Contents

1. Introduction 6
2. Preliminaries and notation 8
   2.1. Notation 8
   2.2. Differential calculus of maps between locally convex spaces 8
   2.3. Fréchet differentiability 9
3. Weighted function spaces 10
   3.1. Definition and examples 10
   3.2. Topological and uniform structure 12
      3.2.1. Reduction to lower order 12
      3.2.2. Projective limits and the topology of $\mathcal{C}^{\infty}_{\mathcal{W}}(U, Y)$ 13
      3.2.3. A completeness criterion 14
   3.3. Composition on weighted functions and superposition operators 17
      3.3.1. Composition with a multilinear map 17
      3.3.2. Composition of weighted functions with bounded functions 21
      3.3.3. Composition of weighted functions with an analytic map 26
   3.4. Weighted maps into locally convex spaces 30
      3.4.1. Definition and topological structure 30
      3.4.2. Weighted decreasing maps 33
      3.4.3. Composition and superposition 35
4. Lie groups of weighted diffeomorphisms 41
   4.1. Weighted diffeomorphisms and endomorphisms 41
      4.1.1. Composition of weighted endomorphisms in charts 42
      4.1.2. Smooth monoids of weighted endomorphisms 47
   4.2. Lie group structures on weighted diffeomorphisms 48
      4.2.1. The Lie group structure of $\mathrm{Diff}_{\mathcal{W}}(X)$ 48
      4.2.2. On decreasing weighted diffeomorphisms and dense subgroups 54
      4.2.3. On diffeomorphisms that are weighted endomorphisms 55
   4.3. Regularity 57
      4.3.1. The tangent group and the regularity differential equation of $\mathrm{Diff}_{\mathcal{W}}(X)$ 57
      4.3.2. Conclusion and calculation of one-parameter groups 61
5. Integration of certain Lie algebras of vector fields 62
   5.1. On the smoothness of the conjugation action on $\mathrm{Diff}_{\mathcal{W}}(X)_0$ 63
      5.1.1. Contravariant composition on weighted functions 64
   5.2. Conclusion and examples 67
6. Lie group structures on weighted mapping groups 69
   6.1. Weighted maps into Banach Lie groups 70
      6.1.1. Construction of the Lie group 70
      6.1.2. Regularity 74
      6.1.3. Semidirect products with weighted diffeomorphisms 77

6.2. Weighted maps into locally convex Lie groups ... 78
6.2.1. Construction of the Lie group ... 78
6.2.2. A larger Lie group of weighted mappings ... 81
A. Differential calculus ... 95
A.1. Differential calculus of maps between locally convex spaces ... 95
A.1.1. Curves and integrals ... 95
A.1.2. Differentiable maps ... 98
A.2. Fréchet differentiability ... 104
A.3. Relation between the differential calculi ... 108
A.4. Some facts concerning ordinary differential equations ... 111
A.4.1. Maximal solutions of ODEs ... 111
A.4.2. Flows and dependence on parameters and initial values ... 113
B. Locally convex Lie groups ... 116
B.1. Locally convex manifolds ... 116
B.2. Lie groups ... 117
B.2.1. Generation of Lie groups ... 118
B.2.2. Regularity ... 118
B.2.3. Group actions ... 119
C. Quasi-inversion in algebras ... 120
C.1. Definition ... 121
C.2. Topological monoids and algebras with continuous quasi-inversion ... 121
References ... 124
Notation ... 126
Index ... 128

Abstract

In this work, we construct and study certain classes of infinite-dimensional Lie groups that are modelled on weighted function spaces. In particular, we construct a Lie group $\mathrm{Diff}_{\mathcal{W}}(X)$ of diffeomorphisms, for each Banach space X and each set $\mathcal{W}$ of weights on X containing the constant weights. We also construct certain types of "weighted mapping groups". These are Lie groups modelled on weighted function spaces of the form $\mathcal{C}^k_{\mathcal{W}}(U, \mathbf{L}(G))$, where G is a given (finite- or infinite-dimensional) Lie group. Both the weighted diffeomorphism groups and the weighted mapping groups are shown to be regular Lie groups in Milnor's sense.

We also discuss semidirect products of such groups. Moreover, we study the integrability of Lie algebras of vector fields of the form $\mathcal{C}^\infty_{\mathcal{W}}(X, X) \rtimes \mathbf{L}(G)$, where X is a Banach space and G a Lie group acting smoothly on X.

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1. Introduction

Diffeomorphism groups of compact manifolds, as well as groups $\mathcal{C}^k(K, G)$ of Lie group-valued mappings on compact manifolds are among the most important and well-studied examples of infinite-dimensional Lie groups (see for example [Les67], [Mil84], [Ham82], [Omo97], [PS86] and [KM97]). While the diffeomorphism group $\mathrm{Diff}(K)$ of a compact manifold is modelled on the Fréchet space $\mathcal{C}^\infty(K, \mathbf{T}K)$ of smooth vector fields on K, for a noncompact smooth manifold M, it is not possible to make $\mathrm{Diff}(M)$ a Lie group modelled on the space of all smooth vector fields in a satisfactory way (see [Mil82]). We mention that the LF-space $\mathcal{C}^\infty_c(M, \mathbf{T}M)$ of compactly supported smooth vector fields can be used as the modelling space for a Lie group structure on $\mathrm{Diff}(M)$. But the topology on this Lie group is too fine for many purposes; the group $\mathrm{Diff}_c(M)$ of compactly supported diffeomorphisms (which coincide with the identity map outside some compact set) is an open subgroup (see [Mic80] and [Mil82]). Likewise, it is no problem to turn the groups $\mathcal{C}^k_c(M, G)$ of compactly supported Lie group-valued maps into Lie groups (cf. [Mil84], [AHKM$^+$93], [Glö02b]). However, only in special cases does there exist a Lie group structure on $\mathcal{C}^\infty(M, G)$, equipped with its natural group topology, the smooth compact-open topology (see [NW08]).

In view of these limitations, it is natural to look for Lie groups of diffeomorphisms which are larger than $\mathrm{Diff}_c(M)$ and modelled on larger Lie algebras of vector fields than $\mathcal{C}^\infty_c(M, \mathbf{T}M)$. In the same vein, one would like to find mapping groups modelled on larger spaces than $\mathcal{C}^k_c(M, \mathbf{L}(G))$.

In this work, we construct such groups in the important case where the noncompact manifold M is a vector space (or an open subset thereof, in the case of mapping groups). For most of the results, the vector space is even allowed to be a Banach space X. The groups we consider are modelled on spaces of weighted functions on X. For example, we are able to construct a Lie group structure on the group $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$ of diffeomorphisms differing from $\mathrm{id}_{\mathbb{R}^n}$ by a rapidly decreasing $\mathbb{R}^n$-valued map. Considered as a topological group, this group has been used in quantum physics ([Gol04]). For $n = 1$, another Lie group structure (in the setting of convenient differential calculus) has been given by P. Michor ([Mic06, §6.4]), and applied to the Burgers equation. The general case was treated in the author's unpublished diploma thesis [Wal06].

To explain our results, let X and Y be Banach spaces, $U \subseteq X$ open and nonempty, $k \in \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$, and $\mathcal{W}$ a set of functions f on U taking values in the extended real line $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty, -\infty\}$ called weights. As usual, we let $\mathcal{C}^k_{\mathcal{W}}(U, Y)$ be the set of all k-times continuously Fréchet differentiable functions $\gamma : U \to Y$ such that $f \cdot \|D^{(\ell)}\gamma\|_{\mathrm{op}}$

is bounded for all integers $\ell \leq k$ and all $f \in \mathcal{W}$. Then $\mathcal{C}^k_{\mathcal{W}}(U,Y)$ is a locally convex topological vector space in a natural way. We prove (see Theorems 4.2.17 and 4.3.11)

THEOREM. *Let X be a Banach space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. Then* $\mathrm{Diff}_{\mathcal{W}}(X) := \{\phi \in \mathrm{Diff}(X) : \phi - \mathrm{id}_X, \phi^{-1} - \mathrm{id}_X \in \mathcal{C}^\infty_{\mathcal{W}}(X,X)\}$ *is a regular Lie group modelled on* $\mathcal{C}^\infty_{\mathcal{W}}(X,X)$.

Replacing $\mathcal{C}^\infty_{\mathcal{W}}(X,X)$ by the subspace of functions γ such that $f(x)\cdot\|D^{(\ell)}\gamma(x)\|_{\mathrm{op}} \to 0$ as $\|x\| \to \infty$, we obtain a subgroup $\mathrm{Diff}_{\mathcal{W}}(X)^\circ$ of $\mathrm{Diff}_{\mathcal{W}}(X)$ which is also a Lie group (see Proposition 4.2.19).

As for mapping groups, we first consider mappings into Banach Lie groups. In Section 6.1 we show

THEOREM. *Let X be a normed space, $U \subseteq X$ an open nonempty subset, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $k \in \overline{\mathbb{N}}$ and G a Banach Lie group. Then there exists a connected Lie group $\mathcal{C}^k_{\mathcal{W}}(U,G) \subseteq G^U$ modelled on $\mathcal{C}^k_{\mathcal{W}}(U,\mathbf{L}(G))$, and this Lie group is regular.*

Using the natural action of diffeomorphisms on functions, we can form the semidirect product $\mathcal{C}^\infty_{\mathcal{W}}(X,G) \rtimes \mathrm{Diff}_{\mathcal{W}}(X)$ and make it a Lie group.

In the case of finite-dimensional domains, we can even discuss mappings into arbitrary Lie groups modelled on locally convex spaces. To this end, given a locally convex space Y and an open subset U in a finite-dimensional vector space X we define a certain space $\mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet$ of $\mathcal{C}^k$-maps which decay together with their derivatives as we approach the boundary of U (see Definition 3.4.8 for details). We obtain the following result:

THEOREM. *Let X be a finite-dimensional space, $U \subseteq X$ an open nonempty subset, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $k \in \overline{\mathbb{N}}$ and G a locally convex Lie group. Then there exists a connected Lie group $\mathcal{C}^k_{\mathcal{W}}(U,G)^\bullet \subseteq G^U$ modelled on $\mathcal{C}^k_{\mathcal{W}}(U,\mathbf{L}(G))^\bullet$.*

We also discuss certain larger subgroups of G^U admitting Lie group structures that make $\mathcal{C}^k_{\mathcal{W}}(U,G)^\bullet$ an open normal subgroup (see Section 6.2.2).

Finally, we consider Lie groups G acting smoothly on a Banach space X. We investigate when the G-action leaves the identity component $\mathrm{Diff}_{\mathcal{W}}(X)_0$ of $\mathrm{Diff}_{\mathcal{W}}(X)$ invariant and whether $\mathrm{Diff}_{\mathcal{W}}(X)_0 \rtimes G$ can be made a Lie group in this case. In particular, we show that $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)_0 \rtimes \mathrm{GL}(\mathbb{R}^n)$ is a Lie group for each n (Example 5.2.4). By contrast, $\mathrm{GL}(\mathbb{R}^n)$ does not leave $\mathrm{Diff}_{\{1_{\mathbb{R}^n}\}}(\mathbb{R}^n)$ invariant (Example 5.2.5).

We mention that certain weighted mapping groups on finite-dimensional spaces (consisting of smooth mappings) have already been discussed in [BCR81, §4.2] assuming additional hypotheses on the range group (cf. Remark 6.2.29). Besides the added generality, we provide a more complete discussion of superposition operators on weighted function spaces.

In the case where $\mathcal{W} = \{1_X\}$, our group $\mathrm{Diff}_{\mathcal{W}}(X)$ also has a counterpart in the studies of Jürgen Eichhorn and collaborators ([Eic96], [ES96], [Eic07]), who investigated certain diffeomorphism groups on noncompact manifolds with bounded geometry.

Semidirect products of diffeomorphism groups and function spaces on compact manifolds arise in ideal magnetohydrodynamics (see [KW09, II.3.4]). Further, the group

$\mathcal{S}(\mathbb{R}^n) \rtimes \mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$ and its continuous unitary representations are encountered in quantum physics (see [Gol04]; cf. also [Ism96, §34] and the references therein).

2. Preliminaries and notation

We give some notation and basic definitions. More details are provided in the appendix, as is a list of symbols used in this work.

2.1. Notation. We write $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$, $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ and $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$. Further we denote norms by $\|\cdot\|$.

Definition 2.1.1. Let A, B be subsets of the normed space X. As usual, the *distance* of A and B is defined as

$$\operatorname{dist}(A, B) := \inf\{\|a - b\| : a \in A,\, b \in B\} \in [0, \infty].$$

Thus $\operatorname{dist}(A, B) = \infty$ iff $A = \emptyset$ or $B = \emptyset$.

Further, for $x \in X$ and $r \in \mathbb{R}$ we define

$$B_X(x, r) := \{y \in X : \|y - x\| < r\}$$

Occasionally, we just write $B_r(x)$ instead of $B_X(x, r)$. For the closed ball, we write $\overline{B}_r(x)$ and the like.

Further, we define

$$\mathbb{D} := \overline{B}_{\mathbb{K}}(0, 1),$$

where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. No confusion will arise from this abuse of notation.

2.2. Differential calculus of maps between locally convex spaces. We give basic definitions for the differential calculus for maps between locally convex spaces that is known as Keller's C^k_c-theory. More results can be found in Section A.1.

Definition 2.2.1 (Directional derivatives). Let X and Y be locally convex spaces, $U \subseteq X$ an open nonempty set, $u \in U$, $x \in X$ and $f : U \to Y$ a map. The *derivative of f at u in the direction x* is defined as

$$\lim_{\substack{t \to 0 \\ t \in \mathbb{K}^*}} \frac{f(u + tx) - f(u)}{t} =: (D_x f)(u) =: df(u; x),$$

whenever that limit exists.

Definition 2.2.2. Let X and Y be locally convex spaces, $U \subseteq X$ an open nonempty set, and $f : U \to Y$ be a map.

We call f a *$\mathcal{C}^1_{\mathbb{K}}$-map* or just $\mathcal{C}^1_{\mathbb{K}}$ if f is continuous, the derivative $df(u; x)$ exists for all $(u, x) \in U \times X$ and the map $df : U \times X \to Y$ is continuous.

Inductively, for a $k \in \mathbb{N}$ we call f a *$\mathcal{C}^k_{\mathbb{K}}$-map* or just $\mathcal{C}^k_{\mathbb{K}}$ if f is a $\mathcal{C}^1_{\mathbb{K}}$-map and $d^1 f := df : U \times X \to Y$ is a $\mathcal{C}^{k-1}_{\mathbb{K}}$-map. In this case, the *kth iterated differential* of f is defined by

$$d^k f := d^{k-1}(df) : U \times X^{2^k - 1} \to Y.$$

If f is a *$\mathcal{C}^k_{\mathbb{K}}$-map* for each $k \in \mathbb{N}$, we call f a *$\mathcal{C}^\infty_{\mathbb{K}}$-map* or just $\mathcal{C}^\infty_{\mathbb{K}}$ or *smooth*.

Further, for each $k \in \overline{\mathbb{N}}$ we define

$$\mathcal{C}^k_{\mathbb{K}}(U,Y) := \{f : U \to Y : f \text{ is } \mathcal{C}^k_{\mathbb{K}}\}.$$

Often, we shall simply write $\mathcal{C}^k(U,Y)$, $\mathcal{C}^k$ etc.

It is obvious from the definition of differentiability that iterated directional derivatives exist and depend continuously on the directions. The converse of this assertion also holds.

Proposition 2.2.3. *Let $f : U \to Y$ be a continuous map and $r \in \overline{\mathbb{N}}$. Then $f \in \mathcal{C}^r(U,Y)$ iff for all $u \in U$, $k \in \mathbb{N}$ with $k \leq r$ and $x_1, \dots, x_k \in X$ the iterated directional derivative*

$$d^{(k)}f(u; x_1, \dots, x_k) := (D_{x_k} \cdots D_{x_1} f)(u)$$

exists and the map

$$U \times X^k \to Y : (u, x_1, \dots, x_k) \mapsto d^{(k)}f(u; x_1, \dots, x_k)$$

is continuous. We call $d^{(k)}f$ the kth derivative *of f.*

2.3. Fréchet differentiability. We give basic definitions for Fréchet differentiability for maps between normed spaces. More results can be found in Section A.2.

Definition 2.3.1 (Fréchet differentiability). Let X and Y be normed spaces and U an open nonempty subset of X. We call a map $\gamma : U \to Y$ *Fréchet differentiable* or $\mathcal{FC}^1$ if it is a $\mathcal{C}^1$-map and the map

$$D\gamma : U \to \mathrm{L}(X,Y) : x \mapsto d\gamma(x; \cdot)$$

is continuous. Inductively, for $k \in \mathbb{N}^*$ we call γ a $\mathcal{FC}^{k+1}$-map if it is Fréchet differentiable and $D\gamma$ is an $\mathcal{FC}^k$-map. We denote the set of all k-times Fréchet differentiable maps from U to Y with $\mathcal{FC}^k(U,Y)$. Additionally, we define the *smooth* maps by

$$\mathcal{FC}^\infty(U,Y) := \bigcap_{k \in \mathbb{N}^*} \mathcal{FC}^k(U,Y)$$

and $\mathcal{FC}^0(U,Y) := \mathcal{C}^0(U,Y)$. The map

$$D : \mathcal{FC}^{k+1}(U,Y) \to \mathcal{FC}^k(U, \mathrm{L}(X,Y)) : \gamma \mapsto D\gamma$$

is called the *derivative operator*.

Remark 2.3.2. Let X and Y be normed spaces, U an open nonempty subset of X, $k \in \mathbb{N}^*$ and $\gamma \in \mathcal{FC}^k(U,Y)$. Then for each $\ell \in \mathbb{N}^*$ with $\ell \leq k$ there exists a continuous map

$$D^{(\ell)}\gamma : U \to \mathrm{L}^\ell(X,Y),$$

where $\mathrm{L}^\ell(X,Y)$ denotes the space of ℓ-linear maps $X^\ell \to Y$, endowed with the operator topology. The map $D^{(\ell)}\gamma$ can be described more explicitly. If $\gamma \in \mathcal{FC}^k(U,Y)$, then also $\gamma \in \mathcal{C}^k(U,Y)$, and for each $x \in U$ we have the relation

$$D^{(k)}\gamma(x) = d^{(k)}\gamma(x; \cdot).$$

3. Weighted function spaces

In this chapter we give the definition of some locally convex vector spaces consisting of weighted functions. The Lie groups that are constructed in this work will be modelled on these spaces. We first discuss maps between normed spaces. In Section 3.4, we will also look at maps that take values in arbitrary locally convex spaces. The treatment of the latter spaces requires some rather technical effort. Since these function spaces are only needed in Section 6.2, the reader may possibly skip this section.

3.1. Definition and examples

Definition 3.1.1. Let X and Y be normed spaces and $U \subseteq X$ an open nonempty set. For $k \in \mathbb{N}$ and a map $f : U \to \overline{\mathbb{R}}$ we define the quasinorm

$$\|\cdot\|_{f,k} : \mathcal{FC}^k(U,Y) \to [0,\infty] : \phi \mapsto \sup\{|f(x)|\,\|D^{(k)}\phi(x)\|_{\mathrm{op}} : x \in U\}.$$

Furthermore, for any nonempty set $\mathcal{W}$ of maps $U \to \overline{\mathbb{R}}$ and $k \in \overline{\mathbb{N}}$ we define the vector space

$$\mathcal{C}^k_{\mathcal{W}}(U,Y) := \{\gamma \in \mathcal{FC}^k(U,Y) : (\forall f \in \mathcal{W}, \ell \in \mathbb{N}, \ell \leq k)\ \|\gamma\|_{f,\ell} < \infty\}$$

and notice that the seminorms $\|\cdot\|_{f,\ell}$ induce a locally convex vector space topology on $\mathcal{C}^k_{\mathcal{W}}(U,Y)$.

We call the elements of $\mathcal{W}$ *weights* and $\mathcal{C}^k_{\mathcal{W}}(U,Y)$ a *space of weighted maps* or *space of weighted functions*.

An important example is the space of bounded functions with bounded derivatives:

Example 3.1.2. Let $k \in \overline{\mathbb{N}}$. We define

$$\mathcal{BC}^k(U,Y) := \mathcal{C}^k_{\{1_U\}}(U,Y).$$

Remark 3.1.3. Let U and V be nonempty open subsets of a normed space X and $U \subseteq V$. For a set $\mathcal{W} \subseteq \overline{\mathbb{R}}^V$, we define

$$\mathcal{W}|_U := \{f|_U : f \in \mathcal{W}\}.$$

Further we write with an abuse of notation

$$\mathcal{C}^k_{\mathcal{W}}(U,Y) := \mathcal{C}^k_{\mathcal{W}|_U}(U,Y).$$

Remark 3.1.4. As is clear, for any set $T \subseteq 2^{\mathcal{W}}$ with $\mathcal{W} = \bigcup_{\mathcal{F}\in T} \mathcal{F}$ we have

$$\mathcal{C}^k_{\mathcal{W}}(U,Y) = \bigcap_{\substack{\mathcal{F}\in T\\ \ell\in\mathbb{N},\, \ell\leq k}} \mathcal{C}^\ell_{\mathcal{F}}(U,Y).$$

We define some subsets of $\mathcal{C}^k_{\mathcal{W}}(U,Y)$:

Definition 3.1.5. Let X and Y be normed spaces, $U \subseteq X$ and $V \subseteq Y$ open nonempty sets and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$. For $k \in \overline{\mathbb{N}}$ we set

$$\mathcal{C}^k_{\mathcal{W}}(U,V) := \{\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,Y) : \gamma(U) \subseteq V\},$$
$$\mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V) := \{\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,V) : (\exists r > 0)\ \gamma(U) + B_Y(0,r) \subseteq V\}.$$

Obviously $\mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V) \subseteq \mathcal{C}^k_{\mathcal{W}}(U,V)$, and if $1_U \in \mathcal{W}$, then $\mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V)$ is open in $\mathcal{C}^k_{\mathcal{W}}(U,Y)$. The symbol $\mathcal{BC}^{\partial,k}(U,V)$ is defined analogously.

If $U \subseteq X$ is an open neighborhood of 0, we set

$$\mathcal{C}^k_{\mathcal{W}}(U,Y)_0 := \{\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,Y) : \gamma(0) = 0\}.$$

Analogously, we define $\mathcal{C}^k_{\mathcal{W}}(U,V)_0$, $\mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V)_0$ and $\mathcal{BC}^0(U,V)_0$ as the corresponding sets of functions vanishing at 0.

Furthermore, we define the set of *decreasing weighted maps* as

$$\mathcal{C}^k_{\mathcal{W}}(U,Y)^o := \{\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,Y) : (\forall f \in \mathcal{W}, \ell \in \mathbb{N}, \ell \leq k, \varepsilon > 0)(\exists r > 0)\ \|\gamma|_{U\setminus B_r(0)}\|_{f,\ell} < \varepsilon\}.$$

Note that we are primarily interested in the spaces $\mathcal{C}^k_{\mathcal{W}}(X,Y)^o$, but for technical reasons it is useful to have the spaces $\mathcal{C}^k_{\mathcal{W}}(U,Y)^o$ available for $U \subset X$.

LEMMA 3.1.6. *$\mathcal{C}^k_{\mathcal{W}}(U,Y)^o$ is a closed vector subspace of $\mathcal{C}^k_{\mathcal{W}}(U,Y)$.*

Proof. It is obvious from the definition of $\mathcal{C}^k_{\mathcal{W}}(U,Y)^o$ that it is a vector subspace. It remains to show that it is closed. To this end, let $(\gamma_i)_{i\in I}$ be a net in $\mathcal{C}^k_{\mathcal{W}}(U,Y)^o$ that converges to $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,Y)$ in the topology of $\mathcal{C}^k_{\mathcal{W}}(U,Y)$. Let $f \in \mathcal{W}$, $\ell \in \mathbb{N}$ with $\ell \leq k$ and $\varepsilon > 0$. Then there exists an $i_\varepsilon \in I$ such that

$$i \geq i_\varepsilon \Rightarrow \|\gamma - \gamma_i\|_{f,\ell} < \varepsilon/2.$$

Further there exists an $r > 0$ such that

$$\|\gamma_{i_\varepsilon}|_{U\setminus B_r(0)}\|_{f,\ell} < \varepsilon/2.$$

Hence

$$\|\gamma|_{U\setminus B_r(0)}\|_{f,\ell} \leq \|\gamma|_{U\setminus B_r(0)} - \gamma_{i_\varepsilon}|_{U\setminus B_r(0)}\|_{f,\ell} + \|\gamma_{i_\varepsilon}|_{U\setminus B_r(0)}\|_{f,\ell} < \varepsilon. \blacksquare$$

Examples involving finite-dimensional spaces. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $n \in \mathbb{N}$. In the following, let U be an open nonempty subset of $\mathbb{K}^n$. For a map $f : U \to \overline{\mathbb{R}}$ and a multiindex $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq k$ we define

$$\|\cdot\|_{f,\alpha} : \mathcal{C}^k_{\mathbb{K}}(U,Y) \to [0,\infty] : \phi \mapsto \sup\{|f(x)|\,\|\partial^\alpha\phi(x)\| : x \in U\}.$$

We conclude from identity (A.3.6.1) in Proposition A.3.6 that for a set $\mathcal{W}$ of maps $U \to \overline{\mathbb{R}}$ and $k \in \overline{\mathbb{N}}$

$$\mathcal{C}^k_{\mathcal{W}}(U,Y) = \{\phi \in \mathcal{C}^k_{\mathbb{K}}(U,Y) : (\forall f \in \mathcal{W}, \alpha \in \mathbb{N}_0^n, |\alpha| \leq k)\ \|\phi\|_{f,\alpha} < \infty\},$$

and the topology defined by the seminorms $\|\cdot\|_{f,\alpha}$ coincides with the one defined above using the seminorms $\|\cdot\|_{f,\ell}$. This characterization of $\mathcal{C}^k_{\mathcal{W}}(U,Y)$ allows us to recover well-known spaces as special cases:

- If $\mathcal{W}$ is the space $\mathcal{C}^0(U,\mathbb{R}^m)$ of all continuous functions, then

  $$\mathcal{C}^\infty_{\mathcal{W}}(U,\mathbb{R}^m) = \mathcal{D}(U,\mathbb{R}^m) = \mathcal{C}^\infty_c(U,\mathbb{R}^m)$$

  where $\mathcal{D}(U,\mathbb{R}^m)$ denotes the space of compactly supported smooth functions from U to $\mathbb{R}^m$; it should be noticed that $\mathcal{C}^\infty_{\mathcal{C}^0(U,\mathbb{R}^m)}(U,\mathbb{R}^m)$ is *not* endowed with the ordinary inductive limit topology $\varinjlim_K \mathcal{D}_K(U,\mathbb{R}^m)$, but instead the (coarser) topology making it the projective limit

$$\varprojlim_{p\in\mathbb{N}}(\varinjlim_{K}\mathcal{D}^p_K(U,\mathbb{R}^m)) = \varprojlim_{p\in\mathbb{N}}\mathcal{D}^p(U,\mathbb{R}^m),$$

where $\mathcal{D}^p_K(U,\mathbb{R}^m)$ denotes the $\mathcal{C}^p$-maps with support in the compact set K, endowed with the topology of uniform convergence of derivatives up to order p; and $\mathcal{D}^p(U,\mathbb{R}^m)$ the compactly supported $\mathcal{C}^p$-maps endowed with the inductive limit topology of the sets $\mathcal{D}^p_K(U,\mathbb{R}^m)$.

- The vector-valued *Schwartz space* $\mathcal{S}(\mathbb{R}^n,\mathbb{R}^n)$. Here $U = Y = \mathbb{R}^n$, $k = \infty$ and $\mathcal{W}$ is the set of polynomial functions on $\mathbb{R}^n$.
- The space $\mathcal{BC}^k(U,\mathbb{K}^m)$ of all bounded $\mathcal{C}^k$-functions from $U \subseteq \mathbb{K}^n$ to $\mathbb{K}^m$ whose partial derivatives are bounded (for $\mathcal{W} = \{1_U\}$); see Example 3.1.2.
- If $\mathcal{W} = \{1_X, \infty\cdot 1_{X\setminus U}\}$, then the space $\mathcal{C}^k_{\mathcal{W}}(X,Y)$ consists of $\mathcal{BC}^k(X,Y)$ functions that are defined on X and vanish on the complement of U.

3.2. Topological and uniform structure. We analyze the topology of the weighted function spaces defined above. In Proposition 3.2.3 we shall provide a method that greatly simplifies the treatment of these spaces; it will be used throughout this work. We will also describe the spaces $\mathcal{C}^k_{\mathcal{W}}(U,Y)$ as the projective limits of suitable larger spaces. In particular, this will simplify the treatment of the spaces $\mathcal{C}^\infty_{\mathcal{W}}(U,Y)$. Further we give a sufficient criterion on the set $\mathcal{W}$ which ensures that $\mathcal{C}^k_{\mathcal{W}}(U,Y)$ is complete.

3.2.1. Reduction to lower order. For $\ell > 1$, it is hard to estimate the seminorms $\|\cdot\|_{f,\ell}$ because in most cases the higher order derivatives $D^{(\ell)}\cdot$ cannot be computed. We develop a technique that allows us to avoid the computation.

First, we show that $\mathcal{C}^k_{\mathcal{W}}(U,Y)$ is endowed with the initial topology of the derivative maps.

LEMMA 3.2.1. *Let X and Y be normed spaces, $U \subseteq X$ an open nonempty set, $k \in \overline{\mathbb{N}}$, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ and $\gamma \in \mathcal{FC}^k(U,Y)$. Then*

$$\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,Y) \Leftrightarrow (\forall \ell\in\mathbb{N}, \ell\leq k)\ D^{(\ell)}\gamma \in \mathcal{C}^0_{\mathcal{W}}(U,\mathrm{L}^\ell(X,Y)),$$

and the map

$$\mathcal{C}^k_{\mathcal{W}}(U,Y) \to \prod_{\substack{\ell\in\mathbb{N}\\ \ell\leq k}}\mathcal{C}^0_{\mathcal{W}}(U,\mathrm{L}^\ell(X,Y)) : \gamma \mapsto (D^{(\ell)}\gamma)_{\ell\in\mathbb{N},\ell\leq k}$$

is a topological embedding.

Proof. Both assertions are clear from the definition of $\mathcal{C}^k_{\mathcal{W}}(U,Y)$ and $\mathcal{C}^0_{\mathcal{W}}(U,\mathrm{L}^\ell(X,Y))$. ∎

The next lemma states a relation between the higher order derivatives of γ and those of $D\gamma$.

LEMMA 3.2.2. *Let X and Y be normed spaces, $U \subseteq X$ an open nonempty set, $k \in \mathbb{N}$ and $\gamma \in \mathcal{FC}^{k+1}(U,Y)$. Then*

$$\|D^{(\ell)}D\gamma(x)\|_{\mathrm{op}} = \|D^{(\ell+1)}\gamma(x)\|_{\mathrm{op}} \tag{3.2.2.1}$$

for each $x \in U$ and $\ell < k$. In particular, for each map $f \in \overline{\mathbb{R}}^U$, $\ell < k$ and subset $V \subseteq U$,

$$\|\gamma|_V\|_{f,\ell+1} = \|(D\gamma)|_V\|_{f,\ell}. \tag{3.2.2.2}$$

Proof. In Lemma A.2.14 the identity

$$D^{(\ell+1)}\gamma = \mathcal{E}_{\ell,1} \circ (D^{(\ell)} D\gamma)$$

is proved, where $\mathcal{E}_{\ell,1} : \mathrm{L}(X, \mathrm{L}^\ell(X,Y)) \to \mathrm{L}^{\ell+1}(X,Y)$ is an isometric isomorphism (see Lemma A.2.5). The asserted identities follow immediately. ∎

We can state the main tool for the treatment of weighted function spaces $\mathcal{C}^k_{\mathcal{W}}(U,Y)$ with $k \geq 1$. It is useful because it allows induction arguments of the following kind: Suppose we want to show that $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,Y)$. First, we have to show that $\gamma \in \mathcal{C}^0_{\mathcal{W}}(U,Y)$. Then, we suppose $\gamma \in \mathcal{C}^\ell_{\mathcal{W}}(U,Y)$ and show that $D\gamma$ in $\mathcal{C}^\ell_{\mathcal{W}}(U, \mathrm{L}(X,Y))$ by expressing it in terms of γ. This finishes the induction argument.

PROPOSITION 3.2.3 (Reduction to lower order). *Let X and Y be normed spaces, $U \subseteq X$ an open nonempty set, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$, $k \in \mathbb{N}$ and $\gamma \in \mathcal{FC}^1(U,Y)$. Then*

$$\gamma \in \mathcal{C}^{k+1}_{\mathcal{W}}(U,Y) \Leftrightarrow (D\gamma, \gamma) \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Y)) \times \mathcal{C}^0_{\mathcal{W}}(U,Y).$$

Moreover, the map

$$\mathcal{C}^{k+1}_{\mathcal{W}}(U,Y) \to \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Y)) \times \mathcal{C}^0_{\mathcal{W}}(U,Y) : \gamma \mapsto (D\gamma, \gamma)$$

is a topological embedding. In particular, the map

$$D : \mathcal{C}^{k+1}_{\mathcal{W}}(U,Y) \to \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Y))$$

is continuous.

Proof. The first relation follows immediately from the definition of $\mathcal{FC}^k$-maps and identity (3.2.2.2). This identity, together with Lemma 3.2.1, also implies that $\mathcal{C}^{k+1}_{\mathcal{W}}(U,Y)$ is endowed with the initial topology with respect to

$$D : \mathcal{C}^{k+1}_{\mathcal{W}}(U,Y) \to \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Y))$$

and the inclusion map

$$\mathcal{C}^{k+1}_{\mathcal{W}}(U,Y) \to \mathcal{C}^0_{\mathcal{W}}(U,Y).$$

This proves the second assertion. ∎

The same argument can be made for the vanishing weighted functions.

COROLLARY 3.2.4. *Let X and Y be normed spaces, $U \subseteq X$ an open nonempty set, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$, $k \in \mathbb{N}$ and $\gamma \in \mathcal{FC}^1(U,Y)$. Then*

$$\gamma \in \mathcal{C}^{k+1}_{\mathcal{W}}(U,Y)^o \Leftrightarrow (D\gamma, \gamma) \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Y))^o \times \mathcal{C}^0_{\mathcal{W}}(U,Y)^o.$$

Proof. This is also an immediate consequence of Proposition 3.2.3 and (3.2.2.2). ∎

3.2.2. Projective limits and the topology of $\mathcal{C}^\infty_{\mathcal{W}}(U,Y)$. Sometimes it is useful that $\mathcal{C}^k_{\mathcal{W}}(U,Y)$ can be written as the projective limit of larger weighted functions spaces.

PROPOSITION 3.2.5. *Let X and Y be normed spaces, $U \subseteq X$ an open nonempty set, $k \in \overline{\mathbb{N}}$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ a nonempty set. Further let $(\mathcal{F}_i)_{i \in I}$ be a directed family of nonempty* *subsets of* $\mathcal{W}$ *such that* $\bigcup_{i\in I}\mathcal{F}_i=\mathcal{W}$. *Consider* $I\times\{\ell\in\mathbb{N}:\ell\leq k\}$ *as a directed set via*

$$((i_1,\ell_1)\leq(i_2,\ell_2)) \;\Leftrightarrow\; i_1\leq i_2 \text{ and } \ell_1\leq\ell_2.$$

Then $\mathcal{C}^k_{\mathcal{W}}(U,Y)$ *is the projective limit of*

$$\{\mathcal{C}^{\ell}_{\mathcal{F}_i}(U,Y):\ell\in\mathbb{N},\,\ell\leq k,\,i\in I\}$$

in the category of topological (vector) spaces, with the inclusion maps as morphisms.

Proof. Since

$$\mathcal{C}^k_{\mathcal{W}}(U,Y)=\bigcap_{\substack{i\in I\\ \ell\in\mathbb{N},\,\ell\leq k}}\mathcal{C}^{\ell}_{\mathcal{F}_i}(U,Y),$$

the set $\mathcal{C}^k_{\mathcal{W}}(U,Y)$ is the desired projective limit as a set, and hence also as a vector space. Moreover, it is well known that the initial topology with respect to the limit maps $\mathcal{C}^k_{\mathcal{W}}(U,Y)\to\mathcal{C}^{\ell}_{\mathcal{F}_i}(U,Y)$ makes $\mathcal{C}^k_{\mathcal{W}}(U,Y)$ the projective limit as a topological space, and also as a topological vector space. But it is clear from the definition that the given topology on $\mathcal{C}^k_{\mathcal{W}}(U,Y)$ coincides with this initial topology. ■

Corollary 3.2.6. *Let* X *and* Y *be normed spaces,* $U\subseteq X$ *an open nonempty set and* $\mathcal{W}\subseteq\overline{\mathbb{R}}^U$. *The space* $\mathcal{C}^{\infty}_{\mathcal{W}}(U,Y)$ *is endowed with the initial topology with respect to the inclusion maps*

$$\mathcal{C}^{\infty}_{\mathcal{W}}(U,Y)\to\mathcal{C}^k_{\mathcal{W}}(U,Y).$$

Moreover, $\mathcal{C}^{\infty}_{\mathcal{W}}(U,Y)$ *is the projective limit of the spaces* $\mathcal{C}^k_{\mathcal{W}}(U,Y)$ *with* $k\in\mathbb{N}$, *together with the inclusion maps.*

Proof. This is an immediate consequence of Proposition 3.2.5. ■

3.2.3. A completeness criterion. We describe a condition on $\mathcal{W}$ that ensures that $\mathcal{C}^k_{\mathcal{W}}(U,Y)$ is complete, provided that Y is a Banach space. The proof uses Proposition 3.2.3. We start with the following observation concerning the continuity of evaluation maps.

Lemma 3.2.7. *Let* X *and* Y *be normed spaces,* $U\subseteq X$ *an open nonempty set,* $k\in\overline{\mathbb{N}}$ *and* $x\in U$. *Suppose that* $\mathcal{W}\subseteq\overline{\mathbb{R}}^U$ *contains a weight* $f_x\in\mathcal{W}$ *with* $f_x(x)\neq 0$. *Then*

$$\mathrm{ev}_x:\mathcal{C}^k_{\mathcal{W}}(U,Y)\to Y:\gamma\mapsto\gamma(x)$$

is a continuous linear map.

Proof. If there exists $f\in\mathcal{W}$ with $f(x)\in\{-\infty,\infty\}$, then for each $\gamma\in\mathcal{C}^k_{\mathcal{W}}(U,Y)$,

$$\|\mathrm{ev}_x(\gamma)\|=0\leq\|\gamma\|_{f,0}.$$

Otherwise, for each $f\in\mathcal{W}$ with $f(x)\neq 0$ and $\gamma\in\mathcal{C}^k_{\mathcal{W}}(U,Y)$, we have

$$\|\mathrm{ev}_x(\gamma)\|=\|\gamma(x)\|\leq\frac{1}{|f(x)|}\|\gamma\|_{f,0}.$$

In both cases, these estimates ensure the continuity of ev_x. ■

We examine when the image of $\mathcal{C}^{k+1}_{\mathcal{W}}(U,Y)$ under the embedding described in Proposition 3.2.3 is closed.

Proposition 3.2.8. *Let X and Y be normed spaces, $U \subseteq X$ an open nonempty set and $k \in \mathbb{N}$. Further let $\mathcal{W} \subseteq \overline{\mathbb{R}}$ such that for each compact line segment $S \subseteq U$ there exists $f_S \in \mathcal{W}$ with $\inf_{x\in S}|f_S(x)| > 0$. Then the image of $\mathcal{C}^{k+1}_{\mathcal{W}}(U,Y)$ under the embedding described in Proposition 3.2.3 is closed.*

Proof. Let $(\gamma_i)_{i\in I}$ be a net in $\mathcal{C}^{k+1}_{\mathcal{W}}(U,Y)$ such that $(\gamma_i)_{i\in I}$ converges to γ in $\mathcal{C}^{0}_{\mathcal{W}}(U,Y)$ and the net $(D\gamma_i)_{i\in I}$ converges to Γ in $\mathcal{C}^{k}_{\mathcal{W}}(U,\mathrm{L}(X,Y))$. We have to show that $\gamma \in \mathcal{C}^{k+1}_{\mathcal{W}}(U,Y)$ with $D\gamma = \Gamma$.

To this end, consider $x \in U$, $h \in X$ and $t \in \mathbb{R}^*$ such that the line segment $S_{x,t,h} := \{x + sth : s \in [0,1]\}$ is contained in U. Since evaluation maps and weak integration are continuous (see Lemmas 3.2.7 and A.1.7, respectively) and the hypothesis on $\mathcal{W}$ implies that $(D\gamma_i)_{i\in I}$ converges to Γ uniformly on $S_{x,t,h}$, we derive

$$\begin{aligned}\frac{\gamma(x+th)-\gamma(x)}{t} &= \lim_{i\in I}\frac{\gamma_i(x+th)-\gamma_i(x)}{t}\\ &= \lim_{i\in I}\frac{\int_0^1 D\gamma_i(x+sth)\cdot(th)\,ds}{t} = \int_0^1 \Gamma(x+sth)\cdot h\,ds.\end{aligned}$$

Since Γ is continuous, we can apply Proposition A.1.8 and get

$$\lim_{t\to 0}\frac{\gamma(x+th)-\gamma(x)}{t} = \int_0^1 \lim_{t\to 0}(\Gamma(x+sth)\cdot h)\,ds = \Gamma(x)\cdot h.$$

Because Γ and the evaluation of linear maps are continuous (Lemma A.2.3), γ is a $\mathcal{C}^1$-map with $d\gamma(x;\cdot) = \Gamma(x)$, and another application of the continuity of Γ shows that $\gamma \in \mathcal{FC}^1(U,Y)$ with $D\gamma = \Gamma$. Finally we conclude from Proposition 3.2.3 that $\gamma \in \mathcal{C}^{k+1}_{\mathcal{W}}(U,Y)$. ∎

The last proposition allows us to deduce the completeness of $\mathcal{C}^{k}_{\mathcal{W}}(U,Y)$ from that of $\mathcal{C}^{0}_{\mathcal{W}}(U,Y)$.

Corollary 3.2.9. *In the situation of Proposiion 3.2.8, assume that $\mathcal{C}^{0}_{\mathcal{W}}(U,Y)$ is complete for each Banach space Y. Then also $\mathcal{C}^{k}_{\mathcal{W}}(U,Y)$ is complete, for each $k \in \overline{\mathbb{N}}$.*

Proof. The proof for $k < \infty$ is by induction.

$k = 0$: This holds by our hypothesis.

$k \to k+1$: We conclude from Proposition 3.2.8 that $\mathcal{C}^{k+1}_{\mathcal{W}}(U,Y)$ is isomorphic to a closed vector subspace of $\mathcal{C}^{k}_{\mathcal{W}}(U,\mathrm{L}(X,Y)) \times \mathcal{C}^{0}_{\mathcal{W}}(U,Y)$, which is complete by induction hypothesis.

$k = \infty$: This follows from Corollary 3.2.6 and the fact that $\mathcal{C}^{k}_{\mathcal{W}}(U,Y)$ is complete for all $k \in \mathbb{N}$ because projective limits of complete topological vector spaces are complete. ∎

We give a sufficient condition for the completeness of $\mathcal{C}^{0}_{\mathcal{W}}(U,Y)$.

Proposition 3.2.10. *Let X be a normed space, $U \subseteq X$ an open nonempty set and Y a Banach space. Further let $\mathcal{W} \subseteq \overline{\mathbb{R}}$ such that for each compact set $K \subseteq U$ there exists $f_K \in \mathcal{W}$ with $\inf_{x\in K}|f_K(x)| > 0$. Then $\mathcal{C}^{0}_{\mathcal{W}}(U,Y)$ is complete.*

Proof. Let $(\gamma_i)_{i\in I}$ be a Cauchy net in $\mathcal{C}^{0}_{\mathcal{W}}(U,Y)$. The hypothesis on $\mathcal{W}$ implies that the topology of $\mathcal{C}^{0}_{\mathcal{W}}(U,Y)$ is finer than the topology of uniform convergence on compact sets. Hence we deduce from the completeness of Y that there exists a map $\gamma : U \to Y$

to which $(\gamma_i)_{i\in I}$ converges uniformly on each compact subset of U; and since each γ_i is continuous, the restriction of γ to each compact subset is continuous. Hence γ is sequentially continuous since the union of a convergent sequence with its limit is compact; but U is first countable, so γ is continuous.

It remains to show that $\gamma \in \mathcal{C}^0_{\mathcal{W}}(U,Y)$ and $(\gamma_i)_{i\in I}$ converges to γ in $\mathcal{C}^0_{\mathcal{W}}(U,Y)$. To see this, let $f \in \mathcal{W}$ and $\varepsilon > 0$. Then there exists an $\ell \in I$ such that

$$(\forall i, j \geq \ell)\ \|\gamma_i \ - \gamma_j\|_{f,0} \leq \varepsilon,$$

which is equivalent to

$$(\forall x \in U, i, j \geq \ell)\ |f(x)|\,\|\gamma_i(x) - \gamma_j(x)\| \leq \varepsilon.$$

If we fix i in this estimate and let $\gamma_j(x)$ pass to its limit, then we get

$$(\forall x \in U, i \geq \ell)\ |f(x)|\,\|\gamma_i(x) - \gamma(x)\| \leq \varepsilon. \tag{$*$}$$

The triangle inequality now shows that

$$(\forall x \in U)\ |f(x)|\,\|\gamma(x)\| \leq \varepsilon + |f(x)|\,\|\gamma_\ell(x)\| \leq \varepsilon + \|\gamma_\ell\|_{f,0},$$

so $\gamma \in \mathcal{C}^0_{\mathcal{W}}(U,Y)$. Finally we conclude from $(*)$ that $\|\gamma_i - \gamma\|_{f,0} \leq \varepsilon$ for all $i \geq \ell$, so $(\gamma_i)_{i\in I}$ converges to γ in $\mathcal{C}^0_{\mathcal{W}}(U,Y)$. ■

Finally, we state the derived criterion in a citable form.

Corollary 3.2.11. *Let X be a normed space, $U \subseteq X$ an open nonempty set, Y a Banach space and $k \in \overline{\mathbb{N}}$. Further let $\mathcal{W} \subseteq \overline{\mathbb{R}}$ such that for each compact set $K \subseteq U$ there exists $f_K \in \mathcal{W}$ with $\inf_{x\in K}|f_K(x)| > 0$. Then $\mathcal{C}^k_{\mathcal{W}}(U,Y)$ is complete.*

Proof. This is an immediate consequence of Corollary 3.2.9 and Proposition 3.2.10. ■

Corollary 3.2.12. *Let X be a normed space, $U \subseteq X$ an open nonempty set, Y a Banach space and $k \in \overline{\mathbb{N}}$. Further let $\mathcal{W}$ be a set of weights with $1_U \in \mathcal{W}$. Then $\mathcal{C}^k_{\mathcal{W}}(U,Y)$ is complete; in particular, $\mathcal{BC}^k(U,Y)$ is complete.*

3.2.3.1. *An integrability criterion.* The given completeness criterion entails a criterion for the existence of the weak integral of a continuous curve to a space $\mathcal{C}^k_{\mathcal{W}}(U,Y)$ where Y is not necessarily complete.

Lemma 3.2.13. *Let X and Y be normed spaces, $U \subseteq X$ a nonempty open set, $k \in \overline{\mathbb{N}}$, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$, $\Gamma : [a,b] \to \mathcal{C}^k_{\mathcal{W}}(U,Y)$ a map and $R \in \mathcal{C}^k_{\mathcal{W}}(U,Y)$.*

(a) *Assume that Γ is weakly integrable and that for each $x \in U$ there exists $f_x \in \mathcal{W}$ with $f_x(x) \neq 0$. Then*

$$\int_a^b \Gamma(s)\,ds = R \ \Leftrightarrow\ (\forall x \in U)\ \operatorname{ev}_x\left(\int_a^b \Gamma(s)\,ds\right) = R(x),$$

and for each $x \in U$ we have

$$\operatorname{ev}_x\left(\int_a^b \Gamma(s)\,ds\right) = \int_a^b \operatorname{ev}_x(\Gamma(s))\,ds. \tag{$*$}$$

(b) *Assume that for each compact set $K \subseteq U$, there exists a weight $f_K \in \mathcal{W}$ with $\inf_{x \in K} |f_K(x)| > 0$, that Γ is continuous and*

$$\int_a^b \mathrm{ev}_x(\Gamma(s))\, ds = \mathrm{ev}_x(R) \tag{$**$}$$

for all $x \in U$. Then Γ is weakly integrable with

$$\int_a^b \Gamma(s)\, ds = R.$$

Proof. (a) Since $\{\mathrm{ev}_x : x \in U\}$ separates the points of $\mathcal{C}^k_{\mathcal{W}}(U, Y)$, the stated equivalence is obvious. Further, we proved in Lemma 3.2.7 that the condition on $\mathcal{W}$ implies that $\{\mathrm{ev}_x : x \in U\} \subseteq \mathrm{L}(\mathcal{C}^k_{\mathcal{W}}(U, Y), Y)$, so $(*)$ follows from Lemma A.1.4.

(b) Let $\widetilde{Y}$ be the completion of Y. Then $\mathcal{C}^k_{\mathcal{W}}(U, Y) \subseteq \mathcal{C}^k_{\mathcal{W}}(U, \widetilde{Y})$, and we denote the inclusion map by ι. It is obvious that ι is a topological embedding. In the following, we denote the evaluation of $\mathcal{C}^k_{\mathcal{W}}(U, \widetilde{Y})$ at $x \in U$ with $\widetilde{\mathrm{ev}}_x$.

Since we proved in Corollary 3.2.11 that the condition on $\mathcal{W}$ ensures that $\mathcal{C}^k_{\mathcal{W}}(U, \widetilde{Y})$ is complete, $\iota \circ \Gamma$ is weakly integrable. Since $\widetilde{\mathrm{ev}}_x \circ \iota = \mathrm{ev}_x$ for $x \in U$, we conclude from (a) (using $(*)$ and $(**)$) that

$$\int_a^b (\iota \circ \Gamma)(s)\, ds = \iota(R).$$

This identity ensures the integrability of Γ: By the Hahn–Banach theorem, each $\lambda \in \mathcal{C}^k_{\mathcal{W}}(U, Y)'$ extends to a $\widetilde{\lambda} \in \mathcal{C}^k_{\mathcal{W}}(U, \widetilde{Y})'$, that is $\widetilde{\lambda} \circ \iota = \lambda$. Hence

$$\int_a^b (\lambda \circ \Gamma)(s)\, ds = \int_a^b (\widetilde{\lambda} \circ \iota \circ \Gamma)(s)\, ds = \widetilde{\lambda}(\iota(R)) = \lambda(R). \quad \blacksquare$$

3.3. Composition on weighted functions and superposition operators. In this subsection we discuss the behaviour of weighted functions when they are composed with certain functions. In particular, we show that a continuous multilinear or a suitable analytic map induce a superposition operator between weighted function spaces. Moreover, we examine the composition between bounded functions and between bounded functions mapping 0 to 0 and weighted functions.

3.3.1. Composition with a multilinear map. We prove that a continuous multilinear map from a normed space $Y_1 \times \cdots \times Y_m$ to a normed space Z induces a continuous multilinear map from $\mathcal{C}^k_{\mathcal{W}}(U, Y_1) \times \cdots \times \mathcal{C}^k_{\mathcal{W}}(U, Y_m)$ to $\mathcal{C}^k_{\mathcal{W}}(U, Z)$. As a preparation, we calculate the differential of a composition of a multilinear map and other differentiable maps. The following definition is quite useful to do that.

DEFINITION 3.3.1. Let $Y_1, \ldots, Y_m$, X and Z be normed spaces and $b : Y_1 \times \cdots \times Y_m \to Z$ a continuous m-linear map. For each $i \in \{1, \ldots, m\}$ we define the m-linear continuous map

$$b^{(i)} : Y_1 \times \cdots \times Y_{i-1} \times \mathrm{L}(X, Y_i) \times Y_{i+1} \times \cdots \times Y_m \to \mathrm{L}(X, Z),$$
$$(y_1, \ldots, y_{i-1}, T, y_{i+1}, \ldots, y_m) \mapsto (h \mapsto b(y_1, \ldots, y_{i-1}, T \cdot h, y_{i+1}, \ldots, y_m)).$$

LEMMA 3.3.2. *Let $Y_1, \ldots, Y_m$ and Z be normed spaces, U be an open nonempty subset of the normed space X and $k \in \overline{\mathbb{N}}$. Further let $b : Y_1 \times \cdots \times Y_m \to Z$ be a continuous*

m-linear map and $\gamma_1 \in \mathcal{FC}^k(U,Y_1),\ldots,\gamma_m \in \mathcal{FC}^k(U,Y_m)$. Then

$$b \circ (\gamma_1,\ldots,\gamma_m) \in \mathcal{FC}^k(U,Z)$$

with

$$D(b \circ (\gamma_1,\ldots,\gamma_m)) = \sum_{i=1}^m b^{(i)} \circ (\gamma_1,\ldots,\gamma_{i-1}, D\gamma_i, \gamma_{i+1},\ldots,\gamma_m). \tag{3.3.2.1}$$

Proof. To calculate the derivative of $b \circ (\gamma_1,\ldots,\gamma_m)$, we apply the chain rule to get

$$\begin{aligned} D(b \circ (\gamma_1,\ldots,\gamma_m))(x)\cdot h &= \sum_{i=1}^m b(\gamma_1(x),\ldots,\gamma_{i-1}(x), D\gamma_i(x)\cdot h, \gamma_{i+1}(x),\ldots,\gamma_m(x)) \\ &= \sum_{i=1}^m b^{(i)}(\gamma_1(x),\ldots,\gamma_{i-1}(x), D\gamma_i(x), \gamma_{i+1}(x),\ldots,\gamma_m(x))\cdot h. \quad \blacksquare \end{aligned}$$

We are ready to prove the result about the superposition.

