_G \overline{\mathcal{Z}}_k^G} \subset \overline{\mathcal{Z}}_n^{\mathbb{Z}}}$$

Because of the division formulas, it is important to detect spaces $X \notin \bigcup_G \mathcal{Z}_n^G$. It turns out that this happens for every metrizable space $X$ with $\dim X \le n$, or more generally with transfinite separation dimension $\mathsf{trt}(X) < n+1$. The latter dimension can be introduced inductively (see [2]):

- $\mathsf{trt}(X) = -1$ iff $X = \emptyset$;
- $\mathsf{trt}(X) \le \alpha$ for an ordinal $\alpha$ if each closed subset $A \subset X$ with $|A| > 1$ contains a closed subset $B \subset A$ such that $\mathsf{trt}(B) < \alpha$ and $A \setminus B$ is disconnected.

A space $X$ is called $\mathsf{trt}$-*dimensional* if $\mathsf{trt}(X) \le \alpha$ for some ordinal $\alpha$. For a $\mathsf{trt}$-dimensional space $X$ we let $\mathsf{trt}(X)$ be the smallest ordinal $\alpha$ with $\mathsf{trt}(X) \le \alpha$.

By [2], each compact metrizable $\mathsf{trt}$-dimensional space is a $C$-space. On the other hand, a Čech-complete space is $\mathsf{trt}$-dimensional if it can be written as the countable union of hereditarily disconnected subspaces (see [59]). It is easy to see that for a finite-dimensional metrizable separable space $X$ we get $\mathsf{trt}(X) \le \dim(X)$. Moreover, if $X$ is finite-dimensional and compact, then $\mathsf{trt}(X) = \dim(X)$ (see [65]).

The following theorem was proved in [4] and [3].

THEOREM 10.9. *Let $X \in \bigcup_G \mathcal{Z}_n^G$ for some $n \in \omega \cup \{\infty\}$.*

(1) *If $n < \infty$, then $\mathsf{trt}(X) > n$.*
(2) *If $n = \infty$, then $X$ is not $\mathsf{trt}$-dimensional.*
(3) *If $n = \infty$ and $X$ is locally compact and locally contractible, then $X$ is not a $C$-space.*

Consequently, for any metrizable separable space $X \in \bigcup_G \mathcal{Z}_n^G$ we have

$$\dim X \ge \mathsf{trt}(X) > n.$$

A similar inequality holds for cohomological and extension dimensions of $X$. We recall their definitions.

For a space $X$ and a $CW$-complex $L$ we write $\text{e-dim}\, X \le L$ if each map $f : A \to L$ defined on a closed subset $A \subset X$ admits a continuous extension $\bar{f} : X \to L$; see [25] and [26] for more information on extension dimension theory. It follows from the classical Hurewicz–Wallman Theorem [32, 1.9.3] that $\text{e-dim}\, X \le S^n$ iff $\dim X \le n$. The cohomological dimension with respect to a given Abelian group $G$ can be expressed via extension dimension as follows: define

$$\dim_G X \le n \quad \text{if } \text{e-dim}\, X \le K(G, n),$$

where $K(G, n)$ is the Eilenberg–MacLane complex of type $(G, n)$, and let $\dim_G X$ be the smallest non-negative integer with $\dim_G X \le n$. If there is no such integer $n$, we put $\dim_G X = \infty$.

THEOREM 10.10. *Let $n \in \omega$ and $X \in \mathcal{Z}_n^{\mathbb{Z}}$ be a locally compact* $\mathrm{LC}^n$*-space. Then:*

(1) $\dim_G X > n$ *for any Abelian group $G$.*
(2) $\text{e-dim}\, X \not\le L$ *for any $CW$-complex $L$ with a non-trivial homotopy group $\pi_k(L)$ for some $k \le n$.*

## 11. Homological $Z_n$-sets and $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-properties

In this subsection we discuss the interplay between the classes $\mathcal{Z}_n^G$ and $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$. The following two theorems present homological counterparts of the formulas

$$n\text{-}\overline{\mathrm{DD}}^{\{n,0\}} \subset \mathcal{Z}_n \subset \bigcap_{m+k\le n} m\text{-}\overline{\mathrm{DD}}^{\{0,k\}} \quad \text{and}$$

$$\mathsf{Br} \cap \mathrm{LC}^n \cap 0\text{-}\overline{\mathrm{DD}}^{\{0,n\}} \subset \mathrm{LC}^0 \cap \overline{\mathcal{Z}}_n \subset 0\text{-}\overline{\mathrm{DD}}^{\{0,n\}}$$

from Theorem 7.1.

THEOREM 11.1. *Let $X$ be a Tychonoff space and $n \in \omega \cup \{\infty\}$.*

(1) *If an $\mathrm{LC}^1$-space $X$ has the $2$-$\overline{\mathrm{DD}}^{\{0,2\}}$-property, then each homological $Z_n$-point in $X$ is a homotopical $Z_n$-point:*

$$\boxed{\mathrm{LC}^1 \cap 2\text{-}\overline{\mathrm{DD}}^{\{0,2\}} \cap \mathcal{Z}_n^{\mathbb{Z}} \subset \mathcal{Z}_n}$$

(2) *If a metrizable separable Baire $\mathrm{LC}^n$-space $X$ has the $0$-$\overline{\mathrm{DD}}^{\{0,2\}}$-property and contains a dense set of homological $Z_n$-points, then $X$ contains a dense set of homotopical $Z_n$-points and $X \in 0$-$\overline{\mathrm{DD}}^{\{0,n\}}$:*

$$\boxed{\mathrm{LC}^n \cap \mathsf{Br} \cap 0\text{-}\overline{\mathrm{DD}}^{\{0,2\}} \cap \overline{\mathcal{Z}}_n^{\mathbb{Z}} \subset \overline{\mathcal{Z}}_n \cap 0\text{-}\overline{\mathrm{DD}}^{\{0,n\}}}$$

(3) *If $X$ has the $(2n+1)$-$\overline{\mathrm{DD}}^{\{0,0\}}$-property, then each point of $X$ is a homological $Z_n$-point:*

$$\boxed{(2n+1)\text{-}\overline{\mathrm{DD}}^{\{0,0\}} \subset \mathcal{Z}_n^{\mathbb{Z}}}$$

(4) *If $X$ has the $2n$-$\overline{\mathrm{DD}}^{\{0,0\}}$-property, then each point of $X$ is a $G$-homological $Z_n$-point for any group $G$ with divisible quotient $G/\mathrm{Tor}(G)$. Consequently,*

$$\boxed{2n\text{-}\overline{\mathrm{DD}}^{\{0,0\}} \subset \mathcal{Z}_n^{\mathbb{Q}} \cap \textstyle\bigcap_p (\mathcal{Z}_n^{\mathbb{Z}_p} \cap \mathcal{Z}_n^{\mathbb{Q}_p})}$$

THEOREM 11.2. *Let $m, m, k$ be non-negative integers or infinity.*

(1) *If each point of an $\mathrm{LC}^1$-space $X$ with the $2$-$\overline{\mathrm{DD}}^{\{0,2\}}$-property is a homological $Z_{m+k}$-point, then $X$ has the $m$-$\overline{\mathrm{DD}}^{\{0,k\}}$-property. This can be written as*

$$\boxed{\mathrm{LC}^1 \cap \mathcal{Z}_{m+k}^{\mathbb{Z}} \cap 2\text{-}\overline{\mathrm{DD}}^{\{0,2\}} \subset m\text{-}\overline{\mathrm{DD}}^{\{0,k\}}}$$

(2) *If a metrizable separable $\mathrm{LC}^k$-space $X$ has the $0$-$\overline{\mathrm{DD}}^{\{0,2\}}$-property and contains a dense set of homological $Z_k$-points, then $X$ has the $0$-$\overline{\mathrm{DD}}^{\{0,k\}}$-property. This can be written as*

$$\boxed{\mathrm{LC}^k \cap \overline{\mathcal{Z}}_k^{\mathbb{Z}} \cap 0\text{-}\overline{\mathrm{DD}}^{\{0,2\}} \subset 0\text{-}\overline{\mathrm{DD}}^{\{0,k\}}}$$

(3) *If each point of a metrizable separable $\mathrm{LC}^2$-space $X$ is a homological $Z_{m+n+k}$-point and $X$ has the $m$-$\overline{\mathrm{DD}}^{\{n,\max\{n,2\}\}}$-property, then $X$ has the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property. This can be written as*

$$\boxed{\mathrm{LC}^2 \cap \mathcal{Z}^{\mathbb{Z}}_{m+n+k} \cap m\text{-}\overline{\mathrm{DD}}^{\{n,\max\{2,n\}\}} \subset m\text{-}\overline{\mathrm{DD}}^{\{n,k\}}}$$

(4) *If each point of an* $\mathrm{LC}^1$*-space* $X$ *is a homological* $Z_m$*-point and* $X \in 2\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$*, then* $X$ *has the* $m\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$*-property. This can be written as*

$$\boxed{\mathrm{LC}^1 \cap \mathcal{Z}^{\mathbb{Z}}_{m} \cap 2\text{-}\overline{\mathrm{DD}}^{\{0,0\}} \subset m\text{-}\overline{\mathrm{DD}}^{\{0,0\}}}$$

(5) *If each point of an* $\mathrm{LC}^0$*-space* $X$ *is a* $G$*-homological* $Z_2$*-point for some group* $G$*, then* $X$ *has the* $\mathrm{DD}^1\mathrm{P}$*. If in addition* $X$ *is a metrizable* $\mathrm{LC}^1$*-space containing a dense set of homotopical* $Z_2$*-points, then* $X \in 2\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$*.*

$$\boxed{\mathrm{LC}^0 \cap (\textstyle\bigcup_G \mathcal{Z}^G_2) \subset 0\text{-}\overline{\mathrm{DD}}^{\{1,1\}} \quad \textit{and} \quad \mathrm{LC}^1 \cap \overline{\mathcal{Z}}_2 \cap (\textstyle\bigcup_G \mathcal{Z}^G_2) \subset 2\text{-}\overline{\mathrm{DD}}^{\{0,0\}}}$$

## 12. Homological characterization of the $0\text{-}\overline{\mathrm{DD}}^{\{n,k\}}$-property

In this subsection we prove a quantified version of the homological characterization of the $0\text{-}\overline{\mathrm{DD}}^{\{\infty,\infty\}}$-property due to Daverman and Walsh [19].

First, we provide a homotopical version of the Daverman–Walsh result.

THEOREM 12.1. *Let* $n, k$ *be finite or infinite integers (with* $n \le k$*). A Polish* $(\mathrm{LC}^k)$*-space* $X$ *has the* $0\text{-}\overline{\mathrm{DD}}^{\{n,k\}}$*-property if (and only if) there is a countable family* $\mathcal{F}$ *of (*$n$*-dimensional compact) homotopical* $Z_k$*-sets in* $X$ *such that each compact subset* $K \subset X \setminus \bigcup \mathcal{F}$ *is a homotopical* $Z_n$*-set in* $X$*.*

Under some mild assumptions on $X$ it is possible to replace the homotopical conditions in Theorem 12.1 by homological ones.

THEOREM 12.2. *A Polish* $\mathrm{LC}^{\max\{n,k\}}$*-space* $X \in 0\text{-}\overline{\mathrm{DD}}^{\{2,2\}}$ *has the* $0\text{-}\overline{\mathrm{DD}}^{\{n,k\}}$*-property provided each point of* $X$ *is a homological* $Z_{2+\max\{n,k\}}$*-point and there is a countable family* $\mathcal{F}$ *of homological* $Z_k$*-sets in* $X$ *such that each compact subset* $K \subset X \setminus \bigcup \mathcal{F}$ *is a homological* $Z_n$*-set in* $X$*.*

This theorem implies another characterization of $0\text{-}\overline{\mathrm{DD}}^{\{n,k\}}$-properties in terms of approximation properties defined as follows. We shall say that a topological space has the *n-dimensional approximation property* (briefly, $\mathrm{AP}[n]$) if for any open cover $\mathcal{U}$ of $X$ and a map $f : \mathbb{I}^n \to X$ there is a map $g : \mathbb{I}^n \to X$ such that $g$ is $\mathcal{U}$-homotopic to $f$ and $\mathsf{trt}(g(\mathbb{I}^n)) < n+1$. Here we assume that $\alpha < \infty + 1$ for each ordinal $\alpha$ (which is essential if $n = \infty$).

Observe that each $\mathrm{LC}^0$-space has $\mathrm{AP}[0]$ and each $\mathrm{LC}^1$-space has $\mathrm{AP}[1]$.

THEOREM 12.3. *If each point of a Polish* $\mathrm{LC}^{\max\{n,k\}}$*-space* $X$ *is a homological* $Z_{n+k}$*-point and* $X$ *has the properties* $\mathrm{AP}[n]$ *and* $0\text{-}\overline{\mathrm{DD}}^{\{2,\min\{2,n\}\}}$*, then* $X$ *has the* $0\text{-}\overline{\mathrm{DD}}^{\{n,k\}}$*-property. This can be written as*

$$\boxed{\Pi^0_2 \cap \mathrm{LC}^{\max\{n,k\}} \cap \mathcal{Z}^{\mathbb{Z}}_{n+k} \cap \mathrm{AP}[n] \cap 0\text{-}\overline{\mathrm{DD}}^{\{2,\min\{2,n\}\}} \subset 0\text{-}\overline{\mathrm{DD}}^{\{n,k\}}}$$

## 13. $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-properties of locally rectifiable spaces

There is a non-trivial interplay between $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-properties for spaces having a kind of homogeneity property. We recall that a space $X$ is *topologically homogeneous* if for any two points $x_0, x \in X$ there is a homeomorphism $h_x : X \to X$ such that $h_x(x_0) = x$. If the homeomorphism $h_x$ can be chosen to depend continuously on $x$ then $X$ is called rectifiable at $x_0$.

More precisely, we define a topological space $X$ to be *locally rectifiable at a point* $x_0 \in X$ if there exists a neighborhood $U$ of $x_0$ such that for every $x \in U$ there is a homeomorphism $h_x : X \to X$ such that $h_x(x_0) = x$ and $h_x$ continuously depends on $x$ in the sense that the map $H : U \times X \to U \times X$, $(x, z) \mapsto (x, h_x(z))$, is a homeomorphism. If $U = X$, then the space $X$ is called *rectifiable* at $x_0$.

A space $X$ is called (*locally*) *rectifiable* if it is (locally) rectifiable at each point $x \in X$. Rectifiable spaces were introduced in [37] and studied in [48]. The class of rectifiable spaces contains the underlying spaces of topological groups but also contains spaces not homeomorphic to topological groups. A simplest such example is the 7-dimensional sphere $S^7$ (see [74]). It should be mentioned that all finite-dimensional spheres $S^n$ are locally rectifiable but only $S^1$, $S^3$ and $S^7$ are rectifiable (this follows from Adams' famous result [1] detecting $H$-spaces among the spheres). It can be shown that each connected locally rectifiable space is topologically homogeneous. On the other hand, the Hilbert cube is topologically homogeneous but fails to be (locally) rectifiable (see [37]).

By $\mathcal{LR}$ we denote the class of Tychonoff locally rectifiable spaces.

THEOREM 13.1. *Let $X$ be a locally rectifiable Tychonoff space.*

(1) *If $X$ has the $m$-$\overline{\mathrm{DD}}^{\{0,k\}}$-property, then each point of $X$ is a homotopical $Z_{m+k}$-point:*

$$\boxed{\mathcal{LR} \cap m\text{-}\overline{\mathrm{DD}}^{\{0,k\}} \subset \mathcal{Z}_{m+k}}$$

(2) *If $X$ has the $m$-$\overline{\mathrm{DD}}^{\{0,k\}}$-property, then $X$ has $i$-$\overline{\mathrm{DD}}^{\{0,j\}}$-properties for all $i, j$ with $i + j \le m + k$:*

$$\boxed{\mathcal{LR} \cap m\text{-}\overline{\mathrm{DD}}^{\{0,k\}} \subset \textstyle\bigcap_{i+j\le m+k} i\text{-}\overline{\mathrm{DD}}^{\{0,j\}}}$$

(3) *If either $X \in \overline{\mathcal{Z}}_{m+p}$ or $X \in \overline{\mathcal{Z}}^{\mathbb{Z}}_{m+p} \cap \mathrm{LC}^1$, then the product $X \times Y$ has the $m$-$\overline{\mathrm{DD}}^{\{n,k+p+1\}}$-property for each separable metrizable $\mathrm{LC}^k$-space $Y$ possessing the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property with $n \le k$. This can be written as*

$$\boxed{\big(\mathcal{LR} \cap \mathrm{LC}^1 \cap \overline{\mathcal{Z}}^{\mathbb{Z}}_{m+p}\big) \times \big(\mathrm{LC}^k \cap m\text{-}\overline{\mathrm{DD}}^{\{n,k\}}\big) \subset m\text{-}\overline{\mathrm{DD}}^{\{n,k+p+1\}}}$$

REMARK 13.2. Since $\mathbb{R}^q \in \mathcal{Z}_{q-1}$ is rectifiable, Theorem 13.1(3) implies that the product $X \times \mathbb{R}^{m+p}$ has the $m$-$\overline{\mathrm{DD}}^{\{n,k+p\}}$-property for any separable metrizable $\mathrm{LC}^k$-space having the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property with $n \le k$. This result was established by W. Mitchell [54, Theorem 4.3(1)] in the case $X$ is a compact ANR and $m = 0$. Moreover, a particular case of Theorem 13.1(1) when $X$ is an ANR and $m = 0$ was also established in [54].

## 14. $k$-Root and division formulas for the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-properties

In this section we discuss $k$-root and division formulas for the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-properties, one of the most surprising features of these properties.

Theorem 14.1 ($k$-Root formulas). *Let $n$ be a non-negative integer or infinity and $k$ be a positive integer.*

(1) *If $X$ is an $\mathrm{LC}^1$-space with $X \in 2$-$\overline{\mathrm{DD}}^{\{0,0\}}$ and $X^k \in (kn+k-1)$-$\overline{\mathrm{DD}}^{\{0,0\}}$, then $X$ has the $n$-$\overline{\mathrm{DD}}^{\{0,0\}}$-property. This can be written as*

$$\boxed{\mathrm{LC}^1 \cap 2\text{-}\overline{\mathrm{DD}}^{\{0,0\}} \cap \sqrt[k]{(kn+k-1)\text{-}\overline{\mathrm{DD}}^{\{0,0\}}} \subset n\text{-}\overline{\mathrm{DD}}^{\{0,0\}}}$$

(2) *If $X$ is a separable metrizable $\mathrm{LC}^{kn+k-1}$-space with the $0$-$\overline{\mathrm{DD}}^{\{0,2\}}$-property and $X^k$ has the $0$-$\overline{\mathrm{DD}}^{\{0,kn+k-1\}}$-property, then $X$ has the $0$-$\overline{\mathrm{DD}}^{\{0,n\}}$-property. This can be written as*

$$\boxed{\mathrm{LC}^{kn+k-1} \cap 0\text{-}\overline{\mathrm{DD}}^{\{0,2\}} \cap \sqrt[k]{0\text{-}\overline{\mathrm{DD}}^{\{0,kn+k-1\}}} \subset 0\text{-}\overline{\mathrm{DD}}^{\{0,n\}}}$$

To write down division formulas for the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property, let us introduce two new classes in addition to the classes $\bigcup_G \mathcal{Z}_n^G$ and $\exists_\circ\bigcup_G \overline{\mathcal{Z}}_n^G$:

- $\mathcal{Z}_n^{\exists G}$, the class of spaces $X$ with all $x \in X$ being $\exists G$-homological $Z_n$-points in $X$;
- $\Delta\mathcal{Z}_n^{\exists G}$, the class of spaces $X$ whose diagonal $\Delta_X$ is an $\exists G$-homological $Z_n$-set in $X^2$.

Note that any at most $n$-dimensional polyhedron belongs to none of the last two classes.

Theorem 14.2 (Division formulas). *Let $n \le k$ be non-negative integers or infinity and $m$ a non-negative integer.*

(1) *An $\mathrm{LC}^1$-space with the $2$-$\overline{\mathrm{DD}}^{\{0,0\}}$-property has the $n$-$\overline{\mathrm{DD}}^{\{0,0\}}$-property provided $X \times Y$ has the $(n+m)$-$\overline{\mathrm{DD}}^{\{0,0\}}$-property for some space $Y$ whose diagonal $\Delta_Y$ fails to be a $\exists G$-homological $Z_m$-set in $Y^2$. This can be written as*

$$\boxed{\mathrm{LC}^1 \cap 2\text{-}\overline{\mathrm{DD}}^{\{0,0\}} \cap \frac{(n+m)\text{-}\overline{\mathrm{DD}}^{\{0,0\}}}{Top \setminus \Delta\mathcal{Z}_m^{\exists G}} \subset n\text{-}\overline{\mathrm{DD}}^{\{0,0\}}}$$

(2) *A separable metrizable $\mathrm{LC}^{n+m}$-space $X \in 0$-$\overline{\mathrm{DD}}^{\{0,2\}}$ has the $0$-$\overline{\mathrm{DD}}^{\{0,n\}}$-property provided $X \times Y$ has the $0$-$\overline{\mathrm{DD}}^{\{0,n+m\}}$-property for some metrizable separable Baire $\mathrm{LC}^{n+m}$-space $Y$ that contains no non-empty open set $U \in \bigcup_G \overline{\mathcal{Z}}_m^G$.*

$$\boxed{\mathrm{LC}^{n+m} \cap 0\text{-}\overline{\mathrm{DD}}^{\{0,2\}} \cap \frac{0\text{-}\overline{\mathrm{DD}}^{\{0,n+m\}}}{\mathsf{Br} \cap \mathrm{LC}^{n+m} \setminus \exists_\circ\bigcup_G \overline{\mathcal{Z}}_m^G} \subset 0\text{-}\overline{\mathrm{DD}}^{\{0,n\}}}$$

(3) *A separable metrizable $\mathrm{LC}^{k+m}$-space $X \in \mathcal{Z}_{k+2}^{\mathbb{Z}}$ with $X \in 0$-$\overline{\mathrm{DD}}^{\{2,2\}}$ has the $0$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property provided $X \times Y \in 0$-$\overline{\mathrm{DD}}^{\{n+m,k+m\}}$ for some metrizable separable $\mathrm{LC}^{k+m}$-space $Y \notin \mathcal{Z}_m^{\exists G}$. This can be written as*

$$\boxed{\mathrm{LC}^{k+m} \cap 0\text{-}\overline{\mathrm{DD}}^{\{2,2\}} \cap \mathcal{Z}^{\mathbb{Z}}_{k+2} \cap \frac{0\text{-}\overline{\mathrm{DD}}^{\{n+m,k+m\}}}{\mathrm{LC}^{k+m} \setminus \mathcal{Z}^{\exists G}_m} \subset 0\text{-}\overline{\mathrm{DD}}^{\{n,k\}}}$$

(4) *A separable metrizable* $\mathrm{LC}^{k+m}$*-space* $X \in \mathcal{Z}^{\mathbb{Z}}_{n+k+m}$ *with* $X \in 0\text{-}\overline{\mathrm{DD}}^{\{2,2\}}$ *has the* $0\text{-}\overline{\mathrm{DD}}^{\{n,k\}}$*-property provided* $X \times Y \in 0\text{-}\overline{\mathrm{DD}}^{\{n+m,n+m\}}$ *for some metrizable separable* $\mathrm{LC}^{n+m}$*-space* $Y \notin \bigcup_G \mathcal{Z}^G_m$*. This can be written as*

$$\boxed{\mathrm{LC}^{k+m} \cap 0\text{-}\overline{\mathrm{DD}}^{\{2,2\}} \cap \mathcal{Z}^{\mathbb{Z}}_{n+k+m} \cap \frac{0\text{-}\overline{\mathrm{DD}}^{\{n+m,n+m\}}}{\mathrm{LC}^{n+m} \setminus \bigcup_G \mathcal{Z}^G_m} \subset 0\text{-}\overline{\mathrm{DD}}^{\{n,k\}}}$$

## 15. Characterizing $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-properties with $m,n,k \in \{0,\infty\}$

In this subsection we apply the results obtained in the preceding subsections to the case of $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-properties with $m,n,k \in \{0,\infty\}$. Let us note that $0\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$ has been characterized in Proposition 5.6 while $\infty\text{-}\overline{\mathrm{DD}}^{\{\infty,\infty\}}$ is equivalent to $0\text{-}\overline{\mathrm{DD}}^{\{\infty,\infty\}}$. So, it suffices to consider only the properties: $0\text{-}\overline{\mathrm{DD}}^{\{0,\infty\}}$, $\infty\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$, $\infty\text{-}\overline{\mathrm{DD}}^{\{0,\infty\}}$, and $0\text{-}\overline{\mathrm{DD}}^{\{\infty,\infty\}}$. These can be characterized in terms of homotopical or homological $Z_\infty$-points as follows:

Corollary 15.1.

(1) *A topological* ($\mathrm{LC}^1$-)*space* $X$ *has the* $\infty\text{-}\overline{\mathrm{DD}}^{\{0,\infty\}}$*-property if and only if all points of* $X$ *are homotopical* $Z_\infty$*-points (resp., if and only if all points of* $X$ *are homotopical* $Z_2$*-points and* $X \in \infty\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$*):*

$$\boxed{\mathcal{Z}_2 \cap \infty\text{-}\overline{\mathrm{DD}}^{\{0,0\}} \cap \mathrm{LC}^1 \subset \infty\text{-}\overline{\mathrm{DD}}^{\{0,\infty\}} = \mathcal{Z}_\infty}$$

(2) *An* $\mathrm{LC}^1$*-space* $X$ *has the* $\infty\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$*-property if and only if* $X \in 2\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$ *and all points of* $X$ *are homological* $Z_\infty$*-points:*

$$\boxed{\mathcal{Z}^{\mathbb{Z}}_\infty \cap 2\text{-}\overline{\mathrm{DD}}^{\{0,0\}} \cap \mathrm{LC}^1 \subset \infty\text{-}\overline{\mathrm{DD}}^{\{0,0\}} \subset \mathcal{Z}^{\mathbb{Z}}_\infty}$$

(3) *A Polish* $\mathrm{LC}^\infty$*-space* $X$ *has the* $0\text{-}\overline{\mathrm{DD}}^{\{0,\infty\}}$*-property if and only if* $X$ *has a dense set of homotopical* $Z_\infty$*-points if and only if* $X \in 0\text{-}\overline{\mathrm{DD}}^{\{0,2\}}$ *and* $X$ *has a dense set of homological* $Z_\infty$*-points:*

$$\boxed{\overline{\mathcal{Z}}^{\mathbb{Z}}_\infty \cap 0\text{-}\overline{\mathrm{DD}}^{\{0,2\}} \cap \mathrm{LC}^\infty \subset 0\text{-}\overline{\mathrm{DD}}^{\{0,\infty\}} \text{ and } 0\text{-}\overline{\mathrm{DD}}^{\{0,\infty\}} \cap \mathrm{LC}^\infty \cap \Pi^0_2 \subset \overline{\mathcal{Z}}_\infty}$$

(4) *If each point of a metrizable separable* $\mathrm{LC}^\infty$*-space* $X$ *is a homological* $Z_\infty$*-point and* $X$ *has the properties* $\mathrm{AP}[\infty]$ *and* $0\text{-}\overline{\mathrm{DD}}^{\{2,2\}}$*, then* $X$ *has the* $0\text{-}\overline{\mathrm{DD}}^{\{\infty,\infty\}}$*-property:*

$$\boxed{\mathcal{Z}^{\mathbb{Z}}_\infty \cap 0\text{-}\overline{\mathrm{DD}}^{\{2,2\}} \cap \mathrm{LC}^\infty \cap \mathrm{AP}[\infty] \subset \infty\text{-}\overline{\mathrm{DD}}^{\{\infty,\infty\}} = 0\text{-}\overline{\mathrm{DD}}^{\{\infty,\infty\}}}$$

According to the famous characterization of Hilbert cube manifolds due to Toruńczyk [68], a locally compact ANR-space $X$ is a $Q$-manifold if and only if $X$ has the $0\text{-}\overline{\mathrm{DD}}^{\{\infty,\infty\}}$-

property. Combining this characterization with the last item of Corollary 15.1 we obtain a new characterization of Q-manifolds.

Corollary 15.2. *A locally compact ANR is a Q-manifold if and only if*

- *$X$ has the disjoint disk property;*
- *each point of $X$ is a homological $Z_\infty$-point;*
- *each map $f : \mathbb{I}^\infty \to X$ can be uniformly approximated by maps with* $\mathsf{trt}$*-dimensional image.*

Next, we discuss $k$-root formulas for $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-properties with $m, n, k \in \{0, \infty\}$.

Corollary 15.3 ($k$-Root formulas).

(1) *An* $\mathrm{LC}^1$*-space $X$ has the* $\infty$-$\overline{\mathrm{DD}}^{\{0,0\}}$*-property if and only if* $X \in 2$-$\overline{\mathrm{DD}}^{\{0,0\}}$ *and* $X^k \in \infty$-$\overline{\mathrm{DD}}^{\{0,0\}}$ *for some finite $k$. This can be written as*

$$\boxed{\infty\text{-}\overline{\mathrm{DD}}^{\{0,0\}} \supset \sqrt[k]{\infty\text{-}\overline{\mathrm{DD}}^{\{0,0\}}} \cap 2\text{-}\overline{\mathrm{DD}}^{\{0,0\}} \cap \mathrm{LC}^1}$$

(2) *A metrizable separable* $\mathrm{LC}^\infty$*-space $X$ has the* 0-$\overline{\mathrm{DD}}^{\{0,\infty\}}$*-property if and only if* $X \in$ 0-$\overline{\mathrm{DD}}^{\{0,2\}}$ *and* $X^k \in$ 0-$\overline{\mathrm{DD}}^{\{0,\infty\}}$ *for some finite $k$. This can be written as*

$$\boxed{0\text{-}\overline{\mathrm{DD}}^{\{0,\infty\}} \supset \sqrt[k]{0\text{-}\overline{\mathrm{DD}}^{\{0,\infty\}}} \cap 0\text{-}\overline{\mathrm{DD}}^{\{0,2\}} \cap \mathrm{LC}^\infty}$$

(3) *An* $\mathrm{LC}^1$*-space $X$ has the* $\infty$-$\overline{\mathrm{DD}}^{\{0,\infty\}}$*-property iff $X$ has the* 2-$\overline{\mathrm{DD}}^{\{0,2\}}$*-property and* $X^k \in \infty$-$\overline{\mathrm{DD}}^{\{0,\infty\}}$ *for some finite $k$. This can be written as*

$$\boxed{\infty\text{-}\overline{\mathrm{DD}}^{\{0,\infty\}} \supset \sqrt[k]{\infty\text{-}\overline{\mathrm{DD}}^{\{0,\infty\}}} \cap 2\text{-}\overline{\mathrm{DD}}^{\{0,2\}} \cap \mathrm{LC}^1}$$

(4) *A metrizable separable* $\mathrm{LC}^\infty$*-space $X$ has the* 0-$\overline{\mathrm{DD}}^{\{\infty,\infty\}}$*-property if $X$ has the* 0-$\overline{\mathrm{DD}}^{\{2,2\}}$*-property,* $X \in \mathrm{AP}[\infty]$ *and* $X^k \in$ 0-$\overline{\mathrm{DD}}^{\{\infty,\infty\}}$ *for some finite $k$. This can be written as*

$$\boxed{0\text{-}\overline{\mathrm{DD}}^{\{\infty,\infty\}} \supset \sqrt[k]{0\text{-}\overline{\mathrm{DD}}^{\{\infty,\infty\}}} \cap 0\text{-}\overline{\mathrm{DD}}^{\{2,2\}} \cap \mathrm{LC}^\infty \cap \mathrm{AP}[\infty]}$$

Finally, we turn to division formulas for the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-properties with $m, n, k \in \{0, \infty\}$.

Corollary 15.4 (Division formulas).

(1) *An* $\mathrm{LC}^1$*-space $X$ has the* $\infty$-$\overline{\mathrm{DD}}^{\{0,0\}}$*-property provided* $X \in 2$-$\overline{\mathrm{DD}}^{\{0,0\}}$ *and the product $X \times Y$ has the* $\infty$-$\overline{\mathrm{DD}}^{\{0,0\}}$*-property for some space* $Y \notin \bigcup_G \mathcal{Z}^G_\infty$*. This can be written as*

$$\boxed{\infty\text{-}\overline{\mathrm{DD}}^{\{0,0\}} \supset \frac{\infty\text{-}\overline{\mathrm{DD}}^{\{0,0\}}}{\mathit{Top} \setminus \bigcup_G \mathcal{Z}^G_\infty} \cap 2\text{-}\overline{\mathrm{DD}}^{\{0,0\}} \cap \mathrm{LC}^1}$$

(2) *An* $\mathrm{LC}^1$*-space $X$ has the* $\infty$-$\overline{\mathrm{DD}}^{\{0,\infty\}}$*-property provided* $X \in 2$-$\overline{\mathrm{DD}}^{\{0,2\}}$ *and the product $X \times Y$ has the* $\infty$-$\overline{\mathrm{DD}}^{\{0,\infty\}}$*-property for some space* $Y \notin \bigcup_G \mathcal{Z}^G_\infty$*. This can be written as*

$$\boxed{\infty\text{-}\overline{\mathrm{DD}}^{\{0,\infty\}} \supset \frac{\infty\text{-}\overline{\mathrm{DD}}^{\{0,\infty\}}}{Top \setminus \bigcup_G \mathcal{Z}_\infty^G} \cap 2\text{-}\overline{\mathrm{DD}}^{\{0,2\}} \cap \mathrm{LC}^1}$$

(3) *A metrizable separable* $\mathrm{LC}^\infty$*-space* $X$ *has the* $0\text{-}\overline{\mathrm{DD}}^{\{\infty,\infty\}}$*-property provided* $X \in 0\text{-}\overline{\mathrm{DD}}^{\{2,2\}}$ *and* $X \times Y$ *has the* $0\text{-}\overline{\mathrm{DD}}^{\{\infty,\infty\}}$*-property for some separable metrizable* $\mathrm{LC}^\infty$*-space* $Y \notin \bigcup_G \mathcal{Z}_\infty^G$*. This can be written as*

$$\boxed{0\text{-}\overline{\mathrm{DD}}^{\{\infty,\infty\}} \supset \frac{\mathrm{LC}^\infty \cap 0\text{-}\overline{\mathrm{DD}}^{\{\infty,\infty\}}}{Top \setminus \bigcup_G \mathcal{Z}_\infty^G} \cap 0\text{-}\overline{\mathrm{DD}}^{\{2,2\}}}$$

(4) *A metrizable separable* $\mathrm{LC}^\infty$*-space* $X$ *has the* $0\text{-}\overline{\mathrm{DD}}^{\{0,\infty\}}$*-property provided* $X \in 0\text{-}\overline{\mathrm{DD}}^{\{0,2\}}$ *and the product* $X \times Y$ *has the* $0\text{-}\overline{\mathrm{DD}}^{\{0,\infty\}}$*-property for some metrizable separable* $\mathrm{LC}^\infty$*-space* $Y \notin \exists_\circ \bigcup_G \overline{\mathcal{Z}}_\infty^G$*. This can be written as*

$$\boxed{0\text{-}\overline{\mathrm{DD}}^{\{0,\infty\}} \supset \frac{\mathrm{LC}^\infty \cap 0\text{-}\overline{\mathrm{DD}}^{\{0,\infty\}}}{Top \setminus \exists_\circ \bigcup_G \overline{\mathcal{Z}}_\infty^G} \cap 0\text{-}\overline{\mathrm{DD}}^{\{0,2\}}}$$

The third item of Corollary 15.4 combined with Toruńczyk's characterization of $Q$-manifolds implies the following division theorem for $Q$-manifolds proven in [3] and implicitly in [19].

Corollary 15.5. *A space* $X$ *is a* $Q$*-manifold if and only if the product* $X \times Y$ *is a* $Q$*-manifold for some space* $Y \notin \bigcup_G \mathcal{Z}_\infty^G$*.*

## 16. Dimension of spaces with the $m$-$\overline{\mathrm{DD}}^n$-property

In this section we study the dimensional properties of spaces possessing the $m$-$\overline{\mathrm{DD}}^n$-property.

Theorem 16.1. *If a metrizable separable space* $X$ *has the* $m$-$\overline{\mathrm{DD}}^n$*-property, then* $\dim X \geq n + (m+1)/2$*.*

This theorem combined with Theorem 9.1 allows us to calculate the smallest possible dimension of a space $X$ with $m$-$\overline{\mathrm{DD}}^n$. For a real number $r$ let

$$\lfloor r \rfloor = \max\{n \in \mathbb{Z} : n \leq r\}, \qquad \lceil r \rceil = \min\{n \in \mathbb{Z} : n \geq r\}.$$

Corollary 16.2. *Let* $n, m$ *be non-negative integers and* $D$ *be a dendrite with a dense set of end-points.*

(1) *If* $m$ *is odd and* $d = n + (m+1)/2$*, then the power* $D^d$ *is a* $d$*-dimensional absolute retract with the* $m$-$\overline{\mathrm{DD}}^n$*-property.*
(2) *If* $m$ *is even and* $d = n + (m+2)/2$*, then the product* $D^{d-1} \times \mathbb{I}$ *is a* $d$*-dimensional absolute retract with the* $m$-$\overline{\mathrm{DD}}^n$*-property.*

*Consequently,* $n + \lceil (m+1)/2 \rceil$ *is the smallest possible dimension of a compact absolute retract with the* $m$-$\mathrm{DD}^n$*-property.*

Theorem 16.1 implies that $\dim(X) \geq m+1$ for each metrizable separable space $X$ with the $(2m+1)$-$\overline{\mathrm{DD}}^0$-property. A similar inequality also holds also for cohomological dimension.

THEOREM 16.3. *Let $X$ be a locally compact metrizable* $\mathrm{LC}^m$*-space having the property* $(2m+1)$-$\overline{\mathrm{DD}}^0$. *Then* $\dim_G X \geq m+1$ *for any non-trivial Abelian group $G$.*

In some cases the condition $X \in (2m+1)$-$\overline{\mathrm{DD}}^0$ from Theorem 16.3 can be weakened to $X \in 2m$-$\overline{\mathrm{DD}}^0$.

THEOREM 16.4. *Let $X$ be a locally compact* $\mathrm{LC}^{2m}$*-space with the* $2m$-$\overline{\mathrm{DD}}^0$*-property and let $G$ be a non-trivial Abelian group. The inequality* $\dim_G X \geq m+1$ *holds in each of the following cases:*

(1) *$G$ fails to be both divisible and periodic;*
(2) *$G$ is a field;*
(3) *$X$ is an* ANR*-space.*

Theorem 16.4 implies the following estimation for the extension dimension of spaces $X \in m$-$\overline{\mathrm{DD}}^{\{0,0\}}$:

THEOREM 16.5. *Let $X$ be a locally compact* $\mathrm{LC}^m$*-space such that* $\text{e-dim}\, X \leq L$ *for some CW-complex $L$. If* $X \in m$-$\overline{\mathrm{DD}}^0$, *then we have:*

(1) *The homotopy groups $\pi_i(L)$ are trivial for all $i < m/2$.*
(2) *For $n = \lfloor m/2 \rfloor$ the group $\pi_n(L)$ is both divisible and periodic, and $\pi_n(L) = \widetilde{H}_n(L)$.*
(3) *$\pi_i(L) = 0$ for all $i \leq m/2$ provided $X$ is an ANR-space.*

Finally, we discuss the dimension properties of spaces $X \in \infty$-$\overline{\mathrm{DD}}^{\{0,0\}}$.

THEOREM 16.6. *Let $X$ be a locally compact metrizable* $\mathrm{LC}^\infty$*-space possessing the property* $\infty$-$\overline{\mathrm{DD}}^{\{0,0\}}$.

(1) *All points of $X$ are homological $Z_\infty$-points.*
(2) *$X$ fails to be* trt*-dimensional.*
(3) *If* $\text{e-dim}\, X \leq L$ *for some CW-complex $L$, then $L$ is contractible.*
(4) *If $X$ is locally contractible, then $X$ is not a $C$-space.*

The first item of this theorem follows from Theorem 7.1(7). The last three items follow from Theorems 10.9(1) and 10.10.

## 17. Some examples and open problems

First, we discuss the problem of distinguishing between the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-properties for various $m, n, k$. Let us note that if an Euclidean space $E$ has the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property for some $m, n, k$, then $E$ has the $a$-$\overline{\mathrm{DD}}^{\{b,c\}}$-property for all non-negative integers $a, b, c$ with $a+b+c \leq n+m+k$. This feature is specific for Euclidean spaces and does not hold in the general case. For example, each dendrite $D$ with a dense set of end-points has the 0-$\overline{\mathrm{DD}}^{\{0,2\}}$-property (and in fact, 0-$\overline{\mathrm{DD}}^{\{0,\infty\}}$) but does not have the 0-$\overline{\mathrm{DD}}^{\{1,1\}}$-property.

The next example from Daverman's book [17] shows that the properties $0\text{-}\overline{\mathrm{DD}}^{\{0,2\}}$ and $0\text{-}\overline{\mathrm{DD}}^{\{1,1\}}$ are completely incomparable.

Example 17.1. *There is a 2-dimensional absolute retract $\Lambda \subset \mathbb{R}^3$ with $0\text{-}\overline{\mathrm{DD}}^{\{1,1\}}$-property that fails to have the $0\text{-}\overline{\mathrm{DD}}^{\{0,2\}}$-property.*

Question 17.2. *Does the space $\Lambda$ from Example 17.1 have the $2\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$-property?*

It follows from Theorem 14.2(1) that a Polish $\mathrm{LC}^\infty$-space $X$ has the $\infty\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$-property provided $X \times \mathbb{R} \in \infty\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$ and $X \in 2\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$. We do not know if the latter condition is essential.

Question 17.3. *Does a compact absolute retract $X$ possess the $\infty\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$-property provided $X \times \mathbb{I}$ has that property? (Let us observe that $X \times \mathbb{I} \in \infty\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$ implies $X \times \mathbb{I} \in \infty\text{-}\overline{\mathrm{DD}}^{\{1,\infty\}}$.)*

This question is equivalent to another intriguing one.

Question 17.4. *Does a compact absolute retract $X$ contain a $Z_2$-point provided all points of the product $X \times \mathbb{I}$ are $Z_\infty$-points?*

Problem 17.5. *Let $X$ be a compact* AR *with the $\infty\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$-property.*

(1) *Is there any $Z_2$-point in $X$?*
(2) *Is $X$ strongly infinite-dimensional?*
(3) *Is $X \times \mathbb{I}$ homeomorphic to the Hilbert cube?*

Problem 17.6. *Is a space $X \in 0\text{-}\overline{\mathrm{DD}}^{\{2,2\}}$ homeomorphic to the Hilbert cube $Q$ provided some finite power of $X$ is homeomorphic to $Q$?*

There are three interesting examples relevant to these questions. The first of them was constructed by Singh in [63], the second by Daverman and Walsh in [19] and the third by Banakh and Repovš in [6].

Example 17.7 (Singh). *There is a space $X$ with the following properties:*

(1) *$X$ is a compact absolute retract;*
(2) *$X$ contains no topological copy of the 2-disk $\mathbb{I}^2$;*
(3) *$X \times \mathbb{I}$ is homeomorphic to the Hilbert cube;*
(4) *all but countably many points of $X$ are $Z_2$-points;*
(5) $X \in \infty\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$;
(6) $X \notin 2\text{-}\overline{\mathrm{DD}}^{\{0,2\}} \cup 0\text{-}\overline{\mathrm{DD}}^{\{2,2\}}$;
(7) $X \times \mathbb{I} \in \infty\text{-}\overline{\mathrm{DD}}^{\{\infty,\infty\}}$.

Example 17.8 (Daverman–Walsh). *There is a space $X$ with the following properties:*

(1) *$X$ is a compact absolute retract;*
(2) *$X \times \mathbb{I}$ is homeomorphic to the Hilbert cube;*
(3) *each point of $X$ is a $Z_\infty$-point;*
(4) $X \in \infty\text{-}\overline{\mathrm{DD}}^{\{0,\infty\}} \cap 0\text{-}\overline{\mathrm{DD}}^{\{1,\infty\}}$;
(5) $X \notin 0\text{-}\overline{\mathrm{DD}}^{\{2,2\}}$;
(6) $X \times \mathbb{I} \in \infty\text{-}\overline{\mathrm{DD}}^{\{\infty,\infty\}}$.

Example 17.9 (Banakh–Repovš). *There is a countable family $\mathcal{X}$ of spaces such that*

(1) *the product $X \times Y$ of any two different spaces $X, Y \in \mathcal{X}$ is homeomorphic to the Hilbert cube;*
(2) *no finite power $X^k$ of any space $X \in \mathcal{X}$ is homeomorphic to $Q$.*

Note that there is no uncountable family $\mathcal{X}$ possessing the properties (1) and (2) from Example 17.9 (see [6]).

It may be convenient to describe the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-properties of a space $X$ using the following sets:

$$*\text{-}\overline{\mathrm{DD}}^{\{*,*\}}(X) = \{(m,n,k) \in \omega^3 : X \text{ has the } m\text{-}\overline{\mathrm{DD}}^{\{n,k\}}\text{-property}\},$$
$$0\text{-}\overline{\mathrm{DD}}^{\{*,*\}}(X) = \{(n,k) \in \omega^2 : X \text{ has the } 0\text{-}\overline{\mathrm{DD}}^{\{n,k\}}\text{-property}\},$$
$$*\text{-}\overline{\mathrm{DD}}^{*}(X) = \{(m,n) \in \omega^2 : X \text{ has the } m\text{-}\overline{\mathrm{DD}}^{n}\text{-property}\}.$$

Problem 17.10. *Describe the geometry of the sets $*$-$\overline{\mathrm{DD}}^{\{*,*\}}(X)$, $0$-$\overline{\mathrm{DD}}^{\{*,*\}}(X)$ and $*$-$\overline{\mathrm{DD}}^{*}(X)$ for a given space $X$. Which subsets of $\omega^3$ or $\omega^2$ can be realized as the sets $*$-$\overline{\mathrm{DD}}^{\{*,*\}}(X)$, $0$-$\overline{\mathrm{DD}}^{\{*,*\}}(X)$ or $*$-$\overline{\mathrm{DD}}^{*}(X)$ for a suitable $X$?*

In fact, we can consider the following partial pre-order $\underset{\mathrm{DD}}{\Rightarrow}$ on $\omega^3$: $(m,n,k) \underset{\mathrm{DD}}{\Rightarrow} (a,b,c)$ if each space $X$ with the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property also has the $a$-$\overline{\mathrm{DD}}^{\{b,c\}}$-property.

Problem 17.11. *Describe the properties of the partial preorder $\underset{\mathrm{DD}}{\Rightarrow}$ on $\omega^3$.*

By Proposition 3.4(2), a paracompact space $X$ is Lefschetz ANE$[n]$ for a finite $n$ if and only if $X$ is an LC$^{n-1}$-space. Consequently, the product of two paracompact ANE$[n]$-spaces is an ANE$[n]$-space for every finite $n$.

Problem 17.12. *Is the product of two (paracompact) Lefschetz* ANE$[\infty]$*-spaces a Lefschetz* ANE$[\infty]$*-space?*

# Part II. PROOFS

## 18. Preliminaries

This section is of a preliminary character and collects some notion, conventions and auxiliary results.

**18.1. Topological spaces.** Since we often deal with perfect maps, let us describe their interplay with the class of proper maps. We recall that a map $p : X \to Y$ between topological spaces is called *perfect* if $p$ is closed and the preimages $p^{-1}(y)$, $y \in Y$, are compact; $p$ is *proper* if the preimage $p^{-1}(K)$ of each compact subset $K \subset Y$ is compact. Each perfect map is proper [33, 3.7.2]. Conversely, each proper map $f : X \to Y$ into a $k$-space $Y$ is perfect [33, 3.7.18].

We recall that a topological space $X$ is a *$k$-space* if a subset $F \subset X$ is closed if and only if its intersection $F \cap K$ with each compact subset $K \subset X$ is closed in $X$. It is well-known that each first countable (in particular, metrizable) space is a $k$-space. On the other hand, $CW$-complexes also are $k$-spaces. Therefore, to check that a map $f : X \to K$ into a $CW$-complex is perfect it suffices to check that the preimage of each compact subset of $K$ is compact.

For a subset $A \subset X$ and a cover $\mathcal{U}$ of $X$ we write $\operatorname{diam} A < \mathcal{U}$ if $A \subset U$ for some $U \in \mathcal{U}$. Sometimes, we also write $A \prec \mathcal{U}$ to denote that $\operatorname{diam} A < \mathcal{U}$. For two families $\mathcal{U}, \mathcal{V}$ of subsets of $X$ we write $\mathcal{U} \prec \mathcal{V}$ if $\operatorname{diam} U < \mathcal{V}$ for all $U \in \mathcal{U}$. In this case we say that $\mathcal{U}$ *refines* $\mathcal{V}$, or $\mathcal{U}$ is *inscribed* in $\mathcal{V}$. For two covers $\mathcal{U}, \mathcal{V}$ of $X$ let $\mathcal{U} \wedge \mathcal{V} = \{U \cap V : U \in \mathcal{U},\ V \in \mathcal{V}\}$.

A *partition of unity* on a space $X$ is a family of continuous functions $\{\lambda_i : X \to [0,1]\}_{i \in \mathcal{I}}$ such that the family $\{\lambda_i^{-1}(0,1]\}_{i \in \mathcal{I}}$ is locally finite and $\sum_{i \in \mathcal{I}} \lambda_i(x) = 1$ for all $x \in X$. Let $\mathcal{U}$ be a cover of $X$. A partition of unity $\{\lambda_U : X \to [0,1]\}_{U \in \mathcal{U}}$ is *subordinated* to $\mathcal{U}$ if $\lambda_U^{-1}(0,1] \subset U$ for all $U \in \mathcal{U}$. We shall often use the fact that for each open cover $\mathcal{U}$ of a paracompact space $X$ there is a partition of unity $\{\lambda_U\}_{U \in \mathcal{U}}$ subordinated to $\mathcal{U}$ and such that for any $x \in X$ at most $\dim X + 1$ values $\lambda_U(x)$ are strictly positive.

A subset $A$ of a topological space $X$ is called *functionally open* (resp., *functionally closed*) if $A = f^{-1}(B)$ for some continuous function $f : X \to \mathbb{R}$ and some open (resp., closed) subset $B \subset \mathbb{R}$. A subset $A$ of a normal space is functionally open if and only if $A$ is an open $F_\sigma$-set in $X$.

**18.2. Uniform covers.** For a cover $\mathcal{U}$ of $X$ and a subset $A \subset X$ let $St(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$ be the star of $A$ and $St(\mathcal{U}) = \{St(U,\mathcal{U}) : U \in \mathcal{U}\}$ be the star of the cover $\mathcal{U}$. Also we put $St^0(\mathcal{U}) = \mathcal{U}$ and $St^{n+1}(\mathcal{U}) = St(St^n(\mathcal{U}))$ for $n \geq 0$.

Given a pseudometric $\rho$ on a set $X$, by $B_\rho(x_0,\varepsilon)=\{x\in X:\rho(x,x_0)<\varepsilon\}$ we denote the open $\varepsilon$-ball centered at a point $x_0\in X$. For a family $\mathcal{U}$ of subsets of a set $X$ and a pseudometric $\rho$ on $X$ we let $\operatorname{mesh}_\rho\mathcal{U}=\sup\{\operatorname{diam}_\rho U : U\in\mathcal{U}\}$, where $\operatorname{diam}_\rho U=\sup_{x,y\in U}\rho(x,y)$ is the *$\rho$-diameter* of $U$.

An open cover $\mathcal{U}$ of a topological space $X$ is called a *uniform cover* of $X$ if there exists a continuous pseudometric $\rho$ on $X$ such that $\mathcal{U}$ is refined by the cover $\{B_\rho(x,1):x\in X\}$.

The following fundamental result is due to J. Tukey (see [33, 5.4.H]).

LEMMA 18.1. *Each open cover of a paracompact spaces $X$ is uniform.*

This result can be partly generalized for Tychonoff spaces.

LEMMA 18.2. *For any open cover $\mathcal{U}$ of a Tychonoff space $X$ and any compact set $K\subset X$ there is a continuous pseudometric $\rho$ on $X$ such that the family $\{B_\rho(x,1):x\in K\}$ is inscribed in the cover $\mathcal{U}$.*

*Proof.* Embed $X$ into a Tychonoff cube $\mathbb{I}^\kappa$ for a suitable cardinal $\kappa$. For each $x\in K$ find a finite index set $F(x)\subset\kappa$ and an open set $W_x\subset\mathbb{I}^{F(x)}$ whose preimage $V_x=\operatorname{pr}^{-1}_{F(x)}(W_x)$ under the projection $\operatorname{pr}_{F(x)}:X\to\mathbb{I}^{F(x)}$ contains the point $x$ and lies in some $U\in\mathcal{U}$. By the compactness of $K$, the open cover $\{V_x:x\in K\}$ of $K$ has a finite subcover $\{V_{x_1},\dots,V_{x_m}\}$. Now, consider the finite set $F=\bigcup_{i=1}^m F(x_i)$ and note that each set $V_{x_i}$ is the preimage of some open set $W_i$ under the projection $\operatorname{pr}_F:X\to\mathbb{I}^F$. Let $d$ be any metric on the finite-dimensional cube $\mathbb{I}^F$. By the compactness of $C=\operatorname{pr}_F(K)\subset\bigcup_{i=1}^m W_i$, there is $\varepsilon>0$ such that each $\varepsilon$-ball centered at a point $z\in C$ lies in some $W_i$. Finally, define the pseudometric $\rho$ on $X$ letting $\rho(x,x')=(1/\varepsilon)\cdot d(\operatorname{pr}_F(x),\operatorname{pr}_F(x'))$ for $x,x'\in X$. It is easy to see that each 1-ball centered at any point $x\in K$ lies in some $U\in\mathcal{U}$. ■

**18.3. Homotopies.** When working with $\mathcal{U}$-homotopies it is convenient to consider their metric counterpart, $\varepsilon$-homotopies. If $\rho$ is a continuous pseudometric on a space $M$, then two maps $f,g:X\to M$ are called *$\varepsilon$-homotopic* if there is a homotopy $h:X\times[0,1]\to M$ linking $f$ and $g$ such that for each $x\in X$ the set $h(\{x\}\times[0,1])$ has diameter $<\varepsilon$. Thus $\varepsilon$-homotopies are precisely $\mathcal{U}$-homotopies for the cover of $M$ by open sets of diameter $<\varepsilon$. Conversely, for any open cover $\mathcal{U}$ of a Tychonoff space $M$ and a map $f:X\to M$ with $f(X)\subset K$ for some compact set $K\subset M$, we can find a continuous pseudometric $\rho$ on $X$ such that each map $g:X\to M$ which is 1-homotopic to $f$ is $\mathcal{U}$-homotopic to $f$ (see Lemma 18.2).

The following standard fact from the theory of retracts allows us to extend homotopies.

LEMMA 18.3 (Borsuk's homotopy extension lemma). *Let $K$ be a normal space and $L$ be a neighborhood retract of $K$. Let $f:K\to X$ be a map into a space $X$ and $\mathcal{U}$ be an open cover of $X$. Then any map $g:L\to X$ $\mathcal{U}$-homotopic to $f|L$ can be extended to a map $\bar g:K\to X$ $\mathcal{U}$-homotopic to $f$.*

Usually, this lemma will be applied to pairs $(K,L)$ consisting of a simplicial complex $K$ and its subcomplex $L$.

**18.4. Function spaces.** In this subsection we collect some information concerning the function spaces $C(K,X)$. Let us observe that in the definition of the source limitation topology we can consider only positive functions $\varepsilon$ with $\varepsilon(x)\le 1$ for all $x\in K$, i.e., $\varepsilon\in$

$C(K,(0,1])$. This restriction implies another one: we can suppose that all pseudometrics $\rho$ on $X$ are $\leq 1$. In the case of metrizable $X$ and paracompact $K$ there is an equivalent description of the source limitation topology on $C(K,X)$.

LEMMA 18.4. *Let $f \in C(K,X)$, where $X$ is a metrizable space and $K$ paracompact. Then, for every metric $d$ on $X$ generating its topology, the neighborhood base at $f$ in the source limitation topology consists of the sets $B_d(f,\varepsilon)$, $\varepsilon \in C(K,(0,1])$.*

*Proof.* Let $Metr(X)$ denote all compatible metrics $\rho$ for $X$ with $\rho \leq 1$ and $\mathcal{E}(K)$ be the collection of all continuous positive functions $\varepsilon \leq 1$ on $K$. We are going first to show that the sets $B_\rho(f,\varepsilon)$, $\rho \in Metr(X)$ and $\varepsilon \in \mathcal{E}(K)$, form a base for the source limitation topology on $C(K,X)$. It suffices to prove that for any continuous pseudometric $\rho_1 \leq 1$ on $X$ there exists a metric $\rho \in Metr(X)$ such that $B_\rho(f,\varepsilon) \subset B_{\rho_1}(f,\varepsilon)$ for all $f \in C(K,X)$ and all $\varepsilon \in \mathcal{E}(K)$. Indeed, just take $\rho = \max\{\rho_1, d\}$, where $d$ is any metric from $Metr(X)$.

Let us prove now that, under our hypotheses, the source limitation topology coincides with the graph topology $\tau_\Gamma$ on $C(K,X)$. The graph topology was introduced in [56] and its base consists of all sets of the form $U_G = \{f \in C(K,X) : \Gamma(f) \subset G\}$, where $G$ is an open set in $K \times X$ and $\Gamma(f)$ denotes the graph of $f$. Let $V \subset C(K,X)$ be open with respect to the source limitation topology and $f \in V$. Then there exists a metric $\rho \in Metr(X)$ and $\varepsilon \in \mathcal{E}(K)$ with $B_\rho(f,\varepsilon) \subset V$. Obviously, $G = \{(x,y) \in K \times X : \rho(y,f(x)) < \varepsilon(x)\}$ is open in $K \times X$ and $f \in U_G \subset B_\rho(f,\varepsilon)$. Next, suppose $V \subset C(K,X)$ is open with respect to $\tau_\Gamma$ and $f \in V$. So, there is an open set $G$ in $K \times X$ with $f \in U_G \subset V$. We fix a metric $\rho \in Metr(X)$ and, as in the proof of Theorem 2.11 from [21], we can find a function $\varepsilon \in \mathcal{E}(K)$ such that $B_\rho(f,\varepsilon) \subset U_G$.

Hence, the source limitation topology on $C(K,X)$ coincides with the topology $\tau_\Gamma$. On the other hand, according to [21, Theorem 2.11], for any metric $d \in Metr(X)$, the topology $\tau_d$ on $C(K,X)$ whose base consists of all sets $B_d(f,\varepsilon)$, $\varepsilon \in \mathcal{E}(K)$, coincides with $\tau_\Gamma$. Therefore, for any metric $d \in Metr(X)$ the family of all sets $B_d(f,\varepsilon)$, $\varepsilon \in \mathcal{E}(K)$, is a base for the source limitation topology on $C(K,X)$. ■

Lemma 18.4 implies that, for a metric space $(X,d)$ and a compact space $K$, the source limitation topology on $C(K,X)$ coincides with the uniform convergence topology generated by the metric $d$. In particular, we have the following lemma.

LEMMA 18.5. *For a compact* (*metrizable*) *space $K$ and a metrizable* (*separable*) *space $X$ the function space $C(K,X)$ is metrizable* (*and separable*).

**18.5. $\mathcal{V}$-maps.** This subsection contains some information on $\mathcal{V}$-maps.

A map $f : X \to Y$ between topological spaces is called a (uniform) *$\mathcal{V}$-map*, where $\mathcal{V}$ is a cover of $X$, if there is a (uniform) open cover $\mathcal{U}$ of $Y$ such that $f^{-1}(\mathcal{U}) = \{f^{-1}(U) : U \in \mathcal{U}\}$ refines the cover $\mathcal{V}$. This notion has a metric counterpart: a map $f : M \to Y$ from a metric space $(M,d)$ is an *$\varepsilon$-map* if each point $y \in Y$ has a neighborhood $U_y$ whose preimage $f^{-1}(U_y)$ is of diameter $< \varepsilon$.