PROPOSITION 3.3.3. *Let U be an open nonempty subset of the normed space X. Let $Y_1,\ldots,Y_m$ be normed spaces, $k \in \overline{\mathbb{N}}$ and $\mathcal{W},\mathcal{W}_1,\ldots,\mathcal{W}_m \subseteq \overline{\mathbb{R}}^U$ nonempty sets such that*

$$(\forall f \in \mathcal{W})\ (\exists g_{f,1} \in \mathcal{W}_1,\ldots,g_{f,m} \in \mathcal{W}_m)\ |f| \le |g_{f,1}|\cdots|g_{f,m}|.$$

Further let Z be another normed space and $b : Y_1 \times\cdots\times Y_m \to Z$ a continuous m-linear map. Then

$$b \circ (\gamma_1,\ldots,\gamma_m) \in \mathcal{C}^k_{\mathcal{W}}(U,Z)$$

for all $\gamma_1 \in \mathcal{C}^k_{\mathcal{W}_1}(U,Y_1),\ldots,\gamma_m \in \mathcal{C}^k_{\mathcal{W}_m}(U,Y_m)$. The map

$$M_k(b) : \mathcal{C}^k_{\mathcal{W}_1}(U,Y_1) \times\cdots\times \mathcal{C}^k_{\mathcal{W}_m}(U,Y_m) \to \mathcal{C}^k_{\mathcal{W}}(U,Z) : (\gamma_1,\ldots,\gamma_m) \mapsto b \circ (\gamma_1,\ldots,\gamma_m)$$

is m-linear and continuous.

Proof. For $k < \infty$, we proceed by induction on k.

$k = 0$: For $f \in \mathcal{W}$, $x \in U$ and $\gamma_1 \in \mathcal{C}^k_{\mathcal{W}_1}(U,Y_1),\ldots,\gamma_m \in \mathcal{C}^k_{\mathcal{W}_m}(U,Y_m)$ we compute

$$|f(x)|\,\|b \circ (\gamma_1,\ldots,\gamma_m)(x)\| \le \|b\|_{\mathrm{op}} \prod_{i=1}^m |g_{f,i}|\|\gamma_i(x)\| \le \|b\|_{\mathrm{op}} \prod_{i=1}^m \|\gamma_i\|_{g_{f,i},0},$$

so $b \circ (\gamma_1,\ldots,\gamma_m) \in \mathcal{C}^0_{\mathcal{W}}(U,Z)$. From this estimate we also conclude

$$\|M_0(b)(\gamma_1,\ldots,\gamma_m)\|_{f,0} = \|b \circ (\gamma_1,\ldots,\gamma_m)\|_{f,0} \le \|b\|_{\mathrm{op}} \prod_{i=1}^m \|\gamma_i\|_{g_{f,i},0},$$

so $M_0(b)$ is continuous at 0. Since the m-linearity of $M_0(b)$ is obvious, this implies the continuity of $M_0(b)$ (see [Bou87, Chapter I, §1, no. 6]).

$k \to k+1$: From Proposition 3.2.3 (together with the induction base) we know that for $\gamma_1 \in \mathcal{C}^{k+1}_{\mathcal{W}_1}(U,Y_1),\ldots,\gamma_m \in \mathcal{C}^{k+1}_{\mathcal{W}_m}(U,Y_m)$,

$$b \circ (\gamma_1,\ldots,\gamma_m) \in \mathcal{C}^{k+1}_{\mathcal{W}}(U,Z) \iff D(b \circ (\gamma_1,\ldots,\gamma_m)) \in \mathcal{C}^k_{\mathcal{W}}(U,\mathrm{L}(X,Z))$$

and that $M_{k+1}(b)$ is continuous if

$$D \circ M_{k+1}(b) : \mathcal{C}^{k+1}_{\mathcal{W}_1}(U,Y_1) \times\cdots\times \mathcal{C}^{k+1}_{\mathcal{W}_m}(U,Y_m) \to \mathcal{C}^k_{\mathcal{W}}(U,\mathrm{L}(X,Z))$$

is. We know from (3.3.2.1) in Lemma 3.3.2 that

$$D(b \circ (\gamma_1, \ldots, \gamma_m)) = \sum_{i=1}^{m} b^{(i)} \circ (\gamma_1, \ldots, \gamma_{i-1}, D\gamma_i, \gamma_{i+1}, \ldots, \gamma_m).$$

By the inductive hypothesis,

$$b^{(i)} \circ (\gamma_1, \ldots, \gamma_{i-1}, D\gamma_i, \gamma_{i+1}, \ldots, \gamma_m) \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Z))$$

and hence

$$D(b \circ (\gamma_1, \ldots, \gamma_m)) \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Z)).$$

Since

$$M_k(b^{(i)}) : \mathcal{C}^k_{\mathcal{W}_1}(U, Y_1) \times \cdots \times \mathcal{C}^k_{\mathcal{W}_i}(U, \mathrm{L}(X, Y_i)) \times \cdots \times \mathcal{C}^k_{\mathcal{W}_m}(U, Y_m) \to \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Z))$$

is continuous by the inductive hypothesis, it follows that $D \circ M_{k+1}(b)$ is continuous as

$$(D \circ M_{k+1}(b))(\gamma_1, \ldots, \gamma_m) = \sum_{i=1}^{m} M_k(b^{(i)})(\gamma_1, \ldots, \gamma_{i-1}, D\gamma_i, \gamma_{i+1}, \ldots, \gamma_m).$$

Furthermore, $M_{k+1}(b)$ is obviously m-linear, so the induction step is finished.

$k = \infty$: From the assertions already established, we derive the commutative diagram

$$\begin{tikzcd}
\mathcal{C}^\infty_{\mathcal{W}_1}(U, Y_1) \times \cdots \times \mathcal{C}^\infty_{\mathcal{W}_m}(U, Y_m) \arrow[r, "M_\infty(b)"] \arrow[d, hook] & \mathcal{C}^\infty_{\mathcal{W}}(U, Z) \arrow[d, hook] \\
\mathcal{C}^n_{\mathcal{W}_1}(U, Y_1) \times \cdots \times \mathcal{C}^n_{\mathcal{W}_m}(U, Y_m) \arrow[r, "M_n(b)"] & \mathcal{C}^n_{\mathcal{W}}(U, Z)
\end{tikzcd}$$

for each $n \in \mathbb{N}$, where the vertical arrows represent inclusion maps. Using Corollary 3.2.6 we easily deduce the continuity of $M_\infty(b)$ from that of $M_n(b)$. ■

We prove an analogous result for decreasing functions.

Corollary 3.3.4. *Let $Y_1, \ldots, Y_m$ be normed spaces, U an open nonempty subset of the normed space X, $k \in \overline{\mathbb{N}}$ and $\mathcal{W}, \mathcal{W}_1, \ldots, \mathcal{W}_m \subseteq \overline{\mathbb{R}}^U$ nonempty such that*

$$(\forall f \in \mathcal{W})\ (\exists g_{f,1} \in \mathcal{W}_1, \ldots, g_{f,m} \in \mathcal{W}_m)\ |f| \leq |g_{f,1}| \cdots |g_{f,m}|.$$

Further let Z be another normed space, $b : Y_1 \times \cdots \times Y_m \to Z$ a continuous m-linear map and $j \in \{1, \ldots, m\}$. Then

$$b \circ (\gamma_1, \ldots, \gamma_j, \ldots, \gamma_m) \in \mathcal{C}^k_{\mathcal{W}}(U, Z)^o \tag{†}$$

for all $\gamma_i \in \mathcal{C}^k_{\mathcal{W}_i}(U, Y_i)$ $(i \neq j)$ and $\gamma_j \in \mathcal{C}^k_{\mathcal{W}_j}(U, Y_j)^o$. Moreover, the map

$$M_k(b) : \mathcal{C}^k_{\mathcal{W}_1}(U, Y_1) \times \cdots \times \mathcal{C}^k_{\mathcal{W}_j}(U, Y_j)^o \times \cdots \times \mathcal{C}^k_{\mathcal{W}_m}(U, Y_m) \to \mathcal{C}^k_{\mathcal{W}}(U, Z)^o,$$
$$(\gamma_1, \ldots, \gamma_j, \ldots, \gamma_m) \mapsto b \circ (\gamma_1, \ldots, \gamma_j, \ldots, \gamma_m),$$

is m-linear and continuous.

Proof. Using Proposition 3.3.3 and Lemma 3.1.6, we only have to prove that (†) holds. This is done by induction on k (which we may assume finite).

$k=0$: For $f\in\mathcal{W}$, $x\in U$ and $\gamma_1\in\mathcal{C}^0_{\mathcal{W}_1}(U,Y_1),\dots,\gamma_j\in\mathcal{C}^0_{\mathcal{W}_j}(U,Y_j)^o,\dots,\gamma_m\in\mathcal{C}^0_{\mathcal{W}_m}(U,Y_m)$ we compute

$$\begin{aligned}&|f(x)|\,\|b\circ(\gamma_1,\dots,\gamma_j,\dots,\gamma_m)(x)\|\\&\qquad\le\|b\|_{\mathrm{op}}\prod_{i=1}^m|g_{f,i}(x)|\|\gamma_i(x)\|\le\Big(\|b\|_{\mathrm{op}}\prod_{i\neq j}\|\gamma_i\|_{g_{f,i},0}\Big)|g_{f,j}(x)|\,\|\gamma_j(x)\|.\end{aligned}$$

From this estimate we easily see that $b\circ(\gamma_1,\dots,\gamma_j,\dots,\gamma_m)\in\mathcal{C}^0_{\mathcal{W}_j}(U,Z)^o$.

$k\to k+1$: From Corollary 3.2.4 (together with the induction base) we know that for $\gamma_1\in\mathcal{C}^{k+1}_{\mathcal{W}_1}(U,Y_1),\dots,\gamma_j\in\mathcal{C}^{k+1}_{\mathcal{W}_j}(U,Y_j)^o,\dots,\gamma_m\in\mathcal{C}^{k+1}_{\mathcal{W}_m}(U,Y_m)$

$$b\circ(\gamma_1,\dots,\gamma_j,\dots,\gamma_m)\in\mathcal{C}^{k+1}_{\mathcal{W}}(U,Z)^o\Leftrightarrow D(b\circ(\gamma_1,\dots,\gamma_j,\dots,\gamma_m))\in\mathcal{C}^k_{\mathcal{W}}(U,\mathrm{L}(X,Z))^o.$$

We know from (3.3.2.1) in Lemma 3.3.2 that

$$\begin{aligned}D(b\circ(\gamma_1,\dots,\gamma_j,\dots,\gamma_m))=&\sum_{\substack{i=1\\i\neq j}}^m b^{(i)}\circ(\gamma_1,\dots,\gamma_j,\dots,\gamma_{i-1},D\gamma_i,\gamma_{i+1},\dots,\gamma_m)\\&+b^{(j)}\circ(\gamma_1,\dots,\gamma_{j-1},D\gamma_j,\gamma_{j+1},\dots,\gamma_m).\end{aligned}$$

Because $\gamma_j\in\mathcal{C}^k_{\mathcal{W}_j}(U,Y_j)^o$ and $D\gamma_j\in\mathcal{C}^k_{\mathcal{W}_j}(U,\mathrm{L}(X,Y_j))^o$, we can apply the inductive hypothesis to all $b^{(i)}$ and the $\mathcal{C}^k$-maps $\gamma_1,\dots,\gamma_m$ and $D\gamma_1,\dots,D\gamma_m$ to see that this is an element of $\mathcal{C}^k_{\mathcal{W}}(U,\mathrm{L}(X,Z))^o$. ■

We list some applications of Proposition 3.3.3. In the following corollaries, $k\in\overline{\mathbb{N}}$, U is an open nonempty subset of the normed space X and $\mathcal{W}\subseteq\overline{\mathbb{R}}^U$ always contains the constant map 1_U.

Corollary 3.3.5. *Let A be a normed algebra with the continuous multiplication $*$. Then $\mathcal{C}^k_{\mathcal{W}}(U,A)$ is an algebra with the continuous multiplication*

$$\begin{gathered}M(*):\mathcal{C}^k_{\mathcal{W}}(U,A)\times\mathcal{C}^k_{\mathcal{W}}(U,A)\to\mathcal{C}^k_{\mathcal{W}}(U,A),\\M(*)(\gamma,\eta)(x)=\gamma(x)*\eta(x).\end{gathered}$$

We shall often write $*$ instead of $M(*)$.

Corollary 3.3.6. *If E, F and G are normed spaces, then the composition*

$$\cdot:\mathrm{L}(F,G)\times\mathrm{L}(E,F)\to\mathrm{L}(E,G)$$

is bilinear and continuous and therefore induces the continuous bilinear maps

$$\begin{gathered}M(\cdot):\mathcal{C}^k_{\mathcal{W}}(U,\mathrm{L}(F,G))\times\mathcal{C}^k_{\mathcal{W}}(U,\mathrm{L}(E,F))\to\mathcal{C}^k_{\mathcal{W}}(U,\mathrm{L}(E,G)),\\M(\cdot)(\gamma,\eta)(x)=\gamma(x)\cdot\eta(x)\end{gathered}$$

and

$$\begin{gathered}M_{\mathcal{BC}}(\cdot):\mathcal{C}^k_{\mathcal{W}}(U,\mathrm{L}(F,G))\times\mathcal{BC}^k(U,\mathrm{L}(E,F))\to\mathcal{C}^k_{\mathcal{W}}(U,\mathrm{L}(E,G)),\\M_{\mathcal{BC}}(\cdot)(\gamma,\eta)(x)=\gamma(x)\cdot\eta(x).\end{gathered}$$

We shall often denote $M(\cdot)$ just by $\cdot$.

COROLLARY 3.3.7. *Let E and F be normed spaces. Then the evaluation of linear maps*

$$\cdot : \mathrm{L}(E,F) \times E \to F : (T,w) \mapsto T \cdot w$$

is bilinear und continuous (see Lemma A.2.3) and hence induces the continuous bilinear map

$$M(\cdot) : \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(E,F)) \times \mathcal{C}^k_{\mathcal{W}}(U,E) \to \mathcal{C}^k_{\mathcal{W}}(U,F),$$
$$M(\cdot)(\Gamma,\eta)(x) = \Gamma(x) \cdot \eta(x).$$

Instead of $M(\cdot)$ we will often write $\cdot$.

3.3.2. Composition of weighted functions with bounded functions. We explore the composition between spaces of bounded functions and spaces of weighted functions. A case of particular interest is the composition between certain subsets of the spaces $\mathcal{BC}^k(U,Y)$.

3.3.2.1. *Composition of bounded functions.* We discuss under which conditions the composition is continuous or differentiable.

LEMMA 3.3.8. *Let X, Y and Z be normed spaces, $U \subseteq X$ and $V \subseteq Y$ open nonempty subsets and $k \in \overline{\mathbb{N}}$. Then for $\gamma \in \mathcal{BC}^{k+1}(V,Z)$ and $\eta \in \mathcal{BC}^{\partial,k}(U,V)$,*

$$\gamma \circ \eta \in \mathcal{BC}^k(U,Z),$$

and the map

$$\mathcal{BC}^{k+1}(V,Z) \times \mathcal{BC}^{\partial,k}(U,V) \to \mathcal{BC}^k(U,Z) : (\gamma,\eta) \mapsto \gamma \circ \eta \tag{$*$}$$

is continuous.

Proof. For $k < \infty$ this is proved by induction.

$k = 0$: Obviously

$$\mathcal{BC}^1(V,Z) \circ \mathcal{BC}^{\partial,0}(U,V) \subseteq \mathcal{BC}^0(U,Z),$$

so it remains to show that the composition is continuous. To this end, let $\gamma, \gamma_0 \in \mathcal{BC}^1(V,Z)$, $\eta, \eta_0 \in \mathcal{BC}^{\partial,0}(U,V)$ with $\|\eta - \eta_0\|_{1_U,0} < \operatorname{dist}(\eta_0(U), \partial V)$ and $x \in U$. Then

$$\begin{aligned}
&\|(\gamma \circ \eta)(x) - (\gamma_0 \circ \eta_0)(x)\| \\
&\qquad = \|\gamma(\eta(x)) - \gamma(\eta_0(x)) + \gamma(\eta_0(x)) - \gamma_0(\eta_0(x))\| \\
&\qquad \leq \left\| \int_0^1 D\gamma(t\eta(x) + (1-t)\eta_0(x)) \cdot (\eta(x) - \eta_0(x))\, dt \right\| + \|(\gamma - \gamma_0)(\eta_0(x))\| \\
&\qquad \leq \|D\gamma\|_{1_V,0} \|\eta(x) - \eta_0(x)\| + \|(\gamma - \gamma_0)(\eta_0(x))\|;
\end{aligned}$$

in this estimate we used $\|\eta - \eta_0\|_{1_U,0} < \operatorname{dist}(\eta_0(U), \partial V)$ to ensure that the line segment between $\eta(x)$ und $\eta_0(x)$ is contained in V. The estimate yields

$$\|\gamma \circ \eta - \gamma_0 \circ \eta_0\|_{1_U,0} \leq \|\gamma\|_{1_V,1} \|\eta - \eta_0\|_{1_U,0} + \|\gamma - \gamma_0\|_{1_U,0},$$

whence the composition is continuous.

$k \to k+1$: In the following, we denote the composition map $(*)$ with $g_{k,Z}$. We know from Proposition 3.2.3 (and the induction base) that

$$g_{k+1,Z}(\mathcal{BC}^{k+2}(V,Z) \times \mathcal{BC}^{\partial,k+1}(U,V)) \subseteq \mathcal{BC}^{k+1}(U,Z)$$
$$\Leftrightarrow \ (D \circ g_{k+1,Z})(\mathcal{BC}^{k+2}(V,Z) \times \mathcal{BC}^{\partial,k+1}(U,V)) \subseteq \mathcal{BC}^{k}(U,\mathrm{L}(X,Z))$$

and $g_{k+1,Z}$ is continuous iff so is $D \circ g_{k+1,Z}$, as a map to $\mathcal{BC}^k(U,\mathrm{L}(X,Z))$. An application of the chain rule gives

$$(D \circ g_{k+1,Z})(\gamma,\eta) = g_{k,\mathrm{L}(Y,Z)}(D\gamma,\eta) \cdot D\eta \tag{$**$}$$

for $\gamma \in \mathcal{BC}^{k+2}(V,Z)$ and $\eta \in \mathcal{BC}^{\partial,k+1}(U,V)$, where $\cdot$ denotes composition of linear maps (see Corollary 3.3.6). Since $D\gamma \in \mathcal{BC}^{k+1}(V,\mathrm{L}(Y,Z))$, we deduce from the inductive hypothesis that

$$g_{k,\mathrm{L}(Y,Z)}(D\gamma,\eta) \in \mathcal{BC}^k(U,\mathrm{L}(Y,Z)),$$

and using Corollary 3.3.6 we get

$$(D \circ g_{k+1,Z})(\gamma,\eta) \in \mathcal{BC}^k(U,\mathrm{L}(Y,Z)).$$

The continuity of $D \circ g_{k+1,Z}$ follows with identity $(**)$ from the continuity of $g_{k,\mathrm{L}(Y,Z)}$ (by the inductive hypothesis), $\cdot$ (by Corollary 3.3.6) and D (by Proposition 3.2.3).

$k = \infty$: From the assertions already established, we derive the commutative diagram

$$\begin{tikzcd}
\mathcal{BC}^{\infty}(V,Z) \times \mathcal{BC}^{\partial,\infty}(U,V) \arrow[r, "g_{\infty,Z}"] \arrow[d, hook] & \mathcal{BC}^{\infty}(U,Z) \arrow[d, hook] \\
\mathcal{BC}^{n+1}(V,Z) \times \mathcal{BC}^{\partial,n}(U,V) \arrow[r, "g_{n,Z}"] & \mathcal{BC}^{n}(U,Z)
\end{tikzcd}$$

for each $n \in \mathbb{N}$, where the vertical arrows represent inclusion maps. Using Corollary 3.2.6 we easily deduce the continuity of $g_{\infty,Z}$ from that of $g_{n,Z}$. ■

As a preparation for discussing the differentiable properties of composition, we prove a nice identity for its differential quotient.

Lemma 3.3.9. *Let X, Y and Z be normed spaces and $U \subseteq X$, $V \subseteq Y$ be open subsets. Further, let $\gamma \in \mathcal{FC}^1(V,Z)$, $\tilde\gamma \in \mathcal{C}^0(V,Z)$, $\tilde\eta \in \mathcal{BC}^0(U,Y)$ and $\eta \in \mathcal{C}^0(U,V)$ such that* $\operatorname{dist}(\eta(U),\partial V) > 0$. *Then, for all $x \in U$ and $t \in \mathbb{R}^*$ with*

$$|t| \leq \frac{\operatorname{dist}(\eta(U),\partial V)}{\|\tilde\eta\|_{1_U,0}+1},$$

we have the identity

$$\mathrm{ev}_x\left(\frac{(\gamma+t\tilde\gamma)\circ(\eta+t\tilde\eta)-\gamma\circ\eta}{t}\right) = \mathrm{ev}_x(\tilde\gamma\circ(\eta+t\tilde\eta)) + \int_0^1 \mathrm{ev}_x((D\gamma\circ(\eta+st\tilde\eta))\cdot\tilde\eta)\,ds, \tag{3.3.9.1}$$

where ev_x *denotes the evaluation at x.*

Proof. For t as above,

$$(\gamma+t\tilde\gamma)\circ(\eta+t\tilde\eta)-\gamma\circ\eta = \gamma\circ(\eta+t\tilde\eta)+t\tilde\gamma\circ(\eta+t\tilde\eta)-\gamma\circ\eta,$$

and an application of the mean value theorem gives

$$\mathrm{ev}_x(\gamma \circ (\eta + t\tilde{\eta}) - \gamma \circ \eta) = \int_0^1 \mathrm{ev}_x((D\gamma \circ (\eta + st\tilde{\eta})) \cdot t\tilde{\eta})\, ds.$$

Division by t leads to the desired result. ∎

So we are ready to discuss when the composition is differentiable.

PROPOSITION 3.3.10. *Let X, Y and Z be normed spaces, $U \subseteq X$ and $V \subseteq Y$ open subsets and $k \in \overline{\mathbb{N}}$, $\ell \in \overline{\mathbb{N}}^*$. Then the continuous map*

$$g_{\mathcal{BC},Z}^{k+\ell+1} : \mathcal{BC}^{k+\ell+1}(V, Z) \times \mathcal{BC}^{\partial,k}(U, V) \to \mathcal{BC}^k(U, Z) : (\gamma, \eta) \mapsto \gamma \circ \eta$$

(cf. Lemma 3.3.8) is a $\mathcal{C}^\ell$-map with

$$dg_{\mathcal{BC},Z}^{k+\ell+1}(\gamma_0, \eta_0; \gamma, \eta) = g_{\mathcal{BC},Z}^{k+\ell+1}(\gamma, \eta_0) + g_{\mathcal{BC},\mathrm{L}(Y,Z)}^{k+\ell}(D\gamma_0, \eta_0) \cdot \eta. \tag{3.3.10.1}$$

Proof. For $k < \infty$, the proof is by induction on ℓ which we may assume finite because the inclusion maps $\mathcal{BC}^\infty(V, Z) \to \mathcal{BC}^{k+\ell+1}(V, Z)$ are continuous linear (and hence smooth).

$\ell = 1$: Let $\gamma_0, \gamma \in \mathcal{BC}^{k+\ell+1}(V, Z)$, $\eta_0 \in \mathcal{BC}^{\partial,k}(U, V)$ and $\eta \in \mathcal{BC}^k(U, Y)$. From Lemmas 3.3.9 and 3.2.13 we conclude that for $t \in \mathbb{K}$ with $|t| \leq \frac{\mathrm{dist}(\eta_0(U), \partial V)}{\|\eta\|_{1_U,0}+1}$, the integral

$$\int_0^1 (D\gamma_0 \circ (\eta_0 + st\eta)) \cdot \eta\, ds$$

exists in $\mathcal{BC}^k(U, Z)$. Using identity (3.3.9.1) we derive

$$\begin{aligned}\frac{g_{\mathcal{BC},Z}^{k+\ell+1}(\gamma_0 + t\gamma, \eta_0 + t\eta) - g_{\mathcal{BC},Z}^{k+\ell+1}(\gamma_0, \eta_0)}{t} &= g_{\mathcal{BC},Z}^{k+\ell+1}(\gamma, \eta_0 + t\eta) \\ &\quad + \int_0^1 g_{\mathcal{BC},\mathrm{L}(Y,Z)}^{k+\ell}(D\gamma_0, \eta_0 + st\eta) \cdot \eta\, ds.\end{aligned}$$

We use Proposition A.1.8 and the continuity of $g_{\mathcal{BC},Z}^{k+\ell+1}$, $g_{\mathcal{BC},\mathrm{L}(Y,Z)}^{k+\ell}$ and $\cdot$ (cf. Lemma 3.3.8, and Corollary 3.3.7) to see that the right hand side above converges to

$$g_{\mathcal{BC},Z}^{k+\ell+1}(\gamma, \eta_0) + g_{\mathcal{BC},\mathrm{L}(Y,Z)}^{k+\ell}(D\gamma_0, \eta_0) \cdot \eta$$

in $\mathcal{BC}^k(U, Z)$ as $t \to 0$. Hence $g_{\mathcal{BC},Z}^{k+\ell+1}$ is differentiable and its differential is given by (3.3.10.1) and thus continuous.

$\ell - 1 \to \ell$: The map $g_{\mathcal{BC},Z}^{k+\ell+1}$ is $\mathcal{C}^\ell$ if $dg_{\mathcal{BC},Z}^{k+\ell+1}$ is $\mathcal{C}^{\ell-1}$. The latter follows easily from (3.3.10.1), since the inductive hypothesis ensures that $g_{\mathcal{BC},Z}^{k+\ell+1}$ and $g_{\mathcal{BC},\mathrm{L}(Y,Z)}^{k+\ell}$ are $\mathcal{C}^{\ell-1}$; and $\cdot$ and D are smooth.

If $k = \infty$, then in view of Corollary 3.2.6 and Proposition A.1.12, $g_{\mathcal{BC},Z}^{\infty}$ is smooth as a map to $\mathcal{BC}^\infty(U, Z)$ iff it is smooth as a map to $\mathcal{BC}^j(U, Z)$ for each $j \in \mathbb{N}$. This was already proved in the case where $k = j$ and $\ell = \infty$. ∎

3.3.2.2. *Composition of weighted functions with bounded functions.* Generally, we cannot expect that the composition of a weighted function with a bounded function is a weighted function for the same weights. As an example, the composition of the constant 1 function and a Schwartz function is not a Schwartz function. However, if we compose a bounded function mapping 0 to 0 with a weighted function, we get good results.

Lemma 3.3.11. *Let X, Y and Z be normed spaces, $U \subseteq X$ and $V \subseteq Y$ open subsets such that V is star-shaped with center 0, $k \in \overline{\mathbb{N}}$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$. Then for $\gamma \in \mathcal{BC}^{k+1}(V, Z)_0$ and $\eta \in \mathcal{C}^{\partial,k}_{\mathcal{W}}(U, V)$,*

$$\gamma \circ \eta \in \mathcal{C}^k_{\mathcal{W}}(U, Z),$$

and the composition map

$$\mathcal{BC}^{k+1}(V, Z)_0 \times \mathcal{C}^{\partial,k}_{\mathcal{W}}(U, V) \to \mathcal{C}^k_{\mathcal{W}}(U, Z) : (\gamma, \eta) \mapsto \gamma \circ \eta \tag{$*$}$$

is continuous.

Proof. We distinguish the cases $k < \infty$ and $k = \infty$:

$k < \infty$: To prove that for $\gamma \in \mathcal{BC}^{k+1}(V, Z)_0$ and $\eta \in \mathcal{C}^{\partial,k}_{\mathcal{W}}(U, V)$ the composition $\gamma \circ \eta$ is in $\mathcal{C}^k_{\mathcal{W}}(U, Z)$, in view of Proposition 3.2.3 it suffices to show that

$$\gamma \circ \eta \in \mathcal{C}^0_{\mathcal{W}}(U, Z) \text{ and for } k > 0 \text{ also } D(\gamma \circ \eta) \in \mathcal{C}^{k-1}_{\mathcal{W}}(U, \mathrm{L}(X, Z)).$$

Similarly the continuity of the composition $(*)$, which is denoted by g_k in the remainder of this proof, is equivalent to the continuity of $\iota_0 \circ g_k$ and for $k > 0$ also of $D \circ g_k$, where $\iota_0 : \mathcal{C}^k_{\mathcal{W}}(U, Z) \to \mathcal{C}^0_{\mathcal{W}}(U, Z)$ denotes the inclusion map.

First we show $\gamma \circ \eta \in \mathcal{C}^0_{\mathcal{W}}(U, Z)$. To this end, let $f \in \mathcal{W}$ and $x \in U$. Then

$$\begin{aligned}|f(x)|\,\|\gamma(\eta(x))\| &= |f(x)|\,\|\gamma(\eta(x)) - \gamma(0)\| \\ &= |f(x)| \left\| \int_0^1 D\gamma(t\eta(x)) \cdot \eta(x)\, dt \right\| \leq \|D\gamma\|_{1_V,0} \|\eta\|_{f,0};\end{aligned}$$

here we used that the line segment from 0 to $\eta(x)$ is contained in V. Hence

$$\|\gamma \circ \eta\|_{f,0} \leq \|\gamma\|_{1_V,1} \|\eta\|_{f,0} < \infty.$$

To check the continuity of $\iota_0 \circ g_k$, let $\gamma, \gamma_0 \in \mathcal{BC}^{k+1}(V, Z)_0$ and $\eta, \eta_0 \in \mathcal{C}^{\partial,k}_{\mathcal{W}}(U, V)$ such that $\|\eta - \eta_0\|_{1_U,0} < \operatorname{dist}(\eta_0(U), \partial V)$, $f \in \mathcal{W}$ and $x \in U$. Then

$$\begin{aligned}&|f(x)|\,\|(\gamma \circ \eta)(x) - (\gamma_0 \circ \eta_0)(x)\| \\ &\quad= |f(x)|\,\|\gamma(\eta(x)) - \gamma(\eta_0(x)) + \gamma(\eta_0(x)) - \gamma_0(\eta_0(x))\| \\ &\quad\leq |f(x)|\,\|\gamma(\eta(x)) - \gamma(\eta_0(x))\| + |f(x)|\,\|(\gamma - \gamma_0)(\eta_0(x))\| \\ &\quad= |f(x)| \left\| \int_0^1 D\gamma(t\eta(x) + (1-t)\eta_0(x)) \cdot (\eta(x) - \eta_0(x))\, dt \right\| \\ &\qquad + |f(x)|\,\|(\gamma - \gamma_0)(\eta_0(x)) - (\gamma - \gamma_0)(0)\| \\ &\quad\leq |f(x)|\,\|D\gamma\|_{1_V,0} \|\eta(x) - \eta_0(x)\| + |f(x)| \left\| \int_0^1 D(\gamma - \gamma_0)(t\eta_0(x)) \cdot \eta_0(x)\, dt \right\| \\ &\quad\leq |f(x)|\,\|D\gamma\|_{1_V,0} \|\eta(x) - \eta_0(x)\| + |f(x)|\,\|D(\gamma - \gamma_0)\|_{1_V,0} \|\eta_0(x)\|.\end{aligned}$$

Therefore

$$\|\gamma \circ \eta - \gamma_0 \circ \eta_0\|_{f,0} \leq \|\gamma\|_{1_V,1} \|\eta - \eta_0\|_{f,0} + \|\gamma - \gamma_0\|_{1_V,1} \|\eta_0\|_{f,0},$$

from which the continuity of $\iota_0 \circ g_k$ in (γ_0, η_0) is easily concluded.

For $k > 0$, $\gamma \in \mathcal{BC}^{k+1}(V, Z)_0$ and $\eta \in \mathcal{C}^{\partial,k}_{\mathcal{W}}(U, V)$ we apply the chain rule to get

$$(D \circ g_k)(\gamma, \eta) = D(\gamma \circ \eta) = (D\gamma \circ \eta) \cdot D\eta = g^k_{\mathcal{BC},\mathrm{L}(Y,Z)}(D\gamma, \eta) \cdot D\eta; \tag{$**$}$$

here we used that $\eta \in \mathcal{BC}^k(U,V)$ because 1_U is in $\mathcal{W}$. Since $D\eta \in \mathcal{C}^{k-1}_{\mathcal{W}}(U,\mathrm{L}(X,Y))$ and $g^k_{\mathcal{BC},\mathrm{L}(Y,Z)}(D\gamma,\eta) \in \mathcal{BC}^{k-1}(U,\mathrm{L}(Y,Z))$ (see Lemma 3.3.8), $(D \circ g_k)(\gamma,\eta)$ is in $\mathcal{C}^{k-1}_{\mathcal{W}}(U,\mathrm{L}(Y,Z))$ (see Corollary 3.3.6). Using that D, $\cdot$ and $g^k_{\mathcal{BC},\mathrm{L}(Y,Z)}$ are continuous (see Proposition 3.2.3, Corollary 3.3.6 and Lemma 3.3.8, respectively), we deduce the continuity of $D \circ g_k$ from $(**)$.

$k = \infty$: From the assertions already established, we derive the commutative diagram

$$\begin{tikzcd}
\mathcal{BC}^\infty(V,Z)_0 \times \mathcal{C}^{\partial,\infty}_{\mathcal{W}}(U,V) \arrow[r, "g_\infty"] \arrow[d, tail] & \mathcal{C}^{\partial,\infty}_{\mathcal{W}}(U,Z) \arrow[d, tail] \\
\mathcal{BC}^{n+1}(V,Z)_0 \times \mathcal{C}^{\partial,n}_{\mathcal{W}}(U,V) \arrow[r, "g_n"] & \mathcal{C}^{\partial,n}_{\mathcal{W}}(U,Z)
\end{tikzcd}$$

for each $n \in \mathbb{N}$, where the vertical arrows represent inclusion maps. Using Corollary 3.2.6 we easily deduce the continuity of g_∞ from that of g_n. ∎

Proposition 3.3.12. *Let X, Y and Z be normed spaces, $U \subseteq X$ and $V \subseteq Y$ open subsets such that V is star-shaped with center 0, $k \in \overline{\mathbb{N}}$, $\ell \in \overline{\mathbb{N}}^*$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$. Then the map*

$$g^{k+\ell+1}_{\mathcal{W},Z} : \mathcal{BC}^{k+\ell+1}(V,Z)_0 \times \mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V) \to \mathcal{C}^k_{\mathcal{W}}(U,Z) : (\gamma,\eta) \mapsto \gamma \circ \eta$$

whose existence was stated in Lemma 3.3.11 is a $\mathcal{C}^\ell$-map with

$$dg^{k+\ell+1}_{\mathcal{W},Z}(\gamma_0,\eta_0;\gamma,\eta) = g^{k+\ell+1}_{\mathcal{W},Z}(\gamma,\eta_0) + g^{k+\ell}_{\mathcal{BC},\mathrm{L}(Y,Z)}(D\gamma_0,\eta_0)\cdot\eta. \tag{3.3.12.1}$$

Proof. For $k < \infty$, the proof is by induction on ℓ which we may assume finite because the inclusion maps $\mathcal{BC}^\infty(V,Z)_0 \to \mathcal{BC}^{k+\ell+1}(V,Z)_0$ are continuous linear (and hence smooth).

$\ell = 1$: Let $\gamma_0,\gamma \in \mathcal{BC}^{k+\ell+1}(V,Z)_0$, $\eta_0 \in \mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V)$ and $\eta \in \mathcal{C}^k_{\mathcal{W}}(U,Y)$. From Lemmas 3.3.9 and 3.2.13 we conclude that for $t \in \mathbb{K}$ with $|t| \leq \frac{\operatorname{dist}(\eta_0(U),\partial V)}{\|\eta\|_{1_U,0}+1}$, the integral

$$\int_0^1 (D\gamma_0 \circ (\eta_0 + st\eta))\cdot\eta\,ds$$

exists in $\mathcal{C}^k_{\mathcal{W}}(U,Z)$. Using identity (3.3.9.1) we derive

$$\begin{aligned}\frac{g^{k+\ell+1}_{\mathcal{W},Z}(\gamma_0+t\gamma,\eta_0+t\eta) - g^{k+\ell+1}_{\mathcal{W},Z}(\gamma_0,\eta_0)}{t} &= g^{k+\ell+1}_{\mathcal{W},Z}(\gamma,\eta_0+t\eta) \\ &\quad + \int_0^1 g^{k+\ell}_{\mathcal{BC},\mathrm{L}(Y,Z)}(D\gamma_0,\eta_0+st\eta)\cdot\eta\,ds.\end{aligned}$$

We use Proposition A.1.8 and the continuity of $g^{k+\ell+1}_{\mathcal{W},Z}$, $g^{k+\ell}_{\mathcal{BC},\mathrm{L}(Y,Z)}$ and $\cdot$ (cf. Lemmas 3.3.11 and 3.3.8, and Corollary 3.3.7) to see that the right hand side above converges to

$$g^{k+\ell+1}_{\mathcal{W},Z}(\gamma,\eta_0) + g^{k+\ell}_{\mathcal{BC},\mathrm{L}(Y,Z)}(D\gamma_0,\eta_0)\cdot\eta$$

in $\mathcal{C}^k_{\mathcal{W}}(U,Z)$ as $t \to 0$. Hence $g^{k+\ell+1}_{\mathcal{W},Z}$ is differentiable and its differential is given by (3.3.12.1) and thus continuous.

$\ell - 1 \to \ell$: The map $g^{k+\ell+1}_{\mathcal{W},Z}$ is $\mathcal{C}^\ell$ if $dg^{k+\ell+1}_{\mathcal{W},Z}$ is $\mathcal{C}^{\ell-1}$. The latter follows easily from (3.3.12.1), since the inductive hypothesis and Proposition 3.3.10 ensure that $g^{k+\ell+1}_{\mathcal{W},Z}$ and $g^{k+\ell}_{\mathcal{BC},\mathrm{L}(Y,Z)}$ are $\mathcal{C}^{\ell-1}$; and $\cdot$ and D are smooth.

If $k = \infty$, then in view of Corollary 3.2.6 and Proposition A.1.12, $g^{\infty}_{\mathcal{W},Z}$ is smooth as a map to $\mathcal{C}^{\infty}_{\mathcal{W}}(U,Z)$ iff it is smooth as a map to $\mathcal{C}^{j}_{\mathcal{W}}(U,Z)$ for each $j \in \mathbb{N}$. This was already proved in the case where $k = j$ and $\ell = \infty$. ∎

3.3.3. Composition of weighted functions with an analytic map. We discuss a sufficient criterion for an analytic map to operate on $\mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V)$ through (covariant) composition. First, we state a result about superposition of weighted functions that is a direct consequence of Proposition 3.3.12. Then we have to treat real and complex analytic functions separately. While the complex case is straightforward, in the real case we have to deal with complexifications.

LEMMA 3.3.13. *Let X, Y and Z be normed spaces, $U \subseteq X$ and $V \subseteq Y$ open subsets such that V is star-shaped with center 0, $k \in \overline{\mathbb{N}}$, $\ell \in \overline{\mathbb{N}}^*$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$. Suppose further that $\Phi : V \to Z$ is a map that satisfies*

$$W \text{ open in } V\text{, bounded and star-shaped with center } 0,\ \operatorname{dist}(W,\partial V) > 0 \\ \Rightarrow\ \Phi|_W \in \mathcal{BC}^{k+\ell+1}(W,Z)_0.$$

Then $\Phi \circ \gamma \in \mathcal{C}^{k}_{\mathcal{W}}(U,Z)$ for all $\gamma \in \mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V)$, and the map

$$\mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V) \to \mathcal{C}^{k}_{\mathcal{W}}(U,Z) : \gamma \mapsto \Phi \circ \gamma$$

is $\mathcal{C}^{\ell}$.

Proof. For $r > 0$ we define

$$M_r := [0,1] \cdot (\{y \in V : \operatorname{dist}(y,\partial V) > r\} \cap B_{1/r}(0)).$$

It is obvious that M_r is open, bounded and star-shaped with center 0. Further, using that V is star-shaped with center 0 and M_r is bounded, we see that $\operatorname{dist}(M_r,\partial V) > 0$. Hence we know from Proposition 3.3.12 that

$$\mathcal{C}^{\partial,k}_{\mathcal{W}}(U,M_r) \to \mathcal{C}^{k}_{\mathcal{W}}(U,Z) : \gamma \mapsto \Phi \circ \gamma$$

is defined and $\mathcal{C}^{\ell}$ since $\Phi \in \mathcal{BC}^{k+\ell+1}(M_r,Z)_0$ by our assumption. But

$$\mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V) = \bigcup_{r>0} \mathcal{C}^{\partial,k}_{\mathcal{W}}(U,M_r),$$

and $1_U \in \mathcal{W}$ implies that each $\mathcal{C}^{\partial,k}_{\mathcal{W}}(U,M_r)$ is open in $\mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V)$, hence the assertion is proved. ∎

LEMMA 3.3.14. *Let Y and Z be complex normed spaces, $V \subseteq Y$ an open nonempty subset and $\Phi : V \to Z$ a complex analytic map that satisfies the following condition:*

$$W \subseteq V,\ W \text{ open in } V,\ \operatorname{dist}(W,\partial V) > 0\ \Rightarrow\ \Phi|_W \in \mathcal{BC}^0(W,Z). \tag{3.3.14.1}$$

Then $\Phi|_W \in \mathcal{BC}^{\infty}(W,Z)$ for all open subsets $W \subseteq V$ with $\operatorname{dist}(W,\partial V) > 0$.

Proof. Let $W \subseteq V$ be an open subset of V such that there exists $r > 0$ with $2r < \operatorname{dist}(W,\partial V)$. Then for each $x \in W$ and $h \in Y$ with $\|h\| \leq 1$ we get an analytic map

$$\Phi_{x,h} : B_{\mathbb{C}}(0,2r) \to Z : z \mapsto \Phi(x + zh),$$

by restricting Φ (see Theorem A.1.23). By applying the Cauchy estimates (stated in Corollary A.1.26) to this map, for each $k \in \mathbb{N}$ we get the estimate

$$\|\Phi_{x,h}^{(k)}(0)\| \leq \frac{k!}{(3r/2)^k}\|\Phi|_{V+B_Y(0,r)}\|_\infty.$$

From Lemma A.1.25 and the chain rule we know that $\Phi_{x,h}^{(k)}(0) = D^{(k)}\Phi(x)(h,\dots,h)$, so we conclude with the Polarization Formula (Proposition A.1.20) that

$$\|D^{(k)}\Phi(x)\|_{\mathrm{op}} \leq \frac{(2k)^k}{(3r/2)^k}\|\Phi|_{V+B_Y(0,r)}\|_\infty,$$

and the assertion follows immediately since $\|\Phi|_{V+B_Y(0,r)}\|_\infty < \infty$ by (3.3.14.1). ■

3.3.3.1. *On real analytic maps and good complexifications.* The previous two lemmas would allow us to state the desired result concerning covariant composition, but only for complex analytic maps. There are examples of real analytic maps for which the assertion of Lemma 3.3.14 is false. We define a class of real analytic maps that is suited to our needs. First, we state the following small result concerning complexifications.

LEMMA 3.3.15. *Let X and Y be real normed spaces, $U \subseteq X$ an open nonempty set, $k \in \overline{\mathbb{N}}$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$. Further let $\iota : Y \to Y_{\mathbb{C}}$ denote the canonical inclusion into $Y_{\mathbb{C}}$.*

(a) *$\mathcal{C}^k_{\mathcal{W}}(U, Y_{\mathbb{C}})$ is the complexification of $\mathcal{C}^k_{\mathcal{W}}(U, Y)$, and the canonical inclusion map is given by*

$$\mathcal{C}^k_{\mathcal{W}}(U, Y) \to \mathcal{C}^k_{\mathcal{W}}(U, Y_{\mathbb{C}}) : \gamma \mapsto \iota \circ \gamma.$$

(b) *Let $V \subseteq Y$ be an open nonempty set and $\widetilde{V} \subseteq Y_{\mathbb{C}}$ an open neighborhood of $\iota(V)$ such that*

$$(\forall M \subseteq V)\ \mathrm{dist}(M, Y \setminus V) > 0 \Rightarrow \mathrm{dist}(\iota(M), Y_{\mathbb{C}} \setminus \widetilde{V}) > 0. \tag{3.3.15.1}$$

Then

$$\iota \circ \mathcal{C}^{\partial,k}_{\mathcal{W}}(U, V) \subseteq \mathcal{C}^{\partial,k}_{\mathcal{W}}(U, \widetilde{V}).$$

Proof. (a) It is well known that $Y_{\mathbb{C}} \cong Y \times Y$ and $\iota(y) = (y, 0)$ for each $y \in Y$. Hence

$$\mathcal{C}^k_{\mathcal{W}}(U, Y_{\mathbb{C}}) \cong \mathcal{C}^k_{\mathcal{W}}(U, Y \times Y) \cong \mathcal{C}^k_{\mathcal{W}}(U, Y) \times \mathcal{C}^k_{\mathcal{W}}(U, Y)$$

by Lemma 3.4.16 (and Proposition 3.3.3), and

$$\iota \circ \gamma = (\gamma, 0) \in \mathcal{C}^k_{\mathcal{W}}(U, Y) \times \mathcal{C}^k_{\mathcal{W}}(U, Y) \cong \mathcal{C}^k_{\mathcal{W}}(U, Y_{\mathbb{C}})$$

for $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U, Y)$.

(b) This is an immediate consequence of (a) and condition (3.3.15.1). ■

DEFINITION 3.3.16. Let Y and Z be real normed spaces, $V \subseteq Y$ an open nonempty set, and $\Phi : V \to Z$ a real analytic map. We say that Φ has a *good complexification* if there exists a complexification $\widetilde{\Phi} : \widetilde{V} \subseteq Y_{\mathbb{C}} \to Z_{\mathbb{C}}$ of Φ which satisfies (3.3.14.1) and whose domain satisfies (3.3.15.1). In this case, we call $\widetilde{\Phi}$ a good complexification.

The following lemma states that good complexifications always exist at least locally. It is not needed in the further discussion.

LEMMA 3.3.17. *Let Y and Z be real normed spaces, $V \subseteq Y$ an open nonempty set and $\Phi : V \to Z$ a real analytic map. Then for each $x \in V$ there exists an open neighborhood $W_x \subseteq Y$ of x such that $\Phi|_{W_x}$ has a good complexification.*

Proof. Let $\widetilde{\Phi} : \widetilde{V} \subseteq Y_{\mathbb{C}} \to Z_{\mathbb{C}}$ be a complexification of Φ and $\iota : V \to \widetilde{V}$ the canonical inclusion. Then there exists an $r > 0$ such that $B_{Y_{\mathbb{C}}}(\iota(x), r) \subseteq \widetilde{V}$ and $\widetilde{\Phi}$ is bounded on $B_{Y_{\mathbb{C}}}(\iota(x), r)$. Then it is obvious that $W_x := \iota^{-1}(B_{Y_{\mathbb{C}}}(\iota(x), r)) = B_Y(x, r)$ has the stated property. ■

Power series. We present a class of analytic maps which have good complexifications: absolutely convergent power series in Banach algebras.

LEMMA 3.3.18. *Let A be a Banach algebra and $\sum_{\ell=0}^{\infty} a_\ell z^\ell$ a power series with $a_\ell \in \mathbb{K}$ and the radius of convergence $R \in]0, \infty]$. For $x \in A$ define*

$$P_x : B_A(x, R) \to A : y \mapsto \sum_{\ell=0}^{\infty} a_\ell (y - x)^\ell.$$

(a) *The map P_x is analytic.*
(b) *If $\mathbb{K} = \mathbb{C}$ then P_x satisfies* (3.3.14.1).
(c) *If $\mathbb{K} = \mathbb{R}$ then P_x has a good complexification.*

Proof. The map P_x is defined since $\sum_{\ell=0}^{\infty} a_\ell (y - x)^\ell$ is absolutely convergent on $B_R(x)$ and A is complete.

(a) This is a special case of [Bou67, §3.2.9].

(b) If $V \subseteq B_A(x, R)$ is open and $\operatorname{dist}(V, \partial B_A(x, R)) > 0$, there exists $r \in \mathbb{R}$ with $0 < r < R$ such that $V \subseteq B_A(x, r)$. So for $y \in V$,

$$\Big\| \sum_{\ell=0}^{\infty} a_\ell (y - x)^\ell \Big\| \leq \sum_{\ell=0}^{\infty} |a_\ell| \, \|y - x\|^\ell \leq \sum_{\ell=0}^{\infty} |a_\ell| r^\ell < \infty.$$

(c) It is well known that the complexification of a Banach algebra is a Banach algebra as well, and a complexification of P_x is given by

$$\tilde{P}_x : B_{A_{\mathbb{C}}}(x, R) \to A : y \mapsto \sum_{\ell=0}^{\infty} a_\ell (y - x)^\ell. \quad ■$$

3.3.3.2. *Main result.* Finally, we state the desired result for complex analytic maps and real analytic maps with good complexifications.

PROPOSITION 3.3.19. *Let X, Y and Z be normed spaces, $U \subseteq X$ and $V \subseteq Y$ open nonempty sets such that V is star-shaped with center 0, $k \in \overline{\mathbb{N}}$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$. Further, let $\Phi : V \to Z$ with $\Phi(0) = 0$ be either a complex analytic map that satisfies* (3.3.14.1) *or a real analytic map that has a good complexification. Then for each $\gamma \in \mathcal{C}^{\partial,k}_{\mathcal{W}}(U, V)$,*

$$\Phi \circ \gamma \in \mathcal{C}^k_{\mathcal{W}}(U, Z),$$

and the map

$$\Phi_* : \mathcal{C}^{\partial,k}_{\mathcal{W}}(U, V) \to \mathcal{C}^k_{\mathcal{W}}(U, Z) : \gamma \mapsto \Phi \circ \gamma$$

is analytic.

Proof. If Φ is complex analytic, this is an immediate consequence of Lemma 3.3.13 and Lemma 3.3.14.

If Φ is real analytic, by our assumptions there exists a good complexification $\widetilde{\Phi} : \widetilde{V} \subseteq Y_{\mathbb{C}} \to Z$. We know from the first part that $\widetilde{\Phi}$ induces a complex analytic map

$$\widetilde{\Phi}_* : \mathcal{C}^{\partial,k}_{\mathcal{W}}(U, \widetilde{V}) \to \mathcal{C}^k_{\mathcal{W}}(U, Z_{\mathbb{C}}) : \gamma \mapsto \widetilde{\Phi} \circ \gamma.$$

Since $\mathcal{C}^{\partial,k}_{\mathcal{W}}(U, V) \subseteq \mathcal{C}^{\partial,k}_{\mathcal{W}}(U, \widetilde{V})$ by Lemma 3.3.15 and Φ_* coincides with the restriction of $\widetilde{\Phi}_*$ to $\mathcal{C}^{\partial,k}_{\mathcal{W}}(U, V)$, it follows that Φ_* is real analytic. ∎

3.3.3.3. *Quasi-inversion algebras of weighted functions.* As an application, we see that for a set $\mathcal{W}$ of weights with $1_U \in \mathcal{W}$ and a Banach algebra A, the space $\mathcal{C}^k_{\mathcal{W}}(U, A)$ can be turned into a topological algebra with continuous quasi-inversion. Details on algebras with quasi-inversion can be found in Chapter C.

Proposition 3.3.20. *Let A be a Banach algebra, X a normed space, $U \subseteq X$ an open nonempty subset, $k \in \overline{\mathbb{N}}$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$. Then the locally convex space $\mathcal{C}^k_{\mathcal{W}}(U, A)$ endowed with the multiplication described in Corollary* 3.3.5 *becomes a complete topological algebra with continuous quasi-inversion in the sense of Definition* C.2.1. *For each $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U, A)^q$,*

$$QI_{\mathcal{C}^k_{\mathcal{W}}(U,A)}(\gamma) = QI_A \circ \gamma,$$

and

$$\mathcal{C}^{\partial,k}_{\mathcal{W}}(U, B_A(0,1)) = \{\gamma \in \mathcal{C}^k_{\mathcal{W}}(U, A) : \|\gamma\|_{1_U,0} < 1\} \subseteq \mathcal{C}^k_{\mathcal{W}}(U, A)^q.$$

Proof. The relation $QI_{\mathcal{C}^k_{\mathcal{W}}(U,A)}(\gamma) = QI_A \circ \gamma$ is an immediate consequence of the definition of the multiplication, so it only remains to show that $\mathcal{C}^k_{\mathcal{W}}(U, A)^q$ is open and $QI_{\mathcal{C}^k_{\mathcal{W}}(U,A)}$ is continuous. We proved in Lemma C.2.4 that it suffices to find a neighborhood of 0 that consists of quasi-invertible elements such that the restriction of $QI_{\mathcal{C}^k_{\mathcal{W}}(U,A)}$ to it is continuous. We show that $\mathcal{C}^{\partial,k}_{\mathcal{W}}(U, B_A(0,1))$ is such a neighborhood. The map

$$G : B_1(0) \to A : x \mapsto \sum_{i=1}^{\infty} x^i$$

is given by a power series and maps 0 to 0, hence we know from Lemma 3.3.18 and Proposition 3.3.19 that the map

$$\mathcal{C}^{\partial,k}_{\mathcal{W}}(U, B_A(0,1)) \to \mathcal{C}^k_{\mathcal{W}}(U, A) : \gamma \mapsto G \circ \gamma$$

is defined and analytic. Since $G \circ \gamma = \sum_{i=1}^{\infty} \gamma^i$ for each $\gamma \in \mathcal{C}^{\partial,k}_{\mathcal{W}}(U, B_A(0,1))$, we conclude from Lemma C.2.5 that γ is quasi-invertible with

$$QI_{\mathcal{C}^k_{\mathcal{W}}(U,A)}(\gamma) = -G \circ \gamma \ ∎$$

Example 3.3.21. Let Y be a Banach space, $U \subseteq X$ an open nonempty subset, $k \in \overline{\mathbb{N}}$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$. Then the locally convex space $\mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(Y))$ endowed with the multiplication described in Corollary 3.3.6 becomes a complete topological algebra with continuous quasi-inversion.

3.4. Weighted maps into locally convex spaces. We define and examine weighted functions with values in arbitrary locally convex spaces. In order to do this, we use tools and definitions provided in A.1.2.2. The material of this section is only needed for later discussions of weighted mapping groups with values in arbitrary locally convex Lie groups in Section 6.2; readers primarily interested in diffeomorphism groups may want to skip this section.

3.4.1. Definition and topological structure. The definition of weighted function with values in locally convex spaces relies on the one with values in normed spaces.

DEFINITION 3.4.1. Let X be a normed space, $U \subseteq X$ an open nonempty set, Y a locally convex space, $k \in \overline{\mathbb{N}}$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$. We define

$$\mathcal{C}^k_{\mathcal{W}}(U,Y) := \{\gamma \in \mathcal{C}^k(U,Y) : (\forall p \in \mathcal{N}(Y))\ \pi_p \circ \gamma \in \mathcal{C}^k_{\mathcal{W}}(U,Y_p)\},$$

using notation as in Definition A.1.28. For $p \in \mathcal{N}(Y)$, $f \in \mathcal{W}$ and $\ell \in \mathbb{N}$ with $\ell \leq k$,

$$\|\cdot\|_{p,f,\ell} : \mathcal{C}^k_{\mathcal{W}}(U,Y) \to \mathbb{R} : \gamma \mapsto \|\pi_p \circ \gamma\|_{f,\ell}$$

is a seminorm on $\mathcal{C}^k_{\mathcal{W}}(U,Y)$. We endow $\mathcal{C}^k_{\mathcal{W}}(U,Y)$ with the locally convex vector space topology generated by these seminorms.

We show that the structure of $\mathcal{C}^k_{\mathcal{W}}(U,Y)$ is already determined by $\{\|\cdot\|_{p,f,\ell} : p \in \mathcal{P}$, and f,ℓ are as usual$\}$, where $\mathcal{P}$ is just a generator of $\mathcal{N}(Y)$. This can be useful in some cases.

LEMMA 3.4.2. *Let X be a normed space, $U \subseteq X$ an open nonempty set, Y a locally convex space, $k \in \overline{\mathbb{N}}$, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ and $\mathcal{P} \subseteq \mathcal{N}(Y)$ a set that generates $\mathcal{N}(Y)$. Then for $\gamma \in \mathcal{C}^k(U,Y)$,*

$$\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,Y) \Leftrightarrow (\forall p \in \mathcal{P})\, \pi_p \circ \gamma \in \mathcal{C}^k_{\mathcal{W}}(U,Y_p),$$

and the map

$$\mathcal{C}^k_{\mathcal{W}}(U,Y) \to \prod_{p\in\mathcal{P}} \mathcal{C}^k_{\mathcal{W}}(U,Y_p) : \gamma \mapsto (\pi_p \circ \gamma)_{p\in\mathcal{P}} \tag{†}$$

is a topological embedding.

Proof. Let $q \in \mathcal{N}(Y)$. Then there exist $p_1,\dots,p_n \in \mathcal{P}$ and $C > 0$ such that

$$q \leq C \max_{i=1,\dots,n} p_i.$$

Further we know that for each $\ell \in \mathbb{N}$ with $\ell \leq k$ and $x \in U$, $h_1,\dots,h_\ell \in X$,

$$d^{(\ell)}(\pi_q \circ \gamma)(x;h_1,\dots,h_\ell) = (\pi_q \circ d^{(\ell)}\gamma)(x,h_1,\dots,h_\ell),$$

so for $y \in U$ we get

$$\begin{aligned}
\|d^{(\ell)}(\pi_q \circ \gamma)(x;h_1,\dots,h_\ell) - d^{(\ell)}(\pi_q \circ \gamma)(y;h_1,\dots,h_\ell)\|_q \\
&\leq \|d^{(\ell)}\gamma(x;h_1,\dots,h_\ell) - d^{(\ell)}\gamma(y;h_1,\dots,h_\ell)\|_q \\
&\leq C \max_{i=1,\dots,n} \|d^{(\ell)}\gamma(x;h_1,\dots,h_\ell) - d^{(\ell)}\gamma(y;h_1,\dots,h_\ell)\|_{p_i}.
\end{aligned}$$

Since we assumed that $\pi_{p_i} \circ \gamma \in \mathcal{FC}^k(U, Y_{p_i})$, from this estimate we conclude, applying Proposition A.3.2, that $\pi_q \circ \gamma \in \mathcal{FC}^k(U, Y_q)$ with

$$\|D^{(\ell)}(\pi_q \circ \gamma)(x)\|_{\mathrm{op}} \leq C \max_{i=1,\dots,n} \|D^{(\ell)}(\pi_{p_i} \circ \gamma)(x)\|_{\mathrm{op}}$$

for all $\ell \in \mathbb{N}$ with $\ell \leq k$ and $x \in U$. This implies that

$$\|\gamma\|_{q,f,\ell} \leq C \max_{i=1,\dots,n} \|\gamma\|_{p_i,f,\ell}$$

for each $f \in \mathcal{W}$ and $\ell \in \mathbb{N}$ with $\ell \leq k$. Hence

$$\pi_q \circ \gamma \in \mathcal{C}^k_{\mathcal{W}}(U, Y_q),$$

and $\|\cdot\|_{q,f,\ell}$ is continuous with respect to the initial topology induced by (†). Since q was arbitrary, the proof is complete. ∎

An integrability criterion. We generalize the assertion of Lemma 3.2.13.

LEMMA 3.4.3. *Let X be a normed space, $U \subseteq X$ a nonempty open set, Y a locally convex space, $k \in \overline{\mathbb{N}}$, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ such that for each compact set $K \subseteq U$, there exists an $f_K \in \mathcal{W}$ with $\inf_{x\in K} |f_K(x)| > 0$. Further, let $\Gamma : [a, b] \to \mathcal{C}^k_{\mathcal{W}}(U, Y)$ a continuous curve and $R \in \mathcal{C}^k_{\mathcal{W}}(U, Y)$. Assume that*

$$\int_a^b \mathrm{ev}_x(\Gamma(s))\, ds = \mathrm{ev}_x(R) \tag{$*$}$$

for all $x \in U$. Then Γ is weakly integrable with

$$\int_a^b \Gamma(s)\, ds = R.$$

Proof. We derive from Lemma 3.4.2 that the dual space of $\mathcal{C}^k_{\mathcal{W}}(U, Y)$ coincides with the set of functionals $\{\lambda \circ \pi_{p*} : p \in \mathcal{N}(Y),\ \lambda \in \mathcal{C}^k_{\mathcal{W}}(U, Y_p)'\}$. Hence Γ is weakly integrable with the integral R iff

$$\int_a^b \lambda(\pi_p \circ \Gamma)(s)\, ds = \lambda(\pi_p \circ R)$$

for all $p \in \mathcal{N}(Y)$ and $\lambda \in \mathcal{C}^k_{\mathcal{W}}(U, Y_p)'$; this is clearly equivalent to the weak integrability of $\pi_p \circ \Gamma$ with integral $\pi_p \circ R$ for all $p \in \mathcal{N}(Y)$. But we derive this assertion from identity $(*)$ and Lemma 3.2.13. ∎

3.4.1.1. *Reduction to lower order.* We prove a generalization of Proposition 3.2.3. To this end, we need a locally convex topology on $\mathrm{L}(X, Y)$, where X is a normed and Y a locally convex space. We define such a topology and show that it arises as the initial topology with respect to the embedding $\mathrm{L}(X, Y) \to \prod_{p\in\mathcal{N}(Y)} \mathrm{L}(X, Y_p)$.

Topology on linear operators

DEFINITION 3.4.4 (Topology on linear operators). Let X be a normed space and Y a locally convex space. For each $p \in \mathcal{N}(Y)$ and $T \in \mathrm{L}(X, Y)$, we set

$$\|T\|_{\mathrm{op},p} := \sup_{x\neq 0} \frac{\|Tx\|_p}{\|x\|} = \|\pi_p \circ T\|_{\mathrm{op}}.$$

This obviously defines a seminorm on $\mathrm{L}(X, Y)$, and henceforth we endow $\mathrm{L}(X, Y)$ with the locally convex topology that is generated by these seminorms. Further we define $\mathrm{L}(X, Y)_{\mathrm{op},p} := \mathrm{L}(X, Y)_{\|\cdot\|_{\mathrm{op},p}}$.

Lemma 3.4.5. *Let X be a normed space, Y a locally convex space and $p \in \mathcal{N}(Y)$. Then the map induced by*

$$(\pi_p)_* : \mathrm{L}(X, Y) \to \mathrm{L}(X, Y_p) : T \mapsto \pi_p \circ T$$

that makes

$$\begin{tikzcd}
(\mathrm{L}(X,Y), \|\cdot\|_{\mathrm{op},p}) \arrow[rr, "(\pi_p)_*"] \arrow[dr, two heads, "\pi_{\mathrm{op},p}"] & & \mathrm{L}(X, Y_p) \\
& \mathrm{L}(X,Y)_{\mathrm{op},p} \arrow[ur, dashed] &
\end{tikzcd}$$

a commutative diagram is an isometric isomorphism onto the image of $(\pi_p)_*$. The map

$$\mathrm{L}(X, Y) \to \prod_{p \in \mathcal{N}(Y)} \mathrm{L}(X, Y_p) : T \mapsto (\pi_p \circ T)_{p \in \mathcal{N}(Y)}$$

is a topological embedding.

Proof. Since $\|T\|_{\mathrm{op},p} = \|\pi_p \circ T\|_{\mathrm{op}}$ for each $T \in \mathrm{L}(X, Y)$, the induced map is an isometry. By the definition of the topology of $\mathrm{L}(X, Y)$,

$$\mathrm{L}(X, Y) \to \prod_{p \in \mathcal{N}(Y)} \mathrm{L}(X, Y)_{\mathrm{op},p} : T \mapsto (\pi_{\mathrm{op},p} \circ T)_{p \in \mathcal{N}(Y)}$$

is an embedding, so by the transitivity of initial topologies, the proof is finished. ■

Weighted maps into spaces of linear operators and the main result. Before we can prove the main result, we have to look at the structure of $\mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Y))$.

Lemma 3.4.6. *Let X be a normed space, Y a locally convex space, $U \subseteq X$ an open nonempty subset and $k \in \overline{\mathbb{N}}$. Then for $\Gamma \in \mathcal{C}^k(U, \mathrm{L}(X, Y))$, $\ell \in \mathbb{N}$ with $\ell \leq k$ and $f \in \overline{\mathbb{R}}^U$,*

$$\|\Gamma\|_{\|\cdot\|_{\mathrm{op},p}, f, \ell} = \|(\pi_p)_* \circ \Gamma\|_{f,\ell}. \tag{3.4.6.1}$$

Further for $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ and $k \in \overline{\mathbb{N}}$,

$$\Gamma \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Y)) \;\Leftrightarrow\; (\forall p \in \mathcal{N}(Y))\ (\pi_p)_* \circ \Gamma \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Y_p)),$$

and the map

$$\mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Y)) \to \prod_{p \in \mathcal{N}(Y)} \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Y_p)) : \Gamma \mapsto ((\pi_p)_* \circ \Gamma)_{p \in \mathcal{P}}$$

is a topological embedding.

Proof. Note first that $\pi_{\mathrm{op},p} \circ \Gamma$ is $\mathcal{FC}^k$ iff $(\pi_p)_* \circ \Gamma$ is $\mathcal{FC}^k$ as a consequence of Lemma 3.4.5 and Proposition A.3.2. Using Lemma 3.4.5 it is easy to see that (3.4.6.1) is satisfied. This implies that for each $p \in \mathcal{N}(Y)$ the equivalence

$$(\pi_p)_* \circ \Gamma \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Y_p)) \;\Leftrightarrow\; \pi_{\mathrm{op},p} \circ \Gamma \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Y)_{\mathrm{op},p})$$

holds and that the isometry whose existence was stated in Lemma 3.4.5 induces an embedding

$$\mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Y)_{\mathrm{op},p}) \to \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Y_p)).$$

Further we proved in Lemma 3.4.2 that

$$\mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Y)) \to \prod_{p\in\mathcal{N}(Y)} \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Y)_{\mathrm{op},p}) : \Gamma \mapsto ((\pi_{\mathrm{op},p})_* \circ \Gamma)_{p\in\mathcal{P}}$$

is an embedding, so we are done. ∎

Proposition 3.4.7 (Reduction to lower order). *Let X be a normed space, Y a locally convex space, $U \subseteq X$ an open nonempty set, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ and $k \in \mathbb{N}$. Let $\gamma \in \mathcal{C}^1(U,Y)$. Then*

$$\gamma \in \mathcal{C}^{k+1}_{\mathcal{W}}(U,Y) \;\Leftrightarrow\; (D\gamma,\gamma) \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Y)) \times \mathcal{C}^0_{\mathcal{W}}(U,Y).$$

Furthermore, the map

$$\mathcal{C}^{k+1}_{\mathcal{W}}(U,Y) \to \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Y)) \times \mathcal{C}^0_{\mathcal{W}}(U,Y) : \gamma \mapsto (D\gamma,\gamma)$$

is a topological embedding.

Proof. The definition of $\mathcal{C}^{k+1}_{\mathcal{W}}(U,Y)$, Proposition 3.2.3 and Lemma 3.4.6 give the equivalences

$$\begin{aligned}\gamma \in \mathcal{C}^{k+1}_{\mathcal{W}}(U,Y) \;&\Leftrightarrow\; (\forall p \in \mathcal{N}(Y))\ \pi_p \circ \gamma \in \mathcal{C}^{k+1}_{\mathcal{W}}(U,Y_p)\\ &\Leftrightarrow\; (\forall p \in \mathcal{N}(Y))\ (D(\pi_p\circ\gamma), \pi_p\circ\gamma) \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Y_p)) \times \mathcal{C}^0_{\mathcal{W}}(U,Y_p)\\ &\Leftrightarrow\; (D\gamma,\gamma) \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Y)) \times \mathcal{C}^0_{\mathcal{W}}(U,Y).\end{aligned}$$

Furthermore, we have the commutative diagram

$$\begin{tikzcd} \mathcal{C}^{k+1}_{\mathcal{W}}(U,Y) \arrow[r] \arrow[d, tail] & \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Y)) \times \mathcal{C}^0_{\mathcal{W}}(U,Y) \arrow[d, tail] \\ \prod_{p\in\mathcal{N}(Y)} \mathcal{C}^{k+1}_{\mathcal{W}}(U,Y_p) \arrow[r, tail] & \prod_{p\in\mathcal{N}(Y)} \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Y_p)) \times \mathcal{C}^0_{\mathcal{W}}(U,Y_p) \end{tikzcd}$$

and since the maps represented by the three lower arrows are embeddings, so is the map at the top. ∎

3.4.2. Weighted decreasing maps. We give another definition for weighted maps that decay at infinity. Here, the domain of the maps is contained in a finite-dimensional vector space.

Definition 3.4.8. Let Y be a normed space, U an open nonempty subset of the *finite-dimensional* space X and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$. For $k \in \overline{\mathbb{N}}$ we define

$$\begin{aligned}\mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet := \{\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,Y) : (\forall f \in \mathcal{W}, \ell \in \mathbb{N}, \ell \le k)&\\ (\forall \varepsilon > 0)(\exists K \subseteq U \text{ compact})\ \|\gamma|_{U\setminus K}\|_{f,\ell} < \varepsilon\}.&\end{aligned}$$

For a locally convex space Y we set

$$\mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet := \{\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,Y) : (\forall p \in \mathcal{N}(Y))\ \pi_p\circ\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,Y_p)^\bullet\}.$$

For a subset $V \subseteq Y$, we define

$$\mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet := \{\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet : \gamma(U) \subseteq V\}.$$

As in Lemma 3.1.6, we can prove that $\mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet$ is closed in $\mathcal{C}^k_{\mathcal{W}}(U,Y)$.

LEMMA 3.4.9. *Let Y be a locally convex space, U an open nonempty subset of the finite-dimensional space X, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ and $k \in \overline{\mathbb{N}}$. Then $\mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet$ is a closed vector subspace of $\mathcal{C}^k_{\mathcal{W}}(U,Y)$.*

Proof. It is obvious from the definition of $\mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet$ that it is a vector subspace. It remains to show that it is closed. To this end, let $(\gamma_i)_{i\in I}$ be a net in $\mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet$ that converges to $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,Y)$ in the topology of $\mathcal{C}^k_{\mathcal{W}}(U,Y)$. Let $p \in \mathcal{N}(Y)$, $f \in \mathcal{W}$, $\ell \in \mathbb{N}$ with $\ell \leq k$ and $\varepsilon > 0$. Then there exists an $i_\varepsilon \in I$ such that

$$i \geq i_\varepsilon \Rightarrow \|\gamma - \gamma_i\|_{p,f,\ell} < \varepsilon/2.$$

Further there exists a compact set K such that

$$\|\gamma_{i_\varepsilon}|_{U\setminus K}\|_{p,f,\ell} < \varepsilon/2.$$

Hence

$$\|\gamma|_{U\setminus K}\|_{p,f,\ell} \leq \|\gamma|_{U\setminus K} - \gamma_{i_\varepsilon}|_{U\setminus K}\|_{p,f,\ell} + \|\gamma_{i_\varepsilon}|_{U\setminus K}\|_{p,f,\ell} < \varepsilon,$$

so $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet$. ∎

Further, we prove the following convexity criterion.

LEMMA 3.4.10. *Let X be a finite-dimensional space, $U \subseteq X$ an open nonempty subset, Y a locally convex space, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $\ell \in \overline{\mathbb{N}}$ and $V \subseteq Y$ convex. Then the set $\mathcal{C}^\ell_{\mathcal{W}}(U,V)^\bullet$ is convex.*

Proof. It is obvious that $\mathcal{C}^\ell_{\mathcal{W}}(U,V)$—whose definition is straightforward—is convex since V is so. But then

$$\mathcal{C}^\ell_{\mathcal{W}}(U,V)^\bullet = \mathcal{C}^\ell_{\mathcal{W}}(U,V) \cap \mathcal{C}^\ell_{\mathcal{W}}(U,Y)^\bullet$$

is convex as intersection of convex sets. ∎

As in Corollary 3.2.4, we prove a reduction to lower order for $\mathcal{C}^{k+1}_{\mathcal{W}}(U,Y)^\bullet$.

PROPOSITION 3.4.11. *Let X be a finite-dimensional space, Y a locally convex space, $U \subseteq X$ an open nonempty set, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$, $k \in \mathbb{N}$ and $\gamma \in \mathcal{C}^1(U,Y)$. Then*

$$\gamma \in \mathcal{C}^{k+1}_{\mathcal{W}}(U,Y)^\bullet \;\Leftrightarrow\; (D\gamma, \gamma) \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Y))^\bullet \times \mathcal{C}^0_{\mathcal{W}}(U,Y)^\bullet,$$

and the map

$$\mathcal{C}^{k+1}_{\mathcal{W}}(U,Y)^\bullet \to \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Y))^\bullet \times \mathcal{C}^0_{\mathcal{W}}(U,Y)^\bullet : \gamma \mapsto (D\gamma, \gamma)$$

is a topological embedding.

Proof. It is a consequence of identity (3.2.2.2) in Lemma 3.2.2 that for each $p \in \mathcal{N}(Y)$,

$$\pi_p \circ \gamma \in \mathcal{C}^{k+1}_{\mathcal{W}}(U,Y_p)^\bullet \;\Leftrightarrow\; (D(\pi_p \circ \gamma), \pi_p \circ \gamma) \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Y_p))^\bullet \times \mathcal{C}^0_{\mathcal{W}}(U,Y_p)^\bullet.$$

Further it is a consequence of identity (3.4.6.1) in Lemma 3.4.6 that

$$D\gamma \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Y))^\bullet \;\Leftrightarrow\; (\forall p \in \mathcal{N}(Y)) D(\pi_p \circ \gamma) \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Y_p))^\bullet,$$

so the equivalence is proved. The assertion on the embedding is a consequence of Proposition 3.4.7 and Lemma 3.4.9. ■

3.4.3. Composition and superposition. As in Section 3.3, we examine which kind of maps induce superposition operators on $\mathcal{C}^k_{\mathcal{W}}(U,Y)$ or $\mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet$. We show that continuous multilinear maps induce superposition operators on both function spaces. For $\mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet$, we can prove a much stronger result: A smooth function mapping 0 on 0 induces a superposition operator between these spaces.

3.4.3.1. *Composition with a multilinear map.* The following definition and lemma are mostly the same as in Section 3.3.1, but here Z denotes a locally convex space.

Definition 3.4.12. Let X be a normed space, $Y_1,\dots,Y_m$ and Z locally convex spaces and $b : Y_1\times\cdots\times Y_m\to Z$ a continuous m-linear map. For each $i\in\{1,\dots,m\}$, we define the m-linear continuous map

$$b^{(i)} : Y_1\times\cdots\times Y_{i-1}\times \mathrm{L}(X,Y_i)\times Y_{i+1}\times\cdots\times Y_m\to \mathrm{L}(X,Z),$$
$$(y_1,\dots,y_{i-1},T,y_{i+1},\dots,y_m)\mapsto (h\mapsto b(y_1,\dots,y_{i-1},T\cdot h,y_{i+1},\dots,y_m)).$$

Lemma 3.4.13. *Let $Y_1,\dots,Y_m$ and Z be locally convex spaces, U be an open nonempty subset of the normed space X and $k\in\overline{\mathbb{N}}$. Further let $b : Y_1\times\cdots\times Y_m\to Z$ be a continuous m-linear map and $\gamma_1\in\mathcal{C}^k(U,Y_1),\dots,\gamma_m\in\mathcal{C}^k(U,Y_m)$. Then*

$$b\circ(\gamma_1,\dots,\gamma_m)\in\mathcal{C}^k(U,Z)$$

with

$$D(b\circ(\gamma_1,\dots,\gamma_m))=\sum_{i=1}^m b^{(i)}\circ(\gamma_1,\dots,\gamma_{i-1},D\gamma_i,\gamma_{i+1},\dots,\gamma_m). \tag{3.4.13.1}$$

Proof. To calculate the derivative of $b\circ(\gamma_1,\dots,\gamma_m)$, we apply the chain rule to get

$$\begin{aligned} d(b\circ(\gamma_1,\dots,\gamma_m))(x;h) &= \sum_{i=1}^m b(\gamma_1(x),\dots,\gamma_{i-1}(x),d\gamma_i(x;h),\gamma_{i+1}(x),\dots,\gamma_m(x))\\ &= \sum_{i=1}^m b^{(i)}(\gamma_1(x),\dots,\gamma_{i-1}(x),D\gamma_i(x),\gamma_{i+1}(x),\dots,\gamma_m(x))\cdot h.\end{aligned}$$

This implies (3.4.13.1). ■

Now we can prove the results about multilinear superposition.

Proposition 3.4.14. *Let U be an open nonempty subset of the normed space X. Let $Y_1,\dots,Y_m$ be locally convex spaces, $k\in\overline{\mathbb{N}}$ and $\mathcal{W},\mathcal{W}_1,\dots,\mathcal{W}_m\subseteq\overline{\mathbb{R}}^U$ sets such that*

$$(\forall f\in\mathcal{W})(\exists g_{f,1}\in\mathcal{W}_1,\dots,g_{f,m}\in\mathcal{W}_m)\ |f|\leq|g_{f,1}|\cdots|g_{f,m}|.$$

Further let Z be another locally convex space and $b : Y_1\times\cdots\times Y_m\to Z$ a continuous m-linear map. Then

$$b\circ(\gamma_1,\dots,\gamma_m)\in\mathcal{C}^k_{\mathcal{W}}(U,Z)$$

for all $\gamma_1\in\mathcal{C}^k_{\mathcal{W}_1}(U,Y_1),\dots,\gamma_m\in\mathcal{C}^k_{\mathcal{W}_m}(U,Y_m)$. The map

$$b_* : \mathcal{C}^k_{\mathcal{W}_1}(U,Y_1)\times\cdots\times\mathcal{C}^k_{\mathcal{W}_m}(U,Y_m)\to\mathcal{C}^k_{\mathcal{W}}(U,Z) : (\gamma_1,\dots,\gamma_m)\mapsto b\circ(\gamma_1,\dots,\gamma_m)$$

is m-linear and continuous.

Proof. Let p be a continuous seminorm on Z. Then there exist $q_1 \in \mathcal{N}(Y_1), \ldots, q_m \in \mathcal{N}(Y_m)$ such that, for all $y_1 \in Y_1, \ldots, y_m \in Y_m$,

$$\|b(y_1, \ldots, y_m)\|_p \leq \|y_1\|_{q_1} \cdots \|y_m\|_{q_m}.$$

Hence there exists an m-linear map $\widetilde{b}$ that makes the diagram

$$\begin{tikzcd}
Y_1 \times \cdots \times Y_m \arrow[r, "b"] \arrow[d, two heads, "\pi_{q_1} \times \cdots \times \pi_{q_m}"'] & Z \arrow[d, two heads, "\pi_p"] \\
Y_{1,q_1} \times \cdots \times Y_{m,q_m} \arrow[r, "\widetilde{b}"] & Z_p
\end{tikzcd}$$

commutative. For $\gamma_1 \in \mathcal{C}^k_{\mathcal{W}_1}(U, Y_1), \ldots, \gamma_m \in \mathcal{C}^k_{\mathcal{W}_1}(U, Y_m)$ we know from Proposition 3.3.3 that

$$\widetilde{b} \circ (\pi_{q_1} \circ \gamma_1, \ldots, \pi_{q_m} \circ \gamma_m) \in \mathcal{C}^k_{\mathcal{W}}(U, Z_p)$$

and the map $\widetilde{b}_*$ is continuous. Since

$$\widetilde{b}_* \circ ((\pi_{q_1})_* \times \cdots \times (\pi_{q_m})_*) = (\pi_p)_* \circ b_*$$

and the left hand side is continuous, we conclude using Lemma 3.4.2 that b_* is well-defined and continuous since p was arbitrary. ∎

Corollary 3.4.15. *Let $Y_1, \ldots, Y_m$ be locally convex spaces, U be an open nonempty subset of the finite-dimensional space X, $k \in \overline{\mathbb{N}}$ and $\mathcal{W}, \mathcal{W}_1, \ldots, \mathcal{W}_m \subseteq \overline{\mathbb{R}}^U$ such that*

$$(\forall f \in \mathcal{W})(\exists g_{f,1} \in \mathcal{W}_1, \ldots, g_{f,m} \in \mathcal{W}_m)\ |f| \leq |g_{f,1}| \cdots |g_{f,m}|.$$

Further let Z be another locally convex space, $b : Y_1 \times \cdots \times Y_m \to Z$ a continuous m-linear map, and $j \in \{1, \ldots, m\}$. Then

$$b \circ (\gamma_1, \ldots, \gamma_j, \ldots, \gamma_m) \in \mathcal{C}^k_{\mathcal{W}}(U, Z)^\bullet \tag{†}$$

for all $\gamma_i \in \mathcal{C}^k_{\mathcal{W}_i}(U, Y_i)$ $(i \neq j)$ and $\gamma_j \in \mathcal{C}^k_{\mathcal{W}_j}(U, Y_j)^\bullet$. The map

$$\mathcal{C}^k_{\mathcal{W}_1}(U, Y_1) \times \cdots \times \mathcal{C}^k_{\mathcal{W}_j}(U, Y_j)^\bullet \times \cdots \times \mathcal{C}^k_{\mathcal{W}_m}(U, Y_m) \to \mathcal{C}^k_{\mathcal{W}}(U, Z)^\bullet,$$
$$(\gamma_1, \ldots, \gamma_j, \ldots, \gamma_m) \mapsto b \circ (\gamma_1, \ldots, \gamma_j, \ldots, \gamma_m)$$

is m-linear and continuous.

Proof. Using Proposition 3.4.14 and Lemma 3.4.9, we only have to prove that (†) holds. This is done by induction on k.

$k = 0$: Let $p \in \mathcal{N}(Z)$. Then there exist $q_1 \in \mathcal{N}(Y_1), \ldots, q_m \in \mathcal{N}(Y_m)$ such that

$$\|b(y_1, \ldots, y_m)\|_p \leq \|y_1\|_{q_1} \cdots \|y_m\|_{q_m}$$

for all $y_1 \in Y_1, \ldots, y_m \in Y_m$. So for $f \in \mathcal{W}$, $x \in U$ and $\gamma_1 \in \mathcal{C}^0_{\mathcal{W}_1}(U, Y_1), \ldots, \gamma_j \in \mathcal{C}^0_{\mathcal{W}_j}(U, Y_j)^\bullet, \ldots, \gamma_m \in \mathcal{C}^0_{\mathcal{W}_m}(U, Y_m)$ we compute

$$\begin{aligned}
&|f(x)|\, \|b \circ (\gamma_1, \ldots, \gamma_j, \ldots, \gamma_m)(x)\|_p \\
&\qquad \leq \prod_{i=1}^m |g_{f,i}(x)|\, \|\gamma_i(x)\|_{q_i} \leq \Big(\prod_{i \neq j} \|\gamma_i\|_{q_i, g_{f,i}, 0}\Big) |g_{f,j}(x)|\, \|\gamma_j(x)\|_{q_j}.
\end{aligned}$$

From this estimate we easily deduce that $b \circ (\gamma_1, \ldots, \gamma_j, \ldots, \gamma_m) \in \mathcal{C}^0_{\mathcal{W}_j}(U, Z)^\bullet$.

$k \to k+1$: From Proposition 3.4.11 (together with the induction base) we know that for $\gamma_1 \in \mathcal{C}^{k+1}_{\mathcal{W}_1}(U, Y_1), \dots, \gamma_j \in \mathcal{C}^{k+1}_{\mathcal{W}_j}(U, Y_j)^\bullet, \dots, \gamma_m \in \mathcal{C}^{k+1}_{\mathcal{W}_m}(U, Y_m)$

$$b \circ (\gamma_1, \dots, \gamma_j, \dots, \gamma_m) \in \mathcal{C}^{k+1}_{\mathcal{W}}(U, Z)^\bullet \Leftrightarrow D(b \circ (\gamma_1, \dots, \gamma_j, \dots, \gamma_m)) \in \mathcal{C}^{k}_{\mathcal{W}}(U, \mathrm{L}(X, Z))^\bullet.$$

We know from (3.4.13.1) in Lemma 3.4.13 that

$$\begin{aligned} D(b \circ (\gamma_1, \dots, \gamma_j, \dots, \gamma_m)) = &\sum_{\substack{i=1 \\ i \neq j}}^{m} b^{(i)} \circ (\gamma_1, \dots, \gamma_j, \dots, \gamma_{i-1}, D\gamma_i, \gamma_{i+1}, \dots, \gamma_m) \\ &+ b^{(j)} \circ (\gamma_1, \dots, \gamma_{j-1}, D\gamma_j, \gamma_{j+1}, \dots, \gamma_m). \end{aligned}$$

Noticing that $\gamma_j \in \mathcal{C}^{k}_{\mathcal{W}_j}(U, Y_j)^\bullet$ and $D\gamma_j \in \mathcal{C}^{k}_{\mathcal{W}_j}(U, \mathrm{L}(X, Y_j))^\bullet$, we can apply the inductive hypothesis to all $b^{(i)}$ and the $\mathcal{C}^k$-maps $\gamma_1, \dots, \gamma_m$ and $D\gamma_1, \dots, D\gamma_m$. Hence $D(b \circ (\gamma_1, \dots, \gamma_j, \dots, \gamma_m)) \in \mathcal{C}^{k}_{\mathcal{W}}(U, \mathrm{L}(X, Z))^\bullet$. ∎

As an application, we prove that the space of weighted functions into a product is canonically isomorphic to the product of the weighted function spaces.

LEMMA 3.4.16. *Let X be a normed space, $U \subseteq X$ an open nonempty set, $(Y_i)_{i \in I}$ a family of locally convex spaces, $k \in \overline{\mathbb{N}}$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$. Then for each $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U, \prod_{i \in I} Y_i)$ and $j \in I$,*

$$\pi_j \circ \gamma \in \mathcal{C}^k_{\mathcal{W}}(U, Y_j),$$

and the map

$$\mathcal{C}^k_{\mathcal{W}}\Big(U, \prod_{i \in I} Y_i\Big) \to \prod_{i \in I} \mathcal{C}^k_{\mathcal{W}}(U, Y_i) : \gamma \mapsto (\pi_i \circ \gamma)_{i \in I} \qquad (\dagger)$$

is an isomorphism of locally convex topological vector spaces.

The same statement holds for $\mathcal{C}^k_{\mathcal{W}}(U, \prod_{i \in I} Y_i)^\bullet$:

$$\mathcal{C}^k_{\mathcal{W}}\Big(U, \prod_{i \in I} Y_i\Big)^\bullet \to \prod_{i \in I} \mathcal{C}^k_{\mathcal{W}}(U, Y_i)^\bullet : \gamma \mapsto (\pi_i \circ \gamma)_{i \in I} \qquad (\dagger\dagger)$$

is an isomorphism of locally convex topological vector spaces.

Proof. We proved in Proposition 3.4.14 that for $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U, \prod_{i \in I} Y_i)$ and $j \in I$, $\pi_j \circ \gamma \in \mathcal{C}^k_{\mathcal{W}}(U, Y_j)$ and the map (†) is linear and continuous. Since a function to a product is determined by its components, the map (†) is also injective. What remains to be shown is the surjectivity, and the continuity of the inverse mapping. To this end, we notice that for each $j \in I$ and $p \in \mathcal{N}(Y_j)$, the map

$$P_{j,p} : \prod_{i \in I} Y_i \to \mathbb{R} : (y_i)_{i \in I} \mapsto \|y_j\|_p$$

is a continuous seminorm, and the set $\{P_{j,p} : j \in I, p \in \mathcal{N}(Y_j)\}$ generates $\mathcal{N}(\prod_{i \in I} Y_i)$. Now, for each $i \in I$ let $\gamma_i \in \mathcal{C}^k_{\mathcal{W}}(U, Y_i)$. We define the map

$$\gamma : U \to \prod_{i \in I} Y_i : x \mapsto (\gamma_i(x))_{i \in I}.$$

Then γ is a $\mathcal{C}^k$-map, and $P_{j,p} \circ \gamma = p \circ \gamma_j$. We see with Proposition A.3.2 that this implies that $\pi_{P_{j,p}} \circ \gamma$ is an $\mathcal{FC}^k$-map, and for each $f \in \mathcal{W}$ and $\ell \in \mathbb{N}$ with $\ell \leq k$ we derive the

identity
$$\|\pi_{P_{j,p}} \circ \gamma\|_{P_{j,p},f,\ell} = \|\pi_p \circ \gamma_j\|_{p,f,\ell}.$$
We proved in Lemma 3.4.2 that this identity implies that $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U, \prod_{i\in I} Y_i)$. Further it also proves that the inverse map of (†) is continuous using that it is linear.

The assertions about (††) follow from Corollary 3.4.15 and the assertions about (†). ∎

3.4.3.2. *Superposition with differentiable functions on weighted decreasing maps.* We show that a smooth function mapping 0 to 0 induces a superposition operator on $\mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet$, provided that $1_U \in \mathcal{W}$. The proof uses that the image of decreasing maps is (almost) compact, and so composition with a smooth map can be described in terms of compositions with bounded maps taking values in normed spaces.

On the image of decreasing maps

LEMMA 3.4.17. *Let U be an open nonempty subset of the finite-dimensional space X, Y a locally convex space, $k \in \overline{\mathbb{N}}$, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, and $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet$. Then*
$$\gamma(U) \cup \{0\}$$
is compact.

Proof. Since $1_U \in \mathcal{W}$, $\gamma \in \mathcal{C}^0_{\{1_U\}}(U,Y)^\bullet$. By the definition of this space, γ extends to a continuous map $\widetilde{\gamma} : U \cup \{\infty\} \to Y$ defined on the Alexandroff compactification of U by setting $\widetilde{\gamma}(\infty) := 0$. Hence $\widetilde{\gamma}(U \cup \{\infty\}) = \gamma(U) \cup \{0\}$ is compact. ∎

We give two easy consequences of the last lemma.

LEMMA 3.4.18. *Let U be an open nonempty subset of the finite-dimensional space X, V an open nonempty zero neighborhood of the* normed *space Y, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, and $k \in \overline{\mathbb{N}}$. Then $\mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet \subseteq \mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V)$.*

Proof. This is an immediate consequence of Lemma 3.4.17. ∎

LEMMA 3.4.19. *Let U be an open nonempty subset of the finite-dimensional space X, Y a normed space, $V \subseteq Y$ an open zero neighborhood, $k \in \mathbb{N}$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$. Then $\mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet$ is open in $\mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet$.*

Proof. We proved in Lemma 3.4.18 that $\mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet \subseteq \mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V)$. Hence $\mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet = \mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V) \cap \mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet$ is open in $\mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet$. ∎

Superposition with a bounded map. As a preparation, we prove a version of Lemma 3.3.11 for decreasing functions. Further, we calculate the differentials of the superposition operator.

LEMMA 3.4.20. *Let U be an open nonempty subset of the finite-dimensional space X, Y and Z normed spaces, $V \subseteq Y$ open and star-shaped with center 0, $k, \ell \in \mathbb{N}$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$. Further let $\phi \in \mathcal{BC}^{k+\ell+1}(V,Z)$ with $\phi(0) = 0$. Then*
$$\phi \circ \mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet \subseteq \mathcal{C}^k_{\mathcal{W}}(U,Z)^\bullet,$$
and
$$\mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet \to \mathcal{C}^k_{\mathcal{W}}(U,Z)^\bullet : \gamma \mapsto \phi \circ \gamma$$
is a $\mathcal{C}^\ell$-map.

Proof. We proved in Lemma 3.4.18 that $\mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet \subseteq \mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V)$. Hence we can apply Proposition 3.3.12 to see that

$$\phi \circ \mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet \subseteq \mathcal{C}^k_{\mathcal{W}}(U,Z)$$

and the map

$$\mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet \to \mathcal{C}^k_{\mathcal{W}}(U,Z) : \gamma \mapsto \phi \circ \gamma$$

is $\mathcal{C}^\ell$; here we used that $\mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet = \mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V) \cap \mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet$. Because $\mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet$ is closed in $\mathcal{C}^k_{\mathcal{W}}(U,Y)$ by Lemma 3.4.9, it only remains to show that for each $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet$, we have $\phi \circ \gamma \in \mathcal{C}^k_{\mathcal{W}}(U,Z)^\bullet$. This is done by induction on k:

$k = 0$: Let $f \in \mathcal{W}$ and $x \in U$. Then

$$\begin{aligned}|f(x)|\,\|\phi(\gamma(x))\| &= |f(x)|\,\|\phi(\gamma(x)) - \phi(0)\| \\ &= |f(x)|\left\|\int_0^1 D\phi(t\gamma(x)) \cdot \gamma(x)\,dt\right\| \leq \|D\phi\|_{\mathrm{op},\infty}|f(x)|\,\|\gamma(x)\|;\end{aligned}$$

here we used that the line segment from 0 to $\gamma(x)$ is contained in V. From this estimate we conclude that $\phi \circ \gamma \in \mathcal{C}^0_{\mathcal{W}}(U,Z)^\bullet$.

$k \to k+1$: By the chain rule

$$D(\phi \circ \gamma) = (D\phi \circ \gamma) \cdot D\gamma.$$

Now $D\phi \circ \gamma \in \mathcal{BC}^{k+1}(U, \mathrm{L}(Y,Z))$ by Lemma 3.3.8, since $\gamma \in \mathcal{BC}^{k+1}(U,V)$. Further $D\gamma \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Y))^\bullet$, so Corollary 3.4.15 yields $(D\phi \circ \gamma) \cdot D\gamma \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Z))^\bullet$. By Proposition 3.4.11, the case $k+1$ follows from the inductive hypothesis. ■

LEMMA 3.4.21. *Let X, Y and Z be normed spaces, $U \subseteq X$ and $V \subseteq Y$ open subsets such that V is star-shaped with center 0, $k \in \overline{\mathbb{N}}$, $m \in \mathbb{N}^*$, $\phi \in \mathcal{BC}^{k+m+1}(V,Z)_0$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$. By Lemma 3.3.11,*

$$\phi_* : \mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V) \to \mathcal{C}^k_{\mathcal{W}}(U,Z) : \gamma \mapsto \phi \circ \gamma$$

is defined and $\mathcal{C}^m$. For its ℓth differential, we have

$$d^{(\ell)}\phi_*(\gamma;\gamma_1,\ldots,\gamma_\ell) = d^{(\ell)}\phi \circ (\gamma,\gamma_1,\ldots,\gamma_\ell) \qquad (\ell \leq m).$$

Proof. Let $x \in U$. Using the identity $\mathrm{ev}^Z_x \circ \phi_* = \phi \circ \mathrm{ev}^Y_x$ (with self-explanatory notation for point evaluations), we calculate

$$\begin{aligned}(\mathrm{ev}^Z_x \circ d^{(\ell)}\phi_*)(\gamma;\gamma_1,\ldots,\gamma_\ell) &= d^{(\ell)}(\mathrm{ev}^Z_x \circ \phi_*)(\gamma;\gamma_1,\ldots,\gamma_\ell) = d^{(\ell)}(\phi \circ \mathrm{ev}^Y_x)(\gamma;\gamma_1,\ldots,\gamma_\ell) \\ &= (d^{(\ell)}\phi \circ (\mathrm{ev}^Y_x)^{\ell+1})(\gamma,\gamma_1,\ldots,\gamma_\ell) = \mathrm{ev}^Z_x(d^{(\ell)}\phi \circ (\gamma,\gamma_1,\ldots,\gamma_\ell));\end{aligned}$$

here we used Lemmas A.1.16 and A.1.17. ■

The main result. Before we can prove the main result, we need the following facts concerning compact and star-shaped sets in topological vector spaces.

LEMMA 3.4.22. *Let Z be a locally convex space and $K \subseteq Z$ a compact set.*

(a) *The set $[0,1] \cdot K$ is compact and star-shaped with center 0.*
(b) *Let K be star-shaped and V an open neighborhood of K. Then there exists an open star-shaped set W such that $K \subseteq W \subseteq V$.*

Proof. (a) $[0,1] \cdot K$ is compact since it is the image of a compact set under a continuous map.

(b) The set $K \times \{0\}$ is compact, hence using the continuity of the addition and the Wallace lemma, we find an open 0-neighborhood U such that $K+U \subseteq V$. We may assume that U is absolutely convex. Then $K+U$ is open, star-shaped and contained in V. ∎

PROPOSITION 3.4.23. *Let U be an open nonempty subset of the finite-dimensional space X, Y and Z locally convex spaces, $V \subseteq Y$ open and star-shaped with center 0, $k, m \in \mathbb{N}$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$. Let $\phi \in \mathcal{C}^{k+m+2}(V, Z)$ with $\phi(0) = 0$. Then for $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U, V)^\bullet$,*

$$\phi \circ \gamma \in \mathcal{C}^k_{\mathcal{W}}(U, Z)^\bullet,$$

and the map

$$\phi_* : \mathcal{C}^k_{\mathcal{W}}(U, V)^\bullet \to \mathcal{C}^k_{\mathcal{W}}(U, Z)^\bullet : \gamma \mapsto \phi \circ \gamma$$

is $\mathcal{C}^m$ with

$$d^{(\ell)}\phi_*(\gamma; \gamma_1, \dots, \gamma_\ell) = d^{(\ell)}\phi \circ (\gamma, \gamma_1, \dots, \gamma_\ell) \quad \text{for all } \ell \leq m.$$

Proof. Let $\widetilde{\gamma} \in \mathcal{C}^k_{\mathcal{W}}(U, V)^\bullet$. By Lemmas 3.4.17 and 3.4.22, the set

$$K := [0,1] \cdot (\widetilde{\gamma}(U) \cup \{0\})$$

is compact and star-shaped with center 0. Hence by Lemma A.3.4, for each $p \in \mathcal{N}(Z)$ there exists a $q \in \mathcal{N}(Y)$ and an open set $W \supseteq K$ with respect to q such that $\tilde{\phi} \in \mathcal{BC}^{k+m+1}(W_q, Z_p)$. In view of Lemma 3.4.22, we may assume that W (and hence W_q) is star-shaped with center 0. We know from Lemma 3.4.19 that $\mathcal{C}^k_{\mathcal{W}}(U, W_q)^\bullet$ is a neighborhood of $\pi_q \circ \widetilde{\gamma}$ in $\mathcal{C}^k_{\mathcal{W}}(U, Y_q)^\bullet$. In Lemma 3.4.20 we stated that

$$\tilde{\phi}_* : \mathcal{C}^k_{\mathcal{W}}(U, W_q)^\bullet \to \mathcal{C}^k_{\mathcal{W}}(U, Z_p)^\bullet : \gamma \mapsto \tilde{\phi} \circ \gamma$$

is $\mathcal{C}^m$. The diagram

$$\begin{tikzcd}
\mathcal{C}^k_{\mathcal{W}}(U, W)^\bullet \arrow[rr, "\pi_{q*}"] \arrow[dr, "(\pi_p \circ \phi)_*"'] & & \mathcal{C}^k_{\mathcal{W}}(U, W_q)^\bullet \arrow[dl, "\tilde{\phi}_*"] \\
& \mathcal{C}^k_{\mathcal{W}}(U, Z_p)^\bullet &
\end{tikzcd}$$

is commutative. This implies that $(\pi_p \circ \phi)_*$ is $\mathcal{C}^m$ on $\mathcal{C}^k_{\mathcal{W}}(U, W)^\bullet$ since it is the composition of $\tilde{\phi}_*$ and the smooth map π_{q*} (see Corollary 3.4.15). By Lemmas A.1.17 and 3.4.21 we can calculate its higher derivatives:

$$\begin{aligned}
d^{(\ell)}(\pi_p \circ \phi)_*|_{\mathcal{C}^k_{\mathcal{W}}(U,W)^\bullet}(\gamma; \gamma_1, \dots, \gamma_\ell) &= d^{(\ell)}(\tilde{\phi} \circ \pi_q)_*|_{\mathcal{C}^k_{\mathcal{W}}(U,W)^\bullet}(\gamma; \gamma_1, \dots, \gamma_\ell) \\
&= d^{(\ell)}\tilde{\phi}_*(\pi_q \circ \gamma; \pi_q \circ \gamma_1, \dots, \pi_q \circ \gamma_\ell) = d^{(\ell)}\tilde{\phi} \circ (\pi_q \circ \gamma, \pi_q \circ \gamma_1, \dots, \pi_q \circ \gamma_\ell) \\
&= d^{(\ell)}(\tilde{\phi} \circ \pi_q) \circ (\gamma, \gamma_1, \dots, \gamma_\ell) = d^{(\ell)}(\pi_p \circ \phi) \circ (\gamma, \gamma_1, \dots, \gamma_\ell) \\
&= \pi_p \circ d^{(\ell)}\phi \circ (\gamma, \gamma_1, \dots, \gamma_\ell)
\end{aligned}$$

for $\ell \in \mathbb{N}$ with $\ell \leq m$.

Since $\widetilde{\gamma}$ and p were arbitrary, we conclude that the map

$$\mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet \to \prod_{p\in\mathcal{N}(Z)} \mathcal{C}^k_{\mathcal{W}}(U,Z_p)^\bullet : \gamma \mapsto (\pi_p \circ \phi \circ \gamma)_{p\in\mathcal{N}(Z)}$$

is $\mathcal{C}^m$. Since its image and all directional derivatives are contained in $\mathcal{C}^k_{\mathcal{W}}(U,Z)^\bullet$ (in the sense of Lemma 3.4.2), we conclude that it is $\mathcal{C}^m$ as a map to $\mathcal{C}^k_{\mathcal{W}}(U,Z)^\bullet$. ∎

4. Lie groups of weighted diffeomorphisms

In this chapter, we prove that for each Banach space X appropriate subgroups of the diffeomorphism group $\mathrm{Diff}(X)$ can be turned into Lie groups that are modelled on some weighted function space described earlier. Further, we show that these Lie groups are regular. Here

$$\mathrm{Diff}(X) := \{\phi \in \mathcal{FC}^\infty(X,X) : \phi \text{ is bijective and } \phi^{-1} \in \mathcal{FC}^\infty(X,X)\};$$

the chain rule ensures that $\mathrm{Diff}(X)$ is actually a group with composition and inversion as group operations.

4.1. Weighted diffeomorphisms and endomorphisms. In this section, we define and examine sets of *weighted endomorphisms* $\mathrm{End}_{\mathcal{W}}(X)$ and *weighted diffeomorphisms* $\mathrm{Diff}_{\mathcal{W}}(X)$. We show that if $1_X \in \mathcal{W}$, then $\mathrm{End}_{\mathcal{W}}(X)$ is a smooth monoid and $\mathrm{Diff}_{\mathcal{W}}(X)$ is its group of units that can be turned into a Lie group. Further, we discuss certain subsets of these, the *decreasing weighted diffeomorphisms* respectively *endomorphisms.*

For nonempty $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$, we define

$$\mathrm{Diff}_{\mathcal{W}}(X) := \{\phi \in \mathrm{Diff}(X) : \phi - \mathrm{id}_X,\ \phi^{-1} - \mathrm{id}_X \in \mathcal{C}^\infty_{\mathcal{W}}(X,X)\},$$
$$\mathrm{End}_{\mathcal{W}}(X) := \{\gamma + \mathrm{id}_X : \gamma \in \mathcal{C}^\infty_{\mathcal{W}}(X,X)\}.$$

The set $\mathrm{End}_{\mathcal{W}}(X)$ can be turned into a smooth manifold using the differentiable structure generated by the bijective map

$$\kappa_{\mathcal{W}} : \mathcal{C}^\infty_{\mathcal{W}}(X,X) \to \mathrm{End}_{\mathcal{W}}(X) : \gamma \mapsto \gamma + \mathrm{id}_X . \tag{4.1.0.1}$$

We clarify the relation between $\mathrm{End}_{\mathcal{W}}(X)$ and $\mathrm{Diff}_{\mathcal{W}}(X)$. The following is obvious:

Lemma 4.1.1. *Let $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ and $\phi \in \mathrm{Diff}(X)$. Then*

$$\phi \in \mathrm{Diff}_{\mathcal{W}}(X) \iff \phi, \phi^{-1} \in \mathrm{End}_{\mathcal{W}}(X).$$

Furthermore, we have

Lemma 4.1.2. *Let $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ such that $\mathrm{End}_{\mathcal{W}}(X)$ is a monoid with respect to the composition of maps. Then the group of units is given by*

$$\mathrm{End}_{\mathcal{W}}(X)^\times = \mathrm{Diff}_{\mathcal{W}}(X);$$

in particular $\mathrm{Diff}_{\mathcal{W}}(X)$ is a subgroup of $\mathrm{Diff}(X)$.

Proof. Obviously

$$\phi \in \mathrm{End}_{\mathcal{W}}(X)^\times \iff \phi \text{ is bijective and } \phi, \phi^{-1} \in \mathrm{End}_{\mathcal{W}}(X).$$

Since $\mathrm{End}_{\mathcal{W}}(X)$ consists of smooth maps, the assertion follows from Lemma 4.1.1. ∎

In the rest of this section, we prove that $\operatorname{End}_{\mathcal{W}}(X)$ is a smooth monoid if $1_X \in \mathcal{W}$; thus $\operatorname{Diff}_{\mathcal{W}}(X)$ is a group by Lemma 4.1.2. Further, we define the set of *weighted decreasing endomorphisms* and show that it is a closed submonoid of $\operatorname{End}_{\mathcal{W}}(X)$. The main part is to show that the monoid multiplication

$$\circ : \operatorname{End}_{\mathcal{W}}(X) \times \operatorname{End}_{\mathcal{W}}(X) \to \operatorname{End}_{\mathcal{W}}(X)$$

is defined and smooth, so we elaborate on this.

4.1.1. Composition of weighted endomorphisms in charts. We study what composition looks like in the global chart $\kappa_{\mathcal{W}}^{-1}$ (from (4.1.0.1)). For $\eta, \gamma \in \mathcal{C}^{\infty}_{\mathcal{W}}(X, X)$,

$$\kappa_{\mathcal{W}}(\gamma) \circ \kappa_{\mathcal{W}}(\eta) = (\gamma + \mathrm{id}_X) \circ (\eta + \mathrm{id}_X) = \gamma \circ (\eta + \mathrm{id}_X) + \eta + \mathrm{id}_X\,. \tag{4.1.2.1}$$

Obviously $\kappa_{\mathcal{W}}(\gamma) \circ \kappa_{\mathcal{W}}(\eta) \in \operatorname{End}_{\mathcal{W}}(X)$ if and only if $\gamma \circ (\eta + \mathrm{id}_X) \in \mathcal{C}^{\infty}_{\mathcal{W}}(X, X)$; and the smoothness of $\circ$ is equivalent to that of

$$\mathcal{C}^{\infty}_{\mathcal{W}}(X, X) \times \mathcal{C}^{\infty}_{\mathcal{W}}(X, X) \to \mathcal{C}^{\infty}_{\mathcal{W}}(X, X) : (\gamma, \eta) \mapsto \gamma \circ (\eta + \mathrm{id}_X).$$

4.1.1.1. *Important maps.* For technical reasons we look at more general maps

$$g_Y : \mathcal{C}^0(W, Y) \times \mathcal{C}^0(U, V) \to \mathcal{C}^0(U, Y) : (\gamma, \eta) \mapsto \gamma \circ (\eta + \mathrm{id}_X); \tag{4.1.2.2}$$

here $U, V, W \subseteq X$ are open nonempty subsets with $V + U \subseteq W$ and Y is a normed space. These maps play an important role in further discussions.

Continuity properties. We discuss when the restriction of g_Y to weighted function spaces has values in a weighted function space and is continuous. We start with the following lemma whose assertion is used as the base case for Lemma 4.1.4.

LEMMA 4.1.3. *Let X and Y be normed spaces, $U, V, W \subseteq X$ open nonempty subsets such that $V + U \subseteq W$ and V is balanced, and $\mathcal{W} \subseteq \overline{\mathbb{R}}^W$.*

(a) *For $\gamma \in \mathcal{C}^0_{\mathcal{W}}(W, Y) \cap \mathcal{BC}^1(W, Y)$, $\eta \in \mathcal{C}^0_{\mathcal{W}}(U, V)$, $f \in \mathcal{W}$ and $x \in U$,*

$$|f(x)|\, \|g_Y(\gamma, \eta)(x)\| \le |f(x)|\, (\|\gamma\|_{1_{\{x\}+\mathbb{D}\eta(U)},1}\, \|\eta(x)\| + \|\gamma(x)\|). \tag{4.1.3.1}$$

In particular

$$g_Y(\gamma, \eta) = \gamma \circ (\eta + \mathrm{id}_X) \in \mathcal{C}^0_{\mathcal{W}}(U, Y).$$

(b) *Let $\gamma, \gamma_0 \in \mathcal{C}^0_{\mathcal{W}}(W, Y) \cap \mathcal{BC}^1(W, Y)$ and $\eta, \eta_0 \in \mathcal{C}^0_{\mathcal{W}}(U, V)$ such that*

$$\{t\eta(x) + (1 - t)\eta_0(x) : t \in [0, 1], x \in U\} \subseteq V.$$

Then for each $f \in \mathcal{W}$,

$$\begin{aligned}\|g_Y(\gamma, \eta) - g_Y(\gamma_0, \eta_0)\|_{f,0} \le{}& \|\gamma\|_{1_W,1}\|\eta - \eta_0\|_{f,0} \\ &+ \|\gamma - \gamma_0\|_{1_W,1}\|\eta_0\|_{f,0} + \|\gamma - \gamma_0\|_{f,0}.\end{aligned} \tag{4.1.3.2}$$

In particular, if $1_W \in \mathcal{W}$ then the map

$$g_{Y,0} : \mathcal{C}^1_{\mathcal{W}}(W, Y) \times \mathcal{C}^{\partial,0}_{\mathcal{W}}(U, V) \to \mathcal{C}^0_{\mathcal{W}}(U, Y) : (\gamma, \eta) \mapsto g_Y(\gamma, \eta)$$

is continuous.