The following characterization of closed $\mathcal{V}$-maps is well known.

LEMMA 18.6. *A closed map $f : X \to Y$ is a $\mathcal{V}$-map with respect to an open cover $\mathcal{V}$ of $X$ if and only if* $\operatorname{diam} f^{-1}(y) < \mathcal{V}$ *for any point $y \in Y$.*

The next lemma shows that, in some situations, $\mathcal{V}$-maps between topological spaces $X$ and $Y$ form an open set in the function space $C(X,Y)$ endowed with the *uniform topology*. A neighborhood base of this topology at a given $f \in C(X,Y)$ consists of the sets

$$B_\rho(f,\varepsilon) = \{g \in C(X,Y) : \rho(f,g) < \varepsilon\}$$

where $\rho$ runs over the continuous pseudometrics on $Y$ and $\varepsilon$ over the positive real numbers. It is clear that the source limitation topology on $C(X,Y)$ is stronger than the uniform topology. So, a subset of $C(X,Y)$ is open in the source limitation topology if it is open is the uniform topology.

LEMMA 18.7. *Let $\mathcal{V}$ be an open cover of a space $X$. Then, for any space $Y$, the set of all uniform $\mathcal{V}$-maps from $X$ into $Y$ is open in $C(X,Y)$ equipped with the uniform topology. In particular, if $Y$ is paracompact, the same conclusion is true for the set of all $\mathcal{V}$-maps.*

*Proof.* Suppose that $f : X \to Y$ is a uniform $\mathcal{V}$-map. Then there exists a uniform open cover $\mathcal{U}$ of $Y$ with $f^{-1}(\mathcal{U})$ refining the cover $\mathcal{V}$. Since $\mathcal{U}$ is uniform, there is a continuous pseudometric $\rho$ on $Y$ such that $\operatorname{diam} B_\rho(y,1) < \mathcal{U}$ for any point $y \in Y$. We claim that $B_\rho(f,1/2)$ consists of uniform $\mathcal{V}$-maps. Obviously, it suffices to prove that for any $g \in B_\rho(f,1/2)$ each set $g^{-1}(B_\rho(y,1/2))$, $y \in Y$, is contained in some $V_y \in \mathcal{V}$. By the choice of $\rho$, for every $y \in Y$ there exists $U_y \in \mathcal{U}$ containing $B_\rho(y,1)$. Consequently, $f^{-1}(B_\rho(y,1)) \subset f^{-1}(U_y) \subset V_y$ for some $V_y \in \mathcal{V}$. To show that $g^{-1}(B_\rho(y,1/2)) \subset V_y$, take any point $x \in g^{-1}(B_\rho(y,1/2))$ and note that $\rho(f(x),y) \le \rho(f(x),g(x)) + \rho(g(x),y) < 1/2 + 1/2 = 1$, i.e., $x \in f^{-1}(B_\rho(y,1)) \subset V_y$.

The second half of the lemma follows from the fact that every open cover of $Y$ is uniform provided $Y$ is paracompact (see Lemma 18.1). ■

**18.6. Abelian groups.** In this subsection we collect some information on Abelian groups. Recall that an Abelian group $G$ is *divisible* by a prime number $p$ if for any $g \in G$ the equation $p \cdot x = g$ has a solution in $G$. A group $G$ is *divisible* if it is divisible by any prime number $p$. By $\operatorname{Tor}(G) = \{x \in G : nx = 0 \text{ for some } n \in \mathbb{N}\}$ we denote the *torsion part* of $G$. A group $G$ is *periodic* if $G = \operatorname{Tor}(G)$. The torsion part decomposes into the direct sum $\operatorname{Tor}(G) = \bigoplus_p p\text{-}\operatorname{Tor}(G)$, where $p\text{-}\operatorname{Tor}(G) = \{x \in G : p^k x = 0 \text{ for some } k \in \mathbb{N}\}$ is the $p$-torsion part of $G$. It is easy to see that each group $p$-$\operatorname{Tor}(G)$ is divisible by any prime number $q \ne p$.

A subgroup $H$ of an Abelian group $G$ is *complemented* in $G$ if there is a subgroup $H^\perp \subset G$ such that $H \cap H^\perp = \{0\}$ and $H + H^\perp = G$. A subgroup $H \subset G$ is *servant* provided, for any $h \in H$ and $n \in \mathbb{N}$, the equation $nx = h$ has a solution in $H$ if and only if it has a solution in $G$. By [34, 27.5], if $G$ is an Abelian group and $H$ its cyclic subgroup generated by any element from $\operatorname{Tor}(G)$, then $H$ is complemented in $G$ provided $H$ is servant.

In the following notations of some standard Abelian groups, $p$ is always a prime number:

- $\mathbb{Z}$ denotes the group of integer numbers;
- $\mathbb{Q}$ is the group of rational numbers;
- $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ is the cyclic group of order $p$;

- $\mathbb{Q}_p = \{z \in \mathbb{C} : z^{p^k} = 1 \text{ for some } k \in \mathbb{N}\}$ is the quasicyclic $p$-group;
- $R_p = \{m/n \in \mathbb{Q} : n \text{ is not divisible by } p\}$.

Next, we need to recall some information on the tensor product $G \otimes H$ and the torsion product $G * H$ of Abelian groups $G, H$. The definitions of these operations can be found in any textbook on homological algebra or algebraic topology (see e.g. [64] or [40]). In some textbooks (e.g. [40]) the torsion product $G * H$ is denoted by $\mathrm{Tor}(G, H)$.

The only information about torsion products we need is that the torsion product $G*H$ contains an element of order $n$ iff both groups $G$ and $H$ contain such an element (see Exercise 6 on [40, p. 267]). We need a bit more information on tensor products.

LEMMA 18.8. *For any non-trivial Abelian groups $G$ and $H$ we have:*

(1) *$G \otimes \mathbb{Z}$ is isomorphic to $G$;*
(2) *$G \otimes H$ is periodic if either $G$ or $H$ is periodic;*
(3) *$G \otimes H \neq 0$ if both $G$ and $H$ contain elements of infinite order;*
(4) *$G \otimes H \neq 0$ if $G, H$ are cyclic groups of the same order;*
(5) *if $\varphi\colon G \to H$ is an epimorphism with a periodic kernel and $H$ contains an element of a prime order $p$, then $G$ also contains an element of order $p$;*
(6) *if $H \otimes H = 0$, then $H$ is a periodic divisible group;*
(7) *if $G/\mathrm{Tor}(G)$ is divisible and $H$ is periodic and divisible, then $H \otimes G = 0$.*

*Proof.* The first four items are well known and can be found in [40, Exercises, p. 267]. The fifth item follows easily from the definitions. To prove (6), let $H \otimes H = 0$. Then $H$ is periodic according to (3). Assuming $H = \mathrm{Tor}(H) = \bigoplus_p p\text{-}\mathrm{Tor}(H)$ is not divisible, we conclude that the $p$-torsion group $p$-$\mathrm{Tor}(H)$ is not divisible by $p$ for some prime $p$. This means that there is an element $h \in p\text{-}\mathrm{Tor}(H)$ such that the equation $h = px$ has no solution in $H$. Then the cyclic group $C$ generated by $h$ is servant in $H$, and consequently complemented in $p$-$\mathrm{Tor}(H)$ by [34, 27.5]. Since $C$ is a quotient group of $H$, the equality $H \otimes H = 0$ would imply that $C \otimes C = 0$, which contradicts (4).

To prove (7), suppose that $H$ and $G/\mathrm{Tor}(G)$ are divisible and $H$ is periodic. To show that $H \otimes G = 0$, take any $x \in H$, $y \in G$ and consider their tensor product $x \otimes y$. If $y \in \mathrm{Tor}(G)$, then $ny = 0$ for some $n \in \mathbb{N}$. Since $H$ is divisible, there exists $z \in H$ with $nz = x$. Then $x \otimes y = (nz) \otimes y = z \otimes (ny) = 0$. If $y \notin \mathrm{Tor}(G)$, then the periodicity of $H$ implies the existence of $m \in \mathbb{N}$ with $mx = 0$. Because $G/\mathrm{Tor}(G)$ is divisible, there exists $z \in G$ such that $mz - y = g \in \mathrm{Tor}(G)$. Repeating the preceding argument, we can show that $x \otimes g = 0$. Then $x \otimes y = x \otimes (mz - g) = x \otimes (mz) - x \otimes g = (mx) \otimes z - 0 = 0$. ∎

## 19. $\triangle$-dimension of maps

In this section we discuss dimensional properties of maps and will prove Proposition 2.1. This proof is divided into several lemmas. Our first lemma provides the proof of item (1) of Proposition 2.1.

LEMMA 19.1. $\dim(f) \le \dim_\triangle(f)$ *for any perfect map $f : X \to Y$ between Hausdorff topological spaces.*

*Proof.* The inequality is trivial if $\dim_\triangle(f) \geq \omega$. So assume that $\dim_\triangle(f) = n < \omega$ and take a map $g : X \to I^n$ such that the diagonal map $f \triangle g : X \to Y \times I^n$ is light. Then for every $y \in Y$ the preimage $f^{-1}(y) = (f \triangle g)^{-1}(\{y\} \times I^n)$ is a compact space of dimension $\leq \dim(\{0\} \times I^n) + \dim(f \triangle g) = n$ by Theorem 3.3.10 on dimension-lowering mappings from [32]. This yields the desired inequality $\dim(f) = \sup_{y \in Y} \dim f^{-1}(y) \leq n = \dim_\triangle(f)$. ■

The second item of Proposition 2.1 is trivial and follows from the definition of $\dim_\triangle(f)$. The proof of the third item is provided by the following lemma [58].

LEMMA 19.2. *For any perfect map $f : X \to Y$ defined on a paracompact submetrizable space $X$ we have* $\dim_\triangle(f) \leq \omega$.

*Proof.* Pasynkov [58, Proposition 9.1] proved this fact in the case $X$ is metrizable, but his proof remains valid also for paracompact submetrizable spaces. ■

The fourth item of Proposition 2.1 follows from a result of M. Tuncali and V. Valov [71, Theorem 1.3]. This result concerns the so-called $\sigma$-perfect maps. Following [71], we call a map $f : K \to M$ *$\sigma$-perfect* if $K$ can be written as the countable union $K = \bigcup_{n=1}^{\infty} K_n$ of closed subspaces such that each restriction $f|K_n : K_n \to M$, $n \in \mathbb{N}$, is a perfect map. In particular, every map defined on a $\sigma$-compact space is $\sigma$-perfect.

LEMMA 19.3. *Let $f : K \to M$ be a $\sigma$-perfect $n$-dimensional map from a paracompact submetrizable space $K$ onto a paracompact $C$-space $M$. Then the function space $C(K, \mathbb{I}^n)$ contains a dense $G_\delta$-set of maps $g : K \to \mathbb{I}^n$ such that $f \triangle g : K \to M \times \mathbb{I}^n$ is* 0*-dimensional.*

This lemma has been established in [71] for metrizable space $K$. But the proof works for submetrizable $K$ as well.

The final item of Proposition 2.1 can be easily derived from the following result of M. Levin [47]:

LEMMA 19.4 (Levin). *Let $f : K \to M$ be an $n$-dimensional map between metrizable compacta. Then $C(K, \mathbb{I}^{n+1})$ contains a dense $G_\delta$-set of maps $g : K \to \mathbb{I}^{n+1}$ such that $f \triangle g : K \to M \times \mathbb{I}^{n+1}$ is light.*

## 20. Simplicial complexes and PL-maps

Since simplicial complexes and PL-maps play a significant role in our further considerations, we collect the necessary information on this topic.

By an *abstract simplicial complex* we understand any set $K$ such that each element $\sigma \in K$ is a finite non-empty set with all non-empty subsets of $\sigma$ being also in $K$. Elements of the set $K$ are called *simplexes* while elements of the set $\bigcup K$ are called *vertices* of $K$. A subset $L \subset K$ is a *subcomplex* of $K$ if $L$ itself is an abstract simplicial complex. The *$n$-skeleton*, $n \geq 0$, of the abstract complex $K$ is its subcomplex $K^{(n)} = \{\sigma \in K : \operatorname{card}(\sigma) \leq n+1\}$. Identifying each vertex $v \in \bigcup K$ with the singleton $\{v\} \in K^{(0)}$, we can identify $\bigcup K$ with the 0-skeleton $K^{(0)}$ of $K$.

The *geometric realization* $|K|$ of a simplicial complex $K$ is the set $|K|$ of all functions $x : \bigcup K \to [0,1]$ such that there is a simplex $\sigma \in K$ with $\sum_{v\in\sigma} x(v) = 1$ and $x(v) = 0$ if $v \notin \sigma$. Hence, each function $x \in |K|$ takes non-zero values only on a finite subset of $\bigcup K$ which necessarily is a simplex of $K$. Because of that, the geometric realization $|K|$ of $K$ can be considered as a subset of the Banach space $l_1(\bigcup K)$ of all functions $x : \bigcup K \to \mathbb{R}$ with norm $\|x\| = \sum_{v\in\bigcup K} |x(v)| < \infty$. For each vertex $v \in \bigcup K$ of $K$ we denote by $\mathrm{pr}_v : |K| \to [0,1]$ the canonical projection assigning to a function $x \in |K|$ its value $x(v)$ at $v$. It is clear that each projection $\mathrm{pr}_v : |K| \to [0,1]$ is continuous with respect to the metric topology on $|K|$ inherited from the Banach space $l_1(\bigcup K)$. The set $St(v) = \mathrm{pr}_v^{-1}(0,1]$ is called the *open star* of $v$ in $|K|$.

If $L$ is a subcomplex of $K$, then we can identify the geometric realization $|L|$ of $L$ with the set $\{x \in |K| : x(v) = 0 \text{ for all } v \in \bigcup K \setminus \bigcup L\}$. Thus, for each simplex $\sigma \in K$ we can consider its geometric realization $|\sigma| \subset |K|$ which is called a *geometric simplex*. It is clear that $|\sigma|$ is a compact subset of $|K|$ with respect to the metric topology inherited from $l_1(\bigcup K)$. The *combinatorial interior* of a geometric simplex $|\sigma|$ is the set

$$|\mathring{\sigma}| = \{x \in |\sigma| : x(v) > 0 \text{ for all } v \in \sigma\}.$$

The complement $\partial|\sigma| = |\sigma| \setminus |\mathring{\sigma}|$ is called the *combinatorial boundary* of $|\sigma|$. Let us observe that the star of any vertex $v \in \bigcup K$ is the union of the combinatorial interiors of all simplexes containing $v$, i.e., $St(v) = \bigcup_{v\in\sigma\in K} |\mathring{\sigma}|$.

Each geometric simplicial complex $|K|$ contains a canonical dense set $Q_K = |K| \cap \mathbb{Q}^{\bigcup K}$ consisting of all functions $x \in |K|$ with rational values. Such functions $x \in |K|$ will be called *rational points* of $|K|$. Observe that, for any countable subcomplex $L \subset K$, the set $|L| \cap Q_M = Q_L$ is countable and dense in $|L|$.

The geometric realization of any abstract simplicial complex $K$ can be described more geometrically as follows. Identifying finite subsets of $\bigcup K$ with their characteristic functions, we can consider the complex $K$ as a subset of $l^1(\bigcup K)$. In this description, the 0-skeleton $K^{(0)}$ of $K$ coincides with the standard unit basis of the Banach space $l_1(\bigcup K)$. Now, it is easy to see that

$$|K| = \bigcup_{\sigma\in K} \mathrm{conv}(\sigma^{(0)}),$$

where $\mathrm{conv}(A)$ stands for the convex hull of a given set $A \subset l_1(\bigcup K)$.

Besides the metric topology, every geometric simplicial complex $|K|$ carries the *CW-topology* which is the strongest topology on $|K|$ inducing the original metric topology on each geometric simplex $|\sigma| \subset |K|$, $\sigma \subset K$.

Unless stated otherwise, all geometric simplicial complexes $|K|$ will always be equipped with the $CW$-topology. This topology has many nice properties. For example:

- $|K|$ is a stratifiable space [35], and hence it is hereditarily paracompact;
- $|K|$ is submetrizable (because the $l_1$-metric is continuous on $|K|$);
- $|K|$ is a $k$-space;
- each compact subset of $|K|$ is contained in the geometric realization $|L|$ of some finite subcomplex $L \subset K$.

A typical example of an abstract simplicial complex is the *nerve* $N(\mathcal{U})$ of a cover $\mathcal{U}$ for a given space $X$. By definition, $N(\mathcal{U})$ consists of all finite subsets $\mathcal{F} \subset \mathcal{U}$ such that $\bigcap \mathcal{F} \neq \emptyset$. Thus, the non-empty sets $U \in \mathcal{U}$ are the vertices of the complex $N(\mathcal{U})$. Every partition of unity $\{\lambda_U : X \to [0,1]\}_{U \in \mathcal{U}}$ subordinated to the cover $\mathcal{U}$ determines a map

$$\lambda : X \to |N(\mathcal{U})|, \qquad x \mapsto (\lambda_U(x))_{U \in \mathcal{U}},$$

called the *canonical map into the nerve* of $\mathcal{U}$. This map is continuous with respect to the $CW$-topology on $|N(\mathcal{U})|$ because $\{\lambda_U^{-1}(0,1]\}_{U \in \mathcal{U}}$ is a locally finite cover of $X$. Moreover, since $\lambda^{-1}(St(U)) \subset U$ for all $U \in \mathcal{U}$, the canonical map $\lambda : X \to N(\mathcal{U})$ is a $\mathcal{U}$-map.

A map $f : |K| \to L$ from a geometric simplicial complex to a linear space $L$ is called a PL-*map* if it is linear on each geometric simplex $|\sigma| \subset |K|$, i.e.,

$$f(x) = \sum_{v \in \sigma} x(v) f(v)$$

for all $x \in |\sigma|$. We recall that each vertex $v \in \bigcup K$ is identified with the characteristic function of the singleton $\{v\}$. In particular, the identity inclusion $i : |K| \to l_1(\bigcup K)$ is a PL-map. Every PL-map is uniquely determined by its values on the set $|K^{(0)}|$ of geometric vertices of $|K|$. Conversely, each map $f : \bigcup K \to L$ to a linear space induces the *canonical PL-map* $|f| : |K| \to L$, $x \mapsto \sum_{v \in \bigcup K} x(v) \cdot f(v)$.

A map $f : |K| \to |M|$ between two geometric complexes is said to be a *PL-map* if its composition with the embedding $|M| \subset l_1(\bigcup M)$ is a PL-map. It can be shown that if $f : |K| \to |M|$ is a PL-map and $|\sigma|$ a geometric simplex of $K$, then $f(|\sigma|)$ is contained in some geometric simplex $|\tau|$ of $M$. A PL-map $f : |K| \to |M|$ is called *rational* if $f(Q_K) \subset Q_M$, where $Q_K$ and $Q_M$ are the rational points of $|K|$ and $|M|$, respectively.

The next lemma is very useful when working with PL-maps.

LEMMA 20.1. *Let $f : |K| \to |M|$ be a PL-map and $L$ a subcomplex of $M$. Then $f^{-1}(|L|)$ is a subcomplex of $|K|$.*

*Proof.* It suffices to show that $f^{-1}(|L|)$ is the union of all simplexes $|\sigma|$ from $|K|$ with $f(|\sigma|) \subset |L|$. So, let $x \in f^{-1}(|L|)$ and $|\sigma| \in |K|$, $|\tau| \in |L|$ be the unique simplexes such that $x$ belongs to the combinatorial interior of $|\sigma|$ and $f(x)$ is contained in the combinatorial interior of $|\tau|$. Since $f$ is a PL-map, there exists a simplex $|\tau'|$ from $|M|$ which contains $f(|\sigma|)$. Observe that $|\tau'|$ contains also $|\tau|$. Taking into account that $x$ lies in the combinatorial interior of $|\sigma|$, we conclude that $f(v) \in |\tau|$ for every vertex $v$ of $|\sigma|$, and hence $f(|\sigma|) \subset |\tau|$, which completes the proof. ■

A PL-map $f : |K| \to |M|$ is called a *simplicial map* if $f(|\sigma|)$ is a geometric simplex of $|M|$ for each geometric simplex $|\sigma|$ of $K$, i.e., $f$ maps each geometric simplex of $|K|$ onto a geometric simplex of $|M|$. It can be shown that a PL-map $f : |K| \to |M|$ is simplicial if and only if $f(|K^{(0)}|) \subset |M^{(0)}|$. Hence, every simplicial map is uniquely determined by its restriction $f^{(0)} : |K^{(0)}| \to |M^{(0)}|$. Obviously, $f^{(0)}$ can be identified with a map $f^{(0)} : \bigcup K \to \bigcup M$. Moreover, a map $f^{(0)} : \bigcup K \to \bigcup M$ induces a simplicial map

$f : |K| \to |M|$ if and only if $f^{(0)}(\sigma) \in M$ for each simplex $\sigma \in K$. Clearly, any simplicial map is a rational PL-map.

Next, we discuss some formalism related to subdivisions of simplicial complexes. By a *subdivision* of a geometric simplicial complex $|K|$ we understand an abstract simplicial complex $M$ such that $\bigcup M \subset |K|$ and the canonical PL-map $|e| : |M| \to l_1(\bigcup K)$ induced by the embedding $e : \bigcup M \subset |K| \subset l_1(\bigcup K)$ is a homeomorphism between the geometric complexes $|M|$ and $|K|$ endowed with the $CW$-topologies. Using the properties of these topologies, one can check that the preimage $|e|^{-1}(|\sigma|)$ of each geometric simplex of $K$ coincides with the geometric realization of some finite subcomplex of $M$. A subdivision $M$ of $|K|$ is called a *rational subdivision* if $\bigcup M \subset Q_K$.

We shall say that the triangulation of a geometric simplicial complex $|K|$ *refines* a cover $\mathcal{U}$ of $|K|$ if each geometric simplex $|\sigma|$ of $K$ is contained in some set $U \in \mathcal{U}$.

We need the following well-known fact concerning the existence of fine subdivisions.

LEMMA 20.2. *For each open cover $\mathcal{U}$ of a geometric simplicial complex $|K|$ there is a rational subdivision $M$ of $|K|$ such that the cover $\{St(v) : v \in \bigcup M\}$ refines the cover $h^{-1}(\mathcal{U}) = \{h^{-1}(U) : U \in \mathcal{U}\}$, where $h : |M| \to |K|$ is the canonical homeomorphism.*

We recall some information on dual skeleta of geometric simplicial complexes. To this end, we need the notion of barycentric subdivision.

The *barycenter* of a geometric simplex $|\sigma|$ is the function $b_\sigma = \frac{1}{n}\sum_{v\in\sigma}\chi_v$, where $\mathrm{card}(\sigma) = n$ and $\chi_v$ are the characteristic functions of the singletons $\{v\}$. The *barycentric subdivision* $B_\sigma$ of a geometric simplex $|\sigma|$ is defined by induction on the cardinality of $\sigma$. If $\mathrm{card}(\sigma) \le 1$, then $B_\sigma = \{|\sigma^{(0)}|\}$. Assume that the barycentric subdivision $B_\sigma$ is defined for all simplexes of size $\mathrm{card}(\sigma) \le n$. Given a simplex $\sigma$ with $\mathrm{card}(\sigma) = n+1$, let

$$B_\sigma = \{\tau, \tau \cup \{b_\sigma\} : \tau \in B_\varsigma \text{ for a proper subset } \varsigma \subset \sigma\}.$$

Let $|\sigma|$ be a geometric simplex and $n \in \omega$. The *dual skeleton* to the $n$-skeleton $\sigma^{(n)}$ of $\sigma$ is the complex

$$\sigma_{(n)} = \{\tau \in B_\sigma : \tau \cap |\sigma^{(n)}| = \emptyset\}$$

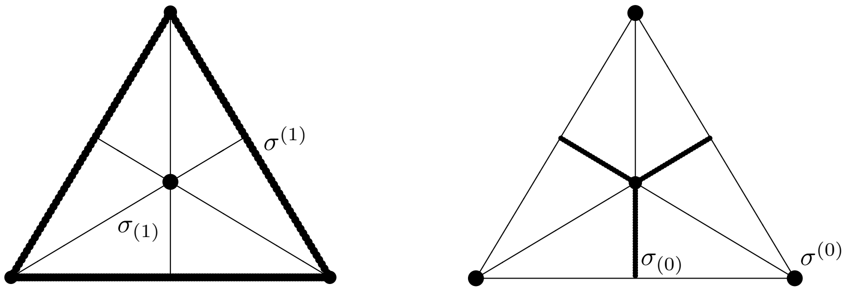
consisting of all simplexes of the barycentric subdivision of $\sigma$ which are disjoint from the geometric $n$-skeleton of $\sigma$.

The *dual skeleton* $K_{(n)}$ to the $n$-skeleton $K^{(n)}$ of a simplicial complex $K$ is the complex $K_{(n)} = \bigcup_{\sigma\in K}\sigma_{(n)}$. The *geometric dual skeleton* to the $n$-skeleton $K^{(n)}$ of $K$ is the subcomplex

$$|K_{(n)}| = \bigcup_{\sigma\in K_{(n)}} \mathrm{conv}(\sigma)$$

of the barycentric subdivision (the latter is simplicially homeomorphic to the geometric realization of $K_{(n)}$—that is the reason we use the same symbol $|K_{(n)}|$ for both of them).

In the following figure we draw two possible pairs of dual (non-empty) skeleta of the two-dimensional simplex.




The importance of dual skeleta $|\sigma^{(n)}|$ and $|\sigma_{(n)}|$ lies in the possibility to write each point $z \in |\sigma|$ as a convex combination $z = (1-t)z^n + tz_n$, where $z^n \in |\sigma^{(n)}|$ and $z_n \in |\sigma_{(n)}|$. The parameter $t$ is uniquely determined, $t = 0$ iff $z \in |\sigma^{(n)}|$, and $t = 1$ iff $z \in |\sigma_{(n)}|$. Moreover the point $z_n$ (resp. $z^n$) is also uniquely determined iff $z \notin |\sigma^{(n)}|$ (resp. $z \notin |\sigma_{(n)}|$). This means that the simplex $|\sigma|$ has the join structure $|\sigma^{(n)}| * |\sigma_{(n)}|$.

We recall that the *join* $X * Y$ of two topological spaces is the quotient space $X \times Y \times [0,1]/_\sim$, where $\sim$ is the equivalence relation whose non-degenerate equivalence classes are the set $\{x\} \times Y \times \{0\}$ and $X \times \{y\} \times \{1\}$ for $x, y \in Y$.

We finish this section with a lemma describing a property of the $l_1$-metric on simplicial complexes.

LEMMA 20.3. *Let $f : |\Delta| \to |\sigma|$ be a surjective simplicial map between geometric simplexes. Then*

(1) *the map $f$ is non-expanding, i.e., $\mathrm{dist}(f(x), f(x')) \le \mathrm{dist}(x, x')$ for all $x, x' \in |\Delta|$;*
(2) *for any points $x \in |\Delta|$ and $y, y' \in |\sigma|$ with $y = f(x)$ there is a point $x' \in |\Delta|$ such that $f(x') = y'$ and $\mathrm{dist}(x, x') = \mathrm{dist}(y, y')$.*

*Proof.* Denote by $f_0 : \bigcup \Delta \to \bigcup \sigma$ the map on vertices of the simplexes $\Delta$ and $\sigma$, induced by the simplicial map $f$.

The map $f$, being simplicial, maps a point $x \in |\Delta|$ to the point $y \in |\sigma|$ such that $y(w) = \sum_{v \in f_0^{-1}(w)} x(v)$ for $w \in \bigcup \sigma$. Then for any points $x, x' \in \Delta$ we have

$$\begin{aligned}\mathrm{dist}(f(x), f(x')) &= \sum_{w \in \bigcup \sigma} |y(w) - y'(w)| = \sum_{w \in \bigcup \sigma} \Big| \sum_{v \in f_0^{-1}(w)} x(v) - \sum_{v \in f_0^{-1}(w)} x'(v) \Big| \\ &\le \sum_{w \in \bigcup \sigma} \sum_{v \in f_0^{-1}(w)} |x(v) - x'(v)| = \sum_{v \in \bigcup \Delta} |x(v) - x'(v)| = \mathrm{dist}(x, x'),\end{aligned}$$

which proves the non-expanding property of $f$.

Now, take any points $x \in |\Delta|$ and $y, y' \in |\sigma|$ with $f(x) = y$. Since $f(x) = y$, $y(w) = \sum_{v \in f_0^{-1}(w)} x(v)$ for all $w \in \bigcup \sigma$. For each $w \in \bigcup \sigma$ find non-negative real numbers $x'(v)$, $v \in f_0^{-1}(w)$, such that

- $\sum_{v \in f_0^{-1}(w)} x'(v) = y'(v)$;
- $x'(v) \le x(v)$ for $v \in f_0^{-1}(w)$ iff $y'(w) \le y(w)$.

Since $\sum_{v\in\bigcup\Delta} x'(v) = \sum_{w\in\bigcup\sigma}\sum_{v\in f_0^{-1}(w)} x'(v) = \sum_{w\in\bigcup\sigma} y'(w) = 1$, the function $x' : \bigcup\Delta \to [0,1]$ belongs to the geometric simplex $|\Delta|$. Then

$$\begin{aligned}\operatorname{dist}(x',x) &= \sum_{v\in\bigcup\Delta} |x'(v)-x(v)| = \sum_{w\in\bigcup\sigma}\sum_{v\in f_0^{-1}(w)} |x'(v)-x(v)|\\ &= \sum_{w\in\bigcup\sigma}\Big|\sum_{v\in f_0^{-1}(w)} (x'(v)-x(v))\Big| = \sum_{w\in\bigcup\sigma} |y'(w)-y(w)| = \operatorname{dist}(y',y). \ \blacksquare\end{aligned}$$

A topological space $X$ is called a *polyhedron* if for some simplicial complex $K$ there is a homeomorphism $h : |K| \to X$. In this case the family $T = \{h(|\sigma|) : \sigma \in K\}$ is called a *triangulation* of $X$ while $h(|K^{(n)}|)$ is the $n$-skeleton of the triangulation.

A map $f : X \to Y$ between topological spaces is called a *PL-map* if there is a PL-map $g : |K| \to |M|$ between two simplicial complexes and two homeomorphisms $\alpha : X \to |K|$ and $\beta : Y \to |M|$ such that $g\circ\alpha = \beta\circ f$.

In the proofs below we shall not distinguish between abstract simplicial complexes and their geometric realizations. So, by a *simplicial complex* we shall always understand the geometric realization $|K|$ of an abstract simplicial complex $K$ equipped with the $CW$-topology. Let us also mention the following well-known fact: each (rational) PL-map $p : K \to M$ between finite simplicial complexes is simplicial with respect to suitable (rational) subdivisions of the complexes $K, M$.

## 21. LC$^n$-spaces

In this subsection we provide the necessary information on LC$^n$-spaces. $\mathcal{U}$-near and $\mathcal{U}$-homotopic maps into LC$^n$-spaces are closely related according to the following standard lemma that can be found in [41, V.5.1] (recall that two maps $f, g : K \to X$ are $\mathcal{U}$-*near* with respect to a cover $\mathcal{U}$ of $X$ if $\operatorname{diam}\{f(x), g(x)\} < \mathcal{U}$ for any $x \in X$).

LEMMA 21.1. *For any open cover $\mathcal{U}$ of a metrizable* LC$^n$*-space $X$ there is an open cover $\mathcal{V}$ of $X$ such that any two $\mathcal{V}$-near maps $f, g : K \to X$ defined on a metrizable space $K$ with $\dim K \le n$ are $\mathcal{U}$-homotopic.*

The following Tychonoff version of Lemma 21.1 is a key ingredient of the proof of Proposition 3.2.

LEMMA 21.2. *For any open cover $\mathcal{U}$ of a Tychonoff* LC$^n$*-space $X$ and a map $f : K \to X$, where $K$ is a compact polyhedron with $\dim K \le n$, there is an open cover $\mathcal{V}$ of $X$ such that any $\mathcal{V}$-near map $g : K \to X$ to $f$ is $\mathcal{U}$-homotopic to $f$.*

*Proof.* Let $\mathcal{U}_0 = \mathcal{U}$ and $k = \dim K$. Since $X$ has the LC$^k$-property, there is an open cover $\mathcal{V}_0$ of $X$ such that each map $\alpha : \partial\mathbb{I}^{k+1} \to X$ with $\operatorname{diam}\alpha(\partial\mathbb{I}^{k+1}) < \mathcal{V}_0$ has a continuous extension $\bar\alpha : \mathbb{I}^{k+1} \to X$ with $\operatorname{diam}\bar\alpha(\mathbb{I}^{k+1}) < \mathcal{U}_0$. By Lemma 18.2, there is a continuous pseudometric $\rho_0$ on $X$ such that $\operatorname{diam} B_{\rho_0}(x,1) < \mathcal{V}_0$ for all $x \in f(K)$. Let $\mathcal{U}_1$ be the cover of $X$ by open $\rho_0$-balls of radius $1/8$.

By a finite induction of length $k$, we can construct sequences $(\mathcal{U}_i)_{i\le k}$, $(\mathcal{V}_i)_{i\le k}$ of open covers of $X$ and a sequence $(\rho_i)_{i\le k}$ of continuous pseudometrics on $X$ such that the following conditions hold for every $i \le k$:

$(1_i)$ $\mathcal{U}_{i+1} = \{B_{\rho_i}(x, 1/8) : x \in X\}$;
$(2_i)$ each map $\alpha : \partial\mathbb{I}^{k+1-i} \to X$ with $\operatorname{diam}\alpha(\partial\mathbb{I}^{k+1-i}) < \mathcal{V}_i$ has a continuous extension $\bar{\alpha} : \mathbb{I}^{k+1-i} \to X$ with $\operatorname{diam}\bar{\alpha}(\mathbb{I}^{k+1-i}) < \mathcal{U}_i$;
$(3_i)$ $\operatorname{diam} B_{\rho_i}(x, 1) < \mathcal{V}_i$ for all $x \in f(K)$.

We claim that the cover $\mathcal{V} = \mathcal{V}_k$ satisfies our requirements. Let $g : K \to X$ be a map $\mathcal{V}$-near to $f$. We should prove that $g$ is $\mathcal{U}$-homotopic to $f$.

Select a triangulation $T$ of the complex $K$ so fine that for any simplex $\sigma$ of this triangulation and for every $i \le k$ we have $\operatorname{diam}_{\rho_i} f(\sigma) < 1/4$ and $\operatorname{diam}_{\rho_i} g(\sigma) < 1/4$. By $K^{(i)}$ we denote the $i$-skeleton of $K$ (with respect to the triangulation $T$). It is convenient to assume that $K^{(-1)} = \emptyset$. Consider the map $H^{(-1)} : K \times \{0,1\} \to X$ defined by $H^{(-1)}(x, 0) = f(x)$ and $H^{(-1)}(x, 1) = g(x)$ for $x \in K$.

We shall construct by induction a sequence of maps

$$H^{(i)} : (K^{(i)} \times \mathbb{I}) \cup (K \times \{0,1\}) \to X, \qquad i \le k,$$

such that

$(4_i)$ $H^{(i)}(x,t) = H^{(i-1)}(x,t)$ for any $(x,t) \in (K^{(i-1)} \times \mathbb{I}) \cup (K \times \{0,1\})$;
$(5_i)$ $\operatorname{diam} H^{(i)}(\sigma \times \mathbb{I}) < \mathcal{U}_{k-i}$ for any $i$-dimensional simplex $\sigma$ of $K$.

Assume that for some $i \le k$ the map $H^{(i-1)}$ has been constructed. We need to extend this map to a map $H^{(i)}$ defined on $(K^{(i)} \times \mathbb{I}) \cup (K \times \{0,1\})$. Take any $i$-dimensional simplex $\sigma \in K^{(i)}$ and let $\sigma^{(i-1)} = \sigma \cap K^{(i-1)}$ be the $(i-1)$-dimensional skeleton of $\sigma$. It is the union of all $(i-1)$-dimensional faces of $\sigma$. By the inductive assumption, for each $(i-1)$-dimensional face $\tau$ of $\sigma$ we have $\operatorname{diam} H^{(i-1)}(\tau \times \mathbb{I}) < \mathcal{U}_{k-(i-1)}$ and hence

$$\operatorname{diam}_{\rho_{k-i}} H^{(i-1)}(\tau \times \mathbb{I}) < 1/4$$

according to condition $(1_{k-i})$. The product $\sigma \times \mathbb{I}$ can be considered as an $(i+1)$-dimensional cube with boundary $\partial(\sigma \times \mathbb{I}) = (\sigma^{(i-1)} \times \mathbb{I}) \cup (\sigma \times \{0,1\})$. Observe that

$$\begin{aligned}\operatorname{diam}_{\rho_{k-i}} H^{(i-1)}(\partial(\sigma \times \mathbb{I})) &\le \operatorname{diam}_{\rho_{k-i}} H^{(i-1)}(\sigma \times \{0\}) + \operatorname{diam}_{\rho_{k-i}} H^{(i-1)}(\sigma \times \{1\}) \\ &\quad + 2 \max_{\tau \subset \sigma^{(i-1)}} \operatorname{diam}_{\rho_{k-i}} H^{(i-1)}(\tau \times \mathbb{I}) \\ &\le \operatorname{diam}_{\rho_{k-i}} f(\sigma) + \operatorname{diam}_{\rho_{k-i}} g(\sigma) + 2\tfrac{1}{4} \le \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{2}{4} = 1.\end{aligned}$$

Then, by condition $(3_{k-i})$, $\operatorname{diam} H^{(i-1)}\big(\partial(\sigma \times \mathbb{I})\big) < \mathcal{V}_{k-i}$. So, condition $(2_{k-i})$ shows that the map $H^{(i-1)}|\partial(\sigma \times \mathbb{I})$ admits a continuous extension $H^{(i)}_\sigma : \sigma \times \mathbb{I} \to X$ with $\operatorname{diam} H^{(i)}_\sigma(\sigma \times \mathbb{I}) < \mathcal{U}_{k-i}$.

Finally, define a map $H^{(i)} : (K^{(i)} \times \mathbb{I}) \cup (K \times \{0,1\}) \to X$ letting

$$H^{(i)}(x,t) = \begin{cases} f(x) & \text{if } t = 0, \\ g(x) & \text{if } t = 1, \\ H^{(i)}_\sigma(x,t) & \text{if } (x,t) \in \sigma \times \mathbb{I}. \end{cases}$$

It is clear that the map $H^{(i)}$ satisfies conditions $(4_i)$ and $(5_i)$.

Completing the inductive construction, we obtain a map $H = H^{(k)} : K \times \mathbb{I} \to X$ such that

- $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for $x \in K$ and
- for any $x \in K$ $\operatorname{diam} H(\{x\} \times \mathbb{I}) < \mathcal{U}_0 = \mathcal{U}$.

Hence, $H$ is a $\mathcal{U}$-homotopy linking the maps $f$ and $g$. ∎

We also need the following completion result (see [22, Theorem 2.8]).

Lemma 21.3. *For any metrizable* $\mathrm{LC}^n$*-space* $X$ *there is a completely metrizable* $\mathrm{LC}^n$*-space* $\tilde{X}$ *containing* $X$ *as a relative* $\mathrm{LC}^n$*-set.*

Relative $\mathrm{LC}^n$-sets are tightly connected with locally $n$-negligible sets in the sense of H. Toruńczyk [67]. A subset $A \subset X$ is called *locally $n$-negligible* in $X$ if given $k < n+1$, $x \in X$, and a neighborhood $U \subset X$ of $x$ there is another neighborhood $V \subset U$ of $x$ such that for each map $f : (\mathbb{I}^k, \partial\mathbb{I}^k) \to (V, V \setminus A)$ there is a homotopy $(h_t) : (\mathbb{I}^k, \partial\mathbb{I}^k) \to (U, U \setminus A)$ such that $h_0 = f$ and $h_1(\mathbb{I}^k) \subset U \setminus A$. The metrizable case of the next result is due to Toruńczyk [67, Theorem 2.8]. The general case has a similar proof.

Lemma 21.4. *If* $X$ *is a dense relative* $\mathrm{LC}^n$*-subset of a Tychonoff space* $\tilde{X}$*, then the complement* $\tilde{X} \setminus X$ *is locally $n$-negligible in* $\tilde{X}$*.*

The following property of locally $n$-negligible sets, established in [67, Theorem 2.3], indicates their importance.

Lemma 21.5. *A subset* $A$ *of a Tychonoff space* $X$ *is locally $n$-negligible if and only if given a simplicial pair* $(K, L)$ *with* $\dim K \le n$*, a continuous pseudometric* $\rho$ *on* $X$*, a continuous function* $\varepsilon : K \to (0,1]$ *and a map* $f : K \times \{0\} \cup L \times \mathbb{I} \to X$ *with* $\rho(f(x,0), f(x,t)) < \varepsilon(x)$ *and* $f(x,1) \in X \setminus A$ *for all* $(x,t) \in L \times \mathbb{I}$*, there is a map* $\tilde{f} : K \times \mathbb{I} \to X$ *which extends* $f$ *and satisfies* $\rho(\tilde{f}(x,0), \tilde{f}(x,t)) < \varepsilon(x)$ *and* $\tilde{f}(x,1) \notin A$ *for all* $(x,t) \in K \times \mathbb{I}$*.*

This lemma implies the following one which is the second part of Proposition 5.5.

Lemma 21.6. *Each dense relative* $\mathrm{LC}^n$*-subset* $X$ *of a Tychonoff space* $\tilde{X}$ *is homotopically $n$-dense in* $\tilde{X}$*.*

## 22. Constructing pseudometrics with nice local properties

In this section we shall establish that paracompact spaces with nice local properties also admit nice pseudometrics (see also [22] for a similar result in the realm of metrizable spaces).

For a metric space $(X, \rho)$ and a real number $\varepsilon > 0$ let

- $\mathcal{B}_\rho(\varepsilon) = \{B_\rho(x,\varepsilon) : x \in X\}$ be the cover of $X$ by open $\varepsilon$-balls;
- $\mathcal{D}_\rho(\varepsilon)$ be the cover of $X$ by open subsets of diameter $< \varepsilon$.

Let $X$ be a space and $\ll$ be a binary relation on the set $\operatorname{cov}(X)$ of open covers of $X$. We say that $\ll$ is *admissible* if it satisfies the following conditions:

- $\mathcal{V} \ll \mathcal{U}$ implies $\mathcal{V} \prec \mathcal{U}$ for any covers $\mathcal{U}, \mathcal{V} \in \operatorname{cov}(X)$;
- if $\mathcal{V} \prec \mathcal{V}' \ll \mathcal{U}' \prec \mathcal{U}$, then $\mathcal{V} \ll \mathcal{U}$ for any $\mathcal{U}, \mathcal{U}', \mathcal{V}', \mathcal{V} \in \operatorname{cov}(X)$;
- for every $\mathcal{U} \in \operatorname{cov}(X)$ there exists $\mathcal{V} \in \operatorname{cov}(X)$ such that $\mathcal{V} \ll \mathcal{U}$.

Lemma 22.1. *Let $X$ be a paracompact space and $\ll$ be an admissible binary relation on* $\mathrm{cov}(X)$*. Then, for any continuous pseudometric $\eta$ on $X$ there is a continuous pseudometric $\rho \geq \eta$ such that $\mathcal{B}_\rho(r/4) \ll \mathcal{D}_\rho(r)$ for every $r \in (0, 1/2]$.*

*Proof.* Using the paracompactness of $X$ and the properties of the relation $\ll$, we construct inductively a sequence $(\mathcal{V}_i)_{i=0}^\infty$ of open covers of $X$ so that the following conditions are satisfied:

(1) $\mathrm{mesh}_\eta\, \mathcal{V}_i < 2^{-i-1}$, $i \geq 0$;
(2) $St(\mathcal{V}_i) \ll \mathcal{V}_{i-1}$, $i \geq 1$.

Let $V_0 = X \times X$ and $V_i = \bigcup\{V \times V : V \in \mathcal{V}_i\}$ for $i > 0$. Since $St(\mathcal{V}_i) \prec \mathcal{V}_{i-1}$, we have $3V_i \subset V_{i-1}$, where

$$3V_i = \{(x, y) \in X^2 : \exists a, b \in X \text{ with } (x, a), (a, b), (b, y) \in V_i\}.$$

By Theorem 8.1.10 of [33], there is a continuous pseudometric $d$ on $X$ such that

$$\{(x, y) \in X^2 : d(x, y) < 2^{-i}\} \subset V_i \subset \{(x, y) \in X^2 : d(x, y) \leq 2^{-i}\}$$

for all $i \in \omega$.

We claim that $\mathcal{B}_d(r/4) \ll \mathcal{B}_d(r)$ for every $r \in (0, 1]$. Choose $i \in \mathbb{N}$ with

$$2^{-i-1} < r/4 \leq 2^{-i}$$

and note that $i \geq 2$ because $2^{-i-1} < r/4 \leq 1/4$. It follows from the inclusion $\{(x, y) \in X^2 : d(x, y) < 2^{-i}\} \subset V_i$ that $B_d(x, r/4) \subset B_d(x, 2^{-i}) \subset St(x, \mathcal{V}_i)$ for every $x \in X$. So, by (2), we have $\mathcal{B}_d(r/4) \prec St(\mathcal{V}_i) \ll \mathcal{V}_{i-1}$. On the other hand, the inclusion

$$V_{i-1} \subset \{(x, y) \in X^2 : d(x, y) \leq 2^{-(i-1)}\} \subset \{(x, y) \in X^2 : d(x, y) < r\}$$

implies that $\mathcal{V}_{i-1} \prec \mathcal{D}_d(r)$. Hence, $\mathcal{B}_d(r/4) \prec St(\mathcal{V}_i) \ll \mathcal{V}_{i-1} \prec \mathcal{D}_d(r)$.

Therefore, from the properties of $\ll$, we obtain

(3) $\mathcal{B}_d(r/4) \ll \mathcal{D}_d(r)$.

We claim that the pseudometric $\rho = \max\{d, \eta\}$ satisfies our requirements. This will follow from (3) if $\rho(x, y) = d(x, y)$ for any points $x, y \in X$ with $\rho(x, y) \leq 1/2$ (indeed, in such a situation we would have $\mathcal{B}_\rho(r/4) = \mathcal{B}_d(r/4) \ll \mathcal{D}_d(r) = \mathcal{D}_\rho(r)$ for every positive $r \leq 1/2$). Assume there are two points $x, y \in X$ with $d(x, y) < \rho(x, y) = \eta(x, y) \leq 1/2$ and choose $i \in \omega$ with $2^{-i} < \eta(x, y) \leq 2^{-i+1}$. Note that $i > 1$. Then $d(x, y) < 2^{-i+1}$, so $(x, y) \in V_{i-1}$ and hence $\mathrm{diam}\{x, y\} < \mathcal{V}_{i-1}$. Since $\mathrm{mesh}_\eta(\mathcal{V}_{i-1}) < 2^{-i}$, we conclude that $\eta(x, y) < 2^{-i}$. The last inequality is not possible by the choice of $i$. Therefore, $\rho(x, y) = d(x, y)$ for all $x, y \in X$ with $\rho(x, y) \leq 1/2$. ■

Applying Lemma 22.1 to $\mathrm{LC}^n$-spaces or locally contractible spaces, we obtain

Lemma 22.2. *For any continuous pseudometric $\eta$ on a paracompact $\mathrm{LC}^n$-space there exists a continuous pseudometric $\rho \geq \eta$ such that any map $f\colon \partial\mathbb{I}^k \to X$ with $k < n+1$ and $\mathrm{diam}_\rho f(\partial\mathbb{I}^k) \leq 1/10$ extends to a map $\bar{f}\colon \mathbb{I}^k \to X$ with $\mathrm{diam}_\rho \bar{f}(\mathbb{I}^k) < 5\,\mathrm{diam}_\rho f(\partial\mathbb{I}^k)$. If, in addition, $X$ is locally contractible, $\rho$ can be chosen such that $B_\rho(x, r/4)$ is contractible in $B_\rho(x, r)$ for every $x \in X$ and every $r \leq 1/2$.*

*Proof.* Suppose $X$ is $\mathrm{LC}^n$ and consider the following binary relation $\ll_n$ on $\mathrm{cov}(X)$: $\mathcal{V} \ll_n \mathcal{U}$ iff $\mathcal{V} \prec \mathcal{U}$ and any map $f\colon \partial\mathbb{I}^k \to X$ with $k < n+1$ and $\operatorname{diam} f(\partial\mathbb{I}^k) < \mathcal{V}$ extends to a map $\bar{f}\colon \mathbb{I}^k \to X$ with $\operatorname{diam} \bar{f}(\mathbb{I}^k) < \mathcal{U}$. Obviously, $\ll_n$ is admissible. So, we can apply Lemma 22.1 to find a continuous pseudometric $\rho \geq \eta$ on $X$ such that $\mathcal{B}_\rho(r/4) \ll_n \mathcal{D}_\rho(r)$ for every positive $r \leq 1/2$. Let us show that this pseudometric satisfies our requirements. Take any map $f\colon \partial\mathbb{I}^k \to X$ with $k < n+1$ and $\operatorname{diam}_\rho f(\partial\mathbb{I}^k) \leq 1/10$. If $\operatorname{diam}_\rho f(\partial\mathbb{I}^k) = 0$, then $f$ is a constant map and hence admits a constant extension. So we can assume that $\operatorname{diam}_\rho f(\partial\mathbb{I}^k) > 0$ and choose a real number $r < 1/2$ with

$$\operatorname{diam}_\rho f(\partial\mathbb{I}^k) < r < \tfrac{5}{4} \operatorname{diam}_\rho f(\partial\mathbb{I}^k) \leq \tfrac{1}{8}.$$

Then $\operatorname{diam} f(\partial\mathbb{I}^k) < \mathcal{B}_\rho(r)$. Since $\mathcal{B}_\rho(r) \ll_n \mathcal{D}_\rho(4r)$, the map $f$ admits a continuous extension $\bar{f} : \mathbb{I}^k \to X$ with

$$\operatorname{diam}_\rho \bar{f}(\mathbb{I}^k) < 4r < 5 \operatorname{diam}_\rho f(\partial\mathbb{I}^k).$$

If $X$ is locally contractible, then we can produce the pseudometric $\rho$ applying Theorem 22.1 to the relation $\mathcal{U} \ll_c \mathcal{V}$ on $\mathrm{cov}(X)$ defined by $\mathcal{U} \ll_c \mathcal{V}$ iff each set $U \in \mathcal{U}$ is contractible in some set $V \in \mathcal{V}$. ■

## 23. Lefschetz ANE$[n]$-spaces

In this section we study Lefschetz ANE$[n]$-spaces and prove Proposition 3.4. We recall that a topological space $X$ is called a *Lefschetz* ANE$[n]$ if for every open cover $\mathcal{U}$ of $X$ there is an open cover $\mathcal{V}$ of $X$ such that each partial $\mathcal{V}$-realization $f : L \to X$ of a simplicial complex $K$ with $\dim K \leq n$ can be extended to a full $\mathcal{U}$-realization $\tilde{f} : K \to X$ of $K$.

The items of Proposition 3.4 will be established in the next 12 lemmas (Lemma 23.7 and Lemma 23.11 are auxiliary for the proof of Lemma 23.8 and Lemma 23.12, respectively).

LEMMA 23.1. *A metrizable space $X$ is a Lefschetz* ANE$[n]$ *if and only if $X$ is an* ANE$[n]$ *for the class of metrizable spaces.*

*Proof.* For $n$ finite this lemma follows from Theorems 2.1 and 4.1 of [41, Ch. V] characterizing metrizable $\mathrm{LC}^{n-1}$-spaces as both ANE$[n]$'s and Lefschetz ANE$[n]$'s.

For $n = \infty$ this lemma is due to Lefschetz and can be found in [41, Theorem IV.4.1]. ■

LEMMA 23.2. *A regular* (*paracompact*) *space $X$ is a Lefschetz* ANE$[n]$ *for a finite $n$* (*if and*) *only if $X$ is* $\mathrm{LC}^{n-1}$.

In the realm of metrizable spaces this lemma has been proved in [41, V.4.1] but the proof remains true for regular (paracompact) spaces as well.

The following lemma (establishing the third item of Proposition 3.4) is also known and can be proved by a standard method due to J. Dugundji (see [41, §II.14]).

LEMMA 23.3. *Each convex subset of a* (*locally convex*) *linear topological space is a Lefschetz* ANE$[n]$ *for every $n$* (*a Lefschetz* ANE$[\infty]$).

Answering an old problem of Borsuk in the negative, R. Cauty [15] has constructed a $\sigma$-compact metrizable linear topological space $L$ which is not an ANR. By Lemma 23.1, Cauty's space is not ANE[$\infty$]. Thus, we have

LEMMA 23.4. *There exists a metrizable $\sigma$-compact linear topological space that fails to be a Lefschetz* ANE[$\infty$].

Following [12] we define a subset $A$ of a space $X$ to be a *neighborhood retract* of $X$ if $A$ is closed in $X$ and $A$ is a retract of some open set $U \subset X$ containing $A$.

LEMMA 23.5. *Each neighborhood retract of a Lefschetz* ANE[$n$]*-space is a Lefschetz* ANE[$n$]*-space.*

*Proof.* Let $X$ be a Lefschetz ANE[$n$]-space and $Y$ a neighborhood retract of $X$. Fix a retraction $r : O_Y \to Y$ of an open neighborhood $O_Y$ of $Y$ in $X$. Given an open cover $\mathcal{U}$ of $Y$ consider the open cover $\tilde{\mathcal{U}} = \{X\setminus Y\}\cup r^{-1}(\mathcal{U})$ of $X$, where $r^{-1}(\mathcal{U}) = \{r^{-1}(U) : U \in \mathcal{U}\}$. Since $X$ is a Lefschetz ANE[$n$], there is an open cover $\tilde{\mathcal{V}}$ of $X$ such that any partial $\tilde{\mathcal{V}}$-realization $f : L \to X$ of a simplicial complex $K$ with $\dim K \le n$ extends to a full $\tilde{\mathcal{U}}$-realization of $K$ in $X$. Consider the open cover $\mathcal{V} = \{Y \cap V : V \in \tilde{\mathcal{V}}\}$ of $Y$. The proof will be completed if we check that every partial $\mathcal{V}$-realization $f : L \to Y$ of a simplicial complex $K$ with $\dim K \le n$ can be extended to a full $\mathcal{U}$-realization $\tilde{f} : K \to Y$ of $K$ in $Y$. We may consider the map $f$ as a partial $\tilde{\mathcal{V}}$-realization of $K$ in $X$. Then the choice of the cover $\tilde{\mathcal{V}}$ guarantees that it extends to a full $\tilde{\mathcal{U}}$-realization $\tilde{f} : K \to X$ of $K$ in $X$. It easy to see that $\tilde{f}(K) \subset O_Y$ and $\bar{f} = r \circ \tilde{f} : K \to Y$ is a full $\mathcal{U}$-realization of $K$ in $Y$ extending the partial realization $f$. ∎

LEMMA 23.6. *A functionally open subspace of a Lefschetz* ANE[$n$]*-spaces is a Lefschetz* ANE[$n$]*-space.*

*Proof.* Given two open covers $\mathcal{U}, \mathcal{V}$ of a Lefschetz ANE[$n$]-space $X$ we write $\mathcal{V} \ll \mathcal{U}$ if $\mathcal{V} \prec \mathcal{U}$ and any partial $\mathcal{V}$-realization $f : L \to X$ of a simplicial complex $K$ with $\dim K \le n$ extends to a full $\mathcal{U}$-realization $\bar{f} : K \to X$ of $K$.

Let $U$ be a functionally open subspace of $X$ and $\mathcal{U}$ be an open cover of $U$. Since $U$ is functionally open, there is a sequence $(U_i)_{i\ge 0}$ of open subsets of $X$ such that $U = \bigcup_{i\in\omega} U_i$ and $\mathrm{cl}_X(U_i) \subset U_{i+1}$ for all $i$. It will also be convenient to put $U_i = \emptyset$ for negative $i$. For any $i < j$ consider the "ring" $U_i^j = U_j \setminus \overline{U_i}$ and note that $\mathcal{R} = \{U_{i-1}^{i+1} : i \in \omega\}$ is an open cover of $U$.

For every $i \in \omega$ consider the open cover

$$\mathcal{U}_i = \{X \setminus \overline{U}_{i+2}\} \cup (\mathcal{U} \wedge \mathcal{R})$$

of $X$ and find an open cover $\mathcal{V}_i$ of $X$ with $\mathcal{V}_i \ll \mathcal{U}_i$. The covers $\mathcal{V}_i$ can be chosen so that $\mathcal{V}_{i+1} \prec \mathcal{V}_i$ for all $i \in \omega$. Using the Lefschetz ANE[$n$]-property of $X$, find an open cover $\mathcal{W}_i$ of $X$ with $\mathcal{W}_i \ll \mathcal{V}_{i+3}$. Finally, take an open cover $\mathcal{V} \prec \mathcal{R}$ of $U$ such that $\{V \in \mathcal{V} : V \subset U_{i-4}^{i+4}\} \prec \mathcal{W}_i$ for all $i \in \omega$.

We claim that each partial $\mathcal{V}$-realization $f : L \to U$ of a simplicial complex $K$ with $\dim K \le n$ extends to a full $\mathcal{U}$-realization $\bar{f} : K \to U$ of $K$ in $U$. For every $i < j$ consider the subcomplex $K_i^j = \{\sigma \in K : f(\sigma^{(0)}) \subset U_i^j\}$ and observe that $K = \bigcup_{i\in\omega} K_{i-1}^{i+1}$ because $f$ is a partial $\mathcal{R}$-realization of $K$.

Next, we show that for every $i \in \omega$ the map $f$ restricted to $L \cap K^{i+3}_{i-3}$ is a partial $\mathcal{W}_i$-realization of the complex $K^{i+3}_{i-3}$ in $X$. Given any simplex $\sigma \in K^{i+3}_{i-3}$, note that $f(\sigma \cap L) \subset V$ for some $V \in \mathcal{V}$ because $f$ is a partial $\mathcal{V}$-realization of $K$. Since $f(\sigma^{(0)}) \subset V \cap U^{i+3}_{i-3}$ and $\mathcal{V} \prec \mathcal{R}$, we conclude that $V \subset U^{i+4}_{i-4}$. Now, the choice of $\mathcal{V}$ implies that $V \subset W$ for some $W \in \mathcal{W}_i$, which means that $f$ restricted to $L \cap K^{i+3}_{i-3}$ is a partial $\mathcal{W}_i$-realization of $K^{i+3}_{i-3}$ in $X$.

Since $\mathcal{W}_i \ll \mathcal{V}_{i+3}$, the partial $\mathcal{W}_i$-realization $f|(L \cap K^{i+3}_{i-3})$ extends to a full $\mathcal{V}_{i+3}$-realization $f_i : K^{i+3}_{i-3} \to X$ of $K^{i+3}_{i-3}$ in $X$ for every $i \in \omega$. Now let $L' = L \cup \bigcup_{i\in\omega} K^{5i+2}_{5i-2}$ and consider the map $g : L' \to X$ defined by $g(x) = f(x)$ if $x \in L$ and $g(x) = f_{5i}(x)$ if $x \in K^{5i+2}_{5i-2}$, $i \in \omega$. Because $K^{5i+2}_{5i-2} \subset K^{5i+3}_{5i-3}$ for every $i$ and $K^{5i+2}_{5i-2} \cap K^{5j+2}_{5j-2} = \emptyset$ for distinct $i, j$, the map $g$ is well-defined.

We claim that for every $i \in \omega$ the map $g$ restricted to $K^{i+1}_{i-2} \cap L'$ is a partial $\mathcal{V}_i$-realization of $K^{i+1}_{i-2}$ in $X$. Indeed, given any simplex $\sigma \in K^{i+1}_{i-2}$ there exists a number $j \in \omega$ with $\sigma \in K^{j+1}_{j-1}$ (recall that $K = \bigcup_{j\in\omega} K^{j+1}_{j-1}$). Observe that $|i-j| \le 1$ and there is a unique number $m \in \omega$ such that $5m - 2 \le j \le 5m + 2$. Then $5m - 3 \le j - 1 < j + 1 \le 5m + 3$. So, $\sigma \in K^{5m+3}_{5m-3}$. Since $L'$ is the union of $L$ and all $K^{5p+2}_{5p-2}$ and $p \in \omega$, we have

$$K^{5m+3}_{5m-3} \cap L' = (K^{5m+3}_{5m-3} \cap L) \cup (K^{5m+3}_{5m-3} \cap \cup_{p\in\omega} K^{5p+2}_{5p-2}) = (K^{5m+3}_{5m-3} \cap L) \cup K^{5m+2}_{5m-2}.$$

Consequently, $g(\sigma \cap L') = f_{5m}(\sigma \cap L') \subset f_{5m}(\sigma)$ with $f_{5m}(\sigma)$ being a subset of an element of $\mathcal{V}_{5m+3}$. Because $i \le j + 1 \le 5m + 3$, $\mathcal{V}_{5m+3} \prec \mathcal{V}_i$. Hence, $g$ restricted to $K^{i+1}_{i-2} \cap L'$ is a partial $\mathcal{V}_i$-realization of $K^{i+1}_{i-2}$ in $X$.

By the choice of $\mathcal{V}_i$, each partial realization $g|K^{i+1}_{i-2} \cap L'$ extends to a full $\mathcal{U}_i$-realization $g_i : K^{i+1}_{i-2} \to X$, $i \in \omega$. It follows from $K = \bigcup_{j\in\omega} K^{j+1}_{j-1}$ that $K = L' \cup \bigcup_{i\in\omega} K^{5i+4}_{5i+1}$. Moreover, for distinct $i, j$ the complexes $K^{5i+4}_{5i+1}$ and $K^{5j+4}_{5j+1}$ are disjoint. So, the map $\bar{f} : K \to U$, $\bar{f}(x) = g(x)$ for every $x \in L'$ and $\bar{f}(x) = g_{5i+3}(x)$ for every $x \in K^{5i+4}_{5i+1}$, is well-defined.

It remains to prove that $\bar{f}$ is a full $\mathcal{U}$-realization of $K$ in $U$. Take any simplex $\sigma \in K$. Since $K = \bigcup_{i\in\omega} K^{i+1}_{i-1}$ there is $i \in \omega$ such that either $\sigma \in K^{5i+2}_{5i-2}$ or $\sigma \in K^{5i+4}_{5i+1}$. In the first case

$$\bar{f}(\sigma) = f_{5i}(\sigma) \prec \mathcal{V}_{5i+3} \prec \mathcal{U}_{5i+3} = \{X \setminus \overline{U}_{5i+5}\} \cup (\mathcal{R} \wedge \mathcal{U}).$$

On the other hand, $\sigma \in K^{5i+2}_{5i-2}$ implies $f(\sigma^{(0)}) \subset U^{5i+2}_{5i-2} \subset U_{5i+5}$. Hence, $\bar{f}(\sigma) \prec \mathcal{R} \wedge \mathcal{U} \prec \mathcal{U}$. In the second case $\bar{f}(\sigma) = g_{5i+3}(\sigma) \prec \mathcal{U}_{5i+3}$ and $f(\sigma^{(0)}) \subset U^{5i+4}_{5i+1} \subset U_{5i+5}$, which again implies $\bar{f}(\sigma) \prec \mathcal{U}$. ∎

Lemma 23.7. *A space $X$ is a Lefschetz* ANE$[n]$ *if $X = X_0 \cup X_1$ is the union of two functionally open subspaces that are Lefschetz* ANE$[n]$*-spaces.*

*Proof.* The complements $F_i = X \setminus X_{1-i}$, $i \in \{0, 1\}$, are disjoint functionally closed subsets of $X$. So, we can find continuous functions $\xi_i : X \to [0, 1]$ such that $F_i = \xi_i^{-1}(0)$ for $i \in \{0, 1\}$. Letting

$$\xi(x) = \frac{\xi_0(x)}{\xi_0(x) + \xi_1(x)},$$

we obtain a continuous function $\xi : X \to [0, 1]$ such that $F_i = \xi^{-1}(i)$ for $i \in \{0, 1\}$. For a real number $t \in \mathbb{R}$ let $U_t = \xi^{-1}\big([0, t)\big)$. Define also a continuous pseudometric $\rho$ on $X$ by $\rho(x, y) = |\xi(x) - \xi(y)|$, $x, y \in X$.

To show that $X$ is a Lefschetz ANE$[n]$, fix any open cover $\mathcal{U}$ of $X$. Take an open cover $\mathcal{U}_1$ of $X_1$ such that $\mathcal{U}_1 \prec \mathcal{U}$ and $\operatorname{mesh}_\rho \mathcal{U}_1 < 1/8$. Since the space $X_1$ is a Lefschetz ANE$[n]$, there is an open cover $\mathcal{V}_1$ of $X_1$ such that each partial $\mathcal{V}_1$-realization $f : L \to X_1$ of a simplicial complex $K$ with $\dim K \le n$ extends to a full $\mathcal{U}_1$-realization $\tilde{f} : K \to X_1$. Next, take an open cover $\mathcal{U}_0$ of $X_0$ such that $\mathcal{U}_0 \prec \mathcal{U}$, $\operatorname{mesh}_\rho \mathcal{U}_0 < 1/8$, and $\mathcal{U}_0 \prec \mathcal{V}_1 \cup \{U_{1/8}\}$. Choose an open cover $\mathcal{V}_0$ of $X_0$ such that each partial $\mathcal{V}_0$-realization $f : L \to X_0$ of a simplicial complex $K$ with $\dim K \le n$ extends to a full $\mathcal{U}_0$-realization $\tilde{f} : K \to X_0$ of $K$ in $X_0$.

Finally choose an open cover $\mathcal{V}$ of $X$ such that

- $\operatorname{mesh}_\rho \mathcal{V} < 1/8$;
- $\mathcal{V} \prec \mathcal{V}_0 \cup \{X \setminus \overline{U}_{7/8}\}$;
- $\mathcal{V} \prec \mathcal{V}_1 \cup \{U_{1/8}\}$.

We claim that each partial $\mathcal{V}$-realization $f : L \to X$ of a simplicial complex $K$ with $\dim K \le n$ extends to a full $\mathcal{U}$-realization $\tilde{f} : K \to X$ of $K$ in $X$. Let

$$K_0 = \{\sigma \in K : f(\sigma^{(0)}) \subset U_{1/2}\},$$
$$K_0^+ = \{\sigma \in K : f(\sigma^{(0)}) \subset U_{3/4}\},$$
$$K_1 = \{\sigma \in K : f(\sigma^{(0)}) \subset X \setminus \overline{U}_{1/4}\}.$$

It follows from $\operatorname{mesh}_\rho \mathcal{V} < 1/8$ that $K = K_0 \cup K_1$ and $f(L \cap K_0^+) \subset U_{7/8}$. Since $\mathcal{V} \prec \mathcal{V}_0 \cup \{X \setminus \overline{U}_{7/8}\}$, the map $f$ restricted to $L \cap K_0^+$ is a partial $\mathcal{V}_0$-realization of $K_0^+$ in $X_0$. By the choice of the cover $\mathcal{V}_0$, this map extends to a full $\mathcal{U}_0$-realization $g : K_0^+ \to X_0$ of $K_0^+$ in $X_0$. Now let $L' = L \cup K_0$ and $h : L' \to X$ be a map defined by $h(x) = f(x)$ for $x \in L$ and $h(x) = g(x)$ for $x \in K_0$.

It can be shown that $h$ restricted to $L' \cap K_1$ is a partial $\mathcal{V}_1$-realization of $K_1$ in $X_1$. By the choice of the cover $\mathcal{V}_1$, this map extends to a full $\mathcal{U}_1$-realization $\tilde{h} : K_1 \to X_1$ of $K_1$ in $X_1$. Finally, define a full $\mathcal{U}$-realization $\tilde{f} : K \to X$ of $K$ in $X$ letting $\tilde{f}(x) = g(x)$ for $x \in K_0$ and $\tilde{f}(x) = \tilde{h}(x)$ for $x \in K_1$. ∎

Lemma 23.8. *A topological space $X$ is a Lefschetz* ANE$[n]$*-space provided $X$ has a uniform open cover by Lefschetz* ANE$[n]$*-spaces.*

*Proof.* Assume that $\mathcal{W}$ is a uniform open cover of $X$ by Lefschetz ANE$[n]$-spaces. Then there exists a continuous pseudometric $\rho$ on $X$ such that the cover $\{B_\rho(x,1) : x \in X\}$ refines $\mathcal{W}$. Consequently, there is a metric space $(M, \tilde{\rho})$ and a continuous map $p\colon X \to M$ with $\rho(x,y) = \tilde{\rho}(p(x), p(y))$ for all $x, y \in X$. So, without loss of generality, we can assume that every $W \in \mathcal{W}$ is of the form $W = p^{-1}(U_W)$, where $U_W$ is open in $(M, \tilde{\rho})$. Moreover, since the cover $\{U_W : W \in \mathcal{W}\} \in \operatorname{cov}(M)$ admits a $\sigma$-discrete open refinement, according to Lemma 23.6, we can additionally assume that the cover $\mathcal{W}$ is $\sigma$-discrete in $X$ and consists of functionally open subsets. Write $\mathcal{W} = \bigcup_{i\in\omega} \mathcal{W}_i$ as the countable union of discrete collections of functionally open sets. It is easy to see that the union $W_i = \bigcup \mathcal{W}_i$, being a topological sum of Lefschetz ANE$[n]$-spaces, is a Lefschetz ANE$[n]$-space. Hence, $X = \bigcup_{i\in\omega} W_i$ is a countable union of functionally open subspaces $W_i$ that are Lefschetz ANE$[n]$-spaces. Then Lemma 23.7 guarantees that for every $i \in \omega$ the space $X_i = \bigcup_{j\le i} W_j$ is a Lefschetz ANE$[n]$. Since the sets $X_i$ are functionally open, we can find an increasing sequence $(U_i)_{i\in\omega}$ of functionally open subspaces of $X$ such that $U_i \subset X_i$,

$\overline{U}_{i+1} \subset U_i$, $i \in \omega$, and $X = \bigcup_{i\in\omega} U_i$. Arguing as in Lemma 23.6, we prove that $X$ is a Lefschetz ANE[$n$]-space. ■

The following lemma due to Lefschetz is proved in [12, V.8.1] for compact metric spaces. The proof still remains true for arbitrary metric spaces.

LEMMA 23.9. *A metric space $X$ is a Lefschetz* ANE[$n$]*-space if for every $\varepsilon > 0$ there is $\delta$ such that each partial $\mathcal{D}_\rho(\delta)$-realization $f : L \to X$ of a simplicial complex $K$ with $\dim K \le n$ extends to a full $\mathcal{D}_\rho(\varepsilon)$-realization $\bar f : K \to X$ of $K$ in $X$.*

We recall that $\mathcal{D}_\rho(\varepsilon)$ stands for the cover of $X$ by all open subsets of $\rho$-diameter $< \varepsilon$. According to Lemma 22.1, the above lemma can be reversed:

LEMMA 23.10. *Let $\eta$ be a continuous pseudometric on a paracompact Lefschetz* ANE[$n$]*-space $X$. Then there is a continuous pseudometric $\rho \ge \eta$ such that any partial $\mathcal{D}_\rho(r/4)$-realization $f : L \to X$ of a simplicial complex $K$ with $r \in (0, 1/2]$ and $\dim K \le n$ extends to a full $\mathcal{D}_\rho(r)$-realization $\bar f : K \to X$ of $K$.*

To prove the last item of Proposition 3.4 we shall need the following technical lemma.

LEMMA 23.11. *Let $\rho$ be a continuous pseudometric on a topological space $X$ such that for every $r \in (0, 1/2]$ any partial $\mathcal{D}_\rho(r/4)$-realization of an $n$-dimensional simplicial complex into $X$ extends to a full $\mathcal{D}_\rho(r)$-realization. Suppose $K$ is a simplicial complex with $\dim K \le n$, $\varepsilon : K^{(0)} \to (0, 2^{-8})$ a function, and $f : K^{(0)} \to X$ a partial realization of $K$ such that*

$$\sup \varepsilon(\sigma^{(0)}) < 2 \inf \varepsilon(\sigma^{(0)}) \quad \text{and} \quad \operatorname{diam}_\rho f(\sigma^{(0)}) < 4 \sup \varepsilon(\sigma^{(0)})$$

*for every simplex $\sigma$ of $K$. Then $f$ extends to a full realization $\bar f : K \to X$ such that*

$$\operatorname{diam}_\rho \bar f(\sigma) < 2^{13} \sup \varepsilon(\sigma^{(0)})$$

*for every simplex $\sigma$ of $K$.*

*Proof.* For every $i < j$ consider the subcomplex

$$K_i^j = \{\sigma \in K : \varepsilon(\sigma^{(0)}) \subset (2^{-j}, 2^{-i})\}$$

and note that $K = K_8^\infty = \bigcup_{i\ge 9} K_{i-1}^{i+1}$ because $\sup \varepsilon(\sigma^{(0)}) < 2 \inf \varepsilon(\sigma^{(0)})$ for all $\sigma \in K$.

According to our hypothesis, for every simplex $\sigma$ of $K_i^j$ we have

$$\operatorname{diam}_\rho f(\sigma^{(0)}) \le 4 \sup \varepsilon(\sigma^{(0)}) < 4 \cdot 2^{-i} = 2^{-i+2}.$$

This means that the map $f$ restricted to $K^{(0)} \cap K_i^j$ is a partial $\mathcal{D}_\rho(2^{-i+2})$-realization of the complex $K_i^j$. In particular, $f|K^{(0)} \cap K_{i-3}^{i+3}$ is a partial $\mathcal{D}_\rho(2^{-i+5})$-realization of $K_{i-3}^{i+3}$ for all $i \ge 8$. Hence, by the choice of the pseudometric $\rho$, $f|K^{(0)} \cap K_{i-3}^{i+3}$ extends to a full $\mathcal{D}_\rho(2^{-i+7})$-realization $g_i : K_{i-3}^{i+3} \to X$ of $K_{i-3}^{i+3}$, $i \ge 8$. Let

$$L = K^{(0)} \cup \bigcup_{m=2}^{\infty} K_{5m-2}^{5m+2}$$

and consider the map $g : L \to X$ defined by $g(x) = f(x)$ if $x \in K^{(0)}$ and $g(x) = g_{5m}(x)$ if $x \in K_{5m-2}^{5m+2}$. Since $K_{5i-2}^{5i+2} \cap K_{5j-2}^{5j+2} = \emptyset$ for distinct $i, j$, the map $g$ is well-defined.

We claim that for every $i \ge 10$ the map $g$ restricted to $L \cap K_{i+1}^{i+4}$ is a partial $\mathcal{D}_\rho(2^{-i+7})$-realization of $K_{i+1}^{i+4}$. Indeed, any simplex $\sigma$ of $K_{i+1}^{i+4}$ lies in $K_{j-1}^{j+1}$ for some $j \in \omega$ with

$i+1<j<i+4$. Find a number $m\ge 2$ such that $5m-2\le j\le 5m+2$. Then $\sigma\subset K^{5m+3}_{5m-3}$ and $g$ coincides with $g_{5m}$ on $\sigma\cap L$. Since $g_{5m}$ is a full $\mathcal{D}_\rho(2^{-5m+8})$-realization of $K^{5m+3}_{5m-3}$, we have

$$\operatorname{diam}_\rho g(\sigma\cap L)\le \operatorname{diam}_\rho g_{5m}(\sigma)<2^{-5m+7}\le 2^{-j+9}\le 2^{-i+7}.$$

Now we can extend the partial $\mathcal{D}_\rho(2^{-i+7})$-realization $g|L\cap K^{i+4}_{i+1}\to X$ of $K^{i+4}_{i+1}$ to a full $\mathcal{D}_\rho(2^{-i+9})$-realization $f_i:K^{i+4}_{i+1}\to X$. It follows that $K=L\cup\bigcup_{m=2}^\infty K^{5m+4}_{5m+1}$. Since for distinct $i,j$ the complexes $K^{5i+4}_{5i+1}$ and $K^{5j+4}_{5j+1}$ are disjoint, the map $\bar f:K\to X$ defined by $\bar f(x)=g(x)$ if $x\in L$ and $\bar f(x)=f_{5i}(x)$ if $x\in K^{5i+4}_{5i+1}$ is well-defined.

It remains to check that $\operatorname{diam}_\rho \bar f(\sigma)<2^{13}\sup\varepsilon(\sigma^{(0)})$ for every simplex $\sigma$ of $K$. To this end, choose $i\ge 9$ with $\sigma\subset K^{i+1}_{i-1}$, which implies $\varepsilon(\sigma^{(0)})\subset(2^{-i-1},2^{-i+1})$. There exists $m\ge 2$ such that either $5m+1\le i-1\le i+1\le 5m+4$ or $5m-2\le i-1\le i+1\le 5m+2$. Then

$$\operatorname{diam}_\rho \bar f(\sigma)=\operatorname{diam}_\rho f_{5m}(\sigma)\le 2^{-5m+9}\le 2^{-i+12}<2^{13}\sup\varepsilon(\sigma^{(0)})$$

in the first case, and

$$\operatorname{diam}_\rho \bar f(\sigma)=\operatorname{diam}_\rho g_{5m}(\sigma)\le 2^{-5m+7}\le 2^{-i+8}<2^{9}\sup\varepsilon(\sigma^{(0)})$$

in the second case. ■

Finally, we can prove the last item of Proposition 3.4.

LEMMA 23.12. *Let $g:X\to Y$ be a map from a paracompact Lefschetz* ANE$[n]$*-space to a metric space $Y$. Then there exist a metrizable Lefschetz* ANE$[n]$*-space $\tilde X$ and maps $\pi:X\to\tilde X$, $\tilde g:\tilde X\to Y$ such that $g=\tilde g\circ\pi$.*

*Proof.* The metric of $Y$ induces a continuous pseudometric $\eta$ on $X$ defined by $\eta(x,y)=\operatorname{dist}(g(x),g(y))$, $x,y\in X$. By Lemma 23.10, there is a continuous pseudometric $\rho\ge\eta$ such that any partial $\mathcal{D}_\rho(r/4)$-realization $f:L\to X$ of a simplicial complex $K$, where $r\in(0,1/2]$ and $\dim K\le n$, extends to a full $\mathcal{D}_\rho(r)$-realization $\bar f:K\to X$ of $K$ in $X$. Consider the metric space $(\tilde X,\tilde\rho)$, where $\tilde X=X/_\sim$ is the quotient set with respect to the equivalence relation $x\sim y$ iff $\rho(x,y)=0$ and $\tilde\rho$ is the quotient metric. The metric topology on $\tilde X$ may not coincide with the quotient topology, but the quotient map $\pi:X\to\tilde X$ is continuous. Since $\rho\ge\eta$, the map $g:X\to Y$ induces a non-expanding (and hence continuous) map $\tilde g:\tilde X\to Y$ such that $g=\tilde g\circ\pi$.

It remains to prove that the metric space $(\tilde X,\tilde\rho)$ is a Lefschetz ANE$[n]$. According to Lemma 23.1, this is equivalent to showing that $\tilde X$ is an ANE$[n]$-space for metrizable spaces. Let $A$ be a metrizable space with $\dim A\le n$ and $h:B\to\tilde X$ be a continuous map defined on a closed subspace $B$ of $A$. Using the Hausdorff Theorem [33, 4.5.20(c)] on extension of metrics, we can choose a metric $d$ on $A$ turning $f:B\to\tilde X$ into a non-expanding map. Let $\tilde h:B\to X$ be any (not necessarily continuous) function with $\pi\circ\tilde h=h$. Then $\tilde h$ is still non-expanding with respect to the pseudometric $\rho$ on $X$.

To construct a neighborhood extension $\bar h$ of $h$ we use the classical approach of Dugundji (see [28]). Choose a locally finite open cover $\mathcal{U}$ of $A\setminus B$ of order $\le n+1=\dim A+1$ such that for every $U\in\mathcal{U}$,

$$\operatorname{diam} U<\tfrac12\operatorname{dist}(U,B)\quad\text{and}\quad \operatorname{dist}(U,b_U)<\tfrac32\operatorname{dist}(U,B)$$

for some point $b_U \in B$. Consider the subcollection $\mathcal{V} = \{U \in \mathcal{U} : \operatorname{dist}(U, B) < 2^{-8}\}$ and note that the union $O(B) = \bigcup \mathcal{V}$ is an open neighborhood of $B$ in $A$. Since the cover $\mathcal{U}$ has order $\le n+1$, the nerve $K = N(\mathcal{V})$ of $\mathcal{V}$ has $\dim K \le n$. Let $\lambda : O(B) \to K$ be the canonical map induced by some partition of unity $\{\lambda_V : O(B) \to [0,1]\}_{V\in\mathcal{V}}$ subordinated to $\mathcal{V}$. Next, define two functions $f : K^{(0)} \to X$ and $\varepsilon : K^{(0)} \to (0, 2^{-8})$ letting $f(U) = \tilde{h}(b_U)$ and $\varepsilon(U) = \operatorname{dist}(U, B)$ for every $U \in K^{(0)} = \mathcal{V}$.

We claim that the functions $f$ and $\varepsilon$ satisfy the conditions of Lemma 23.11. Take any simplex $\sigma$ in $K$ and fix a point $x \in \bigcap_{U\in\sigma^{(0)}} U$. Choose two vertices $V, W \in \sigma^{(0)}$ with $\varepsilon(V) = \inf \varepsilon(\sigma^{(0)})$ and $\varepsilon(W) = \sup \varepsilon(\sigma^{(0)})$ and observe that

$$\begin{aligned}\sup \varepsilon(\sigma^{(0)}) &= \varepsilon(W) = \operatorname{dist}(W, B) \le \operatorname{dist}(x, B) \le \operatorname{dist}(V, B) + \operatorname{diam} V \\ &\le \operatorname{dist}(V, B) + \tfrac{1}{2}\operatorname{dist}(V, B) < 2\operatorname{dist}(V, B) = 2\varepsilon(V) = 2\inf \varepsilon(\sigma^{(0)}).\end{aligned}$$

Observe also that for every vertex $U \in \sigma^{(0)}$ we have

$$\operatorname{dist}(x, b_U) \le \operatorname{dist}(U, b_U) + \operatorname{diam} U \le \tfrac{3}{2}\operatorname{dist}(U, B) + \tfrac{1}{2}\operatorname{dist}(U, B) = 2\varepsilon(U) \le 2\sup \varepsilon(\sigma^{(0)}).$$

Thus,

$$\operatorname{diam}_d\{b_U : U \in \sigma^{(0)}\} \le 4\sup \varepsilon(\sigma^{(0)}).$$

Taking into account that the map $\tilde{h}$ is non-expanding, we conclude that

$$\operatorname{diam}_\rho f(\sigma^{(0)}) = \operatorname{diam}_\rho\{\tilde{h}(b_U) : U \in \sigma^{(0)}\} \le \operatorname{diam}_d\{b_U : U \in \sigma^{(0)}\} \le 4\sup \varepsilon(\sigma^{(0)}).$$

Now, we can apply Lemma 23.11 to find a full realization $\bar{f} : K \to X$ of $K$ such that

$$\operatorname{diam}_\rho \bar{f}(\sigma) \le 2^{13}\sup \varepsilon(\sigma^{(0)})$$

for every simplex $\sigma$ of $K$. Finally, define a map $\bar{h} : O(B) \to \tilde{X}$ letting $\bar{h}(x) = h(x)$ for $x \in B$ and $\bar{h}(x) = \pi \circ \bar{f} \circ \lambda(x)$ for $x \in O(B) \setminus B$. It is clear that $\bar{h}$ is continuous at the points of $O(B) \setminus B$. It remains to check that $\bar{h}$ is also continuous at each point $b \in B$. Given an arbitrary positive $\delta < 3^{-1}2^{-8}$, it suffices to show that $\tilde{\rho}(\bar{h}(x), h(b)) < (2^{14}+5)\delta$ for any point $x \in O(B)$ with $d(x, b) < \delta$. To this end, fix $x \in O(B)$ with $d(x, b) < \delta$ and consider the simplex $\sigma = \{U \in \mathcal{V} : \lambda_U(x) > 0\}$ of $K$ whose geometric realization contains the point $\lambda(x)$. Observe that

$$\inf \varepsilon(\sigma^{(0)}) = \inf_{U\in\sigma} \operatorname{dist}_d(U, B) \le d(x, b) < \delta \quad\text{and}\quad \sup \varepsilon(\sigma^{(0)}) < 2\delta.$$

For any $U \in \sigma$ the choice of $b_U$ guarantees that

$$\begin{aligned}\operatorname{dist}(x, b_U) &\le \operatorname{dist}(U, b_U) + \operatorname{diam} U < \tfrac{3}{2}\operatorname{dist}(U, B) + \tfrac{1}{2}\operatorname{dist}(U, B) \\ &= 2\varepsilon(U) \le 2\sup \varepsilon(\sigma^{(0)}) < 4\inf \varepsilon(\sigma^{(0)}) < 4\delta.\end{aligned}$$

Then

$$\begin{aligned}\tilde{\rho}(\bar{h}(x), h(b)) &= \rho(\bar{f} \circ \lambda(x), \tilde{h}(b)) \le \rho(\bar{f} \circ \lambda(x), \tilde{h}(b_U)) + \rho(\tilde{h}(b_U), \tilde{h}(b)) \\ &\le \operatorname{diam}_\rho \bar{f}(\sigma) + d(b_U, b) \le 2^{13}\sup \varepsilon(\sigma^{(0)}) + d(x, b) + d(x, b_U) \\ &\le 2^{13} \cdot 2 \cdot \delta + \delta + 4\delta = (2^{14}+5)\delta. \ \blacksquare\end{aligned}$$

## 24. Density of simplicially factorizable maps in function spaces

The aim of this section is to prove Proposition 3.5. The first item of this proposition will be derived from

LEMMA 24.1. *Let $f : Z \to X$ be a map from a space $Z$ into a Lefschetz* ANE$[n]$*-space $X$ and $O(f)$ be its neighborhood in $C(Z, X)$. Then there is an open cover $\mathcal{V}$ of $Z$ such that for any $\mathcal{V}$-map $\alpha : Z \to P$ into a paracompact space $P$ with* $\dim P \le n$ *there is a map $\beta : G \to X$ with $\beta \circ \alpha \in O(f)$, where $G \subset P$ is an open neighborhood of the closure of $\alpha(Z)$ in $P$.*

*Proof.* Since $O(f)$ is a neighborhood of $f$ in $C(Z, X)$, there exist a continuous pseudometric $\rho$ on $X$ and a positive function $\varepsilon : Z \to (0, 1]$ such that $B_\rho(f, \varepsilon) = \{g \in C(Z, X) : \rho(f, g) < \varepsilon\} \subset O(f)$. We may assume that $\varepsilon(x) < 1$ for all $x \in X$. Using the Lefschetz ANE$[n]$-property of $X$ construct inductively a sequence $(\mathcal{U}_k)_{k\in\omega}$ of open covers of $X$ such that:

- $\mathrm{mesh}_\rho\, \mathcal{U}_k < 2^{-k-2}$;
- $\mathcal{U}_{k+1} \prec \mathcal{U}_k$;
- each partial $\mathcal{U}_{k+1}$-realization of a simplicial complex $K$ with $\dim K \le n$ extends to a full $\mathcal{U}_k$-realization of $K$.

Let $\mathcal{V} = \bigcup_{k\in\omega} \mathcal{V}_k$ be an open cover of $Z$, where the $\mathcal{V}_k$ are open families in $Z$ satisfying:

- $\bigcup_{V\in\mathcal{V}_k} V = \varepsilon^{-1}((2^{-k-1}, 2^{-k+1}))$, $k \in \omega$;
- $f(\mathcal{V}_k) \prec \mathcal{U}_{k+8}$ and $\mathrm{mesh}_\rho\, f(\mathcal{V}_k) < 2^{-k-2}$ for every $k \in \omega$.

We claim that the cover $\mathcal{V}$ of $Z$ satisfies our requirements. Take any $\mathcal{V}$-map $\alpha : Z \to P$ to a paracompact space $P$ with $\dim P \le n$ and find a locally finite open cover $\mathcal{W}$ of $P$ such that $\alpha^{-1}(St(\mathcal{W})) \prec \mathcal{V}$. Since $\dim P \le n$, we may additionally assume that the order of $\mathcal{W}$ is $\le n + 1$. Hence, the nerve $N(\mathcal{W})$ of $\mathcal{W}$ is of dimension $\le n$. Replacing $P$ by a suitable closed neighborhood of $\overline{\alpha(Z)}$, we can also assume that each set $W \in \mathcal{W}$ meets the image $\alpha(Z)$. So, there exists a point $z_W \in \alpha^{-1}(W)$ for each $W \in \mathcal{W}$.

Let $\{\lambda_W : P \to [0, 1]\}_{W\in\mathcal{W}}$ be a partition of unity subordinated to the cover $\mathcal{W}$ and let $\lambda = (\lambda_W) : P \to N(\mathcal{W})$ be the canonical map. It suffices to construct a map $\beta : N(\mathcal{W}) \to X$ with $\rho(f, \beta \circ \lambda \circ \alpha) < \varepsilon$.

Denote by $K$ the nerve $N(\mathcal{W})$, and for any integers $i < j$ let

$$K_i^j = \big\{\sigma = \{W_1, \ldots, W_m\} \in K : \varepsilon(\alpha^{-1}(\textstyle\bigcup \sigma)) \subset (2^{-j}, 2^{-i})\big\},$$

where $\bigcup \sigma = \bigcup_{k=1}^m W_k$. It follows from $\alpha^{-1}(St(\mathcal{W})) \prec \mathcal{V} \prec \{\varepsilon^{-1}(2^{-i-1}, 2^{-i+1}) : i \in \omega\}$ that $K = \bigcup_{i\in\omega} K_{i-1}^{i+1}$.

Letting $r(\{W\}) = f(z_W)$ for $W \in \mathcal{W}$, we define a partial realization $r : K^{(0)} \to X$ of the complex $K = N(\mathcal{W})$. We shall extend this partial realization to a full realization $\beta : K \to X$ of $K$ in $X$ such that for every $i \in \omega$ the map $\beta$ restricted to $K_{i-1}^{i+1}$ is a full $\mathcal{U}_i$-realization of $K_{i-1}^{i+1}$ in $X$.

First, we show that for every $i \in \omega$ the map $r$ restricted to $K^{(0)} \cap K_{i-3}^{i+3}$ is a partial $\mathcal{U}_{i+5}$-realization of the complex $K_{i-3}^{i+3}$ in $X$. Given any simplex $\sigma \in K_{i-3}^{i+3}$, note that $\alpha^{-1}(\bigcup \sigma) \prec \alpha^{-1}(St(\mathcal{W})) \prec \mathcal{V}$ and hence $\alpha^{-1}(\bigcup \sigma) \subset V_\sigma$ for some $V_\sigma \in \mathcal{V}$. So, $\varepsilon(V_\sigma)$

intersects the interval $(2^{-i-3},2^{-i+3})$. Since $\bigcup_{k=i-2}^{i+2}\{V : V\in\mathcal{V}_k\}=\varepsilon^{-1}((2^{-i-3},2^{-i+3}))$ and $\varepsilon(\mathcal{V}_k)\subset(2^{-k-1},2^{-k+1})$ for every $V\in\mathcal{V}_k$ and $k\in\omega$, $\varepsilon(V_\sigma)\subset(2^{-i-4},2^{-i+4})$. Then $V_\sigma\in\bigcup_{k=i-3}^{i+3}\mathcal{V}_k$ and, by the choice of the covers $\mathcal{U}_k$ and $\mathcal{V}$, we have $f(V_\sigma)\prec\mathcal{U}_{i+5}$. Consequently,

$$r(\sigma^{(0)})=f(\{z_W : W\in\sigma\})\subset f\big(\alpha^{-1}(\textstyle\bigcup\sigma)\big)\subset f(V_\sigma)\prec\mathcal{U}_{i+5}.$$

Hence, $r$ restricted to $K_{i-3}^{i+3}\cap K^{(0)}=(K_{i-3}^{i+3})^{(0)}$ is a partial $\mathcal{U}_{i+5}$-realization of the complex $K_{i-3}^{i+3}$ in $X$. So, by the choice of $\mathcal{U}_{i+5}$, $r|(K_{i-3}^{i+3})^{(0)}$ can be extended to a full $\mathcal{U}_{i+4}$-realization $r_i : K_{i-3}^{i+3}\to X$ of $K_{i-3}^{i+3}$. Let $L=K^{(0)}\cup\bigcup_{i\in\omega}K_{5i-2}^{5i+2}$ and consider the map $\tilde r : L\to X$ defined by $\tilde r(x)=r(x)$ if $x\in K^{(0)}$ and $\tilde r(x)=r_{5i}(x)$ if $x\in K_{5i-2}^{5i+2}$. This map is well-defined because $K_{5i-2}^{5i+2}\subset K_{5i-3}^{5i+3}$ and $K_{5i-2}^{5i+2}\cap K_{5j-2}^{5j+2}=\emptyset$ for distinct $i,j$.

Let us show that for every $i\in\omega$ the map $\tilde r$ restricted to $K_{i-2}^{i+1}\cap L$ is a partial $\mathcal{U}_{i+1}$-realization of $K_{i-2}^{i+1}$ in $X$. Indeed, given any simplex $\sigma\in K_{i-2}^{i+1}$ there exists a number $j\in\omega$ with $\sigma\in K_{j-1}^{j+1}$ (recall that $K=\bigcup_{j\in\omega}K_{j-1}^{j+1}$). Observe that $|i-j|\le 1$ and there is a unique number $m\in\omega$ such that $5m-2\le j\le 5m+2$. Then $5m-3\le j-1<j+1\le 5m+3$. So, $\sigma\in K_{5m-3}^{5m+3}$. Since $L$ is the union of $K^{(0)}$ and all $K_{5k-2}^{5k+2}$, $k\in\omega$, $K_{5m-3}^{5m+3}\cap L=(K_{5m-3}^{5m+3}\cap K^{(0)})\cup(K_{5m-3}^{5m+3}\cap\bigcup_{k\in\omega}K_{5k-2}^{5k+2})=(K_{5m-3}^{5m+3})^{(0)}\cup K_{5m-2}^{5m+2}$. Consequently, $\tilde r(\sigma\cap L)=r_{5m}(\sigma\cap L)\subset r_{5m}(\sigma)\prec\mathcal{U}_{5m+4}$ (recall that $r_{5m}$ is a full $\mathcal{U}_{5m+4}$-realization of $K_{5m-3}^{5m+3}$). Because $i+1\le j+2\le 5m+4$, $\mathcal{U}_{5m+4}\prec\mathcal{U}_{i+1}$. Hence, $\tilde r$ restricted to $K_{i-2}^{i+1}\cap L$ is a partial $\mathcal{U}_{i+1}$-realization of $K_{i-2}^{i+1}$ in $X$.

Therefore, $\tilde r|K_{i-2}^{i+1}\cap L$ extends to a full $\mathcal{U}_i$-realization $\tilde r_i : K_{i-2}^{i+1}\to X$ of $K_{i-2}^{i+1}$. Finally define a map $\beta : K\to X$ by $\beta(x)=\tilde r(x)$ for every $x\in L$ and $\beta(x)=\tilde r_{5i+3}(x)$ for every $x\in K_{5i+1}^{5i+4}$. Since $K=L\cup\bigcup_{i\in\omega}K_{5i+1}^{5i+4}$ and the complexes $K_{5i+1}^{5i+4}$ and $K_{5j+1}^{5j+4}$ are disjoint for distinct $i,j$, the map $\beta$ is well-defined. It is easy to check that for every $i\in\omega$ the map $\beta$ restricted to $K_{i-1}^{i+1}$ is a full $\mathcal{U}_i$-realization of the complex $K_{i-1}^{i+1}$ in $X$.

We claim that the map $\gamma=\beta\circ\lambda : P\to X$ has the required property: $\rho(\gamma\circ\alpha,f)<\varepsilon$. Indeed, take any point $z\in Z$ and put $a=\alpha(z)\in P$, $b=\lambda(a)\in K$. Let $\sigma=\{W\in\mathcal{W} : a\in W\}$ be the simplex of $K=N(\mathcal{W})$ whose geometric realization contains the point $b$. Since $\alpha^{-1}(St(\mathcal{W}))\prec\mathcal{V}$, there is a set $V_0\in\mathcal{V}$ with $\alpha^{-1}(\bigcup\sigma)\subset V_0$. By the choice of the cover $\mathcal{V}$ there is a number $k\in\omega$ such that $\varepsilon(V_0)\subset(2^{-k-1},2^{-k+1})$. For this $k$ we also have $\sigma\in K_{k-1}^{k+1}$ and $\operatorname{diam}_\rho f(V_0)<2^{-k-2}$. Since $\operatorname{mesh}_\rho\mathcal{U}_k<2^{-k-2}$ and $\beta(\sigma)\prec\mathcal{U}_k$, $\operatorname{diam}_\rho\beta(\sigma)<2^{-k-2}$.

Take any vertex $W\in\sigma$ of $\sigma$ and let $w\in\sigma^{(0)}$ be its geometric realization. Note that $\{z,z_W\}\subset\alpha^{-1}(W)\subset V_0$ and hence $\rho(f(z),f(z_W))\le\operatorname{diam}_\rho f(V_0)<2^{-k-2}$. Since $\beta(w)=r(w)=f(z_W)$ and $\gamma\circ\alpha(z)=\beta(b)$, we obtain

$$\begin{aligned}\rho(f(z),\gamma\circ\alpha(z))&\le\rho(f(z),f(z_W))+\rho(\beta(w),\beta(b))\le 2^{-k-2}+\operatorname{diam}_\rho\bar r(\sigma)\\&<2^{-k-2}+2^{-k-2}=2^{-k-1}\le\inf\varepsilon(V_0)\le\varepsilon(z).\ \blacksquare\end{aligned}$$

We are now able to prove the first item of Proposition 3.5.

LEMMA 24.2. *If $X$ is a paracompact space and $Y$ is a Lefschetz* ANE$[k]$*-space with $k=\dim X$, then the simplicially factorizable maps $g\in C(X,Y)$ form a dense set in $C(X,Y)$.*

*Proof.* Take any map $f\in C(X,Y)$ and a neighborhood $O(f)$ of $f$ in $C(X,Y)$. By Lemma 24.1, there exists an open cover $\mathcal{V}$ of $X$ such that for any $\mathcal{V}$-map $\alpha : X\to K$

into a paracompact space $K$ we can find a map $\beta : G \to Y$ defined on an open neighborhood $G$ of $\alpha(X)$ in $K$ with $\beta \circ \alpha \in O(f)$. Because of the paracompactness of $X$, the cover $\mathcal{V}$ can be assumed locally finite and of order $\leq k+1$, where $k = \dim X$. Take any partition of unity $\{\lambda_V : X \to [0,1]\}_{V\in\mathcal{V}}$ subordinated to the cover $\mathcal{V}$ and consider the canonical map $\lambda : X \to N(\mathcal{V})$ to the nerve of $\mathcal{V}$. It is clear that $\lambda$ is a $\mathcal{V}$-map. Now, the choice of the cover $\mathcal{V}$ implies the existence of a map $\beta : G \to Y$ defined on an open neighborhood $G$ of $\alpha(X)$ in $N(\mathcal{V})$ such that $\beta \circ \alpha \in O(f)$. The space $G$, being an open subspace of a simplicial complex, is homeomorphic to a simplicial complex [49, p. 473]. So, $\beta \circ \alpha$ is a simplicially factorizable map, which completes the proof. ∎

Our final lemma in this section provides a proof of the second item of Proposition 3.5.

LEMMA 24.3. *Simplicially factorizable maps from a paracompact $C$-space $X$ into a locally contractible paracompact space $Y$ form a dense subset in $C(X,Y)$.*

*Proof.* We fix a map $f \in C(X,Y)$ and its neighborhood $B_\rho(f,\varepsilon)$ in $C(X,Y)$, where $\varepsilon\colon X \to (0,1]$ is a continuous function and $\rho$ is a continuous pseudometric on $Y$. By Lemma 22.2, we can suppose that $\rho$ has the following property: for every $y \in Y$ and every $r \in (0,1/2]$ the ball $B_\rho(y,r/4)$ is contractible in $B_\rho(y,r)$ to the point $y$. For any point $x\in X$ choose a neighborhood $G_x \subset X$ of $x$ such that $\sup\varepsilon(G_x) < 2\varepsilon_x$, where $\varepsilon_x = \inf\varepsilon(G_x) > 0$. Now, for every $n \geq 1$ consider the open families $\alpha'_n = \{B_\rho(f(x),\varepsilon_x/12^{n-1}) : x \in X\}$ and $\beta'_n = \{B_\rho(f(x),3\varepsilon_x/12^n) : x \in X\}$ in $Y$, and the open covers $\alpha_n = \{G_x \cap f^{-1}(B_\rho(f(x),\varepsilon_x/12^{n-1})) : x \in X\}$ and $\beta_n = \{G_x \cap f^{-1}(B_\rho(f(x),3\varepsilon_x/12^n)) : x \in X\}$ of $X$.

For simplicity, we denote the balls $B_\rho(f(x),\varepsilon_x/12^{n-1})$ and $B_\rho(f(x),3\varepsilon_x/12^n)$ by $U_n(x)$ and $V_n(x)$, respectively. By the choice of the pseudometric $\rho$ , for every $x \in X$ and every $n \geq 1$ there exists a homotopy $H_{x,n}\colon V_n(x) \times \mathbb{I} \to U_n(x)$ such that:

- each $H_{x,n}$ contracts $V_n(x)$ in $U_n(x)$ to the point $f(x)$, i.e., $H_{x,n}(y,0) = y$ and $H_{x,n}(y,1) = f(x)$ for all $y \in V_n(x)$.

Since $X$ is a paracompact $C$-space, there exists a sequence $(\mu_n)_{n\geq 3}$ of open disjoint families in $X$ such that

- each $\mu_n$ refines $\beta_n$;
- $\mu = \bigcup_{n\geq 3}\mu_n$ is a locally finite open cover of $X$.

For every $n \geq 3$ and $W \in \mu_n$ fix a point $x_W \in X$ such that $W \subset G_{x_W} \cap f^{-1}(V_n(x_W))$.

CLAIM 24.4. *Let $W \in \mu_n$ and $W' \in \mu_{n'}$, where $n < n'$. If $W \cap W' \neq \emptyset$, then*

$$\varepsilon_{x_{W'}} \leq 2\varepsilon_{x_W} \quad \textit{and} \quad \rho(f(x_{W'}), f(x_W)) \leq \frac{4\varepsilon_{x_W}}{12^n}.$$

*Proof.* Indeed, suppose that $z \in W \cap W'$. Then $z \in G_{x_W} \cap G_{x_{W'}}$. Hence, $\varepsilon_{x_{W'}} \leq \varepsilon(z) \leq \sup\,\varepsilon(G_{x_W}) < 2\varepsilon_{x_W}$.

To show the second part of this claim, observe that $z \in f^{-1}(V_n(x_W)) \cap f^{-1}(V_{n'}(x_{W'}))$. Thus, $f(z) \in V_n(x_W) \cap V_{n'}(x_{W'})$. The last inclusion implies that $\rho(f(z), f(x_W)) \leq$

$3\varepsilon_{x_W}/12^n$ and $\rho(f(z), f(x_{W'})) \le 3\varepsilon_{x_{W'}}/12^{n'}$. Since $\varepsilon_{x_{W'}} \le 2\varepsilon_{x_W}$ and $n' \ge n+1$, we finally obtain

$$\rho\big(f(x_W), f(x_{W'})\big) \le \frac{3\varepsilon_{x_W}}{12^n} + \frac{3\varepsilon_{x_{W'}}}{12^{n'}} \le \frac{3\varepsilon_{x_W}}{12^n} + \frac{6\varepsilon_{x_W}}{12^{n+1}} < \frac{4\varepsilon_{x_W}}{12^n}. \ ■$$

Now, consider the nerve $N(\mu)$ of the cover $\mu$ and the canonical map $\theta\colon X \to |N(\mu)|$ generated by some partition of unity subordinated to $\mu$. Observe that any simplex $\sigma = \{W_0, W_1, \ldots, W_k\}$ from $N(\mu)$, where $W_i \in \mu_{n_i}$, can be ordered so that $n_0 < n_1 < \cdots < n_k$. This is possible because $\bigcap_{i=0}^k W_i \ne \emptyset$, so $n_i \ne n_j$ for $i \ne j$ (recall that each $\mu_n$ is disjoint). We are going to define a map $g\colon |N(\mu)| \to Y$ with $g \circ \theta \in B_\rho(f, \varepsilon)$. To this end, we define by induction maps $g_k\colon |N(\mu)^{(k)}| \to Y$ such that $g_k\big||N(\mu)^{(k-1)}| = g_{k-1}$ and for every simplex $\sigma = \{W_0, W_1, \ldots, W_k\} \subset N(\mu)^{(k)}$ and its face $\sigma_0 = \sigma \setminus \{W_0\}$ we get:

- $f(W_0) \cup g_k(|\sigma|) \subset U_{n_0-1}(x_{W_0})$;
- $g_k(|\sigma_0|) \subset V_{n_0-1}(x_{W_0})$;
- for any $z \in |\sigma_0|$ and $t \in [0,1]$ we get $g_k\big(t\{W_0\} + (1-t)z\big) = H_{x_{W_0}, n_0-1}(g_{k-1}(z), t)$.

To start the inductive construction, define a map $g_0\colon |N(\mu)^{(0)}| \to Y$ by $g_0(\{W\}) = f(x_W)$ for each vertex $W \in \mu$ of $N(\mu)$. Suppose that for some $k \in \omega$ a map $g_k$ satisfying the above conditions has been defined. Fix a simplex $\sigma = \{W_0, W_1, \ldots, W_k, W_{k+1}\} \subset N(\mu)^{(k+1)}$ with $W_i \in \mu_{n_i}$, $i \in \{0, \ldots, k+1\}$. Then its combinatorial boundary $\sigma \cap N(\mu)^{(k)}$ consists of the simplexes $\sigma_i = \{W_0, \ldots, W_{i-1}, W_{i+1}, \ldots, W_{k+1}\}$, $1 \le i \le k+1$, and the simplex $\sigma_0 = \{W_1, W_2, \ldots, W_{k+1}\}$.

Claim 24.5. *$f(W_0) \cup g_k(|\sigma_i|) \subset U_{n_0-1}(x_W)$ for every $i \in \{1, 2, \ldots, k+1\}$ and $f(W) \cup g_k(\sigma_0) \subset V_{n_0-1}(x_{W_0})$.*

*Proof.* The first part of the claim follows from the inductive hypotheses. By the same reason, $f(W_1) \cup g_k(|\sigma_0|) \subset U_{n_1-1}(x_{W_1})$. To prove the second part, observe that $W_0 \subset f^{-1}(V_{n_0}(x_{W_0}))$. Hence, $f(W_0) \subset V_{n_0}(x_{W_0}) \subset V_{n_0-1}(x_{W_0})$.

So, it remains to show that $g_k(|\sigma_0|) \subset V_{n_0-1}(x_{W_0})$. To this end, let $y \in g_k(|\sigma_0|)$. Then $y \in U_{n_1-1}(x_{W_1})$ and hence $\rho(y, f(x_{W_1})) \le \varepsilon_{x_{W_1}}/12^{n_1-2}$. Since $W_0 \cap W_1 \ne \emptyset$, Claim 24.4 yields $\varepsilon_{x_{W_1}} \le 2\varepsilon_{x_{W_0}}$. Consequently, $\rho(y, f(x_{W_1})) \le 2\varepsilon_{x_{W_0}}/12^{n_1-2}$. Moreover, again by Claim 24.4, we have $\rho(f(x_{W_1}), f(x_{W_0})) \le 4\varepsilon_{x_{W_0}}/12^{n_0}$. Combining the last two inequalities and taking into account that $n_0 \le n_1 - 1$, we obtain

$$\rho(y, f(x_{W_0})) \le \frac{2\varepsilon_{x_{W_0}}}{12^{n_1-2}} + \frac{4\varepsilon_{x_{W_0}}}{12^{n_0}} \le \frac{24\varepsilon_{x_{W_0}}}{12^{n_0}} + \frac{4\varepsilon_{x_{W_0}}}{12^{n_0}} = \frac{28\varepsilon_{x_{W_0}}}{12^{n_0}} < \frac{3\varepsilon_{x_{W_0}}}{12^{n_0-1}}.$$

Therefore, $y \in V_{n_0-1}(x_{W_0})$. ■

Let us return to the definition of the map $g_{k+1}$. It suffices to define $g_{k+1}$ on the geometric realization of every $(k+1)$-dimensional simplex $\sigma = \{W_0, W_1, \ldots, W_{k+1}\}$. We fix such a geometric simplex and observe that its points are of the form $t\{W_0\} + (1-t)z$ for some $t \in [0,1]$ and $z \in |\sigma_0|$, where $\sigma_0 = \{W_1, \ldots, W_{k+1}\}$. Now, we define

$$g_{k+1}(t\{W_0\} + (1-t)z) = H_{x_{W_0}, n_0-1}\big(g_k(z), t\big).$$

This definition is correct because $g_k(|\sigma_0|) \subset V_{n_0-1}(x_{W_0})$ according to Claim 24.5. Since $H_{x_{W_0}, n_0-1}(V_{n_0-1}(x_{W_0}) \times \mathbb{I}) \subset U_{n_0-1}(x_{W_0})$, we get $f(W_0) \cup g_{k+1}(|\sigma|) \subset U_{n_0-1}(x_{W_0})$.

To complete the inductive step, it remains to show that $g_{k+1}\big||\sigma^{(k)}| = g_k\big||\sigma^{(k)}|$ and $g_{k+1}(|\sigma_0|) \subset V_{n_0-1}(x_{W_0})$. Fix any point $x \in |\sigma^{(k)}|$. If $x \in |\sigma_0|$, then $x = 0 \cdot \{W_0\} + 1 \cdot x$ and hence $g_{k+1}(x) = H_{x_{W_0},n_0-1}(g_k(x), 0) = g_k(x) \in V_{n_0-1}(x_{W_0})$ according to Claim 24.5. This implies $g_{k+1}(|\sigma_0|) \subset V_{n_0-1}(x_{W_0})$ and $g_{k+1}\big||\sigma_0| = g_k$.

If $x \notin |\sigma_0|$, then $x = t\{W_0\} + (1-t)z \in \sigma_0$ for some $t \in (0,1]$ and some $z \in |\sigma_0 \cap \sigma_i| \subset |\sigma^{(k-2)}|$ where $i \in \{1,\dots,k+1\}$. The inductive assumption guarantees that $g_k(z) = g_{k-1}(z)$ and hence

$$g_{k+1}(x) = H_{x_{W_0},n_0-1}(g_k(z), t) = H_{x_{W_0},n_0-1}(g_{k-1}(z), t) = g_k(x).$$

Therefore, $g_{k+1}\big||\sigma^{(k-1)} = g_k\big||\sigma^{(k-1)}$.

After completing the inductive construction we obtain a sequence of maps $(g_k)_{k\in\omega}$ composing a map $g : |N(\mu)| \to Y$ defined by $g\big||N(\mu)^{(k)}| = g_k$ for $k \in \omega$. We claim that $g \circ \theta \in B_\rho(f, \varepsilon)$. This is equivalent to

$$\rho(g_k(\theta(x)), f(x)) < \varepsilon(x)$$

for any $k \geq 0$ and $x \in \theta^{-1}(N(\mu)^{(k)})$. Let $\sigma = \{W_0, W_1, \dots, W_k\} = \{W \in \mu : \theta(x) \in W\}$ be the largest simplex of the nerve $N(\mu)$ whose geometric realization $|\sigma|$ contains the point $\theta(x)$. The inclusion $W_0 \subset G_{x_{W_0}} \cap f^{-1}(V_{n_0}(x_{W_0}))$ implies $f(x) \in V_{n_0}(x_{W_0})$ and $\rho(f(x), f(x_{W_0})) < 3\varepsilon_{x_{W_0}}/12^{n_0}$. On the other hand, according to our construction, $g_k(\sigma) \subset U_{n_0-1}(x_{W_0})$ and hence, $\rho(g_k(\theta(x)), f(x_{W_0})) < \varepsilon_{x_{W_0}}/12^{n_0-2}$. Combining the last two inequalities, we get

$$\rho(f(x), g_k(\theta(x))) < \frac{\varepsilon_{x_{W_0}}(3 + 12^2)}{12^{n_0}} < \varepsilon(x)$$

because $n_0 \geq 3$ and $x \in G_{x_{W_0}}$. ■

## 25. Proof of Theorem 4.1

This section is devoted to the proof of Theorem 4.1 on approximation of $n$-dimensional maps by PL-maps.

Recall that a *PL-map* (resp., a *simplicial map*) is a map $f : K \to M$ between simplicial complexes that maps each simplex $\sigma$ of $K$ into (resp., onto) a simplex $\tau$ of $M$ and is linear on $\sigma$. A PL-map $f : K \to M$ is *rational* if $f(K \cap \mathbb{Q}^{\bigcup K}) \subset M \cap \mathbb{Q}^{\bigcup M}$.

The proof of Theorem 4.1 will be established in three steps: for light maps $p$, then for maps $p$ with $\dim_\triangle(p) < \omega$ and finally for maps $p$ with $\dim_\triangle(p) \geq \omega$.

To prove the initial step we need a simplicial version of the notion of $\mathcal{V}$-map. Let $\mathcal{V}$ be a cover of a space $X$. A map $f : X \to K$ to a simplicial complex $K$ is called a *strong $\mathcal{V}$-map* if for each vertex $v \in \bigcup K$ of $K$ the preimage $f^{-1}(St(v))$ of the open star of $v$ lies in some $V \in \mathcal{V}$.

Lemma 25.1. *Let $p : X \to Y$ be a perfect $\mathcal{U}$-disjoint map between paracompact spaces, where $\mathcal{U}$ is an open cover of $X$. Then there exists an open cover $\mathcal{V}$ of $Y$ satisfying the following condition: for any strong $\mathcal{V}$-map $\beta : Y \to M$ into a simplicial complex $M$ there are a strong $\mathcal{U}$-map $\alpha : X \to K$ into a simplicial complex $K$ and a perfect light simplicial map $f : K \to M$ such that $f \circ \alpha = \beta \circ p$.*

*Proof.* Since $p$ is $\mathcal{U}$-disjoint, every $y \in Y$ has a neighborhood $G_y$ in $Y$ such that $p^{-1}(G_y)$ is the union of a disjoint open in $X$ family $\mathcal{U}_y$ refining $\mathcal{U}$. Since the fibers $p^{-1}(y)$, $y \in Y$, are compact and $p$ is a perfect map, we can assume that $\mathcal{U}_y$ is finite. Now, using the paracompactness of $Y$, we choose a locally finite open cover $\mathcal{V}$ of $Y$ refining the cover $\{G_y : y \in Y\}$. Obviously, the preimage $p^{-1}(V)$ of each set $V \in \mathcal{V}$ is the union $\bigcup \mathcal{U}_V$ of a finite disjoint collection $\mathcal{U}_V$ of open subsets of $X$ which refines the cover $\mathcal{U}$.

We are going to show that the cover $\mathcal{V}$ satisfies our requirements. Take any strong $\mathcal{V}$-map $\beta : Y \to M$ into a simplicial complex $M$. Then for any vertex $v$ of $M$ we have $V_v = \beta^{-1}(St(v)) \subset V$ for some $V \in \mathcal{V}$. The choice of $\mathcal{V}$ guarantees that $p^{-1}(V_v)$ is the union of a finite disjoint collection $\mathcal{U}_v$ of open subsets of $X$ inscribed in the cover $\mathcal{U}$. Then $\mathcal{U}' = \bigcup_{v \in \bigcup M} \mathcal{U}_v$ is a cover of $X$ refining $\mathcal{U}$. Write $\mathcal{U}'$ as the disjoint union $\mathcal{U}' = \bigcup_{v \in \cup M} \mathcal{U}'_v$ of finite families $\mathcal{U}'_v \subset \mathcal{U}_v$ and consider the finite-to-one map $f_* : \mathcal{U}' \to \bigcup M$ assigning to each set $U \in \mathcal{U}'$ the vertex $v \in \bigcup M$ such that $U \in \mathcal{U}'_v$.

Now, let $K = N(\mathcal{U}')$ be the nerve of the cover $\mathcal{U}'$. We claim that if $\{U_1, \ldots, U_n\}$ form a simplex in $N(\mathcal{U}')$, then $\{f_*(U_1), \ldots, f_*(U_n)\}$ form a simplex in $M$. For every $i \le n$ let $v_i = f_*(U_i)$. The sets $U_1, \ldots, U_n$ form a simplex in $N(\mathcal{U}')$ and hence, have a common point $x \in U_1 \cap \cdots \cap U_n$. Consider the point $z = \beta(p(x)) \in M$ and note that $p(x) \in \bigcap_{j \le n} p(U_j) \subset \beta^{-1}(St(v_i))$ for all $i \le n$. Thus, $z \in St(v_i)$, $i \le n$. Moreover, $z$ can be treated as a function $z : \bigcup M \to [0,1]$ such that there is a simplex $\sigma \subset M$ with $v \in \sigma$ provided $z(v) \ne 0$. Since $z \in St(v_i)$ is equivalent to $z(v_i) > 0$, all vertices $v_1, \ldots, v_n$ lie in the simplex $\sigma$. Therefore, $\{f_*(U_1), \ldots, f_*(U_n)\}) = \{v_1, \ldots, v_n\}$ form a simplex in $M$ and the map $f_* : \mathcal{U}' \to \bigcup M$ determines a simplicial map $f : N(\mathcal{U}') \to M$.

The map $f$ is perfect since $f_*$ is finite-to-one. Let us show that $f$ is light. This will follow as soon as we show that $f_*$ is injective on the vertices of any simplex of $N(\mathcal{U}')$. Take any simplex $\sigma$ of $N(\mathcal{U}')$ with vertices $U_1, \ldots, U_n$ and let $v_i = f_*(U_i)$ for $i \le n$. Assuming that $v_i = v_j$ for some indices $i \ne j$, we deduce that $U_i, U_j$ are two intersecting elements of the disjoint family $\mathcal{U}'_{v_i}$, a contradiction. So, $f : N(\mathcal{U}') \to M$ is a perfect light simplicial map between the simplicial complexes $N(\mathcal{U}')$ and $M$.

It remains to define a map $\alpha : X \to K = N(\mathcal{U}')$ so that $f \circ \alpha = \beta \circ p$. For every vertex $U \in \mathcal{U}'$ of $K$ let $v = f_*(U)$ and consider the map $\mu_U : X \to [0,1]$ defined by $\mu_U(x) = \mathrm{pr}_v(\beta(p(x)))$ if $x \in U$ and $\mu_U(x) = 0$ otherwise (here $\mathrm{pr}_v : M \to [0,1]$ is the coordinate projection). It can be shown that $\mu_U$ is a well-defined function and the cover $\{\mu_U^{-1}(0,1]\}_{U \in \mathcal{U}'}$ of $X$ is point-finite and subordinated to the cover $\mathcal{U}'$. Moreover, $\sum_{U \in \mathcal{U}'} \mu_U \equiv 1$. So, we can consider the map

$$\alpha : X \to N(\mathcal{U}'), \qquad x \mapsto (\mu_U(x))_{U \in \mathcal{U}'}.$$

It is easy to check that $f \circ \alpha = \beta \circ p$.

Let us show that the map $\alpha$ is continuous. For any $z \in N(\mathcal{U}')$ consider the canonical open neighborhood $N(z) = \bigcap\{St(U) : U \in \mathcal{U}' \text{ and } z \in St(U)\}$ and note that $\alpha^{-1}(N(z)) = \bigcap\{U \in \mathcal{U}' : z \in St(U)\}$ is an open subset of $X$. Fix any point $x_0 \in X$ and take any open neighborhood $O(z_0)$ of $z_0 = \alpha(x_0)$ in $K$. Replacing $O(z_0)$ with a smaller neighborhood, if necessary, we can assume that $O(z_0) \subset N(z_0)$. Consider the open neighborhood $N = O(z_0) \cup \bigcup\{N(z) : z \in f^{-1}(f(z_0)) \text{ and } z \ne z_0\}$ of the fiber $f^{-1}(f(z_0))$. Then $K \setminus N$ is a closed subset of $K$ and $f(K \setminus N)$ is a closed subset of $M$, not containing

$t_0 = f(z_0)$. The complement $O(t_0) = M \setminus f(K \setminus N)$ is an open neighborhood of $t_0$. Now, the continuity of the map $\beta : Y \to M$ at $y_0 = p(x_0)$ yields the existence of a neighborhood $O(y_0) \subset Y$ of $y_0$ such that $\beta(O(y_0)) \subset O(t_0)$. Then $O(x_0) = p^{-1}(O(y_0)) \cap \alpha^{-1}(N(z_0))$ is an open neighborhood of $x_0$ with $\alpha(O(x_0)) \subset N(z_0) \cap f^{-1}(O(t_0)) \subset N(z_0) \cap N \subset O(z_0)$ because the sets $N(z)$ are pairwise disjoint for distinct points $z \in f^{-1}(f(z_0))$.