Proof. (a) For $x \in U$, using the triangle inequality and the mean value theorem we derive

$$\begin{aligned}
|f(x)|\,\|g_Y(\gamma,\eta)(x)\| &= |f(x)|\,\|\gamma(\eta(x)+x)\| \\
&\leq |f(x)|\,\|\gamma(\eta(x)+x)-\gamma(x)\| + |f(x)|\,\|\gamma(x)\| \\
&= |f(x)| \left\| \int_0^1 D\gamma(x+t\eta(x))\cdot\eta(x)\,dt \right\| + |f(x)|\,\|\gamma(x)\| \\
&\leq |f(x)|\,\|D\gamma|_{\{x\}+\mathbb{D}\eta(U)}\|_{\mathrm{op},\infty}\|\eta(x)\| + |f(x)|\,\|\gamma(x)\|
\end{aligned}$$

and from this we easily deduce the assertion. We could apply the mean value theorem because the line segment $\{x+t\eta(x) : t\in[0,1]\}$ is contained in $U+V$ since V is balanced.

(b) For $x \in U$ we have

$$|f(x)|\,\|g_{Y,0}(\gamma,\eta)(x) - g_{Y,0}(\gamma_0,\eta_0)(x)\| = |f(x)|\,\|\gamma(\eta(x)+x)-\gamma_0(\eta_0(x)+x)\|.$$

We add $0 = \gamma(\eta_0(x)+x) - \gamma(\eta_0(x)+x)$ and apply the triangle inequality to see that

$$\begin{aligned}
&|f(x)|\,\|\gamma(\eta(x)+x)-\gamma(\eta_0(x)+x)+\gamma(\eta_0(x)+x)-\gamma_0(\eta_0(x)+x)\| \\
&\qquad\leq |f(x)|\,\|\gamma(\eta(x)+x)-\gamma(\eta_0(x)+x)\| + |f(x)|\,\|(\gamma-\gamma_0)(\eta_0(x)+x)\|.
\end{aligned}$$

We discuss the two summands separately. For the first summand, we can apply the mean value theorem (Proposition A.2.11) because we assumed that the line segment $\{t\eta(x)+(1-t)\eta_0(x) : t\in[0,1]\}$ is contained in V, and get

$$\begin{aligned}
&|f(x)|\,\|\gamma(\eta(x)+x)-\gamma(\eta_0(x)+x)\| \\
&\qquad= |f(x)| \left\| \int_0^1 D\gamma(t\eta(x)+(1-t)\eta_0(x)+x)\cdot(\eta(x)-\eta_0(x))\,dt \right\| \\
&\qquad\leq |f(x)|\,\|\gamma\|_{1_W,1}\|\eta(x)-\eta_0(x)\|.
\end{aligned}$$

By applying the mean value theorem, which is possible because V is balanced, the second summand becomes

$$\begin{aligned}
&|f(x)|\,\|(\gamma-\gamma_0)(\eta_0(x)+x)\| \\
&\qquad= |f(x)|\,\|(\gamma-\gamma_0)(\eta_0(x)+x)-(\gamma-\gamma_0)(x)+(\gamma-\gamma_0)(x)\| \\
&\qquad\leq |f(x)| \left(\left\| \int_0^1 D(\gamma-\gamma_0)(t\eta_0(x)+x)\cdot\eta_0(x)\,dt \right\| + \|(\gamma-\gamma_0)(x)\| \right) \\
&\qquad\leq |f(x)| \left(\|\gamma-\gamma_0\|_{1_W,1}\|\eta_0(x)\| + \|(\gamma-\gamma_0)(x)\| \right).
\end{aligned}$$

Combining these two estimates gives (4.1.3.2).

The continuity of $g_{Y,0}$ follows from this estimate: For each $\eta \in \mathcal{C}^{\partial,0}_{\mathcal{W}}(U,V)$, there exists an $r>0$ such that

$$\eta(U) + B_r(0) \subseteq V,$$

and since $1_W \in \mathcal{W}$,

$$F_\eta := \{\widetilde{\eta} \in \mathcal{C}^0_{\mathcal{W}}(U,X) : \|\eta-\widetilde{\eta}\|_{1_W,0} < r\}$$

is a neighborhood of η in $\mathcal{C}^{\partial,0}_{\mathcal{W}}(U,V)$. The estimate (4.1.3.2) ensures that $g_{Y,0}$ is continuous on $\mathcal{C}^1_{\mathcal{W}}(W,Y)\times F_\eta$. ∎

Lemma 4.1.4. *Let X and Y be normed spaces, $U, V, W \subseteq X$ open nonempty subsets such that $V + U \subseteq W$ and V is balanced, $k \in \mathbb{N}$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^W$ with $1_W \in \mathcal{W}$. Then*

$$g_Y(\mathcal{C}^{k+1}_{\mathcal{W}}(W,Y) \times \mathcal{C}^k_{\mathcal{W}}(U,V)) \subseteq \mathcal{C}^k_{\mathcal{W}}(U,Y),$$

and the map

$$g_{Y,k} : \mathcal{C}^{k+1}_{\mathcal{W}}(W,Y) \times \mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V) \to \mathcal{C}^k_{\mathcal{W}}(U,Y) : (\gamma,\eta) \mapsto g_Y(\gamma,\eta)$$

which arises by restricting g_Y is continuous.

Proof. The proof is by induction. The case $k = 0$ was treated in Lemma 4.1.3.

$k \to k+1$: We use Proposition 3.2.3 (and Lemma 4.1.3) to see that

$$g_Y(\mathcal{C}^{k+2}_{\mathcal{W}}(W,Y) \times \mathcal{C}^{k+1}_{\mathcal{W}}(U,V)) \subseteq \mathcal{C}^{k+1}_{\mathcal{W}}(U,Y)$$

is equivalent to

$$(D \circ g_Y)(\mathcal{C}^{k+2}_{\mathcal{W}}(W,Y) \times \mathcal{C}^{k+1}_{\mathcal{W}}(U,V)) \subseteq \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Y));$$

and that the continuity of $g_{Y,k+1}$ is equivalent to that of $D \circ g_{Y,k+1}$.

Applying the chain rule to g_Y shows that for $\gamma \in \mathcal{C}^{k+2}_{\mathcal{W}}(W,Y)$ and $\eta \in \mathcal{C}^{k+1}_{\mathcal{W}}(U,V)$,

$$(D \circ g_Y)(\gamma,\eta) = g_{\mathrm{L}(X,Y),k}(D\gamma,\eta) \cdot (D\eta + \mathrm{id}), \tag{$*$}$$

where $\cdot$ denotes the composition of linear maps (see Corollary 3.3.6) and id denotes the constant map $x \mapsto \mathrm{id}_X$. Since $D\gamma \in \mathcal{C}^{k+1}_{\mathcal{W}}(W, \mathrm{L}(X,Y))$, we derive from the induction hypothesis that

$$g_{\mathrm{L}(X,Y),k}(D\gamma,\eta) \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Y)).$$

Hence we conclude from Corollary 3.3.6 and $D\eta + \mathrm{id} \in \mathcal{BC}^k(U, \mathrm{L}(X))$ that

$$(D \circ g_Y)(\gamma,\eta) \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Y)).$$

The continuity of $D \circ g_{Y,k+1}$ follows easily from $(*)$: We use the inductive hypothesis to conclude that $g_{\mathrm{L}(X,Y),k}$ is continuous. Since D and

$$\cdot : \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Y)) \times \mathcal{BC}^k(U, \mathrm{L}(X)) \to \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X,Y))$$

are smooth (see Proposition 3.2.3 and Corollary 3.3.6) as well as the translation with id in $\mathcal{BC}^k(U, \mathrm{L}(X))$, the continuity of $g_{Y,k+1}$ is proved. ∎

Restriction to decreasing functions. Finally, we study the restriction of $g_{Y,k}$ to decreasing functions.

Lemma 4.1.5. *Let X and Y be normed spaces, $U, V, W \subseteq X$ open nonempty subsets such that $V + U \subseteq W$ and V is balanced, $k \in \mathbb{N}$ and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. Then*

$$g_{Y,k}(\mathcal{C}^{k+1}_{\mathcal{W}}(W,Y)^o \times \mathcal{C}^k_{\mathcal{W}}(U,V)) \subseteq \mathcal{C}^k_{\mathcal{W}}(U,Y)^o.$$

Proof. The proof is by induction on k:

$k = 0$: We use estimate (4.1.3.1) in Lemma 4.1.3. Let $f \in \mathcal{W}$, $\gamma \in \mathcal{C}^1_{\mathcal{W}}(W,Y)^o$ and $\eta \in \mathcal{C}^0_{\mathcal{W}}(U,V)$. Then for every $\varepsilon > 0$ there exists $r > 0$ such that

$$\|\gamma|_{W \setminus B_r(0)}\|_{f,0} < \varepsilon/2$$

and (as $1_X \in \mathcal{W}$)

$$\|\gamma|_{W\setminus B_r(0)}\|_{1_W,1} < \frac{\varepsilon}{2(\|\eta\|_{f,0}+1)}.$$

Since $1_X \in \mathcal{W}$, we have $K := \|\eta\|_{1_U,0} < \infty$. Let $R \in \mathbb{R}$ such that $R > r + K$. Then for each $x \in U \setminus B_R(0)$

$$x + \mathbb{D}\eta(x) \subseteq W \setminus B_r(0),$$

so we conclude from estimate (4.1.3.1) that

$$|f(x)|\,\|g_{Y,k}(\gamma,\eta)(x)\| \leq \|\gamma\|_{1_{\{x\}+\mathbb{D}\eta(U)},1}\,\|\eta\|_{f,0} + |f(x)|\,\|\gamma(x)\| < \frac{\varepsilon}{2(\|\eta\|_{f,0}+1)}\|\eta\|_{f,0} + \frac{\varepsilon}{2}$$

for $x \in U \setminus B_R(0)$. Thus $g_{Y,k}(\gamma,\eta) \in \mathcal{C}^0_{\mathcal{W}}(U,Y)^o$.

$k \to k+1$: We calculate using the chain rule that

$$(D \circ g_{Y,k+1})(\gamma,\eta) = g_{\mathrm{L}(X,Y),k}(D\gamma,\eta)\cdot(D\eta + \mathrm{id}).$$

Since $D\gamma \in \mathcal{C}^{k+1}_{\mathcal{W}}(W,\mathrm{L}(X,Y))^o$ (see Corollary 3.2.4),

$$g_{\mathrm{L}(X,Y),k}(D\gamma,\eta) \in \mathcal{C}^k_{\mathcal{W}}(U,\mathrm{L}(X,Y))^o$$

by the inductive hypothesis. Further, $D\eta + \mathrm{id} \in \mathcal{BC}^k(U,\mathrm{L}(X))$, so we conclude from Corollary 3.3.4 that

$$(D \circ g_{Y,k+1})(\gamma,\eta) \in \mathcal{C}^k_{\mathcal{W}}(U,\mathrm{L}(X,Y))^o.$$

From this (and the base case $k = 0$) we conclude by Corollary 3.2.4 that

$$g_{Y,k+1}(\gamma,\eta) \in \mathcal{C}^{k+1}_{\mathcal{W}}(U,Y)^o. \blacksquare$$

Differentiability properties. We discuss whether restrictions of $g_{Y,k}$ to certain weighted function spaces are differentiable. First, we provide a nice identity for the differential quotient of $g_{Y,k}$.

LEMMA 4.1.6. *Let X and Y be normed spaces, $U, V, W \subseteq X$ open nonempty subsets such that $V + U \subseteq W$ and V is balanced, $\mathcal{W} \subseteq \overline{\mathbb{R}}^W$ with $1_W \in \mathcal{W}$, $\gamma, \gamma_1 \in \mathcal{C}^2_{\mathcal{W}}(W,Y)$, $\eta \in \mathcal{C}^0_{\mathcal{W}}(U,V)$, $\eta_1 \in \mathcal{C}^0_{\mathcal{W}}(U,X)$ and $t \in \mathbb{K}^*$. Further, suppose that*

$$\{\eta + s t\eta_1 : s \in [0,1]\} \subseteq \mathcal{C}^0_{\mathcal{W}}(U,V).$$

Then for each $x \in U$,

$$\mathrm{ev}_x\left(\frac{g_{Y,1}(\gamma + t\gamma_1, \eta + t\eta_1) - g_{Y,1}(\gamma,\eta)}{t}\right) = \int_0^1 \mathrm{ev}_x\big(g_{\mathrm{L}(X,Y),1}(D(\gamma + st\gamma_1), \eta + st\eta_1)\cdot\eta_1 + g_{Y,1}(\gamma_1, \eta + st\eta_1)\big)\,ds.$$

Proof. We first prove that the relevant weak integral exists. To this end, we take a closer look at the integrand. Since $\{\eta + st\eta_1 : s \in [0,1]\} \subseteq \mathcal{C}^0_{\mathcal{W}}(U,V)$, we have

$$\begin{aligned}&\mathrm{ev}_x\big(g_{\mathrm{L}(X,Y),1}(D(\gamma + st\gamma_1), \eta + st\eta_1)\cdot\eta_1 + g_{Y,1}(\gamma_1, \eta + st\eta_1)\big)\\ &= D\gamma(\eta(x)+st\eta_1(x)+x)\cdot\eta_1(x) + stD\gamma_1(\eta(x)+st\eta_1(x)+x)\cdot\eta_1(x) + \gamma_1(\eta(x)+st\eta_1(x)+x).\end{aligned}$$

The mean value theorem yields

$$\int_0^1 D\gamma(\eta(x) + st\eta_1(x) + x)\cdot\eta_1(x)\,ds = \frac{\gamma(\eta(x) + t\eta_1(x) + x) - \gamma(\eta(x) + x)}{t}$$

and

$$\int_0^1 \big(stD\gamma_1(\eta(x)+st\eta_1(x)+x)\cdot\eta_1(x)+\gamma_1(\eta(x)+st\eta_1(x)+x)\big)\,ds \\ = \gamma_1(\eta(x)+t\eta_1(x)+x);$$

the latter identity follows from the fact that

$$\frac{d}{ds}s\gamma_1(\eta(x)+st\eta_1(x)+x) = stD\gamma_1(\eta(x)+st\eta_1(x)+x)\cdot\eta_1(x)+\gamma_1(\eta(x)+st\eta_1(x)+x).$$

So the integral exists and has the value

$$\frac{\gamma(\eta(x)+t\eta_1(x)+x)-\gamma(\eta(x)+x)}{t}+\gamma_1(\eta(x)+t\eta_1(x)+x) \\ = \frac{g_{Y,1}(\gamma+t\gamma_1,\eta+t\eta_1)(x)-g_{Y,1}(\gamma,\eta)(x)}{t}. \qquad \blacksquare$$

PROPOSITION 4.1.7. *Let X and Y be normed spaces, $U, V, W \subseteq X$ open nonempty subsets such that $V+U\subseteq W$ and V is balanced, $\mathcal{W}\subseteq\overline{\mathbb{R}}^W$ with $1_W\in\mathcal{W}$, $k\in\mathbb{N}$ and $\ell\in\mathbb{N}^*$. Then*

$$g_{Y,k,\ell}:\mathcal{C}^{k+\ell+1}_{\mathcal{W}}(W,Y)\times\mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V)\to\mathcal{C}^k_{\mathcal{W}}(U,Y):(\gamma,\eta)\mapsto\gamma\circ(\eta+\mathrm{id}_X)$$

is a $\mathcal{C}^\ell$-map with the directional derivative

$$dg_{Y,k,\ell}(\gamma,\eta;\gamma_1,\eta_1) = g_{\mathrm{L}(X,Y),k,\ell-1}(D\gamma,\eta)\cdot\eta_1+g_{Y,k,\ell}(\gamma_1,\eta). \tag{4.1.7.1}$$

Proof. This is proved by induction.

$\ell=1$: From Lemmas 4.1.6 and 3.2.13 we conclude that for $\gamma,\gamma_1\in\mathcal{C}^{k+\ell+1}_{\mathcal{W}}(W,Y)$, $\eta\in\mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V)$, $\eta_1\in\mathcal{C}^k_{\mathcal{W}}(U,X)$ and for all $t\in\mathbb{R}^*$ in a suitable neighborhood of 0 we have the identity

$$\frac{g_{Y,k,\ell}(\gamma+t\gamma_1,\eta+t\eta_1)-g_{Y,k,\ell}(\gamma,\eta)}{t} = \int_0^1 g_{\mathrm{L}(X,Y),k,\ell-1}(D(\gamma+st\gamma_1),\eta+st\eta_1)\cdot\eta_1\,ds \\ + \int_0^1 g_{Y,k,\ell}(\gamma_1,\eta+st\eta_1)\,ds.$$

The theorem about parameter dependent integrals (Proposition A.1.8) yields the assertions if we let $t\to 0$ in the above expression.

$\ell-1\to\ell$: This follows easily from (4.1.7.1). Since D and $\cdot$ are smooth (see Proposition 3.2.3 and Corollary 3.3.7) and $g_{\mathrm{L}(X,Y),k,\ell-1}$ resp. $g_{Y,k,\ell}$ are $\mathcal{C}^{\ell-1}$ by the inductive hypothesis, $dg_{Y,k,\ell}$ is $\mathcal{C}^{\ell-1}$ and hence $g_{Y,k,\ell}$ is $\mathcal{C}^\ell$. $\blacksquare$

COROLLARY 4.1.8. *Let X and Y be normed spaces, $U, V, W\subseteq X$ open nonempty subsets such that $V+U\subseteq W$ and V is balanced, $\mathcal{W}\subseteq\overline{\mathbb{R}}^W$ with $1_W\in\mathcal{W}$ and $k\in\overline{\mathbb{N}}$. Then the map*

$$g_{Y,k,\infty}:\mathcal{C}^{\infty}_{\mathcal{W}}(W,Y)\times\mathcal{C}^{k}_{\mathcal{W}}(U,V)\to\mathcal{C}^k_{\mathcal{W}}(U,Y):(\gamma,\eta)\mapsto\gamma\circ(\eta+\mathrm{id}_X)$$

(which is definable due to Lemma 4.1.4*) is smooth. In particular, $g_{Y,\infty}:=g_{Y,\infty,\infty}$ is smooth.*

Proof. For $k<\infty$, this follows from Proposition 4.1.7 since the inclusion maps

$$\mathcal{C}^{\infty}_{\mathcal{W}}(W,Y)\to\mathcal{C}^{k+\ell+1}_{\mathcal{W}}(W,Y)$$

are smooth. Now let $k = \infty$. From the assertions already established, we derive the commutative diagram

$$\begin{tikzcd} \mathcal{C}^{\infty}_{\mathcal{W}}(W,Y) \times \mathcal{C}^{\infty}_{\mathcal{W}}(U,V) \arrow[r, "g_{Y,\infty}"] \arrow[d, hook] & \mathcal{C}^{\infty}_{\mathcal{W}}(U,Y) \arrow[d, hook] \\ \mathcal{C}^{\infty}_{\mathcal{W}}(W,Y) \times \mathcal{C}^{n}_{\mathcal{W}}(U,V) \arrow[r, "g_{Y,n,\infty}"] & \mathcal{C}^{n}_{\mathcal{W}}(U,Y) \end{tikzcd}$$

for each $n \in \mathbb{N}$, where the vertical arrows represent the inclusion maps. Using Corollary 3.2.6 we easily deduce the smoothness of $g_{Y,\infty}$ from the one of $g_{Y,n,\infty}$. ■

Restriction to decreasing functions. We examine the restriction of $g_{Y,k,\infty}$ to decreasing functions. We show that it takes values in the decreasing functions and is also smooth.

Corollary 4.1.9. *Let X and Y be normed spaces, $U, V, W \subseteq X$ open nonempty subsets such that $V + U \subseteq W$ and V is balanced, $\mathcal{W} \subseteq \overline{\mathbb{R}}^W$ with $1_W \in \mathcal{W}$ and $k \in \overline{\mathbb{N}}$. Then*

$$g_{Y,k,\infty}(\mathcal{C}^{\infty}_{\mathcal{W}}(W,Y)^o \times \mathcal{C}^{k}_{\mathcal{W}}(U,V)^o) \subseteq \mathcal{C}^{k}_{\mathcal{W}}(U,Y)^o,$$

and the restriction $g_{Y,k,\infty}|^{\mathcal{C}^{k}_{\mathcal{W}}(U,Y)^o}_{\mathcal{C}^{\infty}_{\mathcal{W}}(W,Y)^o \times \mathcal{C}^{k}_{\mathcal{W}}(U,V)^o}$ is smooth.

Proof. We deduce this from Lemma 4.1.5, the smoothness of the unrestricted map (Corollary 4.1.8) and Proposition A.1.12 that can be used because $\mathcal{C}^{k}_{\mathcal{W}}(U,Y)^o$ is closed by Lemma 3.1.6. ■

4.1.2. Smooth monoids of weighted endomorphisms. We are able to prove that $\mathrm{End}_{\mathcal{W}}(X)$ and the set $\mathrm{End}_{\mathcal{W}}(X)^{\circ}$—which is defined below—are smooth monoids, provided that $1_X \in \mathcal{W}$.

Corollary 4.1.10. *For $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$, $\mathrm{End}_{\mathcal{W}}(X)$ is a smooth monoid with the group of units*

$$\mathrm{End}_{\mathcal{W}}(X)^{\times} = \mathrm{Diff}_{\mathcal{W}}(X).$$

Further, the set

$$\mathrm{End}_{\mathcal{W}}(X)^{\circ} := \{\gamma + \mathrm{id}_X : \gamma \in \mathcal{C}^{\infty}_{\mathcal{W}}(X,X)^o\} \tag{4.1.10.1}$$

is a closed submonoid of $\mathrm{End}_{\mathcal{W}}(X)$ that is a smooth monoid.

Proof. We first show that $\mathrm{End}_{\mathcal{W}}(X)$ is a monoid. Since $\mathrm{id}_X \in \mathrm{End}_{\mathcal{W}}(X)$ is obviously satisfied, it remains to show that it is closed under composition. Since every element of $\mathrm{End}_{\mathcal{W}}(X)$ can uniquely be written as $\phi + \mathrm{id}_X$ with $\phi \in \mathcal{C}^{\infty}_{\mathcal{W}}(X,X)$, we have to show that for $\gamma, \eta \in \mathcal{C}^{\infty}_{\mathcal{W}}(X,X)$,

$$\kappa_{\mathcal{W}}(\gamma) \circ \kappa_{\mathcal{W}}(\eta) - \mathrm{id}_X \in \mathcal{C}^{\infty}_{\mathcal{W}}(X,X).$$

But we know from identity (4.1.2.1) that

$$\kappa_{\mathcal{W}}(\gamma) \circ \kappa_{\mathcal{W}}(\eta) - \mathrm{id}_X = g_{X,\infty}(\gamma,\eta) + \eta,$$

so we see with Corollary 4.1.8 that this assertion holds, hence $\mathrm{End}_{\mathcal{W}}(X)$ is a monoid. Further, from this identity we easily deduce the smoothness of the composition from that of $g_{X,\infty}$, which was also proved in Proposition 4.1.7.

$\mathrm{End}_{\mathcal{W}}(X)^{\circ}$ is a closed subset of $\mathrm{End}_{\mathcal{W}}(X)$ since $\kappa_{\mathcal{W}}$ is a homeomorphism and by Lemma 3.1.6, $\mathcal{C}^{\infty}_{\mathcal{W}}(X,X)^{o}$ is a closed vector subspace of $\mathcal{C}^{\infty}_{\mathcal{W}}(X,X)$. We know from Corollary 4.1.9 and the fact that $\mathcal{C}^{\infty}_{\mathcal{W}}(X,X)^{o}$ is a vector space that for $\gamma,\eta \in \mathcal{C}^{\infty}_{\mathcal{W}}(X,X)^{o}$,

$$\kappa_{\mathcal{W}}(\gamma) \circ \kappa_{\mathcal{W}}(\eta) - \mathrm{id}_X = g_{X,\infty}(\gamma,\eta) + \eta \in \mathcal{C}^{\infty}_{\mathcal{W}}(X,X)^{o},$$

and that this map is smooth, hence $\mathrm{End}_{\mathcal{W}}(X)^{\circ}$ is a smooth submonoid of $\mathrm{End}_{\mathcal{W}}(X)$.

The relation $\mathrm{End}_{\mathcal{W}}(X)^{\times} = \mathrm{Diff}_{\mathcal{W}}(X)$ was proved in Lemma 4.1.2. ∎

4.2. Lie group structures on weighted diffeomorphisms. In this section, we first prove that $\mathrm{Diff}_{\mathcal{W}}(X)$—which was already proved to be a group in Corollary 4.1.10—is in fact a Lie group. Also we define and discuss the set of *decreasing weighted diffeomorphisms*, $\mathrm{Diff}_{\mathcal{W}}(X)^{\circ}$. We show that it is a normal subgroup of $\mathrm{Diff}_{\mathcal{W}}(X)$ that can be turned into a Lie group. Finally, we elaborate on when diffeomorphisms that are weighted endomorphisms are weighted diffeomorphisms.

4.2.1. The Lie group structure of $\mathrm{Diff}_{\mathcal{W}}(X)$**.** We show that $\mathrm{Diff}_{\mathcal{W}}(X)$ is an open subset of $\mathrm{End}_{\mathcal{W}}(X)$ and the group inversion is smooth, whence $\mathrm{Diff}_{\mathcal{W}}(X)$ is a Lie group. In order to do this, we have to examine the inversion map on $\mathrm{Diff}(X) \cap \mathrm{End}_{\mathcal{W}}(X)$. First, we give some basic definitions and state some easy results.

Definition 4.2.1. Let X be a normed space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$. We set

$$H_{\mathcal{W}} := \{\phi \in \mathcal{C}^{\infty}_{\mathcal{W}}(X,X) : \phi + \mathrm{id}_X \in \mathrm{Diff}(X)\}$$

and

$$I : H_{\mathcal{W}} \to \mathcal{FC}^{\infty}(X,X) : \phi \mapsto (\phi + \mathrm{id}_X)^{-1} - \mathrm{id}_X\,. \tag{4.2.1.1}$$

Lemma 4.2.2. *Let X be a normed space, $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ and $\phi \in H_{\mathcal{W}}$. Then*

$$(I(\phi) + \mathrm{id}_X) \circ (\phi + \mathrm{id}_X) = (\phi + \mathrm{id}_X) \circ (I(\phi) + \mathrm{id}_X) = \mathrm{id}_X, \tag{4.2.2.1}$$

$$I(\phi) \circ (\phi + \mathrm{id}_X) = -\phi, \tag{4.2.2.2}$$

$$\phi \circ (I(\phi) + \mathrm{id}_X) = -I(\phi). \tag{4.2.2.3}$$

Proof. This is obvious. ∎

In the following, it will be useful to define $B_R(0) = \emptyset$ if $R < 0$. This will allow us to avoid distinction of cases.

Lemma 4.2.3. *Let X be a normed space and $R, r \in \mathbb{R}$ with $r > 0$. Then*

$$(X \setminus \overline{B}_R(0)) + B_r(0) \subseteq X \setminus \overline{B}_{R-r}(0).$$

Proof. Let $x \in X \setminus \overline{B}_R(0)$ and $y \in B_r(0)$. We apply the triangle inequality:

$$\|x + y\| \geq \|x\| - \|y\| > R - r. \ ∎$$

4.2.1.1. *On the range of the inversion map.* We discuss whether the range of I consists of weighted functions. More precisely, for suitable functions $\phi \in H_{\mathcal{W}}$ we provide an estimate for $\|I(\phi)\|_{f,0}$ and an identity for $DI(\phi)$. Further, for a decreasing map $\phi \in H_{\mathcal{W}}$, we want to consider $I(\phi)|_{X \setminus \overline{B}_R(0)}$ for $R > 0$. To avoid case distinctions, in the following R mostly denotes an arbitrary real number.

LEMMA 4.2.4. *Let X be a normed space, $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$ and $\phi \in H_{\mathcal{W}}$. Then $I(\phi) \in \mathcal{BC}^0(X, X)$.*

Proof. This is an immediate consequence of identity (4.2.2.3). ∎

LEMMA 4.2.5. *Let X be a Banach space, $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$, $\phi \in H_{\mathcal{W}}$ and r a real number such that $\|\phi\|_{1_{X\setminus \overline{B}_r(0)},1} = \sup_{x \in X \setminus \overline{B}_r(0)} \|D\phi(x)\|_{\mathrm{op}} < 1$. Let $R \in \mathbb{R}$ be such that $R > r + \|I(\phi)\|_{1_X,0}$ (note that $\|I(\phi)\|_{1_X,0} < \infty$ by Lemma 4.2.4). Then for all $f \in \mathcal{W}$ and $x \in X \setminus \overline{B}_R(0)$,*

$$|f(x)|\,\|I(\phi)(x)\| \leq \frac{|f(x)|\,\|\phi(x)\|}{1 - \|\phi\|_{1_{X\setminus \overline{B}_r(0)},1}}. \tag{4.2.5.1}$$

Proof. We set $\psi := I(\phi)$. Then for $f \in \mathcal{W}$ and $x \in X \setminus \overline{B}_R(0)$, by (4.2.2.3),

$$\begin{aligned}|f(x)|\,\|\psi(x)\| &= |f(x)|\,\|\phi(\psi(x)+x) - \phi(x) + \phi(x)\| \\ &\leq |f(x)| \left(\int_0^1 \|D\phi(x + s\psi(x)) \cdot \psi(x)\|\, ds + \|\phi(x)\| \right) \\ &\leq \|D\phi|_{X\setminus \overline{B}_r(0)}\|_{1_{X\setminus \overline{B}_r(0)},0} |f(x)|\,\|\psi(x)\| + |f(x)|\,\|\phi(x)\|;\end{aligned}$$

here we used that $\{x + s\psi(x) : s \in [0,1]\}$ is contained in $X \setminus \overline{B}_r(0)$ by the choice of R and Lemma 4.2.3. From this we can derive the desired estimate since $\|D\phi\|_{1_{X\setminus \overline{B}_r(0)},0} < 1$. ∎

We now state a formula for $DI(\phi)$.

LEMMA 4.2.6. *Let X be a Banach space, $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$, $\phi \in H_{\mathcal{W}}$ and $x \in X$. If $\|D\phi(x)\|_{\mathrm{op}} < 1$, then*

$$D(I(\phi))((\phi + \mathrm{id}_X)(x)) = D\phi(x) \cdot QI_{\mathrm{L}(X)}(-D\phi(x)) - D\phi(x), \tag{4.2.6.1}$$

where $QI_{\mathrm{L}(X)}$ denotes the quasi-inversion (discussed in Chapter C).

Proof. We set $\psi := I(\phi)$. From identity (4.2.2.2) and the chain rule, one gets

$$D\psi((\phi + \mathrm{id}_X)(x)) \cdot (D\phi(x) + \mathrm{id}_X) = -D\phi(x). \tag{$*$}$$

Since $\|D\phi(x)\|_{\mathrm{op}} < 1$, the linear map $D\phi(x) + \mathrm{id}_X$ is bijective with

$$(D\phi(x) + \mathrm{id}_X)^{-1} = \sum_{k=0}^{\infty} (-D\phi(x))^k = \sum_{k=1}^{\infty} (-D\phi(x))^k + \mathrm{id}_X = -QI_{\mathrm{L}(X)}(-D\phi(x)) + \mathrm{id}_X;$$

(cf. Lemma C.2.6). Using this identity we can easily derive (4.2.6.1) from $(*)$. ∎

We show that for suitable maps $\phi \in H_{\mathcal{W}}$, at least the restriction of $I(\phi)$ to the complement of a ball is a smooth weighted function.

PROPOSITION 4.2.7. *Let X be a Banach space, $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$, $\phi \in H_{\mathcal{W}}$ and $r \in \mathbb{R}$ such that* $\sup_{x \in X \setminus \overline{B}_r(0)} \|D\phi(x)\|_{\mathrm{op}} < 1$. *Then for each $R \in \mathbb{R}$ with $R > r + \|I(\phi)\|_{1_X,0}$ (by Lemma 4.2.4, $\|I(\phi)\|_{1_X,0} < \infty$),*

$$I(\phi)|_{X \setminus \overline{B}_R(0)} \in \mathcal{C}^\infty_{\mathcal{W}}(X \setminus \overline{B}_R(0), X).$$

Proof. We prove by induction that $I(\phi)|_{X \setminus \overline{B}_R(0)} \in \mathcal{C}^k_{\mathcal{W}}(X \setminus \overline{B}_R(0), X)$ for all $k \in \mathbb{N}$. In this proof, we will identify maps with their restrictions; no confusion will arise.

$k = 0$: This case was treated in Lemma 4.2.5.

$k \to k+1$: Using Proposition 3.2.3 (and the induction base), we see that

$$I(\phi) \in \mathcal{C}^{k+1}_{\mathcal{W}}(X \setminus \overline{B}_R(0), X) \;\Leftrightarrow\; D\,I(\phi) \in \mathcal{C}^k_{\mathcal{W}}(X \setminus \overline{B}_R(0), \mathrm{L}(X));$$

the second condition will be verified now. Since $\|\phi\|_{1_{X \setminus \overline{B}_r(0)},1} < 1$, the map $-D\phi$ is quasi-invertible in $\mathcal{C}^\infty_{\mathcal{W}}(X \setminus \overline{B}_r(0), \mathrm{L}(X))$ with

$$QI(-D\phi) = QI_{\mathrm{L}(X)} \circ (-D\phi),$$

by Proposition 3.3.20, here $QI := QI_{\mathcal{C}^\infty_{\mathcal{W}}(X \setminus \overline{B}_r(0), \mathrm{L}(X))}$. From this, identity (4.2.6.1) and the fact that $\phi + \mathrm{id}_X$ is a diffeomorphism with $(\phi + \mathrm{id}_X)^{-1} = I(\phi) + \mathrm{id}_X$ (see identity (4.2.2.1)), we deduce that

$$D\,I(\phi) = (D\phi \cdot QI(-D\phi) - D\phi) \circ (I(\phi) + \mathrm{id}_X) \tag{$*$}$$

on $X \setminus \overline{B}_R(0)$. We use Proposition 3.3.20 and Corollary 3.3.6 to see that

$$D\phi \cdot QI(-D\phi) \in \mathcal{C}^\infty_{\mathcal{W}}(X \setminus \overline{B}_r(0), \mathrm{L}(X)).$$

Choose $s > \|I(\phi)\|_{1_X,0}$ such that $R > r + s$. Then $(X \setminus \overline{B}_R(0)) + B_s(0) \subseteq X \setminus \overline{B}_r(0)$, by Lemma 4.2.3. Since we know from the induction hypothesis that $I(\phi) \in \mathcal{C}^k_{\mathcal{W}}(X \setminus \overline{B}_R(0), X)$, we derive from identity $(*)$ and Corollary 4.1.8 (applied with $U = X \setminus \overline{B}_R(0)$, $V = B_s(0)$ and $W = X \setminus \overline{B}_r(0)$) that

$$D\,I(\phi) = g_{\mathrm{L}(X),\infty,k}(D\phi \cdot QI(-D\phi) - D\phi,\, I(\phi)).$$

Hence $D\,I(\phi) \in \mathcal{C}^k_{\mathcal{W}}(X \setminus \overline{B}_R(0), \mathrm{L}(X))$. ∎

Finally, we examine I on decreasing maps.

COROLLARY 4.2.8. *Let X be a Banach space, $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$ and $\phi \in H_{\mathcal{W}} \cap \mathcal{C}^\infty_{\mathcal{W}}(X, X)^o$. Then there exists an $R \in \mathbb{R}$ such that*

$$I(\phi)|_{X \setminus \overline{B}_R(0)} \in \mathcal{C}^\infty_{\mathcal{W}}(X \setminus \overline{B}_R(0), X)^o.$$

Proof. Since $\phi \in \mathcal{C}^\infty_{\mathcal{W}}(X, X)^o$, there exists an $r \in \mathbb{R}$ such that $\sup_{x \in X \setminus \overline{B}_r(0)} \|D\phi(x)\|_{\mathrm{op}} < 1$. By Proposition 4.2.7, there exists $R \in \mathbb{R}$ such that $I(\phi)|_{X \setminus \overline{B}_R(0)} \in \mathcal{C}^\infty_{\mathcal{W}}(X \setminus \overline{B}_R(0), X)$. Further, by (4.2.2.3),

$$I(\phi)|_{X \setminus \overline{B}_R(0)} = -\phi \circ (I(\phi)|_{X \setminus \overline{B}_R(0)} + \mathrm{id}_{X \setminus \overline{B}_R(0)}) = g_{X,\infty}(-\phi, I(\phi)|_{X \setminus \overline{B}_R(0)}),$$

hence an application of Lemma 4.1.5 finishes the proof. ∎

4.2.1.2. *An open set of weighted diffeomorphisms.* We describe an open neighborhood of 0 in $\mathcal{C}^\infty_{\mathcal{W}}(X, X)$ whose image under $\kappa_{\mathcal{W}}$ consists of diffeomorphisms.

DEFINITION 4.2.9. Let X be a normed space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. We set

$$U_{\mathcal{W}} := \{\phi \in \mathcal{C}^\infty_{\mathcal{W}}(X,X) : \|\phi\|_{1_X,1} < 1\}.$$

Since $1_X \in \mathcal{W}$, $U_{\mathcal{W}}$ is open.

The following fact shows that $\kappa_{\mathcal{W}}(U_{\mathcal{W}}) \subseteq \mathrm{Diff}(X)$.

PROPOSITION 4.2.10. *Let E and F be Banach spaces and $\phi \in \mathcal{FC}^1(E,F)$ such that for all $x \in E$ the linear map $D\phi(x) \in \mathrm{L}(E,F)$ is invertible and there exists some $K \in \mathbb{R}$ with $\|D\phi(x)^{-1}\|_{\mathrm{op}} \leq K$ for all $x \in E$. Then ϕ is a surjective homeomorphism.*

Proof. A proof can be found in [CH82, Chapter 2.3, Theorem 3.9]. ∎

COROLLARY 4.2.11. *Let X be a Banach space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$.*

(a) $\kappa_{\mathcal{W}}(U_{\mathcal{W}}) \subseteq \mathrm{Diff}(X)$.
(b) $I(U_{\mathcal{W}}) \subseteq \mathcal{C}^\infty_{\mathcal{W}}(X,X)$.
(c) $\kappa_{\mathcal{W}}(U_{\mathcal{W}}) \subseteq \mathrm{Diff}_{\mathcal{W}}(X)$.

Proof. Let $\phi \in U_{\mathcal{W}}$.

(a) The map $D\phi(x) + \mathrm{id}_X$ is invertible for all $x \in X$ with

$$(D\phi(x) + \mathrm{id}_X)^{-1} = \sum_{\ell=0}^{\infty} (-D\phi(x))^\ell,$$

and from this we get the estimate

$$\|(D(\phi + \mathrm{id}_X)(x))^{-1}\|_{\mathrm{op}} \leq \frac{1}{1 - \|D\phi\|_{\mathrm{op},\infty}}.$$

We conclude from Proposition 4.2.10 that $\phi + \mathrm{id}_X$ is a bijection of X, and the classical inverse function theorem shows that $(\phi + \mathrm{id}_X)^{-1}$ is smooth. Hence $\phi + \mathrm{id}_X$ is a diffeomorphism.

(b) From (a) we conclude that $\phi \in H_{\mathcal{W}}$, so we can apply Proposition 4.2.7 with $R < 0$ and a sufficiently small *negative* real number r to see that $I(\phi) \in \mathcal{C}^\infty_{\mathcal{W}}(X,X)$.

(c) From the previous assertions we conclude that

$$\phi + \mathrm{id}_X,\ (\phi + \mathrm{id}_X)^{-1} \in \mathrm{End}_{\mathcal{W}}(X) \cap \mathrm{Diff}(X).$$

By Lemma 4.1.1 this is equivalent to $\kappa_{\mathcal{W}}(\phi) = \phi + \mathrm{id}_X \in \mathrm{Diff}_{\mathcal{W}}(X)$. ∎

Continuity of the inversion map. We show that the inversion is continuous on $\kappa_{\mathcal{W}}(U_{\mathcal{W}})$. Since this is an identity neighborhood, we deduce that $\mathrm{Diff}_{\mathcal{W}}(X)$ is a topological group.

PROPOSITION 4.2.12. *Let X be a Banach space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. Then the map*

$$U_{\mathcal{W}} \to \mathcal{C}^\infty_{\mathcal{W}}(X,X) : \phi \mapsto I(\phi) = (\phi + \mathrm{id}_X)^{-1} - \mathrm{id}_X$$

(*defined by Corollary* 4.2.11) *is continuous.*

Proof. By Corollary 3.2.6, the above map is continuous iff so are the maps

$$I_\ell : U_{\mathcal{W}} \to \mathcal{C}^\ell_{\mathcal{W}}(X,X)$$

for each $\ell \in \mathbb{N}$. We shall verify this condition by induction on ℓ.

$\ell = 0$: For $\phi, \phi_1 \in U_{\mathcal{W}}$ we set $\psi := I(\phi)$ and $\psi_1 := I(\phi_1)$. For $x \in X$, using identity (4.2.2.3), the mean value theorem and by adding $0 = \phi_1(\psi(x)+x) - \phi_1(\psi(x)+x)$ we compute

$$\begin{aligned}\psi_1(x) - \psi(x) &= \phi_1(\psi(x)+x) - \phi_1(\psi_1(x)+x) + \phi(\psi(x)+x) - \phi_1(\psi(x)+x)\\ &= \int_0^1 D\phi_1(t\psi(x) + (1-t)\psi_1(x) + x)\cdot(\psi(x) - \psi_1(x))\,dt + g_{X,\infty}(\phi - \phi_1, \psi)(x).\end{aligned}$$

Let $f \in \mathcal{W}$. For the integral above, we have

$$|f(x)|\left\|\int_0^1 D\phi_1(t\psi(x) + (1-t)\psi_1(x) + x)\cdot(\psi(x) - \psi_1(x))\,dt\right\| \leq \|\phi_1\|_{1_X,1}\|\psi - \psi_1\|_{f,0},$$

whence

$$\|\psi_1 - \psi\|_{f,0} \leq \|\phi_1\|_{1_X,1}\|\psi - \psi_1\|_{f,0} + \|g_{X,\infty}(\phi - \phi_1, \psi)\|_{f,0}. \tag{$*$}$$

We have to estimate the last summand in $(*)$. Fix $\phi \in U_{\mathcal{W}}$ and choose $\xi \in \mathbb{R}$ such that $\|\phi\|_{1_X,1} < \xi < 1$. Since $g_{X,\infty}$ is continuous (Corollary 4.1.8) and $g_{X,\infty}(0, \psi) = 0$, for each $\varepsilon > 0$ there exists a neighborhood V of ϕ in $U_{\mathcal{W}}$ such that for all $\phi_1 \in V$,

$$\|g_{\mathcal{W},0,X}(\phi - \phi_1, \psi)\|_{f,0} < \varepsilon.$$

Shrinking V, we may assume that each $\phi_1 \in V$ satisfies $\|\phi_1\|_{1_X,1} \leq \xi$. We conclude from $(*)$ that

$$\|\psi_1 - \psi\|_{f,0} \leq \frac{\varepsilon}{1 - \|\phi_1\|_{1_X,1}} \leq \frac{\varepsilon}{1-\xi}$$

for $\phi_1 \in V$, from which we infer that I_0 is continuous in ϕ.

$\ell \to \ell + 1$: Because of Proposition 3.2.3 (and the induction base) $I_{\ell+1}$ is continuous iff so is $D \circ I_{\ell+1} : U_{\mathcal{W}} \to \mathcal{C}^{\ell}_{\mathcal{W}}(X, \mathrm{L}(X))$. Using identity (4.2.6.1), we see that for $\phi \in U_{\mathcal{W}}$,

$$(D \circ I_{\ell+1})(\phi) = g_{\mathrm{L}(X),\ell,\infty}(D\phi \cdot QI(-D\phi) - D\phi,\, I_\ell(\phi)),$$

where $QI := QI_{\mathcal{C}^\infty_{\mathcal{W}}(X,\mathrm{L}(X))}$. Since $g_{\mathrm{L}(X),\ell,\infty}$, D, $\cdot$, QI and I_ℓ are continuous (see Corollary 4.1.8, Proposition 3.2.3, Corollary 3.3.6, Proposition 3.3.20 and the inductive hypothesis, respectively), we conclude that $D \circ I_{\ell+1}$ is continuous. ■

Corollary 4.2.13. *Let X be a Banach space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. Then* $\mathrm{Diff}_{\mathcal{W}}(X)$ *is an open submanifold of* $\mathrm{End}_{\mathcal{W}}(X)$. *Further, the inversion map of* $\mathrm{Diff}_{\mathcal{W}}(X)$ *is continuous.*

Proof. We established in Corollary 4.1.10 that $\mathrm{End}_{\mathcal{W}}(X)$ is a topological monoid with the unit group $\mathrm{Diff}_{\mathcal{W}}(X)$. To show that $\mathrm{Diff}_{\mathcal{W}}(X)$ is open we just need to find an open neighborhood of id_X in $\mathrm{End}_{\mathcal{W}}(X)$ that is contained in $\mathrm{Diff}_{\mathcal{W}}(X)$, and the inversion is continuous if it is so on this neighborhood (see Lemma C.2.3). But we proved in Corollary 4.2.11 that $\kappa_{\mathcal{W}}(U_{\mathcal{W}}) \subseteq \mathrm{Diff}_{\mathcal{W}}(X)$, and in Proposition 4.2.12 that the inversion map is continuous on $\kappa_{\mathcal{W}}(U_{\mathcal{W}})$; see the commutative diagram

$$\begin{tikzcd} \kappa_{\mathcal{W}}(U_{\mathcal{W}}) \arrow[r, "-1"] & \mathrm{Diff}_{\mathcal{W}}(X) \\ U_{\mathcal{W}} \arrow[u, two heads, hook, "\kappa_{\mathcal{W}}"] \arrow[r, "I"] & \kappa_{\mathcal{W}}^{-1}(\mathrm{Diff}_{\mathcal{W}}(X)) \arrow[u, two heads, hook, "\kappa_{\mathcal{W}}"'] \end{tikzcd}$$

■

Smoothness of inversion. Because of Corollary 4.2.13, we can give

DEFINITION 4.2.14. Let X be a normed space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. We define

$$I_{\mathcal{W}} : \kappa_{\mathcal{W}}^{-1}(\mathrm{Diff}_{\mathcal{W}}(X)) \to \kappa_{\mathcal{W}}^{-1}(\mathrm{Diff}_{\mathcal{W}}(X)) : \phi \mapsto \kappa_{\mathcal{W}}^{-1}(\kappa_{\mathcal{W}}(\phi)^{-1}) = (\phi + \mathrm{id}_X)^{-1} - \mathrm{id}_X\,.$$

It remains to show that this map is smooth. To this end, we calculate a nice identity for the differential quotient of $I_{\mathcal{W}}$.

LEMMA 4.2.15. *Let X be a Banach space, $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$, $\phi \in \kappa_{\mathcal{W}}^{-1}(\mathrm{Diff}_{\mathcal{W}}(X))$, $\psi \in \mathcal{C}^{\infty}_{\mathcal{W}}(X, X)$ and $t \in \mathbb{K}^*$ such that $\phi + t\psi \in \kappa_{\mathcal{W}}^{-1}(\mathrm{Diff}_{\mathcal{W}}(X))$. Then*

$$\frac{I_{\mathcal{W}}(\phi + t\psi) - I_{\mathcal{W}}(\phi)}{t} = -\int_0^1 g_{X,\infty}(\psi + g_{\mathrm{L}(X),\infty}(D(I_{\mathcal{W}}(\phi + t\psi)), \phi + st\psi) \cdot \psi, I_{\mathcal{W}}(\phi))\, ds.$$

Proof. The existence of the integral follows from Lemma A.1.6 since $g_{X,\infty}$, $g_{\mathrm{L}(X),\infty}$, D, $\cdot$ and $I_{\mathcal{W}}$ are continuous and $\mathcal{C}^{\infty}_{\mathcal{W}}(X, X)$ is complete (see Corollary 4.1.8, Proposition 3.2.3, Corollary 3.3.7, Corollary 4.2.13 and Corollary 3.2.12, respectively). To prove the stated identity, we use evaluation maps (see Lemma 3.2.13). Since $\phi + \mathrm{id}_X$ is a diffeomorphism, all points of X can be represented as $\phi(x) + x$, where $x \in X$. For any point of this form we compute

$$\begin{aligned} &\mathrm{ev}_{\phi(x)+x}\left(-\int_0^1 g_{X,\infty}(\psi + g_{\mathrm{L}(X),\infty}(D(I_{\mathcal{W}}(\phi + t\psi)), \phi + st\psi) \cdot \psi, I_{\mathcal{W}}(\phi))\, ds \right) \\ &\qquad = -\int_0^1 g_{X,\infty}(\psi + g_{\mathrm{L}(X),\infty}(D(I_{\mathcal{W}}(\phi + t\psi)), \phi + st\psi) \cdot \psi, I_{\mathcal{W}}(\phi))(\phi(x) + x)\, ds, \end{aligned}$$

where we used Lemma A.1.4. In view of the definition of $g_{X,\infty}$ and replacing $I_{\mathcal{W}}(\phi)$ with $(\phi + \mathrm{id}_X)^{-1} - \mathrm{id}_X$, the preceding integral equals

$$-\int_0^1 \psi(x) + g_{\mathrm{L}(X),\infty}(D(I_{\mathcal{W}}(\phi + t\psi)), \phi + st\psi)(x) \cdot \psi(x)\, ds.$$

We factor out $\psi(x)$, put in the definition of $g_{\mathrm{L}(X),\infty}$ and multiply with $1 = \frac{t}{t}$ to obtain

$$\begin{aligned} &= -\int_0^1 (g_{\mathrm{L}(X),\infty}(D(I_{\mathcal{W}}(\phi + t\psi)), \phi + st\psi)(x) + \mathrm{id}_X) \cdot \psi(x)\, ds \\ &= -\frac{1}{t}\int_0^1 D(I_{\mathcal{W}}(\phi + t\psi) + \mathrm{id}_X)(\phi(x) + st\psi(x) + x) \cdot (t\psi(x))\, ds \end{aligned}$$

using that $D\,\mathrm{id}_X(y) = \mathrm{id}_X$ for all $y \in X$. The mean value theorem gives

$$= \frac{(I_{\mathcal{W}}(\phi + t\psi) + \mathrm{id}_X)(\phi(x) + x) - (I_{\mathcal{W}}(\phi + t\psi) + \mathrm{id}_X)(\phi(x) + t\psi(x) + x)}{t}.$$

We plug in the definition of $I_{\mathcal{W}}$ and obtain

$$\begin{aligned} &= \frac{(\phi + t\psi + \mathrm{id}_X)^{-1}(\phi(x) + x) - (\phi + t\psi + \mathrm{id}_X)^{-1}(\phi(x) + t\psi(x) + x)}{t} \\ &= \frac{(\phi + t\psi + \mathrm{id}_X)^{-1}(\phi(x) + x) - (\phi + \mathrm{id}_X)^{-1}(\phi(x) + x)}{t}. \end{aligned}$$

This can be rewritten as

$$= \frac{I_{\mathcal{W}}(\phi + t\psi)(\phi(x) + x) - I_{\mathcal{W}}(\phi)(\phi(x) + x)}{t},$$

so finally we get

$$= \mathrm{ev}_{\phi(x)+x}\left(\frac{I_{\mathcal{W}}(\phi + t\psi) - I_{\mathcal{W}}(\phi)}{t}\right). \blacksquare$$

Having proved this identity, we easily show that the inversion is smooth and conclude that $\mathrm{Diff}_{\mathcal{W}}(X)$ is a Lie group.

PROPOSITION 4.2.16. *Let X be a Banach space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. Then $I_{\mathcal{W}}$ is a smooth map with*

$$dI_{\mathcal{W}}(\phi;\phi_1) = -g_{X,\infty}(\phi_1 + g_{\mathrm{L}(X),\infty}(D(I_{\mathcal{W}}(\phi)),\phi)\cdot\phi_1, I_{\mathcal{W}}(\phi)) \tag{4.2.16.1}$$

using the notation of Corollary 4.1.8*.*

Proof. We prove by induction that $I_{\mathcal{W}}$ is a $\mathcal{C}^k$-map for all $k \in \mathbb{N}$.

$k = 1$: We just have to use Lemma 4.2.15 and Proposition A.1.8 to obtain the differentiability of $I_{\mathcal{W}}$ with the derivative (4.2.16.1).

$k \to k+1$: If $I_{\mathcal{W}}$ is $\mathcal{C}^k$, we conclude from (4.2.16.1) and the fact that D, $\cdot$, $g_{\mathrm{L}(X),\infty}$ and $g_{X,\infty}$ are smooth (see Proposition 3.2.3, Corollary 3.3.7 (together with Example A.1.15) and Proposition 4.1.7, respectively) that $dI_{\mathcal{W}}$ is $\mathcal{C}^k$, so $I_{\mathcal{W}}$ is $\mathcal{C}^{k+1}$ by definition. $\blacksquare$

THEOREM 4.2.17. *Let X be a Banach space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ such that $1_X \in \mathcal{W}$. Then $\mathrm{Diff}_{\mathcal{W}}(X)$ is a Lie group.*

Proof. In Corollary 4.1.10 we showed that $\mathrm{Diff}_{\mathcal{W}}(X)$ is a group and that the composition of $\mathrm{End}_{\mathcal{W}}(X)$ is smooth. Since $\mathrm{Diff}_{\mathcal{W}}(X)$ is an open subset of $\mathrm{End}_{\mathcal{W}}(X)$ by Corollary 4.2.13, the composition of $\mathrm{Diff}_{\mathcal{W}}(X)$ is also smooth. Further, the group inversion of $\mathrm{Diff}_{\mathcal{W}}(X)$ is smooth by Proposition 4.2.16 since for $\phi \in \mathrm{Diff}_{\mathcal{W}}(X)$,

$$\phi^{-1} = (\kappa_{\mathcal{W}} \circ I_{\mathcal{W}} \circ \kappa_{\mathcal{W}}^{-1})(\phi). \blacksquare$$

4.2.2. On decreasing weighted diffeomorphisms and dense subgroups. We define the set $\mathrm{Diff}_{\mathcal{W}}(X)^\circ$ of *decreasing weighted diffeomorphisms* and show that it is a closed normal subgroup of $\mathrm{Diff}_{\mathcal{W}}(X)$ which can be turned into a Lie group. Further, we give sufficient conditions on $\mathcal{W}$ ensuring that the group $\mathrm{Diff}_c(X)$ of compactly supported diffeomorphisms is dense in $\mathrm{Diff}_{\mathcal{W}}(X)^\circ$.