The continuity of the map $\alpha$ is established. Finally, since $\alpha^{-1}(St(U)) \subset U$ for any $U \in \mathcal{U}'$, $\alpha : X \to K = N(\mathcal{U}')$ is a strong $\mathcal{U}$-map. ∎

The following PL-version of Lemma 25.1 provides a proof of Theorem 4.1 for light maps.

Lemma 25.2. *Let $p : X \to Y$ be a perfect $\mathcal{U}$-disjoint map between paracompact spaces, where $\mathcal{U}$ is an open cover of $X$. Then there is an open cover $\mathcal{V}$ of $Y$ satisfying the following condition: for any $\mathcal{V}$-map $\beta : Y \to M$ into a simplicial complex $M$ there exist a $\mathcal{U}$-map $\alpha : X \to K$ into a simplicial complex $K$ and a perfect light rational PL-map $f : K \to M$ such that $f \circ \alpha = \beta \circ p$.*

*Proof.* We apply Lemma 25.1 to find a cover $\mathcal{V}$ of $Y$ such that for any strong $\mathcal{V}$-map $\beta' : Y \to M'$ into a simplicial complex $M'$ there exist a $\mathcal{U}$-map $\alpha : X \to K$ into a simplicial complex $K$ and perfect light simplicial map $f' : K \to M'$ with $f' \circ \alpha = \beta' \circ p$.

We claim that the cover $\mathcal{V}$ satisfies our requirements. Indeed, if $\beta : Y \to M$ is a $\mathcal{V}$-map into a simplicial complex $M$, then there exists an open cover $\mathcal{W}$ of $M$ whose preimage $\beta^{-1}(\mathcal{W})$ refines $\mathcal{V}$. By Lemma 20.2, the complex $M$ admits a rational subdivision $M'$ such that the cover $\{St(v) : v \in \bigcup M'\}$ of $M'$ by open stars refines $h^{-1}(\mathcal{W})$, where $h : M' \to M$ is the canonical homeomorphism. Then $\beta' = h^{-1} \circ \beta : Y \to M'$ is a strong $\mathcal{V}$-map. So, the choice of $\mathcal{V}$ guarantees the existence of a $\mathcal{U}$-map $\alpha : X \to K$ into a simplicial complex $K$ and perfect light simplicial map $f' : K \to M'$ such that $f' \circ \alpha = \beta' \circ p$. Then $f = h \circ f' : K \to M$ is a perfect light rational PL-map with $f \circ \alpha = \beta \circ p$. ∎

Next, we prove Theorem 4.1 for maps $p$ with $\dim_\triangle(p) < \omega$.

Lemma 25.3. *Let $p : X \to Y$ be a perfect map between paracompact spaces with $\dim_\triangle(p) = n < \omega$. Then for every open cover $\mathcal{U}$ of $X$ there is an open cover $\mathcal{V}$ of $Y$ satisfying the following condition: for any $\mathcal{V}$-map $\beta : Y \to M$ into a simplicial complex $M$ there exist a $\mathcal{U}$-map $\alpha : X \to K$ into a simplicial complex $K$ and a perfect rational PL-map $f : K \to M$ with $p \circ \alpha = \beta \circ f$ and $\dim(f) = \dim_\triangle(f) \le \dim_\triangle(p)$.*

*Proof.* Let $\mathcal{U}$ be an open cover of $X$. According to the definition of $\dim_\triangle(p)$, there is a map $g : X \to \mathbb{I}^n$ such that the diagonal product $p \triangle g : X \to Y \times \mathbb{I}^n$ is $\mathcal{U}$-disjoint.

Applying Lemma 25.2 for the perfect map $p \triangle g : X \to Y \times \mathbb{I}^n$, we find an open cover $\mathcal{V}'$ of $Y \times \mathbb{I}^n$ satisfying the following property: for any $\mathcal{V}'$-map $\beta' : Y \times \mathbb{I}^n \to M$ into a simplicial complex $M$ there exist an $\mathcal{U}$-map $\alpha : X \to K$ into a simplicial complex $K$ and a perfect light rational PL-map $f : K \to M$ such that $f \circ \alpha = \beta' \circ (p \triangle g)$.

Since for every $y \in Y$ the set $\{y\} \times \mathbb{I}^n$ is compact, there exists a neighborhood $V_y \subset Y$ of $y$ such that every point $z \in \mathbb{I}^n$ has a neighborhood $W_z$ with $V_y \times W_z$ being a subset of some $V \in \mathcal{V}'$. Choose $\mathcal{V}$ to be a locally finite open cover of $Y$ refining $\{V_y : y \in Y\}$.

Let us show that the cover $\mathcal{V}$ satisfies our requirements. Take any $\mathcal{V}$-map $\beta : Y \to M$ to a simplicial complex $M$ and consider the map $\beta \times \mathrm{id} : Y \times \mathbb{I}^n \to M \times \mathbb{I}^n$, $\beta \times \mathrm{id} : (y,t) \mapsto (\beta(y), t)$. It is easy to see that $\beta \times \mathrm{id}$ is a $\mathcal{V}'$-map. Next, choose a triangulation of the product $M \times \mathbb{I}^n$ such that the projection $\mathrm{pr} : M \times \mathbb{I}^n \to M$ is a rational PL-map. Then, by the choice of the cover $\mathcal{V}'$, there exist a $\mathcal{U}$-map $\alpha : X \to K$ into a simplicial complex $K$ and a perfect light rational PL-map $h : K \to M \times \mathbb{I}^n$ such that $h \circ \alpha = (\beta \times \mathrm{id}) \circ (p \triangle g)$. Finally, let $f = \mathrm{pr} \circ h : K \to M$. It is clear that $f \circ \alpha = \beta \circ p$. Since $f$ is a composition of the light map $h$ and the perfect $n$-dimensional map $\mathrm{pr} : M \times \mathbb{I}^n \to M$, $\dim(f) \le n$. According to Proposition 1(4), $\dim_\triangle(f) \le n$ (we can apply Proposition 1(4) because $K$ is submetrizable and $M$ is a $C$-space). Moreover, $f$, being a composition of two rational PL-maps, is a rational PL-map as well. ∎

Finally, we treat the case $\dim_\triangle(p) \ge \omega$.

LEMMA 25.4. *Let $p : X \to Y$ be a perfect map between paracompact spaces. Then for every open cover $\mathcal{U}$ of $X$ there is an open cover $\mathcal{V}$ of $Y$ satisfying the following condition: for any $\mathcal{V}$-map $\beta : Y \to M$ into a simplicial complex $M$ there exist a $\mathcal{U}$-map $\alpha : X \to K$ into a simplicial complex $K$ and a perfect rational PL-map $f : K \to M$ with $p \circ \alpha = \beta \circ f$.*

*Proof.* Let $\mathcal{U}$ be an open cover of $X$. Fix a cardinal $\tau$ and an injective continuous map $g : X \to \mathbb{I}^\tau$. Then $p \triangle g : X \to Y \times \mathbb{I}^\tau$ is a perfect embedding. So, $p \triangle g$ is a $U$-disjoint map. By Lemma 25.2, there is an open cover $\mathcal{W}$ of $Y \times \mathbb{I}^\tau$ such that for any $\mathcal{W}$-map $\beta' : Y \times \mathbb{I}^\tau \to M$ into a simplicial complex $M$ there exist a $\mathcal{U}$-map $\alpha : X \to K$ into a simplicial complex $K$ and a perfect rational PL-map $h : K \to M$ with $h \circ \alpha = \beta' \circ (p \triangle g)$.

Next, for any $y \in Y$ take an open neighborhood $V_y \subset Y$ of $y$ and a finite open cover $\mathcal{W}_y$ of $\mathbb{I}^\tau$ such that the family $\{V_y \times W : W \in \mathcal{W}_y\}$ refines $\mathcal{W}$. Let $\mathcal{V}'$ be a locally finite open cover of $Y$ refining $\{V_y : y \in Y\}$. For every $V \in \mathcal{V}'$ there exists $y(V) \in Y$ with $V \subset V_{y(V)}$ and let $\mathcal{W}_V = \mathcal{W}_{y(V)}$. Then $\mathcal{W}' = \{V \times W : W \in \mathcal{W}_V, V \in \mathcal{V}'\}$ is a locally finite open cover of $Y \times \mathbb{I}^\tau$ refining $\mathcal{W}$. Let $\{\lambda_W : Y \times \mathbb{I}^\tau \to [0,1]\}_{W \in \mathcal{W}'}$ be a partition of unity subordinated to $\mathcal{W}'$ and let

$$\lambda : Y \times \mathbb{I}^\tau \to N(\mathcal{W}'), \qquad (y,t) \mapsto (\lambda_W(y,t))_{W \in \mathcal{W}'},$$

be the canonical map into the nerve of $\mathcal{W}'$. It is easy to check that $\lambda$ is a $\mathcal{W}$-map.

Let $\mathcal{V}$ be a locally finite open cover of $Y$ refining $\mathcal{V}'$ and such that each $V \in \mathcal{V}$ meets only finitely many sets $V' \in \mathcal{V}'$.

We claim that the cover $\mathcal{V}$ satisfies our requirements. Take any $\mathcal{V}$-map $\beta : Y \to M$ into a simplicial complex $M$. According to Lemma 20.2, we may assume that the triangulation of $M$ is so fine that the preimage $\beta^{-1}(\sigma)$ of any simplex $\sigma \subset M$ lies in some $V \in \mathcal{V}$. Then

$$\beta' : Y \times \mathbb{I}^\tau \to M \times N(\mathcal{W}'), \qquad (y,t) \mapsto (\beta(y), \lambda(y,t)),$$

is a $\mathcal{W}$-map because so is $\lambda$. Now, triangulate the product $M \times N(\mathcal{W}')$ such that the projections onto $M$ and onto $N(\mathcal{W}')$ are both simplicial maps, and consider the subcomplex $L \subset M \times N(\mathcal{W}')$ of all simplexes intersecting $\beta'(Y \times \mathbb{I}^\tau)$. Let us show that the projection $\mathrm{pr} : L \to M$ is a perfect map. It suffices to check that the preimage $\mathrm{pr}^{-1}(\sigma)$ of any simplex $\sigma \subset M$ is a finite subcomplex of $L$. By assumption, $\beta^{-1}(\sigma)$ is contained in some $V_0 \in \mathcal{V}$, so it meets only finitely many sets $V \in \mathcal{V}'$. Consequently,

$\mathcal{W}_\sigma = \{V \times \mathcal{W}_V : V \in \mathcal{V}', \beta^{-1}(\sigma) \cap V \neq \emptyset\}$ is finite and so is the subcomplex $N(\mathcal{W}_\sigma)$ of $N(\mathcal{W}')$ consisting of all simplexes whose vertices belong to $\mathcal{W}_\sigma$. Now, it is easy to see that $\mathrm{pr}^{-1}(\sigma) \subset \sigma \times N(\mathcal{W}_\sigma)$, which implies that $\mathrm{pr} : L \to M$ is perfect.

Since $\beta' : Y \times \mathbb{I}^\tau \to L$ is a $\mathcal{W}$-map, the choice of $\mathcal{W}$ implies the existence of a $\mathcal{U}$-map $\alpha : X \to K$ to a simplicial complex $K$ and perfect rational PL-map $h : K \to L$ such that $h \circ \alpha = \beta' \circ (p \triangle g)$. Finally, consider the perfect rational PL-map $f = \mathrm{pr} \circ h : K \to M$ and note that $f \circ \alpha = \mathrm{pr} \circ h \circ \alpha = \mathrm{pr} \circ \beta' \circ (p \triangle g) = \beta \circ p$. ∎

## 26. Simplicial characterization of the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property

This section is devoted to the proof of Theorem 5.3. The "if" part is trivial. To prove the "only if" part, we suppose that $X$ is a submetrizable space having the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property, $\mathcal{U}$ is an open cover of $X$, $p_N : N \to M$ and $p_K : K \to M$ are simplicial maps between compact simplicial complexes such that $m = \dim M$, $n = \dim(p_N)$ and $k = \dim(p_K)$. For any maps $f : N \to X$, $g : K \to X$ we are going to construct two maps $f' : N \to X$, $g' : K' \to X$ such that $f' \underset{\mathcal{U}}{\sim} f$, $g' \underset{\mathcal{U}}{\sim} g$ and $f'(p_N^{-1}(z)) \cap g'(p_K^{-1}(z)) = \emptyset$ for all $z \in M$.

By Lemma 18.2, we can assume that $\mathcal{U}$ consists of open $\varepsilon$-balls with respect to a suitable continuous pseudometric $\rho$ on $X$. Since $X$ is submetrizable, we can also assume that $\rho$ is a metric. So, it suffices to construct maps $f' : N \to M$ and $g' : K \to M$, $\varepsilon$-homotopic to $f$ and $g$, respectively, such that $f'(p_N^{-1}(z)) \cap g'(p_K^{-1}(z)) = \emptyset$ for all $z \in M$. This will be done in three steps.

First, we assume that $M, N, K$ are simplexes and $p_N, p_K$ are affine functions mapping vertices of $N, K$ to vertices of $M$. Let $M^{(0)}$, $N^{(0)}$, $K^{(0)}$ be the sets of vertices of the simplexes $M, N, K$, respectively, and $\sigma_n$, $\sigma_k$ simplexes with $\dim \sigma_n = \dim(p_N) = n$ and $\dim \sigma_k = \dim(p_K) = k$.

Since $\dim(p_N) = n$, the preimage $p_N^{-1}(z) \cap N^{(0)}$ of each vertex $z \in M^{(0)}$ contains at most $n + 1$ points. Consequently, we can find a map $e_n : N^{(0)} \to \sigma_n^{(0)}$ which is injective on each set $p_N^{-1}(z) \cap N^{(0)}$, $z \in M^{(0)}$. This map induces an affine map $\bar{e}_n : N \to \sigma_n$. Then $\bar{e}_N = p_N \triangle \bar{e}_n : N \to M \times \sigma_n$ is an affine embedding. So, $\bar{e}_N(N)$ is a retract of $M \times \sigma_n$. Hence, there exists a map $r_N : M \times \sigma_n \to N$ such that $r_N \circ \bar{e}_N$ is the identity.

We can do the same for the simplex $K$ to find an affine map $\bar{e}_k : K \to \sigma_k$ such that the diagonal map $\bar{e}_K = p_K \triangle \bar{e}_k : K \to M \times \sigma_k$ is an affine embedding, and a map $r_K : M \times \sigma_k \to K$ with $r_K \circ \bar{e}_K$ being the identity.

Now, consider the maps $f \circ r_N : M \times \sigma_n \to X$ and $g \circ r_K : M \times \sigma_k \to X$. The $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property of $X$ allows us to find two maps $\tilde{f} : M \times \sigma_n \to X$ and $\tilde{g} : M \times \sigma_k \to X$ such that $\tilde{f} \underset{\mathcal{U}}{\sim} f \circ r_N$, $\tilde{g} \underset{\mathcal{U}}{\sim} g \circ r_K$ and $\tilde{f}(\{z\} \times \sigma_n) \cap \tilde{g}(\{z\} \times \sigma_k) = \emptyset$ for all $z \in M$. Then the maps $f' = \tilde{f} \circ (p_N \triangle \bar{e}_n) : N \to X$ and $g' = \tilde{g} \circ (p_K \triangle \bar{e}_k) : K \to X$ have the desired properties: $f' \underset{\mathcal{U}}{\sim} f$, $g' \underset{\mathcal{U}}{\sim} g$ and $\tilde{f}(p_N^{-1}(z)) \cap g'(p_K^{-1}(z)) = \emptyset$ for all $z \in M$. This completes the proof of the particular case with $M, N, K$ being simplexes.

At the second step we consider the case when $M, N, K$ are disjoint unions of simplexes. Write $M = \bigsqcup \mathcal{M}$ as a disjoint union of a family $\mathcal{M}$ of simplexes. We can find similar representations for the complexes $N$ and $K$: $N = \bigsqcup \mathcal{N}$ and $K = \bigsqcup \mathcal{K}$. Enumerate the product $\mathcal{M}\times\mathcal{N}\times\mathcal{K}$ as $\mathcal{M}\times\mathcal{N}\times\mathcal{K} = \{(M_i, N_i, K_i) : 1 \le i \le l\}$, where $l = |\mathcal{M}\times\mathcal{N}\times\mathcal{K}|$.

Let $\varepsilon_1 = \varepsilon/2$, $f_0 = f$ and $g_0 = g$. Using the implication of the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property of $X$ established at the first step, we can construct (by finite induction of length $l$) sequences of maps $f_i : N \to X$, $g_i : K \to X$, $i \le l$, and a sequence $(\varepsilon_i)_{i \le l+1}$ of real numbers such that

(1) $f_i$ is $\varepsilon_i$-homotopic to $f_{i-1}$ and $g_i$ is $\varepsilon_i$-homotopic to $g_{i-1}$;
(2) $\min_{z\in M_i} \operatorname{dist}\big(f_i(p_N^{-1}(z) \cap N_i), g_i(p_K^{-1}(z) \cap K_i)\big) \ge 5\varepsilon_{i+1}$;
(3) $\varepsilon_{i+1} < \varepsilon_i/2$.

Then the final maps $f_l : N \to X$, $g_l : K \to X$ have the desired properties:

(4) $f_l$ is $\varepsilon$-homotopic to $f_0$ and $g_l$ is $\varepsilon$-homotopic to $g_0$;
(5) $\min_{z\in M_i} \operatorname{dist}\big(f_l(p_N^{-1}(z) \cap N_i), g_l(p_K^{-1}(z) \cap K_i)\big) \ge \varepsilon_{i+1}$ for all $i \le l$.

Since for each triple $(z, x, y) \in M \times N \times K$ there is a number $i \le l$ with $(z, x, y) \in (M_i, N_i, K_i)$, the latter condition implies $f_l(p_N^{-1}(z)) \cap g_l(p_K^{-1}(z)) = \emptyset$ for all $z \in M$.

At the third step we prove the general case by induction on $s = \dim M + \dim N + \dim K$. The second step allows us to start the induction with $s = 0$.

Now, suppose the assertion has already been proved for all triples $M, N, K$ with $\dim M + \dim N + \dim K < s$ for some $s$. Assume that $\dim M + \dim N + \dim K = s > 0$ and $X$ has the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property for $m = \dim M$, $n = \dim(p_N)$, $k = \dim(p_K)$. Given any $\varepsilon > 0$ we should construct maps $f' : N \to X$, $g' : K \to X$ such that $f'$ is $\varepsilon$-homotopic to $f$, $g'$ is $\varepsilon$-homotopic to $g$ and $f(p_N^{-1}(z)) \cap g(p_K^{-1}(z)) = \emptyset$ for all $z \in M$.

Let $\varepsilon_1 = \varepsilon/2$ and $M', N', K'$ be the codimension one skeleta of the simplicial complexes $M, N, K$, respectively. If one of the complexes, say $M$, is zero-dimensional, then we set $M' = \emptyset$.

In case $M' = \emptyset$, let $f_1 = f$ and $g_1 = g$. Otherwise apply the inductive hypothesis and the fact that $\dim M' + \dim N + \dim K < s$ to find two maps $f_1 : p_N^{-1}(M') \to X$, $g_1 : p_K^{-1}(M') \to X$ such that

(6) $f_1$ is $\varepsilon_1$-homotopic to $f|p_N^{-1}(M')$,
(7) $g_1$ is $\varepsilon_1$-homotopic to $g|p_K^{-1}(M')$ and
(8) $f_1(p_N^{-1}(z)) \cap g_1(p_K^{-1}(z)) = \emptyset$ for all $z \in M'$.

By the Borsuk homotopy extension lemma 18.3, the maps $f_1, g_1$ can be extended to maps $\bar f_1 : N \to X$, $\bar g_1 : K \to X$ such that $\bar f_1$ is $\varepsilon_1$-homotopic to $f$ and $\bar g_1$ is $\varepsilon_1$-homotopic to $g$. Take any positive real number $\varepsilon_2 \le \varepsilon_1/2$ with

(9) $5\varepsilon_2 < \min_{z\in M'} \operatorname{dist}\big(f_1(p_N^{-1}(z), g_1(p_K^{-1}(z))\big)$.

Applying the inductive hypothesis to the maps $\bar f_1|N' : N' \to X$, $\bar g_1 : K \to X$ combined with Lemma 18.3, we find two maps $\bar f_2 : N \to X$, $\bar g_2 : K \to X$ such that

(10) $\bar f_2$ is $\varepsilon_2$-homotopic to $\bar f_1$, $\bar g_2$ is $\varepsilon_2$-homotopic to $\bar g_1$ and
(11) $\bar f_2(p_N^{-1}(z) \cap N') \cap \bar g_2(p_K^{-1}(z)) = \emptyset$ for all $z \in M$.

Next, take any positive real number $\varepsilon_3 \le \varepsilon_2/2$ with

(12) $5\varepsilon_3 < \min_{z\in M} \operatorname{dist}\big(\bar f_2(p_N^{-1}(z)\cap N'), \bar g_2(p_K^{-1}(z))\big)$.

Applying the inductive hypothesis to the maps $\bar f_2$ and $g_2|K'$ and using Lemma 18.3, we find two maps $\bar f_3 : N \to X$, $\bar g_3 : K \to X$ such that

(13) $\bar f_3$ is $\varepsilon_3$-homotopic to $\bar f_2$, $\bar g_3$ is $\varepsilon_3$-homotopic to $\bar g_2$ and
(14) $\bar f_3(p_N^{-1}(z)) \cap \bar g_3(p_K^{-1}(z)\cap K') = \emptyset$ for all $z \in M$.

Now, take any positive real number $\varepsilon_4 < \varepsilon_3/2$ with

(15) $5\varepsilon_4 < \min_{z\in M} \operatorname{dist}\big(\bar f_3(p_N^{-1}(z)), \bar g_3(p_K^{-1}(z)\cap K')\big)$.

Let $\mathcal M$ be the family of open simplexes of dimension $\dim M$ in the complex $M$. Observe that the simplexes from $\mathcal M$ are disjoint and $\bigcup\mathcal M = M\setminus M'$. For each $\sigma\in\mathcal M$ denote by $\bar\sigma$ its closure in $M$ and set $\overline{\mathcal M} = \{\bar\sigma : \sigma\in\mathcal M\}$. Let $\bigsqcup\mathcal M$ be the disjoint topological sum of the closed simplexes from $\overline{\mathcal M}$ and $\mathrm{pr}_M : \bigsqcup\overline{\mathcal M} \to M$ be the natural surjective map (whose restriction to $\bigsqcup\mathcal M$ is a homeomorphism between $\bigsqcup\mathcal M \subset \bigsqcup\overline{\mathcal M}$ and $M\setminus M'$).

Having defined the family $\mathcal M$, we define families of simplexes $\mathcal N$ and $\mathcal K$:

$$\mathcal N = \{\sigma\subset N : \sigma\not\subset N',\ p_N(\sigma)\not\subset M'\},$$
$$\mathcal K = \{\sigma\subset K : \sigma\not\subset K',\ p_K(\sigma)\not\subset M'\}.$$

Let also $\overline{\mathcal N} = \{\bar\sigma : \sigma\in\mathcal N\}$, $\overline{\mathcal K} = \{\bar\sigma : \sigma\in\mathcal K\}$, and $\mathrm{pr}_N : \bigsqcup\mathcal N \to N$, $\mathrm{pr}_K : \bigsqcup\mathcal K\to K$ be the natural maps.

The simplicial maps $p_N : N\to M$ and $p_K : K\to M$ induce simplicial maps $\bar p_N : \bigsqcup\overline{\mathcal N} \to \bigsqcup\overline{\mathcal M}$ and $\bar p_K : \bigsqcup\overline{\mathcal K}\to\bigsqcup\overline{\mathcal M}$ making the following diagrams commutative:

$$\begin{array}{ccc} \bigsqcup\overline{\mathcal N} & \xrightarrow{\mathrm{pr}_N} & N \\ {\scriptstyle \bar p_N}\big\downarrow & & \big\downarrow{\scriptstyle p_N} \\ \bigsqcup\overline{\mathcal M} & \xrightarrow{\mathrm{pr}_M} & M \end{array} \quad\text{and}\quad \begin{array}{ccc} \bigsqcup\overline{\mathcal K} & \xrightarrow{\mathrm{pr}_K} & K \\ {\scriptstyle \bar p_K}\big\downarrow & & \big\downarrow{\scriptstyle p_K} \\ \bigsqcup\overline{\mathcal M} & \xrightarrow{\mathrm{pr}_M} & M \end{array}$$

Since $\bigsqcup\overline{\mathcal M}$, $\bigsqcup\overline{\mathcal N}$, $\bigsqcup\overline{\mathcal K}$ are disjoint unions of cells, we may apply the implication of the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property, established at the second step, to find maps $f_4 : \bigsqcup\overline{\mathcal N}\to X$, $g_4 : \bigsqcup\overline{\mathcal K}\to X$ such that

(16) $f_4$ is $\varepsilon_4$-homotopic to $\bar f_3\circ\mathrm{pr}_N$,
(17) $g_4$ is $\varepsilon_4$-homotopic to $\bar g_3\circ\mathrm{pr}_K$,
(18) $f_4(\bar p_N^{-1}(z))\cap g_4(\bar p_K^{-1}(z)) = \emptyset$ for all $z\in\bigsqcup\overline{\mathcal M}$.

Let $\rho_M,\rho_N,\rho_K$ be the canonical $l_1$-metrics on the geometric simplicial complexes $M,N,K$, respectively. By the uniform continuity of the maps $\bar f_i,\bar g_i$, $i\le 3$, there is $\delta\in(0,1/2)$ such that

$$\max_{i\le 3}\{\operatorname{dist}(\bar f_i(x),\bar f_i(x')), \operatorname{dist}(\bar g_i(y),\bar g_i(y'))\} < \varepsilon_4$$

for any points $x,x'\in N$ and $y,y'\in K$ with $\rho_N(x,x')<4\delta$, $\rho_K(y,y')<4\delta$. Consider the $\delta$-neighborhood $O_\delta(M\setminus\bigcup\mathcal M) = \{z\in M : \exists z'\in M\setminus\bigcup\mathcal M \text{ with } \rho_M(z,z')<\delta\}$. Similarly, we define neighborhoods $O_\delta(N\setminus\bigcup\mathcal N)$ and $O_\delta(K\setminus\bigcup\mathcal K)$.

Using (16), (17) we construct maps $f' : N \to X$ and $g' : K \to X$ such that

(19) $f'$ is $\varepsilon_4$-homotopic to $\bar{f}_3$;
(20) $f'|N \setminus \bigcup \mathcal{N} = \bar{f}_3|N \setminus \bigcup \mathcal{N}$;
(21) $f'|N \setminus O_\delta(N \setminus \bigcup \mathcal{N}) = f_4|N \setminus O_\delta(N \setminus \bigcup \mathcal{N})$;

and

(22) $g'$ is $\varepsilon_4$-homotopic to $\bar{g}_3$;
(23) $g'|K \setminus \bigcup \mathcal{K} = \bar{g}_3|K \setminus \bigcup \mathcal{K}$;
(24) $g'|K \setminus O_\delta(K \setminus \bigcup \mathcal{K}) = g_4|K \setminus O_\delta(K \setminus \bigcup \mathcal{K})$.

The choice of the numbers $\varepsilon_i$ and the conditions (6), (10), (13), (16), (19) imply that the map $f'$ is $\varepsilon$-homotopic to $f$. Similarly, $g'$ is $\varepsilon$-homotopic to $g$. It remains to prove that $f'(p_N^{-1}(z)) \cap g'(p_K^{-1}(z)) = \emptyset$ for any $z \in M$. Take any points $x \in p_N^{-1}(z)$ and $y \in p_K^{-1}(z)$ and find maximal simplexes $\sigma_z \subset M$, $\sigma_x \subset N$, $\sigma_y \subset K$ containing the points $z, x, y$, respectively. Now, consider the following cases:

(i) $z \in M'$. Then

$$\begin{aligned}\operatorname{dist}(f'(x), g'(y)) &\geq \operatorname{dist}(f_1(p_N^{-1}(z)), g_1(p_K^{-1}(z))) - \operatorname{dist}(f', f_1) - \operatorname{dist}(g', g_1)\\ &\geq 5\varepsilon_2 - 2(\varepsilon_2 + \varepsilon_3 + \varepsilon_4) \geq \varepsilon_2 > 0\end{aligned}$$

(ii) $z \in O_\delta(M') \setminus M'$. Then the maximal simplex $\sigma_z$ containing $z$ belongs to the family $\mathcal{M}$ and $\rho_M(z, z') < \delta$ for some $z' \in \sigma_z \cap M'$. Taking into account that $p_N(x) = z \in \sigma_z \setminus M'$ we conclude that $p_N(\sigma_x^{(0)}) = \sigma_z^{(0)}$. By Lemma 20.3, there exists a point $x' \in \sigma_x$ such that $p_N(x') = z'$ and $\rho_N(x, x') = \rho_M(z, z') < \delta$. Analogously, there exists a point $y' \in \sigma_y$ such that $p_K(y') = z'$ and $\rho_K(y, y') = \rho_M(z, z') < \delta$. Then $\max\{\operatorname{dist}(\bar{f}_1(x), \bar{f}_1(x')), \operatorname{dist}(\bar{g}_1(y), \bar{g}_1(y'))\} < \varepsilon_4$ by the choice of $\delta$. Now,

$$\begin{aligned}\operatorname{dist}(f'(x), g'(y)) &\geq \operatorname{dist}(\bar{f}_1(x), \bar{g}_1(y)) - \operatorname{dist}(f', \bar{f}_1) - \operatorname{dist}(g', \bar{g}_1)\\ &\geq \operatorname{dist}(\bar{f}_1(x'), \bar{g}_1(y')) - \operatorname{dist}(\bar{f}_1(x'), \bar{f}_1(x)) - \operatorname{dist}(\bar{g}_1(y'), \bar{g}_1(y)) - 2(\varepsilon_4 + \varepsilon_3 + \varepsilon_2)\\ &\geq \operatorname{dist}\big(\bar{f}_1(p_N^{-1}(z')), \bar{g}_1(p_K^{-1}(z'))\big) - 2\varepsilon_4 - 2(\varepsilon_2 + \varepsilon_3 + \varepsilon_4) \geq 5\varepsilon_2 - 4\varepsilon_2 > 0.\end{aligned}$$

(iii) $z \notin O_\delta(M')$ and $x \in N \setminus \bigcup \mathcal{N}$. Then $x \in N'$ (because $p_N(x) = z \notin M'$) and

$$\begin{aligned}\operatorname{dist}(f'(x), g'(y)) &\geq \operatorname{dist}(\bar{f}_2(x), \bar{g}_2(y)) - \operatorname{dist}(f', \bar{f}_2) - \operatorname{dist}(g', \bar{g}_2)\\ &\geq 5\varepsilon_3 - 2(\varepsilon_3 + \varepsilon_4) > 0.\end{aligned}$$

(iv) $z \notin O_\delta(M')$ and $x \in O_\delta(N \setminus \cup \mathcal{N})$. In this case we can find $x' \in N'$ with $\rho_N(x, x') < \delta$. Then letting $z' = p_N(x')$, Lemma 20.3 implies $\rho_M(z, z') = \rho_M(p_N(x), p_N(x')) \leq \rho_N(x, x') < \delta$. Applying again Lemma 20.3, we find a point $y' \in \sigma_y$ with $\rho_K(y, y') = \rho_M(z, z') < \delta$. Hence,

$$\max\{\operatorname{dist}(\bar{f}_2(x'), \bar{f}_2(x)), \operatorname{dist}(\bar{g}_2(y), \bar{g}_2(y'))\} < \varepsilon_4$$

and

$$\begin{aligned}\operatorname{dist}(f'(x), g'(y)) &\geq \operatorname{dist}(\bar{f}_2(x), \bar{g}_2(y)) - \operatorname{dist}(f', \bar{f}_2) - \operatorname{dist}(g', \bar{g}_2)\\ &\geq \operatorname{dist}(\bar{f}_2(x'), \bar{f}_2(y')) - \operatorname{dist}(\bar{f}_2(x'), \bar{f}_2(x)) - \operatorname{dist}(\bar{g}_2(y'), \bar{g}_2(y)) - 2(\varepsilon_4 + \varepsilon_3)\\ &\geq \operatorname{dist}(\bar{f}_2(p_N^{-1}(z') \cap N'), \bar{g}_2(p_K^{-1}(z'))) - 2\varepsilon_4 - 2(\varepsilon_3 + \varepsilon_4) \geq 5\varepsilon_3 - 4\varepsilon_3 > 0.\end{aligned}$$

(v) $z \notin O_\delta(M')$, $x \notin O_\delta(N \setminus \bigcup \mathcal{N})$, $y \in O_\delta(K \setminus \bigcup \mathcal{K})$. In this case, we can choose $y' \in K'$ with $\rho_K(y', y') < \delta$. Then letting $z' = p_K(y')$, we have $\rho_M(z, z') = \rho_M(p_K(y), p_K(y')) \leq \rho_K(y, y') < \delta$. According to Lemma 20.3, there exists a point $x' \in \sigma_x$ with $\rho_N(x, x') = \rho_M(z, z') < \delta$. Therefore,

$$\begin{aligned}\operatorname{dist}(f'(x), g'(y)) &\geq \operatorname{dist}(\bar f_3(x), \bar g_3(y)) - \operatorname{dist}(f', \bar f_3) - \operatorname{dist}(g', \bar g_3)\\ &\geq \operatorname{dist}(\bar f_3(x'), \bar f_3(y')) - \operatorname{dist}(\bar f_3(x'), \bar f_3(x)) - \operatorname{dist}(\bar g_3(y'), \bar g_3(y)) - 2\varepsilon_4\\ &\geq \operatorname{dist}\big(\bar f_3(p_N^{-1}(z')), \bar g_3(p_K^{-1}(z') \cap K')\big) - 4\varepsilon_4 \geq 5\varepsilon_4 - 4\varepsilon_4 > 0.\end{aligned}$$

(vi) $z \notin O_\delta(M')$, $x \notin O_\delta(N \setminus \bigcup \mathcal{N})$, $y \notin O_\delta(K \setminus \bigcup \mathcal{K})$. In this case $f'(x) = f_4(x) \neq g_4(y) = g'(y)$ by (18), (21) and (24).

## 27. Approximations by simplicially factorizable $\mathcal{V}$-maps

The main result of this section is Lemma 27.4 which is the principal (and technically the most difficult) ingredient of the proof of Theorem 3.3. This lemma asserts that if $p : K \to M$ is a perfect map between paracompact spaces, $\mathcal{V}$ is an open cover of $K$ and $X$ is a submetrizable space possessing the $m$-$\overline{\mathrm{DD}}^n$-property with $m = \dim M$ and $n = \dim_\triangle(p)$, then each simplicially factorizable map $f : K \to X$ can be approximated by simplicially factorizable $\mathcal{V}$-maps.

Lemma 27.4 has a technical proof preceded by three other lemmas.

Lemma 27.1. *Let $p : K \to M$ be a PL-map between compact polyhedra and let $f : K \to X$ be a map into a submetrizable space $X$ possessing the $m$-$\overline{\mathrm{DD}}^n$-property with $m = \dim M$ and $n = \dim(p)$. Then for any continuous pseudometric $\rho$ on $X$ and any open cover $\mathcal{V}$ of $K$ there is a map $\tilde f : K \to X$ such that $\tilde f$ is $1$-homotopic to $f$ (with respect to $\rho$) and $p \triangle \tilde f : K \to M \times X$ is a $\mathcal{V}$-map.*

*Proof.* Since $X$ is submetrizable, we can suppose that $\rho$ is a continuous metric on $X$. Fix a continuous metric $d$ on $K$ such that the cover $\{B_d(y, 1) : y \in K\}$ refines $\mathcal{V}$. We assume that $T$ is a triangulation of the polyhedron $K$ so fine that each simplex $\sigma \in T$ is of diameter $< 1/3$ with respect to the metric $d$. Moreover, we also assume that $p$ is a simplicial map, where $K$ carries the triangulation $T$ and $M$ is equipped with a suitable triangulation. Let $\{\sigma_i : 1 \leq i \leq l\}$ be an enumeration of maximal closed simplexes of $K$. The star $St(\sigma)$ of any simplex $\sigma \in T$ in $K$ is the set $St(\sigma) = \bigcup_{v \in \sigma} St(v)$. Since the diameter of each $\sigma \in T$ is $< 1/3$, $St(\sigma)$ is of diameter $< 1$.

Let $f_0 = h_0 = g_0 = f$ and $\varepsilon_0 = 1$, $\varepsilon_1 = 1/2$. Using Theorem 5.3 and Lemma 18.3, by finite induction of length $l$, one can construct three sequences $(f_i, g_i, h_i : K \to X)_{i \leq l}$ of maps and a sequence $(\varepsilon_i)_{i \leq l}$ of positive real numbers satisfying the following conditions for each $i \geq 1$:

- $f_i$ and $g_i$ are $\varepsilon_i$-homotopic to $h_{i-1}$;
- $\min_{z \in M} \rho\big(f_i(p^{-1}(z), g_i(p^{-1}(z))\big) > 5\varepsilon_{i+1}$;
- $\varepsilon_{i+1} \leq \varepsilon_i/2$;
- $h_i(x) = f_i(x)$ for each $x \in \sigma_i$;
- $h_i(x) = g_i(x)$ for each $x \in K \setminus St(\sigma_i)$;
- $h_i$ is $\varepsilon_i$-homotopic to $h_{i-1}$.

Then $\tilde{f} = h_l : K \to X$ is $\varepsilon_0$-homotopic to $f_0 = f$. It remains to show that $p \triangle \tilde{f} : K \to M \times X$ is a $\mathcal{V}$-map. This will follow as soon as we show that it is a 1-map with respect to $d$. Assuming that this is not the case, we could find $x, x' \in K$ with $\mathrm{dist}_d(x, x') \geq 1$ and $(p(x), \tilde{f}(x)) = (p(x'), \tilde{f}(x')) = (z, y)$ for some $(z, y) \in M \times X$. The point $x$ lies in some simplex $\sigma_i$, $i \leq l$. Since $\mathrm{diam}_d(St(\sigma_i)) < 1 \leq \mathrm{dist}_d(x, x')$, $x' \notin St(\sigma_i)$. Then $h_i(x) = f_i(x)$ and $h_i(x') = g_i(x')$. It follows from the inductive construction that $\mathrm{dist}_\rho(h_i(x), h_i(x')) = \mathrm{dist}_\rho(f_i(x), g_i(x')) \geq \mathrm{dist}_\rho(f_i(p^{-1}(z)), g_i(p^{-1}(z))) > 5\varepsilon_{i+1}$. Hence,

$$\rho(\tilde{f}(x), \tilde{f}(x')) = \rho(h_l(x), h_l(x')) \geq \rho(h_i(x), h_i(x')) - 2\rho(h_l, h_i) \geq 5\varepsilon_{i+1} - 4\varepsilon_{i+1} > 0.$$

On the other hand, $\rho(\tilde{f}(x), \tilde{f}(x')) = 0$ because $\tilde{f}(x) = \tilde{f}(x')$. This contradiction completes the proof. ∎

Lemma 27.2. *Let $p : K \to M$ be a PL-map between compact polyhedra and let $f : K \to X$ be a map into a submetrizable space $X$ possessing the $m$-$\overline{\mathrm{DD}}^n$-property with $m = \dim M$ and $n = \dim(p)$. Assume that for some open cover $\mathcal{V}$ of $K$ and some closed subset $F \subset K$ with $F = p^{-1}(p(F))$ the restriction $p \triangle f|F : F \to M \times X$ is a $\mathcal{V}$-map. Then, for any open cover $\mathcal{U}$ of $X$, there is a map $\tilde{f} : K \to X$ such that $\tilde{f}$ is $\mathcal{U}$-homotopic to $f$, $\tilde{f}|F = f|F$, and $p \triangle \tilde{f} : K \to M \times X$ is a $\mathcal{V}$-map.*

*Proof.* Since $p \triangle f|F$ is a $\mathcal{V}$-map, there exists a closed neighborhood $O(F) \subset K$ of $F$ such that the restriction $(p \triangle f)|O(F)$ is also a $\mathcal{V}$-map (this follows from the closedness of the map $p \triangle f$). Because $F = p^{-1}(p(F))$ is a complete preimage, we can assume that so is $O(F) = p^{-1}(p(O(F))$. By Lemma 18.7, the $\mathcal{V}$-maps form an open set in the function space $C(O(F), M \times X)$. Consequently, we can find a continuous metric $\rho$ on $M \times X$ such that any map $g : O(F) \to M \times X$ with $\rho(g, p \triangle f|O(F)) < 1$ is a $\mathcal{V}$-map. By the compactness of $O(F)$, there is an open cover $\mathcal{U}'$ of $X$ such that for any $z \in O(F)$ and any $U \in \mathcal{U}'$ the set $\{p(z)\} \times U$ has $\rho$-diameter $< 1$. Without loss of generality, we may assume that the initial cover $\mathcal{U}$ of $X$ refines $\mathcal{U}'$.

Now, using Lemma 18.2 and Lemma 27.1, we find a $\mathcal{V}$-map $f' : K \to X$, $\mathcal{U}$-homotopic to $f$ and such that $p \triangle f' : K \to M \times X$ is a $\mathcal{V}$-map. Let $h : K \times [0,1] \to X$ be an $\mathcal{U}$-homotopy linking the maps $f$ and $f'$. Take any continuous function $\lambda : K \to [0,1]$ such that $\lambda(F) = \{0\}$ and $\lambda(K \setminus O(F)) \subset \{1\}$, and define a $\mathcal{U}$-homotopy $\tilde{h} : K \times [0,1] \to X$ by the formula $\tilde{h}(x,t) = h(x, \lambda(x)t)$. Finally, consider the map $\tilde{f} : K \to X$, $\tilde{f}(x) = \tilde{h}(x, 1)$. It is clear that $\tilde{f}$ is $\mathcal{U}$-homotopic to $f$ and $\tilde{f}|F = f|F$. It remains to show that $p \triangle \tilde{f} : K \to M \times X$ is a $\mathcal{V}$-map. Since $K$ is compact, it suffices to check that for any $(z, x) \in M \times X$ the set $(p \triangle \tilde{f})^{-1}(z, x) = p^{-1}(z) \cap \tilde{f}^{-1}(x)$ lies in some $V \in \mathcal{V}$.

Since $f$ and $\tilde{f}$ are $\mathcal{U}$-homotopic, $(p \triangle \tilde{f})|O(F)$ is 1-near to $(p \triangle f)|O(F)$ with respect to $\rho$. Hence, by the choice of $\rho$, $(p \triangle \tilde{f})|O(F)$ is a $\mathcal{V}$-map. Consequently, $(p \triangle \tilde{f})^{-1}(z, x) \subset O(F)$ lies in some $V \in \mathcal{V}$ provided $z \in p(O(F))$. If $z \notin p(O(F))$, then $(p \triangle \tilde{f})^{-1}(z, x) = (p \triangle f')^{-1}(z, x)$ is also contained in some $V \in \mathcal{V}$ because $p \triangle f'$ is a $\mathcal{V}$-map. ∎

Lemma 27.3. *Let $p : K \to M$ be a perfect PL-map between polyhedra and $f : K \to X$ be a map into a submetrizable space $X$ possessing the $m$-$\overline{\mathrm{DD}}^n$-property with $m = \dim M$ and $n = \dim(p)$. Then, for any continuous pseudometric $\rho$ on $X$, any continuous map $\varepsilon : K \to (0, 1]$ and any open cover $\mathcal{V}$ of $K$, there is a map $\tilde{f} : K \to X$ which is $\varepsilon$-homotopic to $f$ and $p \triangle \tilde{f} : K \to M \times X$ is a $\mathcal{V}$-map.*

*Proof.* Let $M^{(i)}$, $i \geq 0$, denote the $i$-dimensional skeleton of $M$ and $M^{(-1)} = \emptyset$. We put $f_{-1} = f$ and construct inductively a sequence $(f_i : K \to X)_{i\geq 0}$ of maps such that

- $f_i|p^{-1}(M^{(i-1)}) = f_{i-1}|p^{-1}(M^{(i-1)})$;
- $f_i$ is $\varepsilon/2^{i+2}$-homotopic to $f_{i-1}$;
- $p \triangle f_i$ restricted to $p^{-1}(M^{(i)})$ is a $\mathcal{V}$-map.

Assuming that the map $f_{i-1} : K \to X$ has been constructed, consider the complement $M^{(i)} \setminus M^{(i-1)} = \bigsqcup_{j\in J_i} \mathring{\sigma}_j$, which is the discrete union of open $i$-dimensional simplexes. According to Lemma 20.1 the preimage $p^{-1}(\sigma_j)$ of any simplex $\sigma_j$ is a compact subpolyhedron of $K$. Therefore, we can apply Lemma 27.2 to find maps $g_j : p^{-1}(\sigma_j) \to X$, $j \in J_i$, such that

- $g_j$ coincides with $f_{i-1}$ on the set $p^{-1}(\sigma_j^{(i-1)})$;
- $g_j$ is $\varepsilon/2^{i+2}$-homotopic to $f_{i-1}|p^{-1}(\sigma_j)$;
- $\big(p|p^{-1}(\sigma_j)\big) \triangle g_j$ is a $\mathcal{V}$-map.

Now, define a map $g : p^{-1}(M^{(i)}) \to X$ by the formula

$$g(x) = \begin{cases} f_{i-1}(x) & \text{if } x \in p^{-1}(M^{(i-1)}), \\ g_j(x) & \text{if } x \in p^{-1}(\sigma_j) \text{ for some } j \in J_i. \end{cases}$$

It can be shown that $g$ is $\varepsilon/2^{i+2}$-homotopic to $f_{i-1}|p^{-1}(M^{(i)})$. Moreover, it follows from Lemma 18.6 that the diagonal product $\big(p|p^{-1}(M^{(i)})\big) \triangle g$ is a $\mathcal{V}$-map. Since $p^{-1}(M^{(i)})$ is a subpolyhedron of $K$ (see Lemma 20.1), it is a neighborhood retract of $K$. So, we can apply the Borsuk homotopy extension lemma 18.3 to find a continuous extension $f_i : K \to X$ of $g$ which is $\varepsilon/2^{i+2}$-homotopic to $f_{i-1}$. This completes the inductive step.

Then the limit map $\tilde{f} = \lim_{i\to\infty} f_i : K \to X$ is well-defined, continuous and $\varepsilon$-homotopic to $f$ (the last two properties of $f$ hold because $f|\sigma$ has these properties for any simplex $\sigma \subset K$ and because of the definition of the CW-topology on $K$). Moreover, $p \triangle \tilde{f} : K \to M \times X$ is a $\mathcal{V}$-map since it is perfect and, for each point $(z, x) \in M^{(i)} \times X \subset M \times X$, the preimage $(p \triangle \tilde{f})^{-1}(z, x) = (p \triangle f_i)^{-1}(z, x)$ lies in some set $V \in \mathcal{V}$. ∎

LEMMA 27.4. *Let $p : K \to M$ be a perfect map between paracompact spaces, $\mathcal{V}$ be an open cover of $K$ and $X$ be a submetrizable space possessing the $m$-$\overline{\mathrm{DD}}^n$-property with $m = \dim M$ and $n = \dim_\triangle(p)$. Then, for any simplicially factorizable map $f : K \to X$, any continuous pseudometric $\rho$ on $X$ and any function $\varepsilon \in C(K, (0, 1])$ there exists a simplicially factorizable map $\tilde{f} : K \to X$ such that $p \triangle \tilde{f} : K \to M \times X$ is a $\mathcal{V}$-map and $\tilde{f}$ is $\varepsilon$-homotopic to $f$.*

*Proof.* Since $f$ is simplicially factorizable, it can be written as $f = f^L \circ f_L$ for some maps $f_L : K \to L$ and $f^L : L \to X$ and a polyhedron $L$. The pseudometric $\rho$ induces a continuous pseudometric $d : (x, y) \mapsto \rho(f^L(x), f^L(y))$ on $L$.

By [62] and [14], the polyhedron $L$ is a neighborhood retract of a locally convex space. Hence, $L$ is a Lefschetz ANE$[\infty]$. Taking into account that sufficiently near maps to $L$ can be linked by a small homotopy and applying Lemma 24.1, we can find an open cover $\mathcal{V}_1 \prec \mathcal{V}$ of $K$ such that for each $\mathcal{V}_1$-map $\alpha : K \to K'$ into a paracompact space $K'$ there is a map $\gamma : O(\overline{\alpha(K)}) \to L$ defined on a neighborhood of the closure of $\alpha(K)$ in $K'$ such that $\gamma \circ \alpha$ is $\varepsilon/2$-homotopic to $f_L$ with respect to the pseudometric $d$. Moreover, the cover $\mathcal{V}_1$ can be chosen so fine that $\inf \varepsilon(V) > 0$ for each $V \in \mathcal{V}_1$.

Next, applying Corollary 4.2, we choose a $\mathcal{V}_1$-map $\alpha : K \to K'$ into a polyhedron $K'$, a map $\beta : M \to M'$ into a polyhedron $M'$ with $\dim M' \leq \dim M$, and a perfect PL-map $p' : K' \to M'$ with $\dim(p') \leq \dim_\triangle(p)$ such that $p' \circ \alpha = \beta \circ p$. The choice of the cover $\mathcal{V}_1$ guarantees the existence of a map $\gamma : O(\overline{\alpha(K)}) \to L$ defined on a neighborhood of the closure of $\alpha(K)$ in $K'$ such that $\gamma \circ \alpha$ is $\varepsilon/2$-homotopic to $f_L$. Replacing the triangulation of $K'$ by a suitable subdivision, we may assume that no simplex of the triangulation of $K'$ meets both the set $\overline{\alpha(K)}$ and the complement of $O(\overline{\alpha(K)})$. Then the union $N$ of all simplexes of $K'$ meeting $\overline{\alpha(K)}$ is a subpolyhedron of $K'$ containing $\alpha(K)$ and lying in $O(\overline{\alpha(K)})$.

Now, we are going to construct a continuous function $\delta : N \to (0, 1]$ with $\delta \circ \alpha \leq \varepsilon$. Since $\alpha$ is a $\mathcal{V}_1$-map, there is an open cover $\mathcal{V}'$ of $K'$ such that the cover $\alpha^{-1}(\mathcal{V}') = \{\alpha^{-1}(V) : V \in \mathcal{V}'\}$ refines $\mathcal{V}_1$. So, for each $V \in \mathcal{V}'$ the number $\inf \varepsilon(\alpha^{-1}(V))$ is strictly positive (we put $\inf \emptyset = 1$). Then, using the paracompactness of the simplicial complex $N$, we may construct a positive continuous function $\delta : N \to (0, 1]$ such that $\delta(y) \leq \inf \varepsilon(\alpha^{-1}(y))$ for all $y \in N$. Indeed, we can assume that $\mathcal{V}'$ is locally finite and consider the lower semicontinuous set-valued map $\varphi \colon N \to (0, 1]$ defined by $\varphi(y) = \bigcup\{(0, \inf \varepsilon(\alpha^{-1}(V))] : y \in V \in \mathcal{V}'\}$. Then, by [60, Theorem 6.2, p. 116], $\varphi$ admits a continuous selection $\delta \colon N \to (0, 1]$. Obviously, $\delta$ is the required function with $\delta \circ \alpha \leq \varepsilon$.

According to Lemma 27.3, we can approximate the map $f^L \circ \gamma : N \to X$ by a map $g : N \to X$ such that $p' \triangle g : N \to M' \times X$ is a $\mathcal{V}'$-map and $g$ is $\delta/2$-homotopic to $f^L \circ \gamma$ (with respect to $\rho$). Since $\delta \circ \alpha \leq \varepsilon$, the maps $\tilde{f} = g \circ \alpha : K \to X$ and $f^L \circ \gamma \circ \alpha$ are $\varepsilon/2$-homotopic. On the other hand, since $f_L$ and $\gamma \circ \alpha$ are $\varepsilon/2$-homotopic with respect to $d$, so are $f = f^L \circ f_L$ and $f^L \circ \gamma \circ \alpha$. Consequently, $\tilde{f}$ and $f$ are $\varepsilon$-homotopic.

$$\begin{tikzcd}
K \arrow[rr, "f_L"] \arrow[dr, "\alpha"] \arrow[dd, "p"'] & & L \arrow[r, "f^L"] & X \\
& N \arrow[ur, "\gamma"] \arrow[urr, "g"'] \arrow[dd, "p'"] & & \\
M \arrow[dr, "\beta"'] & & & \\
& M' & &
\end{tikzcd}$$

It remains to verify that $p \triangle \tilde{f} : K \to M \times X$ is a $\mathcal{V}$-map. Since this map is perfect, it suffices to check that the preimage $(p \triangle \tilde{f})^{-1}(z, x) = p^{-1}(z) \cap \tilde{f}^{-1}(x)$ of any point $(z, x) \in M \times X$ is contained in an element of $\mathcal{V}$. Indeed, since $p' \triangle g$ is a $\mathcal{V}'$-map, $(p' \triangle g)^{-1}(\beta(z), x)$ lies in some set $V' \in \mathcal{V}'$. Then, by the choice of the cover $\mathcal{V}'$, $\alpha^{-1}(V') \subset V$ for some $V \in \mathcal{V}$. So, $\alpha^{-1}((p' \triangle g)^{-1}(\beta(z), x)) \subset \alpha^{-1}(V') \subset V$ and consequently,

$$\begin{aligned}
(p \triangle \tilde{f})^{-1}(z, x) = p^{-1}(z) \cap \tilde{f}^{-1}(x) \subset \\
&\subset (\beta \circ p)^{-1}(\beta(z)) \cap (g \circ \alpha)^{-1}(x) = (p' \circ \alpha)^{-1}(\beta(z)) \cap (g \circ \alpha)^{-1}(x) \\
&= \alpha^{-1}((p')^{-1}(\beta(z)) \cap g^{-1}(x)) = \alpha^{-1}((p' \triangle g)^{-1}(\beta(z), x)) \subset V. \quad \blacksquare
\end{aligned}$$

## 28. Proof of Theorem 3.3

Suppose $p : K \to M$ is a perfect map with $K$ being a submetrizable paracompact space, $Y$ a completely metrizable space and $X \subset Y$ a subspace with the $m$-$\overline{\mathrm{DD}}^n$-property, where $m = \dim Y$ and $n = \dim_\triangle(p)$. We need to prove that the set

$$\mathcal{E}(p, Y) = \{f \in C(K, Y) : p \triangle f : K \to M \times Y \text{ is an embedding}\}$$

is a $G_\delta$-set in $C(K, Y)$ having the approximation property from Theorem 3.3.

Because $K$ is submetrizable, it admits a continuous metric $d$. Everywhere in this proof, when considering $\varepsilon$-maps defined on $K$, we shall always refer to the metric $d$.

Since $p$ is perfect, so is $p \triangle f : K \to M \times Y$ for any $f : K \to Y$. Consequently, $p \triangle f$ is a closed embedding if and only if it is injective. Therefore, $\mathcal{E}(p, Y) = \bigcap_{k=1}^\infty \mathcal{E}_k$, where $\mathcal{E}_k = \{f \in C(K, Y) : p \triangle f \text{ is a } 1/k\text{-map}\}$, $k \geq 1$.

The following lemma implies that each $\mathcal{E}_k$ is open in $C(K, Y)$, so $\mathcal{E}(p, X) = \bigcap_{k=1}^\infty \mathcal{E}_k$ is a $G_\delta$-set in $C(K, Y)$.

Lemma 28.1. *For any open cover $\mathcal{V}$ of $K$ the set $\{f \in C(K, Y) : p \triangle f$ is a $\mathcal{V}$-map$\}$ is open in $C(K, Y)$.*

*Proof.* Let $\rho$ be a metric on $Y$ generating its topology. Fix any map $f : K \to Y$ such that $p \triangle f : K \to M \times Y$ is a $\mathcal{V}$-map and take an open cover $\mathcal{U}$ of $M \times Y$ whose preimage $(p \triangle f)^{-1}(\mathcal{U})$ refines the cover $\mathcal{V}$.

By the continuity of the maps $p$ and $f$, for every point $z \in K$ there exists an open neighborhood $W_z \subset K$ of $z$ and a positive real number $\varepsilon_z$ such that $p(W_z) \times B_\rho(f(W_z), 2\varepsilon_z) \subset U_z$ for some $U_z \in \mathcal{U}$. Let $\mathcal{W}$ be a locally finite open cover of the paracompact space $K$ refining $\{W_z : z \in K\}$. For each $W \in \mathcal{W}$ choose $z(W) \in K$ with $W \subset W_{z(W)}$ and let $\varepsilon_W = \varepsilon_{z(W)}$. Then $p(W) \times B_\rho(f(W), 2\varepsilon_W) \subset U_{z(W)}$. Let $\{\lambda_W : K \to [0, 1]\}_{W \in \mathcal{W}}$ be a partition of unity subordinated to $\mathcal{W}$, i.e., $\sum_{W \in \mathcal{W}} \lambda_W \equiv 1$ and $\lambda_W^{-1}(0, 1] \subset W$ for all $W \in \mathcal{W}$. Finally, consider the continuous function

$$\varepsilon : K \to (0, 1], \qquad x \mapsto \sum_{W \in \mathcal{W}} \lambda_W(x) \cdot \varepsilon_W.$$

We claim that for every $x \in K$ there exists a neighborhood $U(x) \in \mathcal{U}$ containing the product $\{p(x)\} \times B_\rho(f(x), 2\varepsilon(x))$. Indeed, use the definition of $\varepsilon(x)$ to find $W \in \mathcal{W}$ containing $x$ with $\varepsilon(x) \leq \varepsilon_W$. Then $\{p(x)\} \times B_\rho(f(x), 2\varepsilon(x)) \subset p(W) \times B_\rho(f(W), 2\varepsilon_W) \subset U_{z(W)}$.

Let us show that $p \triangle g : K \to M \times X$ is a $\mathcal{V}$-map for any $g \in B_\rho(f, \varepsilon)$. Because of Lemma 18.6, it suffices to check that the preimage $(p \triangle g)^{-1}(z, y) = p^{-1}(z) \cap g^{-1}(y)$ of any $(z, y) \in M \times X$ lies in some $V \in \mathcal{V}$. There is nothing to prove if this preimage is empty. So, assume that $p^{-1}(z) \cap g^{-1}(y) \neq \emptyset$. Since the last set is compact, there exists a point $x_0 \in p^{-1}(z) \cap g^{-1}(y)$ such that $\varepsilon(x_0) = \max\{\varepsilon(x) : x \in p^{-1}(z) \cap g^{-1}(y)\}$. As we already proved, there is $U(x_0) \in \mathcal{U}$ such that $\{p(x_0)\} \times B_\rho(f(x_0), 2\varepsilon(x_0)) \subset U(x_0)$. Then the choice of $\mathcal{U}$ implies that $(p \triangle f)^{-1}(U(x_0))$ lies in some set $V_0 \in \mathcal{V}$. We are going to establish that $(p \triangle g)^{-1}(z, y) \subset V_0$, which will complete the proof. Fix any $x \in p^{-1}(z) \cap g^{-1}(y)$ and observe that $g(x) = g(x_0)$. Then $\rho(f(x), f(x_0)) \leq \rho(f(x), g(x)) + \rho(g(x_0), f(x_0)) <$

$\varepsilon(x)+\varepsilon(x_0) \le 2\varepsilon(x_0)$. Consequently, $(p(x), f(x)) \in \{p(x_0)\} \times B_\rho(f(x_0), 2\varepsilon(x_0)) \subset U(x_0)$. Hence, $x \in (p \triangle f)^{-1}(U(x_0)) \subset V_0$. ∎

Therefore the set $\mathcal{E}(p, Y)$ is of type $G_\delta$ in $C(K, Y)$. It remains to prove that for any continuous pseudometric $\rho$ on $Y$, a continuous map $\varepsilon : K \to (0, 1]$ and a simplicially factorizable map $f : K \to X$ there is a map $f_\infty \in \mathcal{E}(p, Y)$ and an $\varepsilon$-homotopy $h : K \times [0, 1] \to Y$ connecting $f$ and $f_\infty$ so that $h(K \times [0, 1)) \subset X$. Since $Y$ is completely metrizable, replacing the pseudometric $\rho$ by a larger complete metric, we may assume that $\rho$ is a complete metric on $Y$ generating its topology.

Let $f_0 = f$ and $\varepsilon_0 = \varepsilon/3$. We shall construct by induction a sequence $(f_i : K \to X)_{i\ge1}$ of simplicially factorizable maps, a sequence $(\varepsilon_i : K \to (0, 1])_{i\ge1}$ of positive continuous functions, and a sequence $(h_i : K \times [0, 1] \to X)_{i\ge1}$ of $\varepsilon_{i-1}$-homotopies satisfying

- $h_{i+1}(z, 0) = f_i(z)$ and $h_{i+1}(z, 1) = f_{i+1}(z)$ for every $z \in K$;
- $p \triangle f_{i+1} : K \to M \times X$ is a $\frac{1}{i+1}$-map;
- $\varepsilon_{i+1} \le \varepsilon_i/2$;
- for each map $g \in B_\rho(f_{i+1}, 3\varepsilon_{i+1})$ the diagonal product $p \triangle g$ is a $\frac{1}{i+1}$-map.

Suppose for some $i$ we have already constructed simplicially factorizable functions $f_1, \dots, f_i$, positive functions $\varepsilon_1, \dots, \varepsilon_i$, and homotopies $h_1, \dots, h_i$ satisfying the above conditions. By Lemma 27.4, there exists a simplicially factorizable map $f_{i+1} : K \to X$ $\varepsilon_i$-homotopic to $f_i$ such that $p \triangle f_{i+1} : K \to M \times X$ is a $\frac{1}{i+1}$-map. Let $h_{i+1} : K \times [0, 1] \to X$ be an $\varepsilon_i$-homotopy connecting the maps $f_i$ and $f_{i+1}$. According to Lemma 28.1, the set $O(f_{i+1}) = \{g \in C(K, X) : p \triangle g \text{ is a } \frac{1}{i+1}\text{-map}\}$ is open in $C(K, X)$. Consequently, there is a positive function $\varepsilon_{i+1} \le \varepsilon_i/2$ such that $B_\rho(f_{i+1}, 3\varepsilon_{i+1}) \subset O(f_{i+1})$. This completes the inductive step.

It follows from the construction that $(f_i)_{i\ge1}$ converges uniformly to some continuous function $f_\infty : K \to Y$. Obviously, $\rho(f_\infty, f_i) \le \sum_{j=i}^\infty \varepsilon_j \le 2\varepsilon_i$ for every $i$. Hence, by the choice of the sequences $(\varepsilon_i)$ and $(f_i)$, $p \triangle f_\infty : K \to M \times Y$ is a $1/i$-map for every $i \ge 1$. So, $p \triangle f_\infty$ is injective. Moreover, the $\varepsilon_{i-1}$-homotopies $h_i$ lead to an $\varepsilon$-homotopy $h : K \times [0, 1] \to X$,

$$h(z, t) = \begin{cases} h_i\big(z, 2^i(t - 1 + \frac{1}{2^{i-1}})\big) & \text{if } t \in [1 - \frac{1}{2^{i-1}}, 1 - \frac{1}{2^i}] \text{ for some } i \ge 1, \\ f_\infty(z) & \text{if } t = 1, \end{cases}$$

connecting $f$ and $f_\infty$ and having the property $h(K \times [0, 1)) \subset X$.

## 29. Proof of Proposition 5.2

The aim of this section is to prove Proposition 5.2. This proposition asserts that a Tychonoff space $X$ has the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property if and only if $X \in a$-$\overline{\mathrm{DD}}^{\{b,c\}}$ for all $a < m+1$, $b < n+1$ and $c < k+1$. This is trivial if all three numbers $m, n, k$ are finite. The "only if" part of this assertion is also trivial. To prove the "if" part, assume that a space $X$ has the $a$-$\overline{\mathrm{DD}}^{\{b,c\}}$-property for all $a < m+1$, $b < n+1$ and $c < k+1$, and fix maps $f : \mathbb{I}^m \times \mathbb{I}^n \to X$, $g : \mathbb{I}^m \times \mathbb{I}^k \to X$ and an open cover $\mathcal{U}$ of $X$. By Lemma 18.2, there is a continuous pseudometric $\rho$ on $X$ such that $\operatorname{diam} B_\rho(x, 1) < \mathcal{U}$ for any $x \in f(\mathbb{I}^m \times \mathbb{I}^n) \cup g(\mathbb{I}^m \times \mathbb{I}^k)$.

For any integers $i < j+1$ we identify the $i$-dimensional cube $\mathbb{I}^i$ with the subset $\mathbb{I}^i \times \{0\}^{j-i}$ of the $j$-dimensional cube $\mathbb{I}^j$ and denote by $\mathrm{pr}^j_i : \mathbb{I}^j \to \mathbb{I}^i$ the natural projection of $\mathbb{I}^j$ onto $\mathbb{I}^i$.

Fix any continuous metric $d$ in $\mathbb{I}^m \times \mathbb{I}^n$ having convex balls with respect to the natural convex structure of $\mathbb{I}^m \times \mathbb{I}^n \subset \mathbb{R}^m \times \mathbb{R}^n$ and find $\delta > 0$ such that $\rho(f(x), f(y)) < 1/2$ for any $x, y \in \mathbb{I}^m \times \mathbb{I}^n$ with $d(x,y) < \delta$.

If $a < m+1$ and $b < n+1$, we also consider the product map

$$\mathrm{pr}^{mn}_{ab} : \mathbb{I}^m \times \mathbb{I}^n \to \mathbb{I}^a \times \mathbb{I}^b, \qquad (x,y) \mapsto (\mathrm{pr}^m_a(x), \mathrm{pr}^n_b(y)),$$

which tends to the identity map of $\mathbb{I}^m \times \mathbb{I}^n$ in $C(\mathbb{I}^m \times \mathbb{I}^n, \mathbb{I}^m \times \mathbb{I}^n)$ as $a \to m$ and $b \to n$ (recall that $\mathbb{I}^a \times \mathbb{I}^b$ is a subset of $\mathbb{I}^m \times \mathbb{I}^n$). Using this convergence, we find $a_0 < m+1$ and $b < n+1$ such that $\mathrm{dist}(x, \mathrm{pr}^{m,n}_{a,b}(x)) < \delta$ for any $x \in \mathbb{I}^m \times \mathbb{I}^n$ and any $a \in [a_0, m+1)$. Since $\mathrm{pr}^{m,n}_{ab}$ is homotopic to the identity of $\mathbb{I}^m \times \mathbb{I}^n$, the choice of $\delta$ implies that $f \circ \mathrm{pr}^{m,n}_{ab}$ is $1/2$-homotopic to $f$ (with respect to the pseudometric $\rho$ on $X$) for any $a \in [a_0, m+1)$.

By analogy, we find $a \in [a_0, m+1)$ and $c < k+1$ such that $g \circ \mathrm{pr}^{mk}_{ac}$ is $1/2$-homotopic to $g$, where $\mathrm{pr}^{mk}_{ac} = \mathrm{pr}^m_a \times \mathrm{pr}^k_c : \mathbb{I}^m \times \mathbb{I}^k \to \mathbb{I}^a \times \mathbb{I}^c$ is the projection corresponding to $a$ and $c$.

Since $X$ has the $a$-$\overline{\mathrm{DD}}^{\{b,c\}}$-property, there exist maps $f' : \mathbb{I}^a \times \mathbb{I}^b \to X$ and $g' : \mathbb{I}^a \times \mathbb{I}^c \to X$ such that $f'$ and $g'$ are $1/2$-homotopic to $f|\mathbb{I}^a \times \mathbb{I}^b$ and $g|\mathbb{I}^a \times \mathbb{I}^c$, respectively, and $f'(\{z\} \times \mathbb{I}^b) \cap g'(\{z\} \times \mathbb{I}^c) = \emptyset$ for all $z \in \mathbb{I}^a$. Then $\tilde{f} = f' \circ \mathrm{pr}^{mn}_{ab}$ and $\tilde{g} = g' \circ \mathrm{pr}^{mk}_{ac}$ are 1-homotopic to $f, g$, respectively. It remains to prove that $\tilde{f}(\{z\} \times \mathbb{I}^n) \cap \tilde{g}(\{z\} \times \mathbb{I}^k) = \emptyset$ for all $z \in \mathbb{I}^m$. Assuming that this is not the case, we find $x \in \mathbb{I}^n$ and $y \in \mathbb{I}^k$ such that $\tilde{f}(z,x) = \tilde{g}(z,y)$. Let $z' = \mathrm{pr}^m_a(z)$, $x' = \mathrm{pr}^n_b(x)$, and $y' = \mathrm{pr}^k_c(y)$. Then

$$f'(z',x') = f' \circ \mathrm{pr}^{mn}_{ab}(z,x) = \tilde{f}(z,x) = \tilde{g}(z,y) = g' \circ \mathrm{pr}^{mk}_{ac}(z,y) = g'(z',y'),$$

which contradicts the choice of the maps $f', g'$.

## 30. Local nature of the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property

The aim of this section is to prove Proposition 5.4 on the local nature of the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property. According to Michael's theorem on local properties [52] this will be done as soon as we check the following three conditions:

(1) Any open subspace of a space with the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property has this property.
(2) A space has the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property provided it is a discrete topological sum of spaces with the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$.
(3) A paracompact submetrizable space has the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property if it is the union of two open subspaces with the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$.

The first two conditions trivially hold. So, it remains to check the last one.

Assume that a paracompact submetrizable space $X$ is the union $X = X_0 \cup X_1$ of two open subspaces $X_0, X_1 \subset X$ with the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property. To check the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property of $X$, fix an open cover $\mathcal{U}$ of $X$ and two maps $f : \mathbb{I}^m \times \mathbb{I}^n \to X$, $g : \mathbb{I}^m \times \mathbb{I}^k \to X$.

Choose two open sets $W_0, W_1 \subset X$ such that $X = W_0 \cup W_1$ and $W_i \subset \overline{W}_i \subset X_i$ for $i \in \{0,1\}$. Using Lemma 18.1, we can find a continuous metric $\rho$ on $X$ such that

- $\operatorname{dist}(X \setminus W_0, X \setminus W_1) \geq 1$;
- $B_\rho(\overline{W}_i, 1) \subset X_i$ for $i \in \{0,1\}$;
- each set of diameter $< 1$ in $(X, \rho)$ lies in some $U \in \mathcal{U}$.

Let $T_n, T_k$ be triangulations of the cubes $\mathbb{I}^m \times \mathbb{I}^n$ and $\mathbb{I}^m \times \mathbb{I}^k$, respectively, such that $\operatorname{diam}_\rho f(\sigma_n) < 1/2$ and $\operatorname{diam}_\rho g(\sigma_k) < 1/2$ for all $\sigma_n \in T_n$ and $\sigma_k \in T_k$.

For every $i \in \{0,1\}$ consider the sets

$$N_i = \bigcup\{\sigma_n \in T_n : f(\sigma_n) \cap \overline{W}_i \neq \emptyset\}, \qquad K_i = \bigcup\{\sigma_k \in T_k : g(\sigma_k) \cap \overline{W}_i \neq \emptyset\}.$$

It is clear that $N_0 \cup N_1 = \mathbb{I}^m \times \mathbb{I}^n$, $K_0 \cup K_1 = \mathbb{I}^m \times \mathbb{I}^k$, and $f(N_i) \cup g(K_i) \subset X_i$ for $i \in \{0,1\}$. Moreover, $N_i$ and $K_i$, $i \in \{0,1\}$, are polyhedra.

Since $X_0$ possesses the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property, we can apply Theorem 5.3 to find two maps $f_0 : N_0 \to X_0$, $g_0 : K_0 \to X_0$ such that $f_0$ is $1/4$-homotopic to $f|N_0$, $g_0$ is $1/4$-homotopic to $g|K_0$ and $f_0(z,x) \neq g_0(z,y)$ for any $(z,x) \in N_0$, $(z,y) \in K_0$. Take a positive number $\delta < 1/4$ such that

$$3\delta < \min\{\operatorname{dist}(f_0(z,x), g_0(z,y)) : (z,x) \in N_0,\ (z,y) \in K_0\}.$$

Next, using the Borsuk homotopy extension lemma 18.3, we extend the maps $f_0, g_0$ to maps $\bar{f}_0 : \mathbb{I}^m \times \mathbb{I}^n \to X$ and $\bar{g}_0 : \mathbb{I}^m \times \mathbb{I}^k \to X$ such that $\bar{f}_0$ is $1/4$-homotopic to $f$ and $\bar{g}_0$ is $1/4$-homotopic to $g$.

We claim that $\bar{f}_0(N_1) \cup \bar{g}_0(K_1) \subset X_1$. Indeed, given any point $x \in N_1$ we can find a simplex $\sigma_n \in T_n$ containing $x$ such that $f(\sigma_n)$ intersects $\overline{W}_1$. Take a point $z \in \sigma_n$ with $f(z) \in \overline{W}_1$. Then $\operatorname{dist}(\bar{f}_0(x), f(z)) \leq \operatorname{dist}(\bar{f}_0(x), f(x)) + \operatorname{dist}(f(x), f(z)) < \operatorname{dist}(\bar{f}_0, f) + \operatorname{diam} f(\sigma_n) < 1/4 + 1/2 < 1$. Consequently, $\bar{f}_0(x)$ lies in the 1-neighborhood of $\overline{W}_1$. Since $\overline{W}_1 \subset X_1$, $\bar{f}_0(x) \in X_1$. Thus, $\bar{f}_0(N_1) \subset X_1$. By analogy, $\bar{g}_0(K_1) \subset X_1$.

Now, applying Theorem 5.3 to the space $X_1$ and the maps $\bar{f}_0|N_1$ and $\bar{g}_0|K_1$, we find $f_1 : N_1 \to X_1$, $g_1 : K_1 \to X_1$ such that $f_1$ is $\delta$-homotopic to $\bar{f}_0|N_1$, $g_1$ is $\delta$-homotopic to $\bar{g}_0|K_1$ and $f_1(z,x) \neq g_1(z,y)$ for any $(z,x) \in N_1$ and $(z,y) \in K_1$. Using again the Borsuk homotopy extension lemma, we extend $f_1, g_1$ to $f' : \mathbb{I}^m \times \mathbb{I}^n \to X$ and $g' : \mathbb{I}^m \times \mathbb{I}^k \to X$ such that $f'$ is $\delta$-homotopic to $\bar{f}_0$ and $g'$ is $\delta$-homotopic to $\bar{g}_0$.

Then $f'$ is $(\delta + 1/4)$-homotopic to $f$ and $g'$ is $(\delta + 1/4)$-homotopic to $g$. The choice of the metric for $X$ ensures that the map $f'$ is $\mathcal{U}$-homotopic to $f$ while $g'$ is $\mathcal{U}$-homotopic to $g$. It remains to check that $f'(z,x) \neq g'(z,y)$ for all $(x,y,z) \in \mathbb{I}^n \times \mathbb{I}^k \times \mathbb{I}^m$. To this end, we consider all possible cases for $(z,x)$ and $(z,y)$.

1. If $(z,x) \in N_1$, $(z,y) \in K_1$, then $f'(z,x) = f_1(z,x) \neq g_1(z,y) = g'(z,y)$ according to the construction of the maps $f_1, g_1$.

2. If $(z,x) \in N_0$, $(z,y) \in K_0$, then

$$\operatorname{dist}(f'(z,x), g'(z,y)) \geq \operatorname{dist}(\bar{f}_0(z,x), \bar{g}_0(z,y)) - \operatorname{dist}(\bar{f}_0, f') - \operatorname{dist}(\bar{g}_0, g') \geq 3\delta - 2\delta > 0.$$

3. If $(z,x) \notin N_1$ and $(z,y) \notin K_0$, then $f(z,x) \in X \setminus W_1$ and $g(z,y) \in X \setminus W_0$. Thus,

$$\begin{aligned}\operatorname{dist}(f'(z,x), g'(z,x)) &\geq \operatorname{dist}(f(z,x), g(z,y)) - \operatorname{dist}(f', f) - \operatorname{dist}(g', g)\\ &\geq \operatorname{dist}(X \setminus W_1, X \setminus W_0) - \operatorname{dist}(f', f) - \operatorname{dist}(g', g)\\ &\geq 1 - 2(\delta + 1/4) > 0.\end{aligned}$$

4. The case $(z,x) \notin N_0$ and $(z,y) \notin K_1$ is analogous to the third one.

## 31. Proof of Theorem 5.7

The "if" part follows from the definition of the $M$-MAP$^n$-property. To prove the "only if" part, take any finite or infinite numbers $m$ and $k \le n$, and assume that $X$ is a Polish ANE$[m+n+1]$-space possessing the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property. Given a separable simplicial complex $M$ with $\dim M \le m$ we should construct disjoint $\sigma$-compact subsets $E, F \subset M \times X$ such that $E$ has $M$-MAP$^n$ and $F$ has $M$-MAP$^k$.

Fix any complete metric $\rho$ generating the topology of $X$ and consider the open cover

$$\mathcal{W} = \{B_\rho(x, t/18) \times (2t/3, 4t/3) : (x,t) \in X \times (0,1]\}$$

of the product $X \times (0,1]$. Since $X \times (0,1]$ is a metrizable (Lefschetz) ANE$[m+n]$-space, there is a cover $\mathcal{W}_0$ of $X \times (0,1]$ such that each partial $\mathcal{W}_0$-realization $f : L^{(0)} \to X \times (0,1]$ of an at most $(m+n)$-dimensional simplicial complex $L$ extends to a full $\mathcal{W}$-realization $\bar f : L \to X \times (0,1]$ of $L$. Let $\mathcal{W}_1$ be an open cover of $X \times (0,1]$ with $St(\mathcal{W}_1) \prec \mathcal{W}_0$.

Let $Q$ denote the set of rational numbers on $(0,1]$ and let $Q_M = \{\chi \in M : \chi(\bigcup M) \subset \{0\} \cup Q\}$ be the canonical countable dense subset in the complex $M$ (recall that we identify the abstract complex $M$ with its geometrical realization $|M|$). Fix also any countable dense set $Q_X$ in $X$. Let $\mathbb{J} = (0,1]$ and denote by $\mathrm{pr}_M, \mathrm{pr}_X, \mathrm{pr}_{\mathbb{J}}, \mathrm{pr}_{X\times\mathbb{J}}$ the natural projections of $M \times X \times (0,1]$ onto $M$, $X$, $(0,1]$, and $X \times (0,1]$, respectively.

Consider the abstract simplicial complex $L$ of dimension $m+n$ consisting of all at most $(m+n+1)$-element finite subsets $\sigma \subset Q_M \times Q_X \times Q$ such that $\mathrm{pr}_{X\times\mathbb{J}}(\sigma) \prec \mathcal{W}_0$ and $\mathrm{pr}_M(\sigma) \subset \tau$ for some simplex $\tau \in M$. The geometric realization of the abstract simplicial complex $L$ will be denoted by the same letter $L$.

Define a function $\Psi^{(0)} : L^{(0)} \to M \times X \times (0,1]$ assigning to each singleton $\{(y,x,t)\} \in L^{(0)}$ the point $(y,x,t) \in Q_M \times Q_X \times Q$. This map can be represented as $\Psi^{(0)} = (\Psi^{(0)}_M, \Psi^{(0)}_X, \Psi^{(0)}_{\mathbb{J}})$, where $\Psi^{(0)}_M : L \to M$, $\Psi^{(0)}_X : L \to X$ and $\Psi^{(0)}_{\mathbb{J}} : L \to (0,1]$ are the coordinate maps of $\Psi^{(0)}$. Extend the maps $\Psi^{(0)}_M$ and $\Psi^{(0)}_{\mathbb{J}}$ to PL-maps $\Psi_M : L \to M$ and $\Psi_{\mathbb{J}} : L \to (0,1]$ by linearity (which means that the maps $\Psi_M$ and $\Psi_{\mathbb{J}}$ are affine on each geometric simplex of $L$).

Next, we extend the partial realization $\Psi^{(0)}_X$ to a full realization $\Psi_X : L \to X$ of $L$ such that

$$\operatorname{diam}_\rho \Psi_X(\sigma) < \tfrac{1}{6} \min \Psi_{\mathbb{J}}(\sigma) \quad \text{for every simplex } \sigma \text{ of } L. \tag{1}$$

This can be done as follows: According to the definition of the complex $L$, the function $\mathrm{pr}_{X\times\mathbb{J}} \circ \Psi^{(0)} : L^{(0)} \to X \times (0,1]$ is a partial $\mathcal{W}_0$-realization of $L$. So, it can be extended to a full $\mathcal{W}$-realization $\overline{\Psi} : L \to X \times (0,1]$ of $L$. Let $\overline{\Psi}_X : L \to X$ and $\overline{\Psi}_{\mathbb{J}} : L \to (0,1]$ be the coordinate functions of $\Psi$. We claim that the map $\Psi_X = \overline{\Psi}_X$ satisfies (1). Indeed, for every geometric simplex $\sigma$ of $L$, we can find $W \in \mathcal{W}$ containing $\overline{\Psi}(\sigma)$. The element $W$ is of the form $W = B_\rho(x, t/18) \times (2t/3, 4t/3)$ for some $(x,t) \in X \times (0,1]$. Then $\Psi_X(\sigma) = \overline{\Psi}_X(\sigma) \subset B_\rho(x, t/18)$ and hence

$$\operatorname{diam}_\rho \Psi_X(\sigma) < \tfrac{2}{18} t < \tfrac{1}{6} \min \overline{\Psi}_{\mathbb{J}}(\sigma^{(0)}) = \tfrac{1}{6} \min \Psi_{\mathbb{J}}(\sigma^{(0)}) = \tfrac{1}{6} \min \Psi_{\mathbb{J}}(\sigma).$$

Next, we consider the subcomplexes $N$ and $K$ of $L$ defined by

$$N = \{\sigma \in L : \dim \sigma - \dim(\Psi_M(\sigma)) \le n\}, \qquad K = \{\sigma \in L : \dim \sigma - \dim(\Psi_M(\sigma)) \le k\}.$$

Let $p_N : N \to M$ and $p_K : K \to M$ be the restrictions of the PL-map $\Psi_M$ the subcomplexes $N$ and $K$. It is clear that $\dim(p_N) \le n$ and $\dim(p_K) \le k$.

For any two simplexes $\sigma \in N$ and $\tau \in K$ consider the open subset

$$\mathcal{D}_{\sigma,\tau} = \{(f,g) \in C(N,X) \times C(K,X) : (p_N \triangle f)(\sigma) \cap (p_K \triangle g)(\tau) = \emptyset\}$$

of $C(N,X)\times C(K,X)$. By Theorem 5.3, this set is dense in $C(N,X)\times C(K,X)$. The space $C(N,X)\times C(K,X)$, being homeomorphic to $C(N \sqcup K, X)$, is a Baire space. Here, $N \sqcup K$ is the disjoint topological sum of $N$ and $K$. Since the complexes $N, K$ are countable, the intersection $\mathcal{D} = \bigcap_{\sigma\in N,\tau\in K} \mathcal{D}_{\sigma,\tau}$ is dense in $C(N,X)\times C(K,X)$. Consequently, there are functions $\psi_N : N \to X$ and $\psi_K : K \to X$ such that

$$(\psi_N, \psi_K) \in \mathcal{D}, \tag{2}$$

$$\rho(\psi_N(x), \Psi_X(x)) < \tfrac{1}{6}\Psi_{\mathbb{J}}(x) \quad \text{for all } x \in N, \tag{3}$$

$$\rho(\psi_K(x), \Psi_X(x)) < \tfrac{1}{6}\Psi_{\mathbb{J}}(x) \quad \text{for all } x \in K. \tag{4}$$

Finally, let $E = (p_N \triangle \psi_N)(N)$ and $F = (p_K \triangle \psi_K)(K)$. Note that $E, F$ are disjoint $\sigma$-compact subsets of $M \times X$. It remains to check that $E$ has the $M$-MAP$^n$-property in $M \times X$ and $F$ has $M$-MAP$^k$-property in $M \times X$. We shall only prove that $E$ has $M$-MAP$^n$; the $M$-MAP$^k$-property of $F$ can be established similarly.

To prove the $M$-MAP$^n$-property of the set $E$, take any at most $n$-dimensional map $p : A \to M$ of a finite-dimensional metrizable compactum $A$, a closed subset $B \subset A$, a map $f : A \to X$, and an open cover $\mathcal{U}$ of $X$. We need to find a map $\bar{g}: A \to X$, $\mathcal{U}$-homotopic to $f$, such that $\bar{g}|B = f|B$ and $(p \triangle \bar{g})(A \setminus B) \subset E$. The image $p(A)$, being a compact subset of $M$, lies in some compact subcomplex $M'$ of $M$. Observe that $\dim A \le \dim(p) + \dim(M') \le n + m$. Since $X$ is an ANE$[m+n+1]$-space, there exists $\varepsilon \le 1$ such that a map $f' : A \to X$ is $\mathcal{U}$-homotopic to $f$ provided $f'$ is $\varepsilon$-near to $f$.

Claim. *There is a map $\xi : A \to [0,\varepsilon]$ such that $\xi^{-1}(0) = B$ and the diagonal map $p \triangle \xi : A \to M \times [0,\varepsilon]$ restricted to $A \setminus B$ is $(n'-1)$-dimensional, where $n' = \dim(p) \le n$.*

Indeed, take any map $\tilde{\xi} : A \to [0,\varepsilon]$ with $\tilde{\xi}^{-1}(0) = B$ and put $\tilde{\xi}' = \tilde{\xi}|(A \setminus B)$. Since the restriction $p' = p|(A \setminus B)$ is a $\sigma$-perfect $n'$-dimensional map, we may apply Lemma 19.3 to conclude that the function space $C(A \setminus B, [0,\varepsilon])$ contains a dense subset of maps $\xi : A \setminus B \to [0,\varepsilon]$ such that $p' \triangle \xi$ is $(n'-1)$-dimensional. Consequently, there is a map $\xi : A \setminus B \to [0,\varepsilon]$ such that $p' \triangle \xi : A \setminus B \to M \times \mathbb{I}$ is $(n'-1)$-dimensional and $|\xi(a) - \tilde{\xi}'(a)| < \frac{1}{2}\tilde{\xi}'(a)$ for all $a \in A \setminus B$. Then $\xi(A \setminus B) \subset (0,\varepsilon]$ and $\xi$ can be extended to a continuous map from $A$ to $[0,\varepsilon]$ (which is denoted again by $\xi$) with $\xi(B) = \{0\}$. Obviously, $\xi^{-1}((0,\varepsilon]) = A \setminus B$. Hence, $\xi|(A \setminus B)$ is a perfect map.

Being a metrizable ANE$[m+n]$-space, $X$ is a Lefschetz ANE$[m+n]$-space (see Proposition 3.4). So, by Lemma 24.1, there exists an open cover $\mathcal{V}$ of $A \setminus B$ with the following property: for any $\mathcal{V}$-map $\alpha : A \setminus B \to P$ into a paracompact space $P$ with $\dim P \le m+n$

there is a map $\gamma : O(\overline{\alpha(A \setminus B)}) \to X$ defined on a neighborhood of the closure of $\alpha(A \setminus B)$ in $P$ such that $\rho(\gamma \circ \alpha, f|A \setminus B) < \xi/6$.

Because $\xi|(A \setminus B)$ is a perfect map, so is $p \triangle \xi|(A \setminus B) : A \setminus B \to M' \times (0,1]$. Moreover, $(p \triangle \xi)|(A \setminus B)$ is an $(n'-1)$-dimensional map. Consequently, we can apply Lemma 25.3 (for the map $(p \triangle \xi)|(A \setminus B)$ and the identity map on $M' \times (0,1]$) to find a $\mathcal{V}$-map $\alpha : A \setminus B \to S$ to a simplicial complex $S$ and a perfect $(n'-1)$-dimensional rational PL-map $\tilde{p} : S \to M' \times (0,1]$ such that $\tilde{p} \circ \alpha = p \triangle \xi$. Then

$$\dim S \leq \dim(M' \times (0,1]) + \dim(\tilde{p}) \leq m + 1 + (n'-1) \leq m + n.$$

Represent $\tilde{p}$ as $\tilde{p} = q \triangle \delta$ for two rational PL-maps $q : S \to M$ and $\delta : K \to (0,1]$. The choice of $\mathcal{V}$ guarantees that there is a map $\gamma : O(\overline{\alpha(A \setminus B)}) \to X$ defined on a neighborhood of the closure of $\alpha(A \setminus B)$ in $S$ such that

$$\rho(\gamma \circ \alpha, f|A \setminus B) < \xi/6. \tag{5}$$

Replacing $S$ by a polyhedral neighborhood of $\overline{\alpha(A \setminus B)}$ with respect to a suitable rational subdivision of $S$, we can assume that $S = O(\overline{\alpha(A \setminus B)})$. Thus, $\gamma$ is defined on the whole space $S$. Moreover, taking a finer rational subdivision of $S$, we may also assume that for any simplex $\sigma \in S$ we have

$$(\gamma \triangle \delta)(\sigma) \prec \mathcal{W}_1. \tag{6}$$

Since $\mathcal{W}_1 \prec \mathcal{W}$, the condition (6) implies that $(\gamma \triangle \delta)(\sigma) \subset B_\rho(x, t/18) \times (2t/3, 4t/3)$. Hence,

$$\operatorname{diam}_\rho \gamma(\sigma) < 2t/18 < \tfrac{1}{6} \min \delta(\sigma) \quad \text{and} \quad \max \delta(\sigma) < 2 \min \delta(\sigma). \tag{7}$$

Consider the map $\beta^{(0)} : S^{(0)} \to L^{(0)}$ assigning to each vertex $v \in S^{(0)}$ the triple $(q(v), x_v, \delta(v)) \in Q_M \times Q_X \times Q$ (recall that $\delta$ is a rational $PL$-map, so $\delta(v) \in Q$), where a point $x_v \in Q_X$ is chosen so that

$$\{(x_v, \delta(v)), (\gamma(v), \delta(v))\} \prec \mathcal{W}_1. \tag{8}$$

Taking into account that $\mathcal{W}_1 \prec \mathcal{W}$ and repeating the preceding argument we can check that

$$\rho(x_v, \gamma(v)) < \delta(v)/6. \tag{9}$$

We claim that $\beta^{(0)}$ extends to a simplicial map $\beta : S \to L$. To prove this claim, it suffices to check that $\beta^{(0)}(\sigma^{(0)})$ is a simplex of $L$ for any $\sigma \in S$. Since $p_M(\beta^{(0)}(\sigma^{(0)})) = q(\sigma^{(0)})$ lies in some simplex of $M$, we need only show that $\mathrm{pr}_{X \times \mathbb{J}}(\beta^{(0)}(\sigma^{(0)}))$ lies in some element of the cover $\mathcal{W}_0$. This follows from (6), (8) and $St(\mathcal{W}_1) \prec \mathcal{W}_0$. Thus, $\beta^{(0)}$ extends to a simplicial map $\beta : S \to L$. Since both $q \triangle \delta$ and $(\Psi_M \triangle \Psi_{\mathbb{J}}) \circ \beta$ are PL-maps whose restrictions on $S^{(0)}$ coincide, they are identical. Hence,

$$\Psi_{\mathbb{J}} \circ \beta(b) = \delta(b) \quad \text{and} \quad \Psi_M \circ \beta(b) = q(b) \quad \text{for all } b \in S. \tag{10}$$

Recall that $q \triangle \delta$ is $(n'-1)$-dimensional, so $q : S \to M$ is $n'$-dimensional. This, according to the definition of the complex $N$, implies that $\beta(S) \subset N$. Hence, we obtain the following diagram which is commutative except for the arrows going to $X$:

$$\begin{tikzcd}
A\setminus B \arrow[rrr, "f"] \arrow[dr, "\alpha"] \arrow[ddd, "p\triangle\xi"'] & & & X \\
& S \arrow[urr, "\gamma"] \arrow[dr, "\beta"] \arrow[ddl, "q\triangle\delta"] & & \\
& & N \arrow[uur, "\psi_N"] \arrow[dr] & \\
M\times(0,1] & & & L \arrow[uuu, "\Psi_X"'] \arrow[lll, "\Psi_M\triangle\Psi_{\mathbb{J}}"]
\end{tikzcd}$$

Consider the map $g = \psi_N \circ \beta \circ \alpha : A \setminus B \to X$. It is easily seen that (10) and the equality $p|(A \setminus B) = q \circ \alpha$ imply $(p \triangle g)(A \setminus B) \subset (p_N \triangle \psi_N)(N) = E$. It remains to show that $\rho(g(a), f(a)) \le \xi(a)$ for each $a \in A \setminus B$. Indeed, then we could extend $g$ to a continuous map $\bar{g} : A \to X$ letting $\bar{g}|B = f|B$. Moreover, since $\xi(a) \le \varepsilon$ for all $a \in A \setminus B$, the map $\bar{g}$ would be $\mathcal{U}$-homotopic to $f$ by the choice of $\varepsilon$. So, the proof will be complete.

To prove that $\rho(f(a), g(a)) \le \xi(a)$ for a given $a \in A \setminus B$, find a simplex $\sigma \in S$ with $b = \alpha(a) \in \sigma$ and fix any vertex $v$ of $\sigma$. Observe that $\Psi_{\mathbb{J}} \circ \beta(b) = \xi(a)$. Taking into account (5), (1), (3), (7), (10), we obtain

$$\begin{aligned}
\rho(f(a), g(a)) &= \rho(f(a), \psi_N \circ \beta(b) \\
&\le \rho(f(a), \gamma \circ \alpha(a)) + \rho(\gamma(b), \gamma(v)) + \rho(\gamma(v), \Psi_X \circ \beta(v)) \\
&\quad + \rho(\Psi_X \circ \beta(v), \Psi_X \circ \beta(b)) + \rho(\Psi_X \circ \beta(b), \psi_N \circ \beta(b)) \\
&\le \tfrac{1}{6}\xi(a) + \operatorname{diam} \gamma(\sigma) + \rho(\gamma(v), x_v) + \tfrac{1}{6}\Psi_{\mathbb{J}} \circ \beta(b) + \tfrac{1}{6}\Psi_{\mathbb{J}} \circ \beta(b) \\
&\le \tfrac{1}{6}\xi(a) + \tfrac{1}{6}\min \delta(\sigma) + \tfrac{1}{6}\delta(v) + \tfrac{2}{6}\xi(a) \le \\
&\le \tfrac{3}{6}\xi(a) + \tfrac{1}{6}\min \delta(\sigma) + \tfrac{1}{6}\max \delta(\sigma) \\
&\le \tfrac{1}{2}\xi(a) + \tfrac{1}{6}\min \delta(\sigma) + \tfrac{2}{6}\min \delta(\sigma) \le \tfrac{1}{2}\xi(a) + \tfrac{3}{6}\delta(b) = \xi(a).
\end{aligned}$$

## 32. Proof of the multiplication formulas

In this section we shall prove the multiplication formulas from Theorem 8.1. Let $X_0, X_1$ be two metrizable spaces and $\rho_0, \rho_1$ metrics generating their topologies. The metric $\rho((x_0, x_1), (x_0', x_1')) = \max\{\rho_0(x_0, x_1), \rho_1(x_0', x_1')\}$ on $X_0 \times X_1$ will be considered.

The next lemma provides the proof of the first multiplication formula.

Lemma 32.1. *If $X_0$ has the $m$-$\overline{\mathrm{DD}}^{\{n,k_0\}}$-property and $X_1$ has the $m$-$\overline{\mathrm{DD}}^{\{n,k_1\}}$-property, then $X_0 \times X_1$ has the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property for $k = k_0 + k_1 + 1$.*

*Proof.* Given any $\varepsilon > 0$ and maps $f = (f_0, f_1) : \mathbb{I}^m \times \mathbb{I}^n \to X_0 \times X_1$, $g = (g_0, g_1) : \mathbb{I}^m \times \mathbb{I}^k \to X_0 \times X_1$, we need to find $f' = (f_0', f_1') : \mathbb{I}^m \times \mathbb{I}^n \to X_0 \times X_1$ and $g' = (g_0', g_1') : \mathbb{I}^m \times \mathbb{I}^k \to X_0 \times X_1$ such that $f'$ is $\varepsilon$-homotopic to $f$, $g'$ is $\varepsilon$-homotopic to $g$ and $f'(\{z\} \times \mathbb{I}^n) \cap g'(\{z\} \times \mathbb{I}^k) = \emptyset$ for all $z \in \mathbb{I}^m$.

Take a triangulation $T$ of the cube $\mathbb{I}^k$ so fine that, for any simplex $\sigma \in T$ and any points $z \in \mathbb{I}^m$, the image $g(\{z\} \times \sigma)$ is of diameter $< \varepsilon/3$ (with respect to the metric $\rho$).

Let $K_0$ be the $k_0$-dimensional skeleton of the triangulation $T$ and $K_1$ be the dual to $K_0$ skeleton in the barycentric subdivision of $T$. Obviously, $K_1$ is $k_1$-dimensional.

Since each $X_i$, $i \in \{0,1\}$, has the $m$-$\overline{\mathrm{DD}}^{\{n,k_i\}}$-property, by Theorem 5.3 combined with the Borsuk homotopy extension lemma, there exist two maps $f'_i : \mathbb{I}^m \times \mathbb{I}^n \to X_i$ and $g''_i : \mathbb{I}^m \times \mathbb{I}^k \to X_i$ such that $f'_i$ is $\varepsilon$-homotopic to $f_i$, $g''_i$ is $\varepsilon/6$-homotopic to $g_i$ and $f'_i(\{z\} \times \mathbb{I}^n) \times g''_i(\{z\} \times K_i) = \emptyset$ for all $z \in \mathbb{I}^m$.

Since $\mathbb{I}^k$ is a subset of the join $K_0 * K_1$, each point $x \in \mathbb{I}^k$ contained in a maximal (closed) simplex $\sigma \in T$ can be written as $x = (1-\lambda(x))x_0 + \lambda(x)x_1$ for some points $x_i \in K_i \cap \sigma$, $i \in \{0,1\}$, and a unique number $\lambda(x) \in [0,1]$ called the *join-parameter* of $x$.

The numbers $\lambda_0(x) = \lambda(x)$, $\lambda_1(x) = 1 - \lambda(x)$ will be called the *join-coordinates* of $x$. Note that $x = \lambda_0(x)x_1 + \lambda_1(x)x_0$ with $x_i \in K_i$ for $i \in \{0,1\}$. Moreover, a point $x \in \mathbb{I}^k$ belongs to $K_i$ iff $\lambda_0(x) = i$ for $i \in \{0,1\}$.

Consider the piecewise-linear map $\ell : [0,1] \to [0,1]$ determined by the conditions $\ell(0) = 0 = \ell(1/2)$ and $\ell(1) = 1$. For every $i \in \{0,1\}$ define the piecewise-linear map $h_i : \mathbb{I}^k \to \mathbb{I}^k$, assigning to a point $x = \lambda_1 x_0 + \lambda_0 x_1$ with join-coordinates $(\lambda_0, \lambda_1)$ the point $h_i(x) = \mu_1 x_0 + \mu_0 x_1$ with join-coordinates $\mu_i = \ell(\lambda_i)$ and $\mu_{1-i} = 1 - \mu_i$.

The crucial property of the map $h_i$ is that for any point $x$ with $\lambda_i(x) \le 1/2$ its image $h_i(x)$ belongs to $K_i$. Moreover, the map $h_i$ is $\mathcal{S}$-homotopic to the identity map with respect to the cover $\mathcal{S}$ of $\mathbb{I}^k$ consisting of all maximal simplexes of the triangulation $T$. Observe that, for each maximal simplex $\sigma \in T$, we have $\operatorname{diam} g''_i(\sigma) \le 2\operatorname{dist}(g''_i, g_i) + \operatorname{diam} g_i(\sigma) < 2\varepsilon/6 + \varepsilon/3 = 2\varepsilon/3$.

Finally, for every $i \in \{0,1\}$ and $(z,x) \in \mathbb{I}^m \times \mathbb{I}^k$, let $g'_i(z,x) = g''_i(z, h_i(x))$. Then $g'_i$ is $2\varepsilon/3$-homotopic to $g''_i$ because $h_i$ is $\mathcal{S}$-homotopic to the identity map on $I^k$ and $\operatorname{diam} g''_i(\sigma) < 2\varepsilon/3$, $\sigma \in \mathcal{S}$. Since $g''_i$ is $\varepsilon/6$-homotopic to $g_i$, the map $g'_i$ is $\varepsilon$-homotopic to $g_i$. So, the diagonal map $g' = (g'_0, g'_1) : \mathbb{I}^m \times \mathbb{I}^k \to X_0 \times X_1$ is $\varepsilon$-homotopic to $g$. Moreover, by the choice of $f'_0, f'_1$, the diagonal map $f' = (f'_0, f'_1) : \mathbb{I}^m \times \mathbb{I}^n \to X_0 \times X_1$ is also $\varepsilon$-homotopic to $f$.

It remains to show that, for any $z \in \mathbb{I}^m$, the sets $g'(\{z\} \times \mathbb{I}^k)$ and $f'(\{z\} \times \mathbb{I}^n)$ do not intersect. Take any point $x \in \mathbb{I}^k$ and consider its join-parameter $\lambda(x)$. If $\lambda(x) \le 1/2$, then $h_0(x) \in K_0$ which yields $g'_0(z,x) = g''_0(z, h_0(x)) \notin f'_0(\{y\} \times \mathbb{I}^n)$. If $\lambda(x) \ge 1/2$, then $\lambda_1(x) = 1 - \lambda(x) \le 1/2$. Hence, $h_1(x) \subset K_1$ and $g'_1(z,x) = g''_1(z, h_1(x)) \notin f'_1(\{y\} \times \mathbb{I}^n)$. ∎

We turn now to the second multiplication formula.

LEMMA 32.2. *Let $m, k_0, k_1, n_0, n_1$ be non-negative integers with $k = k_0 + k_1 + 1$ and $n = n_0 + n_1 + 1$. Assume that $X_0$ has the $m$-$\overline{\mathrm{DD}}^{\{n_0,k_0\}}$-property and $X_1$ has both the $m$-$\overline{\mathrm{DD}}^{\{n,k_1\}}$- and $m$-$\overline{\mathrm{DD}}^{\{n_1,k\}}$-properties. Then $X_0 \times X_1$ has the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property.*

*Proof.* Given any $\varepsilon > 0$ and two maps $f = (f_0, f_1) : \mathbb{I}^m \times \mathbb{I}^n \to X_0 \times X_1$ and $g = (g_0, g_1) : \mathbb{I}^m \times \mathbb{I}^k \to X_0 \times X_1$, we have to find $f' = (f'_0, f'_1) : \mathbb{I}^m \times \mathbb{I}^n \to X_0 \times X_1$ and $g' = (g'_0, g'_1) : \mathbb{I}^m \times \mathbb{I}^k \to X_0 \times X_1$ such that $f'$ is $\varepsilon$-homotopic to $f$, $g'$ is $\varepsilon$-homotopic to $g$ and $f'(\{z\} \times \mathbb{I}^n) \cap g'(\{z\} \times \mathbb{I}^k) = \emptyset$ for all $z \in \mathbb{I}^m$.

Take triangulations $T_n$, $T_k$ of the cubes $\mathbb{I}^n$, $\mathbb{I}^k$ so fine that, for any simplexes $\sigma_n \in T_n$, $\sigma_k \in T_k$ and for any point $z \in \mathbb{I}^m$, the images $f(\{z\} \times \sigma_n)$ and $g(\{z\} \times \sigma_k)$ have diameter $< \varepsilon/3$ (with respect to the metric $\rho$ on $X_0 \times X_1$). Let $K_0$ be the $k_0$-dimensional skeleton of the triangulation $T_k$ and $K_1$ be the skeleton in the barycentric subdivision of $T_k$, dual to $K_0$. Also, let $N_0$ be the $n_0$-dimensional skeleton of the triangulation $T_n$ and $N_1$ its dual skeleton in the barycentric subdivision of $T_n$. Clearly, $K_1$ is $k_1$-dimensional and $N_1$ is $n_1$-dimensional.

Since $X_0$ has the $m$-$\overline{\mathrm{DD}}^{\{n_0,k_0\}}$-property, by Theorem 5.3 and the Borsuk homotopy extension lemma 18.3, there exist $f_0'' : \mathbb{I}^m \times \mathbb{I}^n \to X_0$ and $g_0'' : \mathbb{I}^m \times \mathbb{I}^k \to X_0$ such that $f_0''$ is $\varepsilon/6$-homotopic to $f_0$, $g_0''$ is $\varepsilon/6$-homotopic to $g_0$ and $f_0''(\{z\} \times N_0) \cap g_0''(\{z\} \times K_0) = \emptyset$ for all $z \in \mathbb{I}^m$.

Similarly, since $X_1$ has the $m$-$\overline{\mathrm{DD}}^{\{n,k_1\}}$-property, we can find $f_1''' : \mathbb{I}^m \times \mathbb{I}^n \to X_1$ and $g_1''' : \mathbb{I}^m \times \mathbb{I}^k \to X_1$ such that $f_1'''$ is $\varepsilon/12$-homotopic to $f_1$, $g_1'''$ is $\varepsilon/12$-homotopic to $g_1$ and

$$\varepsilon_1 = \min_{z \in \mathbb{I}^m} \operatorname{dist}(f_1'''(\{z\} \times \mathbb{I}^n), g_1'''(\{z\} \times K_1)),$$

is strictly positive. Let $\varepsilon_2 = \min\{\varepsilon_1/3, \varepsilon/12\}$. Using again Theorem 5.3, Lemma 18.3 and the $m$-$\overline{\mathrm{DD}}^{\{n_1,k\}}$-property of $X_1$, we choose $f_1'' : \mathbb{I}^m \times \mathbb{I}^n \to X_1$ and $g_1'' : \mathbb{I}^m \times \mathbb{I}^k \to X_1$ such that $f_1''$ is $\varepsilon_2$-homotopic to $f_1'''$, $g_1''$ is $\varepsilon_2$-homotopic to $g_1'''$ and $f_1''(\{z\} \times N_1) \cap g_1''(\{z\} \times \mathbb{I}^k) = \emptyset$ for any $z \in \mathbb{I}^m$. Then $f_1'', g_1''$ are $\varepsilon/6$-homotopic to $f_1, g_1$, respectively, and

$$\min_{z \in \mathbb{I}^m} \operatorname{dist}(f_1''(\{z\} \times \mathbb{I}^n), g_1''(\{z\} \times K_1)) \geq \varepsilon_1 - 2\varepsilon_2 \geq \tfrac{1}{3}\varepsilon_1 > 0.$$

Using the join structure of $\mathbb{I}^n \subset N_0 * N_1$, we represent each point $x \in \mathbb{I}^n$ contained in a maximal simplex $\sigma_n \in T_n$ in the form $x = (1 - \nu(x))x_0 + \nu(x)x_1$, where $x_i \in N_i \cap \sigma_n$, $i = 0, 1$, and $\nu(x) \in [0, 1]$ is the join-parameter of $x$. Let $\nu_0(x) = \nu(x)$ and $\nu_1(x) = 1 - \nu(x)$. Observe that a point $x \in \mathbb{I}^n$ belongs to the subcomplex $N_i$, $i \in \{0, 1\}$ iff $\nu_i(x) = 0$.

Similarly, since $\mathbb{I}^k \subset K_0 * K_1$, we represent each point $y \in \mathbb{I}^k$ which is contained in a maximal simplex $\sigma_k \in T_k$ in the form $y = (1 - \kappa(y))y_0 + \kappa(y)y_1$ for some points $y_i \in K_i \cap \sigma_k$, $i = 0, 1$, and some number $\kappa(y) \in [0, 1]$, the join-parameter of $y$. Let $\kappa_0(y) = \kappa(y)$ and $\kappa_1(y) = 1 - \kappa(y)$. Then a point $y \in \mathbb{I}^k$ belongs to the subcomplex $K_i$, $i \in \{0, 1\}$, iff $\kappa_i(y) = 0$.

Let $\ell : [0, 1] \to [0, 1]$ be the piecewise-linear map determined by the conditions $\ell(0) = 0 = \ell(1/2)$ and $\ell(1) = 1$. For every $i \in \{0, 1\}$ consider the piecewise-linear map $h_i^n : \mathbb{I}^n \to \mathbb{I}^n$ assigning to a point $x = \nu_1 x_0 + \nu_0 x_1$ with join-coordinates $(\nu_0, \nu_1)$ the point $h_i^n(x) = \nu_1' x_0 + \nu_0' x_1$ with join-coordinates $\nu_i' = \ell(\nu_i)$ and $\nu_{1-i}' = 1 - \nu_i'$.

The crucial property of the map $h_i^n$ is that $h_i^n(x)$ belongs to $K_i$ for any $x \in \mathbb{I}^n$ with $\nu_i(x) \leq 1/2$. Moreover, $h_i^n$ is $\mathcal{S}$-homotopic to the identity map with respect to the cover $\mathcal{S}$ of $\mathbb{I}^n$ consisting of all maximal simplexes of the triangulation $T_n$. Observe also that, for each maximal simplex $\sigma \in T_n$, we have $\operatorname{diam} f_i''(\sigma) \leq 2 \operatorname{dist}(f_i'', f_i) + \operatorname{diam} f_i(\sigma) < 2\varepsilon/6 + \varepsilon/3 = 2\varepsilon/3$.

Finally, for every $i \in \{0, 1\}$, let $f_i' = f_i'' \circ (\mathrm{id} \times h_i^n) : \mathbb{I}^m \times \mathbb{I}^n \to X_i$, i.e., $f_i' : (z, x) \mapsto f_i''(z, h_i^n(x))$. Note that $f_i'$ is $2\varepsilon/3$-homotopic to $f_i''$. Since $f_i''$ is $\varepsilon/6$-homotopic to $f_i$, the map $f_i'$ is $\varepsilon$-homotopic to $f_i$. Then the diagonal map $f' = (f_0', f_1') : \mathbb{I}^m \times \mathbb{I}^n \to X_0 \times X_1$ is $\varepsilon$-homotopic to $f$.

As above, for every $i \in \{0, 1\}$, we consider the piecewise-linear map $h_i^k : \mathbb{I}^k \to \mathbb{I}^k$, assigning to a point $y = \kappa_1 y_0 + \kappa_0 y_1$ with join-coordinates $(\kappa_0, \kappa_1)$ the point $h_i^k(y) =$

$\kappa'_1 y_0 + \kappa'_0 y_1$ with join-coordinates $\kappa'_i = \ell(\kappa_i)$ and $\kappa'_{1-i} = 1 - \ell(\kappa_i)$. Put $g'_i = g''_i \circ (\mathrm{id} \times h^k_i) : \mathbb{I}^m \times \mathbb{I}^k \to X_i$, $g'_i : (z, y) \mapsto g''_i(z, h^k_i(y))$, and check that the diagonal map $g' = (g'_0, g'_1) : \mathbb{I}^m \times \mathbb{I}^k \to X_0 \times X_1$ is $\varepsilon$-homotopic to $g$.

It remains to show that, for any $z \in \mathbb{I}^m$, the sets $f'(\{z\} \times \mathbb{I}^n)$ and $g'(\{z\} \times \mathbb{I}^k)$ do not intersect. Take any points $x \in \mathbb{I}^n$, $y \in \mathbb{I}^k$ and consider their join-parameters $\nu(x) = \nu_0(x)$ and $\kappa(y) = \kappa_0(y)$.

If $\nu_0(x) \le 1/2$ and $\kappa_0(y) \le 1/2$, then $h^n_0(x) \in N_0$ and $h^k_0(y) \in K_0$. The definition of $f'_i$ and $g'_i$ yields $f'_0(z, x) = f''_0(z, h^n_0(x)) \ne g''_0(z, h^k_0(y)) = g'_0(z, y)$ because $f''_0(\{z\} \times N_0) \cap g''_0(\{z\} \times K_0) = \emptyset$. Hence, $f'(z, x) \ne f'(z, y)$.

If $\nu_0(x) > 1/2$, then $\nu_1(x) < 1/2$ and $h^n_1(x) \in N_1$. So, $f'_1(z, x) = f''_1(z, h^n_1(x)) \ne g''_1(z, h^k_1(y)) = g'_1(z, y)$ because $f''_1(\{z\} \times N_1) \cap g''_1(\{z\} \times \mathbb{I}^k) = \emptyset$. Similarly, we can show that $f'_1(z, x) \ne g'(z, y)$ provided $\kappa_0(y) > 1/2$. ∎

The next lemma yields the final item of Theorem 8.1.

LEMMA 32.3. *If a submetrizable space $X$ has the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property, then the product $\mathbb{I}^d \times X$ has the $(m-d)$-$\overline{\mathrm{DD}}^{\{d+n,d+k\}}$-property for any $d < m+1$.*

*Proof.* By Proposition 5.2, it suffices to consider the case of finite $m, n, k$. To this end, we fix an open cover $\mathcal{U}$ of $\mathbb{I}^d \times X$ and two maps $f : \mathbb{I}^{m+n} \to \mathbb{I}^d \times X$, $g : \mathbb{I}^{m+k} \to \mathbb{I}^d \times X$. Here, we identify the cubes $\mathbb{I}^{m+n}$ and $\mathbb{I}^{m+k}$ with the products $\mathbb{I}^{m-d} \times \mathbb{I}^{d+n}$ and $\mathbb{I}^{m-d} \times \mathbb{I}^{d+k}$.

Write the maps $f$ and $g$ in the form $f = f_1 \triangle f_2$, $g = g_1 \triangle g_2$ for suitable maps $f_1 : \mathbb{I}^{m+n} \to \mathbb{I}^d$, $g_1 : \mathbb{I}^{m+k} \to \mathbb{I}^d$, $f_2 : \mathbb{I}^{m+n} \to X$, $g_2 : \mathbb{I}^{m+k} \to X$.

By a standard compactness argument, find an open cover $\mathcal{U}_1$ of $\mathbb{I}^d$ and an open cover $\mathcal{U}_2$ of $X$ such that $\mathrm{diam}(U_1 \times U_2) < \mathcal{U}$ for any $U_1 \in \mathcal{U}_1$, $U_2 \in \mathcal{U}_2$. According to Lemma 18.2, there is a continuous metric $\rho$ on $X$ (recall that $X$ is submetrizable) such that $\mathrm{diam}\, B_\rho(x, 1) < \mathcal{U}_2$ for each $x \in f_2(\mathbb{I}^{m+n}) \cup g_2(\mathbb{I}^{m+k})$.

Fix any continuous metric of $\mathbb{I}^{m+n}$ whose balls are convex and find $\varepsilon > 0$ such that the images $f_2(B(x, \varepsilon))$ and $g_2(B(x, \varepsilon))$ of any $\varepsilon$-ball have $\rho$-diameter $< 1/2$. By Lemma 24.1, there is an open cover $\mathcal{V}$ of $\mathbb{I}^{m+n}$ such that for any $\mathcal{V}$-map $\alpha : \mathbb{I}^{m+n} \to K$ into a paracompact space $K$ there is a map $\beta : \alpha(\mathbb{I}^{m+n}) \to \mathbb{I}^{m+n}$ such that the composition $\beta \circ \alpha$ is $\varepsilon$-homotopic to the identity map of $\mathbb{I}^{m+n}$.

Consider the projection $\mathrm{pr}^{m+n}_{m-d} : \mathbb{I}^{m+n} \to \mathbb{I}^{m-d}$. Since $\dim(\mathrm{pr}^{m+n}_{m-d}) = d + n$, we can apply Lemma 19.3 to find a map $f'_1 : \mathbb{I}^{m+n} \to \mathbb{I}^d$ which is $\mathcal{U}_1$-homotopic to $f_1$ and such that the diagonal product $\mathrm{pr}^{m+n}_{m-d} \triangle f'_1 : \mathbb{I}^{m+n} \to \mathbb{I}^{m-d} \times \mathbb{I}^d = \mathbb{I}^m$ is $n$-dimensional.

Fix a triangulation of $\mathbb{I}^m$. By Theorem 4.1 (applied for the $n$-dimensional map $\mathrm{pr}^{m+n}_{m-d} \triangle f'_1 : \mathbb{I}^{m+n} \to \mathbb{I}^m$), there is a $\mathcal{V}$-map $\alpha_f : \mathbb{I}^{m+n} \to K_f$ into a compact polyhedron and an $n$-dimensional PL-map $p_f : K_f \to \mathbb{I}^m$ such that $p_f \circ \alpha_f = \mathrm{pr}^{m+n}_{m-d} \triangle f'_1$. The choice of the cover $\mathcal{V}$ guarantees the existence of a map $\beta : \alpha_f(\mathbb{I}^{m+n}) \to \mathbb{I}^{m+n}$ such that the composition $\beta \circ \alpha_f$ is $\varepsilon$-homotopic to the identity. Since $\mathbb{I}^{m+n}$ is an AR, we may extend the map $\beta$ to a map $\beta_f : K_f \to \mathbb{I}^{m+n}$. Finally, consider the map $\gamma_f = f_2 \circ \beta_f : K_f \to X$. Since $\beta_f \circ \alpha_f$ is $\varepsilon$-homotopic to the identity on $\mathbb{I}^{m+n}$, the choice of $\varepsilon$ implies that $\gamma_f \circ \alpha_f$ is $1/2$-homotopic to $f_2$. In this way, we have constructed:

- a map $f'_1 : \mathbb{I}^{m+n} \to \mathbb{I}^d$ which is $\mathcal{U}_1$-homotopic to $f_1$;
- a map $\alpha_f : \mathbb{I}^{m+n} \to K_f$ into a compact polyhedron $K_f$;

- an $n$-dimensional PL-map $p_f : K_f \to \mathbb{I}^m$ such that $p_f \circ \alpha_f = \mathrm{pr}^{m+n}_{m-d} \triangle f'_1$;
- a map $\gamma_f : K_f \to X$ such that $\gamma_f \circ \alpha_f$ is 1/2-homotopic to $f_2$ with respect to the pseudometric $\rho$.

Similarly, considering the projection $\mathrm{pr}^{m+k}_{m-d} : \mathbb{I}^{m+k} \to \mathbb{I}^{m-d}$ and the maps $g_1$, $g_2$ instead of $\mathrm{pr}^{m+n}_{m-d}$, $f_1$ and $f_2$, respectively, we can construct:

- a map $g'_1 : \mathbb{I}^{m+k} \to \mathbb{I}^d$ which is $\mathcal{U}_1$-homotopic to $g_1$;
- a map $\alpha_g : \mathbb{I}^{m+k} \to K_g$ into a compact polyhedron $K_g$;
- a $k$-dimensional PL-map $p_g : K_g \to \mathbb{I}^m$ such that $p_g \circ \alpha_g = \mathrm{pr}^{m+k}_{m-d} \triangle g'_1$;
- a map $\gamma_g : K_g \to X$ such that $\gamma_g \circ \alpha_g$ is 1/2-homotopic to $g_2$ with respect to $\rho$.

Replacing the triangulations of the polyhedra $K_f$, $K_g$ and $\mathbb{I}^m$ with suitable subdivisions, we may assume that the maps $p_f$ and $p_g$ are simplicial. Hence, we can apply Theorem 5.3 (recall that $X$ has the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property) to find two maps $\gamma'_f : K_f \to X$, $\gamma'_g : K_g \to X$ such that $\gamma'_f$ is 1/2-homotopic to $\gamma_f$, $\gamma'_g$ is 1/2-homotopic to $\gamma_g$ and $\gamma'_f(p_f^{-1}(z)) \cap \gamma'_g(p_g^{-1}(z)) = \emptyset$ for all $z \in \mathbb{I}^m$. Consider the compositions $f'_2 = \gamma'_f \circ \alpha_f$ and $g'_2 = \gamma'_g \circ \alpha_g$. Since $\gamma'_f$ is 1/2-homotopic to $\gamma_f$ and $\gamma_f \circ \alpha_f$ is 1/2-homotopic to $f_2$, $f'_2$ is 1-homotopic to $f_2$. Similarly, $g'_2$ is 1-homotopic to $g_2$. Then, by the choice of the pseudometric $\rho$, $f'_2$ is $\mathcal{U}_2$-homotopic to $f_2$ and $g'_2$ is $\mathcal{U}_2$-homotopic to $g_2$.