LEMMA 4.2.18. *Let X be a Banach space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. Further, let $\phi \in \mathrm{End}_{\mathcal{W}}(X)^\circ$ and $\psi \in \mathrm{Diff}_{\mathcal{W}}(X)$. Then $\psi - \psi \circ \phi \in \mathcal{C}^\infty_{\mathcal{W}}(X,X)^o$.*

Proof. By Lemma 3.2.13 and the mean value theorem,

$$\psi - \psi \circ \phi = \int_0^1 D\psi(\mathrm{id}_X + t(\phi - \mathrm{id}_X)) \cdot (\phi - \mathrm{id}_X)\, dt.$$

Since $D\psi \in \mathcal{BC}^\infty(X, \mathrm{L}(X))$, Corollary 4.1.8 implies that $D\psi(\mathrm{id}_X + t(\phi - \mathrm{id}_X)) \in \mathcal{BC}^\infty(X, \mathrm{L}(X))$. Since $\phi - \mathrm{id}_X \in \mathcal{C}^\infty_{\mathcal{W}}(X,X)^o$, the assertion follows from Corollary 3.3.4 and the fact that $\mathcal{C}^\infty_{\mathcal{W}}(X,X)^o$ is closed in $\mathcal{C}^\infty_{\mathcal{W}}(X,X)$. $\blacksquare$

Proposition 4.2.19. *Let X be a Banach space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. The set*

$$\mathrm{Diff}_{\mathcal{W}}(X)^\circ := \mathrm{Diff}_{\mathcal{W}}(X) \cap \mathrm{End}_{\mathcal{W}}(X)^\circ = \{\phi \in \mathrm{Diff}_{\mathcal{W}}(X) : \phi - \mathrm{id}_X \in \mathcal{C}^\infty_{\mathcal{W}}(X, X)^o\}$$

is a closed normal Lie subgroup of $\mathrm{Diff}_{\mathcal{W}}(X)$.

Proof. By Corollary 4.1.10, $\mathrm{End}_{\mathcal{W}}(X)^\circ$ is a smooth submonoid of $\mathrm{End}_{\mathcal{W}}(X)$ that is closed. Since $\mathrm{Diff}_{\mathcal{W}}(X)$ is open in $\mathrm{End}_{\mathcal{W}}(X)$, we conclude that $\mathrm{Diff}_{\mathcal{W}}(X)^\circ$ is a smooth submonoid of $\mathrm{Diff}_{\mathcal{W}}(X)$ that is closed. Further, it is a direct consequence of Corollary 4.2.8 that the inverse of an element of $\mathrm{Diff}_{\mathcal{W}}(X)^\circ$ is in $\mathrm{Diff}_{\mathcal{W}}(X)^\circ$, whence using Lemma B.1.6 we see that the latter is a closed Lie subgroup of $\mathrm{Diff}_{\mathcal{W}}(X)$.

It remains to show that $\mathrm{Diff}_{\mathcal{W}}(X)^\circ$ is normal. To this end, let $\phi \in \mathrm{Diff}_{\mathcal{W}}(X)^\circ$ and $\psi \in \mathrm{Diff}_{\mathcal{W}}(X)$. Then

$$\psi \circ \phi \circ \psi^{-1} - \mathrm{id}_X = \psi \circ \phi \circ \psi^{-1} - \psi \circ \phi^{-1} \circ \phi \circ \psi^{-1} = (\psi - \psi \circ \phi^{-1}) \circ \phi \circ \psi^{-1},$$

so we derive the assertion from Lemmas 4.2.18 and 4.1.5. ■

Lemma 4.2.20. *Let X and Y be finite-dimensional normed spaces and $U \subseteq X$ an open nonempty set. Further, let $\mathcal{W} \subseteq \mathbb{R}^U$ be a set of weights such that*

$$\begin{aligned}
&\bullet\ \mathcal{W} \subseteq \mathcal{C}^\infty(U, [0, \infty[),\\
&\bullet\ (\forall x \in U)(\exists f \in \mathcal{W})\ f(x) > 0\\
&\bullet\ (\forall f_1, \dots, f_n \in \mathcal{W})(\forall k_1, \dots, k_n \in \mathbb{N})(\exists f \in \mathcal{W}, C > 0)\\
&\quad (\forall x \in U)\ \|D^{(k_1)} f_1(x)\|_{\mathrm{op}} \cdots \|D^{(k_n)} f_n(x)\|_{\mathrm{op}} \leq C f(x).
\end{aligned} \tag{4.2.20.1}$$

Then $\mathcal{C}^\infty_c(U, Y)$ is dense in $\mathcal{C}^r_{\mathcal{W}}(U, Y)^o$.

Proof. The proof can be found in [GDS73, §V, 19 b)]. ■

Lemma 4.2.21. *Let X be a finite-dimensional normed space, $\mathcal{W} \subseteq \mathbb{R}^X$ such that $1_X \in \mathcal{W}$ and* (4.2.20.1) *is satisfied* (*where $U = X$*). *Then the set* $\mathrm{Diff}_c(X)$ *of compactly supported diffeomorphisms is dense in* $\mathrm{Diff}_{\mathcal{W}}(X)^\circ$.

Proof. The set $M^\circ_{\mathcal{W}} := \kappa^{-1}_{\mathcal{W}}(\mathrm{Diff}_{\mathcal{W}}(X)) \cap \mathcal{C}^\infty_{\mathcal{W}}(X, X)^o = \kappa^{-1}_{\mathcal{W}}(\mathrm{Diff}_{\mathcal{W}}(X)^\circ)$ is open in $\mathcal{C}^\infty_{\mathcal{W}}(X, X)^o$, and hence $M_c := \mathcal{C}^\infty_c(X, X) \cap M^\circ_{\mathcal{W}}$ is dense in $M^\circ_{\mathcal{W}}$ by Lemma 4.2.20. But $M_c = \kappa^{-1}_{\mathcal{W}}(\mathrm{Diff}_c(X))$, from which the assertion follows. ■

4.2.3. On diffeomorphisms that are weighted endomorphisms. It is obvious that

$$\mathrm{Diff}_{\mathcal{W}}(X) \subseteq \mathrm{End}_{\mathcal{W}}(X) \cap \mathrm{Diff}(X).$$

We give a sufficient criterion on $\mathcal{W}$ that ensures that these two sets are identical, provided that X is finite-dimensional. Further we show $\mathrm{Diff}_{\{1_{\mathbb{R}}\}}(\mathbb{R}) \neq \mathrm{End}_{\{1_{\mathbb{R}}\}}(\mathbb{R}) \cap \mathrm{Diff}(X)$.

Proposition 4.2.22. *Let X be a* finite-dimensional *Banach space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. If there exists $\widehat{f} \in \mathcal{W}$ such that*

$$(\forall R > 0)(\exists r > 0)\ \|x\| \geq r \ \Rightarrow\ |\widehat{f}(x)| \geq R \tag{4.2.22.1}$$

and if each function in $\mathcal{W}$ is bounded on bounded sets, then

$$\mathrm{Diff}_{\mathcal{W}}(X) = \mathrm{End}_{\mathcal{W}}(X) \cap \mathrm{Diff}(X).$$

Proof. It remains to show that
$$\mathrm{End}_{\mathcal{W}}(X) \cap \mathrm{Diff}(X) \subseteq \mathrm{Diff}_{\mathcal{W}}(X).$$
So let ψ be in $\mathrm{End}_{\mathcal{W}}(X) \cap \mathrm{Diff}(X)$ and set $\phi := \psi - \mathrm{id}_X \in H_{\mathcal{W}}$. Then
$$\psi \in \mathrm{Diff}_{\mathcal{W}}(X) \;\Leftrightarrow\; \psi^{-1} \in \mathrm{End}_{\mathcal{W}}(X) \;\Leftrightarrow\; \psi^{-1} - \mathrm{id}_X \in \mathcal{C}^\infty_{\mathcal{W}}(X,X) \;\Leftrightarrow\; I(\phi) \in \mathcal{C}^\infty_{\mathcal{W}}(X,X)$$
(see Lemma 4.1.1 and the definition of I in (4.2.1.1)). The last statement clearly holds iff
$$(\exists R \in \mathbb{R},\, r > 0)\; I(\phi) \in \mathcal{C}^\infty_{\mathcal{W}}(X \setminus \overline{B}_R(0), X) \text{ and } I(\phi) \in \mathcal{C}^\infty_{\mathcal{W}}(B_{R+r}(0), X),$$
and this will be proved now. Obviously $I(\phi) \in \mathcal{C}^\infty_{\mathcal{W}}(B_R(0), X)$ for each $R \in \mathbb{R}$, because each $f \in \mathcal{W}$ is bounded on bounded sets, all the maps $D^{(\ell)}I(\phi)$ are continuous and each closed bounded subset B of X is compact (as X is finite-dimensional); hence
$$\sup_{x\in B} |f(x)|\, \|(D^{(\ell)}I(\phi))(x)\|_{\mathrm{op}} < \infty.$$
It remains to show that there exists an $R \in \mathbb{R}$ such that $I(\phi) \in \mathcal{C}^\infty_{\mathcal{W}}(X \setminus \overline{B}_R(0), X)$. We set $K_\phi := \|\phi\|_{\widehat{f},1} < \infty$ and conclude from (4.2.22.1) that there exists an r_ϕ with
$$\|x\| \geq r_\phi \;\Rightarrow\; |\widehat{f}(x)| \geq K_\phi + 1.$$
Since $|\widehat{f}(x)|\, \|D\phi(x)\|_{\mathrm{op}} \leq K_\phi$ for each $x \in X$, we conclude that
$$\|\phi|_{X\setminus \overline{B}_{r_\phi}(0)}\|_{1_X,1} \leq \frac{K_\phi}{K_\phi + 1} < 1.$$
But we stated in Proposition 4.2.7 that this implies the existence of an $R \in \mathbb{R}$ such that
$$I(\phi) \in \mathcal{C}^\infty_{\mathcal{W}}(X \setminus \overline{B}_R(0), X). \;\blacksquare$$

We give a positive example.

EXAMPLE 4.2.23. The space $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$ satisfies condition (4.2.22.1). We just have to set $\widehat{f}(x_1, \dots, x_n) = x_1^2 + \cdots + x_n^2$ which is clearly a polynomial function on $\mathbb{R}^n$.

As announced, we give a counterexample. As a preparation, we prove

LEMMA 4.2.24. *Let $\gamma \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ be a* bounded *map that satisfies*
$$(\forall x \in \mathbb{R})\; \gamma'(x) > -1. \tag{$*$}$$
Then $\gamma + \mathrm{id}_{\mathbb{R}} \in \mathrm{Diff}(\mathbb{R})$.

Proof. We conclude from $(*)$ that $(\gamma(x) + \mathrm{id}_{\mathbb{R}})'(x) > 0$ for all $x \in \mathbb{R}$, so $\gamma + \mathrm{id}_{\mathbb{R}}$ is strictly monotone and hence injective. Since γ is bounded, $\gamma + \mathrm{id}_{\mathbb{R}}$ is unbounded above and below and hence surjective (by the mean value theorem). $\blacksquare$

EXAMPLE 4.2.25. We give an example of a map $\gamma \in \mathcal{BC}^\infty(\mathbb{R}, \mathbb{R})$ with $\gamma + \mathrm{id}_{\mathbb{R}} \in \mathrm{Diff}(\mathbb{R})$, but $(\gamma + \mathrm{id}_{\mathbb{R}})^{-1} - \mathrm{id}_{\mathbb{R}} \notin \mathcal{BC}^\infty(\mathbb{R}, \mathbb{R})$. To this end, let ϕ be an antiderivative of the function $x \mapsto \frac{2}{\pi} \arctan(x)$ with $\phi(0) = 0$. Then $\sin \circ \phi$ and $\cos \circ \phi$ are in $\mathcal{BC}^\infty(\mathbb{R}, \mathbb{R})$ by a simple induction since $\cos, \sin, \arctan \in \mathcal{BC}^\infty(\mathbb{R}, \mathbb{R})$,
$$(\sin \circ \phi)'(x) = \frac{2}{\pi} \arctan(x)(\cos \circ \phi)(x), \tag{$*$}$$

and an analogous formula holds for $(\cos\circ\phi)'$. We set $\gamma := \sin\circ\phi$. By $(*)$, we have $\gamma'(x) > -1$ for all $x \in \mathbb{R}$, so $\gamma + \mathrm{id}_{\mathbb{R}} \in \mathrm{Diff}(\mathbb{R})$ (see Lemma 4.2.24). But since

$$((\gamma+\mathrm{id}_{\mathbb{R}})^{-1} - \mathrm{id}_{\mathbb{R}})'(x) = \frac{1}{\gamma'(y)+1} - 1$$

with $y := (\gamma+\mathrm{id}_{\mathbb{R}})^{-1}(x)$ and there exists a sequence $(y_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ with

$$\lim_{n\to\infty} \frac{2}{\pi}\arctan(y_n)(\cos\circ\phi)(y_n) = -1,$$

$((\gamma+\mathrm{id}_{\mathbb{R}})^{-1} - \mathrm{id}_{\mathbb{R}})'$ is clearly unbounded.

4.3. Regularity. We prove that the Lie groups $\mathrm{Diff}_{\mathcal{W}}(X)$ and $\mathrm{Diff}_{\mathcal{W}}(X)^\circ$ are regular. For the definition of regularity, see Section B.2.2.

4.3.1. The tangent group and the regularity differential equation of $\mathrm{Diff}_{\mathcal{W}}(X)$. We examine the general (right) regularity differential equation (stated in the initial value problem (B.2.11.1)) and turn it into a differential equation on $\mathcal{C}^\infty_{\mathcal{W}}(X,X)$. To this end, we first describe the group multiplication of the tangent group $\mathbf{T}\,\mathrm{Diff}_{\mathcal{W}}(X)$ and the right action of $\mathrm{Diff}_{\mathcal{W}}(X)$ on $\mathbf{T}\,\mathrm{Diff}_{\mathcal{W}}(X)$ with respect to the chart $\mathbf{T}\kappa_{\mathcal{W}}^{-1}$.

Lemma 4.3.1. *Let X be a Banach space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. Denote the multiplication on $\mathrm{Diff}_{\mathcal{W}}(X)$ with respect to the chart $\kappa_{\mathcal{W}}^{-1}$ by $m_{\mathcal{W}}$. Note that the tangent group $\mathbf{T}\,\mathrm{Diff}_{\mathcal{W}}(X)$ is canonically isomorphic to $\mathcal{C}^\infty_{\mathcal{W}}(X,X) \rtimes \mathrm{Diff}_{\mathcal{W}}(X)$.*

(a) *The group multiplication $\mathbf{T}m_{\mathcal{W}}$ on $\mathbf{T}\,\mathrm{Diff}_{\mathcal{W}}(X)$ (with respect to $\mathbf{T}\kappa_{\mathcal{W}}^{-1}$) is given by*

$$\mathbf{T}m_{\mathcal{W}}\big((\gamma,\gamma_1),(\eta,\eta_1)\big) = \big(m_{\mathcal{W}}(\gamma,\eta), D\gamma\circ(\eta+\mathrm{id}_X)\cdot\eta_1 + \gamma_1\circ(\eta+\mathrm{id}_X) + \eta_1\big).$$

(b) *Let $\phi \in \mathrm{Diff}_{\mathcal{W}}(X)$. Then the right action $\mathbf{T}\rho_\phi$ of ϕ on $\mathbf{T}\,\mathrm{Diff}_{\mathcal{W}}(X)$ with respect to $\mathbf{T}\kappa_{\mathcal{W}}^{-1}$ is given by*

$$\mathbf{T}(\kappa_{\mathcal{W}}^{-1}\circ\rho_\phi\circ\kappa_{\mathcal{W}})(\gamma,\gamma_1) = \big(m_{\mathcal{W}}(\gamma,\kappa_{\mathcal{W}}^{-1}(\phi)), \gamma_1\circ\phi\big).$$

Proof. (a) We have

$$m_{\mathcal{W}}(\gamma,\eta) = \gamma\circ(\eta+\mathrm{id}_X)+\eta$$

and the commutative diagram

$$\begin{tikzcd}
\mathrm{Diff}_{\mathcal{W}}(X)\times\mathrm{Diff}_{\mathcal{W}}(X) \arrow[r, "\circ"] & \mathrm{Diff}_{\mathcal{W}}(X) \\
\kappa_{\mathcal{W}}^{-1}(\mathrm{Diff}_{\mathcal{W}}(X))\times\kappa_{\mathcal{W}}^{-1}(\mathrm{Diff}_{\mathcal{W}}(X)) \arrow[u, two heads, hook, "\kappa_{\mathcal{W}}\times\kappa_{\mathcal{W}}"] \arrow[r, "m_{\mathcal{W}}"] & \kappa_{\mathcal{W}}^{-1}(\mathrm{Diff}_{\mathcal{W}}(X)) \arrow[u, two heads, hook, "\kappa_{\mathcal{W}}"']
\end{tikzcd}$$

The group multiplication on the tangent group is given by applying the tangent functor $\mathbf{T}$ to the group multiplication on $\mathrm{Diff}_{\mathcal{W}}(X)$, and therefore we obtain the group multiplication on $\mathbf{T}\,\mathrm{Diff}_{\mathcal{W}}(X)$ in charts by applying $\mathbf{T}$ to $m_{\mathcal{W}}$ (up to a permutation). Since

$$\mathbf{T}m_{\mathcal{W}}(\gamma,\eta;\gamma_1,\eta_1) = \big(m_{\mathcal{W}}(\gamma,\eta), D\gamma\circ(\eta+\mathrm{id}_X)\cdot\eta_1 + \gamma_1\circ(\eta+\mathrm{id}_X)+\eta_1\big)$$

by (4.1.7.1), the asserted identity holds.

(b) Obviously $(\kappa_{\mathcal{W}}^{-1}\circ\rho_\phi\circ\kappa_{\mathcal{W}})(\cdot) = m_{\mathcal{W}}(\cdot,\kappa_{\mathcal{W}}^{-1}(\phi))$, so we derive the assertion if we apply the identity proved in (a) with $\eta = \kappa_{\mathcal{W}}^{-1}(\phi)$ and $\eta_1 = 0$. ∎

We aim to turn (B.2.11.1) into an ODE on a vector space. Before we can do this, a definition is useful:

DEFINITION 4.3.2. Let X be a normed space, $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$, $k \in \overline{\mathbb{N}}$ and $\mathcal{F}$ be a subset of $\mathcal{W}$ with $1_X \in \mathcal{F}$. By Corollary 4.1.8, the map

$$F_{\mathcal{F},k} : [0,1] \times \mathcal{C}^k_{\mathcal{F}}(X,X) \times \mathcal{C}^\infty([0,1], \mathcal{C}^\infty_{\mathcal{W}}(X,X)) \to \mathcal{C}^k_{\mathcal{F}}(X,X),$$
$$(t, \gamma, p) \mapsto p(t) \circ (\gamma + \mathrm{id}_X),$$

is well-defined and smooth (since the evaluation of curves is smooth by Lemma A.1.9). For each parameter curve $p \in \mathcal{C}^\infty([0,1], \mathcal{C}^\infty_{\mathcal{W}}(X,X))$, we consider the initial value problem

$$\Gamma'(t) = F_{\mathcal{F},k}(t, \Gamma(t), p), \qquad \Gamma(0) = 0, \tag{4.3.2.1}$$

where $t \in [0,1]$.

LEMMA 4.3.3. *Let X be a Banach space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$.*

(a) *For $\gamma \in \mathcal{C}^\infty([0,1], \mathbf{T}_{\mathrm{id}_X} \mathrm{Diff}_{\mathcal{W}}(X))$, the initial value problem*

$$\eta'(t) = \gamma(t) \cdot \eta(t), \qquad \eta(0) = \mathrm{id}_X$$

has a smooth solution

$$\mathrm{Evol}^\rho_{\mathrm{Diff}_{\mathcal{W}}(X)}(\gamma) : [0,1] \to \mathrm{Diff}_{\mathcal{W}}(X)$$

iff the initial value problem (4.3.2.1) (*in Definition* 4.3.2) *with $\mathcal{F} = \mathcal{W}$, $k = \infty$ and $p = d\kappa_{\mathcal{W}}^{-1} \circ \gamma$ has a smooth solution*

$$\Gamma_p : [0,1] \to \kappa_{\mathcal{W}}^{-1}(\mathrm{Diff}_{\mathcal{W}}(X)).$$

In this case,

$$\mathrm{Evol}^\rho_{\mathrm{Diff}_{\mathcal{W}}(X)}(\gamma) = \kappa_{\mathcal{W}} \circ \Gamma_p.$$

(b) *Let $\Omega \subseteq \mathcal{C}^\infty([0,1], \mathbf{T}_{\mathrm{id}_X} \mathrm{Diff}_{\mathcal{W}}(X))$ be an open set such that for each $\gamma \in \Omega$ there exists a right evolution $\mathrm{Evol}^\rho_{\mathrm{Diff}_{\mathcal{W}}(X)}(\gamma)$. Then $\mathrm{evol}^\rho_{\mathrm{Diff}_{\mathcal{W}}(X)}|_\Omega$ is smooth iff so is the map*

$$(d\kappa_{\mathcal{W}}^{-1} \circ \Omega) \to \mathcal{C}^\infty_{\mathcal{W}}(X,X) : p \mapsto \Gamma_p(1).$$

As above, Γ_p denotes a solution to (4.3.2.1) *with respect to p.*

Proof. This is an easy computation involving the previous results. ∎

4.3.1.1. *Solving the differential equation.* We show that the regularity differential equation for $\mathrm{Diff}_{\mathcal{W}}(X)$ is solvable. In order to do this, we use that $\mathcal{C}^\infty_{\mathcal{W}}(X,X)$ is a projective limit of Banach spaces (see Proposition 3.2.5). We solve the differential equation on each step of the projective limit, see that these solutions are compatible with the bonding morphisms of the projective limit, and thus obtain a solution on the limit. Before we do this, we state the following obvious lemma.

LEMMA 4.3.4. *Let X be a Banach space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. Further, let $\mathcal{F} \subseteq \mathcal{W}$ with $1_X \in \mathcal{F}$ and $k \in \overline{\mathbb{N}}$, $p \in \mathcal{C}^\infty([0,1], \mathcal{C}^\infty_{\mathcal{W}}(X,X))$ and $\Gamma : I \to \mathcal{C}^k_{\mathcal{F}}(X,X)$ a solution to* (4.3.2.1) *corresponding to p. Then Γ solves* (4.3.2.1) *also for all subsets $\mathcal{G} \subseteq \mathcal{F}$ containing 1_X and $\ell \in \overline{\mathbb{N}}$ with $\ell \leq k$.*

Proof. This is an easy calculation since the inclusion map $\mathcal{C}^k_{\mathcal{F}}(X,X) \to \mathcal{C}^\ell_{\mathcal{G}}(X,X)$ is continuous linear. ■

Solving the differential equation on the steps. First, we solve (4.3.2.1) on function spaces that are Banach spaces. To this end, we need tools from the theory of ordinary differential equations on Banach spaces. The required facts are described in Section A.4. The hard part will be to show that the solutions are defined on the whole interval $[0,1]$.

The solution on $\mathcal{C}^0_{\mathcal{F}}(X,X)$. We start with the function space $\mathcal{C}^0_{\mathcal{F}}(X,X)$, where $\mathcal{F} \subseteq \mathcal{W}$ is finite and contains 1_X. Then the initial value problem (4.3.2.1) satisfies a global Lipschitz condition and hence is globally solvable.

Lemma 4.3.5. *Let X be a normed space, $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$, $\mathcal{F} \subseteq \mathcal{W}$ with $1_X \in \mathcal{F}$ and $p \in \mathcal{C}^\infty([0,1], \mathcal{C}^\infty_{\mathcal{W}}(X,X))$. Then there exists $K > 0$ such that for each $f \in \mathcal{F}$, all $t \in [0,1]$ and $\gamma, \gamma_0 \in \mathcal{C}^0_{\mathcal{F}}(X,X)$,*

$$\|F_{\mathcal{F},0}(t,\gamma,p) - F_{\mathcal{F},0}(t,\gamma_0,p)\|_{f,0} \leq K\|\gamma - \gamma_0\|_{f,0}.$$

Proof. We have

$$F_{\mathcal{F},0}(t,\gamma,p) - F_{\mathcal{F},0}(t,\gamma_0,p) = g_X(p(t),\gamma) - g_X(p(t),\gamma_0),$$

and deduce from estimate (4.1.3.2) in Lemma 4.1.3 that

$$\|F_{\mathcal{F},0}(t,\gamma,p) - F_{\mathcal{F},0}(t,\gamma_0,p)\|_{f,0} \leq \|p(t)\|_{1_X,1}\|\gamma - \gamma_0\|_{f,0}.$$

Since $p([0,1])$ is a compact (and therefore bounded) subset of $\mathcal{C}^\infty_{\mathcal{W}}(X,X)$,

$$K := \sup_{t\in[0,1]} \|p(t)\|_{1_X,1} < \infty. \quad ■$$

Lemma 4.3.6. *Let X be a Banach space, $\mathcal{F}, \mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{F} \subseteq \mathcal{W}$ and $|\mathcal{F}| < \infty$, $p \in \mathcal{C}^\infty([0,1], \mathcal{C}^\infty_{\mathcal{W}}(X,X))$ and $k = 0$. Then the initial value problem* (4.3.2.1) *corresponding to p has a unique solution which is defined on the whole interval* $[0,1]$.

Proof. We deduce from Lemma 4.3.5 that we can find a norm on $\mathcal{C}^0_{\mathcal{F}}(X,X)$ such that $F_{\mathcal{F},0}(\cdot,\cdot,p)$ satisfies a global Lipschitz condition with respect to the second argument. Since $\mathcal{C}^0_{\mathcal{F}}(X,X)$ is a Banach space, there exists a unique solution $\Gamma : [0,1] \to \mathcal{C}^0_{\mathcal{F}}(X,X)$ of (4.3.2.1) which is defined on the whole interval $[0,1]$; see [Die60, §10.6.1] or Theorem A.4.7 and Lemma A.4.5. ■

Solutions in spaces of differentiable functions. On the spaces $\mathcal{C}^k_{\mathcal{F}}(X,X)$ with $k \geq 1$, it is harder to show that the maximal solution is defined on the whole of $[0,1]$. To show this, we first verify that the differential curve $D \circ \gamma$ of a solution $\gamma : I \to \mathcal{C}^k_{\mathcal{F}}(X,X)$ to (4.3.2.1) is itself a solution to a linear ODE. We start with the following definition.

Definition 4.3.7. Let X be a Banach space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. Further, let $\mathcal{F}$ be a subset of $\mathcal{W}$ with $1_X \in \mathcal{F}$, $k \in \overline{\mathbb{N}}$ and $\Gamma : [0,1] \to \mathcal{C}^k_{\mathcal{F}}(X,X)$ and $P : [0,1] \to \mathcal{C}^\infty_{\mathcal{W}}(X, \mathrm{L}(X))$ be continuous curves. We define the continuous map

$$G^{\Gamma,P}_{\mathcal{F},k} : [0,1] \times \mathcal{C}^k_{\mathcal{F}}(X, \mathrm{L}(X)) \to \mathcal{C}^k_{\mathcal{F}}(X, \mathrm{L}(X)),$$
$$(t,\gamma) \mapsto \big(P(t) \circ (\Gamma(t) + \mathrm{id}_X)\big) \cdot (\gamma + \mathrm{id}),$$

and consider the initial value problem

$$\Phi'(t) = G^{\Gamma,P}_{\mathcal{F},k}(t,\Phi(t)), \qquad \Phi(0) = 0. \tag{4.3.7.1}$$

LEMMA 4.3.8. *Let X be a Banach space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. Further, let $\mathcal{F}$ be a finite subset of $\mathcal{W}$ with $1_X \in \mathcal{F}$, $k \in \mathbb{N}$ and $p \in \mathcal{C}^\infty([0,1], \mathcal{C}^\infty_{\mathcal{W}}(X,X))$. If*

$$\Gamma_k : [0,1] \to \mathcal{C}^k_{\mathcal{F}}(X,X) \quad \text{ and } \quad \Gamma_{k+1} : I \subseteq [0,1] \to \mathcal{C}^{k+1}_{\mathcal{F}}(X,X)$$

are solutions to (4.3.2.1) *corresponding to p, then the curve $D \circ \Gamma_{k+1} : I \to \mathcal{C}^k_{\mathcal{F}}(X, \mathrm{L}(X))$ is a solution to the problem* (4.3.7.1) *with $\Gamma = \Gamma_k$ and $P = D \circ p$.*

Proof. We have $(D \circ \Gamma_{k+1})' = D \circ \Gamma'_{k+1}$ and therefore for $t \in I$,

$$\begin{aligned}
(D \circ \Gamma_{k+1})'(t) &= D\, F_{\mathcal{F},k+1}(t, \Gamma_{k+1}(t), p) \\
&= \big(Dp(t) \circ (\Gamma_{k+1}(t) + \mathrm{id}_X)\big) \cdot (D\Gamma_{k+1}(t) + \mathrm{id}). \\
&= \big((D \circ p)(t) \circ (\Gamma_{k+1}(t) + \mathrm{id}_X)\big) \cdot \big((D \circ \Gamma_{k+1})(t) + \mathrm{id}\big) \\
&= G^{\Gamma_k, D\circ p}_{\mathcal{F},k}(t, (D \circ \Gamma_{k+1})(t)),
\end{aligned}$$

where we used that $\Gamma_k|_I = \Gamma_{k+1}$ by Lemma 4.3.4 since $\mathcal{C}^k_{\mathcal{F}}(X,X)$ is a Banach space. Obviously $(D \circ \Gamma_{k+1})(0) = 0$, so the assertion is proved. ∎

Now we use the embedding from Proposition 3.2.3 to show that the maximal solution to (4.3.2.1) is defined on $[0,1]$.

LEMMA 4.3.9. *Let X be a Banach space, $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$, $\mathcal{F} \subseteq \mathcal{W}$ finite with $1_X \in \mathcal{F}$, $p \in \mathcal{C}^\infty([0,1], \mathcal{C}^\infty_{\mathcal{W}}(X,X))$ and $k \in \mathbb{N}$. Then the initial value problem* (4.3.2.1) *corresponding to p has a unique solution which is defined on the whole interval* $[0,1]$.

Proof. This is proved by induction on k. The case $k = 0$ was treated in Lemma 4.3.6.

$k \to k+1$: We denote the solutions for k and 0 with Γ_k and Γ_0, respectively. Since the function $F_{\mathcal{F},k+1}$ is smooth and $\mathcal{C}^{k+1}_{\mathcal{F}}(X,X)$ is a Banach space, there exists a unique *maximal* solution $\Gamma_{k+1} : I \to \mathcal{C}^{k+1}_{\mathcal{F}}(X,X)$ to (4.3.2.1) (see Proposition A.4.2). Using Lemma 4.3.8, we conclude that $D \circ \Gamma_{k+1}$ is a solution to (4.3.7.1), where $\Gamma = \Gamma_k$ and $P = D \circ p$; here we used that by the induction hypothesis, Γ_k is defined on $[0,1]$. Since the latter ODE is linear, there exists a unique solution

$$S : [0,1] \to \mathcal{C}^k_{\mathcal{F}}(X, \mathrm{L}(X))$$

that is defined on the whole interval $[0,1]$ (see [Die60, §10.6.3] or Theorem A.4.7). Let

$$\iota : \mathcal{C}^{k+1}_{\mathcal{F}}(X,X) \to \mathcal{C}^0_{\mathcal{F}}(X,X) \times \mathcal{C}^k_{\mathcal{F}}(X, \mathrm{L}(X))$$

be the embedding from Proposition 3.2.3. By Lemma 4.3.4, Γ_{k+1} is a solution to (4.3.2.1) for the right hand side $F_{\mathcal{F},0}$, so $\Gamma_{k+1} = \Gamma_0|_I$ since solutions to initial value problems in Banach spaces are unique. Hence

$$\Gamma_{k+1}(I) \subseteq \iota^{-1}\big(\Gamma_0([0,1]) \times S([0,1])\big).$$

Further, $\Gamma_0([0,1]) \times S([0,1])$ is compact and the image of ι is a closed subset of $\mathcal{C}^0_{\mathcal{F}}(X,X) \times \mathcal{C}^k_{\mathcal{F}}(X, \mathrm{L}(X))$ (by Proposition 3.2.8). Hence, because ι^{-1} is a homeomorphism, the image of Γ_{k+1} is contained in a compact set. Since Γ_{k+1} is maximal, this implies that Γ_{k+1} must be defined on the whole of $[0,1]$ (see Theorem A.4.7). ∎

Smooth dependence on the parameter and taking the solution to the limit. We use the constructed solutions on $\mathcal{C}^k_{\mathcal{F}}(X,X)$ and show that there exists a solution to (4.3.2.1) on $\mathcal{C}^\infty_{\mathcal{W}}(X,X)$, depending smoothly on the parameter curve.

PROPOSITION 4.3.10. *Let X be a Banach space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. For each $p \in \mathcal{C}^\infty([0,1], \mathcal{C}^\infty_{\mathcal{W}}(X,X))$ there exists a solution Γ_p to (4.3.2.1) defined on $[0,1]$ which corresponds to p, $\mathcal{W}$ and ∞. The map*

$$[0,1] \times \mathcal{C}^\infty([0,1], \mathcal{C}^\infty_{\mathcal{W}}(X,X)) \to \mathcal{C}^\infty_{\mathcal{W}}(X,X) : (t,p) \mapsto \Gamma_p(t) \tag{†}$$

is smooth.

Proof. For $p \in \mathcal{C}^\infty([0,1], \mathcal{C}^\infty_{\mathcal{W}}(X,X))$, we denote by Γ_p the solution $[0,1] \to \mathcal{C}^0_{\{1_X\}}(X,X)$ to (4.3.2.1) corresponding to p, 0 and $\{1_X\}$—which exists by Lemma 4.3.9. By Lemma 4.3.4, a solution $\Gamma : [0,1] \to \mathcal{C}^k_{\mathcal{F}}(X,X)$ to (4.3.2.1) corresponding to p, a finite set $\mathcal{F} \subseteq \mathcal{W}$ containing 1_X and $k \in \mathbb{N}$—which exists by Lemma 4.3.9—also solves (4.3.2.1) for p, 0 and $\{1_X\}$. Hence, by the uniqueness of solutions to initial value problems for Banach spaces, $\Gamma_p = \Gamma$. Since $\mathcal{F}$ and k were arbitrary, the image of Γ_p is contained in $\mathcal{C}^\infty_{\mathcal{W}}(X,X)$, and we easily calculate that Γ_p is a solution to (4.3.2.1) corresponding to p, $\mathcal{W}$ and ∞.

It remains to show that (†) is smooth. Since $\mathcal{C}^\infty_{\mathcal{W}}(X,X)$ is the projective limit of

$$\{\mathcal{C}^k_{\mathcal{F}}(X,X) : k \in \mathbb{N},\ \mathcal{F} \subseteq \mathcal{W},\ |\mathcal{F}| < \infty,\ 1_X \in \mathcal{F}\}$$

by Proposition 3.2.5, using the universal property of the projective limit (see Proposition A.1.12), we just have to show the map

$$[0,1] \times \mathcal{C}^\infty([0,1], \mathcal{C}^\infty_{\mathcal{W}}(X,X)) \to \mathcal{C}^k_{\mathcal{F}}(X,X) : (t,p) \mapsto \Gamma_p(t)$$

with a finite set $\mathcal{F} \subseteq \mathcal{W}$ containing 1_X and $k \in \mathbb{N}$ is smooth. We deduce this from Corollary A.4.14 since the map $\mathcal{C}^\infty([0,1], \mathcal{C}^\infty_{\mathcal{W}}(X,X)) \to \mathcal{C}^k_{\mathcal{F}}(X,X) : p \mapsto 0$ is smooth. Here, we used implicitly that the inclusion map $\mathcal{C}^\infty_{\mathcal{W}}(X,X) \to \mathcal{C}^k_{\mathcal{F}}(X,X)$ is smooth. ∎

4.3.2. Conclusion and calculation of one-parameter groups. We are ready to prove the regularity of the Lie groups. As a regular Lie group, $\operatorname{Diff}_{\mathcal{W}}(X)$ has an exponential function. We show that the corresponding one-parameter groups induce flows on certain vector fields.

THEOREM 4.3.11. *Let X be a Banach space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. Then the Lie group $\operatorname{Diff}_{\mathcal{W}}(X)$ is regular.*

Proof. We proved in Proposition 4.3.10 that for each smooth curve $p : [0,1] \to \mathcal{C}^\infty_{\mathcal{W}}(X,X)$ the initial value problem (4.3.2.1) has a solution $\Gamma_p : [0,1] \to \mathcal{C}^\infty_{\mathcal{W}}(X,X)$ and the map

$$\Gamma : [0,1] \times \mathcal{C}^\infty([0,1], \mathcal{C}^\infty_{\mathcal{W}}(X,X)) \to \mathcal{C}^\infty_{\mathcal{W}}(X,X) : (t,p) \mapsto \Gamma_p(t)$$

is smooth. Obviously, Γ maps $[0,1] \times \{0\}$ to 0. Since $\kappa_{\mathcal{W}}^{-1}(\operatorname{Diff}_{\mathcal{W}}(X))$ is an open neighborhood of 0 in $\mathcal{C}^\infty_{\mathcal{W}}(X,X)$ (see Corollary 4.2.13) and Γ is continuous, a compactness argument gives a neighborhood U of 0 such that

$$\Gamma([0,1] \times U) \subseteq \kappa_{\mathcal{W}}^{-1}(\operatorname{Diff}_{\mathcal{W}}(X)).$$

We recorded in Lemma 4.3.3 that this is equivalent to the existence of an open neighborhood V of $0 \in \mathcal{C}^\infty([0,1], \mathcal{C}^\infty_{\mathcal{W}}(X,X))$ such that for each $\gamma \in V$, there exists a right evolution

$\mathrm{Evol}^{\rho}_{\mathrm{Diff}_{\mathcal{W}}(X)}(\gamma)$ and that $\mathrm{evol}^{\rho}_{\mathrm{Diff}_{\mathcal{W}}(X)}|_V$ is smooth. But we know from Lemma B.2.10 that this entails the regularity of $\mathrm{Diff}_{\mathcal{W}}(X)$. ■

COROLLARY 4.3.12. *Let X be a Banach space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. Then $\mathrm{Diff}_{\mathcal{W}}(X)^\circ$ is a regular Lie group.*

Proof. Let $\gamma \in \mathcal{C}^\infty([0,1], \mathbf{T}_{\mathrm{id}_X} \mathrm{Diff}_{\mathcal{W}}(X)^\circ)$. Since $\mathbf{T}_{\mathrm{id}_X} \mathrm{Diff}_{\mathcal{W}}(X)^\circ \subseteq \mathbf{T}_{\mathrm{id}_X} \mathrm{Diff}_{\mathcal{W}}(X)$ and $\mathrm{Diff}_{\mathcal{W}}(X)$ is regular by Theorem 4.3.11, there exists a right evolution $\mathrm{Evol}^{\rho}(\gamma) : [0,1] \to \mathrm{Diff}_{\mathcal{W}}(X)$. By Lemma 4.3.3, the curve $\Gamma := \kappa_{\mathcal{W}} \circ \mathrm{Evol}^{\rho}(\gamma)$ is a solution to the initial value problem (4.3.2.1), where $\mathcal{F} = \mathcal{W}$, $k = \infty$ and $p = d\kappa_{\mathcal{W}}^{-1} \circ \gamma$. So for $t \in [0,1]$,

$$\Gamma(t) = \int_0^t \Gamma'(s)\,ds = \int_0^t p(s) \circ (\Gamma(s) + \mathrm{id}_X)\,ds.$$

Hence Lemma 4.1.5 and the fact that $\mathcal{C}^\infty_{\mathcal{W}}(X,X)^o$ is closed in $\mathcal{C}^\infty_{\mathcal{W}}(X,X)$ by Lemma 3.1.6 show that $\mathrm{Evol}^{\rho}(\gamma)$ takes its values in $\mathrm{Diff}_{\mathcal{W}}(X) \cap \mathrm{End}_{\mathcal{W}}(X)^\circ = \mathrm{Diff}_{\mathcal{W}}(X)^\circ$. From this and the smoothness of $\mathrm{evol}^{\rho}_{\mathrm{Diff}_{\mathcal{W}}(X)}$ we easily conclude that $\mathrm{evol}^{\rho}_{\mathrm{Diff}_{\mathcal{W}}(X)^\circ}$ is smooth. ■

On one-parameter subgroups. We calculate the one-parameter subgroups of $\mathrm{Diff}_{\mathcal{W}}(X)$ (and hence for $\mathrm{Diff}_{\mathcal{W}}(X)^\circ$). These arise as flows of vector fields.

LEMMA 4.3.13. *Let X be a Banach space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. Then for $\gamma \in \mathcal{C}^\infty_{\mathcal{W}}(X,X)$, the associated flow of the one-parameter subgroup of $\mathrm{Diff}_{\mathcal{W}}(X)$ with the right logarithmic derivative $\mathbf{T}_0\kappa_{\mathcal{W}}(\gamma)$ is the flow of γ (as a vector field).*

Proof. We proved in Theorem 4.3.11 that $\mathrm{Diff}_{\mathcal{W}}(X)$ is regular, hence the one-parameter subgroup $\mathcal{P}$ of $\mathrm{Diff}_{\mathcal{W}}(X)$ with $\delta_\rho(\mathcal{P})(t) = \mathbf{T}_0\kappa_{\mathcal{W}}(\gamma)$ for all $t \in \mathbb{R}$ exists. We have to show that for any $x \in X$, the curve $\mathbb{R} \to X : t \mapsto \mathcal{P}(t)(x)$ is the solution to the ODE

$$f'(t) = \gamma(f(t)), \qquad f(0) = x.$$

Obviously, $\mathcal{P}(0)(x) = \mathrm{id}_X(x) = x$. Further, $\mathcal{P}(t)(x) = (\mathrm{ev}_x \circ \kappa_{\mathcal{W}} \circ \kappa_{\mathcal{W}}^{-1} \circ \mathcal{P})(t)$. It is an easy computation to see that $\mathrm{ev}_x \circ \kappa_{\mathcal{W}}$ is $\mathcal{C}^1$ with

$$d(\mathrm{ev}_x \circ \kappa_{\mathcal{W}})(\gamma; \gamma_1) = \mathrm{ev}_x(\gamma_1).$$

By our assumptions, for $t \in \mathbb{R}$,

$$\mathcal{P}'(t) = \mathbf{T}_0\kappa_{\mathcal{W}}(\gamma) \cdot \mathcal{P}(t) = \mathbf{T}\rho_{\mathcal{P}(t)}(\mathbf{T}_0\kappa_{\mathcal{W}}(\gamma)) = \mathbf{T}(\rho_{\mathcal{P}(t)} \circ \kappa_{\mathcal{W}})(0, \gamma).$$

So by using this results and Lemma 4.3.1, we get

$$\begin{aligned}(\mathrm{ev}_x \circ \mathcal{P})'(t) &= (d(\mathrm{ev}_x \circ \kappa_{\mathcal{W}}) \circ \mathbf{T}\kappa_{\mathcal{W}}^{-1})(\mathcal{P}'(t)) \\ &= d(\mathrm{ev}_x \circ \kappa_{\mathcal{W}})(\kappa_{\mathcal{W}}^{-1}(\mathcal{P}(t)); \gamma \circ \mathcal{P}(t)) = \gamma(\mathcal{P}(t)(x)).\end{aligned}$$

This proves that $\mathbb{R} \to X : t \mapsto \mathcal{P}(t)(x)$ is the integral curve of γ for the initial value x. ■

5. Integration of certain Lie algebras of vector fields

The aim of this chapter is the integration of Lie algebras that arise as the semidirect product of a weighted function space $\mathcal{C}^\infty_{\mathcal{W}}(X,X)$ and $\mathbf{L}(G)$, where G is a subgroup of $\mathrm{Diff}(X)$ which is a Lie group with respect to composition and inversion of functions.

The canonical candidate for this purpose is the semidirect product of $\operatorname{Diff}_{\mathcal{W}}(X)$ and G—if it can be constructed. Hence we need criteria for

$$G \times \operatorname{Diff}_{\mathcal{W}}(X) \to \operatorname{Diff}(X) : (T, \phi) \mapsto T \circ \phi \circ T^{-1}$$

to have image in $\operatorname{Diff}_{\mathcal{W}}(X)$ and be smooth.

5.1. On the smoothness of the conjugation action on $\operatorname{Diff}_{\mathcal{W}}(X)_0$**.** We slightly generalize our approach by allowing arbitrary Lie groups to act on $\operatorname{Diff}_{\mathcal{W}}(X)$. We need the following notation.

Definition 5.1.1. Let G be a group and $\omega : G \times M \to M$ an action of G on the set M.

(a) For $g \in G$, we denote the partial map $\omega(g, \cdot) : M \to M$ by ω_g.
(b) Assume that G is a locally convex Lie group with the identity element e, M is a smooth manifold and ω is smooth. We define the linear map

$$\dot{\omega} : \mathbf{L}(G) \to \mathfrak{X}(M) \quad \text{by} \quad \dot{\omega}(x)(m) = -\mathbf{T}_e \omega(\cdot, m)(x).$$

Note that $\dot{\omega}$ takes its values in the smooth vector fields because ω is smooth.

Now we can state a first criterion for smoothness of the conjugation action—however only on the identity component $\operatorname{Diff}_{\mathcal{W}}(X)_0$ of $\operatorname{Diff}_{\mathcal{W}}(X)$.

Lemma 5.1.2. *Let X be a Banach space, $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$, G a Lie group and $\omega : G \times X \to X$ a smooth action. We define the map*

$$\alpha : G \times \operatorname{Diff}_{\mathcal{W}}(X) \to \operatorname{Diff}(X) : (T, \phi) \mapsto \omega_T \circ \phi \circ \omega_{T^{-1}}.$$

Assume that there exists an open set $\Omega \in \mathcal{U}_G(\mathbf{1})$ such that the maps

$$\mathcal{C}^\infty_{\mathcal{W}}(X, X) \times \Omega \to \mathcal{C}^\infty_{\mathcal{W}}(X, X) : (\gamma, T) \mapsto \gamma \circ \omega_T \tag{5.1.2.1}$$

and

$$\mathcal{C}^\infty_{\mathcal{W}}(X, X) \times \Omega \to \mathcal{C}^\infty_{\mathcal{W}}(X, X) : (\gamma, T) \mapsto D\omega_T \cdot \gamma \tag{5.1.2.2}$$

are well-defined and smooth.

(a) *For each open neighborhood $U_{\mathcal{W}} \subseteq \operatorname{Diff}_{\mathcal{W}}(X)$ of the identity such that $[\phi, \operatorname{id}_X] := \{t\phi + (1-t)\operatorname{id}_X : t \in [0,1]\} \subseteq \operatorname{Diff}_{\mathcal{W}}(X)$ for each $\phi \in U_{\mathcal{W}}$, the map*

$$(\Omega \cap \Omega^{-1}) \times U_{\mathcal{W}} \to \operatorname{End}_{\mathcal{W}}(X) : (T, \phi) \mapsto \alpha(T, \phi) \tag{†}$$

is well-defined and smooth.
(b) *Suppose that $\Omega = G$. Then the map*

$$G \times \operatorname{Diff}_{\mathcal{W}}(X)_0 \to \operatorname{Diff}_{\mathcal{W}}(X)_0 : (T, \phi) \mapsto \alpha(T, \phi) \tag{††}$$

is well-defined and smooth.

Proof. (a) Using Corollary 4.1.8, Theorem 4.2.17 and the smoothness of (5.1.2.1) and (5.1.2.2), for each $t \in [0,1]$, $T \in \Omega \cap \Omega^{-1}$ and $\phi \in U_{\mathcal{W}}$ we see that

$$\psi_{t,T,\phi} := (D\omega_T \cdot ((\phi - \operatorname{id}_X) \circ (t\phi + (1-t)\operatorname{id}_X)^{-1})) \circ (t\phi + (1-t)\operatorname{id}_X) \circ \omega_T^{-1} \in \mathcal{C}^\infty_{\mathcal{W}}(X, X),$$

and $\psi_{t,T,\phi}$ is a smooth map. Further, using that $t\phi + (1-t)\,\mathrm{id}_X$ is a diffeomorphism for each $t \in [0,1]$, we calculate

$$\begin{aligned}
&(\omega_T \circ \phi \circ \omega_{T^{-1}})(x) - x \\
&\qquad = (\omega_T \circ \phi \circ \omega_T^{-1})(x) - (\omega_T \circ \omega_T^{-1})(x) \\
&\qquad = \int_0^1 D\omega_T \circ (t\phi + (1-t)\,\mathrm{id}_X)(\omega_T^{-1}(x)) \cdot (\phi - \mathrm{id}_X)(\omega_T^{-1}(x))\,dt \\
&\qquad = \int_0^1 \big(D\omega_T \cdot ((\phi - \mathrm{id}_X) \circ (t\phi + (1-t)\,\mathrm{id}_X)^{-1})\big) \circ (t\phi + (1-t)\,\mathrm{id}_X)(\omega_T^{-1}(x))\,dt.
\end{aligned}$$

Hence $\omega_T \circ \phi \circ \omega_{T^{-1}} - \mathrm{id}_X = \int_0^1 \psi_{t,T,\phi}\,dt \in \mathcal{C}^\infty_{\mathcal{W}}(X,X)$ by Proposition A.1.8, using that we proved in Corollary 3.2.12 that $\mathcal{C}^\infty_{\mathcal{W}}(X,X)$ is complete.

Since $\psi_{t,T,\phi}$ is smooth as a function of t, T and ϕ, we can use Proposition A.1.18 to see that (†) is defined and smooth.

(b) Since $\mathrm{Diff}_{\mathcal{W}}(X)$ is locally convex, we find a symmetric open $U_{\mathcal{W}} \in \mathcal{U}(\mathrm{id}_X)$ such that $[U_{\mathcal{W}}, \mathrm{id}_X] \subseteq \mathrm{Diff}_{\mathcal{W}}(X)$. Using the symmetry of $U_{\mathcal{W}}$ and (a) we see that $\alpha(G \times U_{\mathcal{W}}) \subseteq \mathrm{Diff}_{\mathcal{W}}(X)_0$. Since $U_{\mathcal{W}}$ generates $\mathrm{Diff}_{\mathcal{W}}(X)_0$, we can apply Lemma B.2.13 to conclude that $\alpha(G \times \mathrm{Diff}_{\mathcal{W}}(X)_0) \subseteq \mathrm{Diff}_{\mathcal{W}}(X)_0$. Further (††) is smooth by (a) and Lemma B.2.14. ∎

So all we need are criteria for the smoothness of the maps (5.1.2.1) and (5.1.2.2). However, we will only elaborate on (5.1.2.1) and prove the smoothness of (5.1.2.2) under relatively strong assumptions on the group G. A more detailed examination of (5.1.2.2) can be found in [Wal10, §5.2].

Definition 5.1.3. Let X be a normed space, $U \subseteq X$ an open nonempty subset, and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ a nonempty set of weights. We define $\widetilde{\mathcal{W}} \subseteq \overline{\mathbb{R}}^U$ as the set of functions f for which $\|\cdot\|_{f,0}$ is a continuous seminorm on $\mathcal{C}^0_{\mathcal{W}}(U,Y)$, for each normed space Y. Obviously $\mathcal{W} \subseteq \widetilde{\mathcal{W}}$ and by Lemma 3.2.2, $\|\cdot\|_{f,\ell}$ is a continuous seminorm on $\mathcal{C}^k_{\mathcal{W}}(U,Y)$, provided that $f \in \widetilde{\mathcal{W}}$ and $\ell \leq k$.

5.1.1. Contravariant composition on weighted functions. Here we prove sufficient conditions for (5.1.2.1) to be smooth. Since the second factor of the domain of this map in general is not contained in a vector space, we have to wrestle with certain technical difficulties, leading to the definition of a notion of *logarithmically bounded* identity neighborhoods in Lie groups.

Lemma 5.1.4. *Let G be a Lie group and $\omega : G \times M \to M$ a smooth action of G on the smooth manifold M.*

(a) *For any $g \in G$,*
$$\mathbf{T}\omega = \mathbf{T}\omega_g \circ \mathbf{T}\omega \circ (\mathbf{T}\lambda_{g^{-1}} \times \mathrm{id}_{\mathbf{T}M}),$$
where $\lambda_{g^{-1}} : G \to G$ denotes the left multiplication with g^{-1}.

In the following, let $S, T \in G$ and $W : [0,1] \to G$ be a smooth curve with $W(0) = S$ and $W(1) = T$.

(b) *Let N be another smooth manifold and $\gamma : M \to N$ a $\mathcal{C}^1$-map. Then for $t \in [0,1]$ and $x \in M$,*

$$\mathbf{T}(\gamma \circ \omega \circ (W \times \mathrm{id}_M))(t, 1, 0_x) = \mathbf{T}\gamma \circ \mathbf{T}\omega_{W(t)}(-\dot{\omega}(\delta_\ell(W)(t))(x)). \qquad (\dagger)$$

(c) *Let X and Y be normed spaces. Assume that M is an open nonempty subset of X. Then for $\gamma, \eta \in \mathcal{C}^1(M, Y)$ and $x \in M$,*

$$\begin{aligned}&(\gamma \circ \omega_T)(x) - (\eta \circ \omega_S)(x)\\ &= ((\gamma - \eta) \circ \omega_T)(x) - \int_0^1 D\eta(\omega_{W(t)}(x)) \cdot D\omega_{W(t)}(x) \cdot \dot{\omega}(\delta_\ell(W)(t))(x)\, dt. \qquad (5.1.4.1)\end{aligned}$$

Proof. (a) For $h \in G$ and $m \in M$,

$$\omega(h, m) = \omega(gg^{-1}h, m) = \omega(g, \omega(g^{-1}h, m)) = \omega_g(\omega(\lambda_{g^{-1}}(h), m)).$$

Applying the tangent functor gives the assertion.

(b) We calculate

$$\begin{aligned}\mathbf{T}(\gamma \circ \omega \circ (W \times \mathrm{id}_M))(t, 1, 0_x) &= \mathbf{T}\gamma \circ \mathbf{T}\omega(W'(t), 0_x)\\ &= \mathbf{T}\gamma \circ \mathbf{T}\omega_{W(t)} \circ \mathbf{T}\omega(W(t)^{-1} \cdot W'(t), 0_x) = \mathbf{T}\gamma \circ \mathbf{T}\omega_{W(t)}(-\dot{\omega}(W(t)^{-1}W'(t))(x)).\end{aligned}$$

Here we used (a)

(c) By adding $0 = \eta \circ \omega_T - \eta \circ \omega_T$, we get

$$(\gamma \circ \omega_T)(x) - (\eta \circ \omega_S)(x) = ((\gamma - \eta) \circ \omega_T)(x) + (\eta \circ \omega_T)(x) - (\eta \circ \omega_S)(x)$$

We elaborate on the second summand (using ($\dagger$)):

$$\begin{aligned}(\eta \circ \omega_T)(x) - (\eta \circ \omega_S)(x) &= \eta(\omega(W(1), x)) - \eta(\omega(W(0), x))\\ &= \int_0^1 D(\eta \circ \omega \circ (W \times \mathrm{id}_U))(t, x) \cdot (1, 0)\, dt\\ &= -\int_0^1 D\eta(\omega_{W(t)}(x)) \cdot D\omega_{W(t)}(x) \cdot \dot{\omega}(\delta_\ell(W)(t))(x)\, dt. \qquad \blacksquare\end{aligned}$$

DEFINITION 5.1.5. Let G be a Lie group and $U \subseteq G$, $V \subseteq \mathbf{L}(G)$. We call a path $W \in \mathcal{C}^1([0,1], G)$ *V-logarithmically bounded* if $\delta_\ell(W)([0,1]) \subseteq V$. The set U is called *V-logarithmically bounded* if for all $g, h \in U$ there exists a V-logarithmically bounded $W \in \mathcal{C}^\infty([0,1], V)$ with $W(0) = g$ and $W(1) = h$.

PROPOSITION 5.1.6. *Let X and Y be normed spaces, $U \subseteq X$ an open nonempty set, $k \in \overline{\mathbb{N}}$, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ a nonempty set of weights, G a locally convex Lie group and $\omega : G \times U \to U$ a smooth action. Assume that there exists an open neighborhood Ω of $\mathbf{1}$ in G such that*

$$\begin{aligned}&(\forall f \in \mathcal{W}, T \in \Omega)(\exists g \in \widetilde{\mathcal{W}})(\forall \varepsilon > 0)\\ &\qquad (\exists V \in \mathcal{U}_{\mathbf{L}(G)}(0), \widetilde{\Omega} \in \mathcal{U}_\Omega(T)\ V\text{-logarithmically bounded})\\ &\qquad\qquad (\forall S \in \widetilde{\Omega}, v \in V)\ |f|\, \|D\omega_S \cdot \dot{\omega}(v)\| < \varepsilon |g \circ \omega_S|. \qquad (5.1.6.1)\end{aligned}$$

Further assume that $\mathcal{W} \circ \omega_\Omega^{-1} \subseteq \widetilde{\mathcal{W}}$, and that for all $m \in \mathbb{N}$ with $m < k$ and normed spaces Z, the map

$$\mathcal{C}^m_{\mathcal{W}}(U, \mathrm{L}(X, Z)) \times \Omega \to \mathcal{C}^m_{\mathcal{W}}(U, \mathrm{L}(X, Z)) : (\Gamma, T) \mapsto \Gamma \cdot D\omega_T \qquad (5.1.6.2)$$

is defined and continuous.

(a) *The map*

$$\mathcal{C}^{k+1}_{\mathcal{W}}(U,Y) \times \Omega \to \mathcal{C}^{k}_{\mathcal{W}}(U,Y) : (\gamma, T) \mapsto \gamma \circ \omega_T$$

is well-defined and continuous.

(b) *Let $\ell \in \mathbb{N}^*$. Additionally assume that the maps*

$$\mathcal{C}^{k}_{\mathcal{W}}(U,\mathrm{L}(X,Y)) \times \Omega \to \mathcal{C}^{k}_{\mathcal{W}}(U,\mathrm{L}(X,Y)) : (\Gamma, T) \mapsto \Gamma \cdot D\omega_T \tag{5.1.6.3}$$

and

$$\mathcal{C}^{k}_{\mathcal{W}}(U,\mathrm{L}(X,Y)) \times \mathbf{L}(G) \to \mathcal{C}^{k}_{\mathcal{W}}(U,Y) : (\Gamma, v) \mapsto \Gamma \cdot \dot{\omega}(v) \tag{5.1.6.4}$$

are well-defined and $\mathcal{C}^{\ell-1}$. Then the map

$$\mathfrak{c} : \mathcal{C}^{k+\ell+1}_{\mathcal{W}}(U,Y) \times \Omega \to \mathcal{C}^{k}_{\mathcal{W}}(U,Y) : (\gamma, T) \mapsto \gamma \circ \omega_T$$

is $\mathcal{C}^\ell$ with the derivative

$$d\mathfrak{c}((\gamma, S);(\gamma_1, S_1)) = -(D\gamma \circ \omega_S) \cdot D\omega_S \cdot \dot{\omega}(S^{-1} \cdot S_1) + \gamma_1 \circ \omega_S. \tag{$\dagger$}$$

Proof. (a) For $k < \infty$, this is proved by induction on k.

$k = 0$: Let $\gamma, \eta \in \mathcal{C}^1_{\mathcal{W}}(U,Y)$, $T \in \Omega$ and $f \in \mathcal{W}$. Let $g \in \widetilde{\mathcal{W}}$ be as in (5.1.6.1). Given $\varepsilon > 0$, we find a neighborhood $\widetilde{\Omega}$ of T and $V \in \mathcal{U}_{\mathbf{L}(G)}(0)$ such that (5.1.6.1) is satisfied. Using identity (5.1.4.1), for $S \in \widetilde{\Omega}$, a V-logarithmically bounded path $W : [0,1] \to \widetilde{\Omega}$ connecting S and T, and $x \in U$ we calculate

$$\begin{aligned}
&|f(x)|\,\|(\gamma \circ \omega_T)(x) - (\eta \circ \omega_S)(x)\| \\
&\quad \le |f(x)| \left(\|((\gamma - \eta) \circ \omega_T)(x)\| + \left\| \int_0^1 D\eta(\omega_{W(t)}(x)) \cdot D\omega_{W(t)}(x) \cdot \dot{\omega}(\delta_\ell(W)(t))(x)\, dt \right\| \right) \\
&\quad \le \|\gamma - \eta\|_{f \circ \omega_T^{-1}, 0} + \int_0^1 |f(x)|\, \|D\eta(\omega_{W(t)}(x))\|_{\mathrm{op}} \cdot \|D\omega_{W(t)}(x) \cdot \dot{\omega}(\delta_\ell(W)(t))(x)\|\, dt \\
&\quad \le \|\gamma - \eta\|_{f \circ \omega_T^{-1}, 0} + \varepsilon \int_0^1 |(g \circ \omega_{W(t)})(x)|\, \|D\eta(\omega_{W(t)}(x))\|_{\mathrm{op}}\, dt \\
&\quad \le \|\gamma - \eta\|_{f \circ \omega_T^{-1}, 0} + \varepsilon \|\eta\|_{g,1}.
\end{aligned}$$

The continuity at (γ, η) follows from this estimate.

$k \to k+1$: By Proposition 3.2.3 and the inductive hypothesis, we just need to check that the map

$$\mathcal{C}^{k+2}_{\mathcal{W}}(U,Y) \times \Omega \to \mathcal{C}^{k}_{\mathcal{W}}(U,\mathrm{L}(X,Y)) : (\gamma, T) \mapsto D(\gamma \circ \omega_T)$$

is well-defined and continuous. For $\gamma \in \mathcal{C}^{k+2}_{\mathcal{W}}(U,Y)$ and $T \in \Omega$, we have

$$D(\gamma \circ \omega_T) = (D\gamma \circ \omega_T) \cdot D\omega_T.$$

Hence by the inductive hypothesis and the continuity of (5.1.6.2), the induction is finished.

$k = \infty$: This is an easy consequence of the case $k < \infty$ and Corollary 3.2.6.

(b) We prove this by induction on ℓ.

$\ell = 1$: Let $\gamma, \gamma_1 \in \mathcal{C}^{k+\ell+1}_{\mathcal{W}}(U, Y)$, $S \in \Omega$ and $S_1 \in \mathbf{T}_S\Omega$. Further, let $\Gamma :]-\delta, \delta[\to \Omega$ be a smooth curve with $\Gamma(0) = S$ and $\Gamma'(0) = S_1$. Then for sufficiently small $t \neq 0$,

$$\frac{1}{t}((\gamma + t\gamma_1) \circ \omega_{\Gamma(t)} - \gamma \circ \omega_S) = \frac{1}{t}(\gamma \circ w_{\Gamma(t)} - \gamma \circ \omega_S) + \gamma_1 \circ \omega_{\Gamma(t)}.$$

Using identity (5.1.4.1) we elaborate on the first summand:

$$\frac{1}{t}(\gamma \circ w_{\Gamma(t)} - \gamma \circ \omega_S)(x) = -\frac{1}{t}\int_0^1 D\gamma(\omega_{\Gamma(st)}(x)) \cdot D\omega_{\Gamma(st)}(x) \cdot \dot{\omega}(t\delta_\ell(\Gamma)(st))(x)\, ds.$$

Hence

$$\frac{1}{t}(\gamma \circ w_{\Gamma(t)} - \gamma \circ \omega_S) = -\int_0^1 (D\gamma \circ \omega_{\Gamma(st)}) \cdot D\omega_{\Gamma(st)} \cdot \dot{\omega}(\delta_\ell(\Gamma)(st))\, ds;$$

note that the integral on the right hand side exists by Lemma 3.2.13 since the curve

$$[0, 1] \to \mathcal{C}^k_{\mathcal{W}}(U, Y) : s \mapsto (D\gamma \circ \omega_{\Gamma(st)}) \cdot D\omega_{\Gamma(st)} \cdot \dot{\omega}(\delta_\ell(\Gamma)(st))$$

is well-defined and continuous by (a) and the continuity of (5.1.6.3) and (5.1.6.4). Hence by Proposition A.1.8,

$$\lim_{t\to 0} \frac{1}{t}((\gamma + t\gamma_1) \circ \omega_{\Gamma(t)} - \gamma \circ \omega_S) = -(D\gamma \circ \omega_S) \cdot D\omega_S \cdot \dot{\omega}(S^{-1} \cdot S_1) + \gamma_1 \circ \omega_S,$$

so the directional derivatives of $\mathfrak{c}$ exist, are of the form (†) and depend continuously on the directions by (a) and the continuity of (5.1.6.3) and (5.1.6.4).

$\ell \to \ell + 1$: Since (5.1.6.3) and (5.1.6.4) are $\mathcal{C}^\ell$ by assumption, we conclude from (†) and the inductive hypothesis that $d\mathfrak{c}$ is $\mathcal{C}^\ell$, whence $\mathfrak{c}$ is $\mathcal{C}^{\ell+1}$. ■

5.2. Conclusion and examples. Finally, we prove a sufficient criterion for the smoothness of the conjugation action of a Lie group G acting on X and $\mathrm{Diff}_{\mathcal{W}}(X)_0$.

Theorem 5.2.1. *Let X be a Banach space, G a Lie group, $\omega : G \times X \to X$ a smooth action and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. Assume that $\{f \circ \omega_T : f \in \mathcal{W}, T \in G\} \subseteq \widetilde{\mathcal{W}}$ (see Definition* 5.1.3*), $\{D\omega_T : T \in G\} \subseteq \mathcal{BC}^\infty(X, \mathrm{L}(X))$, the maps*

$$D : G \to \mathcal{BC}^\infty(X, \mathrm{L}(X)) : T \mapsto D\omega_T \tag{†}$$

and (5.1.6.4) *are well-defined and smooth and* (5.1.6.1) *is satisfied. Then the map*

$$G \times \mathrm{Diff}_{\mathcal{W}}(X)_0 \to \mathrm{Diff}_{\mathcal{W}}(X)_0 : (T, \phi) \mapsto \omega_T \circ \phi \circ \omega_T^{-1}$$

is well-defined and smooth.

Proof. Since (†) is well-defined and smooth, we can apply Corollary 3.3.7 to see that (5.1.2.2) is also well-defined and smooth. Similarly, using Corollary 3.3.6, we see that (5.1.6.2) and (5.1.6.3) are well-defined and smooth. Hence Proposition 5.1.6 shows that (5.1.2.1) is smooth. The assertion follows from Lemma 5.1.2. ■

Finally, we give a positive and a negative example. The first example shows that we can form the semidirect product $\mathrm{Diff}_{\mathcal{S}}(X)_0 \rtimes \mathrm{GL}(X)$ with respect to conjugation.

Lemma 5.2.2. *Let X, Y and Z be normed spaces, $U \subseteq X$ an open nonempty set, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ nonempty such that for each $f \in \mathcal{W}$, $f\|\cdot\| \in \widetilde{\mathcal{W}}$. Further, let $k \in \overline{\mathbb{N}}$ and $b : Y \times X \to Z$*

be a continuous bilinear map. Then

$$\mathcal{C}^k_{\mathcal{W}}(U,Y)\times \mathrm{L}(X)\to \mathcal{C}^k_{\mathcal{W}}(U,Z) : (\gamma,T)\mapsto b\circ(\gamma,T) \tag{†}$$

is well-defined and smooth.

Proof. The assertion holds for $k=\infty$ if it holds for all $k\in\mathbb{N}$. For $k\neq\infty$, the proof is by induction on k.

$k=0$: Since (†) is bilinear, it is smooth iff it is continuous at 0. So we only prove that. Let $f\in\mathcal{W}$, $\gamma\in\mathcal{C}^k_{\mathcal{W}}(U,Y)$, $T\in\mathrm{L}(X)$ and $x\in U$. Then

$$|f(x)|\,\|b(\gamma(x),T(x))\|\leq \|b\|_{\mathrm{op}}|f(x)|\,\|x\|\,\|\gamma(x)\|\,\|T\|_{\mathrm{op}}\leq \|b\|_{\mathrm{op}}\|\gamma\|_{f\|\cdot\|,0}\|T\|_{\mathrm{op}}.$$

We conclude that $b\circ(\gamma,T)\in\mathcal{C}^0_{\mathcal{W}}(U,Z)$ and that (†) is continuous at 0.

$k\to k+1$: By Lemma 3.3.2, for $\gamma\in\mathcal{C}^k_{\mathcal{W}}(U,Y)$ and $T\in\mathrm{L}(X)$ we have

$$D(b\circ(\gamma,T)) = b^{(1)}\circ(D\gamma,T)+b^{(2)}\circ(\gamma,DT).$$

Since $DT\in\mathcal{BC}^\infty(X,\mathrm{L}(X))$, by Proposition 3.3.3 $b^{(2)}\circ(\gamma,DT)\in\mathcal{C}^{k+1}_{\mathcal{W}}(U,\mathrm{L}(X,Z))$ and the map $(\gamma,T)\mapsto b^{(2)}\circ(\gamma,DT)$ is smooth (here we use that $\mathrm{L}(X)\to\mathcal{BC}^\infty(X,\mathrm{L}(X)) : T\mapsto DT$ is smooth). By the induction hypothesis, the same holds for $(\gamma,T)\mapsto b^{(1)}\circ(D\gamma,T)$. So using Proposition 3.2.3, the proof is finished. ■

LEMMA 5.2.3. *Let X be a Banach space and $G:=\mathrm{GL}(X)$. Define the action*

$$\omega : G\times X\to X : (g,x)\mapsto g(x)$$

and set $\mathcal{W}:=\{x\mapsto\|x\|^n : n\in\mathbb{N}\}$.

(a) *The map* (5.1.6.4) *is smooth.*
(b) *The condition* (5.1.6.1) *is satisfied.*

Proof. We easily see that $\dot\omega = -\operatorname{id}_{\mathrm{L}(X)}$ (since $\mathbf{L}(G)=\mathrm{L}(X)$), and for each $S\in G$ and $x\in X$, $\omega_S=S$ and $DS(x)=S$.

(a) Let Y be another normed space. Since for $\Gamma\in\mathcal{C}^k_{\mathcal{W}}(X,\mathrm{L}(X,Y))$ and $S\in\mathbf{L}(G)$, $\Gamma\cdot\dot\omega(S)=\mathrm{ev}_{\mathrm{L}(X,Y)}\circ(\Gamma,-S)$ and $\mathrm{ev}_{\mathrm{L}(X,Y)}$ is bilinear and continuous, this is a consequence of Lemma 5.2.2.

(b) Let $f=\|\cdot\|^n\in\mathcal{W}$, $T\in G$ and $\varepsilon>0$. There exists an open convex $U\in\mathcal{U}_G(T)$ such that for all $S\in U$,

- $\|S-T\|_{\mathrm{op}}<\varepsilon$,
- $\|S^{-1}\|_{\mathrm{op}}<2\|T^{-1}\|_{\mathrm{op}}$,
- $\|S\|_{\mathrm{op}}<2\|T\|_{\mathrm{op}}$,

Then the path $W:[0,1]\to G : t\mapsto tT+(1-t)S$ has the left logarithmic derivative $\delta_\ell(W)(t)=W(t)^{-1}(T-S)$, hence U is $\overline{B}_{\mathrm{L}(X)}(0,2\|T\|_{\mathrm{op}}\varepsilon)$-logarithmically bounded. We calculate for $x\in X$, $S\in U$ and $A\in\overline{B}_{\mathrm{L}(X)}(0,2\|T\|_{\mathrm{op}}\varepsilon)$ that

$$\begin{aligned}|f(x)|\,\|D\omega_S(x)\cdot\dot\omega(A)(x)\| &= \|x\|^n\|(S\circ A)(x)\|\leq\|S\|_{\mathrm{op}}\|A\|_{\mathrm{op}}\|x\|^{n+1}\\ &\leq 4\|T\|^2_{\mathrm{op}}\varepsilon\|S^{-1}Sx\|^{n+1}\leq\varepsilon 2^{n+3}\|T\|^2_{\mathrm{op}}\|T^{-1}\|^{n+1}_{\mathrm{op}}\|Sx\|^{n+1}.\end{aligned}$$

Since $x\mapsto 2^{n+3}\|T\|^2_{\mathrm{op}}\|T^{-1}\|^{n+1}_{\mathrm{op}}\|x\|^{n+1}\in\widetilde{\mathcal{W}}$, we see that condition (5.1.6.1) is satisfied. ■

EXAMPLE 5.2.4. Let X, G, ω and $\mathcal{W}$ be as in Lemma 5.2.3. For each $S \in G$ and $x \in X$, $DS(x) = S$. Hence the map

$$D : G \to \mathcal{BC}^\infty(X, \mathrm{L}(X)) : S \mapsto DS$$

is smooth. By Lemma 5.2.3, the assumptions of Theorem 5.2.1 hold (since $\mathcal{W} \circ G \subseteq \widetilde{\mathcal{W}}$ is obviously true), hence the map

$$\mathrm{GL}(X) \times \mathrm{Diff}_{\mathcal{W}}(X)_0 \to \mathrm{Diff}_{\mathcal{W}}(X)_0 : (T, \phi) \mapsto T \circ \phi \circ T^{-1}$$

is smooth. So using Lemma B.2.15, we can form the semidirect product

$$\mathrm{Diff}_{\mathcal{W}}(X)_0 \rtimes \mathrm{GL}(X)$$

with respect to the inner automorphisms on $\mathrm{Diff}_{\mathcal{W}}(X)_0$ that are induced by $\mathrm{GL}(X)$.

Finally, we show that the conjugation of $\mathrm{GL}(\mathbb{R})$ on $\mathrm{Diff}_{\{1_\mathbb{R}\}}(X)_0$, if defined, may not be continuous.

EXAMPLE 5.2.5. For each $n \in \mathbb{N}$, $\sin((1 + \frac{1}{2n})n\pi) = \pm 1$, but $\sin(n\pi) = 0$. Hence

$$\| \sin(t_n \cdot) - \sin \|_{1_\mathbb{R}, 0} \geq 1$$

for each $n \in \mathbb{N}$, where $t_n := 1 + \frac{1}{2n}$. By Corollary 4.2.11, $\frac{1}{2} \sin \in \kappa_{\{1_\mathbb{R}\}}^{-1}(\mathrm{Diff}_{\{1_\mathbb{R}\}}(\mathbb{R}))$, and obviously $\kappa_{\{1_\mathbb{R}\}}(\frac{1}{2}\sin) \in \mathrm{Diff}_{\{1_\mathbb{R}\}}(X)_0$. If the conjugation of $\mathrm{GL}(\mathbb{R})$ on $\mathrm{Diff}_{\{1_\mathbb{R}\}}(X)_0$ was defined and continuous, then the map

$$\mathbb{R} \setminus \{0\} \times \mathcal{BC}^\infty(\mathbb{R}, \mathbb{R}) \to \mathcal{BC}^\infty(\mathbb{R}, \mathbb{R}) : (t, \gamma) \mapsto t^{-1}\gamma(t\cdot)$$

would be continuous in $(1, \frac{1}{2}\sin)$. But it is not since for $t > 0$ and $x \in \mathbb{R}$,

$$\begin{aligned}\|t^{-1}\sin(tx) - \sin(x)\| &\geq t^{-1}\|\sin(tx) - \sin(x)\| - \|(t^{-1} - 1)\sin(x)\| \\ &\geq t^{-1}\|\sin(tx) - \sin(x)\| - |t^{-1} - 1|.\end{aligned}$$

Hence we can calculate that for sufficiently large n,

$$\left\| \frac{1}{2} t_n^{-1} \sin(t_n \cdot) - \frac{1}{2} \sin \right\|_{1_\mathbb{R}, 0} \geq \frac{1}{4}.$$

6. Lie group structures on weighted mapping groups

In this chapter we will use the weighted function spaces discussed in Chapter 3 for the construction of locally convex Lie groups, the *weighted mapping groups*. These groups arise as subgroups of G^U, where G is a suitable Lie group and U is an open nonempty subset of a normed space. First, we give some definitions.

DEFINITION 6.0.1. Let U be a nonempty set and G be a group with the multiplication map m_G and the inversion map I_G. Then G^U can be endowed with a group structure: Multiplication is given by

$$((g_u)_{u \in U}, (h_u)_{u \in U}) \mapsto (m_G(g_u, h_u))_{u \in U} = m_G \circ ((g_u)_{u \in U}, (h_u)_{u \in U})$$

and inversion by

$$(g_u)_{u \in U} \mapsto (I_G(g_u))_{u \in U} = I_G \circ (g_u)_{u \in U}.$$

Further we call a set $A \subseteq G$ *symmetric* if

$$A = I_G(A).$$

Inductively, for $n \in \mathbb{N}$ with $n \geq 1$ we define

$$A^{n+1} := m_G(A^n \times A), \quad \text{where } A^1 := A.$$

DEFINITION 6.0.2. Let G be a Lie group and $\phi : V \to \mathbf{L}(G)$ a chart. We call the pair (ϕ, V) *centered around* $\mathbf{1}$ or just *centered* if $V \subseteq G$ is an open identity neighborhood and $\phi(\mathbf{1}) = 0$.

6.1. Weighted maps into Banach Lie groups. In this section, we discuss certain subgroups of G^U, where G is a Banach Lie group and U an open subset of a normed space X. We construct a subgroup $\mathcal{C}^k_{\mathcal{W}}(U, G)$ consisting of *weighted mappings* that can be turned into a (connected) Lie group. Its modelling space is $\mathcal{C}^k_{\mathcal{W}}(U, \mathbf{L}(G))$, where $k \in \overline{\mathbb{N}}$ and $\mathcal{W}$ is a set of weights on U containing 1_U. Later we prove that these groups are regular Lie groups. Finally, we discuss the case when $U = X$. Then $\mathrm{Diff}_{\mathcal{W}}(X)$ acts on $\mathcal{C}^\infty_{\mathcal{W}}(X, G)$, and thus we can turn the semidirect product of these groups into a Lie group.

6.1.1. Construction of the Lie group. We construct the Lie group from local data using Lemma B.2.5. For a chart (ϕ, V) of G, we can endow the set $\phi^{-1} \circ \mathcal{C}^{\partial,k}_{\mathcal{W}}(U, \phi(V)) \subseteq G^U$ with the manifold structure that turns the superposition operator ϕ_* into a chart. We need to check whether the local multiplication and inversion on this set are smooth with respect to this manifold structure. The group operations on G^U arise as the composition of the corresponding operations on G with the mappings (see Definition 6.0.1). Since the group operations of Banach Lie groups are analytic, we will use the results of Section 3.3.3 as our main tools. This allows us to construct $\mathcal{C}^k_{\mathcal{W}}(U, G)$ when G is an analytic Lie group modelled on an arbitrary normed space.

REMARK 6.1.1. We call a Lie group G *normed* if $\mathbf{L}(G)$ is a normable space. A *normed analytic* Lie group is a normed Lie group which is an analytic Lie group.

Local multiplication. The treatment of group multiplication is a simple application of Proposition 3.3.19.

LEMMA 6.1.2. *Let X be a normed space, $U \subseteq X$ an open nonempty subset, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $\ell \in \overline{\mathbb{N}}$, G a normed analytic Lie group with the group multiplication m_G and (ϕ, V) a centered chart of G. Then there exists an open identity neighborhood $W \subseteq V$ such that the map*

$$\mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(W)) \times \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(W)) \to \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(V)) : (\gamma, \eta) \mapsto \phi \circ m_G \circ (\phi^{-1} \circ \gamma, \phi^{-1} \circ \eta) \quad (\dagger)$$

is defined and analytic.

Proof. By Lemma 3.4.16, the map (†) is defined and analytic iff there exists an open identity neighborhood $W \subseteq G$ such that

$$(\phi \circ m_G \circ (\phi^{-1} \times \phi^{-1}))_* : \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(W) \times \phi(W)) \to \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(V))$$

is defined and analytic. There exists an open bounded zero neighborhood $\widetilde{W_L} \subseteq \mathbf{L}(G)$ such that $\widetilde{W_L} + \widetilde{W_L} \subseteq \phi(V)$. By the continuity of the multiplication m_G there exists an open **1**-neighborhood W with $m_G(W \times W) \subseteq \phi^{-1}(\widetilde{W_L})$. We may assume that $\phi(W)$ is star-shaped with center 0. Then

$$(\phi \circ m_G \circ (\phi^{-1} \times \phi^{-1}))(\phi(W) \times \phi(W)) \subseteq \widetilde{W_L}.$$

Further the restriction of $\phi \circ m_G \circ (\phi^{-1} \times \phi^{-1})$ to $\phi(W) \times \phi(W)$ is analytic, takes $(0,0)$ to 0 and has bounded image, since ϕ is centered and $\widetilde{W_L}$ is bounded. In the real case, using Lemma 3.3.17 we can choose $\phi(W)$ sufficiently small such that the restriction of $\phi \circ m_G \circ (\phi^{-1} \times \phi^{-1})$ to $\phi(W)$ has a good complexification. Hence we can apply Proposition 3.3.19 to see that

$$(\phi \circ m_G \circ (\phi^{-1} \times \phi^{-1})) \circ \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(W) \times \phi(W)) \in \mathcal{C}^{\ell}_{\mathcal{W}}(U, \widetilde{W_L})$$

and that the map $(\phi \circ m_G \circ (\phi^{-1} \times \phi^{-1}))_*$ is analytic. But

$$\mathcal{C}^{\ell}_{\mathcal{W}}(U, \widetilde{W_L}) \subseteq \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(V))$$

by the definition of $\widetilde{W_L}$, and this gives the assertion. ∎

Local inversion. The discussion of inversion is more delicate. For a short explanation, let $(\phi, \widetilde{V})$ be a chart for G, $V \subseteq \widetilde{V}$ a symmetric open identity neighborhood and I_G the inversion of G. Then the superposition $\phi \circ I_G \circ \phi^{-1}$ described in Proposition 3.3.19 does not necessarily map $\mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(V))$ into itself; hence we have to construct symmetrical open subsets.

LEMMA 6.1.3. *Let G be a group, $U \subseteq G$ a topological space and $V \subseteq U$ a symmetric subset with $\mathbf{1} \in V^\circ$ such that the inversion $I_G : V \to V$ is continuous. Then*

$$V^\circ \cap I_G(V^\circ)$$

is a symmetric set that is open in U and contains $\mathbf{1}$.

Proof. Let $W := V^\circ \cap I_G(V^\circ)$. Then $\mathbf{1} \in W$, and since

$$W^{-1} = I_G(W) = I_G(V^\circ \cap I_G(V^\circ)) = I_G(V^\circ) \cap I_G(I_G(V^\circ)) = I_G(V^\circ) \cap V^\circ = W,$$

it is a symmetric set. Since I_G is a homeomorphism, $I_G(V^\circ)$ is an open subset of V. Hence $W = I_G(V^\circ) \cap V^\circ$ is an open subset of V° and hence of U. ∎

LEMMA 6.1.4. *Let X be a normed space, $U \subseteq X$ an open nonempty subset, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $\ell \in \overline{\mathbb{N}}$, G a normed analytic Lie group with the group inversion I_G, and (ϕ, V) a centered chart of G such that $\phi(V)$ is bounded and V is symmetric.*

(a) *The map*

$$I_L := \phi \circ I_G \circ \phi^{-1} : \phi(V) \to \phi(V)$$

is an analytic bijective involution. Hence for any open and star-shaped set $W \subseteq \phi(V)$ with center 0, the map

$$\mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, W) \to \mathcal{C}^{\ell}_{\mathcal{W}}(U, \phi(V)) : \gamma \mapsto I_L \circ \gamma$$

is analytic, assuming in the real case that $I_L|_W$ has a good complexification.

(b) *Let $\Omega \subseteq \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(V))$. Then $\phi^{-1} \circ (\Omega \cap I_L \circ \Omega)$ is a symmetric subset of G^U.*

(c) *For any open zero neighborhood* $\widetilde{W} \subseteq \phi(V)$ *there exists an open convex zero neighborhood* $W \subseteq \widetilde{W}$ *such that*

$$\mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, W) \subseteq \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \widetilde{W}) \cap I_L \circ \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \widetilde{W}).$$

(d) *There exists an open convex zero neighborhood* $W \subseteq \phi(V)$ *and a zero neighborhood* $C^{\ell}_{\mathcal{W}} \subseteq \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(V))$ *such that*

$$\mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, W) \subseteq (C^{\ell}_{\mathcal{W}})^{\circ} \cap I_L \circ (C^{\ell}_{\mathcal{W}})^{\circ},$$

$\phi^{-1} \circ C^{\ell}_{\mathcal{W}}$ *is symmetric in* G^U, *the map*

$$C^{\ell}_{\mathcal{W}} \to C^{\ell}_{\mathcal{W}} : \gamma \mapsto I_L \circ \gamma$$

is continuous and its restriction to $(C^{\ell}_{\mathcal{W}})^{\circ}$ *is analytic. The set* W *can be chosen independently of* ℓ *and* $\mathcal{W}$.

Proof. (a) The assertions concerning I_L follow from the fact that V is symmetric and G is an analytic Lie group. The assertion on the superposition map of I_L is a consequence of Proposition 3.3.19 since W is star-shaped with center 0 and $\phi(V)$ is bounded.

(b) This is an easy computation.

(c) By the continuity of addition, we find an open zero neighborhood H with $H + H \subseteq \widetilde{W}$. Since I_L is continuous at 0 there exists an open convex zero neighborhood W with $I_L(W) \subseteq H$ and $W \subseteq \widetilde{W}$. Then

$$\mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, W) \subseteq \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \widetilde{W})$$

and by (a),

$$I_L \circ \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, W) \subseteq \mathcal{C}^{\ell}_{\mathcal{W}}(U, H) \subseteq \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \widetilde{W}).$$

The fact that $I_L \circ I_L = \mathrm{id}_{\phi(V)}$ completes the argument.

(d) Let $W_3 \subseteq \phi(V)$ be an open convex zero neighborhood. Then by (c) we find open convex zero neighborhoods $W_1, W_2 \subseteq \phi(V)$ such that

$$\mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, W_i) \subseteq \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, W_{i+1}) \cap I_L \circ \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, W_{i+1})$$

for $i = 1, 2$. So

$$C^{\ell}_{\mathcal{W}} := \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, W_3) \cap I_L \circ \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, W_3)$$

is a zero neighborhood, and by (b), $\phi^{-1} \circ C^{\ell}_{\mathcal{W}}$ is symmetric. Hence the superposition of I_L maps $C^{\ell}_{\mathcal{W}}$ into itself and is continuous on $C^{\ell}_{\mathcal{W}}$ and analytic on $(C^{\ell}_{\mathcal{W}})^{\circ}$ (see (a)). Further

$$(C^{\ell}_{\mathcal{W}})^{\circ} \cap I_L \circ (C^{\ell}_{\mathcal{W}})^{\circ} \supseteq \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, W_2) \cap I_L \circ \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, W_2) \supseteq \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, W_1),$$

whence (d) is established with $W := W_1$. ∎

Construction of the Lie group structure. After discussing the group operations locally, we turn a subgroup of G^U into a Lie group for each centered chart of G. We will also show that the identity component of this group does not depend on the chosen chart.

LEMMA 6.1.5. *Let* X *and* Y *be normed spaces,* $U \subseteq X$ *an open nonempty subset,* $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ *with* $1_U \in \mathcal{W}$, $\ell \in \overline{\mathbb{N}}$ *and* $V \subseteq Y$ *convex. Then the set* $\mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, V)$ *is convex.*

Proof. It is obvious that $\mathcal{C}^{\ell}_{\mathcal{W}}(U, V)$ is convex since V is. The set $\mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, V)$ is the interior of $\mathcal{C}^{\ell}_{\mathcal{W}}(U, V)$ with respect to the norm $\|\cdot\|_{1_U,0}$, hence it is convex. ∎

Proposition 6.1.6. *Let X be a normed space, $U \subseteq X$ an open nonempty subset, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $\ell \in \overline{\mathbb{N}}$, G a normed analytic Lie group and (ϕ, V) a centered chart. There exist a subgroup $(G,\phi)^U_{\mathcal{W},\ell}$ of G^U that can be turned into an analytic Lie group which is modelled on $\mathcal{C}^\ell_{\mathcal{W}}(U, \mathbf{L}(G))$, and an open $\mathbf{1}$-neighborhood $W \subseteq V$ which is independent of $\mathcal{W}$ and ℓ such that*

$$\mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(W)) \to (G,\phi)^U_{\mathcal{W},\ell} : \gamma \mapsto \phi^{-1} \circ \gamma$$

is an analytic embedding onto an open set. Moreover, for any convex open zero neighborhood $\widetilde{W} \subseteq \phi(W)$, the set $\phi^{-1} \circ \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \widetilde{W})$ generates the identity component of $(G,\phi)^U_{\mathcal{W},\ell}$ as a group.

Proof. Using Lemma 6.1.2 we find an open $\mathbf{1}$-neighborhood $\widetilde{W} \subseteq V$ such that

$$\mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(\widetilde{W})) \times \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(\widetilde{W})) \to \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(V)) : (\gamma, \eta) \mapsto \phi \circ m_G \circ (\phi^{-1} \circ \gamma, \phi^{-1} \circ \eta)$$

is analytic. We may assume that $\widetilde{W}$ is symmetric. Using Lemmas 6.1.4(d) and 6.1.3, we find an open zero neighborhood $H \subseteq \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(\widetilde{W}))$ such that $\phi^{-1} \circ H$ is symmetric, the map

$$H \to H : \gamma \mapsto I_L \circ \gamma$$

is analytic and $\mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(W)) \subseteq H$ for some open $\mathbf{1}$-neighborhood $W \subseteq V$, which is independent of $\mathcal{W}$ and ℓ. We endow $\phi^{-1} \circ H$ with the differential structure which turns the bijection

$$\phi^{-1} \circ H \to H : \gamma \mapsto \phi \circ \gamma$$

into an analytic diffeomorphism. Then we can apply Lemma B.2.5 to construct an analytic Lie group structure on the subgroup $(G,\phi)^U_{\mathcal{W},\ell}$ of G^U which is generated by $\phi^{-1} \circ H$ such that $\phi^{-1} \circ H$ becomes an open subset of $(G,\phi)^U_{\mathcal{W},\ell}$.

Since we may assume that $\phi(W)$ is convex, $\mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(W))$ is open and convex (see Lemma 6.1.5), hence the set

$$\phi^{-1} \circ \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(W))$$

is connected and open by the construction of the differential structure of $(G,\phi)^U_{\mathcal{W},\ell}$. Furthermore it obviously contains the unit element, whence it generates the identity component. ∎

Lemma 6.1.7. *Let X be a normed space, $U \subseteq X$ an open nonempty subset, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $\ell \in \overline{\mathbb{N}}$ and G be a normed analytic Lie group. Then for centered charts (ϕ_1, V_1), (ϕ_2, V_2), the identity component of $(G,\phi_1)^U_{\mathcal{W},\ell}$ coincides with the one of $(G,\phi_2)^U_{\mathcal{W},\ell}$, and the identity map between them is an analytic diffeomorphism.*

Proof. We may assume that $\phi_1(V_1)$ and $\phi_2(V_2)$ are bounded. Using Proposition 6.1.6, we find open $\mathbf{1}$-neighborhoods $W_1 \subseteq V_1$, $W_2 \subseteq V_2$ such that the identity component of $(G,\phi_i)^U_{\mathcal{W},\ell}$ is generated by $\phi_i^{-1} \circ \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi_i(W_i))$ for $i \in \{1,2\}$. Since $\phi_1 \circ \phi_2^{-1}$ is analytic, we find open zero neighborhoods $\widetilde{W}_1^L \subseteq \phi_1(W_1)$ and $\widetilde{W}_2^L \subseteq \phi_2(W_2)$ such that

$$(\phi_1 \circ \phi_2^{-1})(\widetilde{W}_2^L) \subseteq \widetilde{W}_1^L \quad \text{and} \quad \widetilde{W}_1^L + \widetilde{W}_1^L \subseteq \phi_1(W_1),$$

and $\widetilde{W}_2^L$ is convex. Then by Proposition 6.1.6, the identity component of $(G,\phi_2)^U_{\mathcal{W},\ell}$ is generated by

$$\phi_2^{-1} \circ \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \widetilde{W}_2^L),$$

and in the real case we may assume that $\phi_1 \circ \phi_2^{-1}|_{\widetilde{W}_2^L}$ has a good complexification. By Proposition 3.3.19 the map

$$\mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \widetilde{W}_2^L) \to \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi_1(W_1)) : \gamma \mapsto \phi_1 \circ \phi_2^{-1} \circ \gamma$$

is defined and analytic, and this implies that

$$\phi_2^{-1} \circ \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \widetilde{W}_2^L) \subseteq \phi_1^{-1} \circ \mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi_1(W_1)).$$

Hence the identity component of $(G, \phi_2)^U_{\mathcal{W},\ell}$ is contained in the one of $(G, \phi_1)^U_{\mathcal{W},\ell}$, and the inclusion map of the former into the latter is analytic.

Exchanging the roles of ϕ_1 and ϕ_2 in the preceding argument, we get the assertion. ∎

Definition 6.1.8. Let X be a normed space, $U \subseteq X$ an open nonempty subset, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $\ell \in \overline{\mathbb{N}}$ and G be a normed analytic Lie group. We write $\mathcal{C}^\ell_{\mathcal{W}}(U, G)$ for the connected Lie group constructed in Proposition 6.1.6. There and in Lemma 6.1.7 it was proved that for any centered chart (ϕ, V) of G and $W \subseteq V$ such that $\phi(W)$ is convex, the inverse of

$$\mathcal{C}^{\partial,\ell}_{\mathcal{W}}(U, \phi(W)) \to \mathcal{C}^\ell_{\mathcal{W}}(U, G) : \gamma \mapsto \phi^{-1} \circ \gamma$$

is a chart.

6.1.2. Regularity. We show that for a Banach Lie group G, the Lie group $\mathcal{C}^\ell_{\mathcal{W}}(U, G)$ is regular.

Lemma 6.1.9. *Let G, H be Lie groups and $\phi : G \to H$ a Lie group morphism.*

(a) *For each $g \in G$ and $v \in \mathbf{T}_g G$, we have $\mathbf{T}_g\phi(v) = \phi(g) \cdot \mathbf{L}(\phi)(g^{-1} \cdot v)$.*
(b) *Let $\gamma \in \mathcal{C}^1([0,1], G)$. Then $\delta_\ell(\phi \circ \gamma) = \mathbf{L}(\phi) \circ \delta_\ell(\gamma)$.*

Proof. The proof of (a) being straightforward, we turn to (b). We calculate the derivative of $\phi \circ \gamma$ using (a) and the fact that ϕ is a Lie group morphism:

$$(\phi \circ \gamma)'(t) = \mathbf{T}(\phi \circ \gamma)(t, 1) = \mathbf{T}_{\gamma(t)}\phi(\gamma'(t)) = \phi(\gamma(t)) \cdot \mathbf{L}(\phi)(\gamma(t)^{-1} \cdot \gamma'(t)).$$

From this we derive

$$\delta_\ell(\phi \circ \gamma)(t) = (\phi \circ \gamma)(t)^{-1} \cdot (\phi \circ \gamma)'(t) = \mathbf{L}(\phi)(\gamma(t)^{-1} \cdot \gamma'(t)) = \mathbf{L}(\phi)(\delta_\ell(\gamma)(t)).\ ∎$$

The following is well known from the theory of Banach Lie groups.

Lemma 6.1.10. *Let G be a Banach Lie group and $V \in \mathcal{U}(\mathbf{1})$. Then there exists a balanced open $W \in \mathcal{U}_{\mathbf{L}(G)}(0)$ such that*

$$\gamma \in \mathcal{C}^0([0,1], W) \ \Rightarrow\ \mathrm{Evol}^\ell_G(\gamma) \in \mathcal{C}^0([0,1], V). \tag{6.1.10.1}$$

Furthermore, the map $\mathrm{evol}^\ell_G : \mathcal{C}^0([0,1], W) \to G$ is continuous.

We define some terminology needed for the proof.

Definition 6.1.11. Let X be a normed space, $U \subseteq X$ an open nonempty set, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $k \in \overline{\mathbb{N}}$ and G be a Banach Lie group. Further, let $\mathcal{F}_1, \mathcal{F}_2 \subseteq \mathcal{W}$ be such that $1_U \in \mathcal{F}_1 \subseteq \mathcal{F}_2$ and $\ell_1, \ell_2 \in \overline{\mathbb{N}}$ such that $\ell_1 \leq \ell_2 \leq k$. We denote the inclusion

$$\mathcal{C}^{\ell_2}_{\mathcal{F}_2}(U, \mathbf{L}(G)) \to \mathcal{C}^{\ell_1}_{\mathcal{F}_1}(U, \mathbf{L}(G)).$$

by $\iota^L_{(\mathcal{F}_2,\ell_2),(\mathcal{F}_1,\ell_1)}$ and the inclusion

$$\mathcal{C}^{\ell_2}_{\mathcal{F}_2}(U, G) \to \mathcal{C}^{\ell_1}_{\mathcal{F}_1}(U, G)$$

by $\iota^G_{(\mathcal{F}_2,\ell_2),(\mathcal{F}_1,\ell_1)}$. Further, we define $\iota^L_{\mathcal{F}_1,\ell_1} := \iota^L_{(\mathcal{W},k),(\mathcal{F}_1,\ell_1)}$ and $\iota^G_{\mathcal{F}_1,\ell_1} := \iota^G_{(\mathcal{W},k),(\mathcal{F}_1,\ell_1)}$. Then for a suitable centered chart (ϕ, V) of G, the diagram

$$\begin{tikzcd}
\mathcal{C}^{\partial,\ell_2}_{\mathcal{F}_2}(U,\phi(V)) \arrow[r, "\phi_*^{-1}"] \arrow[d, "\iota^L_{(\mathcal{F}_2,\ell_2),(\mathcal{F}_1,\ell_1)}"'] & \mathcal{C}^{\ell_2}_{\mathcal{F}_2}(U,G) \arrow[d, "\iota^G_{(\mathcal{F}_2,\ell_2),(\mathcal{F}_1,\ell_1)}"] \\
\mathcal{C}^{\partial,\ell_1}_{\mathcal{F}_1}(U,\phi(V)) \arrow[r, "\phi_*^{-1}"] & \mathcal{C}^{\ell_1}_{\mathcal{F}_1}(U,G)
\end{tikzcd}$$

commutes. Hence we derive the identity

$$\mathbf{L}(\iota^G_{(\mathcal{F}_2,\ell_2),(\mathcal{F}_1,\ell_1)}) = \mathbf{T}_0\phi_*^{-1} \circ \mathbf{T}_0\iota^L_{(\mathcal{F}_2,\ell_2),(\mathcal{F}_1,\ell_1)} \circ \mathbf{T_1}\phi_*.$$

Let $x \in U$. Consider the maps

$$\mathrm{ev}^G_x : \mathcal{C}^{\partial,\ell_1}_{\mathcal{F}_1}(U,G) \to G : \gamma \mapsto \gamma(x), \qquad \mathrm{ev}^L_x : \mathcal{C}^{\partial,\ell_1}_{\mathcal{F}_1}(U,\mathbf{L}(G)) \to \mathbf{L}(G) : \gamma \mapsto \gamma(x).$$

Obviously, the diagram

$$\begin{tikzcd}
\mathcal{C}^{\partial,\ell_1}_{\mathcal{F}_1}(U,\phi(V)) \arrow[r, "\phi_*^{-1}"] \arrow[d, "\mathrm{ev}^L_x"'] & \mathcal{C}^{\ell_1}_{\mathcal{F}_1}(U,G) \arrow[d, "\mathrm{ev}^G_x"] \\
\phi(V) \arrow[r, "\phi^{-1}"] & G
\end{tikzcd}$$

commutes, so we derive the identity

$$\mathbf{L}(\mathrm{ev}^G_x) = \mathbf{T}_0\phi^{-1} \circ \mathbf{T}_0 \mathrm{ev}^L_x \circ \mathbf{T_1}\phi_*.$$

Remark 6.1.12. In the following, if E is a locally convex vector space, we shall frequently identify $\mathbf{T}_0 E = \{0\} \times E$ with E in the obvious way. Then for a Banach Lie group G and a centered chart (ϕ, V) of G such that $d\phi|_{\mathbf{L}(G)} = \mathrm{id}_{\mathbf{L}(G)}$, we can identify $\mathcal{C}^k_{\mathcal{W}}(U,\mathbf{L}(G))$ with $\mathbf{L}(\mathcal{C}^k_{\mathcal{W}}(U,G))$ via $\mathbf{T}_0\phi_*^{-1}$ and $\mathbf{T_1}\phi_*$, respectively.

Lemma 6.1.13. *Let X be a normed space, $U \subseteq X$ an open nonempty set, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $k \in \overline{\mathbb{N}}$, G a Banach Lie group and (ϕ, V) a centered chart for G such that $d\phi|_{\mathbf{L}(G)} = \mathrm{id}_{\mathbf{L}(G)}$. Further, let $x \in U$ and $\Gamma : [0,1] \to \mathcal{C}^k_{\mathcal{W}}(U,\mathbf{L}(G))$ be a smooth curve whose left evolution exists. Then $\mathrm{ev}^G_x \circ \mathrm{Evol}^\ell(\mathbf{T}_0\phi_*^{-1} \circ \Gamma)$ is the left evolution of $\mathrm{ev}^L_x \circ \Gamma$.*

Proof. We set $\eta := \mathrm{Evol}^\ell(\mathbf{T}_0\phi_*^{-1} \circ \Gamma)$ and calculate using Lemma 6.1.9 and Definition 6.1.11 that

$$\delta_\ell(\mathrm{ev}^G_x \circ \eta) = \mathbf{L}(\mathrm{ev}^G_x) \circ \delta_\ell(\eta) = \mathbf{T}_0\phi^{-1} \circ \mathbf{T}_0 \mathrm{ev}^L_x \circ \mathbf{T_1}\phi_* \circ \mathbf{T}_0\phi_*^{-1} \circ \Gamma = \mathrm{ev}^L_x \circ \Gamma. \blacksquare$$

Proposition 6.1.14. *Let X be a normed space, $U \subseteq X$ an open nonempty set, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $k \in \overline{\mathbb{N}}$ and G a Banach Lie group.*

(a) *$\mathcal{C}^k_{\mathcal{W}}(U,G)$, endowed with the Lie group structure described in Definition* 6.1.8, *is regular.*
(b) *The exponential function of $\mathcal{C}^k_{\mathcal{W}}(U,G)$ is given by*

$$\mathcal{C}^k_{\mathcal{W}}(U,\mathbf{L}(G)) \to \mathcal{C}^k_{\mathcal{W}}(U,G) : \gamma \mapsto \exp_G \circ \gamma,$$

where we identify $\mathcal{C}^k_{\mathcal{W}}(U,\mathbf{L}(G))$ with $\mathbf{L}(\mathcal{C}^k_{\mathcal{W}}(U,G))$.

Proof. (a) Let $(\phi, \widetilde{V})$ be a centered chart of G such that $d\phi|_{\mathbf{L}(G)} = \mathrm{id}_{\mathbf{L}(G)}$. We set

$$\mathbf{F} := \{\mathcal{F} \subseteq \mathcal{W} : 1_U \in \mathcal{F},\, |\mathcal{F}| < \infty\}.$$

After shrinking $\widetilde{V}$, we may assume that the inverse map of

$$\mathcal{C}^{\partial,\ell}_{\mathcal{F}}(U, \widetilde{V}) \to \mathcal{C}^{\ell}_{\mathcal{F}}(U, G) : \Gamma \mapsto \phi^{-1} \circ \Gamma$$

is a chart around the identity for $\mathcal{F} \in \mathbf{F}$ and $\ell \in \mathbb{N}$ with $\ell \leq k$ (see Definition 6.1.8). Let $V \subseteq \widetilde{V}$ be an open $\mathbf{1}$-neighborhood such that $\phi(V) + \phi(V) \subseteq \phi(\widetilde{V})$. We choose an open zero neighborhood $W \subseteq \phi(\widetilde{V})$ such that the implication (6.1.10.1) holds. Let $\Gamma : [0,1] \to \mathcal{C}^{\partial,k}_{\mathcal{W}}(U, W)$ be a smooth curve. Then $\Gamma_{\mathcal{F},\ell} := \iota^{L}_{\mathcal{F},\ell} \circ \Gamma$ is smooth, and since $\mathcal{C}^{\ell}_{\mathcal{F}}(U, G)$ is a Banach Lie group, the curve $\mathbf{T}_0\phi_*^{-1} \circ \Gamma_{\mathcal{F},\ell}$ has a smooth left evolution $\eta_{\mathcal{F},\ell} : [0,1] \to \mathcal{C}^{\ell}_{\mathcal{F}}(U, G)$. Then, for each $x \in U$, $\mathrm{ev}^G_x \circ \eta_{\mathcal{F},\ell}$ is the left evolution of $\mathrm{ev}^L_x \circ \Gamma_{\mathcal{F},\ell}$ by Lemma 6.1.13. Since we assumed that (6.1.10.1) holds, we conclude that for each $t \in [0,1]$, the image of $\eta_{\mathcal{F},\ell}(t)$ is contained in V.

Further, for $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{F}$ such that $\mathcal{F}_1 \subseteq \mathcal{F}_2$ and $\ell_1, \ell_2 \in \mathbb{N}$ such that $\ell_1 \leq \ell_2 \leq k$,

$$\begin{aligned}\delta_\ell(\iota^G_{(\mathcal{F}_2,\ell_2),(\mathcal{F}_1,\ell_1)} \circ \eta_{\mathcal{F}_2,\ell_2}) &= \mathbf{L}(\iota^G_{(\mathcal{F}_2,\ell_2),(\mathcal{F}_1,\ell_1)}) \circ \delta_\ell(\eta_{\mathcal{F}_2,\ell_2}) \\ &= \mathbf{T}_0\phi_*^{-1} \circ \mathbf{T}_0\iota^L_{(\mathcal{F}_2,\ell_2),(\mathcal{F}_1,\ell_1)} \circ \mathbf{T_1}\phi_* \circ \delta_\ell(\eta_{\mathcal{F}_2,\ell_2}) = \mathbf{T}_0\phi_*^{-1} \circ \Gamma_{\mathcal{F}_1,\ell_1} = \delta_\ell(\eta_{\mathcal{F}_1,\ell_1}).\end{aligned}$$

Hence $\eta_{\mathcal{F}_1,\ell_1} = \iota^G_{(\mathcal{F}_2,\ell_2),(\mathcal{F}_1,\ell_1)} \circ \eta_{\mathcal{F}_2,\ell_2}$. So the family $(\phi_* \circ \eta_{\mathcal{F},\ell})_{\mathcal{F}\in\mathbf{F},\ell\leq k}$ is compatible with the inclusion maps, hence using Proposition 3.2.5 and Proposition A.1.12, we derive a smooth curve $\widetilde{\eta} : [0,1] \to \mathcal{C}^{\partial,k}_{\mathcal{W}}(U, \phi(\widetilde{V}))$ such that for all $\mathcal{F} \in \mathbf{F}$ and $\ell \in \mathbb{N}$ with $\ell \leq k$, we have $\iota^L_{\mathcal{F},\ell} \circ \widetilde{\eta} = \phi_* \circ \eta_{\mathcal{F},\ell}$. We set $\eta := \phi_*^{-1} \circ \widetilde{\eta}$. Then

$$\mathbf{T}_0\phi_*^{-1} \circ \mathbf{T}_0\iota^L_{\mathcal{F},\ell} \circ \mathbf{T_1}\phi_* \circ \delta_\ell(\eta) = \mathbf{L}(\iota^G_{\mathcal{F},\ell}) \circ \delta_\ell(\eta) = \delta_\ell(\eta_{\mathcal{F},\ell}) = \mathbf{T}_0\phi_*^{-1} \circ \Gamma_{\mathcal{F},\ell} = \mathbf{T}_0\phi_*^{-1} \circ \iota^L_{\mathcal{F},\ell} \circ \Gamma,$$

and since $\mathcal{F}$ and ℓ were arbitrary, we conclude (using Proposition 3.2.5) that $\mathbf{T_1}\phi_* \circ \delta_\ell(\eta) = \Gamma$ and thus

$$\delta_\ell(\eta) = \mathbf{T}_0\phi_*^{-1} \circ \Gamma.$$

It remains to show that the left evolution is smooth. To this end, we denote the left evolution of $\mathcal{C}^{\ell}_{\mathcal{F}}(U, G)$ with $\mathrm{evol}_{\mathcal{F},\ell}$ and the one of $\mathcal{C}^{k}_{\mathcal{W}}(U, G)$ with evol. From our results above and Definition 6.1.11, we derive the commutative diagram

$$\begin{tikzcd}
\mathcal{C}^\infty([0,1], \mathcal{C}^{\partial,k}_{\mathcal{W}}(U, W)) \arrow[r, "\mathrm{evol}\circ\mathbf{T}_0\phi_*^{-1}"] \arrow[d, "\iota^L_{\mathcal{F},\ell}"'] & \phi_*^{-1} \circ \mathcal{C}^{\partial,k}_{\mathcal{W}}(U, \phi(\widetilde{V})) \arrow[d, "\iota^G_{\mathcal{F},\ell}"] \\
\mathcal{C}^\infty([0,1], \mathcal{C}^{\partial,\ell}_{\mathcal{F}}(U, W)) \arrow[r, "\mathrm{evol}_{\mathcal{F},\ell}\circ\mathbf{T}_0\phi_*^{-1}"] & \phi_*^{-1} \circ \mathcal{C}^{\partial,\ell}_{\mathcal{F}}(U, \phi(\widetilde{V}))
\end{tikzcd}$$

Since the three lower arrows represent smooth maps, the map

$$\phi_* \circ \iota^G_{\mathcal{F},\ell} \circ \mathrm{evol} \circ \mathbf{T}_0\phi_*^{-1} = \iota^L_{\mathcal{F},\ell} \circ \phi_* \circ \mathrm{evol} \circ \mathbf{T}_0\phi_*^{-1}$$

is smooth on $\mathcal{C}^\infty([0,1], \mathcal{C}^{\partial,k}_{\mathcal{W}}(U, W))$. Using Proposition A.1.12 and Section 3.2.2, we conclude that $\phi_* \circ \mathrm{evol} \circ \mathbf{T}_0\phi_*^{-1}$ is smooth, and since ϕ_* and $\mathbf{T}_0\phi_*^{-1}$ are diffeomorphisms, using Lemma B.2.10 we deduce that evol is smooth.