Finally, consider the maps $f' = f'_1 \triangle f'_2 : \mathbb{I}^{m+n} \to \mathbb{I}^d \times X$ and $g' = g'_1 \triangle g'_2 : \mathbb{I}^{m+k} \to \mathbb{I}^d \times X$. Observe that $f'$ is $\mathcal{U}$-homotopic to $f$ and $g'$ is $\mathcal{U}$-homotopic to $g$. To prove that $\mathbb{I}^d \times X$ has the $(m-d)$-$\overline{\mathrm{DD}}^{\{d+n,d+k\}}$-property, we need to show that $f'(\{z\} \times \mathbb{I}^{d+n}) \cap g'(\{z\} \times \mathbb{I}^{d+k}) = \emptyset$ for any $z \in \mathbb{I}^{m-d}$.

Assuming that this is not the case, we find two points $x \in \mathbb{I}^{d+n}$, $y \in \mathbb{I}^{d+k}$ with $f'(z,x) = g'(z,y)$ for some $z \in \mathbb{I}^{m-d}$. So, $f'_1(z,x) = g'_1(z,y)$ and $f'_2(z,x) = g'_2(z,y)$. Let $x' = \alpha_f(z,x) \in K_f$ and $y' = \alpha_g(z,y) \in K_g$. Since

$$(\mathrm{pr}^{m+n}_{m-d} \triangle f'_1)(z,x) = (z, f'_1(z,x)) = (z, g'_1(z,y)) = (\mathrm{pr}^{m+k}_{m-d} \triangle g'_1)(z,y)$$

we have

$$\begin{aligned} p_f(x') &= (p_f \circ \alpha_f)(x,z) = (\mathrm{pr}^{m+n}_{m-d} \triangle f'_1)(z,x) \\ &= (\mathrm{pr}^{m+k}_{m-d} \triangle g'_1)(z,y) = (p_g \circ \alpha_g)(y,z) = p_g(y'). \end{aligned}$$

The last equality and the choice of the maps $\gamma'_f$, $\gamma'_g$ imply that $\gamma'_f(x') \neq \gamma'_g(y')$. Therefore,

$$f'_2(z,x) = (\gamma'_f \circ \alpha_f)(z,x) = \gamma'_f(x') \neq \gamma'_g(y') = (\gamma'_g \circ \alpha_g)(z,x) = g'_2(z,y),$$

which contradicts $f'_2(z,x) = g'_2(z,y)$. ■

## 33. Proof of base enlargement formulas

This section is devoted to the proof of Theorem 8.3. The first item of this theorem follows from the next lemma.

LEMMA 33.1. *Let $n, k, m_0, m_1$ be non-negative integer numbers and $m = m_0 + m_1 + 1$. A metrizable space $X$ has the property $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$ provided $X \in 0$-$\overline{\mathrm{DD}}^{\{n+m_0,k+m_1\}}$, $X \in m_0$-$\overline{\mathrm{DD}}^{\{n,k+1+m_1\}}$ and $X \in m_1$-$\overline{\mathrm{DD}}^{\{n+1+m_0,k\}}$.*

*Proof.* Let $\rho$ be a metric generating the topology of $X$. Given any $\varepsilon > 0$ and maps $f : \mathbb{I}^m \times \mathbb{I}^n \to X$, $g : \mathbb{I}^m \times \mathbb{I}^k \to X$ we are going to construct two other maps $f' : \mathbb{I}^m \times \mathbb{I}^n \to X$, $g' : \mathbb{I}^m \times \mathbb{I}^k \to X$ such that $f'$ is $\varepsilon$-homotopic to $f$, $g'$ is $\varepsilon$-homotopic to $g$ and $f'(\{z\} \times \mathbb{I}^n) \cap g'(\{z\} \times \mathbb{I}^k) = \emptyset$ for all $z \in \mathbb{I}^m$. These maps will be constructed in four steps using the join structure of $\mathbb{I}^m$.

Select a triangulation $T$ of the cube $\mathbb{I}^m$ so fine that, for any simplex $\sigma \in T$ and points $x \in \mathbb{I}^n$, $y \in \mathbb{I}^k$, the sets $f(\sigma \times \{x\})$ and $g(\sigma \times \{y\})$ have diameter $< \varepsilon/3$. Let $M_0$ be the $m_0$-dimensional skeleton of the triangulation $T$ and $M_1$ be the dual ($m_1$-dimensional) skeleton of the barycentric subdivision of $T$. Since $\mathbb{I}^m$ is a subset of the join $M_0 * M_1$, each point $z \in \mathbb{I}^m$ lying in a maximal simplex $\sigma \in T$ can be represented in the form $z = (1 - \mu(z))z_0 + \mu(z)z_1$ for some points $z_i \in M_i \cap \sigma$, $i \in \{0, 1\}$, where $\mu(z) \in [0, 1]$ is the join-parameter of $z$. If $z \notin M_1$, then the point $z_0$ is uniquely determined and will be denoted by $\mathrm{pr}_0(z)$; if $z \notin M_0$, then the point $z_1$ is unique and will be denoted by $\mathrm{pr}_1(z)$. Thus, each point $z \in \mathbb{I}^m \setminus (M_0 \cup M_1)$ can be written as $z = (1-\mu(z))\mathrm{pr}_0(z)+\mu(z)\mathrm{pr}_1(z)$. For a point $z \in \mathbb{I}^m$, let $\mu_0(z) = \mu(z)$ and $\mu_1(z) = 1 - \mu_0(z)$.

Since $X$ has the $0\text{-}\overline{\mathrm{DD}}^{\{m_0+n,m_1+k\}}$-property, by Theorem 5.3 and Lemma 18.3, there exist maps $f_1 : \mathbb{I}^m \times \mathbb{I}^n \to X$ and $g_1 : \mathbb{I}^m \times \mathbb{I}^k \to X$ such that

$$f_1 \text{ is } \varepsilon/18\text{-homotopic to } f, \quad g_1 \text{ is } \varepsilon/18\text{-homotopic to } g \tag{33.1}$$

and $f_1(M_0 \times \mathbb{I}^n) \cap g_1(M_1 \times \mathbb{I}^k) = \emptyset$.

The continuity of $f_1, g_1$ implies the existence of closed neighborhoods $O(M_0)$ and $O(M_1)$ of the subcomplexes $M_0$ and $M_1$ in $\mathbb{I}^m$, respectively, such that

$$\varepsilon_1 = \mathrm{dist}(f_1(O(M_0) \times \mathbb{I}^n), g_1(O(M_1) \times \mathbb{I}^k)) > 0.$$

Put $\varepsilon_2 = \min\{\varepsilon_1/5, \varepsilon/18\}$ and find a positive $\delta < 1/2$ such that each point $z \in \mathbb{I}^m$ with join-parameter $\mu(z) < \delta$ (resp., $\mu(z) > 1-\delta$) belongs to the neighborhood $O(M_0)$ (resp., $O(M_1)$).

Consider the cone $\mathrm{Cone}(M_1) = M_1\times[0, 1-\delta]/(M_1\times\{0\})$ over $M_1$ and observe that the subspace $M_{\le 1-\delta} = \{z \in \mathbb{I}^m : \mu(z) \le 1 - \delta\} \subset \mathbb{I}^m$ is homeomorphic to the product $M_0 \times \mathrm{Cone}(M_1)$ via the homeomorphism $z \mapsto (z_0, (z_1, \mu(z))_\sim)$, where $(z_1, \mu(z))_\sim \in \mathrm{Cone}(M_1)$ stands for the equivalence class of the pair $(z_1, \mu(z))$. This homeomorphism maps the set $M_{\le 1-\delta} \cap \mathrm{pr}_0^{-1}(z_0)$ onto the fiber $\{z_0\} \times \mathrm{Cone}(M_1)$ for each $z_0 \in M_0$.

Now, since $X$ has the $m_0\text{-}\overline{\mathrm{DD}}^{\{n,k+1+m_1\}}$-property, by Theorem 5.3, there exist two maps $\tilde f_2 : M_0 \times \mathbb{I}^n \to X$ and $\tilde g_2 : M_{\le 1-\delta} \times \mathbb{I}^k \to X$ such that

- $\tilde f_2$ is $\varepsilon_2$-homotopic to $f_1|M_0 \times \mathbb{I}^n$;
- $\tilde g_2$ is $\varepsilon_2$-homotopic to $g_1|(M_{\le 1-\delta} \times \mathbb{I}^k)$;
- $\tilde f_2(\{z\} \times \mathbb{I}^n) \cap \tilde g_2\big((M_{\le 1-\delta} \cap \mathrm{pr}_0^{-1}(z)) \times \mathbb{I}^k\big) = \emptyset$ for each $z \in M_0$.

By the Borsuk homotopy extension lemma, the maps $\tilde f_2$, $\tilde g_2$ can be extended to maps $f_2 : \mathbb{I}^m \times \mathbb{I}^n \to X$ and $g_2 : \mathbb{I}^m \times \mathbb{I}^k \to X$ such that

$$f_2 \text{ is } \varepsilon_2\text{-homotopic to } f_1 \quad \text{and} \quad g_2 \text{ is } \varepsilon_2\text{-homotopic to } g_1. \tag{33.2}$$

Then

$$\varepsilon_3 = \min_{z\in M_0} \mathrm{dist}\big(f_2(\{z\} \times \mathbb{I}^n), g_2((M_{\le 1-\delta} \cap \mathrm{pr}_0^{-1}(z)) \times \mathbb{I}^k)\big) > 0$$

and let $\varepsilon_4 = \min\{\varepsilon_3/3, \varepsilon_2\}$.

Consider the subspace $M_{\geq\delta} = \{z \in \mathbb{I}^m : \mu(z) \geq \delta\}$ and note that it is naturally homeomorphic to the product $M_1 \times \mathrm{Cone}(M_0)$. Since $X$ has the $m_1$-$\overline{\mathrm{DD}}^{\{k,n+1+m_0\}}$-property, we can repeat the arguments from the construction of the maps $f_2, g_2$ to find two maps $f_3 : \mathbb{I}^m \times \mathbb{I}^n \to X$, $g_3 : \mathbb{I}^m \times \mathbb{I}^k \to X$ such that

$$f_3 \text{ is } \varepsilon_4\text{-homotopic to } f_2, \quad g_3 \text{ is } \varepsilon_4\text{-homotopic to } g_2 \tag{33.3}$$

and $f_3\big((M_{\geq\delta} \cap \mathrm{pr}_1^{-1}(z)) \times \mathbb{I}^n\big) \cap g_3(\{z\} \times \mathbb{I}^k) = \emptyset$ for all $z \in M_1$.

Now, we estimate the distance from $f_3$ to $f_1$ and $f$. Recall that $\varepsilon_4 + \varepsilon_2 \leq 2\varepsilon_2 \leq 2\min\{\varepsilon_1/5, \varepsilon/18\}$. Taking into account the homotopical nearness from (33.1)–(33.3), we conclude that the map $f_3$ is $(\varepsilon_4 + \varepsilon_2)$-homotopic to $f_1$ and $\varepsilon/6$-homotopic to $f$. Consequently, for any point $x \in \mathbb{I}^n$ and any simplex $\sigma$ of the triangulation of $\mathbb{I}^m$, the image $f_3(\sigma \times \{x\})$ has diameter

$$\operatorname{diam} f_3(\sigma \times \{x\}) \leq \operatorname{diam} f(\sigma \times \{x\}) + 2\operatorname{dist}(f_3, f) < \frac{\varepsilon}{3} + 2\frac{\varepsilon}{6} = \frac{2}{3}\varepsilon. \tag{33.4}$$

Similar estimates hold for the map $g_3$.

Next, consider the piecewise-linear map $\ell : [0,1] \to [0,1]$ determined by the conditions $\ell(0) = \ell(1/2) = 0$ and $\ell(1) = 1$. Define two piecewise-linear maps $h_0, h_1 : \mathbb{I}^m \to \mathbb{I}^m$ such that the map $h_i$ assigns to any point $z = \mu_1 z_0 + \mu_0 z_1 \in \mathbb{I}^m$ with join-coordinates $(\mu_0, \mu_1)$ the point $h_i(z) = \mu_1' z_0 + \mu_0' z_1$, where $\mu_i' = \ell(\mu_i)$ and $\mu_{1-i}' = 1 - \mu_i'$. Note that each map $h_i : \mathbb{I}^m \to \mathbb{I}^m$ is $\mathcal{S}$-homotopic to the identity map with respect to the cover $\mathcal{S}$ of $\mathbb{I}^m$ consisting of all maximal simplexes of the triangulation $T$.

Finally, define maps $f' : \mathbb{I}^m \times \mathbb{I}^n \to X$ and $g' : \mathbb{I}^m \times \mathbb{I}^k \to X$ by letting $f'(z,x) = f_3(h_0(z), x)$ for $(z,x) \in \mathbb{I}^m \times \mathbb{I}^n$ and $g'(z,y) = g_3(h_1(z), y)$ for $(z,y) \in \mathbb{I}^m \times \mathbb{I}^k$. We claim that they have the desired properties. Observe that, according to (33.4), $f'$ is $2\varepsilon/3$-homotopic to $f_3$. Hence, $f'$ is $\varepsilon$-homotopic to $f$ because $f_3$ is $\varepsilon/6$-homotopic to $f$. The same argument yields that $g'$ is $\varepsilon$-homotopic to $g$.

It remains to check that for any $z \in \mathbb{I}^m$ the sets $f'(\{z\} \times \mathbb{I}^n)$ and $g'(\{z\} \times \mathbb{I}^k)$ do not intersect, or equivalently, that $f'(z,x) \neq g'(z,y)$ for any $x \in \mathbb{I}^n$ and $y \in \mathbb{I}^k$. We consider separately two cases.

1. $\mu(z) = \mu_0(z) \leq 1/2$. In this case, since $h_0(z) = (1 - l(\mu(z)))z_0 + l(\mu(z))z_1$, we have $\mu(h_0(z)) = l(\mu(z)) = 0$. So, $h_0(z) = \mathrm{pr}_0(z) = z_0 \in M_0$ and $f'(z,x) = f_3(z_0, x)$.

Next, we consider the point $h_1(z) = l(\mu_1(z))z_0 + (1 - l(\mu_1(z)))z_1$. If $\mu(h_1(z)) \leq 1 - \delta$, then $h_1(z) \in M_{\leq 1-\delta} \cap \mathrm{pr}_0^{-1}(z_0)$. Thus, $\operatorname{dist}(f_2(z_0,x), g_2(h_1(z),y)) \geq \varepsilon_3$. Then

$$\begin{aligned}
\operatorname{dist}(f'(z,x), g'(z,y)) &= \operatorname{dist}(f_3(z_0,x), g_3(h_1(z),y)) \\
&\geq \operatorname{dist}(f_2(z_0,x), g_2(h_1(z),y)) - \operatorname{dist}(f_3,f_2) - \operatorname{dist}(g_2,g_3) \\
&\geq \varepsilon_3 - 2\varepsilon_4 \geq \varepsilon_3/3 > 0.
\end{aligned}$$

If $\mu(h_1(z)) > 1 - \delta$, then $h_1(z) \in O(M_1)$ by the choice of $\delta$. Hence,

$$\begin{aligned}
\operatorname{dist}(f'(z,x), g'(z,y)) &= \operatorname{dist}(f_3(z_0,x), g_3(h_1(z),y)) \\
&\geq \operatorname{dist}(f_1(z_0,x), g_1(h_1(z),y)) - \operatorname{dist}(f_1,f_3) - \operatorname{dist}(g_1,g_3) \\
&\geq \varepsilon_1 - 2(\varepsilon_4 + \varepsilon_2) \geq \varepsilon_1 - 4\varepsilon_2 \geq \varepsilon_1/5 > 0.
\end{aligned}$$

2. $\mu(z) \geq 1/2$. In this case $\mu_1(z) \leq 1/2$, so $l(\mu_1(z)) = 0$ and $h_1(z) = l(\mu_1(z))z_0 + (1 - l(\mu_1(z)))z_1 = z_1 \in M_1$. Hence, $g'(z, y) = g_3(z_1, y)$.

Consider now the point $h_0(z)$. If $\mu(h_0(z)) \geq \delta$, then $h_0(z) \in M_{\geq\delta} \cap \mathrm{pr}_1^{-1}(z_1)$. Thus,

$$\mathrm{dist}(f'(z, x), g'(z, y)) = \mathrm{dist}(f_3(h_0(z), x), g_3(z_1, y)) > 0.$$

If $\mu(h_0(z)) < \delta$, then $h_0(z) \in O(M_0)$ by the choice of $\delta$, and hence

$$\begin{aligned}\mathrm{dist}(f'(z, x), g'(z, y)) &= \mathrm{dist}(f_3(h_0(z), x), g_3(z_1, y)) \\ &\geq \mathrm{dist}(f_1(h_0(z), x), g_1(z_1, y)) - \mathrm{dist}(f_1, f_3) - \mathrm{dist}(g_1, g_3) \\ &\geq \varepsilon_1 - 2(\varepsilon_4 + \varepsilon_2) \geq \varepsilon_1 - 4\varepsilon_2 \geq \varepsilon_1/5 > 0. \ \blacksquare\end{aligned}$$

The particular case of Lemma 33.1 when $m_0 = m$ and $m_1 = 0$ provides the second item of Theorem 8.3.

LEMMA 33.2. *Let $n, k, m$ be non-negative integers. If a metrizable space $X$ has both the $0$-$\overline{\mathrm{DD}}^{\{n+m+1,k\}}$- and $m$-$\overline{\mathrm{DD}}^{\{n,k+1\}}$-properties, then it has the $(m+1)$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property.*

The final item of Theorem 8.3 will be derived from Lemma 33.3 below.

LEMMA 33.3. *If a metrizable space $X$ has the $0$-$\overline{\mathrm{DD}}^{\{n+i,k+j\}}$-property for all $i, j \in \omega$ with $i + j < m + 1$, then it has the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property.*

*Proof.* Because of Proposition 5.2, it suffices to consider the case of finite $m, n, k$. By induction on $m' \leq m$, we shall prove that the space $X$ has the $m'$-$\overline{\mathrm{DD}}^{\{n+i,k+j\}}$-property for all integers $i, j \in \omega$ with $m' + i + j < m + 1$. Observe that our lemma follows from the above statement with $m' = m$ and $i = j = 0$.

Let us prove this statement. For $m' = 0$ it follows from our hypothesis. Assume that for some $m_0 \leq m$ the statement has been established for all $m' < m_0$. We shall prove it for $m' = m_0$. Take any integers $i, j \in \omega$ with $m_0+i+j < m+1$. According to Lemma 33.2, the $m_0$-$\overline{\mathrm{DD}}^{\{n+i,k+j\}}$-property of $X$ will be proved as soon as we check that $X$ has both the $0$-$\overline{\mathrm{DD}}^{\{n+i+m_0,k+j\}}$-property and the $(m_0 - 1)$-$\overline{\mathrm{DD}}^{\{n+i,k+j+1\}}$-property. But this follows from the inductive hypothesis since $i+m_0+j < m+1$ and $(m_0-1)+i+j+1 < m+1$. $\blacksquare$

## 34. Proof of Theorem 9.1

The proof of Theorem 9.1 is divided into a few lemmas.

LEMMA 34.1. *Let $L$ be a metrizable space with the $0$-$\overline{\mathrm{DD}}^{\{0,0\}}$-property. Then, for any non-negative integer $i < l$, $L^l$ has the $0$-$\overline{\mathrm{DD}}^{\{i,l-1-i\}}$-property.*

*Proof.* The statement is obviously true if $l = 1$. Assume that, for some number $l - 1$, the power $L^{l-1}$ has the $0$-$\overline{\mathrm{DD}}^{\{i,l-2-i\}}$-property for all $i < l - 1$. We are going to show that the power $L^l = L \times L^{l-1}$ has the $0$-$\overline{\mathrm{DD}}^{\{i,l-1-i\}}$-property for all $i < l$.

If $i = 0$, then $L$ has the $0$-$\overline{\mathrm{DD}}^{\{0,0\}}$-property and $L^{l-1}$ has the $0$-$\overline{\mathrm{DD}}^{\{0,l-2\}}$-property. So, by the first multiplication formula, $L \times L^{l-1}$ has the $0$-$\overline{\mathrm{DD}}^{\{0,0+(l-2)+1\}}$-property. Because $\{0, l-1\} = \{l-1, 0\}$, $L^l$ has the $0$-$\overline{\mathrm{DD}}^{\{i,l-1-i\}}$ property for $i \in \{0, l-1\}$.

If $0 < i < l-1$, then the 0-$\overline{\mathrm{DD}}^{\{i,l-1-i\}}$-property of the product $L^l = L \times L^{l-1}$ follows from the second multiplication formula and the fact that $L$ has the 0-$\overline{\mathrm{DD}}^{\{0,0\}}$-property and $L^{l-1}$, according to the inductive hypothesis, has both the 0-$\overline{\mathrm{DD}}^{\{i-1,l-i-1\}}$- and 0-$\overline{\mathrm{DD}}^{\{i,l-i-2\}}$-properties. ∎

Lemma 34.2. *Let $d, k$ be non-negative integers with $d > 0$ and $D$ be a metrizable space with the 0-$\overline{\mathrm{DD}}^{\{0,k\}}$-property. Then $D^d$ has the 0-$\overline{\mathrm{DD}}^{\{i,(d-i)(k+1)-1\}}$-property for all $i < d$.*

*Proof.* The proof is by induction with respect to $d$. The statement is trivial for $d = 1$. Assume that, for some $d > 1$, $D^{d-1}$ has the 0-$\overline{\mathrm{DD}}^{\{i,(d-1-i)(k+1)-1\}}$-property for all $i < d-1$.

Let us check that $D^d = D \times D^{d-1}$ has the 0-$\overline{\mathrm{DD}}^{\{i,(d-i)(k+1)-1\}}$-property for all $i < d$.

We shall consider separately the cases $i = 0$, $i = d-1$, and $0 < i < d-1$.

For $i = 0$, we use the inductive assumption to conclude that $D^{d-1}$ possesses the 0-$\overline{\mathrm{DD}}^{\{0,(d-1)(k+1)-1\}}$-property. Since $D$ has the 0-$\overline{\mathrm{DD}}^{\{0,k\}}$-property and $k+(d-1)(k+1) - 1 + 1 = d(k+1) - 1$, by the first multiplication formula, the product $D^d = D \times D^{d-1}$ has the 0-$\overline{\mathrm{DD}}^{\{0,d(k+1)-1\}}$-property.

The case $i = d-1$ is treated similarly. By the inductive assumption, $D^{d-1}$ has the 0-$\overline{\mathrm{DD}}^{\{d-2,k\}}$-property. Taking into account the 0-$\overline{\mathrm{DD}}^{\{0,k\}}$-property of $D$ and applying the first multiplication formula, we conclude that $D^d = D \times D^{d-1}$ has the 0-$\overline{\mathrm{DD}}^{\{d-1,k\}}$-property.

For $0 < i < d-1$, we use the inductive assumption to conclude that $D^{d-1}$ has both the 0-$\overline{\mathrm{DD}}^{\{i,(d-1-i)(k+1)-1\}}$- and 0-$\overline{\mathrm{DD}}^{\{i-1,(d-i)(k+1)-1\}}$-properties. Since $D$ has the 0-$\overline{\mathrm{DD}}^{\{0,k\}}$-property, the second multiplication formula implies that $D^d$ has the property 0-$\overline{\mathrm{DD}}^{\{i,(d-i)(k+1)-1\}}$ (here, we use that $k+((d-1-i)(k+1)-1)+1 = (d-i)(k+1)-1)$. ∎

Lemma 34.3. *Let $d \in \mathbb{N}$, $l \in \omega$ and $L$, $D$ be metrizable spaces with $L \in$ 0-$\overline{\mathrm{DD}}^{\{0,0\}}$ and $D \in$ 0-$\overline{\mathrm{DD}}^{\{0,d+l\}}$. Then, for any non-negative integer $i \le l+1$, the product $D^d \times L^l$ has the 0-$\overline{\mathrm{DD}}^{\{d-1+i,d+l-i\}}$-property.*

*Proof.* The proof is by induction on $l$.

To start the induction, consider the case $l = 0$. By Lemma 34.2 (with $k = d$ and $i = d-1$), the power $D^d$ has the 0-$\overline{\mathrm{DD}}^{\{d-1,d\}}$-property. This completes the proof of the assertion when $l = 0$.

Assume that for some $l > 1$ the product $D^d \times L^{l-1}$ has 0-$\overline{\mathrm{DD}}^{\{d-1+i,d+(l-1)-i\}}$ for all $0 \le i \le l$. We should prove that the product $D^d \times L^l = (D^d \times L^{l-1}) \times L$ has the 0-$\overline{\mathrm{DD}}^{\{d-1+i,d+l-i\}}$-property for all $0 \le i \le l+1$.

Let $i = 0$. Then applying Lemma 34.2 (with $k = d+l$ and $i = d-1$), we find that $D^d$ has the 0-$\overline{\mathrm{DD}}^{\{d-1,d+l\}}$-property. So does the product $D^d \times L^l$. This also yields the 0-$\overline{\mathrm{DD}}^{\{d-1+i,d+l-i\}}$-property of $D^d \times L^l$ for $i = l+1$ because $\{d-1+(l+1), d+l-(l+1)\} = \{d+l, d-1\}$.

Assume that $1 \le i \le l$. Then, by the inductive hypothesis, the product $D^d \times L^{l-1}$ has both 0-$\overline{\mathrm{DD}}^{\{d-1+i,d+(l-1)-i\}}$ and 0-$\overline{\mathrm{DD}}^{\{d-1+(i-1),d+(l-1)-(i-1)\}}$. Since $L$ has the 0-$\overline{\mathrm{DD}}^{\{0,0\}}$-property, applying the second multiplication formula to the product $D^d \times L^l \cong L \times (D^d \times L^{l-1})$, we conclude that $D^d \times L^l$ has the 0-$\overline{\mathrm{DD}}^{\{d-1+i,d+l-i\}}$-property. ∎

Finally, we are in a position to complete the proof of Theorem 9.1.

LEMMA 34.4. *Let $m, n, k, d, l$ be non-negative integers, $L$ be a metrizable space with the $0\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$-property and $D$ be a metrizable space with the $0\text{-}\overline{\mathrm{DD}}^{\{0,d+l\}}$-property. If $m+n+k<2d+l$, then the product $D^d \times L^l$ has the $m\text{-}\overline{\mathrm{DD}}^{\{n,k\}}$-property.*

*Proof.* Suppose first that $d=0$. By Lemma 34.1, $L^l$ has the $0\text{-}\overline{\mathrm{DD}}^{\{i,l-i-1\}}$-property for every $i<l$. Let us show that $L^l$ has the $0\text{-}\overline{\mathrm{DD}}^{\{n+i,k+j\}}$-property for all $i, j$ with $i+j \le m$. Indeed, fix $i, j$ with $i+j \le m$. Then $n+1<l$ and, by Lemma 34.1, $L^l \in 0\text{-}\overline{\mathrm{DD}}^{\{n+i,l-n-i-1\}}$. Since $m+n+k<l$, $k+j \le l-n-i-1$. The last inequality and $L^l \in 0\text{-}\overline{\mathrm{DD}}^{\{n+i,l-n-i-1\}}$ imply that $L^l \in 0\text{-}\overline{\mathrm{DD}}^{\{n+i,k+j\}}$ for all $i, j$ with $i+j \le m$. Consequently, Lemma 33.3 implies that $L^l$ has the $m\text{-}\overline{\mathrm{DD}}^{\{n,k\}}$-property.

If $d \ge 1$, according to Lemma 33.3, it suffices to show that $D^d \times L^l$ has the $0\text{-}\overline{\mathrm{DD}}^{\{n,k\}}$-property for all $n, k \in \omega$ with $n+k<2d+l$. To this end, we fix integers $n, k$ with $n+k<2d+l$ and assume that $k \ge n$. There are two cases: $n \le d-1$ or $n \ge d$.

Suppose $n \le d-1$. By Lemma 34.2, $D^d$ has the $0\text{-}\overline{\mathrm{DD}}^{\{n,(d-n)(d+l+1)-1\}}$-property. This implies that $D^d$ has the $0\text{-}\overline{\mathrm{DD}}^{\{n,k\}}$-property because $k \le 2d+l-1-n \le (d-n)(d+l+1)-1$. Hence, the product $D^d \times L^l$ also has the $0\text{-}\overline{\mathrm{DD}}^{\{n,k\}}$-property for all $n, k$ with $n+k<2d+l$.

When $n \ge d$, we have $n=d-1+i$ for some $i \ge 1$. The last equality yields $i \le l$. Indeed, otherwise we would obtain $n \ge d-1+(l+1)=d+l$. Hence, $k+n \ge 2n \ge 2(d+l)$, which contradicts $n+k<2d+l$. Since $i \le l$, we can apply Lemma 34.3 to conclude that $D^d \times L^l$ has the $0\text{-}\overline{\mathrm{DD}}^{\{d-1+i,d+l-i\}}$-property. Finally, since $k \le 2d+l-1-n = 2d+l-1-(d-1+i)=d+l-i$, $D^d \times L^l$ also has the $0\text{-}\overline{\mathrm{DD}}^{\{n,k\}}$-property. ■

## 35. Proof of Proposition 5.6

The first two items of Proposition 5.6 are trivial and their proofs are left to the reader.

To prove the third item, we need to show the equivalence of the following statements, where $X$ is a given metrizable $\mathrm{LC}^1$-space:

(a) $X$ has the $0\text{-}\overline{\mathrm{DD}}^{\{0,1\}}$-property;
(b) $X$ has the $1\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$-property;
(c) $X$ has no free arc.

The implication (a)⇒(b) follows from the second base enlargement formula, while (b)⇒(c) trivially follows from the fact that the interval $(0,1)$ fails to have the $1\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$-property. To prove that (c)⇒(a), take an open cover $\mathcal{U}$ of $X$ and two maps $f : \mathbb{I}^0 \to X$, $g : \mathbb{I}^1 \to X$. By Lemma 21.1, $X$ admits an open cover $\mathcal{V} \prec \mathcal{U}$ such that every map $g' : \mathbb{I} \to X$ is $\mathcal{U}$-homotopic to $g$ provided it is $\mathcal{V}$-near to $g$. For every point $x \in X$ fix a set $V_x \in \mathcal{V}$ containing $x$ and its open neighborhood $W_x \subset V_x$ such that any two points $y, z \in W_x$ can be linked by an arc in $V_x$ (recall that $X$ being $\mathrm{LC}^1$ is an $\mathrm{LC}^0$-space). By the uniform continuity of $g : \mathbb{I} \to X$, there is a partition $0=t_0<t_1<\dots<t_n=1$ of $[0,1]$ such that for every $i \le n$ the image $g([t_{i-1}, t_i])$ lies in $W_{x_i}$ for some point $x_i \in X$. The choice of the set $W_{x_i}$ guarantees the existence of an embedding $g'_i : [t_{i-1}, t_i] \to V_{x_i}$ such that $g'_i(t_{i-1}) = g(t_{i-1})$ and $g'(t_i) = g(t_i)$. The maps $g'_i$ compose a single continuous map $g' : [0,1] \to X$ equal to $g'_i$ on each interval $[t_{i-1}, t_i]$. It is easy to see that $g'$ is $\mathcal{V}$-near to $g$. Hence, $g'$ is $\mathcal{U}$-homotopic to $g$.

Let us now construct a map $f' : \mathbb{I}^0 \to X$ such that $f'$ is $\mathcal{U}$-homotopic to $f$ and $f'(\mathbb{I}^0) \notin g'(\mathbb{I}^1)$. Observe that $g'(\mathbb{I})$ is a finite union of arcs. Since $X$ contains no free arc, $g'(\mathbb{I})$ is nowhere dense in $X$. Therefore, there exists $x' \in W_x \setminus g'(\mathbb{I})$, where $x$ is the point $f(\mathbb{I}^0)$. Consider the constant map $f' : \mathbb{I}^0 \to \{x'\} \subset X$. The choice of the neighborhood $W_x$ guarantees that $f'$ is $\mathcal{V}$-homotopic to $f$. Moreover, $f'(\mathbb{I}^0) \cap g'(\mathbb{I}^1) = \emptyset$. So, $X$ has the 0-$\overline{\mathrm{DD}}^{\{0,1\}}$-property.

To prove the fourth item, assume that $X$ is a metrizable $n$-dimensional $\mathrm{LC}^n$-space possessing the 0-$\overline{\mathrm{DD}}^{\{0,n\}}$-property. We are going to show that $X$ has the 0-$\overline{\mathrm{DD}}^{\{0,\infty\}}$-property. To this end, take any open cover $\mathcal{U}$ of $X$ and two maps $f : \mathbb{I}^0 \to X$ and $g : \mathbb{I}^\infty \to X$. By Lemma 21.1, $X$ admits an open cover $\mathcal{U}' \prec \mathcal{U}$ such that any two $St(\mathcal{U}')$-near maps from an $n$-dimensional polyhedron into $X$ are $\mathcal{U}$-homotopic.

The space $X$, being an $n$-dimensional $\mathrm{LC}^n$-space, is an ANR (see [41, V.7.1]). Hence, $X$ is a Lefschetz ANE$[\infty]$ (see Proposition 3.4), and we may apply Lemma 24.1 to find an open cover $\mathcal{W}$ of $X$ with the following property: for any $\mathcal{W}$-map $\alpha : X \to K$ into a paracompact space $K$ there is a map $\beta : O(\alpha(X)) \to X$, defined on a neighborhood of $\alpha(X)$, such that the composition $\beta \circ \alpha : X \to X$ is $\mathcal{U}'$-near to the identity map. Choose now a $\mathcal{W}$-map $\alpha : X \to K$ to an $n$-dimensional polyhedron $K$ (recall that $\dim X \le n$, so such a map $\alpha$ exists). The choice of the cover $\mathcal{W}$ guarantees the existence of a map $\beta : O(\alpha(X)) \to X$ with $\beta \circ \alpha$ being $\mathcal{U}'$-near to the identity map. Consider the compact set $C = \alpha \circ g(\mathbb{I}^\infty) \subset K$ and find a compact polyhedral subset $N \subset K$ with $C \subset N \subset O(\alpha(X))$. It is clear that $\dim N \le \dim K \le n$. Since $X$ has the 0-$\overline{\mathrm{DD}}^{\{0,n\}}$-property, we may apply Theorem 5.3 to find maps $f' : \mathbb{I}^0 \to X$ and $\beta' : N \to X$ such that $f'$ is $\mathcal{U}$-homotopic to $f$, $\beta'$ is $\mathcal{U}'$-homotopic to $\beta|N$ and $f'(\mathbb{I}^0) \cap \beta'(N) = \emptyset$. Then the composition $g' = \beta' \circ \alpha \circ g : \mathbb{I}^\infty \to X$ is $\mathcal{U}'$-near to the map $\beta \circ \alpha \circ g$ which is $\mathcal{U}'$-near to $g$. Thus, $g'$ is $St(\mathcal{U}')$-near to $g$ and by the choice of $\mathcal{U}'$, $g'$ is $\mathcal{U}$-homotopic to $g$. Obviously, $f'(\mathbb{I}^0) \notin g'(\mathbb{I}^\infty)$, which completes the proof of the fourth item.

Let us prove the fifth item of Proposition 5.6. The "only if" part follows from Theorem 5.7 with $M$ being a single point.

To prove the "if" part, assume that $A$ and $B$ are disjoint dense subsets of $X$ such that $A$ is relative $\mathrm{LC}^{n-1}$ in $X$ and $B$ is relative $\mathrm{LC}^{k-1}$ in $X$. We also assume that $k \le n$, so $X$ is $\mathrm{LC}^n$. Given any open cover $\mathcal{U}$ of $X$ and two maps $f : \mathbb{I}^n \to X$, $g : \mathbb{I}^k \to X$, it suffices to find $\mathcal{U}$-homotopic maps $f' : \mathbb{I}^n \to A$, $g' : \mathbb{I}^k \to B$ to $f$ and $g$, respectively. Since $X$ is $\mathrm{LC}^n$, there exists an open cover $\mathcal{V}$ of $X$ such that any two $St(\mathcal{V})$-near maps from $\mathbb{I}^n$ to $X$ are $\mathcal{U}$-homotopic. The set $A$, being relative $\mathrm{LC}^{n-1}$ in $X$, is $\mathrm{LC}^{n-1}$. Then, by Proposition 3.4, $A$ is a Lefschetz ANE$[n]$-space. Hence, $A$ has an open cover $\mathcal{V}_A$ refining $\mathcal{V}$ such that if $K_A$ is a simplicial complex of dimension $\le n$, then any partial $\mathcal{V}_A$-realization $h_A\colon L_A \to A$ of $K_A$ extends to a full $\mathcal{V}$-realization $\bar{h}_A\colon K_A \to A$ of $K_A$. Similarly, $B$ has an open cover $\mathcal{V}_B$ refining $\mathcal{V}$ such that if $K_B$ is a simplicial complex of dimension $\le k$, then any partial $\mathcal{V}_B$-realization $h_B\colon L_B \to B$ of $K_B$ can be extended to a full $\mathcal{V}$-realization $\bar{h}_B\colon K_B \to B$ of $K_B$. Now, we choose a triangulation $T_A$ of $\mathbb{I}^n$ so small that $f(\sigma)$ is contained in some $V_\sigma \in \mathcal{V}$ for all $\sigma \in T_A$. Let $K_A$ be $\mathbb{I}^n$ with the triangulation $T_A$ and take any map $h_A\colon K_A^{(0)} \to A$ which is $\mathcal{V}_A$-near to $f|K_A^{(0)}$ (this can be done because $A$ is dense in $X$). Obviously, $h_A$ is a partial $\mathcal{V}_A$-realization of $K_A$. Hence, by the choice of $\mathcal{V}_A$, $h_A$ extends to a full $\mathcal{V}$-realization $\bar{h}_A\colon K_A \to A$. It is easily seen that $f' = \bar{h}_A$ and $f$ are $St(\mathcal{V})$-close. Hence, the choice of $\mathcal{V}$ implies that $f$ and $f'$ are $\mathcal{U}$-homotopic in $X$. The same procedure applied to $\mathbb{I}^k$ and $g$ produces a map $g'\colon \mathbb{I}^k \to B$ which is $\mathcal{U}$-homotopic to $g$.

## 36. Proof of Selection Theorem 6.1

This section is devoted to the proof of Theorem 6.1 for homotopical $Z_n$-sets. Let us recall that a closed subset $A$ of a topological space $X$ is a homotopical $Z_n$-set if for any open cover $\mathcal{U}$ of $X$ and a map $f : \mathbb{I}^n \to X$ there is a map $g : \mathbb{I}^n \to X \setminus A$, $\mathcal{U}$-homotopic to $f$. The following property of homotopical $Z_n$-sets can be proved by a standard inductive argument (see [67]).

Lemma 36.1. *Let $A$ be a homotopical $Z_n$-set in $X$, and $(K, L)$ be a pair of compact polyhedra $L \subset K$ with $\dim K \le n$. Then for any open cover $\mathcal{U}$ of $X$ and any map $f : K \to X$ with $f(L) \cap A = \emptyset$ there is a $\mathcal{U}$-homotopy $h : K \times [0,1] \to X$ such that $h(K \times \{1\}) \cap A = \emptyset$ and $h(x,t) = f(x)$ for all $(x,t) \in K \times \{0\} \cup L \times [0,1]$.*

Everywhere in this section, for a sequence $\mathcal{U}_0, \dots, \mathcal{U}_m$ of open covers of $X$, we denote by $\overline{N}(\mathcal{U}_0, \dots, \mathcal{U}_m)$ the simplicial complex consisting of finite subsets $\sigma \subset \bigcup_{i\le m}\{i\} \times \mathcal{U}_i$ such that

- $|\sigma \cap (\{i\} \times \mathcal{U}_i)| \le 1$ for all $i \le m$ and
- the set $\overline{\mathrm{Rg}}_\cap(\sigma) = \bigcap\{\overline{U} : (i,U) \in \sigma \text{ for some } i \le m\}$ is not empty.

It is convenient to consider the simplexes $\sigma \in \overline{N}(\mathcal{U}_0, \dots, \mathcal{U}_m)$ as functions with domain $\mathrm{dom}(\sigma) \subset \{0, \dots, m\}$, assigning to each number $i \in \mathrm{dom}(\sigma)$ some element $\sigma(i) \in \mathcal{U}_i$. In this case $\overline{\mathrm{Rg}}_\cap(\sigma) = \bigcap_{i\in\mathrm{dom}(\sigma)} \overline{\sigma(i)}$. It is important to observe that the complex $\overline{N}(\mathcal{U}_0, \dots, \mathcal{U}_m)$ has dimension at most $m$.

As usual, for a set-valued map $\Phi : X \multimap Y$ and a set $A \subset X$ we put $\Phi(A) = \bigcup_{x\in A} \Phi(x)$.

Now, we are ready to prove Theorem 6.1. Let $X$ be a paracompact $C$-space and $\Phi : X \multimap Y$ be a compactly semicontinuous set-valued map into a topological space $Y$ assigning to each point $x \in X$ a homotopical $Z_n$-set $\Phi(x)$ in $Y$, where $n = \dim X$ is finite or infinity. Assume that $X$ is a retract of an open subset $W_0 \supset X$ of a locally convex linear topological space $L$ and fix a retraction $r : W_0 \to X$.

Given a map $f : X \to Y$ and a continuous pseudometric $\rho$ on $Y$, we need to find a map $f' : X \to Y$ which is 1-homotopic to $f$ and $f'(x) \notin \Phi(x)$ for all $x \in X$.

Fix a cover $\mathcal{W}$ of $W_0$ by open convex subsets such that $\mathrm{diam}_\rho(f \circ r(W)) < 1/2$ for all $W \in \mathcal{W}$. According to Lemma 18.1, the space $X$ admits a continuous pseudometric $d$ such that the cover $\{B_d(x,1) : x \in X\}$ of $X$ refines $\mathcal{W}$.

By induction, we shall construct for every $m < n+1$, $m \ge 0$, a locally finite open cover $\mathcal{U}_m$ of $X$ and two maps $p_m : K_m \to W_0$ and $h_m : K_m \times [0, m+1] \to Y$, where $K_m = \overline{N}(\mathcal{U}_0, \dots, \mathcal{U}_m)$, such that the following conditions are satisfied:

(1) $p_m : K_m \to W_0$ is a PL-map with $p_m|K_{m-1} = p_{m-1}$;
(2) $\overline{U} \subset B_d(p_m(m,U), 1)$ for every vertex $(m,U)$ of $K_m$;
(3) $h_m(x,t) = h_{m-1}(x, \min\{t,m\})$ for $(x,t) \in K_{m-1} \times [0, m+1]$;

(4) $h_m(x,0) = f \circ r \circ p_m(x)$ for all $x \in K_m$;
(5) $h_m$ is a $(1/2 - 1/2^{m+2})$-homotopy;
(6) $h_m(\sigma \times \{m+1\}) \cap \Phi(\overline{\sigma(m)}) = \emptyset$ for every simplex $\sigma \in K_m$ with $m \in \mathrm{dom}(\sigma)$.

To start the construction, for every $x \in X$ fix a path $h_x : [0,1] \to Y$ with $\mathrm{diam}_\rho\, h_x(\mathbb{I}) < 1/4$, $h_x(0) = f(x)$ and $h_x(1) \notin \Phi(x)$. Such a path exists because $\Phi(x)$ is a homotopical $Z_0$-set. Since $\Phi$ is compactly semicontinuous, every point $x \in X$ has a neighborhood $\overline{O}_x \subset B_d(x,1)$ such that $h_x(1) \notin \Phi(\overline{O}_x)$. Let $\mathcal{U}_0$ be an open locally finite cover of $X$ refining $\{O_x : x \in X\}$. For every $U \in \mathcal{U}_0$ fix a point $x(U)$ with $U \subset O_{x(U)}$ and let $p_0(0,U) = x_U$. Then $\overline{U} \subset \overline{O}_{x(U)} \subset B_d(x(U),1)$, which completes the construction of $\mathcal{U}_0$ and $p_0$.

As for the map $h_0 : K_0 \times [0,1] \to Y$, just let $h_0((0,U),t) = h_{x(U)}(t)$ for $(0,U) \in K_0 = \overline{N}(\mathcal{U}_0)$. Then $h_0((0,U),0) = h_{x(U)}(0) = f(x(U)) = f \circ r \circ p_0(0,U)$, demonstrating (4). Moreover, because $\mathrm{diam}_\rho\, h_x(\mathbb{I}) < 1/4$ for all $x \in X$, $h_0$ is a 1/4-homotopy. So, (5) is also satisfied for $m = 0$. On the other hand, $h_0((0,U),1) = h_{x(U)}(1) \notin \Phi(\overline{O}_{x(U)}) \supset \Phi(\overline{U})$, demonstrating (6).

Now, assume that the covers $\mathcal{U}_0, \ldots, \mathcal{U}_{m-1}$ and the maps $p_{m-1}$, $h_{m-1}$ have been constructed for some $m < n+1$. For every $x \in X$ consider the subcomplex $B_m(x) = \{\sigma \in K_{m-1} : x \in \overline{\mathrm{Rg}}_\cap(\sigma)\}$ of $K_{m-1}$, where $\overline{\mathrm{Rg}}_\cap(\sigma) = \bigcap_{i \in \mathrm{dom}(\sigma)} \overline{\sigma(i)}$. This complex is finite because the covers $\mathcal{U}_0, \ldots, \mathcal{U}_{m-1}$ are locally finite. Moreover, its geometric realization $B_m(x)$ is a compact subset of $K_{m-1}$. Condition (6) from the inductive construction guarantees that $h_{m-1}(B_m(x) \times \{m\}) \cap \Phi(x) = \emptyset$.

Consider the simplicial complex $C_m(x) = B_m(x) \cup \{\{x\} \cup \sigma : \sigma \in B_m(x)\}$ whose geometric realization $C_m(x)$ is a cone over $B_m(x)$ with vertex $\{x\}$. Unifying those cones we obtain the simplicial complex $K_m(X) = K_{m-1} \cup \bigcup_{x \in X} C_m(x)$.

Let $\tilde{p}_m : K_m(X) \to L$ be the PL-map determined by the following conditions:

- $\tilde{p}_m \big| K_{m-1} = p_{m-1}$;
- $\tilde{p}_m(\{x\}) = x$ (here $\{x\}$ with the vertex of the cone $C_m(x)$).

We claim that $\tilde{p}_m(K_m(X)) \subset W_0$, which is equivalent to $\tilde{p}_m(\sigma) \subset W_0$ for every $\sigma \in K_m$. This is true if $\sigma \in K_{m-1}$ because $\tilde{p}_m(\sigma) = p_{m-1}(\sigma)$. If $\sigma \in K_m \setminus K_{m-1}$, then $\sigma \in C_m(x)$ for some $x \in X$. Consequently, $\sigma = \tau \cup \{x\}$ with $\tau \in B_m(x)$. So, $x \in \overline{\mathrm{Rg}}_\cap(\tau)$ and, by (2), for any $(i,U) \in \tau$ we have $x \in \overline{U} \subset B_d(\tilde{p}_m(i,U),1)$ (here we use that $\tilde{p}_m(i,U) = p_{m-1}(i,U)$ for all $(i,U) \in \tau$). Hence, $\tilde{p}_m(i,U) \in B_d(x,1)$ and $\tilde{p}_m(\tau^{(0)}) \subset B_d(x,1) \subset W_\sigma$ for some convex set $W_\sigma \in \mathcal{W}$. Since $\tilde{p}_m(\{x\}) = x$, $\tilde{p}_m(\sigma^{(0)}) \subset W_\sigma$. Then $\tilde{p}_m(\sigma) \subset \mathrm{conv}(\tilde{p}_m(\sigma^{(0)})) \subset W_\sigma \subset W_0$. So, $\tilde{p}_m$ is a map from $K_m(X)$ into $W_0$.

Since $\tilde{p}_m$ coincides with $p_{m-1}$ on $K_{m-1}$, condition (4) implies that $h_{m-1}(z,0) = f \circ r \circ p_{m-1}(z) = f \circ r \circ \tilde{p}_m(z)$ for all $z \in K_{m-1}$. Applying the Borsuk extension lemma to each pair $(C_m(x), C_m(x) \cap K_{m-1})$, we extend the $(1/2 - 1/2^{m+1})$-homotopy $h_{m-1} : K_{m-1} \times [0,m] \to Y$ to a $(1/2 - 1/2^{m+1})$-homotopy $h' : K_m(X) \times [0,m] \to Y$ such that $h'(z,0) = f \circ r \circ \tilde{p}_m(z)$ for all $z \in K_m(X)$.

Condition (6) yields $h'((C_m(x) \cap K_{m-1}) \times \{m\}) \cap \Phi(x) = \emptyset$ for any $x \in X$. Therefore, since each $\Phi(x)$ is a homotopical $Z_n$-set in $Y$ and $K_m(X)$ is a simplicial complex of dimension $\leq m \leq n$, we can apply Lemma 36.1 to any pair $(C_m(x), C_m(x) \cap K_{m-1})$ to obtain a $(1/2 - 1/2^{m+2})$-homotopy $\tilde{h} : K_m(X) \times [0, m+1] \to Y$ such that:

- $\tilde{h}$ extends $h'$;
- $\tilde{h}(z,t) = h'(z,m) = h_{m-1}(z,m)$ for all $(z,t) \in K_{m-1} \times [m, m+1]$;
- $\tilde{h}(C_m(x) \times \{m+1\}) \cap \Phi(x) = \emptyset$.

For every $x \in X$, the compact semicontinuity of $\Phi$ yields a neighborhood $V_x \subset X$ of $x$ with $\tilde{h}(C_m(x) \times \{m+1\}) \cap \Phi(\overline{V}_x) = \emptyset$. We can take $V_x$ so small that $\overline{V}_x \subset B_d(x,1)$ and $\overline{V}_x \cap \overline{\mathrm{Rg}}_\cap(\tau) = \emptyset$ for any simplex $\tau \in K_{m-1}$ with $x \notin \overline{\mathrm{Rg}}_\cap(\tau)$ (at this point we use the local finiteness of the covers $\mathcal{U}_0, \ldots \mathcal{U}_{m-1}$).

Let $\mathcal{U}_m$ be a locally finite open cover of $X$ refining the cover $\{V_x : x \in X\}$. For every $U \in \mathcal{U}_m$ pick a point $x(U) \in X$ with $U \subset V_{x(U)}$ and consider the map $e : K_m^{(0)} \to K_m(X)^{(0)}$ which is identity on $K_{m-1}^{(0)}$ and assigns to each new vertex $(m,U) \in \{m\} \times \mathcal{U}_m$ of $K_m$ the vertex $x(U)$ of $K_m(X)$. Let us verify that the image $e(\sigma)$ of any simplex $\sigma \in K_m$ is a simplex in $K_m(X)$. This is obviously true if $\sigma \in K_{m-1}$. If $\sigma \in K_m \setminus K_{m-1}$, then $\sigma = \tau \cup \{(m,U)\}$, where $\tau$ is a simplex from $K_{m-1}$ and $U \in \mathcal{U}_m$. If suffices to show that $\tau \cup \{x(U)\}$ forms a simplex in $K_m(X)$ because in that case $e(\sigma) = \tau \bigcup \{x(U)\}$. This is equivalent to $x(U) \in \bigcap \overline{\mathrm{Rg}}(\tau)$. Assuming $x(U) \notin \bigcap \overline{\mathrm{Rg}}(\tau)$, we have $\emptyset \neq \overline{\mathrm{Rg}}_\cap(\sigma) = \overline{U} \cap \overline{\mathrm{Rg}}_\cap(\tau) \subset \overline{V}_{x(U)} \cap \overline{\mathrm{Rg}}_\cap(\tau) = \emptyset$, a contradiction. So, $e(\sigma) \in K_m(X)$.

The map $e : K_m^{(0)} \to K_m(X)^{(0)}$ induces a PL-map $\tilde{e} : K_m \to K_m(X)$ between the corresponding geometric realizations. Now, we can define the maps $p_m : K_m \to X$ and $h_m : K_m \times [0, m+1] \to Y$ letting $p_m(z) = \tilde{p}_m \circ \tilde{e}(z)$ and $h_m(z,t) = \tilde{h}(\tilde{e}(z), t)$ for $z \in K_m$.

Let us check that the maps $p_m$ and $h_m$ satisfy conditions (1)–(6). Since $\tilde{p}_m$ maps $K_m(X)$ into $W_0$, $p_m$ is a map into $W_0$. Moreover, $\tilde{p}_m | K_{m-1} = p_{m-1}$ and $\tilde{e}|K_{m-1}$ is the identity. So, the first condition is satisfied. For the second one, take any vertex $(m,U) \in K_m$ and note that $\overline{U} \subset \overline{V}_{x(U)} \subset B_d(x(U),1) = B_d(p_m(m,U),1)$. So, (2) holds as well. Conditions (3)–(5) follow immediately from the definition of $h_m$ and $\tilde{h}$. It remains to check (6). Take any simplex $\sigma \in K_m$ with $m \in \mathrm{dom}(\sigma)$ and observe that $(m,U) \in \sigma$, where $U = \sigma(m)$. Let us first show that $\tilde{e}(\sigma) \subset C_m(x(U))$. This is true if $\sigma$ is the single vertex $(m,U)$ because $\tilde{e}(\sigma) = x(U) \in C_m(x(U))$. Suppose $\tau = \sigma \setminus \{(m,U)\} \neq \emptyset$. It follows from the preceding discussion (concerning the map $e$) that $\tau \in B_m(x(U))$. Then the definition of $\tilde{e}$ yields $\tilde{e}(\sigma) \subset C_m(x(U))$. Hence, in both cases we have

$$\begin{aligned} h_m(\sigma \times \{m+1\}) \cap \Phi(\overline{\sigma(m)}) &= \tilde{h}(\tilde{e}(\sigma) \times \{m+1\}) \cap \Phi(\overline{U}) \\ &\subset \tilde{h}(C_m(x(U))) \times \{m+1\}) \cap \Phi(\overline{V}_{x(U)}) = \emptyset, \end{aligned}$$

which witnesses (6) and completes the inductive step.

Completing the inductive construction, we let

- $K_n = \bigcup_{m<n+1} K_m$;
- $p = \bigcup_{m<n+1} p_m : K_n \to W_0$;
- $h = \bigcup_{m<n+1} h_m : \bigcup_{m<n+1} K_m \times [0, m+1] \to Y$.

Observe that $p$ and $h$ are well defined because $K_{m-1} \subset K_m$, $h_m \big| \big(K_{m-1} \times [0,m]\big) = h_{m-1}$ and $p_m | K_{m-1} = p_{m-1}$ (see (1) and (3)). Moreover, $h$ is defined on $K_n \times J_n$, where $J_n = [0, n+1]$ if $n$ is finite and $J = [0, \infty)$ if $n$ is infinite. The vertices of the complex $K_n$ are pairs $(m,U)$ with $U \in \mathcal{U}_m$ and $m < n+1$.

Since $X$ is a paracompact $C$-space of dimension $n$, according to a generalized version of the Ostrand theorem [38, Theorem 5.2], there exists a sequence $(\mathcal{V}_m)_{m<n+1}$ of discrete families of open sets in $X$ such that each $\mathcal{V}_m$ refines $\mathcal{U}_m$ and $\bigcup_{m<n+1}\mathcal{V}_m$ is a locally finite cover of $X$. The sequence $(\mathcal{V}_m)$ enables us to construct a partition of unity $\{\lambda_{(m,U)} : X \to [0,1] : (m,U) \in K_n^{(0)}\}$ such that

- $\lambda^{-1}_{(m,U)}(0,1] \subset U$ for every $(m,U) \in K_n^{(0)}$;
- the family $\{\lambda^{-1}_{(m,U)}(0,1] : (m,U) \in K_n^{(0)}\}$ is locally finite in $X$;
- for every $m < n+1$ the family $\{\lambda^{-1}_{(m,U)}(0,1] : U \in \mathcal{U}_m\}$ is discrete in $X$.

Observe that for any $x \in X$ the set $\sigma_x = \{(m,U) \in K_n^{(0)} : \lambda_{(m,U)}(x) > 0\}$ is a simplex of $K_n$. Hence, the canonical map

$$\lambda : X \to K_n, \quad x \mapsto (\lambda_{(m,U)}(x))_{(m,U)\in K_n^{(0)}},$$

is well-defined.

Finally we can define the desired map $f' : X \to Y$. If $n$ is finite, we let $f'(x) = h(\lambda(x), n+1)$, $x \in X$.

Suppose $n$ is infinite. First, we show that there exists a continuous function $\xi : X \to [0,\infty)$ with $\sigma_x \in K_{[\xi(x)]}$ for every $x \in X$, where $[\xi(x)]$ stands for the integer part of the real number $\xi(x)$. To this end, for every $x \in X$ we put $m(x) = \max\{m : x \in \overline{\lambda^{-1}_{(m,U)}(0,1]}\}$. Since the cover $\{\lambda^{-1}_{(m,U)}(0,1]\}$ of $X$ is locally finite, so is $\{\overline{\lambda^{-1}_{(m,U)}(0,1]}\}$. Hence, $m(x)$ is a finite integer for every $x \in X$. Next, consider the set-valued map $\phi\colon X \multimap [0,\infty)$ defined by $\phi(x) = [m(x),\infty)$, $x \in X$. It is easily seen that $\phi$ is lower semicontinuous. Hence, by the Michael selection theorem [53], $\phi$ has a continuous selection $\xi\colon X \to [0,\infty)$. Because $\sigma_x \in K_{m(x)}$ and $\xi(x) \geq m(x)$ for every $x \in X$, we have $\sigma_x \in K_{[\xi(x)]}$. Now, we define $f' : X \to Y$ by $f'(x) = h(\lambda(x), \xi(x)+1)$.

It remains to check that the map $f' : X \to Y$ has the desired properties, i.e, $f'$ is 1-homotopic to $f$ and $f'(x) \notin \Phi(x)$ for all $x \in X$. In order to unify both cases ($n < \infty$ and $n = \infty$), we consider the constant function $\xi\colon X \to [0,\infty)$, $\xi(x) = n$, provided $n < \infty$.

To prove that $f'$ is 1-homotopic to $f$, consider the intermediate map $f_0 : X \to Y$ defined by $f_0(x) = h(\lambda(x), 0)$. It follows from (5) that $f_0$ is 1/2-homotopic to $f'$. So, it suffices to check that $f_0$ is 1/2-homotopic to $f$. By (4), $f_0(x) = h(\lambda(x),0) = f\circ r\circ p\circ\lambda(x)$. We define a map $H : X \times [0,1] \to Y$ by

$$H(x,t) = f \circ r((1-t)x + t \cdot p \circ \lambda(x)).$$

Obviously, $H(x,0) = f(x)$ and $H(x,1) = f_0(x)$ for all $x \in X$. So, $H$ is a homotopy connecting $f$ and $f_0$. Let us check that $H$ is a 1/2-homotopy. Given any $x \in X$ consider the simplex $\sigma_x = \{(i,U) \in K_n^{(0)} : \lambda_{(i,U)}(x) > 0\}$, which obviously contains the point $\lambda(x)$. According to (2), for every $(i,U) \in \sigma_x$ we have $x \in U \subset B_d(p(i,U),1)$. Consequently, $p(i,U) \subset B_d(x,1)$ for all $i$ with $(i,U) \in \sigma_x$. Hence, $p(\sigma_x^{(0)}) \subset B_d(x,1) \subset W_x$ for some convex set $W_x \in \mathcal{W}$. Then $p \circ \lambda(x) \in \mathrm{conv}(p(\sigma_x^{(0)})) \subset W_x$. Moreover, $W_x$ contains $x$. So, $(1-t)x + tp \circ \lambda(x) \in W_x$ and $H(\{x\} \times [0,1]) \subset f \circ r(W_x)$. Thus, $H(\{x\} \times [0,1])$ has diameter $< 1/2$ by the choice of $\mathcal{W}$. Therefore, $f$ and $f_0$ are 1/2-homotopic, which implies that $f$ and $f'$ are 1-homotopic.

Finally, we verify that $f'(x) \notin \Phi(x)$ for any $x \in X$. Suppose $m \le \xi(x) < m+1$ for some $m \le n$. Then $K_{[\xi(x)]} = K_m$. Since $\sigma_x$ is a simplex of $K_{[\xi(x)]}$ and contains $\lambda(x)$, $\lambda(x) \in K_m \subset K_{m+1}$. On the other hand $m+1 \le \xi(x)+1 < m+2$. Hence, $f'(x) = h(\lambda(x), \xi(x)+1) = h_{m+1}(\lambda(x), \xi(x)+1)$. Then, by (3), we have $h_{m+1}(\lambda(x), \xi(x)+1) = h_m(\lambda(x), m+1)$. Now, let $m_x = \max \operatorname{dom}(\sigma_x)$. Since $K_{[\xi(x)]} = K_m$ contains $\sigma_x$, $m_x \le m$. Moreover, for every $(i, U) \in \sigma_x$ we have $x \in U$. If $m_x < m$, we choose an element $U_x \in \mathcal{U}_m$ containing $x$ and consider the simplex $\tau_x = \sigma_x \cup \{(m, U_x)\}$ from $K_m$. If $m_x = m$, we just take $\tau_x$ to be $\sigma_x$. In both cases $\tau_x$ is a simplex from $K_m$ containing $\lambda(x)$ such that $m \in \operatorname{dom} \tau_x$ and $x \in \overline{\tau_x(m)}$. Then, by (6), $h_m(\tau_x \times \{m+1\}) \cap \Phi(\overline{\tau_x(m)}) = \emptyset$. Therefore, $f'(x) = h_m(\lambda(x), m+1) \notin \Phi(x)$ for all $x \in X$.

## 37. Proof of Theorem 7.1

In this section we provide a proof of Theorem 7.1.

LEMMA 37.1. *A Hausdorff space $X$ has the $m$-$\overline{\mathrm{DD}}^{\{0,0\}}$-property if and only if the diagonal of $X^2$ is a homotopical $Z_m$-set in $X^2$.*

*Proof.* Assume that $X$ has the $m$-$\overline{\mathrm{DD}}^{\{0,0\}}$-property. To show that the diagonal $\Delta_X$ is a homotopical $Z_m$-set in $X^2$, fix a cover $\mathcal{V}$ of $X^2$ and a map $(f,g) : \mathbb{I}^m \to X^2$. By a standard compactness argument, we can find open covers $\mathcal{U}_1, \mathcal{U}_2$ of $X$ such that for any $U_1 \in \mathcal{U}_1$, $U_2 \in \mathcal{U}_2$ the product $U_1 \times U_2$ lies in some $U \in \mathcal{V}$ provided $U_1 \times U_2$ meets the set $f(\mathbb{I}^m) \times g(\mathbb{I}^m)$. Let $\mathcal{U}$ be an open cover of $X$ refining both $\mathcal{U}_1$ and $\mathcal{U}_2$. The $m$-$\overline{\mathrm{DD}}^{\{0,0\}}$-property of $X$ yields two maps $f', g' : \mathbb{I}^m \to X$ such that $f' \underset{\mathcal{U}}{\sim} f$, $g' \underset{\mathcal{U}}{\sim} g$ and $f'(x) \ne g'(x)$ for all $x \in \mathbb{I}^m$. Then the diagonal product $(f', g') : \mathbb{I}^m \to X^2$ maps $\mathbb{I}^m$ to $X^2 \setminus \Delta_X$ and is $\mathcal{V}$-homotopic to $(f,g)$, demonstrating that $\Delta_X$ is a homotopical $Z_m$-set in $X^2$.

Now, assume that $\Delta_X$ is a homotopical $Z_m$-set in $X^2$. To show that $X$ has the $m$-$\overline{\mathrm{DD}}^{\{0,0\}}$-property, fix an open cover $\mathcal{U}$ of $X$ and two maps $f, g : \mathbb{I}^m \to X$. Consider the open cover $\mathcal{W} = \{U \times V : U, V \in \mathcal{U}\}$ of $X^2$. Since $\Delta_X$ is a homotopical $Z_n$-set in $X^2$, there is a map $(f', g') : \mathbb{I}^m \to X^2 \setminus \Delta_X$ which is $\mathcal{W}$-homotopic to $(f,g)$. Then $f' \underset{\mathcal{U}}{\sim} f$, $g' \underset{\mathcal{U}}{\sim} g$ and $f'(x) \ne g'(x)$ for all $x \in \mathbb{I}^m$, demonstrating the $m$-$\overline{\mathrm{DD}}^{\{0,0\}}$-property of $X$. ∎

LEMMA 37.2. *An $\mathrm{LC}^0$-space $X$ has the $0$-$\overline{\mathrm{DD}}^{\{0,n\}}$-property provided the set of its homotopical $Z_n$-points is dense in $X$.*

*Proof.* To prove that $X$ has the $0$-$\overline{\mathrm{DD}}^{\{0,n\}}$-property, we take an open cover $\mathcal{U}$ of $X$ and two maps $f : \mathbb{I}^n \to X$ and $g : \mathbb{I}^0 \to X$. Fix a set $U \in \mathcal{U}$ containing the singleton $\{x\} = g(\mathbb{I}^0)$. Because $X$ is $\mathrm{LC}^0$, there is a neighborhood $V \subset X$ of $x$ such that any point $x' \in V$ can be linked with $x$ by a path lying in $U$. Since the set of homotopical $Z_n$-points is dense in $X$, there exists a homotopical $Z_n$-point $x' \in V$ in $X$. By the choice of $V$, the constant map $g' : \mathbb{I}^0 \to \{x'\}$ is $\mathcal{U}$-homotopic to $g$. Moreover, $x'$ being a homotopical $Z_n$-point implies that $f$ is $\mathcal{U}$-homotopic to a map $f' : \mathbb{I}^n \to X \setminus \{x'\}$. Then $f'(\mathbb{I}^n) \cap g'(\mathbb{I}^0) = \emptyset$, demonstrating the $0$-$\overline{\mathrm{DD}}^{\{0,n\}}$-property of $X$. ∎

LEMMA 37.3. *If a metrizable separable Baire* (LC$^n$) *space* $X$ *has the property* $0\text{-}\overline{\mathrm{DD}}^{\{0,n\}}$ *then the set of* (*homotopical*) $Z_n$*-points is dense and* $G_\delta$ *in* $X$.

*Proof.* Let $\rho$ be a metric generating the topology of $X$. By Lemma 18.5, the function space $C(\mathbb{I}^n, X)$ is separable, and hence contains a countable dense subset $\{f_k : k \in \mathbb{N}\}$.

For every $m, k \in \mathbb{N}$ consider the set $U_{k,m}$ of all points $x \in X$ satisfying the following condition: there is a map $f' : \mathbb{I}^n \to X \setminus \{x\}$ such that $f'$ is $2^{-m}$-near to $f_k$. It is clear that the $U_{k,m}$ are open in $X$. The $0\text{-}\overline{\mathrm{DD}}^{\{0,n\}}$-property of $X$ implies that each $U_{k,m}$ is dense in $X$. Since $X$ is a Baire space, the intersection $U = \bigcap_{k,m\in\mathbb{N}} U_{k,m}$ is a dense $G_\delta$-set in $X$. It is easily seen that every $x \in U$ is a $Z_n$-point in $X$ and each $Z_n$-point belongs to $U$. Thus the set $U$ of all $Z_n$-points is dense $G_\delta$ in $X$.

If, in addition, $X$ is an LC$^n$-space, then Theorem 10.1(2) implies that each $Z_n$-set is a homotopical $Z_n$-set. So, the set of all homotopical $Z_n$-points in $X$ coincides with the set $U$ of all $Z_n$-points, which completes the proof. ∎

LEMMA 37.4. *A Tychonoff space* $X$ *has the* $m\text{-}\overline{\mathrm{DD}}^{\{0,k\}}$*-property provided each* $x \in X$ *is a homotopical* $Z_{m+k}$*-point.*

*Proof.* Assume that each point $x$ of $X$ is a homotopical $Z_{m+k}$-point. To check the $m\text{-}\overline{\mathrm{DD}}^{\{0,k\}}$-property for $X$, fix an open cover $\mathcal{U}$ of $X$ and two maps $f : \mathbb{I}^m \to X$, $g : \mathbb{I}^m \times \mathbb{I}^k \to X$. By Lemma 18.2, there exists a continuous pseudometric $\rho$ on $X$ such that each ball $B_\rho(x, 1)$, $x \in g(\mathbb{I}^m \times \mathbb{I}^k)$, is contained in some set $U \in \mathcal{U}$.

Consider the map $\Phi : \mathbb{I}^m \times \mathbb{I}^k \to X$ assigning to each $(z, x) \in \mathbb{I}^m \times \mathbb{I}^k$ the singleton $\Phi(z, x) = \{f(z)\}$. Since each $\Phi(z, x)$ is a homotopical $Z_{m+k}$-set in $X$, we may apply Selection Theorem 6.1 to find a map $g' : \mathbb{I}^m \times \mathbb{I}^k \to X$ which is 1-homotopic to $g$ and such that $g'(x, z) \neq \Phi(z, x) = \{f(z)\}$ for all $(z, x) \in \mathbb{I}^m \times \mathbb{I}^k$. The choice of the pseudometric $\rho$ ensures that $g'$ is $\mathcal{U}$-homotopic to $g$. Then the maps $f' = f$ and $g'$ witness the $m\text{-}\overline{\mathrm{DD}}^{\{0,k\}}$-property for $X$. ∎

LEMMA 37.5. *If* $X$ *has either the* $0\text{-}\overline{\mathrm{DD}}^{\{n,n\}}$*-property or the* $n\text{-}\overline{\mathrm{DD}}^{\{n,0\}}$*-property, then each point of* $X$ *is a homotopical* $Z_n$*-point.*

*Proof.* First assume that $X$ has the $0\text{-}\overline{\mathrm{DD}}^{\{n,n\}}$-property. Given a point $x_0 \in X$, an open cover $\mathcal{U}$ of $X$ and a map $f : \mathbb{I}^n \to X$, use the $0\text{-}\overline{\mathrm{DD}}^{\{n,n\}}$-property of $X$ to find two maps $f_0, f_1 : \mathbb{I}^n \to X$ that are $\mathcal{U}$-homotopic to $f$ and have disjoint images. Then for some $i \in \{0, 1\}$ the image $f_i(\mathbb{I}^n)$ does not contain $x_0$, demonstrating that $x_0$ is a homotopical $Z_n$-point.

Next, assume that $X$ has the $n\text{-}\overline{\mathrm{DD}}^{\{0,n\}}$-property. Suppose there exists $x_0 \in X$ which is not a homotopical $Z_n$-point in $X$. So, we can find a map $g\colon \mathbb{I}^n \to X$ and an open cover $\mathcal{U}$ of $X$ such that $x_0 \in g'(\mathbb{I}^n)$ for any map $g'\colon \mathbb{I}^n \to X$ which is $\mathcal{U}$-homotopic to $g$. Consider now the map $f\colon \mathbb{I}^n \times \mathbb{I}^n \to X$ defined by $f(z, y) = g(y)$, $(z, y) \in \mathbb{I}^n \times \mathbb{I}^n$. Since $X$ has the $n\text{-}\overline{\mathrm{DD}}^{\{n,0\}}$-property, there are two maps $f'\colon \mathbb{I}^n \times \mathbb{I}^n \to X$ and $g'\colon \mathbb{I}^n \to X$ which are $\mathcal{U}$-homotopic to $f$ and $g$, respectively, and $g'(z) \notin f'(\{z\} \times \mathbb{I}^n)$ for every $z \in \mathbb{I}^n$. By the choice of $g$ and $\mathcal{U}$, we have $g'(z_0) = x_0$ for some $z_0 \in \mathbb{I}^n$. Hence, $x_0 \notin f'(\{z_0\} \times \mathbb{I}^n)$. On the other hand, the map $h\colon \mathbb{I}^n \to X$, $h(y) = f'(z_0, y)$, is $\mathcal{U}$-homotopic to $g$ because $f$ and $f'$ are $\mathcal{U}$-homotopic. Therefore, $h(\mathbb{I}^n) = f'(\{z_0\} \times \mathbb{I}^n)$ contains $x_0$, a contradiction. ∎

LEMMA 37.6. *If a Tychonoff space $X$ has the $2$-$\overline{\mathrm{DD}}^{\{0,0\}}$-property, then each point of $X$ is a homotopical $Z_1$-point.*

*Proof.* Assume that some point $x_0 \in X$ fails to be a homotopical $Z_1$-point. Then there is a cover $\mathcal{U}$ of $X$ and a map $f_0 : \mathbb{I} \to X$ such that $x_0 \in f'(\mathbb{I})$ for any map $f' : \mathbb{I} \to X$ which is $\mathcal{U}$-homotopic to $f_0$. By Lemma 18.2, we can find a continuous pseudometric $\rho$ on $X$ such that the family $\{B_\rho(x,1) : x \in f_0(\mathbb{I})\}$ refines $\mathcal{U}$.

By the uniform continuity of the map $f_0 : \mathbb{I} \to (X,\rho)$, there is a sequence $0 = t_0 < t_1 < \cdots < t_m = 1$ such that each set $f_0([t_{i-1}, t_i])$, $i \le m$, is of $\rho$-diameter $< 1/4$.

Obviously, $X$, being a space with the $2$-$\overline{\mathrm{DD}}^{\{0,0\}}$-property, also has the $0$-$\overline{\mathrm{DD}}^{\{0,0\}}$-property. So, for every point $x \in X$ and its neighborhood $U$, there exists a path $\gamma : [0,1] \to U$ with $\gamma(0) = x$ and $\gamma(1) \ne x$. Then, we can find a map $f_1 : \mathbb{I} \to X$ such that $f_1$ is $1/16$-homotopic to $f_0$ and $f_1(t_i) \ne x_0$ for all $i \le m$. Choose a pseudometric $d \ge \rho$ on $X$ such that $d(x_0, f_1(t_i)) \ne 0$ for all $i \le m$ and choose a positive number $\varepsilon < 1/16$ with $d(f_1(t_i), x_0) > \varepsilon$, $i \le m$.

Let $\mathrm{pr}_1, \mathrm{pr}_2 : \mathbb{I}^2 \to \mathbb{I}$ be the coordinate projections and $f = f_1 \circ \mathrm{pr}_1 : \mathbb{I}^2 \to X$, $g = f_1 \circ \mathrm{pr}_2 : \mathbb{I}^2 \to X$. The $2$-$\overline{\mathrm{DD}}^{\{0,0\}}$-property of $X$ implies the existence of maps $f', g' : \mathbb{I}^2 \to X$ such that $f'(z) \ne g'(z)$ for all $z \in \mathbb{I}^2$ and $f'$ and $g'$ are $\varepsilon$-homotopic to $f$ and $g$, respectively, with respect to the pseudometric $d$.

For every $i \le m$ consider the sets

$$\begin{aligned} A_i &= \{(x,y) \in \mathbb{I}^2 : x \in [t_{i-1}, t_i],\ f'(x,y) = x_0\}, \\ B_i &= \{(x,y) \in \mathbb{I}^2 : y \in [t_{i-1}, t_i],\ g'(x,y) = x_0\}. \end{aligned}$$

By the choice of $\varepsilon$ and $f', g'$, $A_i \cap (\{t_{i-1}, t_i\} \times \mathbb{I}) = \emptyset$ and $B_i \cap (\mathbb{I} \times \{t_{i-1}, t_i\}) = \emptyset$ for all $i \le m$. So, $A_i \subset (t_{i-1}, t_i) \times \mathbb{I}$ and $B_i \subset \mathbb{I} \times (t_{i-1}, t_i)$, $i \le m$.

Let us show that the set $A_i$ separates the sets $\{t_{i-1}\} \times \mathbb{I}$ and $\{t_i\} \times \mathbb{I}$ for some $i \le m$. If this is not the case, then for every $i \le m$ we could find a path $\gamma_i : [t_{i-1}, t_i] \to [t_{i-1}, t_i] \times \mathbb{I}$ such that $\gamma_i(t_{i-1}) = (t_{i-1}, 0)$, $\gamma_i(t_i) = (t_i, 0)$ and $\gamma_i([t_{i-1}, t_i]) \cap A_i = \emptyset$. The maps $\gamma_i$, $i \le m$ compose a single continuous map $\gamma : [0,1] \to [0,1] \times \mathbb{I}$ such that $\gamma|[t_{i-1}, t_i] = \gamma_i$. Then, the composition $f'' = f' \circ \gamma : [0,1] \to X$ is a continuous map with $f''(\mathbb{I}) \subset X \setminus \{x_0\}$. We claim that $f''$ is $1$-homotopic to $f_0$. Indeed, the map $\gamma : \mathbb{I} \to \mathbb{I} \times \mathbb{I}$ is homotopic to the embedding $e : \mathbb{I} \to \mathbb{I} \times \{0\} \subset \mathbb{I}^2$ via the homotopy $h : \mathbb{I} \times [0,1] \to \mathbb{I}^2$, $h : (x,t) \mapsto (1-t)\gamma(x) + t\, e(x)$. Hence, $f' \circ h : \mathbb{I} \times [0,1] \to X$ is a homotopy between $f''$ and $f' \circ e$. Observe that, for each $x \in [t_{i-1}, t_i]$, the set $h(\{x\} \times \mathbb{I}) \subset [t_{i-1}, t_i] \times \mathbb{I}$ and thus $f' \circ h(\{x\} \times \mathbb{I}) \subset f'([t_{i-1}, t_i] \times \mathbb{I})$. Therefore,

$$\begin{aligned} \operatorname{diam} f' \circ h(\{x\} \times \mathbb{I}) &\le \operatorname{diam} f'([t_{i-1}, t_i] \times \mathbb{I}) \le \operatorname{diam} f([t_{i-1}, t_i] \times \mathbb{I}) + 2\operatorname{dist}(f', f) \\ &= \operatorname{diam} f_1([t_{i-1}, t_i]) + \tfrac{2}{16} < \operatorname{diam} f_0([t_{i-1}, t_i]) + 2\operatorname{dist}(f_1, f_0) + \tfrac{1}{8} \\ &< \tfrac14 + \tfrac{2}{16} + \tfrac18 = \tfrac12. \end{aligned}$$

This means that $f' \circ h$ is a $1/2$-homotopy between $f''$ and $f' \circ e$. Since $f'$ is $1/6$-homotopic to $f_1 \circ \mathrm{pr}_1$, the map $f''$ is $(1/2 + 1/16)$-homotopic to $f_1 \circ \mathrm{pr}_1 \circ e = f_1$. On the other hand, $f_1$ being $1/16$-homotopic to $f_0$ implies that $f''$ is $(1/2 + 1/8)$-homotopic to $f_0$. Hence, $f''$ is $\mathcal{U}$-homotopic to $f_0$. Finally, the choice of $f_0$ and $\mathcal{U}$ implies that $x_0 \in f''(\mathbb{I})$, which is not the case.

This contradiction shows that, for some $i \le m$, the set $A_i$ separates the sets $\{t_{i-1}\} \times \mathbb{I}$ and $\{t_i\} \times \mathbb{I}$. For the same reason, there exists $j \le m$ such that $B_j$ separates the sets $\mathbb{I} \times \{t_{j-1}\}$ and $\mathbb{I} \times \{t_j\}$. It is well known from dimension theory that $A_i$ and $B_j$ must intersect at some point $(x^*, y^*) \in [t_{i-1}, t_i] \times [t_{j-1}, t_j]$. Then $f'(x^*, y^*) = x_0 = g'(x^*, y^*)$, which contradicts the choice of $f', g'$. Therefore, $x_0$ is a homotopical $Z_1$-point in $X$. ■

## 38. Homological $Z_n$-sets

In our subsequent proofs we heavily rely on the machinery of homological $Z_n$-sets. Sometimes it will be necessary to consider homologies with an arbitrary coefficient group $G$. This leads to two specifications of homological $Z_n$-sets, $G$-homological and $\exists G$-homological $Z_n$-sets.

In what follows we consider singular (relative) homologies with coefficients in a non-trivial Abelian group $G$ called a *coefficient group*. If $G = \mathbb{Z}$ we shall write $H_k(X)$ and $H_k(X, Y)$ instead of $H_k(X; G)$ and $H_k(X, Y; G)$. By $\tilde{H}_k(X)$ we denote the homology groups of $X$, reduced in dimension zero. Below, $n$ will stand for a non-negative integer or infinity.

Definition 38.1. A closed subset $A$ of a topological space $X$ is defined to be

- a *$G$-homological $Z_n$-set* in $X$ if for any open set $U \subset X$ and any $k < n+1$ the relative homology group $H_k(U, U \setminus A; G)$ is trivial;
- a *$\exists G$-homological $Z_n$-set* in $X$ if $A$ is a $G$-homological $Z_n$-set for some coefficient group $G$;
- a *homological $Z_n$-set* in $X$ if $A$ is a $\mathbb{Z}$-homological set in $X$ (equivalently, $A$ is a $G$-homological $Z_n$-set in $X$ for all coefficient groups $G$).

The following fact concerning homotopical and homological $Z_n$-sets is of crucial importance (see Theorem 10.1(5)): *a homotopical $Z_2$-set in an* $\mathrm{LC}^1$*-space is a homotopical $Z_n$ if and only if it is a homological $Z_n$-set.*

The following lemma was established in [4, Proposition 3.5, Proposition 3.6, Theorem 4.3 and Theorem 4.4] using the methods of R. Daverman and J. Walsh [19].

Lemma 38.2. *Let $X$ be an arbitrary space.*

1. *Any closed subset $F$ of a $G$-homological $Z_n$-set in $X$ is a $G$-homological $Z_n$-set in $X$.*
2. *The union of any two (homotopical) $G$-homological $Z_n$-sets in $X$ is a (homotopical) $G$-homological $Z_n$-set in $X$.*
3. *A closed set $A \subset X$ is a $G$-homological $Z_n$-set in $X$ if and only if $H_k(U, U \setminus A; G) = 0$ for all $k < n+1$ and all sets $U$ which belong to some fixed base for the topology of $X$.*
4. *A closed* $\mathsf{trt}$*-dimensional subset $A \subset X$ is a $G$-homological $Z_n$-set in $X$ provided each point $a \in A$ is a $G$-homological $Z_{n+d}$-point in $X$ with $d = \mathsf{trt}(A)$.*
5. *If $X$ is a homotopically $n$-dense subset of a space $\tilde{X}$, then a closed subset $A \subset \tilde{X}$ is a (homotopical) $G$-homological $Z_n$-set in $\tilde{X}$ if and only if $A \cap X$ is a (homotopical) $G$-homological $Z_n$-set in $X$.*

Theorem 10.1 reduces the study of homotopical $Z_n$-sets to detecting homological $Z_n$-sets. In the latter case there is a wide arsenal of powerful tools of algebraic topology. Among these tools, two are the most important in our subsequent study: the universal coefficients formula and the Künneth formula.

The universal coefficients formula expresses homology with an arbitrary coefficient group via homology with coefficients in the group $\mathbb{Z}$ of integers. The following form of this formula is taken from [40, 3A.4].

LEMMA 38.3 (Universal coefficients formula). *For each pair $(X, A)$ and all $n \geq 1$ there is a natural exact sequence*

$$0 \to H_n(X, A) \otimes G \to H_n(X, A; G) \to H_{n-1}(X, A) * G \to 0.$$

The relative Künneth formula expresses relative homologies of a product pair via relative homologies of the factors. The following form of this formula is taken from [64, 5.3.10].

LEMMA 38.4 (Relative Künneth formula). *For open sets $U \subset X$, $V \subset Y$ in topological spaces and a non-negative integer $n$ the following exact sequence holds:*

$$0 \to [H(X,U) \otimes H(Y,V)]_n \to H_n(X \times Y, X \times V \cup U \times Y) \to [H(X,U) * H(Y,V)]_{n-1} \to 0.$$

Here,

$$[H(X,U) \otimes H(Y,V)]_n = \bigoplus_{i+j=n} H_i(X,U) \otimes H_j(Y,V),$$

$$[H(X,U) * H(Y,V)]_{n-1} = \bigoplus_{i+j=n-1} H_i(X,U) * H_j(Y,V),$$

where $G \otimes H$ and $G * H$ stand for the tensor and torsion products of the Abelian groups $G$ and $H$, respectively.

We also need the Künneth formula for fields.

LEMMA 38.5 (Künneth formula for fields). *Let $A \subset X$ and $B \subset Y$ be closed subsets of the topological spaces $X$ and $Y$, and let $\mathbb{F}$ be a field. Then for every integer $n$ the homology group $H_n(X \times Y, X \times Y \setminus A \times B; \mathbb{F})$ is isomorphic to*

$$\bigoplus_{i+j=n} H_i(X, X \setminus A; \mathbb{F}) \otimes_{\mathbb{F}} H_j(Y, Y \setminus B; \mathbb{F}),$$

*where each $H_i(X, X \setminus A; \mathbb{F}) \otimes_{\mathbb{F}} H_j(Y, Y \setminus B; \mathbb{F})$ is the tensor product over $\mathbb{F}$.*

Our final lemma, proven in [4, Theorem 9.2], describes the set of all (homotopical or $G$-homological) $Z_n$-points in $\mathrm{LC}^n$-spaces.

LEMMA 38.6. *Let $X$ be a metrizable separable space and $G$ be a coefficient group.*

(1) *The set of $Z_n$-points in $X$ is a $G_\delta$-set in $X$.*
(2) *If $X$ is an $\mathrm{LC}^n$-space, then the set of homotopical (resp., $G$-homological) $Z_n$-points is a $G_\delta$-set in $X$.*

## 39. Proof of Theorem 11.1

The items of Theorem 11.1 follow from the following four lemmas.

LEMMA 39.1. *If each point of an* $\mathrm{LC}^1$*-space* $X$ *is a homological* $Z_n$*-point, and* $X$ *has the* $2\text{-}\overline{\mathrm{DD}}^{\{0,2\}}$*-property, then each point of* $X$ *is a homotopical* $Z_n$*-point.*

*Proof.* By Theorem 7.1(5), the $2\text{-}\overline{\mathrm{DD}}^{\{0,2\}}$-property of $X$ implies that each point of $X$ is a homotopical $Z_2$-point. Theorem 10.1(5) shows that each point of $X$, being a homological $Z_n$-point and a homotopical $Z_2$-point, is a homological $Z_n$-point in $X$. ∎

LEMMA 39.2. *If a metrizable separable Baire* $\mathrm{LC}^n$*-space* $X$ *has the* $0\text{-}\overline{\mathrm{DD}}^{\{0,2\}}$*-property and contains a dense set of homological* $Z_n$*-points, then it contains a dense set of homotopical* $Z_n$*-points and* $X \in 0\text{-}\overline{\mathrm{DD}}^{\{0,n\}}$*.*

*Proof.* For $n \le 2$ the assertion follows from Theorem 7.1(3). So, we assume that $n > 2$. By Theorem 7.1(3), the set $\mathcal{Z}_2(X)$ of homotopical $Z_2$-points is dense and $G_\delta$ in $X$ and by Lemma 38.6, the dense set $\mathcal{Z}_n^{\mathbb{Z}}(X)$ of homological $Z_n$-points is $G_\delta$ in $X$. Moreover, according to Theorem 10.1(5), each point of $\mathcal{Z}_2(X)\cap\mathcal{Z}_n^{\mathbb{Z}}(X)$ is a homotopical $Z_n$-point in $X$. Since $X$ is a Baire space, $\mathcal{Z}_2(X)\cap\mathcal{Z}_n^{\mathbb{Z}}(X)$ is dense in $X$. Hence, $X$ contains a dense set of homotopical $Z_n$-points. Finally, by Theorem 7.1(2), $X$ has the $0\text{-}\overline{\mathrm{DD}}^{\{0,n\}}$-property. ∎

LEMMA 39.3. *If* $X$ *has the* $(2n+1)\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$*-property, then each point of* $X$ *is a homological* $Z_n$*-point.*

*Proof.* By Theorem 7.1(1), the diagonal $\Delta_X$ is a homotopical (and hence homological) $Z_{2n+1}$-set in $X \times X$. Then each $(x,x)$, being a point of the homological $Z_{2n+1}$-set $\Delta_X$ in $X \times X$, is a homological $Z_{2n+1}$-point in $X \times X$. Hence, by Theorem 10.4(1), every $x \in X$ is a homological $Z_n$-point in $X$. ∎

LEMMA 39.4. *If* $X$ *has the* $2n\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$*-property with* $n \ge 1$*, then each point of* $X$ *is a* $G$*-homological* $Z_n$*-point for any group* $G$ *having a divisible quotient* $G/\mathrm{Tor}(G)$*.*

*Proof.* Let $G$ be a non-trivial group with a divisible quotient $G/\mathrm{Tor}(G)$ and $X$ be a space possessing the $2n\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$-property. We need to prove that each $x \in X$ is a $G$-homological $Z_n$-point, i.e., the homology group $H_k(U, U \setminus \{x\}; G)$ is trivial for every $k \le n$ and every open neighborhood $U \subset X$ of $x$.

The triviality of the above groups will follow from the universal coefficients formula as soon as we prove that $H_i(U, U\setminus\{x\}) \otimes G = 0$ and $H_j(U, U\setminus\{x\}) * G = 0$ for all $i \le n$ and $j < n$. Since $X$ has the $(2n-1)\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$-property, according to Lemma 39.3, every $x \in X$ is a homological $Z_{n-1}$-point in $X$. Hence,

$$0 = H_j(U, U\setminus\{x\}) = H_j(U, U\setminus\{x\}) * G = H_j(U, U\setminus\{x\}) \otimes G$$

for all $j < n$. So, it remains to check that $H_n(U, U\setminus\{x\}) \otimes G = 0$.

By Theorem 7.1(1), the $2n\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$-property of $X$ implies that the diagonal $\Delta_X$ is a homotopical $Z_{2n}$-set in $X^2$. Consequently, $(x,x)$ is a homological $Z_{2n}$-point in $X^2$. Hence, $H_n(U, U\setminus\{x\}) \otimes H_n(U, U\setminus\{x\}) = 0$ by the Künneth formula 38.4. Since $G/\mathrm{Tor}(G)$ is divisible, items (6) and (7) of Lemma 18.8 imply $H_n(U, U\setminus\{x\}) \otimes G = 0$. ∎

## 40. Proof of Theorem 11.2

The first item of Theorem 11.2 follows immediately from Theorems 7.1(5), 10.1(5) and 7.1(4). The second item has been proved in Lemma 39.2

The proof of the third item is more complicated and requires some preliminary work.

LEMMA 40.1. *Let $p : K \to M$ be a map between compact polyhedra, $X$ be a metrizable separable space with $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property for $m = \dim M$ and $k \geq n = \dim(p)$, and $(\tilde{X}, \rho)$ be a complete metric space containing $X$. Then for every map $f : K \to X$ there are a map $f_\infty : K \to \tilde{X}$ and a 1-homotopy $(h_t) : K \to X$ such that*

(1) *$h_0 = f$, $h_1 = f_\infty$, $h_t(K) \subset X$ for $t \in [0, 1)$;*
(2) *the restriction $f_\infty | p^{-1}(z)$ is injective and $f_\infty(p^{-1}(z)) \cap X$ is a $Z_k$-set in $X$ for every $z \in M$.*

*Proof.* The proof follows the idea of the proof of Theorem 3.3. Since $X$ is separable, the function space $C(\mathbb{I}^k, X)$ is also separable and contains a countable dense subset $\mathcal{D}$. Let $\mathcal{D} = \{g_i : i \in \omega\}$ be an enumeration of $\mathcal{D}$ such that for every $g \in \mathcal{D}$ the set $\{i \in \omega : g_i = g\}$ is infinite.

Let $f_0 = f$, $\varepsilon_0 = 1/3$, and let $\mathrm{pr} : M \times \mathbb{I}^k \to \mathbb{I}^k$ stand for the natural projection. By induction, we shall construct sequences $\{f_i : K \to X\}_{i \geq 1}$, $\{\bar{g}_i : M \times \mathbb{I}^k \to X\}_{i \geq 1}$, a sequence $\{\varepsilon_i\}_{i \geq 1}$ of positive real numbers and a sequence $\{h_i : K \times [0, 1] \to X\}_{i \geq 1}$ of $\varepsilon_i$-homotopies satisfying the conditions:

(a) $h_{i+1}(z, 0) = f_i(z)$ and $h_{i+1}(z, 1) = f_{i+1}(z)$ for every $z \in K$;
(b) $\bar{g}_{i+1}$ and $g_{i+1} \circ \mathrm{pr}$ are $\varepsilon_i$-homotopic;
(c) $f_{i+1}(p^{-1}(z)) \cap \bar{g}_{i+1}(\{z\} \times \mathbb{I}^k) = \emptyset$ for all $z \in M$;
(d) $p \triangle f_{i+1} : K \to M \times X$ is a $\big(1/(i + 1)\big)$-map;
(e) $\varepsilon_{i+1} \leq \varepsilon_i/2$ and $\varepsilon_{i+1} \leq \frac{1}{2} \min_{z \in M} \mathrm{dist}\big(f_i(p^{-1}(z), \bar{g}_i(\{z\} \times \mathbb{I}^k)\big)$;
(f) for each map $g \in B_\rho(f_{i+1}, 3\varepsilon_{i+1})$ the diagonal product $p \triangle g$ is a $(1/(i + 1))$-map.

Assume that, for some $i$, we have already constructed functions $f_1, \dots, f_i$, $\bar{g}_1, \dots, \bar{g}_i$, positive numbers $\varepsilon_1, \dots, \varepsilon_i$, and homotopies $h_1, \dots, h_i$ satisfying the above conditions. By Theorem 5.3, there are maps $f'_{i+1} : K \to X$, $\bar{g}_{i+1} : M \times \mathbb{I}^k \to X$ which are $\varepsilon_i/2$-homotopic to $f_i$ and $g_{i+1} \circ \mathrm{pr}$, respectively, and are such that $f'_{i+1}(p^{-1}(z)) \cap \bar{g}_{i+1}(\{z\} \times \mathbb{I}^k) = \emptyset$ for every $z \in M$. Pick a positive number $\varepsilon'_i < \varepsilon_i/2$ such that

$$\varepsilon'_i \leq \frac{1}{2} \min_{z \in M} \mathrm{dist}\big(f'_{i+1}(p^{-1}(z)), \bar{g}_{i+1}(\{z\} \times \mathbb{I}^k)\big).$$

Since $k \geq n$, $X$ has the $m$-$\overline{\mathrm{DD}}^{\{n\}}$-property. Then, by Lemma 27.4, there exists a map $f_{i+1} : K \to X$ $\varepsilon'_i$-homotopic to $f'_{i+1}$ such that the diagonal product $p \triangle f_{i+1} : K \to M \times X$ is a $(1/(i+1))$-map. It follows from the choice of $\varepsilon'_i$ that $f_{i+1}$ has property (c) and can be connected with $f_i$ by an $\varepsilon_i$-homotopy $h_{i+1} : K \times [0, 1] \to X$. According to Lemma 28.1, the set $O(f_{i+1}) = \{g \in C(K, X) : p \triangle g$ is a $(1/(i + 1))$-map$\}$ is open in $C(K, X)$. Consequently, there is a positive number $\varepsilon_{i+1} \leq \varepsilon'_i$ such that $B_\rho(f_{i+1}, 3\varepsilon_{i+1}) \subset O(f_{i+1})$. This completes the inductive step.

Repeating the argument from the proof of Theorem 3.3, we can show that the sequence $\{f_i\}_{i \geq 1}$ converges uniformly to some continuous function $f_\infty : K \to \tilde{X}$ which is 1-

homotopic to $f_0$ via a homotopy $(h_t) : K \to \tilde{X}$ satisfying condition (a) of our lemma and such that $f_\infty$ is injective on each fiber $p^{-1}(z)$, $z \in M$. The choice of the numbers $\varepsilon_i$ implies also that $f_\infty(p^{-1}(z)) \cap \bar{g}_{i+1}(\{z\} \times \mathbb{I}^k) = \emptyset$ for all $z \in M$ and $i \in \mathbb{N}$.

We claim that for every $z \in M$ the intersection $X \cap f_\infty(p^{-1}(z))$ is a $Z_k$-set in $X$. Indeed, given $g : \mathbb{I}^k \to X$ and $\varepsilon > 0$, there exists $i \geq 2$ such that $\varepsilon_i < \varepsilon/2$ and $\rho(g_{i+1}, g) < \varepsilon/2$ (such an $i$ exists due to the choice of the enumeration of $\mathcal{D}$). Then the map $\bar{g}_{i+1}$ is $(\varepsilon/2)$-near to $g_{i+1} \circ \mathrm{pr}$. Consequently, the map $g' : \mathbb{I}^k \to X$ defined by $g'(y) = \bar{g}_{i+1}(z, y)$ is $(\varepsilon/2)$-near to $g_{i+1}$. So, $g'$ is $\varepsilon$-near to $g$. Moreover, we have $g'(\mathbb{I}^k) \cap f_\infty(p^{-1}(z)) = \bar{g}_{i+1}(\{z\} \times \mathbb{I}^k) \cap f_\infty(p^{-1}(z)) = \emptyset$, demonstrating the $Z_k$-property of $X \cap f_\infty(p^{-1}(z))$. ■

The following lemma yields the third item of Theorem 11.2.

LEMMA 40.2. *If each point of a metrizable separable* $\mathrm{LC}^2$*-space* $X$ *is a homological* $Z_{m+n+k}$*-point and* $X$ *has the* $m$-$\overline{\mathrm{DD}}^{\{n,\max\{n,2\}\}}$*-property, then it has the* $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$*-property.*

*Proof.* The lemma is trivial if $k \leq \max\{n, 2\}$. So assume that $k > \max\{n, 2\}$. Given an open cover $\mathcal{U}$ and two maps $f : \mathbb{I}^m \times \mathbb{I}^n \to X$ and $g : \mathbb{I}^m \times \mathbb{I}^k \to X$ we have to find maps $f' : \mathbb{I}^m \times \mathbb{I}^n \to X$ and $g' : \mathbb{I}^m \times \mathbb{I}^k \to X$ which are $\mathcal{U}$-homotopic to $f$ and $g$, respectively, and such that $f'(\{z\} \times \mathbb{I}^n) \cap g'(\{z\} \times \mathbb{I}^k) = \emptyset$ for all $z \in \mathbb{I}^m$. By Lemma 18.1, there is a metric $\rho$ on $X$ such that any two 1-homotopic maps are $\mathcal{U}$-homotopic. Let $\tilde{X}$ be the completion of the metric space $(X, \rho)$.

By Lemma 40.1, there is a 1-homotopy $(h_t) : \mathbb{I}^m \times \mathbb{I}^n \times [0, 1] \to \tilde{X}$ such that

- $h_0 = f$, $h_t(\mathbb{I}^m \times \mathbb{I}^n) \subset X$ for all $t \in [0, 1)$;
- the restriction $h_1|\{z\} \times \mathbb{I}^n$ is injective and the intersection $h_1(\{z\} \times \mathbb{I}^n) \cap X$ is a $Z_2$-set in $X$ for every $z \in \mathbb{I}^m$.

Since $X \in \mathrm{LC}^2$, the last condition implies that $h_1(\{z\} \times \mathbb{I}^n) \cap X$ is a homotopical $Z_2$-set in $X$ for all $z \in \mathbb{I}^m$.

Consider the set-valued map $\Phi : \mathbb{I}^m \times \mathbb{I}^k \multimap X$ assigning to each point $(z, y) \in \mathbb{I}^m \times \mathbb{I}^k$ the homotopical $Z_2$-set $\Phi(z, y) = X \cap h_1(\{z\} \times \mathbb{I}^n)$ in $X$. The continuity of $h_1$ implies the compact semicontinuity of $\Phi$. Since $h_1|\{z\} \times \mathbb{I}^n$ is an embedding, $\dim \Phi(z, y) \leq n$ for every $(z, y) \in \mathbb{I}^m \times \mathbb{I}^k$. As all points of $X$ are homological $Z_{n+m+k}$-points, we can apply Lemma 38.2(4) to conclude that $\Phi(z, y)$ is a homological $Z_{m+k}$-set in $X$. Being a homotopical $Z_2$-set and a homological $Z_{m+k}$-set, $\Phi(z, y)$ is a homotopical $Z_{m+k}$-set in $X$ according to Theorem 10.1(5).

Now, we can apply Selection Theorem 6.1 to find a map $g' : \mathbb{I}^m \times \mathbb{I}^k \to X$ which is 1-homotopic to $g$ and such that $g'(z, y) \notin \Phi(z, y)$ for all $(z, y) \in \mathbb{I}^m \times \mathbb{I}^k$. So, $g'(\{z\} \times \mathbb{I}^k) \cap h_1(\{z\} \times \mathbb{I}^n) = \emptyset$ for every $z \in \mathbb{I}^m$ and $\delta = \min_{z \in \mathbb{I}^m} \mathrm{dist}(g'(\{z\} \times \mathbb{I}^k), h_1(\{z\} \times \mathbb{I}^n)) > 0$. Pick $t < 1$ so large that $\rho(h_t, h_1) < \delta/2$. Then the maps $f' = h_t$ and $g'$ are 1-homotopic to $f$ and $g$, respectively, and $f'(\{z\} \times \mathbb{I}^n) \cap g'(\{z\} \times \mathbb{I}^k) = \emptyset$. Thus, $X$ has the $m$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property. ■

Next, we prove the fourth item of Theorem 11.2.

LEMMA 40.3. *Suppose that* $X$ *is an* $\mathrm{LC}^1$*-space with the* 2-$\overline{\mathrm{DD}}^{\{0,0\}}$*-property and each point of* $X$ *is a homological* $Z_n$*-point. Then* $X$ *has the* $n$-$\overline{\mathrm{DD}}^{\{0,0\}}$*-property.*

*Proof.* According to Theorem 7.1(1), it suffices to check that the diagonal $\Delta_X$ is a homotopical $Z_n$-set in $X^2$. By Theorem 10.1(5), this will be done as soon as we prove that $\Delta_X$ is both a homotopical $Z_2$-set and a homological $Z_n$-set in $X^2$. Since $X$ has the 2-$\overline{\mathrm{DD}}^{\{0,0\}}$-property, $\Delta_X$ is a homotopical $Z_2$-set in $X^2$ by Theorem 7.1(1).

So, it remains to prove that $\Delta_X$ is a homological $Z_n$-set in $X^2$. Since each point of $X$ is a homological $Z_n$-point, Theorems 10.1(3), 10.3, and 10.1(5) imply that each point of $X \times \mathbb{I}$ is a homotopical $Z_{n+1}$-point. Then, by Theorem 7.1(4), the space $X \times \mathbb{I}$ has the $(n+1)$-$\overline{\mathrm{DD}}^{\{0,0\}}$-property. So, according to Theorem 7.1(1), the diagonal $\Delta_{X\times\mathbb{I}}$ of the square $(X\times\mathbb{I})^2$ is a homotopical $Z_{n+1}$-set in $(X\times\mathbb{I})^2$. Consequently, $\Delta_{X\times\mathbb{I}}$ is a homological $Z_{n+1}$-set in $X^2 \times \mathbb{I}^2$. Since $((X \times \mathbb{I})^2, \Delta_{X\times\mathbb{I}})$ is homeomorphic to $(X^2 \times \mathbb{I}^2, \Delta_X \times \Delta_{\mathbb{I}})$, the product $\Delta_X \times \Delta_{\mathbb{I}}$ is a homological $Z_{n+1}$-set in $X^2 \times \mathbb{I}^2$.

Since $H_1(\mathbb{I}^2, \mathbb{I}^2 \setminus \Delta_{\mathbb{I}}; G) = H_0(\mathbb{I}^2 \setminus \Delta_{\mathbb{I}}; G) = G$ for every coefficient group $G$, the diagonal $\Delta_{\mathbb{I}}$ fails to be a $\exists G$-homological $Z_1$-set in $X^2$, so we can apply Theorem 10.4(3) to conclude that $\Delta_X$ is a homological $Z_n$-set in $X^2$. ■

Our next lemma yields the first part of Theorem 11.2(5).

LEMMA 40.4. *Let $G$ be a non-trivial Abelian group. If each point of an* $\mathrm{LC}^0$*-space $X$ is a $G$-homological $Z_2$-point, then $X$ has the* $\mathrm{DD}^1\mathrm{P}$.

*Proof.* To show that $X$ has $\mathrm{DD}^1\mathrm{P}$, fix an open cover $\mathcal{U} = \{U_x : x \in X\}$ and two maps $f, g : \mathbb{I} \to X$. Since $X$ is $\mathrm{LC}^0$, it admits an open cover $\mathcal{V} = \{V_x : x \in X\}$ refining $\mathcal{U}$ with the following property: any two points $y, z \in V_x$ can be linked by an arc in $U_x$ for every $x \in X$. Now, take a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0, 1]$ such that the family $\{[t_{i-1}, t_i] : 0 \le i \le n\}$ refines $f^{-1}(\mathcal{V})$. Then $f([t_{i-1}, t_i])$ lies in $V_{x_i}$ for some point $x_i \in X$, $0 \le i \le n$. The choice of the set $V_{x_i}$ guarantees the existence of an embedding $f'_i : [t_{i-1}, t_i] \to U_{x_i}$ such that $f'_i(t_{i-1}) = f(t_{i-1})$ and $f'(t_i) = f(t_i)$. The maps $f'_i$ compose a continuous map $f' : [0, 1] \to X$. Obviously, $f'$ is $\mathcal{U}$-near to $f$ and $f'(\mathbb{I})$ is a finite union of arcs and hence $\dim f'(\mathbb{I}) \le 1$. Since all points of $X$ are $G$-homological $Z_2$-points, we may apply Lemma 38.2(4) to conclude that $f'(\mathbb{I})$ is a $G$-homological $Z_1$-set in $X$. So, by Theorem 10.1(4), $f'(\mathbb{I})$ is a $Z_1$-set in $X$. Consequently, there exists a map $g' : \mathbb{I} \to X \setminus f'(\mathbb{I})$ which is $\mathcal{U}$-near to $g$. ■

Lemma 40.4 yields the following non-trivial statement.

LEMMA 40.5. *Let $X$ be a metrizable* $\mathrm{LC}^1$*-space with the* 0-$\overline{\mathrm{DD}}^{\{0,2\}}$*-property and any point of $X$ is a $G$-homological $Z_2$-point in $X$ for some coefficient group $G$. Then $X$ has the* 2-$\overline{\mathrm{DD}}^{\{0,0\}}$*-property.*

*Proof.* By Lemma 40.4, $X$ has the $\mathrm{DD}^1\mathrm{P}$-property. Then, being an $\mathrm{LC}^1$-space, $X$ has the 0-$\overline{\mathrm{DD}}^{\{1,1\}}$-property. So, $X$ has both the 0-$\overline{\mathrm{DD}}^{\{0,2\}}$-property and 0-$\overline{\mathrm{DD}}^{\{1,1\}}$-property. Now, we can apply the base enlargement formula from Theorem 8.3(3) to conclude that $X$ has the 2-$\overline{\mathrm{DD}}^{\{0,0\}}$-property. ■

Combining Lemma 40.5 with Theorem 7.1(2), we obtain the next lemma providing a proof of the second part of Theorem 11.2(5).

Lemma 40.6. *Let $X$ be a metrizable* $\mathrm{LC}^1$*-space containing a dense set of homotopical $Z_2$-points. If for some group $G$ each point of $X$ is a $G$-homological $Z_2$-point, then $X$ has the* $2$-$\overline{\mathrm{DD}}^{\{0,0\}}$*-property.*

## 41. Proof of Theorem 12.1

The proof of Theorem 12.1 is divided into two lemmas.

Lemma 41.1. *Let $(X,\rho)$ be a complete metric space containing a countable family $\mathcal{F}$ of homotopical $Z_k$-sets such that each compact set $K \subset X \setminus \mathcal{F}$ is a homotopical $Z_n$-set in $X$. Then $X$ has the* $0$-$\overline{\mathrm{DD}}^{\{n,k\}}$*-property.*

*Proof.* Fix two maps $f : \mathbb{I}^k \to X$ and $g : \mathbb{I}^n \to X$. We have to show that these maps are 1-homotopic to maps $f' : \mathbb{I}^k \to X$, $g' : \mathbb{I}^n \to X$ with disjoint images.

Using that $\mathcal{F}$ is a countable family of homotopical $Z_k$-sets in $X$, we can construct a map $f' : \mathbb{I}^k \to X \setminus \bigcup \mathcal{F}$ which is 1-homotopic to $f$. Then $f'(\mathbb{I}^k)$ is a homotopical $Z_n$-set in $X$. Consequently, $g$ is 1-homotopic to a map $g' : \mathbb{I}^n \to X \setminus f'(\mathbb{I}^k)$. ■

Lemma 41.2. *Let $X$ be a separable* (*complete*) *metric* $\mathrm{LC}^k$*-space $X$ possessing the* $0$-$\overline{\mathrm{DD}}^{\{n,k\}}$*-property with $n \le k$. Then there exists a countable family $\mathcal{F}$ of at most $n$-dimensional* (*compact*) *$Z_k$-sets in $X$ such that each compact* (*closed*) *subset $K \setminus \bigcup \mathcal{F}$ is a homotopical $Z_n$-set in $X$.*

*Proof.* The separability of $X$ implies the separability of $C(\mathbb{I}^n, X)$, so we can fix a countable dense subset $\mathcal{D} = \{f_i : i \in \omega\}$ in $C(\mathbb{I}^n, X)$. By Lemma 21.3, $X$ embeds into a Polish $\mathrm{LC}^k$-space $\tilde{X}$ as a dense relative $\mathrm{LC}^k$-subset. If $X$ is complete, then we may assume that $\tilde{X} = X$. Let $\rho$ be a complete metric on $\tilde{X}$. According to Proposition 40.1, for every $i, j \in \omega$ there exists a $2^{-j}$-homotopy $(h_t^{i,j})_{t\in\mathbb{I}} : \mathbb{I}^n \to \tilde{X}$ such that

- $h_0^{i,j} = f_i$;
- $h_t^{i,j}(\mathbb{I}^n) \subset X$ for all $t \in [0,1)$;
- $h_1^{i,j} : \mathbb{I}^n \to \tilde{X}$ is an embedding with $X \cap h_1^{i,j}(\mathbb{I}^n)$ being a $Z_k$-set in $X$.

Since $h_1^{i,j}$ are embeddings, each set $X_{i,j} = X \cap h_1^{i,j}(\mathbb{I}^n)$ is of dimension $\le n$.

We claim that each compact subset $K \subset X \setminus \bigcup_{i,j} X_{i,j}$ is a $Z_n$-set in $X$. Given any map $f : \mathbb{I}^n \to X$ and $\varepsilon > 0$, find $i, j \in \omega$ with $\rho(f, f_i) < \varepsilon/2$ and $2^{-j} < \varepsilon/2$. Since $h_1^{i,j}(\mathbb{I}^n) \cap K = \emptyset$, there is a $\delta \in [0,1)$ such that $h_\delta^{i,j}(\mathbb{I}^n) \cap K = \emptyset$. Then the map $f' = h_\delta^{i,j} : \mathbb{I}^n \to X$ is $\varepsilon$-near to $f$ and the image $f'(\mathbb{I}^n)$ misses $K$. This means that $K$ is a $Z_n$-set. Since $X$ is an $\mathrm{LC}^n$-space, $K$ is a homotopical $Z_n$-set in $X$.

If $X$ is complete, then $\tilde{X} = X$ and the same argument shows that each closed subset $K \subset X$ missing $\bigcup_{i,j} X_{i,j}$ is a homotopical $Z_n$-set in $X$. ■

## 42. Proof of Theorem 12.2

We shall closely follow the proof of the homological characterization of the $0$-$\overline{\mathrm{DD}}^{\{\infty,\infty\}}$-property due to Daverman and Walsh [19]. Their technique is based on the notion of a

Čech carrier for a homology element $z \in H_q(U,V)$ where $V \subset U$ are open subsets of a metrizable $\mathrm{LC}^n$-space $X$. For such a space $X$ every singular homology group $H_q(U,V)$, $q \le n$, coincides with the Čech homology group $\check{H}_q(U,V)$. So, for any compact pair $(C,\partial C) \subset (U,V)$ we have an inclusion homomorphism $i_* : \check{H}_q(C,\partial C) \to \check{H}_q(U,V) = H_q(U,V)$. If $z \in H_q(U,V)$ belongs to $\mathrm{Im}(i_*)$, then the pair $(C,\partial C)$ is called a *Čech carrier* for $z$.

The following lemma is a quantified version of Lemma 3.1 from [19] and its proof is analogous.

LEMMA 42.1. *Let $X$ be a metrizable $\mathrm{LC}^n$-space and $\mathcal{B}$ be a base of the topology of $X$ which is closed under finite unions. A closed subset $A$ of $X$ is a homological $Z_n$-set in $X$ if and only if for any open sets $V \subset U$ from $\mathcal{B}$ each $z \in H_k(U,V)$ with $k \le n$ has a Čech carrier $(C,\partial C)$ with $C \cap A = \emptyset$.*

We also need Lemma 42.2 below which is a counterpart of [19, Lemma 3.3] and has a similar proof.

LEMMA 42.2. *Let $\mathcal{F}$ be a countable family of homological $Z_n$-sets in a Polish $\mathrm{LC}^n$-space $X$. Then for any pair $(U,V)$ of open subsets of $X$ any element $z \in H_k(U,V)$, $k \le n$, has a Čech carrier $(C,\partial C)$ with $C \subset U \setminus \bigcup \mathcal{F}$.*

With Lemmas 42.1 and 42.2 in hand, we are now ready to prove Theorem 12.2. Let $X$ be a Polish $\mathrm{LC}^m$-space each of whose points is a homological $Z_{m+2}$-point and let $n,k \le m$ be any (finite or infinite) numbers. Suppose that $X$ has the $0$-$\overline{\mathrm{DD}}^{\{2,2\}}$-property and $\mathcal{F}$ is a countable family of homological $Z_n$-sets in $X$ such that each compact subset of $X \setminus \bigcup \mathcal{F}$ is a homological $Z_k$-set. We have to prove that $X$ has the $0$-$\overline{\mathrm{DD}}^{\{n,k\}}$-property. There is nothing to prove if $m \le 2$. So, we suppose that $m > 2$.

It follows from Proposition 40.1 (and the $0$-$\overline{\mathrm{DD}}^{\{2,2\}}$-property of $X$) that $C(\mathbb{I}^2,X)$ contains a countable dense subset $\mathcal{D}$ consisting of embeddings such that $f(\mathbb{I}^2)$ is a $Z_2$-set in $X$ for every $f \in \mathcal{D}$. Since each point of $X$ is a homological $Z_{m+2}$-point, any set $f(\mathbb{I}^2)$, $f \in \mathcal{D}$, being a 2-dimensional $Z_2$-set in $X$, is a homological $Z_m$-set in $X$ (see Lemma 38.2(4)). Hence, by Theorem 10.1(5), all $f(\mathbb{I}^2)$, $f \in \mathcal{D}$, are homotopical $Z_m$-sets. Thus, the countable family $\mathcal{E} = \{f(\mathbb{I}^2) : f \in \mathcal{D}\}$ consists of homotopical $Z_m$-sets in $X$.

Let $\mathcal{B}$ be a countable base for the topology of $X$, closed under finite unions. Since $X$ is an $\mathrm{LC}^n$-space, the set $\{(U,V,q,z) : V \subset U,\ V,U \in \mathcal{B},\ q \le n,\ z \in H_q(U,V)\}$ is countable and can be enumerated as $\{(U_i,V_i,q(i),z_i) : i \in \omega\}$. By Lemma 42.2, each $z_i \in H_{q(i)}(U_i,V_i)$ has a Čech carrier $(C_i,\partial C_i) \subset (U_i,V_i)$ such that $C_i$ misses $\bigcup \mathcal{F}$ and $\bigcup \mathcal{E}$. Then every $C_i$ is both a homological $Z_k$-set (because it misses $\bigcup \mathcal{F}$) and a $Z_2$-set (because it misses $\bigcup \mathcal{E}$). Since $X$ is an $\mathrm{LC}^2$-space, each $C_i$ is a homotopical $Z_2$-set in $X$, and hence a homotopical $Z_k$-set (by Theorem 10.1(5)). Thus, $\mathcal{C} = \{C_i : i \in \omega\}$ is a countable family of homotopical $Z_k$-sets in $X$.

We claim that each compact subset $K \subset X \setminus \bigcup \mathcal{C}$ is a homological $Z_n$-set. According to Lemma 42.1, this will follow as soon as we show that, for any open pair $(U,V)$ with $U,V \in \mathcal{B}$ and any $z \in H_q(U,V)$ with $q \le n$, there exists a Čech carrier $(C,\partial C)$ for $z$ such that $C \cap K = \emptyset$. To this end, we choose $i$ with $(U_i,V_i,q(i),z_i) = (U,V,q,z)$. Then $(C_i,\partial C_i)$ is a Čech carrier for $z_i = z$ with $C_i \cap K = \emptyset$.

Now consider the family $\mathcal{C}\cup\mathcal{E}$ consisting of homotopical $Z_k$-sets in $X$. Note that each compact set $K\subset X$ disjoint from $\bigcup(\mathcal{C}\cup\mathcal{E})$ is a homological $Z_n$-set in $X$ (because $K$ is disjoint $\bigcup\mathcal{C}$) and a homotopical $Z_2$-set (because $K$ is disjoint $\bigcup\mathcal{E}$). Hence, $K$ is a homotopical $Z_n$-set in $X$. Finally, Theorem 12.1 implies that $X$ has the 0-$\overline{\mathrm{DD}}^{\{n,k\}}$-property.

## 43. Proof of Theorem 12.3

Assume that each point of a separable metrizable $\mathrm{LC}^{\max\{n,k\}}$-space $X$ is a homological $Z_{n+k}$-point and $X$ has the properties AP[$n$] and 0-$\overline{\mathrm{DD}}^{\{2,\min\{2,n\}\}}$. We have to check that $X$ has the 0-$\overline{\mathrm{DD}}^{\{n,k\}}$-property. Let $m=\max\{n,k\}$. By Lemma 21.3, $X$ embeds as a dense relative $\mathrm{LC}^m$-subset in a Polish $\mathrm{LC}^m$-space $\tilde X$. Fix a complete metric $\rho$ on $\tilde X$.

We consider three cases depending on the value of $n$.

Assume that $n=0$. There is nothing to prove if $k\le 2$. So, we assume that $k>2$. In this case the 0-$\overline{\mathrm{DD}}^{\{0,2\}}$-property of $X$ yields the same property for $\tilde X$ because $X$ is homotopically $m$-dense in $\tilde X$ and $m\ge k\ge 2$. By Theorem 7.1(3), the set $\mathcal{Z}_2(\tilde X)$ of homotopical $Z_2$-points is dense in $\tilde X$. Since $X\in\mathcal{Z}_k^{\mathbb{Z}}$ and $X$ is homotopically $k$-dense in $\tilde X$ (see Lemma 21.6), Lemma 38.2(6) implies that $\tilde X\in\mathcal{Z}_k^{\mathbb{Z}}$. Applying Theorem 11.2(2), we conclude that $\tilde X$ has the 0-$\overline{\mathrm{DD}}^{\{0,k\}}$-property. Then, by Proposition 5.5, the space $X$, being homotopically $k$-dense in $\tilde X$, has also the 0-$\overline{\mathrm{DD}}^{\{0,k\}}$-property.

Next, assume that $n=1$. In this case $X$ has the 0-$\overline{\mathrm{DD}}^{\{2,1\}}$-property. If $k\le 1$, then we are done. So, let $m\ge k>1$. Then $X$ is an $\mathrm{LC}^2$-space. According to Proposition 5.5, the space $\tilde X$ has the 0-$\overline{\mathrm{DD}}^{\{2,1\}}$-property. Consequently, by Theorem 12.1, $\tilde X$ contains a countable family $\mathcal{F}=\{F_i:i\in\omega\}$ of at most one-dimensional closed $Z_2$-subsets such that each compact $K\subset\tilde X\setminus\mathcal{F}$ is a homotopical $Z_1$-set in $\tilde X$. The homotopy 2-negligibility of $X$ in $\tilde X$ implies that any intersection $X\cap F_i$ is a homotopical $Z_2$-set in $X$. Since each point of $X$ is a homological $Z_{1+k}$-point, Lemma 38.2(4) shows that every $F_i\cap X$ is a homological $Z_k$-set in $X$. Therefore, the sets $F_i\cap X$, being homotopical $Z_2$-sets in $X$, are homotopical $Z_k$-sets in $X$. Then, according to Lemma 38.2(6), every $F_i$ is a homotopical $Z_k$-set in $\tilde X$. Because each compact subset of $\tilde X\setminus\bigcup\mathcal{F}$ is a homotopical $Z_1$-set in $\tilde X$, we can apply Theorem 12.1 to conclude that $\tilde X$ has the 0-$\overline{\mathrm{DD}}^{\{1,k\}}$-property. Hence, by Proposition 5.5, $X$ also has the 0-$\overline{\mathrm{DD}}^{\{1,k\}}$-property.