(b) Let (ϕ, V) be a centered chart of G such that $d\phi|_{\mathbf{L}(G)} = \mathrm{id}_{\mathbf{L}(G)}$. We denote the exponential function of $\mathcal{C}^{k}_{\mathcal{W}}(U, G)$ by $\exp_{\mathcal{W}}$. Let $x \in U$ and $\gamma \in \mathcal{C}^{k}_{\mathcal{W}}(U, \mathbf{L}(G))$. We denote the constant, γ-valued curve from $[0,1]$ to $\mathcal{C}^k_{\mathcal{W}}(U, \mathbf{L}(G))$ by Γ. We proved in Lemma 6.1.13 that $\mathrm{ev}^G_x \circ \mathrm{Evol}^\ell(\phi_*^{-1} \circ \Gamma)$ is the left evolution of $\mathrm{ev}^L_x \circ \Gamma$. On the other hand, since Γ is constant, the left evolution of $\mathrm{ev}^L_x \circ \Gamma$ is the restriction of the one-parameter group $\mathbb{R} \to G : t \mapsto \exp_G(t\, \mathrm{ev}^L_x(\gamma))$. Hence

$$\exp_G(\mathrm{ev}^L_x(\gamma)) = (\mathrm{ev}^G_x \circ \mathrm{Evol}^\ell(\phi_*^{-1} \circ \Gamma))(1) = \mathrm{ev}^G_x \circ \mathrm{evol}^\ell(\phi_*^{-1} \circ \Gamma) = \mathrm{ev}^G_x \circ \exp_{\mathcal{W}}(\phi_*^{-1}(\gamma)).$$

Thus $\exp_{\mathcal{W}}(\phi_*^{-1}(\gamma))(x) = \exp_G(\gamma(x))$, from which we deduce the assertion since $x \in U$ was arbitrary. ■

6.1.3. Semidirect products with weighted diffeomorphisms. In this subsection we discuss an action of the diffeomorphism group $\mathrm{Diff}_{\mathcal{W}}(X)$ on the Lie group $\mathcal{C}^\infty_{\mathcal{W}}(X, G)$, where G is a Banach Lie group. This action can be used to construct the semidirect product $\mathcal{C}^\infty_{\mathcal{W}}(X, G) \rtimes \mathrm{Diff}_{\mathcal{W}}(X)$ and turn it into a Lie group. For technical reasons, we first discuss the following action of $\mathrm{Diff}_{\mathcal{W}}(X)$ on G^X.

Definition 6.1.15. Let X be a Banach space, G a Banach Lie group and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. We define the map

$$\widetilde{\omega} : \mathrm{Diff}_{\mathcal{W}}(X) \times G^X \to G^X : (\phi, \gamma) \mapsto \gamma \circ \phi^{-1}.$$

It is easy to see that $\widetilde{\omega}$ is in fact a group action, and moreover that it is a group morphism in its second argument:

Lemma 6.1.16.

(a) *$\widetilde{\omega}$ is a group action of* $\mathrm{Diff}_{\mathcal{W}}(X)$ *on* G^X.
(b) *For each $\phi \in \mathrm{Diff}_{\mathcal{W}}(X)$ the partial map $\widetilde{\omega}(\phi, \cdot)$ is a group homomorphism.*

Proof. These are easy computations. ■

We show that this action leaves $\mathcal{C}^\infty_{\mathcal{W}}(X, G)$ invariant. Since we proved in Lemma 6.1.16 that $\widetilde{\omega}$ is a group morphism in its second argument, it suffices to show that it maps a generating set of $\mathcal{C}^\infty_{\mathcal{W}}(X, G)$ into this space.

Lemma 6.1.17. *Let X be a Banach space, G a Banach Lie group, $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$, $(\phi, \widetilde{V})$ a centered chart of G and V an open identity neighborhood such that $\phi(V)$ is convex. Then*

$$\widetilde{\omega}(\mathrm{Diff}_{\mathcal{W}}(X) \times (\phi^{-1} \circ \mathcal{C}^{\partial,\infty}_{\mathcal{W}}(X, \phi(V)))) \subseteq \phi^{-1} \circ \mathcal{C}^{\partial,\infty}_{\mathcal{W}}(X, \phi(V)),$$

and the map

$$\mathrm{Diff}_{\mathcal{W}}(X) \times \mathcal{C}^{\partial,\infty}_{\mathcal{W}}(X, \phi(V)) \to \mathcal{C}^{\partial,\infty}_{\mathcal{W}}(X, \phi(V)) : (\psi, \gamma) \mapsto \phi \circ \widetilde{\omega}(\psi, \phi^{-1} \circ \gamma)$$

is smooth. Moreover,

$$\widetilde{\omega}(\mathrm{Diff}_{\mathcal{W}}(X) \times \mathcal{C}^\infty_{\mathcal{W}}(X, G)) \subseteq \mathcal{C}^\infty_{\mathcal{W}}(X, G).$$

Proof. Let $\psi \in \mathrm{Diff}_{\mathcal{W}}(X)$ and $\gamma \in \mathcal{C}^{\partial,\infty}_{\mathcal{W}}(X, \phi(V))$. Then

$$\widetilde{\omega}(\psi, \phi^{-1} \circ \gamma) = \phi^{-1} \circ (\gamma \circ \psi^{-1}),$$

and using Corollary 4.1.8 this proves the first and—together with Proposition 4.2.16—the second assertion.

The final assertion follows immediately from the first assertion since we proved in Lemma 6.1.16 that $\widetilde{\omega}$ is a group morphism in its second argument, and according to Definition 6.1.8, $\mathcal{C}^{\infty}_{\mathcal{W}}(X, G)$ is generated by $\phi^{-1} \circ \mathcal{C}^{\partial,k}_{\mathcal{W}}(X, \phi(V))$. ∎

So by restricting $\widetilde{\omega}$ to $\mathrm{Diff}_{\mathcal{W}}(X) \times \mathcal{C}^{\infty}_{\mathcal{W}}(X, G)$, we get a group action of $\mathrm{Diff}_{\mathcal{W}}(X)$ on $\mathcal{C}^{\infty}_{\mathcal{W}}(X, G)$.

Definition 6.1.18. We define

$$\omega := \widetilde{\omega}|_{\mathrm{Diff}_{\mathcal{W}}(X)\times\mathcal{C}^{\infty}_{\mathcal{W}}(X,G)} : \mathrm{Diff}_{\mathcal{W}}(X) \times \mathcal{C}^{\infty}_{\mathcal{W}}(X, G) \to \mathcal{C}^{\infty}_{\mathcal{W}}(X, G) : (\phi, \gamma) \mapsto \gamma \circ \phi^{-1}.$$

Finally, we are able to turn the semidirect product $\mathcal{C}^{\infty}_{\mathcal{W}}(X, G) \rtimes_{\omega} \mathrm{Diff}_{\mathcal{W}}(X)$ into a Lie group.

Theorem 6.1.19. *Let X be a Banach space, G a Banach Lie group and $\mathcal{W} \subseteq \overline{\mathbb{R}}^X$ with $1_X \in \mathcal{W}$. Then $\mathcal{C}^{\infty}_{\mathcal{W}}(X, G) \rtimes_{\omega} \mathrm{Diff}_{\mathcal{W}}(X)$ can be turned into a Lie group modelled on $\mathcal{C}^{\infty}_{\mathcal{W}}(X, \mathbf{L}(G)) \times \mathcal{C}^{\infty}_{\mathcal{W}}(X, X)$.*

Proof. We proved in Lemma 6.1.17 that ω is smooth on a neighborhood of $(\mathrm{id}_X, \mathbf{1})$, and since this neighborhood is the product of generators of $\mathrm{Diff}_{\mathcal{W}}(X)$ resp. $\mathcal{C}^{\infty}_{\mathcal{W}}(X, G)$, we can use Lemma B.2.14 to see that ω is smooth. Hence we can apply Lemma B.2.15 and are done. ∎

6.2. Weighted maps into locally convex Lie groups. In this section, we discuss certain subgroups of G^U, where G is a Lie group and U an open subset of a finite-dimensional space X. We construct a subgroup $\mathcal{C}^k_{\mathcal{W}}(U, G)^{\bullet}$ consisting of *weighted decreasing mappings* that can be turned into a (connected) Lie group. Next, we extend this group to a Lie group $\mathcal{C}^k_{\mathcal{W}}(U, G)^{\bullet}_{\mathrm{ex}}$ which contains $\mathcal{C}^k_{\mathcal{W}}(U, G)^{\bullet}$ as an open normal subgroup, and discuss its relation to "rapidly decreasing mappings".

The modelling space of these groups is $\mathcal{C}^k_{\mathcal{W}}(U, \mathbf{L}(G))^{\bullet}$, where $k \in \overline{\mathbb{N}}$ and $\mathcal{W}$ is a set of weights on U containing 1_U. These spaces were introduced in Section 3.4.

6.2.1. Construction of the Lie group. We construct the Lie group from local data using Lemma B.2.5. For a chart (ϕ, V) of G, we can endow the set $\phi^{-1} \circ \mathcal{C}^k_{\mathcal{W}}(U, \phi(V))^{\bullet} \subseteq G^U$ with the manifold structure that turns the superposition operator ϕ_* into a chart. We then need to check whether multiplication and inversion on G^U are smooth with respect to this manifold structure. The group operations on G^U arise as the composition of the corresponding group operations on G with mappings in G^U (see Definition 6.0.1). The main tool used in this subsection is the superposition with smooth maps that we discussed in Proposition 3.4.23.

Local group operations. We first discuss local multiplication.

Lemma 6.2.1. *Let X be a finite-dimensional space, $U \subseteq X$ an open nonempty subset, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $\ell \in \overline{\mathbb{N}}$, G a locally convex Lie group with the group multiplication m_G and (ϕ, V) a centered chart of G. Then there exists an open identity neighborhood $W \subseteq V$ such that the map*

$$\mathcal{C}^{\ell}_{\mathcal{W}}(U, \phi(W))^{\bullet} \times \mathcal{C}^{\ell}_{\mathcal{W}}(U, \phi(W))^{\bullet} \to \mathcal{C}^{\ell}_{\mathcal{W}}(U, \phi(V))^{\bullet} : (\gamma, \eta) \mapsto \phi \circ m_G \circ (\phi^{-1} \circ \gamma, \phi^{-1} \circ \eta) \quad (\dagger)$$

is defined and smooth.

Proof. By Lemma 3.4.16, the map (†) is defined and smooth iff there exists an open neighborhood $W \subseteq G$ such that

$$(\phi \circ m_G \circ (\phi^{-1} \times \phi^{-1}))_* : \mathcal{C}^\ell_{\mathcal{W}}(U, \phi(W) \times \phi(W))^\bullet \to \mathcal{C}^\ell_{\mathcal{W}}(U, \phi(V))^\bullet$$

is defined and smooth. By the continuity of m_G there exists an open subset $W \subseteq V$ such that $m_G(W \times W) \subseteq V$. We may assume that $\phi(W)$ is star-shaped with center 0. Since $\phi \circ m_G \circ (\phi^{-1} \times \phi^{-1})$ is smooth and maps $(0, 0)$ to 0, we can apply Proposition 3.4.23 to see that

$$(\phi \circ m_G \circ (\phi^{-1} \times \phi^{-1})) \circ \mathcal{C}^\ell_{\mathcal{W}}(U, \phi(W) \times \phi(W))^\bullet \subseteq \mathcal{C}^\ell_{\mathcal{W}}(U, \phi(V))^\bullet$$

and that the map $(\phi \circ m_G \circ (\phi^{-1} \times \phi^{-1}))_*$ is smooth. ∎

Now, we turn to local inversion.

LEMMA 6.2.2. *Let X be a finite-dimensional space, $U \subseteq X$ an open nonempty subset, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $\ell \in \overline{\mathbb{N}}$, G a locally convex Lie group with the group inversion I_G and (ϕ, V) a centered chart such that V is symmetric. Further let $W \subseteq V$ be a symmetric open $\mathbf{1}$-neighborhood such that there exists an open star-shaped set W_L with center 0 and $\phi(W) \subseteq W_L \subseteq \phi(V)$. Then for each $\gamma \in \mathcal{C}^\ell_{\mathcal{W}}(U, \phi(W))^\bullet$,*

$$(\phi \circ I_G \circ \phi^{-1}) \circ \gamma \in \mathcal{C}^\ell_{\mathcal{W}}(U, W)^\bullet,$$

and the map

$$\mathcal{C}^\ell_{\mathcal{W}}(U, \phi(W))^\bullet \to \mathcal{C}^\ell_{\mathcal{W}}(U, \phi(W))^\bullet : \gamma \mapsto (\phi \circ I_G \circ \phi^{-1}) \circ \gamma$$

is smooth.

Proof. Since $I_L := \phi \circ I_G \circ \phi^{-1} : \phi(V) \to \phi(V)$ is smooth and $I_L(0) = 0$, we conclude from Proposition 3.4.23 that

$$\mathcal{C}^\ell_{\mathcal{W}}(U, W_L)^\bullet \to \mathcal{C}^\ell_{\mathcal{W}}(U, \phi(V))^\bullet : \gamma \mapsto I_L \circ \gamma$$

is smooth. Since we proved in Lemma 3.4.19 that $\mathcal{C}^\ell_{\mathcal{W}}(U, \phi(W))^\bullet$ is an open subset of $\mathcal{C}^\ell_{\mathcal{W}}(U, W_L)^\bullet$, the restriction of this map is also smooth, and since W is symmetric, it takes values in this set. ∎

Conclusion. We put everything together to obtain a Lie group for each centered chart of G. We show that the identity component does not depend on the chart used.

LEMMA 6.2.3. *Let X be a finite-dimensional space, $U \subseteq X$ an open nonempty subset, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $\ell \in \overline{\mathbb{N}}$, G a locally convex Lie group and (ϕ, V) a centered chart. Then there exists a subgroup $(G, \phi)^U_{\mathcal{W},\ell}$ of G^U that can be turned into a Lie group. It is modelled on $\mathcal{C}^\ell_{\mathcal{W}}(U, \mathbf{L}(G))^\bullet$ in such a way that there exists an open $\mathbf{1}$-neighborhood $W \subseteq V$ such that*

$$\mathcal{C}^\ell_{\mathcal{W}}(U, \phi(W))^\bullet \to (G, \phi)^U_{\mathcal{W},\ell} : \gamma \mapsto \phi^{-1} \circ \gamma$$

becomes a smooth embedding and its image is open. Further, for any subset $\widetilde{W} \subseteq W$ such that $\phi(\widetilde{W})$ is an open convex zero neighborhood,

$$\phi^{-1} \circ \mathcal{C}^\ell_{\mathcal{W}}(U, \phi(\widetilde{W}))^\bullet$$

generates the identity component of $(G, \phi)^U_{\mathcal{W},\ell}$.

Proof. Using Lemma 6.2.1 we find an open **1**-neighborhood $W \subseteq V$ such that

$$\mathcal{C}^\ell_{\mathcal{W}}(U, \phi(W))^\bullet \times \mathcal{C}^\ell_{\mathcal{W}}(U, \phi(W))^\bullet \to \mathcal{C}^\ell_{\mathcal{W}}(U, \phi(V))^\bullet : (\gamma, \eta) \mapsto \phi \circ m_G \circ (\phi^{-1} \circ \gamma, \phi^{-1} \circ \eta)$$

is smooth. We may assume that W is symmetric and that there exists an open convex set H such that $\phi(W) \subseteq H \subseteq \phi(V)$. We know from Lemma 6.2.2 that the set

$$\phi^{-1} \circ \mathcal{C}^\ell_{\mathcal{W}}(U, \phi(W))^\bullet \subseteq G^U$$

is symmetric and

$$\mathcal{C}^\ell_{\mathcal{W}}(U, \phi(W))^\bullet \to \mathcal{C}^\ell_{\mathcal{W}}(U, \phi(W))^\bullet : \gamma \mapsto \phi \circ I_G \circ \phi^{-1} \circ \gamma$$

is smooth. We endow $\phi^{-1} \circ \mathcal{C}^\ell_{\mathcal{W}}(U, \phi(W))^\bullet$ with the differential structure which turns the bijection

$$\phi^{-1} \circ \mathcal{C}^\ell_{\mathcal{W}}(U, \phi(W))^\bullet \to \mathcal{C}^\ell_{\mathcal{W}}(U, \phi(W))^\bullet : \gamma \mapsto \phi \circ \gamma$$

into a smooth diffeomorphism. Then we can apply Lemma B.2.5 to construct a Lie group structure on the subgroup $(G, \phi)^U_{\mathcal{W},\ell}$ of G^U which is generated by $\phi^{-1} \circ \mathcal{C}^\ell_{\mathcal{W}}(U, \phi(W))^\bullet$, such that $\phi^{-1} \circ \mathcal{C}^\ell_{\mathcal{W}}(U, \phi(W))^\bullet$ becomes an open subset.

Moreover, for each open **1**-neighborhood $\widetilde{W} \subseteq W$ such that $\phi(\widetilde{W})$ is convex, the set $\mathcal{C}^\ell_{\mathcal{W}}(U, \phi(\widetilde{W}))^\bullet$ is convex (Lemma 3.4.10). Hence $\phi^{-1} \circ \mathcal{C}^\ell_{\mathcal{W}}(U, \phi(\widetilde{W}))^\bullet$ is connected, and it is open by the construction of the differential structure of $(G, \phi)^U_{\mathcal{W},\ell}$. Further it obviously contains the unit element, hence it generates the identity component. ∎

LEMMA 6.2.4. *Let X be a finite-dimensional space, $U \subseteq X$ an open nonempty subset, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $\ell \in \overline{\mathbb{N}}$ and G a locally convex Lie group. Then for centered charts (ϕ_1, V_1) and (ϕ_2, V_2), the identity component of $(G, \phi_1)^U_{\mathcal{W},\ell}$ coincides with the one of $(G, \phi_2)^U_{\mathcal{W},\ell}$, and the identity map between them is a smooth diffeomorphism.*

Proof. Using Lemma 6.2.3, we find open **1**-neighborhoods $W_1 \subseteq V_1$, $W_2 \subseteq V_2$ such that the identity component of $(G, \phi_i)^U_{\mathcal{W},\ell}$ is generated by $\phi_i^{-1} \circ \mathcal{C}^\ell_{\mathcal{W}}(U, \phi_i(W_i))^\bullet$ for $i \in \{1, 2\}$. Since $\phi_1 \circ \phi_2^{-1}$ is smooth, we find an open convex zero neighborhood $\widetilde{W_2^L} \subseteq \phi_2(W_1 \cap W_2)$. By Proposition 3.4.23, the map

$$\mathcal{C}^\ell_{\mathcal{W}}(U, \widetilde{W_2^L})^\bullet \to \mathcal{C}^\ell_{\mathcal{W}}(U, \phi_1(W_1))^\bullet : \gamma \mapsto \phi_1 \circ \phi_2^{-1} \circ \gamma$$

is defined and smooth. This implies that

$$\phi_2^{-1} \circ \mathcal{C}^\ell_{\mathcal{W}}(U, \widetilde{W_2^L})^\bullet \subseteq \phi_1^{-1} \circ \mathcal{C}^\ell_{\mathcal{W}}(U, \phi_1(W_1))^\bullet.$$

Hence the identity component of $(G, \phi_2)^U_{\mathcal{W},\ell}$ is contained in the one of $(G, \phi_1)^U_{\mathcal{W},\ell}$, and the inclusion map of the former into the latter is smooth.

Exchanging the roles of ϕ_1 and ϕ_2 in the preceding argument, we get the assertion. ∎

DEFINITION 6.2.5. Let X be a finite-dimensional space, $U \subseteq X$ an open nonempty subset, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $\ell \in \overline{\mathbb{N}}$ and G a locally convex Lie group. Henceforth, we write $\mathcal{C}^\ell_{\mathcal{W}}(U, G)^\bullet$ for the connected Lie group constructed in Lemma 6.2.3. There and in Lemma 6.2.4 it was proved that for any centered chart (ϕ, V) of G there exists an open **1**-neighborhood W such that the inverse of

$$\mathcal{C}^\ell_{\mathcal{W}}(U, \phi(W))^\bullet \to \mathcal{C}^\ell_{\mathcal{W}}(U, G) : \gamma \mapsto \phi^{-1} \circ \gamma$$

is a chart, and that for any convex zero neighborhood $\widetilde{W} \subseteq \phi(W)$, the set

$$\phi^{-1} \circ \mathcal{C}^{\ell}_{\mathcal{W}}(U, \widetilde{W})^{\bullet}$$

generates $\mathcal{C}^{\ell}_{\mathcal{W}}(U, G)^{\bullet}$.

6.2.2. A larger Lie group of weighted mappings. We extend the Lie group described in Definition 6.2.5. Generally, it is possible using Lemma B.2.5 to extend a Lie group G that is a subgroup of a larger group H by looking at its "smooth normalizer", that is, all $h \in H$ that normalize G and for which the inner automorphism, restricted to suitable $\mathbf{1}$-neighborhoods, is smooth. This approach has the disadvantage that we do not really know which maps are contained in the smooth normalizer. So in the following, we will define a subset of G^U and show that it is a group contained in the smooth normalizer of $\mathcal{C}^{\ell}_{\mathcal{W}}(U, G)^{\bullet}$.

Further, we show that this bigger group contains certain groups of *rapidly decreasing mappings* constructed in [BCR81] as open subgroups.

6.2.2.1. *A group of mappings.* We define a set of mappings.

DEFINITION 6.2.6. Let G be a locally convex Lie group, X a finite-dimensional vector space, $U \subseteq X$ a nonempty open subset, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ nonempty and $k \in \overline{\mathbb{N}}$. Then for any centered chart (ϕ, V_ϕ) of G, compact set $K \subseteq U$ and $h \in \mathcal{C}^\infty_c(U, \mathbb{R})$ with $h \equiv 1_U$ on a neighborhood of K we define $M((\phi, V_\phi), K, h)$ as the set

$$\{\gamma \in \mathcal{C}^k(U, G) : \gamma(U \setminus K) \subseteq V_\phi \text{ and } (1_U - h) \cdot (\phi \circ \gamma)|_{U \setminus K} \in \mathcal{C}^k_{\mathcal{W}}(U \setminus K, \mathbf{L}(G))^{\bullet}\}.$$

Further we define

$$\mathcal{C}^k_{\mathcal{W}}(U, G)^{\bullet}_{\text{ex}} := \bigcup_{(\phi, V_\phi), K, h} M((\phi, V_\phi), K, h).$$

In the following, we show that $\mathcal{C}^k_{\mathcal{W}}(U, G)^{\bullet}_{\text{ex}}$ is a subgroup of G^U. In order to do this, we provide some technical tools. First, we show that we can use a cutoff technique to shrink the domain of a decreasing function.

LEMMA 6.2.7. *Let X be a finite-dimensional space, $U \subseteq X$ an open nonempty subset, Y a locally convex space and $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ nonempty. Let $k \in \overline{\mathbb{N}}$ and $\gamma \in \mathcal{C}^k(U, Y)$.*

(a) *Suppose that $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U, Y)^{\bullet}$. Let $A \subseteq U$ be a closed nonempty set such that $\gamma|_{U \setminus A} \equiv 0$ and $V \subseteq U$ an open neighborhood of A. Then $\gamma|_V \in \mathcal{C}^k_{\mathcal{W}}(V, Y)^{\bullet}$.*

(b) *Let $K_1 \subseteq K_2 \subseteq U$ be closed sets such that $\gamma|_{U \setminus K_1} \in \mathcal{C}^k_{\mathcal{W}}(U \setminus K_1, Y)^{\bullet}$ and $h \in \mathcal{BC}^\infty(U, \mathbb{R})$ such that $h \equiv 1$ on a neighborhood of K_2. Then*

$$(1_U - h) \cdot \gamma|_{U \setminus K_2} \in \mathcal{C}^k_{\mathcal{W}}(U \setminus K_2, Y)^{\bullet}.$$

Proof. (a) It is obvious that $\gamma|_V \in \mathcal{C}^k_{\mathcal{W}}(V, Y)$. Let $f \in \mathcal{W}$ and $\ell \in \mathbb{N}$ with $\ell \leq k$. For $\varepsilon > 0$ and $p \in \mathcal{N}(Y)$ there exists a compact set $K \subseteq U$ such that $\|\gamma|_{U \setminus K}\|_{p,f,\ell} < \varepsilon$. The set $\widetilde{K} := K \cap A$ is compact and contained in V. Further $\|\gamma|_{V \setminus \widetilde{K}}\|_{p,f,\ell} < \varepsilon$ since $D^{(\ell)}\gamma|_{U \setminus A} = 0$.

(b) Let $V \supseteq K$ be open in U such that $h|_V \equiv 1$. Then by Corollary 3.4.15,

$$(1_U - h) \cdot \gamma|_{U \setminus K_1} \in \mathcal{C}^k_{\mathcal{W}}(U \setminus K_1, Y)^{\bullet}.$$

Further $(1_U - h) \cdot \gamma|_{U \setminus (U \setminus V)} \equiv 0$. Since $U \setminus K_2$ is an open neighborhood of $U \setminus V$, an application of (a) finishes the proof. ∎

Now we examine $\mathcal{C}^k_{\mathcal{W}}(U, G)^\bullet_{\mathrm{ex}}$. We show that for a mapping in this set, we can change the chart of G, shrink the $\mathbf{1}$-neighborhood and enlarge the compact set.

Lemma 6.2.8. *Let X be a finite-dimensional vector space, $U \subseteq X$ an open nonempty subset, G a locally convex Lie group, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$ and $k \in \overline{\mathbb{N}}$. Further, let $\gamma \in M((\phi, V_\phi), K, h)$.*

(a) *For each $\mathbf{1}$-neighborhood $V \subseteq V_\phi$, there exists a compact set $K_V \subseteq U$ such that for each map $h_V \in \mathcal{C}^\infty_c(U, \mathbb{R})$ with $h_V \equiv 1$ on a neighborhood of K_V, we have $\gamma \in M((\phi|_V, V), K_V, h_V)$.*
(b) *Let (ψ, V_ψ) be a centered chart. Then there exists a compact set $K_\psi \subseteq U$ such that $\gamma \in M((\psi, V_\psi), K_\psi, h_\psi)$ for each $h_\psi \in \mathcal{C}^\infty_c(U, \mathbb{R})$ with $h_\psi \equiv 1$ on a neighborhood of K_ψ.*
(c) *Let $\eta \in M((\phi, V_\phi), \widetilde{K}, \widetilde{h})$. There exists a compact set L such that for each $g \in \mathcal{C}^\infty_c(U, \mathbb{R})$ with $g \equiv 1$ on a neighborhood of L, we have $\gamma, \eta \in M((\phi, V_\phi), L, g)$.*

Proof. (a) Since $(1_U - h) \cdot (\phi \circ \gamma)|_{U \setminus K} \in \mathcal{C}^k_{\mathcal{W}}(U \setminus K, \mathbf{L}(G))^\bullet$ and $1_U \in \mathcal{W}$, there exists a compact set $\widetilde{K} \subseteq U$ such that

$$(1_U - h) \cdot (\phi \circ \gamma)((U \setminus K) \setminus \widetilde{K}) \subseteq \phi(V).$$

We define the compact set $K_V := \widetilde{K} \cup \operatorname{supp}(h)$ and choose $h_V \in \mathcal{C}^\infty_c(U, \mathbb{R})$ with $h_V \equiv 1$ on a neighborhood of K_V. By Lemma 6.2.7 and the fact that $h \equiv 0$ on $U \setminus K_V$, we see that

$$(1_U - h_V) \cdot (\phi \circ \gamma)|_{U \setminus K_V} = (1_U - h_V)(1_U - h) \cdot (\phi \circ \gamma)|_{U \setminus K_V} \in \mathcal{C}^k_{\mathcal{W}}(U \setminus K_V, \mathbf{L}(G))^\bullet.$$

Further we calculate using again that $h \equiv 0$ on $U \setminus K_V$:

$$(\phi \circ \gamma)(U \setminus K_V) = (1_U - h) \cdot (\phi \circ \gamma)((U \setminus K) \setminus K_V) \subseteq \phi(V).$$

(b) There exists an open $\mathbf{1}$-neighborhood $V \subseteq V_\phi \cap V_\psi$ such that $\phi(V)$ is star-shaped with center 0. We know from (a) that there exist a compact set $\widetilde{K} \subseteq U$ and a map $\widetilde{h} \in \mathcal{C}^\infty_c(U, [0,1])$ with $\widetilde{h} \equiv 1$ on a neighborhood of $\widetilde{K}$ such that

$$\gamma \in M((\phi|_V, V), \widetilde{K}, \widetilde{h}).$$

We conclude from Proposition 3.4.23 that

$$(\psi \circ \phi^{-1}) \circ ((1_U - \widetilde{h}) \cdot (\phi \circ \gamma)|_{U \setminus \widetilde{K}} \in \mathcal{C}^k_{\mathcal{W}}(U \setminus \widetilde{K}, \mathbf{L}(G))^\bullet.$$

Let $h_\psi \in \mathcal{C}^\infty_c(U, \mathbb{R})$ be such that $h_\psi \equiv 1$ on a neighborhood of K_ψ, where $K_\psi := \widetilde{K} \cup \operatorname{supp}(\widetilde{h})$. We conclude from Lemma 6.2.7 that

$$(1_U - h_\psi) \cdot (\psi \circ \phi^{-1}) \circ ((1_U - \widetilde{h}) \cdot (\phi \circ \gamma)|_{U \setminus K_\psi} \in \mathcal{C}^k_{\mathcal{W}}(U \setminus K_\psi, \mathbf{L}(G))^\bullet.$$

Since $1_U - \widetilde{h} \equiv 1_U$ on $U \setminus K_\psi$, the proof is finished.

(c) We set $L := \operatorname{supp}(h) \cup \operatorname{supp}(\widetilde{h})$. Then

$$\gamma(U \setminus L) \subseteq \gamma(U \setminus K) \subseteq V_\phi,$$

and for $g \in \mathcal{C}^\infty_c(U, \mathbb{R})$ with $g \equiv 1$ on a neighborhood of L, Lemma 6.2.7 implies that

$$(1_U - g) \cdot (\phi \circ \gamma)|_{U \setminus L} = (1_U - g) \cdot (1_U - h) \cdot (\phi \circ \gamma)|_{U \setminus L} \in \mathcal{C}^k_{\mathcal{W}}(U \setminus L, \mathbf{L}(G))^\bullet.$$

Since the argument for η is the same, we are done. ■

Now we are ready to show that $\mathcal{C}^k_{\mathcal{W}}(U, G)^\bullet_{\mathrm{ex}}$ is a group.

Lemma 6.2.9. *Let X be a finite-dimensional vector space, $U \subseteq X$ an open nonempty subset, G a locally convex Lie group, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$ and $k \in \overline{\mathbb{N}}$. Then the set $\mathcal{C}^k_{\mathcal{W}}(U, G)^\bullet_{\mathrm{ex}}$ is a subgroup of G^U.*

Proof. Let (ϕ, V_ϕ) be a centered chart for G and $V \subseteq V_\phi$ an open neighborhood of $\mathbf{1}$ such that $m_G(V \times I_G(V)) \subseteq V_\phi$ and $\phi(V)$ is star-shaped. We define the map

$$H_G : V \times V \to V_\phi : (x, y) \mapsto m_G(x, I_G(y)).$$

Let $\gamma, \eta \in \mathcal{C}^k_{\mathcal{W}}(U, G)^\bullet_{\mathrm{ex}}$. Using Lemma 6.2.8 we find a compact set $K \subseteq U$ and a map $h \in \mathcal{C}^\infty_c(U, [0, 1])$ with $h \equiv 1_U$ on K such that

$$\gamma, \eta \in M((\phi|_V, V), K, h).$$

We define $H_\phi := \phi \circ H_G \circ (\phi^{-1} \times \phi^{-1})|_{V \times V}$ and want to show that there exists a compact set $\widetilde{K}$ and $\widetilde{h} \in \mathcal{C}^\infty_c(U, \mathbb{R})$ with $\widetilde{h} \equiv 1$ on a neighborhood of $\widetilde{K}$ such that $H_G \circ (\gamma, \eta) \in M((\phi, V_\phi), \widetilde{K}, \widetilde{h})$. It is obvious that

$$(H_G \circ (\gamma, \eta))(U \setminus K) \subseteq m_G(V \times I_G(V)) \subseteq V_\phi.$$

Since we know from Lemma 3.4.16 that

$$(1_U - h) \cdot (\phi \circ \gamma, \phi \circ \eta) = ((1_U - h) \cdot (\phi \circ \gamma), (1_U - h) \cdot (\phi \circ \eta)) \in \mathcal{C}^k_{\mathcal{W}}(U \setminus K, \mathbf{L}(G) \times \mathbf{L}(G))^\bullet,$$

we conclude using Proposition 3.4.23 that

$$H_\phi \circ ((1_U - h) \cdot (\phi \circ \gamma, \phi \circ \eta)) \in \mathcal{C}^k_{\mathcal{W}}(U \setminus K, \mathbf{L}(G))^\bullet.$$

Further, $\widetilde{K} := K \cup \operatorname{supp}(h)$ is a compact set, so by Lemma 6.2.7,

$$(1_U - \widetilde{h}) \cdot H_\phi \circ ((1_U - h) \cdot (\phi \circ \gamma, \phi \circ \eta)) \in \mathcal{C}^k_{\mathcal{W}}(U \setminus \widetilde{K}, \mathbf{L}(G))^\bullet$$

for any $\widetilde{h} \in \mathcal{C}^\infty_c(U, \mathbb{R})$ with $\widetilde{h} \equiv 1$ on a neighborhood of $\widetilde{K}$. Since $(1_U - h) \equiv 0$ on $U \setminus \widetilde{K}$, $(1_U - \widetilde{h}) \cdot (\phi \circ H_G \circ (\gamma, \eta))|_{U \setminus \widetilde{K}} \in \mathcal{C}^k_{\mathcal{W}}(U \setminus \widetilde{K}, \mathbf{L}(G))^\bullet$ and hence

$$H_G \circ (\gamma, \eta) \in M((\phi, V_\phi), \widetilde{K}, \widetilde{h}). \ ■$$

6.2.2.2. *Inclusion in the smooth normalizer.* We show that $\mathcal{C}^k_{\mathcal{W}}(U, G)^\bullet_{\mathrm{ex}}$ is contained in the smooth normalizer of $\mathcal{C}^k_{\mathcal{W}}(U, G)^\bullet$. To this end, we show that each $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U, G)^\bullet_{\mathrm{ex}}$ can be written as a product of a compactly supported $\mathcal{C}^k$-map and a $\mathcal{C}^k$-map that takes values in a chosen chart domain. Next, we show that these two classes of mappings are contained in the smooth normalizer of $\mathcal{C}^k_{\mathcal{W}}(U, G)^\bullet$.

We start with the following technical lemma about extending decreasing functions.

Lemma 6.2.10. *Let X be a finite-dimensional space, $U \subseteq X$ an open nonempty subset, $A \subseteq U$ a closed subset, Y a locally convex space, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $k \in \overline{\mathbb{N}}$ and*

$\gamma \in \mathcal{C}^k_{\mathcal{W}}(U \setminus A, Y)^{\bullet}$. *Then the map*

$$\widetilde{\gamma} : U \to Y : x \mapsto \begin{cases} \gamma(x) & \text{if } x \in U \setminus A, \\ 0 & \text{otherwise} \end{cases}$$

is in $\mathcal{C}^k_{\mathcal{W}}(U, Y)^{\bullet}$.

Proof. Obviously, the assertion holds on $U \setminus A$ and A°, since $\widetilde{\gamma}$ and its derivatives vanish on A°. We show that $\widetilde{\gamma}$ is $\mathcal{C}^k$ on ∂A and $\widetilde{\gamma}$ and its derivatives also vanish there. Since this is true iff for each $p \in \mathcal{N}(Y)$, the map $\pi_p \circ \widetilde{\gamma}$ is $\mathcal{C}^k$ on ∂A and it and its derivatives vanish there, and the identity $\pi_p \circ \widetilde{\gamma} = \widetilde{\pi_p \circ \gamma}$ holds, we may assume that Y is normable.

Since $1_U \in \mathcal{W}$, for each $\ell \in \mathbb{N}$ with $\ell \leq k$, the map $\widetilde{D^{(\ell)}\gamma}$ is continuous and hence

$$\widetilde{D^{(\ell)}\gamma} \in \mathcal{C}^0_{\mathcal{W}}(U, \mathrm{L}^{\ell}(X, Y))^{\bullet}.$$

Using Lemma 3.2.1, it remains to show that $\widetilde{\gamma}$ is $\mathcal{C}^k$ with $D^{(\ell)}\widetilde{\gamma} = \widetilde{D^{(\ell)}\gamma}$ for all $\ell \in \mathbb{N}$ with $\ell \leq k$. We show this by induction on ℓ.

$\ell = 1$: Let $x \in \partial A$ and $h \in X$. If there exists $\delta > 0$ such that $x +]{-\delta}, 0]h \subseteq A$ or $x + [0, \delta]h \subseteq A$, then $D_h\widetilde{\gamma}(x) = 0 = \widetilde{D\gamma}(x)h$.

Otherwise, there exists a null sequence $(t_n)_{n \in \mathbb{N}}$ in $]{-\infty}, 0[$ or $]0, \infty[$ such that for each $n \in \mathbb{N}$, $x + t_n h \in U \setminus A$. Replacing h by $-h$ if necessary, we may assume that all t_n are positive. Since $1_U \in \mathcal{W}$, $\widetilde{D\gamma}$ is continuous and $\widetilde{D\gamma}(x) = 0$, given $\varepsilon > 0$ we find $\delta > 0$ such that for all $s \in]{-\delta}, \delta]$,

$$\|\widetilde{D\gamma}(x + sh)\|_{\mathrm{op}} < \varepsilon.$$

We find an $n \in \mathbb{N}$ such that $t_n \in]{-\delta}, \delta[$. Then we define

$$t := \inf\{\tau > 0 :]\tau, t_n] \subseteq U \setminus A\} > 0.$$

We calculate for $\tau \in]t, t_n[$:

$$\begin{aligned} \left\| \frac{\widetilde{\gamma}(x + t_n h) - \widetilde{\gamma}(x + \tau h)}{t_n} \right\| &< \left\| \frac{\widetilde{\gamma}(x + t_n h) - \widetilde{\gamma}(x + \tau h)}{t_n - \tau} \right\| \\ &= \frac{1}{t_n - \tau} \left\| \int_0^1 D\gamma(x + (st_n + (1 - s)\tau)h) \cdot (t_n - \tau)h \, ds \right\| < \varepsilon \|h\|. \end{aligned}$$

But $\widetilde{\gamma}(x + \tau h) \to 0$ as $\tau \to t$, and hence

$$\left\| \frac{\widetilde{\gamma}(x + t_n h) - \widetilde{\gamma}(x)}{t_n} \right\| = \left\| \frac{\widetilde{\gamma}(x + t_n h)}{t_n} \right\| \leq \varepsilon \|h\|.$$

Since ε was arbitrary, we conclude that $D_h\widetilde{\gamma}(x) = 0 = \widetilde{D\gamma}(x)h$.

$\ell \to \ell+1$: Using the inductive hypothesis, we conclude that $\widetilde{D\gamma}$ is $\mathcal{FC}^{\ell}$, and $D^{(\ell)}\widetilde{D\gamma} = \widetilde{D^{(\ell)}D\gamma}$. Hence $\widetilde{\gamma}$ is $\mathcal{FC}^{\ell+1}$, so by Lemma A.2.14, $D^{(\ell+1)}\widetilde{\gamma} = \widetilde{D^{(\ell+1)}\gamma}$. ∎

Proposition 6.2.11. *Let X be a finite-dimensional space, $U \subseteq X$ an open nonempty subset, G a locally convex Lie group, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $k \in \overline{\mathbb{N}}$, (ϕ, V_ϕ) a centered chart of G and $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U, G)^{\bullet}_{\mathrm{ex}}$. Then there exist maps $\eta \in M((\phi, V_\phi), \emptyset, 0_U)$ and $\chi \in \mathcal{C}^k_c(U, G)$ such that*

$$\gamma = \eta \cdot \chi.$$

Proof. Using Lemma 6.2.8 we find a compact set K and $h \in \mathcal{C}^\infty_c(U,[0,1])$ such that $\gamma \in M((\phi, V_\phi), K, h)$. Using Lemma 6.2.10 we see that

$$\eta := \phi^{-1} \circ \widetilde{(1_U - h) \cdot (\phi \circ \gamma)|_{U\setminus K}} \in M((\phi, V_\phi), \emptyset, 0_U),$$

and it is obvious that $\eta|_{U\setminus \mathrm{supp}(h)} = \gamma|_{U\setminus \mathrm{supp}(h)}$. Hence

$$\chi := \eta^{-1} \cdot \gamma \in \mathcal{C}^k_c(U, G),$$

and obviously $\gamma = \eta \cdot \chi$. ∎

We now show that the weighted maps that take values in a suitable chart domain are contained in the smooth normalizer.

LEMMA 6.2.12. *Let X be a finite-dimensional space, $U \subseteq X$ an open nonempty subset, G a locally convex Lie group, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $k \in \overline{\mathbb{N}}$ and (ϕ, V_ϕ) a centered chart of G. Further let $W_\phi \subseteq V_\phi$ be an open $\mathbf{1}$-neighborhood such that*

$$W_\phi \cdot W_\phi \cdot W_\phi^{-1} \subseteq V_\phi$$

and $\phi(W_\phi)$ is star-shaped with center 0. Then for each $\eta \in M((\phi, W_\phi), \emptyset, 0_U)$, the map

$$\mathcal{C}^k_{\mathcal{W}}(U, \phi(W_\phi))^\bullet \to \mathcal{C}^k_{\mathcal{W}}(U, \phi(V_\phi))^\bullet : \gamma \mapsto \phi \circ (\eta \cdot (\phi^{-1} \circ \gamma) \cdot \eta^{-1})$$

is smooth.

Proof. As a consequence of Proposition 3.4.23 and Lemma 3.4.16, the map

$$\mathcal{C}^k_{\mathcal{W}}(U, \phi(W_\phi))^\bullet \times \mathcal{C}^k_{\mathcal{W}}(U, \phi(W_\phi))^\bullet \times \mathcal{C}^k_{\mathcal{W}}(U, \phi(W_\phi))^\bullet \to \mathcal{C}^k_{\mathcal{W}}(U, \phi(V_\phi))^\bullet,$$
$$(\gamma_1, \gamma_2, \gamma_3) \mapsto \phi \circ ((\phi^{-1} \circ \gamma_1) \cdot (\phi^{-1} \circ \gamma_2) \cdot (\phi^{-1} \circ \gamma_3)^{-1}),$$

is smooth. Hence we easily deduce the desired assertion. ∎

Normalization with compactly supported mappings. While the treatment of $\mathcal{C}^k$-maps with values in a suitable chart domain was straightforward, we need to develop other tools to deal with compactly supported mappings. The main problem is that a compactly supported map may not take values in any chart domain. To get around this problem, we need more technical machinery. As a motivation for the following, let $\chi \in \mathcal{C}^k_c(U, G)$ and (ϕ, V_ϕ) be a centered chart of G. Using that $\chi(U)$ is compact, we can find a symmetrical neighborhood O of $\chi(U)$ and an open $\mathbf{1}$-neighborhood $W_\phi \subseteq V_\phi$ such that $O \cdot W_\phi \cdot O^{-1} \subseteq V_\phi$. Then we can define the "normalization map in charts"

$$N : O \times \phi(W_\phi) \to \phi(V_\phi) : (g, y) \mapsto \phi(g \cdot \phi^{-1}(y) \cdot g^{-1}).$$

We can calculate that for $\gamma \in \phi(W_\phi)^U$, we have the identity

$$\phi \circ (\chi \cdot \gamma \cdot \chi^{-1}) = N \circ (\chi \times \mathrm{id}_{\phi(W_\phi)}) \circ (\mathrm{id}_U, \gamma).$$

In the following two lemmas, we will examine the properties of maps of the form $N \circ (\chi \times \mathrm{id}_{\phi(W_\phi)})$ and whether they induce a kind of superposition operator for decreasing weighted functions.

LEMMA 6.2.13. *Let X, Y and Z be locally convex spaces, $U \subseteq X$, $V \subseteq Y$ and $W \subseteq Z$ open nonempty subsets, M a locally convex manifold and $k \in \overline{\mathbb{N}}$. Let $\Gamma \in \mathcal{C}^\infty(M \times V, W)$ and $\eta \in \mathcal{C}^k(U, M)$. Then the map*

$$\Xi := \Gamma \circ (\eta \times \mathrm{id}_V) : U \times V \to W$$

has the following properties:

(a) *The second partial derivative of* Ξ *is*

$$d_2\Xi = (\pi_2 \circ \mathbf{T}_2\Gamma) \circ (\eta \times \mathrm{id}_{V\times Y})$$

and if $k \geq 1$*, the first partial derivative of* Ξ *is*

$$d_1\Xi = (\pi_2 \circ \mathbf{T}_1\Gamma) \circ (\mathbf{T}\eta \times \mathrm{id}_V) \circ S,$$

where π_2 *denotes the projection* $W \times Z \to Z$ *on the second component, and* $S : U \times V \times X \to U \times X \times V : (x, y, h) \mapsto (x, h, y)$ *denotes the swap map.*

(b) *For all* $x \in U$*, the partial map* $\Xi(x, \cdot) : V \to W$ *is smooth, and for all* $\ell \in \mathbb{N}$ *the map* $d_2^{(\ell)}\Xi : U \times V \times Y^\ell \to W$ *is* $\mathcal{C}^k$*.*

(c) *Assume that* X *has finite dimension. Then for*

$$A_1 : U \times V \to \mathrm{L}(X, Z) : (x, y) \mapsto (h \mapsto d_1\Xi(x, y; h))$$

(*which is only defined if* $k \geq 1$) *and*

$$A_2 : U \times V \times \mathrm{L}(X, Y) \to \mathrm{L}(X, Z) : (x, y, T) \mapsto (h \mapsto d_2\Xi(x, y; T \cdot h)),$$

all partial maps $A_1(x, \cdot)$ *and* $A_2(x, \cdot)$ *are smooth and all partial derivatives* $d_2^{(\ell)}A_1$ *and* $d_2^{(\ell)}A_2$ *are* $\mathcal{C}^{k-1}$*, respectively* $\mathcal{C}^k$*.*

Proof. (a) We calculate for $x \in U$, $y \in V$ and $h \in Y$ that

$$\begin{aligned} d_2\Xi((x, y); h) &= \lim_{t\to 0} \frac{\Xi(x, y + th) - \Xi(x, y)}{t} = \lim_{t\to 0} \frac{\Gamma(\eta(x), y + th) - \Gamma(\eta(x), y)}{t} \\ &= (\pi_2 \circ \mathbf{T}_2\Gamma)(\eta(x), y, h). \end{aligned}$$

This shows the desired identity for $d_2\Xi$. If $k > 0$, we find using the chain rule that

$$d\Xi \circ P = \pi_2 \circ \mathbf{T}\Xi \circ P = \pi_2 \circ \mathbf{T}\Gamma \circ (\mathbf{T}\eta \times \mathrm{id}\,\mathbf{T}V),$$

where $P : U \times X \times V \times Y \to U \times V \times X \times Y$ permutes the middle arguments. Since $d_1\Xi((x, y); h_x) = d\Xi((x, y); (h_x, 0))$, we get the assertion for $d_1\Xi$.

(b) It is obvious that the partial maps are smooth. We prove the second assertion by induction on ℓ.

$\ell = 0$: This is obvious.

$\ell \to \ell + 1$: In (a) we proved that $d_2\Xi$ is of the same form as Ξ. By the inductive hypothesis,

$$d_2^{(\ell)}(d_2\Xi) : U \times V \times Y \times (Y \times Y)^\ell \to W$$

is a $\mathcal{C}^k$-map. But

$$d_2^{(\ell+1)}\Xi(x, y; h_1, h_2, \ldots, h_{\ell+1}) = d_2^{(\ell)}(d_2\Xi)(x, y, h_1; (h_2, 0), \ldots, (h_{\ell+1}, 0)),$$

so $d_2^{(\ell+1)}\Xi$ is $\mathcal{C}^k$.

(c) The partial maps $A_1(x, \cdot)$ and $A_2(x, \cdot)$ are smooth and the maps $d_2^{(\ell)}A_1$ and $d_2^{(\ell)}A_2$ are $\mathcal{C}^{k-1}$ respective $\mathcal{C}^k$ iff for each $h \in X$, the maps $A_1(x, \cdot) \cdot h$ and $A_2(x, \cdot) \cdot h$ have the corresponding properties. By (a),

$$A_1(x, y) \cdot h = d_1\Xi(x, y; h) = (\pi_2 \circ \mathbf{T}_1\Gamma) \circ (\mathbf{T}\eta \times \mathrm{id}_V) \circ S(x, y, h)$$

and

$$\begin{aligned}A_2(x,y,T)\cdot h = d_2\Xi(x,y;T\cdot h) &= (\pi_2\circ \mathbf{T}_2\Gamma)\circ(\eta\times \mathrm{id}_{V\times Y})(x,y,T\cdot h)\\ &= (\pi_2\circ \mathbf{T}_2\Gamma\circ S_1)\circ(\eta\times \mathrm{ev}_h\times \mathrm{id}_V)\circ S_2(x,y,T).\end{aligned}$$

Here S_1 and S_2 denote the swap maps

$$M\times Y\times V\to M\times V\times Y,\qquad U\times V\times \mathrm{L}(X,Y)\to U\times \mathrm{L}(X,Y)\times V$$

respectively. Since S, S_1 and S_2 are restrictions of continuous linear maps, (b) applies to both $A_1(x,\cdot)\cdot h$ and $A_2(x,\cdot)\cdot h$. ∎

Lemma 6.2.14. *Let X be a finite-dimensional space, $U\subseteq X$ an open nonempty subset, Y and Z locally convex spaces, M a locally convex manifold, $V\subseteq Y$ an open zero neighborhood that is star-shaped with center 0, $\mathcal{W}\subseteq \overline{\mathbb{R}}^U$ with $1_U\in\mathcal{W}$ and $k\in\overline{\mathbb{N}}$. Further, let $\Gamma\in\mathcal{C}^\infty(M\times V,Z)$, and $\theta\in\mathcal{C}^k(U,M)$ such that the map*

$$\Xi := \Gamma\circ(\theta\times \mathrm{id}_V) : U\times V\to Z$$

satisfies

- $\Xi(U\times\{0\})=\{0\}$.
- *There exists a compact set $K\subseteq U$ such that $\Xi((U\setminus K)\times V)=\{0\}$.*

Then for any $\gamma\in\mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet$,

$$\Xi\circ(\mathrm{id}_U,\gamma)\in\mathcal{C}^k_{\mathcal{W}}(U,Z)^\bullet, \tag{†}$$

and the map

$$\Xi_* : \mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet\to\mathcal{C}^k_{\mathcal{W}}(U,Z)^\bullet : \gamma\mapsto \Xi\circ(\mathrm{id}_U,\gamma)$$

is smooth.

Proof. We first prove the continuity of Ξ_* by induction on k:

$k=0$: Let $\gamma,\eta\in\mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet$ such that the line segment $\{t\gamma+(1-t)\eta : t\in[0,1]\}\subseteq\mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet$. We easily prove using Lemma 3.4.17 that the set

$$\widetilde{K} := \{t\gamma(x)+(1-t)\eta(x) : t\in[0,1],\, x\in U\}$$

is relatively compact in V. Since $d_2\Xi$ is continuous by Lemma 6.2.13(b) and satisfies $d_2\Xi(U\times V\times\{0\})=\{0\}$, we conclude using the Wallace Lemma that for each $p\in\mathcal{N}(Z)$, there exists $q\in\mathcal{N}(Y)$ such that

$$d_2\Xi(K\times\widetilde{K}\times B_q(0,1))\subseteq B_p(0,1).$$

This relation implies that

$$(\forall x\in K, y\in\widetilde{K}, h\in Y)\ \|d_2\Xi(x,y;h)\|_p\le\|h\|_q.$$

For each $x\in U$, we calculate

$$\Xi(x,\gamma(x))-\Xi(x,\eta(x)) = \int_0^1 d_2\Xi(x,t\gamma(x)+(1-t)\eta(x);\gamma(x)-\eta(x))\,dt.$$

Hence for each $f\in\mathcal{W}$, we have

$$|f(x)|\,\|\Xi(x,\gamma(x))-\Xi(x,\eta(x))\|_p\le|f(x)|\,\|\gamma(x)-\eta(x)\|_q.$$

Taking $\eta=0$, this estimate implies (†). Further, since we proved in Lemma 3.4.18 that $\mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet$ is open, γ has a convex neighborhood in $\mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet$; hence the estimate also implies the continuity of Ξ_* in γ.