Finally, consider the case $n\ge 2$. In this case $X$ has AP[$n$] and 0-$\overline{\mathrm{DD}}^{\{2,2\}}$. Fix a countable dense set $\{f_i:i\in\omega\}$ in $C(\mathbb{I}^n,X)$. Using AP[$n$], for every $i\in\omega$ and $j\in\omega$ we can find a map $f_{i,j}:\mathbb{I}^n\to X$ which is $2^{-j}$-homotopic to $f_i$ and $\mathsf{trt}\,f_{i,j}(\mathbb{I}^n)<n+1$. Since each point of $X$ is a homological $Z_{n+k}$-point, all $f_{i,j}(\mathbb{I}^n)$ are homological $Z_k$-sets in $X$ (see Lemma 38.2(4)). It is clear now that each compact set $K\subset X$ missing $\bigcup_{i,j\in\omega}f_{i,j}(\mathbb{I}^n)$ is a homotopical $Z_n$-set in $X$. Hence, by Theorem 12.2, $X$ has the 0-$\overline{\mathrm{DD}}^{\{n,k\}}$-property.

## 44. Proof of Theorem 13.1

The first item of Theorem 13.1 follows from the next lemma.

Lemma 44.1. *If a Tychonoff space $X$ with the $m$-$\overline{\mathrm{DD}}^{\{0,k\}}$-property is locally rectifiable at a point $x_0\in X$, then $x_0$ is a homotopical $Z_{m+k}$-point in $X$.*

*Proof.* Being locally rectifiable at $x_0$, the space $X$ possesses a homeomorphism $H : U\times X \to U \times X$, where $U$ is a neighborhood of $x_0$, such that for any point $x \in U$ there is a homeomorphism $H_x : X \to X$ with $H_x(x_0) = x$ and $H(x,z) = (x, H_x(z))$. The homeomorphism $H$ generates the map

$$\tilde{H} : U \times U \times X \to X, \quad (x,y,z) \mapsto H_y \circ H_x^{-1}(z),$$

such that $\tilde{H}(x,y,x) = y$ and $\tilde{H}(x,x,z) = z$ for all $(x,y,z) \in U \times U \times X$.

To show that $x_0$ is a homotopical $Z_{m+k}$-point, assume that $X$ has the $m$-$\overline{\mathrm{DD}}^{\{0,k\}}$-property and fix an open cover $\mathcal{U}$ of $X$ and a map $f : \mathbb{I}^m \times \mathbb{I}^k \to X$. We have to construct a map $f' : \mathbb{I}^m \times \mathbb{I}^k \to X \setminus \{x_0\}$ which is $\mathcal{U}$-homotopic to $f$. To this end, consider the constant map $g : \mathbb{I}^m \to \{x_0\}$ and observe that the map

$$h_{g,f} : \mathbb{I}^m \times \mathbb{I}^k \to X, \quad (z,y) \mapsto \tilde{H}(g(z), x_0, f(z,y)),$$

coincides with $f$. By a standard argument, using the continuity of $H$ and the compactness of $g(\mathbb{I}^m) = \{x_0\}$ and $f(\mathbb{I}^m \times \mathbb{I}^k)$, we can find an open cover $\mathcal{V}$ of $X$ with $St(x_0,\mathcal{V}) \subset U$ and such that for any $g' : \mathbb{I}^m \to X$ and $f' : \mathbb{I}^m \times \mathbb{I}^k \to X$ that are $\mathcal{V}$-near to $g$ and $f$, respectively, $h_{g',f'}$ is $\mathcal{U}$-near to $f = h_{g,f}$.

Since $X$ has the $m$-$\overline{\mathrm{DD}}^{\{0,k\}}$-property, $g$ and $f$ can be approximated by $g' : \mathbb{I}^m \to X$ and $f' : \mathbb{I}^m \times \mathbb{I}^k \to X$ that are $\mathcal{V}$-homotopic to $g$ and $f$, respectively, so that $g'(\{z\}) \cap f'(\{z\} \times \mathbb{I}^k) = \emptyset$ for every $z \in \mathbb{I}^m$. Let $(g_t)_{t\in\mathbb{I}} : \mathbb{I}^m \to X$ be a $\mathcal{V}$-homotopy linking $g$ and $g'$ and $(f_t) : \mathbb{I}^m \times \mathbb{I}^k \to X$ be a $\mathcal{V}$-homotopy linking $f$ and $f'$. Then the homotopy $(h_t) : \mathbb{I}^m \times \mathbb{I}^k \to X$ defined by

$$h_t(z,y) = \tilde{H}(g_t(z), x_0, f_t(z,y)) \quad \text{for } (z,y) \in \mathbb{I}^m \times \mathbb{I}^k$$

is a $\mathcal{U}$-homotopy connecting $h_0 = f$ with $h_1 = h_{g',f'}$. The homotopy $(h_t)$ is well-defined because $g_t(\mathbb{I}^n) \subset St(x_0,\mathcal{V}) \subset U$ for all $t \in [0,1]$. We claim that $x_0 \notin h_1(\mathbb{I}^m \times \mathbb{I}^k)$. Indeed, if $x_0 = h_1(z,y)$ for some $(z,y) \in \mathbb{I}^m \times \mathbb{I}^k$, then according to the definition of $\tilde{H}$, we would have $x_0 = \tilde{H}(g'(z), x_0, f'(z,y)) = H_{x_0} \circ H^{-1}_{g'(z)}(f'(z,y))$. Applying to this equality the homeomorphisms $H^{-1}_{x_0}$ and then $H_{g'(z)}$ we obtain $x_0 = H^{-1}_{x_0}(x_0) = H^{-1}_{g'(z)}(f'(z,y))$ and then $g'(z) = H_{g'(z)}(x_0) = H_{g'(z)} \circ H^{-1}_{g'(z)}(f'(z,y)) = f'(z,y)$, which contradicts the choice of $g', f'$. Thus, $h_1 : \mathbb{I}^m \times \mathbb{I}^k \to X$ is $\mathcal{U}$-homotopic to $f$ and its image misses $x_0$, demonstrating that $x_0$ is a homotopical $Z_{m+k}$-point in $X$. ∎

Combining the above lemma with Theorem 7.1(2), we obtain the second item of Theorem 13.1:

LEMMA 44.2. *If a locally rectifiable Tychonoff space $X$ has the $m$-$\overline{\mathrm{DD}}^{\{0,k\}}$-property, then it has the $i$-$\overline{\mathrm{DD}}^{\{0,j\}}$-properties for all $i,j$ with $i + j \le m + k$.*

We need an auxiliary result for the proof of Theorem 13.1(3).

LEMMA 44.3. *Let $\tilde{K}$ be a compact polyhedron in a Tychonoff space $\tilde{X}$ and $f : \tilde{K} \to Y$ be a continuous map into a Tychonoff space $Y$. Assume that $Y$ is locally rectifiable at each point $y \in f(\tilde{K})$ and satisfies one of the following conditions:*

- *each $y \in f(\tilde{K})$ is a homotopical $Z_m$-point in $Y$;*
- *$Y$ is an $\mathrm{LC}^1$-space and each $y \in f(\tilde{K})$ is a homological $Z_m$-point in $Y$.*

*If $X$ is a subset of $\tilde{X}$ such that $K = \tilde{K} \cap X$ is a homotopical $Z_k$-set in $X$, then the graph* $\mathrm{Gr}(f|K) = \{(x, f(y) : x \in K\}$ *of $f|K$ is a homotopical $Z_{k+m+1}$-set in $X \times Y$.*

*Proof.* First we check that $K \times \{y_0\}$ is a homotopical $Z_{k+m+1}$-set in $X \times Y$ for every $y_0 \in f(\tilde{K})$. When $y_0$ is a homotopical $Z_m$-point in $Y$, this follows immediately from Theorem 10.3(1). When $Y$ is an $\mathrm{LC}^1$-space and $y_0$ is a homological $Z_m$-point, we argue as follows. If $k = m = 0$, then $y_0$, being a homological $Z_0$-point in $Y$, is a homotopical $Z_0$-set in $Y$ by Theorem 10.1(3). Consequently, according to Theorem 10.3(1), $K \times \{y_0\}$ is a homotopical $Z_1$-set in $X \times Y$ and we are done. If $k+m \geq 1$, then both $K$ and $\{y_0\}$ are homotopical $Z_0$-sets (being homological $Z_0$-sets) and one of them is a homotopical $Z_1$-set ($K$ is such a set in case $k \geq 1$, and $m \geq 1$ implies that $\{y_0\}$ is a homotopical $Z_1$-set in $Y$ as a homological $Z_1$-point in the $\mathrm{LC}^1$-space $Y$, see Theorem 10.1(2,4)). By Theorem 10.3(1), the product $K \times \{y_0\}$ is a homotopical $Z_2$-set. Taking into account that the latter set is a homological $Z_{k+m+1}$-set (by Theorem 10.3(2)) and applying Theorem 10.1(5), we conclude that $K \times \{y_0\}$ is a homotopical $Z_{k+m+1}$-set in $X \times Y$.

Now, we are ready to prove that $\mathrm{Gr}(f|K)$ is a homotopical $Z_{k+m+1}$-set in $X \times Y$. By Lemma 38.2(2), it suffices to construct a closed finite cover of $\mathrm{Gr}(f|K)$ consisting of homotopical $Z_{n+m+1}$-sets in $X \times Y$. The existence of such a cover will follow from the compactness of $\tilde{K}$ as soon as, for every $x_0 \in \tilde{K}$, we construct a closed neighborhood $\tilde{C} \subset \tilde{K}$ of $x_0$ such that $\mathrm{Gr}(f|C) = \mathrm{Gr}(f|K) \cap \tilde{C} \times Y$ is a homotopical $Z_{n+m+1}$-set in $X \times Y$, where $C = \tilde{C} \cap X$. To this end, fix $x_0 \in \tilde{K}$ and use the continuous homogeneity of $Y$ at $y_0 = f(x_0)$ to find a neighborhood $U \subset Y$ of $y_0$ and a homeomorphism $H : U \times Y \to U \times Y$ such that for every $y \in U$ there is a homeomorphism $H_y : Y \to Y$ with $H_y(y) = y_0$ and $H(y, z) = (y, H_y(z))$ for all $(y, z) \in U \times Y$. By continuity of $f$, the point $x_0$ has a closed neighborhood $\tilde{C} \subset \tilde{K}$ with $f(\tilde{C}) \subset U$. Since $\tilde{K}$ is a polyhedron, we can additionally assume that $\tilde{C}$ is a compact absolute retract, so there is a retraction $r : X \to \tilde{C}$. Obviously, the range of $g = r \circ f : X \to Y$ is contained in $U$. Hence, the homeomorphism $h : X \times Y \to X \times Y$, $(x, y) \mapsto (x, H_{g(x)}(y))$, is well defined. Observe that for every $x \in C$ we have

$$h(x, f(x)) = (x, H_{g(x)}(f(x))) = (x, H_{f(x)}(f(x))) = (x, y_0).$$

Therefore, $h(\mathrm{Gr}(f|C)) \subset C \times \{y_0\} \subset K \times \{y_0\}$. As we already proved, the latter set is a homotopical $Z_{k+m+1}$-set in $X \times Y$.

Since $(X \times Y, \mathrm{Gr}(f|C))$ and $\big(X \times Y, h(\mathrm{Gr}(f|C))\big)$ are homeomorphic, $\mathrm{Gr}(f|C)$ is a homotopical $Z_{k+m+1}$-set in $X \times Y$. ■

Lemma 44.4. *Let $X$ be a Tychonoff locally rectifiable* ($\mathrm{LC}^1$-)*space with $X \in \mathcal{Z}_{m+p}$ (resp., $X \in \mathcal{Z}^{\mathbb{Z}}_{m+p}$). Then, for any $n \leq k$ and any separable metrizable* $\mathrm{LC}^k$*-space $Y \in m\text{-}\overline{\mathrm{DD}}^{\{n,k\}}$, the product $X \times Y$ has the $m\text{-}\overline{\mathrm{DD}}^{\{n,k+p+1\}}$-property.*

*Proof.* To show that $X \times Y$ has the $m\text{-}\overline{\mathrm{DD}}^{\{n,k+p+1\}}$-property, fix an open cover $\mathcal{U}$ and maps $f = (f_X, f_Y) : \mathbb{I}^m \times \mathbb{I}^n \to X \times Y$, $g : \mathbb{I}^m \times \mathbb{I}^{k+p+1} \to X \times Y$. Let $\rho$ be any metric generating the topology of $Y$. The compactness of $\mathbb{I}^m \times \mathbb{I}^n$ implies the existence of an $\varepsilon > 0$ such that for any $f'_Y : \mathbb{I}^m \times \mathbb{I}^n \to Y$ $\varepsilon$-homotopic to $f_Y$ the map $(f_X, f'_Y) : \mathbb{I}^m \times \mathbb{I}^n \to X \times Y$ is $\mathcal{U}$-homotopic to $f = (f_X, f_Y)$.

Let $\tilde{Y}$ be the completion of the metric space $(Y, \rho)$. Since $Y$ has the $m\text{-}\overline{\mathrm{DD}}^{\{n,k\}}$-property, Proposition 40.1 yields an $\varepsilon$-homotopy $(h_t) : \mathbb{I}^m \times \mathbb{I}^n \to \tilde{Y}$ such that

- $h_0 = f_Y$,
- $h_t(\mathbb{I}^m \times \mathbb{I}^n) \subset Y$ for all $t \in [0,1)$;
- $h_1|\{z\} \times \mathbb{I}^n$ is injective and $K_z = Y \cap h_1(\{z\} \times \mathbb{I}^n)$ is a $Z_k$-set in $Y$ for every $z \in \mathbb{I}^m$.

Because $Y$ is an $\mathrm{LC}^k$-space, the latter intersection is a homotopical $Z_k$-set in $Y$.

For every $t \in [0,1]$ consider $f_t = (f_X, h_t) : \mathbb{I}^m \times \mathbb{I}^n \to X \times \tilde{Y}$ and note that $f_0 = f$. We claim that for every $z \in \mathbb{I}^m$ the intersection $B_z = (X \times Y) \cap f_1(\{z\} \times \mathbb{I}^n)$ is a homotopical $Z_{k+m+p+1}$-set in $X \times Y$.

The subspace $\tilde{K}_z = h_1(\{z\} \times \mathbb{I}^n)$ of $\tilde{Y}$, being homeomorphic to $\mathbb{I}^n$, is an absolute retract. Consider the continuous function $\xi = f_X \circ h_1^{-1} : \tilde{K}_z \to X$ and observe that $B_z$ is the intersection of $X \times Y$ with the graph $\mathrm{Gr}(\xi)$. Since $K_z = \tilde{K}_z \cap Y$ is a homotopical $Z_k$-set in $Y$ and $X \in \mathcal{Z}_{m+p} \cup (\mathcal{Z}^{\mathbb{Z}}_{m+p} \cap \mathrm{LC}^1)$, we may apply Lemma 44.3 to conclude that $B_z$ is a homotopical $Z_{k+m+p+1}$-set in $X \times Y$.

Now consider the set-valued map $\Phi : \mathbb{I}^m \times \mathbb{I}^{k+p+1} \multimap X \times Y$ assigning to each pair $(z,t) \in \mathbb{I}^m \times \mathbb{I}^{k+p+1}$ the homotopical $Z_{m+k+p+1}$-set $B_z = (X \times Y) \cap f_1(\{z\} \times \mathbb{I}^n)$. The continuity of $f_1$ implies the compact semicontinuity of $\Phi$. Applying Selection Theorem 6.1, we can find a map $g' : \mathbb{I}^m \times \mathbb{I}^{k+p+1} \to X \times Y$ which is $\mathcal{U}$-homotopic to $g$ and such that $g'(z,t) \notin \Phi(z)$ for every $(z,t) \in \mathbb{I}^m \times \mathbb{I}^{k+p+1}$. This means $g'(\{z\} \times \mathbb{I}^{k+p+1}) \cap f_1(\{z\} \times \mathbb{I}^n) = \emptyset$ for every $z \in \mathbb{I}^n$. By continuity of the homotopy $(h_t)$, there is a $\delta < 1$ such that $g'(\{z\} \times \mathbb{I}^{k+p+1}) \cap f_\delta(\{z\} \times \mathbb{I}^n) = \emptyset$ for every $z \in \mathbb{I}^n$, where $f_\delta = (f_X, h_\delta) : \mathbb{I}^m \times \mathbb{I}^n \to X \times Y$. Then $g'$ and $f' = f_\delta$ are the required maps demonstrating the $m$-$\overline{\mathrm{DD}}^{\{n,k+p+1\}}$-property of $X \times Y$. ■

## 45. Proof of the $k$th root formulas from Theorem 14.1

In this section we establish the $k$-root formulas from Theorem 14.1.

LEMMA 45.1. *Let $X$ be a space whose power $X^k$ has the $(nk+k-1)$-$\overline{\mathrm{DD}}^{\{0,0\}}$-property. Then*

(1) *the diagonal $\Delta_X$ is a homological $Z_n$-set in $X^2$;*
(2) *$X$ has the $n$-$\overline{\mathrm{DD}}^{\{0,0\}}$-property provided $X$ is an $\mathrm{LC}^1$-space having the $2$-$\overline{\mathrm{DD}}^{\{0,0\}}$-property.*

*Proof.* According to Theorem 7.1(1), the $(nk+k-1)$-$\overline{\mathrm{DD}}^{\{0,0\}}$-property of $X^k$ implies that the diagonal $\Delta_{X^k}$ of $X^k \times X^k$ is a homotopical (and hence, homological) $Z_{nk+k-1}$-set in $X^k \times X^k$. Since $(X^k \times X^k, \Delta_{X^k})$ is homeomorphic to $((X \times X)^k, \Delta_X^k)$, $\Delta_X^k$ is a homological $Z_{nk+k-1}$-set in $(X \times X)^k$. Applying Theorem 10.4(1), we conclude that $\Delta_X$ is a homological $Z_n$-set in $X \times X$.

Now, assuming that $X$ is an $\mathrm{LC}^1$-space with the $2$-$\overline{\mathrm{DD}}^{\{0,0\}}$-property, we shall prove that $X$ has the $n$-$\overline{\mathrm{DD}}^{\{0,0\}}$-property. By Theorem 7.1(1), the diagonal $\Delta_X$ is a homotopical $Z_2$-set in $X^2$. Then $\Delta_X$, being a homological $Z_n$-set in $X^2$, is a homotopical $Z_n$-set in $X^2$ (see Theorem 10.1(5)). The last conclusion, according to Theorem 7.1(1), implies that $X$ has the $n$-$\overline{\mathrm{DD}}^{\{0,0\}}$-property. ■

Now, we turn to the second item of Theorem 14.1.

LEMMA 45.2. *Let $k, n \in \mathbb{N}$ and $X$ be a metrizable separable* $\mathrm{LC}^{nk+k-1}$*-space with the* $0\text{-}\overline{\mathrm{DD}}^{\{0,2\}}$*-property such that $X^k$ has the* $0\text{-}\overline{\mathrm{DD}}^{\{0,nk+k-1\}}$*-property. Then $X$ has the* $0\text{-}\overline{\mathrm{DD}}^{\{0,n\}}$*-property.*

*Proof.* The lemma is trivial if $n \le 2$ or $k = 1$. So, let $n > 2$ and $k > 1$. Let us note that, since $X$ is an $\mathrm{LC}^{nk+k-1}$-space, so is $X^k$. By Lemma 21.3, $X$ can be embedded into a Polish $\mathrm{LC}^{nk+k-1}$-space $\tilde{X}$ as a relative $\mathrm{LC}^{nk+k-1}$-subset. Then $X^k$ is a relative $\mathrm{LC}^{nk+k-1}$-subset in $\tilde{X}^k$. Consequently, the Polish space $\tilde{X}^k$ has the $(nk+k-1)\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$-property according to Proposition 5.5. For the same reason, $\tilde{X}$ has the $0\text{-}\overline{\mathrm{DD}}^{\{0,2\}}$-property. Then Theorem 7.1(3) implies the following two facts: the set $\mathcal{Z}_2(\tilde{X})$ of all homotopical $Z_2$-sets is dense $G_\delta$ in $\tilde{X}$, and $\tilde{X}^k \in \overline{\mathcal{Z}}_{nk+k-1}$. Applying Theorem 10.7(2), we conclude that $X \in \overline{\mathcal{Z}}_n^{\mathbb{Z}}$. This means that the set $\mathcal{Z}_n^{\mathbb{Z}}(\tilde{X})$ of homological $Z_n$-points is dense in $\tilde{X}$. By Lemma 38.6, $\mathcal{Z}_n^{\mathbb{Z}}(\tilde{X})$ is of type $G_\delta$ in $\tilde{X}$. Then $\mathcal{Z}_2(\tilde{X}) \cap \mathcal{Z}_n^{\mathbb{Z}}(\tilde{X})$ is also a dense $G_\delta$ in $\tilde{X}$. Since $\tilde{X}$ is an $\mathrm{LC}^1$-space, the latter intersection coincides with the set $\mathcal{Z}_n(\tilde{X})$ of all homotopical $Z_n$-points in $\tilde{X}$. Hence, $\tilde{X} \in \overline{\mathcal{Z}}_n$ and, by Theorem 7.1(2), $\tilde{X}$ has the $0\text{-}\overline{\mathrm{DD}}^{\{0,n\}}$-property. Since $X$ is a relative $\mathrm{LC}^n$-set in $\tilde{X}$, Proposition 5.5 guarantees that $X$ also has the $0\text{-}\overline{\mathrm{DD}}^{\{0,n\}}$-property. ■

## 46. Proof of the division formulas from Theorem 14.2

We divide the proof into four lemmas corresponding to the division formulas from Theorem 14.2.

LEMMA 46.1. *Let $n \in \omega \cup \{\infty\}$, $m \in \omega$, $X$ be a topological space, and $Y$ be a space whose diagonal $\Delta_Y$ fails to be a $\exists G$-homological $Z_m$-set in $Y^2$. If the product $X \times Y$ has the* $(n+m)\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$*-property, then:*

(1) *the diagonal $\Delta_X$ of $X$ is a homological $Z_n$-set in $X^2$;*
(2) *$X$ has the* $n\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$*-property provided $X$ is an* $\mathrm{LC}^1$*-space with* $2\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$*-property.*

*Proof.* Because $X \times Y$ has the $(n+m)\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$-property, its diagonal $\Delta_{X\times Y}$ is a homotopical $Z_{n+m}$-set in $(X \times Y)^2$ (see Theorem 7.1(1)). Since $((X\times Y)^2, \Delta_{X\times Y})$ is homeomorphic to $(X^2 \times Y^2, \Delta_X \times \Delta_Y)$, the product $\Delta_X \times \Delta_Y$ is a homotopical $Z_{n+m}$-set in $X^2 \times Y^2$. Taking into account that $\Delta_Y$ is not an $\exists G$-homological $Z_m$-set in $Y^2$ and applying Theorem 10.4(3), we conclude that $\Delta_X$ is a homological $Z_n$-set in $X^2$. This proves the first item.

To prove the second item, assume that $X$ is an $\mathrm{LC}^1$-space with the $2\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$-property. Consequently, $\Delta_X$ is a homotopical $Z_2$-set in $X^2$ according to Theorem 7.1(1). Then $\Delta_X$, being a homotopical $Z_2$-set and a homological $Z_n$-set in $X^2 \in \mathrm{LC}^1$, is a homotopical $Z_n$-set in $X^2$ by Theorem 10.1. This implies that $X$ has the $n\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$-property, in view of Theorem 7.1(1). ■

We turn now to the proof of the second division formula from Theorem 14.2.

LEMMA 46.2. *Let $X$ be a separable metrizable* $\mathrm{LC}^{n+m}$*-space such that $X \times Y$ has the* $0\text{-}\overline{\mathrm{DD}}^{\{0,n+m\}}$*-property for some metrizable separable Baire* $\mathrm{LC}^{n+m}$*-space $Y$ containing no non-empty open set $U \in \bigcup_G \overline{\mathcal{Z}}_m^G$.*

(1) *If $X$ is a Polish space, then it contains a dense set of homological $Z_n$-points.*
(2) *If $X$ has the* $0$-$\overline{\mathrm{DD}}^{\{0,2\}}$*-property, then it has the* $0$-$\overline{\mathrm{DD}}^{\{0,n\}}$*-property.*

*Proof.* The spaces $X$ and $Y$, being $\mathrm{LC}^{n+m}$, can be embedded as relative $\mathrm{LC}^{n+m}$-subsets into Polish $\mathrm{LC}^{n+m}$-spaces $\tilde{X}$ and $\tilde{Y}$, respectively (we assume that $\tilde{X} = X$ if $X$ is Polish).

Then $X \times Y$ is a relative $\mathrm{LC}^{n+m}$-set in $\tilde{X} \times \tilde{Y}$ and the $0$-$\overline{\mathrm{DD}}^{\{0,n+m\}}$-property of $X \times Y$ implies the same property for $\tilde{X} \times \tilde{Y}$ (see Proposition 5.5). Hence, by Theorem 7.1(3), the set $\mathcal{Z}_{n+k}(\tilde{X} \times \tilde{Y})$ of all homotopical $Z_{n+k}$-points is dense and $G_\delta$ in $\tilde{X} \times \tilde{Y}$.

Next, we shall show that the set $\mathcal{Z}_m^{\exists G}(\tilde{Y})$ of $\exists G$-homological $Z_m$-points is meager in $\tilde{Y}$.

By Theorem 10.2, a point $y \in Y$ is a $\exists G$-homological $Z_m$-point if and only if it is a $G$-homological $Z_m$-point for some Bockstein group $G$ from the countable family $\mathfrak{B} = \{\mathbb{Q}, \mathbb{Z}_p, \mathbb{Q}_p, R_p : p \in \Pi\}$. Thus, $\mathcal{Z}_m^{\exists G}(\tilde{Y}) = \bigcup_{G \in \mathfrak{B}} \mathcal{Z}_m^G(\tilde{Y})$, where $\mathcal{Z}_m^G(\tilde{Y})$ stands for the set of all $G$-homological $Z_m$-points in $\tilde{Y}$. By Lemma 38.6(2), the latter set is $G_\delta$ in $\tilde{Y}$. We claim that $\mathcal{Z}_m^G(\tilde{Y})$ is nowhere dense in $\tilde{Y}$ for any group $G \in \mathfrak{B}$. Indeed, otherwise we could find an open set $U \subset \tilde{Y}$ such that $U \cap \mathcal{Z}_m^G(\tilde{Y})$ is dense $G_\delta$ in $U$. Since $Y$ is a Baire space, $Y \cap U \cap \mathcal{Z}_m^G(\tilde{Y})$ is dense in $U \cap Y$ and consists of $G$-homological $Z_m$-points in $Y$ (see Lemma 38.2(5)). Consequently, $U \in \overline{\mathcal{Z}}_m^G$, which is a contradiction. Therefore, all $\mathcal{Z}_m^G(\tilde{Y})$, $G \in \mathfrak{D}$, are nowhere dense sets in $\tilde{Y}$ and $\mathcal{Z}_m^{\exists G}(\tilde{Y}) = \bigcup_{G \in \mathfrak{B}} \mathcal{Z}_m^G(\tilde{Y})$ is a meager subset of $\tilde{Y}$. Then $\tilde{X} \times \mathcal{Z}_m^{\exists G}(\tilde{Y})$ is meager in $\tilde{X} \times \tilde{Y}$. Hence, $\mathcal{Z}_{n+m}(\tilde{X} \times \tilde{Y}) \setminus (\tilde{X} \times \mathcal{Z}_m^{\exists G}(\tilde{Y}))$ is dense in $\tilde{X} \times \tilde{Y}$.

To show that the set $\mathcal{Z}_n^{\mathbb{Z}}(\tilde{X})$ of homological $Z_n$-points is dense in $\tilde{X}$, fix any non-empty open set $U \subset \tilde{X}$ and pick $(x, y) \in (U \times \tilde{Y}) \cap \mathcal{Z}_{n+m}(\tilde{X} \times \tilde{Y}) \setminus \tilde{X} \times \mathcal{Z}_m^{\exists G}(\tilde{Y})$. Taking into account that $(x, y) \in \mathcal{Z}_{n+m}(\tilde{X} \times \tilde{Y})$ while $y \notin \mathcal{Z}_m^{\exists G}(\tilde{Y})$ and applying Theorem 10.4(3), we find that $x$ is a homological $Z_n$-point in $\tilde{X}$. Thus, the set of homological $Z_n$-points is dense in $\tilde{X}$. This completes the proof of the first item of our lemma because $X = \tilde{X}$.

To prove the second item, assume that $X$ has the $0$-$\overline{\mathrm{DD}}^{\{0,2\}}$-property. If $n \le 2$, then $X$ has the $0$-$\overline{\mathrm{DD}}^{\{0,n\}}$-property and we are done. So, let $n > 2$. Since $X$ is a relative $\mathrm{LC}^2$-set in $\tilde{X}$, by Proposition 5.5, the completion $\tilde{X}$ of $X$ has the $0$-$\overline{\mathrm{DD}}^{\{0,2\}}$-property. Then Theorem 7.1(3) implies that the set $\mathcal{Z}_2(\tilde{X})$ of homotopical $Z_2$-points is dense and $G_\delta$ in $\tilde{X}$. As we have already established, $\mathcal{Z}_n^{\mathbb{Z}}(\tilde{X})$ is also dense in $\tilde{X}$ and $G_\delta$ according to Lemma 38.6. Then the Baire theorem implies that $\mathcal{Z}_2(\tilde{X}) \cap \mathcal{Z}_n^{\mathbb{Z}}(\tilde{X})$ is dense in $\tilde{X}$, while Theorem 10.1(5) ensures that this intersection coincides with the set $\mathcal{Z}_n(\tilde{X})$ of homotopical $Z_n$-points in $\tilde{X}$. Finally, by Theorem 7.1(2), $\tilde{X}$ has the $0$-$\overline{\mathrm{DD}}^{\{0,n\}}$-property. So, $X$ also has the $0$-$\overline{\mathrm{DD}}^{\{0,n\}}$-property (see Proposition 5.5). ■

LEMMA 46.3. *Let $X \in \mathcal{Z}_{k+2}^{\mathbb{Z}}$ be a separable metrizable* $\mathrm{LC}^{k+m}$*-space with the* $0$-$\overline{\mathrm{DD}}^{\{2,2\}}$*-property. Then $X$ has the* $0$-$\overline{\mathrm{DD}}^{\{n,k\}}$*-property with $n \le k$ provided the product $X \times Y$ has the* $0$-$\overline{\mathrm{DD}}^{\{n+m,k+m\}}$*-property for some metrizable separable* $\mathrm{LC}^{k+m}$*-space $Y \notin \mathcal{Z}_m^{\exists G}$.*

*Proof.* In light of Proposition 5.5 and Lemma 38.2(6), we may assume that $X$ and $Y$ are Polish spaces. Since $X \times Y$ has the $0$-$\overline{\mathrm{DD}}^{\{n+m,k+m\}}$-property, we apply Theorem 12.1 to find a countable family $\mathcal{F} = \{F_i : i \in \omega\}$ of homological $Z_{n+m}$-sets in $X \times Y$ such that any compact subset $K \subset X \times Y$ that misses $\bigcup \mathcal{F}$ is a homological $Z_{k+m}$-set in $X \times Y$. By our hypothesis, $Y$ contains a point $y_0$ which is not a $\exists G$-homological $Z_m$-point in $Y$. For every $i \in \omega$ consider the closed subset $E_i = \{x \in X : (x, y_0) \in F_i\}$ which is a homological

$Z_n$-set in $X$ according to Theorem 10.4(3). We claim that each compact subset $K \subset X$ disjoint from $\bigcup_{i\in\omega} E_i$ is a homological $Z_k$-set in $X$. Indeed, otherwise Theorem 10.4(3) would imply that $K \times \{y_0\}$ fails to be a homological $Z_{k+m}$-set in $X \times Y$, which is not the case because $K \times \{y_0\}$ misses $\bigcup \mathcal{F}$. Now, we apply Theorem 12.2 to conclude that $X$ has the $0\text{-}\overline{\mathrm{DD}}^{\{n,k\}}$-property. ∎

LEMMA 46.4. *Let $n, m, k$ be finite or infinite numbers with $n \le k$. A separable metrizable* $\mathrm{LC}^{k+m}$*-space $X \in \mathcal{Z}^{\mathbb{Z}}_{n+k+m}$ with the $0\text{-}\overline{\mathrm{DD}}^{\{2,2\}}$-property has the $0\text{-}\overline{\mathrm{DD}}^{\{n,k\}}$-property provided $X \times Y$ has the $0\text{-}\overline{\mathrm{DD}}^{n+m}$-property for some metrizable separable* $\mathrm{LC}^{n+m}$*-space $Y \notin \bigcup_G \mathcal{Z}^G_m$.*

*Proof.* By Theorem 10.2, $\bigcup_G \mathcal{Z}^G_m$ coincides with the countable union $\bigcup_{G\in\mathfrak{B}} \mathcal{Z}^G_m$, where $\mathfrak{B} = \{\mathbb{Q}, \mathbb{Z}_p, \mathbb{Q}_p, R_p : p \in \Pi\}$. Because of Proposition 5.5 and Lemma 21.3, we may assume that $X$ and $Y$ are Polish. Since $Y \notin \bigcup_G \mathcal{Z}^G_m$, for every group $G \in \mathfrak{B}$ there exists a point $y_G \in Y \setminus \mathcal{Z}^G_m(Y)$.

Since $X \times Y$ has the $0\text{-}\overline{\mathrm{DD}}^{n+m}$-property, we can apply Theorem 12.1 to find a countable family $\{F_i : i \in \omega\}$ of compact $(n+m)$-dimensional subsets in $X \times Y$ such that any compact subset $K \subset X \times Y$ that misses $\bigcup_i F_i$ is a homological $Z_{n+m}$-set in $X \times Y$. For every $i \in \omega$ and $G \in \mathfrak{B}$ consider the set $E_{i,G} = \{x \in X : (x, y_G) \in F_i\}$. Then $\mathcal{E} = \{E_{i,G} : i \in \omega,\ G \in \mathfrak{B}\}$ is a countable family of at most $(n+m)$-dimensional compacta. We claim that each compact $K \subset X \setminus \bigcup \mathcal{E}$ is a homological $Z_n$-set in $X$. This follows from Theorem 10.4(4) and the fact that, for every $G \in \mathfrak{B}$, the product $K \times \{y_G\}$ is a homological $Z_{n+m}$-set in $X \times Y$ (because $K \times \{y_G\}$ is disjoint from $\bigcup_i F_i$) while $y_G$ is not a $G$-homological $Z_m$-point in $Y$.

Since $X \in \mathcal{Z}^{\mathbb{Z}}_{n+m+k}$ and $\dim E_{i,G} \le \dim F_i \le n+m$, Lemma 21.2(4) implies that each set $E_{i,G}$ is a homological $Z_k$-set in $X$. So, $\mathcal{E}$ is a countable family of homological $Z_k$-sets in $X$ such that each compact subset of $X \setminus \bigcup \mathcal{E}$ is a homological $Z_n$-set in $X$. Now, the $0\text{-}\overline{\mathrm{DD}}^{\{n,k\}}$-property of $X$ follows from Theorem 12.2 provided $X \in Z^{\mathbb{Z}}_{k+2}$. The last condition holds if $n+m \ge 2$. If $k+m \ge 2$ and $n \le 2$, then the $0\text{-}\overline{\mathrm{DD}}^{\{n,k\}}$-property of $X$ follows from Theorem 11.2(3). If $k+m \le 1$ and $n \le 2$, then $X \in 0\text{-}\overline{\mathrm{DD}}^{\{2,2\}}$ implies the $0\text{-}\overline{\mathrm{DD}}^{\{n,k\}}$-property of $X$. ∎

## 47. Proof of Corollaries 15.1–15.4

*Proof of Corollary 15.1.* The first item of this corollary follows from Theorems 7.1(4,5), 11.1(3) and 10.1.

The second item follows from Theorems 11.1(3) and 11.2(4).

The third item follows from Theorems 7.1(3) and 11.2(1).

The fourth item can be derived from Theorem 12.2 and Lemma 38.2(5) using the argument from the proof of Theorem 12.3.

*Proof of Corollary 15.3.* The first two items are particular cases of Theorem 14.1(1,2).

To prove the third item, assume that $X$ is an $\mathrm{LC}^1$-space with the $2\text{-}\overline{\mathrm{DD}}^{\{0,2\}}$-property and some finite power $X^k$ of $X$ has the $\infty\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$-property. By Theorem 14.1(1), $X$ has the $\infty\text{-}\overline{\mathrm{DD}}^{\{0,0\}}$-property, and by Theorem 11.1(3), all points of $X$ are homological

$Z_\infty$-points. On the other hand, the 2-$\overline{\mathrm{DD}}^{\{0,2\}}$-property of $X$ implies that all points of $X$ are homotopical $Z_2$-points (see Theorem 7.1(5)). Hence, according to Theorem 10.1(5), all points of $X$ are homotopical $Z_\infty$-points, being homotopical $Z_2$-points and homological $Z_\infty$-points. Finally, Theorem 7.1(4) implies that $X$ has the $\infty$-$\overline{\mathrm{DD}}^{\{0,\infty\}}$-property.

To prove the fourth item, assume that $X$ is a metrizable separable $\mathrm{LC}^\infty$-space with $\mathrm{AP}[\infty]$ and 0-$\overline{\mathrm{DD}}^{\{2,2\}}$ and suppose that some finite power $X^k$ of $X$ has the 0-$\overline{\mathrm{DD}}^{\{\infty,\infty\}}$-property. Then $X^k$ has the $\infty$-$\overline{\mathrm{DD}}^{\{\infty,\infty\}}$-property and, by Theorem 14.1(1), $X$ has the $\infty$-$\overline{\mathrm{DD}}^{\{0,0\}}$-property. Consequently, Theorem 11.1(3) shows that all points of $X$ are homological $Z_\infty$-points. Applying Theorem 15.1(4), we conclude that $X$ has the 0-$\overline{\mathrm{DD}}^{\{\infty,\infty\}}$-property.

*Proof of Corollary 15.4.* The first item of Corollary 15.4 will follow from Theorem 14.2(1) as soon as we prove that if $Y \notin \bigcup_G \mathcal{Z}^G_\infty$ then $Y$ does not belong to $\Delta\mathcal{Z}^{\exists G}_\infty$. Assuming the converse, find a coefficient group $G$ such that $\Delta_Y$ is a $G$-homological $Z_\infty$-set in $Y^2$. By Theorem 10.2, we can assume that $G \in \{\mathbb{Q}, \mathbb{Z}_p : p \text{ is prime}\}$ is a field. Since $Y \notin \mathcal{Z}^G_\infty$, there is a point $y_G$ which is not a $G$-homological $Z_k$-point in $Y$ for some $k$. The Künneth formula 38.5 for coefficients in a field implies that $(y_0, y_0)$ is not a $G$-homological $Z_{2k}$-point in $Y^2$. This is not possible because $(y_0, y_0)$ is a point of the $G$-homological $Z_\infty$-set $\Delta_Y$ in $Y^2$.

To prove the second item, assume that $X$ is an $\mathrm{LC}^1$-space with 2-$\overline{\mathrm{DD}}^{\{0,2\}}$ and $X\times Y$ has the $\infty$-$\overline{\mathrm{DD}}^{\{0,\infty\}}$-property for some $Y \notin \bigcup_G \mathcal{Z}^G_\infty$. Then $X$ has the $\infty$-$\overline{\mathrm{DD}}^{\{0,0\}}$ by the preceding item. So, each point of $X$ is a homological $Z_\infty$-point by Theorem 11.1(3). The 2-$\overline{\mathrm{DD}}^{\{0,2\}}$-property of $X$ implies that all points of $X$ are homotopical $Z_2$-points (see Theorem 7.1(5)). Hence, all points of $X$ are homotopical $Z_\infty$-points according to Theorem 10.1(5). Finally, by Theorem 7.1(4), $X$ has the $\infty$-$\overline{\mathrm{DD}}^{\{0,\infty\}}$-property.

To prove the third item, assume that $X$ is a metrizable separable $\mathrm{LC}^\infty$-space with the 0-$\overline{\mathrm{DD}}^{\{2,2\}}$-property and $X\times Y$ has the 0-$\overline{\mathrm{DD}}^{\{\infty,\infty\}}$-property for some $\mathrm{LC}^\infty$-space $Y \notin \bigcup_G \mathcal{Z}^G_\infty$. Consequently, $X\times Y$ has the $\infty$-$\overline{\mathrm{DD}}^{\{\infty,\infty\}}$ property and $X$ has the 2-$\overline{\mathrm{DD}}^{\{0,0\}}$-property. Then the first item of this corollary implies that $X$ has $\infty$-$\overline{\mathrm{DD}}^{\{0,0\}}$. Hence, $X \in \mathcal{Z}^{\mathbb{Z}}_\infty$ by Theorem 11.1(3). Now, we can apply Theorem 14.2(4) to conclude that $X$ has the 0-$\overline{\mathrm{DD}}^{\{\infty,\infty\}}$-property.

The fourth item is a particular case of Theorem 14.2(2). ■

## 48. Proof of Theorem 16.1

We need to show that any separable metrizable space $X$ with the $m$-$\overline{\mathrm{DD}}^n$-property has $\dim X \geq n + (m+1)/2$. Assuming the converse, find a metrizable compactification $\tilde{X}$ of $X$ with $\dim \tilde{X} = \dim X < n + (m+1)/2$. By Theorem 8.1(3), the $m$-$\overline{\mathrm{DD}}^n$-property of $X$ implies the 0-$\overline{\mathrm{DD}}^{n+m}$-property of the product $X \times \mathbb{I}^m$. Applying Theorem 3.3, we conclude that the product $\tilde{X}\times\mathbb{I}^m$ contains a topological copy of each $(n+m)$-dimensional compactum.

Now consider two cases.

If $m$ is odd, then $\dim \tilde{X} = \dim X \le n + (m+1)/2 - 1 = n + (m-1)/2$ and hence the compactum $\tilde{X}$ embeds into $\mathbb{R}^k$ where $k = 2(n + (m-1)/2) + 1 = 2n + m$. Then $\tilde{X} \times \mathbb{I}^m$ embeds into $\mathbb{R}^{2n+2m}$, and consequently $\mathbb{R}^{2n+2m}$ contains a topological copy of each $(n+m)$-dimensional compactum, which is not true (see [32, 1.11.H]).

If $m = 2k$, them $\dim \tilde{X} \le n + m/2 = n + k$ and hence $\tilde{X}$ embeds into the $(n+k)$-dimensional Menger cube $\mu^{n+k}$. Since $\tilde{X} \times \mathbb{I}^m$ contains a copy of each $(n+m)$-dimensional compactum, so does $\mu^{n+k} \times \mathbb{I}^{2k}$. By [61], the latter product quasi-embeds into $\mathbb{R}^{2n+4k}$ in the sense that for every open cover $\mathcal{V}$ of $\mu^{n+k} \times \mathbb{I}^{2k}$ there is a $\mathcal{V}$-map $f : \mu^{n+k} \times \mathbb{I}^{2k} \to \mathbb{R}^{2n+4k}$. Since $\mu^{n+k} \times \mathbb{I}^{2k}$ contains a topological copy of each $(n+2k)$-dimensional polyhedron, we conclude that all $(n+2k)$-dimensional polyhedra quasi-embed into $\mathbb{R}^{2n+4k}$, which is not true because of a classical example due to Flores (see [32, 1.11.H]).

Remark 48.1. For $n = 0$ the fact that $\mu^{n+2k}$ does not embed into $\mu^{n+k} \times \mathbb{I}^{2k}$ was proved by Melikov and Shchepin in [51, 4.1].

## 49. Cohomological and extension dimension of spaces with $m$-$\overline{\mathrm{DD}}^0$

In this section we shall prove Theorems 16.3–16.5. All spaces in this section are supposed to be metrizable.

In some case the results are true not only for $\mathrm{LC}^n$-spaces but also for the wider class of $\mathrm{lc}^n$-spaces.

We recall that a space $X$ is called an $\mathrm{lc}^n$*-space* if for each $x \in X$ and a neighborhood $U \subset X$ of $x$ there is a neighborhood $V \subset U$ of $x$ such that the homomorphism $H_k(V) \to H_k(U)$ is trivial for every $k \le n$. This property is a homological version of the $\mathrm{LC}^n$-property. It is known that a space (resp., an $\mathrm{LC}^1$-space) $X$ is an $\mathrm{lc}^n$-space if (resp., and only if) $X$ is an $\mathrm{LC}^n$-space (see [73]).

We need the following property of locally compact $\mathrm{lc}^n$-spaces, established in [4, Lemma 1.6].

Lemma 49.1. *Let $X$ be a locally compact $\mathrm{lc}^n$-space and $x$ be a homological $Z_n$-point in $X$. Then, for any neighborhood $U \subset X$ of $x$ and any $k < n+1$, there exists a neighborhood $V \subset U$ of $x$ such that the homomorphism $H_k(X, X \setminus U) \to H_k(X, X \setminus V)$ induced by inclusion is trivial.*

The next lemma provides a proof of a bit more general statement than Theorem 16.3.

Lemma 49.2. *Let $X$ be a locally compact $\mathrm{lc}^m$-space having the $(2m+1)$-$\overline{\mathrm{DD}}^0$-property for some $m \in \omega$. Then $\dim_G X > m$ for any coefficient group $G$.*

*Proof.* Assume that $\dim_G X \le m$ and let $n = \dim_G X$. Then, by Theorem 2 of [45] (or Theorem 1.8 of [24]), the space $X$ contains a point $x^*$ having an open neighborhood $U \subset X$ with compact closure satisfying the following property: for any neighborhood $W \subset U$ of $x^*$ the homomorphism $i_{W,U} : \check{H}^n(X, X \setminus W; G) \to \check{H}^n(X, X \setminus U; G)$ in the relative Čech cohomology groups, induced by the inclusion $(X, X \setminus U) \subset (X, X \setminus W)$, is non-trivial. Since $X$ is an $\mathrm{lc}^n$-space, the Čech cohomology groups in this assertion can be replaced by singular cohomology groups (see [64, VI.§9]).

Singular cohomology groups relate to singular homology groups via the following exact sequence which splits (not naturally, see [40, §3.1]):

$$0 \to \mathrm{Ext}(H_{n-1}(X,A),G) \to H^n(X,A;G) \to \mathrm{Hom}(H_n(X,A),G) \to 0.$$

Since $X$ has the $(2m+1)$-$\overline{\mathrm{DD}}^0$-property, according to Theorem 11.1(3), $x^*$ is a homological $Z_m$-point in $X$. Then, by Lemma 49.1, there are neighborhoods $W \subset V$ of $x^*$ such that $V \subset U$ and the inclusion-induced homomorphisms $H_k(X, X\setminus U) \to H_k(X, X\setminus V)$ and $H_k(X, X\setminus V) \to H_k(X, X\setminus W)$ are trivial for all $k \le m$.

These trivial homomorphisms induce trivial homomorphisms

$$e_{V,U} : \mathrm{Ext}(H_{n-1}(X, X\setminus V), G) \to \mathrm{Ext}(H_{n-1}(X, X\setminus U), G)$$

and

$$h_{W,V} : \mathrm{Hom}(H_n(X, X\setminus W), G) \to \mathrm{Hom}(H_n(X, X\setminus V), G).$$

Now, consider the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Ext}(H_{n-1}(X, X\setminus W), G) & \longrightarrow & H^n(X, X\setminus W; G) & \longrightarrow & \mathrm{Hom}(H_n(X, X\setminus W), G) \\
\downarrow{\scriptstyle e_{W,V}} & & \downarrow{\scriptstyle i_{W,V}} & & \downarrow{\scriptstyle h_{W,V}} \\
\mathrm{Ext}(H_{n-1}(X, X\setminus V), G) & \longrightarrow & H^n(X, X\setminus V; G) & \longrightarrow & \mathrm{Hom}(H_n(X, X\setminus V), G) \\
\downarrow{\scriptstyle e_{V,U}} & & \downarrow{\scriptstyle i_{V,U}} & & \downarrow{\scriptstyle h_{V,U}} \\
\mathrm{Ext}(H_{n-1}(X, X\setminus U), G) & \longrightarrow & H^n(X, X\setminus U; G) & \longrightarrow & \mathrm{Hom}(H_n(X, X\setminus U), G)
\end{array}$$

The rows in this diagram are exact sequences and the homomorphisms $e_{V,U}$ and $h_{W,V}$ are trivial. Therefore, the homomorphism $i_{W,U} = i_{V,U} \circ i_{W,V} : H^n(X, X\setminus W; G) \to H^n(X, X\setminus U; G)$ is also trivial. The last conclusion contradicts the choice of $x^*$ and its neighborhood $U$. ■

Next, we turn to the proof of Theorem 16.4. First, we prove a particular case of this theorem with $G \in \{\mathbb{Q}, \mathbb{Z}_p, R_p : p \in \Pi\}$.

Lemma 49.3. *Let $X$ be a locally compact $\mathrm{lc}^{2m}$-space possessing the $2m$-$\overline{\mathrm{DD}}^0$-property for some $m \in \omega$. Then $\dim_G X > m$ for any group $G \in \{\mathbb{Q}, \mathbb{Z}_p, R_p : p \in \Pi\}$.*

*Proof.* Assume that $\dim_G X \le m$ for some group $G \in \{\mathbb{Q}, \mathbb{Z}_p, R_p : p \in \Pi\}$. Since $X \in (2m-1)$-$\overline{\mathrm{DD}}^0$, Lemma 49.2 implies $\dim_G X > m-1$. Consequently, $\dim_G X = m$. As in the proof of Lemma 49.2, we can find a point $x^* \in X$ and an open neighborhood $U$ of $x^*$ having compact closure in $X$ such that for any neighborhood $W \subset U$ of $x^*$ the inclusion-induced homomorphism $i_{W,U} : H^m(X, X\setminus W; G) \to H^m(X, X\setminus U; G)$ is non-trivial.

By Theorem 11.1(3), the $(2m-1)$-$\overline{\mathrm{DD}}^0$-property of $X$ implies that $x^*$ is a homological $Z_{m-1}$-point in $X$. Since $X$ is a locally compact $\mathrm{lc}^{m-1}$-space, we may apply Lemma 49.1 to find a neighborhood $V \subset U$ so small that the homomorphism $H_k(X, X\setminus U) \to H_k(X, X\setminus V)$ is trivial for every $k < m$.

Because $X \in 2m\text{-}\overline{\mathrm{DD}}^0$, according to Theorem 7.1(1), the diagonal $\Delta_X$ is a $Z_{2m}$-homological set in $X^2$. So is the point $(x^*, x^*)$. Hence, $H_{2m}(X^2, X^2 \setminus \{(x^*, x^*)\}) = 0$. Since $X^2$ is a locally compact $\mathrm{lc}^{2m}$-space, we can apply Lemma 49.1 to find an open neighborhood $W \subset V$ of $x^*$ such that the inclusion-induced homomorphism

$$h_{2m} : H_{2m}(X^2, X^2 \setminus V^2) \to H_{2m}(X^2, X^2 \setminus W^2)$$

is trivial.

Now, let us consider the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Ext}(H_{m-1}(X, X \setminus W), G) & \longrightarrow & H^m(X, X \setminus W; G) & \longrightarrow & \mathrm{Hom}(H_m(X, X \setminus W), G) \\
\downarrow{\scriptstyle e_{W,V}} & & \downarrow{\scriptstyle i_{W,V}} & & \downarrow{\scriptstyle h_{W,V}} \\
\mathrm{Ext}(H_{m-1}(X, X \setminus V), G) & \longrightarrow & H^m(X, X \setminus V; G) & \longrightarrow & \mathrm{Hom}(H_m(X, X \setminus V), G) \\
\downarrow{\scriptstyle e_{V,U}} & & \downarrow{\scriptstyle i_{V,U}} & & \downarrow{\scriptstyle h_{V,U}} \\
\mathrm{Ext}(H_{m-1}(X, X \setminus U), G) & \longrightarrow & H^m(X, X \setminus U; G) & \longrightarrow & \mathrm{Hom}(H_m(X, X \setminus U), G)
\end{array}$$

In this diagram the homomorphism $e_{V,U}$ is trivial because it is induced by the trivial homomorphism $H_{m-1}(X, X \setminus U) \to H_{m-1}(X, X \setminus V)$, while $i_{W,U} = i_{V,U} \circ i_{W,V}$ is non-trivial. Therefore,

$$h_{W,V} : \mathrm{Hom}(H_m(X, X \setminus W), G) \to \mathrm{Hom}(H_m(X, X \setminus V), G)$$

is also non-trivial. This means that, for some homomorphism $\xi : H_m(X, X \setminus W) \to G$, the composition $\xi \circ i$ of $\xi$ with the inclusion-induced homomorphism $i : H_m(X, X \setminus V) \to H_m(X, X \setminus W)$ is not trivial. Since for any non-zero element $x \in G$ the tensor product $x \otimes x$ is non-zero (at this point we use the special feature of the group $G \in \{\mathbb{Q}, \mathbb{Z}_p, R_p : p \in \Pi\}$), the tensor product

$$(\xi \circ i) \otimes (\xi \circ i) : H_m(X, X \setminus V) \otimes H_m(X, X \setminus V) \to G \otimes G$$

of the homomorphism $\xi \circ i$ with itself is non-trivial. Moreover,

$$(\xi \circ i) \otimes (\xi \circ i) = (\xi \otimes \xi) \circ (i \otimes i).$$

So, we will obtain a contradiction as soon as we check that the homomorphism

$$i \otimes i : H_m(X, X \setminus V) \otimes H_m(X, X \setminus V) \to H_m(X, X \setminus W) \otimes H_m(X, X \setminus W)$$

is trivial.

The triviality of this homomorphism follows from the diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & H_m(X, X \setminus V) \otimes H_m(X, X \setminus V) & \longrightarrow & H_{2m}(X^2, X^2 \setminus V^2) \\
\downarrow & & \downarrow{\scriptstyle i \otimes i} & & \downarrow{\scriptstyle h_{2m}} \\
0 & \longrightarrow & H_m(X, X \setminus W) \otimes H_m(X, X \setminus W) & \longrightarrow & H_{2m}(X^2, X^2 \setminus W^2)
\end{array}$$

having exact rows with $h_{2m}$ being a trivial homomorphism. ∎

We are now in a position to derive Theorem 16.4 from Lemma 49.3 using the Bockstein formula for cohomological dimension (see [24]). This formula asserts that

$$\dim_G X = \sup_{H \in \sigma(G)} \dim_H X$$

for any locally compact space $X$, where $\sigma(G)$ is the Bockstein family of the group $G$.

The following lemma provides the proof of a stronger version of Theorem 16.4(1).

LEMMA 49.4. *Let $X$ be a locally compact* $\mathrm{lc}^{2m}$*-space $X$ with the* $2m$-$\overline{\mathrm{DD}}^0$*-property and $G$ be a non-trivial Abelian group. Then* $\dim_G X \geq m+1$ *in each of the following cases:*

(1) *$G$ fails to be both divisible and periodic;*
(2) *$G$ is the additive group of a ring with non-zero multiplication;*
(3) *$X$ is an ANR-space.*

*Proof.* Suppose $G$ is either not divisible or not periodic. If $\dim_G X \leq m$, then applying the Bockstein formula, we obtain $\dim_H X \leq m$ for any group $H \in \sigma(G)$. Now, Lemma 49.3 implies that the Bockstein family $\sigma(G)$ of $G$ contains only quasicyclic groups $\mathbb{Q}_p$. It follows from the definition of $\sigma(G)$ that $\mathrm{Tor}(G) = G$ and all $p$-torsion parts $p$-$\mathrm{Tor}(G)$ of $G$ are divisible by $p$. Since each $p$-group is divisible by any prime $q \neq p$, we infer that $G = \bigoplus_p p\text{-}\mathrm{Tor}(G)$ is both divisible and torsion, a contradiction.

To prove the second item, suppose that $G$ is the additive group of a ring with non-zero multiplication $G \times G \to G$. This multiplication determines a non-trivial bilinear form on $G$, and hence $G \otimes G \neq 0$. Then, by Lemma 18.8, $G$ is either not divisible or not periodic. So, we can apply the preceding item to obtain $\dim_G X \geq m+1$.

If $X$ is an ANR-space, then $\dim_G X \geq \dim_{\mathbb{Q}} X$ by Theorem 12.3 from [24] (Dranishnikov established this theorem in [24] only for ANR-compacta, but without any changes his proof holds also for locally compact ANR-spaces). Since, according to Lemma 49.3, $\dim_{\mathbb{Q}} X \geq m+1$, we are done. ∎

Finally, we shall prove Theorem 16.5 on extension dimension of spaces possessing the $m$-$\overline{\mathrm{DD}}^{\{0,0\}}$-property. We recall its formulation:

LEMMA 49.5. *Let $X$ be a locally compact* $\mathrm{LC}^m$*-space such that* $\text{e-dim}\, X \leq L$ *for some non-contractible CW-complex $L$. If* $X \in m$-$\overline{\mathrm{DD}}^0$*, then:*

(1) *the homotopy groups $\pi_i(L)$ are trivial for all $i < m/2$;*
(2) *the group $\pi_n(L) = \tilde{H}_n(L)$ is divisible and periodic for $n = \lfloor m/2 \rfloor$;*
(3) *$\pi_i(L) = 0$ for all $i \leq m/2$ provided $X$ is an ANR-space.*

*Proof.* We consider separately the cases $m \leq 1$ and $m \geq 2$.

If $m \leq 1$, then it suffices to check that $L$ is connected. By Proposition 5.6(1), $X$ contains an arc $C$ connecting two distinct point $a, b \in X$. Assuming that $L$ is disconnected, consider any map $f : \{a, b\} \to L$ sending $a, b$ to different components of $L$. Because $C \subset X$ is connected, $f$ does not extend to $X$, which contradicts $\text{e-dim}\, X \leq L$.

Next, consider the case of $m \geq 2$. First, we prove that $L$ is simply connected. Since $X$ is an $\mathrm{LC}^2$-space with the 2-$\overline{\mathrm{DD}}^{\{0,0\}}$-property, $\dim X \geq 2$ by Theorem 16.1. Consequently, there is a point $x \in X$ such that any neighborhood $U \subset X$ of $x$ has $\dim U \geq 2$. Since $X$ is a locally compact $\mathrm{LC}^1$-space, $x$ has a closed compact neighborhood $N$ such that any map $f : \mathbb{S}^1 \to N$ is null homotopic in $X$. Moreover, we can assume that $N$ is a Peano continuum because $X$ is an $\mathrm{LC}^0$-space. Then $N$, being a continuum with $\dim N > 1$, is not a dendrite. Consequently, it contains a simple closed curve $S \subset N$ which is null-homotopic in $X$. Assuming that the CW-complex $L$ is not simply connected, we can find a map $f : \mathbb{S}^1 \to L$ that is not homotopic to a constant map. Since $\text{e-dim}\, X \leq L$, $f$ extends to a map $\bar{f} : X \to L$. This fact, combined with the contractibility of $S$ in $X$, implies that $f$ is null-homotopic in $L$. This contradiction shows that $L$ is simply connected.

Since $L$ is not contractible, $\pi_i(L) \neq 0$ for some $i \in \mathbb{N}$. Let $n$ be the smallest number $i$ with this property. The simple connectedness of $L$ yields $n > 1$. Consequently, $H_n(L) = \pi_n(L) \neq 0$ according to the Hurewicz isomorphism theorem. Finally, because $\text{e-dim}\, X \leq L$, we may apply a result of A. Dranishnikov [23, Theorem 9] (see also Theorem 7.14 of [29]) to conclude that $\dim_{H_n(L)} X \leq n$. Then Theorem 16.4 completes the proof. ∎

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