$k \to k+1$: For each $x \in U$, $h \in X$ and $\gamma \in \mathcal{C}^{k+1}_{\mathcal{W}}(U,V)^\bullet$, we calculate

$$\begin{aligned} d(\Xi \circ (\mathrm{id}_U, \gamma))(x;h) &= d\Xi(x,\gamma(x);h, D\gamma(x)\cdot h) \\ &= d_1\Xi(x,\gamma(x);h) + d_2\Xi(x,\gamma(x);D\gamma(x)\cdot h). \end{aligned}$$

Recall the maps A_1 and A_2 defined in Lemma 6.2.13(6.2.13). We get the identity

$$D(\Xi \circ (\mathrm{id}_U,\gamma))(x) = (A_1 \circ (\mathrm{id}_U,\gamma))(x) + (A_2 \circ (\mathrm{id}_U,\gamma,D\gamma))(x).$$

We prove that A_1 and A_2 satisfy the same properties as Ξ does: For $x \in U$, $y \in V$, $h \in X$,

$$A_1(x,0)\cdot h = d_1\Xi(x,0;h) = \lim_{t\to 0} \frac{\Xi(x+th,0) - \Xi(x,0)}{t} = 0,$$

whence $A_1(x,0) = 0$. Let $x \in U \setminus K$. Then

$$A_1(x,y)\cdot h = d_1\Xi(x,y;h) = \lim_{t\to 0} \frac{\Xi(x+th,y) - \Xi(x,y)}{t} = 0$$

since $U \setminus K$ is open, hence $A_1(x,y) = 0$.

As to A_2, for $x \in U$, $y \in V$ and $h \in X$ we calculate

$$A_2(x,y,0)\cdot h = d_2\Xi(x,y;0\cdot h) = 0,$$

whence $A_2(x,y,0) = 0$. Let $x \in U \setminus K$ and $T \in \mathrm{L}(X,Y)$. Then

$$A_2(x,y,T)\cdot h = d_2\Xi(x,y;T\cdot h) = \lim_{t\to 0} \frac{\Xi(x,y+tT\cdot h) - \Xi(x,y)}{t} = 0,$$

hence $A_2(x,y,T) = 0$.

So we can apply the inductive hypothesis to A_1 and A_2 and conclude that

$$A_1 \circ (\mathrm{id}_X,\gamma),\ A_2 \circ (\mathrm{id}_X,\gamma,D\gamma) \in \mathcal{C}^k_{\mathcal{W}}(U,\mathrm{L}(X,Z))^\bullet$$

and the maps $\mathcal{C}^{k+1}_{\mathcal{W}}(U,V)^\bullet \to \mathcal{C}^k_{\mathcal{W}}(U,\mathrm{L}(X,Z))^\bullet$,

$$\gamma \mapsto A_1 \circ (\mathrm{id}_X,\gamma) \quad \text{and} \quad \gamma \mapsto A_2 \circ (\mathrm{id}_X,\gamma,D\gamma),$$

are continuous. In view of Proposition 3.4.11, the continuity of Ξ_* is established.

We pass on to prove the smoothness of Ξ_*. To do this, we have to examine $d_2\Xi$. By Lemma 6.2.13(a), $d_2\Xi = \pi_2 \circ \mathbf{T}_2\Gamma \circ (\theta \times \mathrm{id}_{V\times Y})$, and we easily see that

$$d_2\Xi(U \times \{0\} \times \{0\}) = d_2\Xi((U\setminus K)\times V \times Y) = \{0\}.$$

Hence by the results already established, the map

$$(d_2\Xi)_* : \mathcal{C}^k_{\mathcal{W}}(U, V\times Y)^\bullet \to \mathcal{C}^k_{\mathcal{W}}(U,Z)^\bullet : (\gamma) \mapsto d_2\Xi \circ (\mathrm{id}_U,\gamma)$$

is defined and continuous. Now let $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet$ and $\gamma_1 \in \mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet$. Since $\mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet$ is open, there exists an $r > 0$ such that $\{\gamma + s\gamma_1 : s \in B_{\mathbb{K}}(0,r)\} \subseteq \mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet$. We calculate for $x \in U$ and $t \in B_{\mathbb{K}}(0,r)\setminus\{0\}$ (using Lemma 3.4.16 implicitly) that

$$\begin{aligned} \frac{\Xi_*(\gamma+t\gamma_1)(x) - \Xi_*(\gamma)(x)}{t} &= \frac{\Xi(x,\gamma(x)+t\gamma_1(x)) - \Xi(x,\gamma(x))}{t} \\ &= \int_0^1 d_2\Xi((x,\gamma(x)+st\gamma_1(x));\gamma_1(x))\,ds = \int_0^1 (d_2\Xi)_*(\gamma+st\gamma_1,\gamma_1)(x)\,ds. \end{aligned}$$

Hence by Lemma 3.4.3 and Proposition A.1.8, Ξ_* is $\mathcal{C}^1$ with $d\Xi_*(\gamma;\gamma_1) = (d_2\Xi)_*(\gamma,\gamma_1)$. Using an easy induction argument we conclude from this identity that Ξ_* is $\mathcal{C}^\ell$ for each $\ell \in \mathbb{N}$ and hence smooth. ■

Now we are ready to deal with the inner automorphism induced by a compactly supported map.

Lemma 6.2.15. *Let X be a finite-dimensional space, $U \subseteq X$ an open nonempty subset, G a locally convex Lie group, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $k \in \overline{\mathbb{N}}$ and (ϕ, V_ϕ) a centered chart for G. Let $\chi \in \mathcal{C}^k_c(U,G)$. Then there exists an open $\mathbf{1}$-neighborhood $W_\phi \subseteq V_\phi$ such that the map*

$$\mathcal{C}^k_{\mathcal{W}}(U,\phi(W_\phi))^\bullet \to \mathcal{C}^k_{\mathcal{W}}(U,\mathbf{L}(G))^\bullet : \gamma \mapsto \phi\circ(\chi\cdot(\phi^{-1}\circ\gamma)\cdot\chi^{-1}) \tag{$\dagger$}$$

is defined and smooth.

Proof. Since $\chi(U)$ is compact, we can find an open $\mathbf{1}$-neighborhood $W_\phi \subseteq V_\phi$ and an open symmetrical neighborhood O of $\chi(U)$ such that

$$O\cdot W_\phi\cdot O^{-1} \subseteq V_\phi;$$

we may assume that $\phi(W_\phi)$ is star-shaped with center 0. We define the smooth map

$$N : O\times\phi(W_\phi) \to \mathbf{L}(G) : (g,y) \mapsto \phi(g\cdot\phi^{-1}(y)\cdot g^{-1}) - y.$$

Then it is easy to see that

$$N\circ(\chi\times\mathrm{id}_{\phi(W_\phi)}) : U\times\phi(W_\phi) \to \mathbf{L}(G)$$

satisfies the assumptions of Lemma 6.2.14, and that for $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U,\phi(W_\phi))^\bullet$,

$$(N\circ(\chi\times\mathrm{id}_{\phi(W_\phi)}))\circ(\mathrm{id}_U,\gamma) = \phi\circ(\chi\cdot(\phi^{-1}\circ\gamma)\cdot\chi^{-1}) - \gamma.$$

Hence the map

$$\mathcal{C}^k_{\mathcal{W}}(U,\phi(W_\phi))^\bullet \to \mathcal{C}^k_{\mathcal{W}}(U,\mathbf{L}(G))^\bullet : \gamma \mapsto \phi\circ(\chi\cdot(\phi^{-1}\circ\gamma)\cdot\chi^{-1}) - \gamma$$

is smooth. Since the vector space addition is smooth, ($\dagger$) is defined and smooth. ■

Conclusion and the Lie group structure. Finally, we put everything together and show that $\mathcal{C}^k_{\mathcal{W}}(U,G)^\bullet_{\mathrm{ex}}$ is contained in the smooth normalizer of $\mathcal{C}^k_{\mathcal{W}}(U,G)^\bullet$. As mentioned above, this allows the construction of a Lie group structure on $\mathcal{C}^k_{\mathcal{W}}(U,G)^\bullet_{\mathrm{ex}}$.

Lemma 6.2.16. *Let X be a finite-dimensional space, $U \subseteq X$ an open nonempty subset, G a locally convex Lie group, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$, $k \in \overline{\mathbb{N}}$ and (ϕ, V_ϕ) a centered chart for G. Let $\theta \in \mathcal{C}^k_{\mathcal{W}}(U,G)^\bullet_{\mathrm{ex}}$. Then there exists an open $\mathbf{1}$-neighborhood $W_\phi \subseteq V_\phi$ such that the map*

$$\mathcal{C}^k_{\mathcal{W}}(U,\phi(W_\phi))^\bullet \to \mathcal{C}^k_{\mathcal{W}}(U,\phi(V_\phi))^\bullet : \gamma \mapsto \phi\circ(\theta\cdot(\phi^{-1}\circ\gamma)\cdot\theta^{-1}) \tag{$\dagger$}$$

is defined and smooth.

Proof. Let $\widetilde{V_\phi} \subseteq V_\phi$ be an open $\mathbf{1}$-neighborhood such that

$$\widetilde{V_\phi}\cdot\widetilde{V_\phi}\cdot\widetilde{V_\phi}^{-1} \subseteq V_\phi$$

and $\phi(\widetilde{V_\phi})$ is star-shaped with center 0. According to Proposition 6.2.11 there exist $\eta \in M((\phi,\widetilde{V_\phi}),\emptyset,0_U)$ and $\chi \in \mathcal{C}^k_c(U,G)$ such that $\theta = \eta\cdot\chi$. By Lemma 6.2.15, there exists an open **1**-neighborhood $W_\phi \subseteq V_\phi$ such that

$$\mathcal{C}^k_{\mathcal{W}}(U, \phi(W_\phi))^\bullet \to \mathcal{C}^k_{\mathcal{W}}(U, \phi(\widetilde{V_\phi}))^\bullet : \gamma \mapsto \phi \circ (\chi \cdot (\phi^{-1} \circ \gamma) \cdot \chi^{-1})$$

is smooth, and by Lemma 6.2.12 the map

$$\mathcal{C}^k_{\mathcal{W}}(U, \phi(\widetilde{V_\phi}))^\bullet \to \mathcal{C}^k_{\mathcal{W}}(U, \phi(V_\phi))^\bullet : \gamma \mapsto \phi \circ (\eta \cdot (\phi^{-1} \circ \gamma) \cdot \eta^{-1})$$

is also smooth. Composing these two maps, we obtain the assertion. ■

THEOREM 6.2.17. *Let X be a finite-dimensional space, $U \subseteq X$ an open nonempty subset, G a locally convex Lie group, $\mathcal{W} \subseteq \overline{\mathbb{R}}^U$ with $1_U \in \mathcal{W}$ and $k \in \overline{\mathbb{N}}$. Then $\mathcal{C}^k_{\mathcal{W}}(U, G)^\bullet_{\mathrm{ex}}$ can be made into a Lie group that contains $\mathcal{C}^k_{\mathcal{W}}(U, G)^\bullet$ as an open normal subgroup.*

Proof. We showed in Definition 6.2.5 that $\mathcal{C}^k_{\mathcal{W}}(U, G)^\bullet$ can be turned into a Lie group such that there exists a centered chart (ϕ, V_ϕ) for which

$$\mathcal{C}^k_{\mathcal{W}}(U, \phi(V_\phi))^\bullet \to \mathcal{C}^k_{\mathcal{W}}(U, G)^\bullet : \gamma \mapsto \phi^{-1} \circ \gamma$$

is an embedding and its image generates $\mathcal{C}^k_{\mathcal{W}}(U, G)^\bullet$. Further, we proved in Lemmas 6.2.9 and 6.2.16 that $\mathcal{C}^k_{\mathcal{W}}(U, G)^\bullet_{\mathrm{ex}}$ is a subgroup of G^U and for each $\theta \in \mathcal{C}^k_{\mathcal{W}}(U, G)^\bullet_{\mathrm{ex}}$ there exists an open **1**-neighborhood $W_\phi \subseteq V_\phi$ such that the conjugation operation

$$\mathcal{C}^k_{\mathcal{W}}(U, \phi(W_\phi))^\bullet \to \mathcal{C}^k_{\mathcal{W}}(U, \phi(V_\phi))^\bullet : \gamma \mapsto \phi \circ (\theta \cdot (\phi^{-1} \circ \gamma) \cdot \theta^{-1})$$

is smooth. Hence Lemma B.2.5 gives the assertion. ■

6.2.2.3. *Comparison with groups of* rapidly decreasing mappings*.* In the book [BCR81, Section 4.2.1, pages 111–117], for weights that satisfy conditions described below in Definition 6.2.18, certain Γ-*rapidly decreasing functions* with values in locally convex spaces are defined and used to construct Γ-*rapidly decreasing mappings* that take values in Lie groups. We compare these function spaces with our weighted decreasing functions and will see that they coincide. Further, we will show that the Γ-rapidly decreasing mappings are open subgroups of a certain $\mathcal{C}^k_{\mathcal{W}}(U, G)^\bullet_{\mathrm{ex}}$.

$\mathcal{W}$-rapidly decreasing functions

DEFINITION 6.2.18 (BCR-weights). Let X be a finite-dimensional vector space and $\mathcal{W} \subseteq [1, \infty]^X$ such that

($\mathcal{W}$1) for all $f, g \in \mathcal{W}$, the sets $f^{-1}(\infty)$ and $g^{-1}(\infty) =: M_\infty$ coincide,

($\mathcal{W}$2) $\mathcal{W}$ is upwards directed and contains a smallest element $f_{\min}$ defined by

$$f_{\min}(x) = \begin{cases} 1, & x \notin M_\infty, \\ \infty, & \text{else}, \end{cases}$$

($\mathcal{W}$3) for each $f_1 \in \mathcal{W}$ there exists $f_2 \in \mathcal{W}$ such that

$$(\forall \varepsilon > 0)(\exists n \in \mathbb{N})\ \|x\| \geq n \text{ or } f_1(x) \geq n \Rightarrow f_1(x) \leq \varepsilon f_2(x).$$

Furthermore each $f \in \mathcal{W}$ has to be continuous on the complement of M_∞.

DEFINITION 6.2.19 ($\mathcal{W}$-rapidly decreasing functions). Let $\mathcal{W}$ be a set of weights as in Definition 6.2.18, $U \subseteq \mathbb{R}^m$ open and nonempty and Y a locally convex space. A smooth function $\gamma : U \to Y$ is called $\mathcal{W}$*-rapidly decreasing* if for each $f \in \mathcal{W}$ and $\beta \in \mathbb{N}^m$ we have $\partial^\beta \gamma|_{U \cap M_\infty} \equiv 0$, and the function

$$f \cdot \partial^\beta \gamma : U \to Y$$

is continuous and bounded, where $\infty \cdot 0 = 0$. The set

$$S(U, Y; \mathcal{W}) := \{\gamma \in \mathcal{C}^\infty(U, Y) : \gamma \text{ is } \mathcal{W}\text{-rapidly decreasing}\}$$

endowed with the seminorms

$$\|\gamma\|_{q,f}^k := \sup\{q(f \cdot \partial^\beta \gamma(x)) : x \in U,\ |\beta| \leq k\}$$

(where $q \in \mathcal{N}(Y)$, $k \in \mathbb{N}$ and $f \in \mathcal{W}$) becomes a locally convex space.

Comparison of $S(U, Y; \mathcal{W})$ *and* $\mathcal{C}^\infty_{\mathcal{W}}(U, Y)$. We now show that these function spaces coincide as topological vector spaces. To this end, we need the following technical lemma.

LEMMA 6.2.20. *Let* $\mathcal{W}$ *be a set of weights as in Definition* 6.2.18, $U \subseteq \mathbb{R}^m$ *open and nonempty,* F *a locally convex space,* $\gamma : U \to F$ *a smooth function and* $\beta \in \mathbb{N}^m$. *Suppose that* $\partial^\beta \gamma|_{U \cap M_\infty} \equiv 0$ *and that for each* $f \in \mathcal{W}$ *the function*

$$f \cdot \partial^\beta \gamma : U \to F$$

is bounded. Then for each $f \in \mathcal{W}$, *the function* $f \cdot \partial^\beta \gamma$ *is continuous.*

Proof. Let $f \in \mathcal{W}$ and $x \in U$. If $x \notin \overline{M_\infty \cap U}$, $f \cdot \partial^\beta \gamma$ is continuous on a suitable neighborhood of x since f is so.

Otherwise, $\partial^\beta \gamma(x) = 0$ because $\partial^\beta \gamma$ is continuous. If there exists $V \in \mathcal{U}(x)$ such that f is bounded on $V \setminus M_\infty$, the map $f \cdot \partial^\beta \gamma$ is continuous on V because for $y \in V \setminus M_\infty$ and $q \in \mathcal{N}(F)$,

$$\|f(y)\partial^\beta \gamma(y) - f(x)\partial^\beta \gamma(x)\|_q = \|f(y)\partial^\beta \gamma(y)\|_q \leq \|f|_{V \setminus M_\infty}\|_\infty \|\partial^\beta \gamma(y)\|_q,$$

and this estimate is valid for $y \in M_\infty$.

Otherwise, we choose $g \in \mathcal{W}$ such that ($\mathcal{W}$3) holds. Let $\varepsilon > 0$. There exists an $n \in \mathbb{N}$ such that

$$(\forall y \in U)\ f(y) \geq n \Rightarrow f(y) \leq \frac{\varepsilon}{\|\gamma\|_{q,g}^{|\beta|} + 1} g(y).$$

For $q \in \mathcal{N}(F)$ there exists $V \in \mathcal{U}(x)$ such that for $y \in V$,

$$\|\partial^\beta \gamma(y)\|_q < \varepsilon / n.$$

Let $y \in V$. If $f(y) \geq n$, we calculate

$$\|f(y)\partial^\beta \gamma(y)\|_q = f(y)\|\partial^\beta \gamma(y)\|_q \leq \frac{\varepsilon}{\|\gamma\|_{q,g}^{|\beta|} + 1} g(y)\|\partial^\beta \gamma(y)\|_q < \varepsilon.$$

Otherwise

$$\|f(y)\partial^\beta \gamma(y)\|_q \leq n\|\partial^\beta \gamma(y)\|_q < \varepsilon.$$

So the assertion holds in all cases. ■

LEMMA 6.2.21. *Let* $\mathcal{W}$ *be a set of weights as in Definition* 6.2.18, $U \subseteq \mathbb{R}^m$ *open and nonempty and* F *a locally convex space. Then* $\mathcal{C}^\infty_{\mathcal{W}}(U, Y) = S(U, Y; \mathcal{W})$ *as a topological vector space.*

Proof. We first prove that $\mathcal{C}^\infty_\mathcal{W}(U,Y) = S(U,Y;\mathcal{W})$ as sets. To this end, let $\gamma \in \mathcal{C}^\infty_\mathcal{W}(U,Y)$, $f \in \mathcal{W}$ and $\beta \in \mathbb{N}^m$. We set $k := |\beta|$. We know that for $p \in \mathcal{N}(Y)$, the map $D^{(k)}(\pi_p \circ \gamma)$ vanishes on M_∞, and

$$f \cdot D^{(k)}(\pi_p \circ \gamma) : U \to \mathrm{L}^k(\mathbb{R}^m, Y_p)$$

is bounded. Since the evaluation $\mathrm{L}^k(\mathbb{R}^m, Y_p) \to Y_p$ at a fixed point is continuous linear, the map $f \cdot \partial^\beta(\pi_p \circ \gamma) = \pi_p \circ (f \cdot \partial^\beta \gamma) : U \to Y_p$ is also bounded. Hence $f \cdot \partial^\beta\gamma$ is bounded, so an application of Lemma 6.2.20 gives $\gamma \in S(U,Y;\mathcal{W})$.

On the other hand, let $\gamma \in S(U,Y;\mathcal{W})$ and $k \in \mathbb{N}$. For each $p \in \mathcal{N}(Y)$, we get, by identity (A.3.6.1),

$$D^{(k)}(\pi_p \circ \gamma) = \sum_{\substack{\alpha\in\mathbb{N}^m \\ |\alpha|=k}} S_\alpha \cdot \partial^\alpha(\pi_p \circ \gamma) = \sum_{\substack{\alpha\in\mathbb{N}^m \\ |\alpha|=k}} S_\alpha \cdot (\pi_p \circ \partial^\alpha \gamma).$$

Hence for $f \in \mathcal{W}$,

$$\|\gamma\|_{p,f,k} \leq \|\gamma\|^k_{p,f} \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|=k}} \|S_\alpha\|_{\mathrm{op}} < \infty. \tag{\dagger}$$

So $\gamma \in \mathcal{C}^\infty_\mathcal{W}(U,Y)$.

We see from (†) that for each $p \in \mathcal{N}(Y)$, $f \in \mathcal{W}$ and $k \in \mathbb{N}$ the seminorm $\|\cdot\|_{p,f,k}$ is continuous on $S(U,Y;\mathcal{W})$. Since the seminorms $\|\cdot\|^k_{p,f}$ are obviously continuous on $\mathcal{C}^\infty_\mathcal{W}(U,Y)$, the spaces coincide as topological vector spaces. ∎

Remark 6.2.22. Let $\mathcal{W}$ be a set of weights as in Definition 6.2.18. Then $1_U \in \mathcal{W} \Leftrightarrow M_\infty = \emptyset$. But obviously $\mathcal{C}^k_\mathcal{W}(U,Y) = \mathcal{C}^k_{\mathcal{W}\cup\{1_U\}}(U,Y)$ and $\mathcal{C}^k_\mathcal{W}(U,Y)^\bullet = \mathcal{C}^k_{\mathcal{W}\cup\{1_U\}}(U,Y)^\bullet$ as topological vector spaces.

Rapidly decreasing mappings. In [BCR81, Section 4.2.1, p. 117–118], the set of Γ-*rapidly decreasing mappings* is defined. We will show that these mappings form an open subgroup of $\mathcal{C}^\infty_\mathcal{W}(\mathbb{R}^m, G)^\bullet_{\mathrm{ex}}$.

Definition 6.2.23 ($\mathcal{W}$-rapidly decreasing mappings). Let $m \in \mathbb{N}$, G a locally convex Lie group and $\mathcal{W}$ a set of weights as in Definition 6.2.18. We define $S(\mathbb{R}^m, G;\mathcal{W})$ as the set of smooth functions $\gamma : \mathbb{R}^m \to G$ such that

- $\gamma(x) = \mathbf{1}$ for each $x \in M_\infty$, and $\gamma(x) \to \mathbf{1}$ if $\|x\| \to \infty$.
- For any centered chart $(\phi, \widetilde{V})$ of G and each open $\mathbf{1}$-neighborhood V with $\overline{V} \subseteq \widetilde{V}$, $\phi \circ \gamma|_{\gamma^{-1}(V)} \in S(\gamma^{-1}(V), \mathbf{L}(G);\mathcal{W})$.

In the next lemmas, we provide some tools needed for the further discussion. First, we show that for weights as in Definition 6.2.18, the product of a weighted function with a suitable cutoff function is a weighted decreasing function. We use this result to prove a superposition lemma for the spaces $\mathcal{C}^k_\mathcal{W}(U,Y)$.

Lemma 6.2.24. *Let K be a compact subset of the finite-dimensional vector space X, Y a locally convex space, $k \in \mathbb{N}$, $\mathcal{W}$ a set of weights as in Definition* 6.2.18, *$\gamma \in \mathcal{C}^k_\mathcal{W}(U,Y)$ (where $U := X \setminus K$) and $h \in \mathcal{C}^\infty_c(X,\mathbb{R})$ such that $h \equiv 1$ on a neighborhood V of K. Then*

$$(1-h)|_U \cdot \gamma \in \mathcal{C}^k_\mathcal{W}(U,Y)^\bullet.$$

Proof. We prove this by induction on k.

$k = 0$: Let $f \in \mathcal{W}$, $p \in \mathcal{N}(Y)$ and $\varepsilon > 0$. We use $(\mathcal{W}3)$ to see that there exists $n \in \mathbb{N}$ such that

$$\|\gamma|_{U \setminus \overline{B}_n(0)}\|_{p,f,0} < \frac{\varepsilon}{1 + \|1 - h\|_\infty}.$$

Further, the set

$$A := \left\{ x \in X : |(1-h)(x)| \geq \frac{\varepsilon}{\|\gamma\|_{p,f,0} + 1} \right\} \cap \overline{B}_n(0)$$

is compact and contained in U since $(1 - h) \equiv 0$ on V. Using these two estimates, we easily calculate that $\|(1 - h) \cdot \gamma|_{U \setminus A}\|_{p,f,0} < \varepsilon$.

$k \to k + 1$: We have

$$D((1-h)|_U \cdot \gamma) = (1-h)|_U \cdot D\gamma - Dh|_U \cdot \gamma.$$

By the inductive hypothesis, $(1 - h)|_U \cdot D\gamma \in \mathcal{C}^k_{\mathcal{W}}(U, \mathrm{L}(X, Y))^\bullet$, and since $Dh|_U \in \mathcal{C}^\infty_c(U, \mathrm{L}(X, \mathbb{R}))$, we use Corollary 3.4.15 and Proposition 3.4.11 to finish the proof. ∎

Lemma 6.2.25. *Let $m \in \mathbb{N}$, $k \in \overline{\mathbb{N}}$, $\mathcal{W}$ a set of weights as in Definition* 6.2.18*, Y and Z locally convex spaces, $\Omega \subseteq Y$ open and balanced, $\phi : \Omega \to Z$ a smooth map with $\phi(0) = 0$ and $U \subseteq \mathbb{R}^m$ open and nonempty such that $\mathbb{R}^m \setminus U$ is compact and $\overline{M_\infty} \subseteq U$. Further, let $\gamma \in \mathcal{C}^k_{\mathcal{W}}(U, Y)$ be such that $\gamma(U) \subseteq \Omega$. Then there exists an open set $V \subseteq U$ such that $\mathbb{R}^m \setminus V$ is compact, $\overline{M_\infty} \subseteq V$ and $\phi \circ \gamma|_V \in \mathcal{C}^k_{\mathcal{W}}(V, Z)$.*

Proof. By our assumptions, there exists $h \in \mathcal{C}^\infty_c(\mathbb{R}^m, [0, 1])$ with $h \equiv 1$ on a neighborhood of $\mathbb{R}^m \setminus U$ and $h \equiv 0$ on a neighborhood of $\overline{M_\infty}$. Using Lemma 6.2.24 and Proposition 3.4.23 we see that

$$\phi \circ ((1 - h) \cdot \gamma) \in \mathcal{C}^k_{\mathcal{W}}(U, Z)^\bullet,$$

so $\phi \circ \gamma|_V \in \mathcal{C}^k_{\mathcal{W}}(V, Z)$, where $V := \mathbb{R}^m \setminus \operatorname{supp}(h)$. Further, $\mathbb{R}^m \setminus V$ is compact and $\overline{M_\infty} \subseteq V$, so the proof is finished. ∎

To complete our preparations, we prove a kind of extension lemma for weighted functions.

Lemma 6.2.26. *Let $m \in \mathbb{N}$, $k \in \overline{\mathbb{N}}$, $\mathcal{W}$ a set of weights as in Definition* 6.2.18*, Y a locally convex space, $V \subseteq U$ open and nonempty subsets of $\mathbb{R}^m$ such that $\mathbb{R}^m \setminus V$ is compact and $\overline{M_\infty} \subseteq V$. Further, let $\gamma \in \mathcal{C}^k(U, Y)$ be such that $\gamma|_V \in \mathcal{C}^k_{\mathcal{W}}(V, Y)$. Then for any open set W with $\overline{W} \subseteq U$, the map $\gamma|_W$ is in $\mathcal{C}^k_{\mathcal{W}}(W, Y)$.*

Proof. Obviously $\overline{W \setminus V} \subseteq \overline{W} \cap (\mathbb{R}^m \setminus V)$, hence $\overline{W \setminus V}$ is compact and does not meet M_∞. So for each $f \in \mathcal{W}$ and $\ell \in \mathbb{N}$ with $\ell \leq k$, the map $f \cdot D^{(\ell)}\gamma$ is bounded on $\overline{W \setminus V}$ since f is continuous on this set. But $f \cdot D^{(\ell)}\gamma$ is bounded on V by our assumption. Hence $f \cdot D^{(\ell)}\gamma$ is bounded on all of W and the proof is finished. ∎

Now we are able to prove the main results.

Proposition 6.2.27. *Let $m \in \mathbb{N}$, G a locally convex Lie group and $\mathcal{W}$ a set of weights as in Definition* 6.2.18.

(a) *$S(\mathbb{R}^m, G; \mathcal{W})$ is a group.*
(b) *$\mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m, G)^\bullet \subseteq S(\mathbb{R}^m, G; \mathcal{W})$.*
(c) *$S(\mathbb{R}^m, G; \mathcal{W}) \subseteq \mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m, G)^\bullet_{\text{ex}}$.*

Proof. (a) Let $\gamma_1, \gamma_2 \in S(\mathbb{R}^m, G; \mathcal{W})$. We set $\gamma := \gamma_1 \cdot \gamma_2^{-1}$. Then for $x \in M_\infty$, we have $\gamma(x) = \gamma_1(x) \cdot \gamma_2^{-1}(x) = \mathbf{1}$, and it is easy to see that $\gamma(x) \to \mathbf{1}$ if $\|x\| \to \infty$.

Let $(\phi, \widetilde{V})$ be a centered chart of G and $V \subseteq \widetilde{V}$ an open $\mathbf{1}$-neighborhood with $\overline{V} \subseteq \widetilde{V}$. There exist centered charts (ϕ_1, V_1) and (ϕ_2, V_2) such that $\phi_i \circ \gamma_i \in S(\gamma_i^{-1}(V_i), \mathbf{L}(G); \mathcal{W})$, where $i \in \{1, 2\}$; we may assume that $V_1 \cdot V_2^{-1} \subseteq V$, $V_2 \subseteq V$ and $\phi_1(V_1)$ and $\phi_2(V_2)$ are balanced. We define $W := \bigcap_{i \in \{1,2\}} \gamma_i^{-1}(V_i)$. Then by Lemmas 3.4.16 and 6.2.21,

$$(\phi_1 \circ \gamma_1|_W, \phi_2 \circ \gamma_2|_W) \in \mathcal{C}^\infty_{\mathcal{W}}(W, \phi_1(V_1) \times \phi_2(V_2)).$$

Further $\mathbb{R}^m \setminus W$ is compact, and since there exist closed $A_i \in \mathcal{U}_G(\mathbf{1})$ with $A_i \subseteq V_i$ ($i \in \{1, 2\}$), we have $\overline{M_\infty} \subseteq \bigcap_{i \in \{1,2\}} \gamma_i^{-1}(A_i) \subseteq W$. We now apply Lemma 6.2.25 to $(\phi_1 \circ \gamma_1|_W, \phi_2 \circ \gamma_2|_W)$ and the map

$$\phi \circ \widetilde{m_G} \circ (\phi_1^{-1} \times \phi_2^{-1}) : \phi_1(V_1) \times \phi_2(V_2) \to \mathbf{L}(G)$$

(where $\widetilde{m_G}$ denotes the map $G \times G \to G : (g, h) \mapsto g \cdot h^{-1}$) and find an open set $W' \subseteq W$ such that $\overline{M_\infty} \subseteq W'$, $\mathbb{R}^m \setminus W'$ is compact and $\phi \circ \gamma|_{W'} \in \mathcal{C}^\infty_{\mathcal{W}}(W', \mathbf{L}(G))$. Applying Lemma 6.2.26 with the open sets $W' \subseteq \gamma^{-1}(\widetilde{V})$ and $\gamma^{-1}(V) \subseteq \gamma^{-1}(\widetilde{V})$, we obtain

$$\phi \circ \gamma|_{\gamma^{-1}(V)} \in \mathcal{C}^\infty_{\mathcal{W}}(\gamma^{-1}(V), \mathbf{L}(G)) = S(\gamma^{-1}(V), \mathbf{L}(G); \mathcal{W}).$$

(b) Since we proved that $S(\mathbb{R}^m, G; \mathcal{W})$ is a group, we just have to show that it contains a generating set of $\mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m, G)^\bullet$. We know from Definition 6.2.5 that $\mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m, G)^\bullet$ is generated by $\phi^{-1} \circ \mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m, W)^\bullet$, where $(\phi, \widetilde{W})$ is a centered chart of G and $W \subseteq \phi(\widetilde{W})$ is an open convex zero neighborhood. Let $\gamma \in \mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m, W)^\bullet$. Then $\gamma|_{M_\infty} \equiv 0$, hence $\phi^{-1} \circ \gamma|_{M_\infty} \equiv \mathbf{1}$. Further, since $1_{\mathbb{R}^m} \in \mathcal{W}$, $\gamma(x) \to 0$ if $\|x\| \to \infty$, and thus $(\phi^{-1} \circ \gamma)(x) \to \mathbf{1}$ if $\|x\| \to \infty$. Now let $(\psi, \widetilde{V})$ be a centered chart of G and $V \subseteq \widetilde{V}$ an open $\mathbf{1}$-neighborhood with $\overline{V} \subseteq \widetilde{V}$. There exists an open balanced set $\Omega \subseteq W$ such that $\phi^{-1}(\Omega) \subseteq V$. We set $U := \gamma^{-1}(\Omega)$. Then $\gamma|_U \in \mathcal{C}^\infty_{\mathcal{W}}(U, \mathbf{L}(G))$, $\mathbb{R}^m \setminus U$ is compact, and $\overline{M_\infty} \subseteq \gamma^{-1}(\{0\}) \subseteq U$. Hence we can apply Lemma 6.2.25 to $\gamma|_U$ and $\psi \circ \phi^{-1}|_\Omega$ to see that $\psi \circ \phi^{-1} \circ \gamma|_U \in \mathcal{C}^\infty_{\mathcal{W}}(U, \mathbf{L}(G))$ Applying Lemma 6.2.26 with the open sets $U \subseteq (\psi \circ \phi^{-1} \circ \gamma)^{-1}(\widetilde{V})$ and $(\psi \circ \phi^{-1} \circ \gamma)^{-1}(V) \subseteq (\psi \circ \phi^{-1} \circ \gamma)^{-1}(\widetilde{V})$, we obtain

$$\psi \circ \phi^{-1} \circ \gamma|_{(\psi \circ \phi^{-1} \circ \gamma)^{-1}(V)} \in \mathcal{C}^\infty_{\mathcal{W}}((\psi \circ \phi^{-1} \circ \gamma)^{-1}(V), \mathbf{L}(G)) = S((\psi \circ \phi^{-1} \circ \gamma)^{-1}(V), \mathbf{L}(G); \mathcal{W}).$$

(c) Let $\gamma \in S(\mathbb{R}^m, G; \mathcal{W})$, $(\phi, \widetilde{V})$ be a centered chart of G and V an open $\mathbf{1}$-neighborhood with $\overline{V} \subseteq \widetilde{V}$. Then $K := \mathbb{R}^m \setminus \gamma^{-1}(V)$ is closed and bounded, hence compact, and

$$\phi \circ \gamma|_{\mathbb{R}^m \setminus K} \in S(\mathbb{R}^m \setminus K, \mathbf{L}(G); \mathcal{W}) = \mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m \setminus K, \mathbf{L}(G));$$

the last identity is by Lemma 6.2.21. Let $h \in \mathcal{C}^\infty_c(\mathbb{R}^m, \mathbb{R})$ such that $h \equiv 1$ on a neighborhood of K. Then by Lemma 6.2.24,

$$(1_{\mathbb{R}^m} - h) \cdot \phi \circ \gamma|_{\mathbb{R}^m \setminus K} \in \mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m \setminus K, \mathbf{L}(G))^\bullet.$$

Hence $\gamma \in \mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m, G)^\bullet_{\text{ex}}$. ∎

We characterize when $\mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m, G)^\bullet_{\mathrm{ex}}$ consists entirely of $\mathcal{W}$-rapidly decreasing mappings.

LEMMA 6.2.28. *Let $m \in \mathbb{N}$, G a locally convex Lie group and $\mathcal{W}$ a set of weights as in Definition* 6.2.18. *Then*

$$\mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m, G)^\bullet_{\mathrm{ex}} = S(\mathbb{R}^m, G; \mathcal{W}) \;\Leftrightarrow\; M_\infty = \emptyset.$$

Proof. Suppose that $M_\infty = \emptyset$. Let $\gamma \in \mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m, G)^\bullet_{\mathrm{ex}}$, $(\psi, \widetilde{V})$ a centered chart of G and V a $\mathbf{1}$-neighborhood with $\overline{V} \subseteq \widetilde{V}$. By Lemma 6.2.8, there exist a compact set $K \subseteq \mathbb{R}^m$ and $h \in \mathcal{C}^\infty_c(\mathbb{R}^m, \mathbb{R})$ with $h \equiv 1$ on a neighborhood of K such that $\gamma(\mathbb{R}^m \setminus K) \subseteq \widetilde{V}$ and $(1-h) \cdot (\psi \circ \gamma)|_{\mathbb{R}^m \setminus K} \in \mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m \setminus K, \mathbf{L}(G))^\bullet$. Since $1_{\mathbb{R}^m} \in \mathcal{W}$ and K and $\operatorname{supp}(h)$ are compact, $(\psi \circ \gamma)(x) \to 0$ if $\|x\| \to \infty$, hence $\gamma(x) \to \mathbf{1}$ if $\|x\| \to \infty$. Further $\psi \circ \gamma|_{\mathbb{R}^m \setminus \operatorname{supp}(h)} \in \mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m \setminus \operatorname{supp}(h), \mathbf{L}(G))$, so we apply Lemma 6.2.26 with the open sets $\mathbb{R}^m \setminus \operatorname{supp}(h) \subseteq \gamma^{-1}(\widetilde{V})$ and $\gamma^{-1}(V) \subseteq \gamma^{-1}(\widetilde{V})$ and get $\psi \circ \gamma|_{\gamma^{-1}(V)} \in \mathcal{C}^\infty_{\mathcal{W}}(\gamma^{-1}(V), \mathbf{L}(G))$. Hence $\gamma \in S(\mathbb{R}^m, G; \mathcal{W})$, so in view of Proposition 6.2.27, the implication holds.

Now let $M_\infty \neq \emptyset$. By definition, $\mathcal{C}^\infty_c(\mathbb{R}^m, G) \subseteq \mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m, G)^\bullet_{\mathrm{ex}}$, so there exists a $\gamma \in \mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m, G)^\bullet_{\mathrm{ex}}$ such that $\gamma \not\equiv \mathbf{1}$ on M_∞. Then $\gamma \notin S(\mathbb{R}^m, G; \mathcal{W})$. ■

REMARK 6.2.29. In the book [BCR81], the groups $S(\mathbb{R}^m, G; \mathcal{W})$ are only defined if G is a so-called *LE-Lie group*. Since we do not need this concept, we do not discuss it further. In Proposition 6.2.27 we proved that $S(\mathbb{R}^m, G; \mathcal{W})$ is an open subgroup of $\mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m, G)^\bullet_{\mathrm{ex}}$ and hence a Lie group. Further, for a set $\mathcal{W}$ of weights as in Definition 6.2.18 obviously $\mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m, \mathbf{L}(G))^\bullet = \mathcal{C}^\infty_{\mathcal{W}}(\mathbb{R}^m, \mathbf{L}(G))$, whence the results derived by [BCR81] concerning the Lie group structure of $S(\mathbb{R}^m, G; \mathcal{W})$ are special cases of our more general construction.

It should be noted that the proof of [BCR81, Lemma 4.2.1.9] (whose assertion resembles Proposition 3.4.23) is not really complete: The boundedness of $\gamma \cdot \partial^\beta(g \circ f)$, where $|\beta| > 0$, is hardly discussed. In the finite-dimensional case, compactness arguments similar to the one in Lemma 3.4.17 and the Faà di Bruno formula should save the day, but the infinite-dimensional case requires more work.

A. Differential calculus

In this chapter, we present the tools of Michal-Bastiani and Fréchet differential calculus used in this work. For proofs, we refer the reader to [Mil84], [Ham82], or [Mic80]. Further, we state some facts about ordinary differential equations.

In the following, X, Y and Z denote locally convex topological vector spaces over the same field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

A.1. Differential calculus of maps between locally convex spaces

A.1.1. Curves and integrals

DEFINITION A.1.1 (Curves). A continuous map $\gamma : I \to X$ that is defined on a proper interval $I \subseteq \mathbb{R}$ is called a $\mathcal{C}^0$-*curve*. A $\mathcal{C}^0$-curve $\gamma : I \to X$ is called a $\mathcal{C}^1$-*curve* if the limit

$$\gamma^{(1)}(s) := \lim_{t \to 0} \frac{\gamma(s+t) - \gamma(s)}{t}$$

exists for all $s \in I$ and the map $\gamma^{(1)} : I \to X$ is a $\mathcal{C}^0$-curve.

Inductively, for $k \in \mathbb{N}$ a map $\gamma : I \to X$ is called a $\mathcal{C}^k$-*curve* if it is a $\mathcal{C}^1$-curve and the map $\gamma^{(1)}$ is a $\mathcal{C}^{k-1}$-curve. We then define $\gamma^{(k)} := (\gamma^{(1)})^{(k-1)}$.

If γ is a $\mathcal{C}^k$-curve for each $k \in \mathbb{N}$, we call γ a $\mathcal{C}^\infty$- or *smooth* curve.

Definition A.1.2 (Weak integral). Let $\gamma : [a,b] \to X$ be a map. If there exists $x \in X$ such that

$$\lambda(x) = \int_a^b (\lambda \circ \gamma)(t)\, dt \qquad \text{for all } \lambda \in X',$$

we call γ *weakly integrable* with the *weak integral* x and write

$$\int_a^b \gamma(t)\, dt := x.$$

Definition A.1.3 (Line integral). Let $\gamma : [a,b] \to X$ be a $\mathcal{C}^1$-curve and $f : \gamma([a,b]) \to Y$ a continuous map. We define the *line integral* of f on γ by

$$\int_\gamma f(\zeta)\, d\zeta := \int_a^b f(\gamma(t)) \cdot \gamma^{(1)}(t)\, dt$$

if the weak integral on the right hand side exists.

We record some properties of weak integrals.

Lemma A.1.4. *Let $\gamma : [a,b] \to X$ be a weakly integrable curve and $A : X \to Y$ a continuous linear map. Then the map $A \circ \gamma$ is weakly integrable with the integral*

$$\int_a^b (A \circ \gamma)(t)\, dt = A\biggl(\int_a^b \gamma(t)\, dt\biggr).$$

Proposition A.1.5 (Fundamental theorem of calculus). *Let $\gamma : [a,b] \to X$ be a $\mathcal{C}^1$-curve. Then $\gamma^{(1)}$ is weakly integrable with the integral*

$$\int_a^b \gamma^{(1)}(t)\, dt = \gamma(b) - \gamma(a).$$

Lemma A.1.6. *If X is sequentially complete, each continuous curve in X is weakly integrable.*

Lemma A.1.7. *Endow the set of weakly integrable continuous curves from $[a,b]$ to X with the topology of uniform convergence. The weak integral defines a continuous linear map from this space to X. In particular, for each continuous seminorm $p : X \to \mathbb{R}$ and each weakly integrable continuous curve $\gamma : [a,b] \to X$,*

$$\left\| \int_a^b \gamma(t)\, dt \right\|_p \leq \int_a^b \|\gamma(t)\|_p\, dt,$$

where we define $\|\cdot\|_p := p(\cdot)$.

Proposition A.1.8 (Continuity of parameter-dependent integrals). *Let P be a topological space, $I \subseteq \mathbb{R}$ a proper interval and $a, b \in I$. Further, let $f : P \times I \to X$ be a continuous map such that the weak integral*

$$\int_a^b f(p,t)\,dt =: g(p)$$

exists for all $p \in P$. Then the map $g : P \to X$ is continuous.

Evaluation of curves. We prove that the (simultaneous) evaluation of smooth curves is smooth.

Lemma A.1.9. *Let Y be a locally convex topological vector space and $m \in \overline{\mathbb{N}}$. Then the* evaluation function

$$\mathrm{ev} : \mathcal{C}^m([0,1], Y) \times [0,1] \to Y : (\Gamma, t) \mapsto \Gamma(t)$$

is a $\mathcal{C}^m$-map. For $m \geq 1$, we have

$$d\,\mathrm{ev}((\Gamma, t); (\Gamma_1, s)) = s \cdot \mathrm{ev}(\Gamma', t) + \mathrm{ev}(\Gamma_1, t) \tag{$\dagger$}$$

(*using the same symbol,* ev, *for the evaluation of $\mathcal{C}^{m-1}$-curves*).

Proof. The proof is by induction.

$m = 0$: Let $\Gamma \in \mathcal{C}^0([0,1], Y)$ and $t \in [0,1]$. For a continuous seminorm $\|\cdot\|$ on Y and $\varepsilon > 0$ let U be a neighborhood of Γ in $\mathcal{C}^0([0,1], Y)$ such that for all $\Phi \in U$,

$$\|\Phi - \Gamma\|_\infty < \varepsilon/2,$$

where $\|\cdot\|_\infty$ is defined by

$$\mathcal{C}^0([0,1], Y) \to \mathbb{R} : \Phi \mapsto \sup_{t \in [0,1]} \|\Phi(t)\|.$$

By the continuity of Γ, there exists $\delta > 0$ such that for all $s \in [0,1]$ with $|s - t| < \delta$ we have

$$\|\Gamma(s) - \Gamma(t)\| < \varepsilon/2.$$

Then

$$\|\mathrm{ev}(\Gamma, t) - \mathrm{ev}(\Phi, s)\| \leq \|\Gamma(t) - \Gamma(s)\| + \|\Gamma(s) - \Phi(s)\| < \varepsilon,$$

whence ev is continuous in (Γ, t).

$m = 1$: Let $\Gamma, \Gamma_1 \in \mathcal{C}^1([0,1], Y)$, $t \in]0,1[$, $h \in \mathbb{R}^*$ and $s \in \mathbb{R}$ be such that $t + hs \in [0,1]$. Then

$$\frac{\mathrm{ev}((\Gamma, t) + h(\Gamma_1, s)) - \mathrm{ev}(\Gamma, t)}{h} = \frac{\Gamma(t + hs) - \Gamma(t)}{h} + \mathrm{ev}(\Gamma_1, t + hs),$$

and because Γ is differentiable and ev is continuous, this term converges to

$$s \cdot \mathrm{ev}(\Gamma', t) + \mathrm{ev}(\Gamma_1, t)$$

for $h \to 0$. Since this term has an obvious continuous extension to $\mathcal{C}^1([0,1], Y) \times [0,1] \times \mathcal{C}^1([0,1], Y) \times \mathbb{R}$, ev is differentiable with the directional derivative ($\dagger$), which is continuous.

$m \to m + 1$: The map

$$\mathcal{C}^{m+1}([0,1], Y) \to \mathcal{C}^m([0,1], Y) : \Gamma \mapsto \Gamma'$$

is continuous linear and thus smooth. Using the inductive hypothesis, we therefore deduce from (†) that $d\,\mathrm{ev}$ is $\mathcal{C}^m$. Hence ev is $\mathcal{C}^{m+1}$. ∎

A.1.2. Differentiable maps. We give a short introduction to differential calculus for maps between locally convex spaces. It was first developed by A. Bastiani [Bas64] and is also known as Keller's C_c^k-theory.

Recall the definitions given in Section 2.2. In the following, let X and Y be locally convex spaces and $U \subseteq X$ an open nonempty set.

Proposition A.1.10 (Mean value theorem). *Let $f \in \mathcal{C}^1(U,Y)$ and $v, u \in U$ be such that the line segment $\{tu + (1-t)v : t \in [0,1]\}$ is contained in U. Then*

$$f(v) - f(u) = \int_0^1 df(u + t(v-u); v-u)\,dt.$$

Proposition A.1.11 (Chain rule). *Let $k \in \overline{\mathbb{N}}$, $f \in \mathcal{C}^k(U,Y)$ and $g \in \mathcal{C}^k(V,Z)$ be such that $f(U) \subseteq V$. Then the composition $g \circ f : U \to Z$ is a $\mathcal{C}^k$-map with*

$$d(g \circ f)(u;x) = dg(f(u); df(u;x)) \quad \text{for all } (u,x) \in U \times X.$$

Proposition A.1.12. *Let X and Y be locally convex spaces, $U \subseteq X$ be open and nonempty and $k \in \overline{\mathbb{N}}$.*

(a) *A map*

$$f = (f_i)_{i \in I} : U \to \prod_{i \in I} Y_i$$

to a direct product of locally convex spaces is $\mathcal{C}^k$ iff each component f_i is $\mathcal{C}^k$.

(b) *A map $f : U \to Y$ with values in a closed vector subspace Z is $\mathcal{C}^k$ iff $f|^Z : U \to Z$ is $\mathcal{C}^k$.*

(c) *If Y is the projective limit of locally convex spaces $\{Y_i : i \in I\}$ with limit maps $\pi_i : Y \to Y_i$, then a map $f : U \to Y$ is $\mathcal{C}^k$ iff $\pi_i \circ f : U \to Y_i$ is $\mathcal{C}^k$ for all $i \in I$.*

Characterization of differentiability of higher order. In Proposition 2.2.3, we stated that a map is $\mathcal{C}^k$ iff all iterated directional derivatives up to order k exist and depend continuously on the directions. Here, we present some facts about the iterated directional derivatives.

Remark A.1.13. We give a more explicit formula for the kth derivative. Obviously, $d^{(1)}f(u;x_1) = df(u;x_1)$ and

$$d^{(k)}f(u;x_1,\ldots,x_k) = \lim_{t \to 0} \frac{d^{(k-1)}f(u+tx_k;x_1,\ldots,x_{k-1}) - d^{(k-1)}f(u;x_1,\ldots,x_{k-1}))}{t}.$$

The Schwarz theorem extends to the present situation:

Proposition A.1.14 (Schwarz' theorem). *Let $r \in \overline{\mathbb{N}}$, $f \in \mathcal{C}^r_{\mathbb{K}}(U,Y)$, $k \in \mathbb{N}$ with $k \leq r$ and $u \in U$. The map*

$$d^{(k)}f(u;\cdot) : X^k \to Y : (x_1,\ldots,x_k) \mapsto d^{(k)}f(u;x_1,\ldots,x_k)$$

is continuous, symmetric and k-linear (over the field $\mathbb{K}$).

Examples. We give some examples of $\mathcal{C}^k$-maps and calculate the higher-order differentials of some maps.

EXAMPLE A.1.15.

(a) A map $\gamma : I \to X$ is a $\mathcal{C}^k$-curve iff it is a $\mathcal{C}^k_{\mathbb{R}}$-map, and $d\gamma(x; h) = h \cdot \gamma^{(1)}(x)$.
(b) A continuous linear map $A : X \to Y$ is smooth with $dA(x; h) = A \cdot h$.
(c) More generally, a k-linear continuous map $b : X_1 \times \cdots \times X_k \to Y$ is smooth with

$$db(x_1, \ldots, x_k; h_1, \ldots, h_k) = \sum_{i=1}^{k} b(x_1, \ldots, x_{i-1}, h_i, x_{i+1}, \ldots, x_k).$$

LEMMA A.1.16. *Let X, Y and Z be locally convex topological vector spaces, $U \subseteq X$ an open nonempty set, $k \in \overline{\mathbb{N}}$ and $A : Y \to Z$ a continuous linear map. Then for $\gamma \in \mathcal{C}^k(U, Y)$,*

$$A \circ \gamma \in \mathcal{C}^k(U, Z).$$

Moreover, for each $\ell \in \mathbb{N}$ with $\ell \leq k$,

$$d^{(\ell)}(A \circ \gamma) = A \circ d^{(\ell)}\gamma. \tag{†}$$

Proof. This is proved by induction on ℓ. The chain rule (Proposition A.1.11) ensures $A \circ \gamma \in \mathcal{C}^k(U, Z)$ and

$$d(A \circ \gamma)(x; h) = dA(\gamma(x); d\gamma(x; h)) = A(d\gamma(x; h))$$

for $x \in U$ and $h \in X$, hence (†) is satisfied for $\ell = 1$.

If we assume that (†) holds for an $\ell \in \mathbb{N}$, we conclude for $x \in U$ and $h_1, \ldots, h_\ell, h_{\ell+1} \in X$,

$$\begin{aligned}
&d^{(\ell+1)}(A \circ \gamma)(x; h_1, \ldots, h_\ell, h_{\ell+1}) \\
&\quad = \lim_{t \to 0} \frac{d^{(\ell)}(A \circ \gamma)(x + th_{\ell+1}; h_1, \ldots, h_\ell) - d^{(\ell)}(A \circ \gamma)(x; h_1, \ldots, h_\ell)}{t} \\
&\quad = \lim_{t \to 0} \frac{A(d^{(\ell)}\gamma(x + th_{\ell+1}; h_1, \ldots, h_\ell)) - A(d^{(\ell)}\gamma(x; h_1, \ldots, h_\ell))}{t} \\
&\quad = A\left(\lim_{t \to 0} \frac{d^{(\ell)}\gamma(x + th_{\ell+1}; h_1, \ldots, h_\ell) - d^{(\ell)}\gamma(x; h_1, \ldots, h_\ell)}{t} \right) \\
&\quad = (A \circ d^{(\ell+1)}\gamma)(x; h_1, \ldots, h_\ell, h_{\ell+1}),
\end{aligned}$$

so (†) holds for $\ell + 1$ as well. ■

LEMMA A.1.17. *Let X, Y and Z be locally convex topological vector spaces, $k \in \overline{\mathbb{N}}$ and $A : X \to Y$ a continuous linear map. Then for $\gamma \in \mathcal{C}^k(Y, Z)$,*

$$\gamma \circ A \in \mathcal{C}^k(X, Z).$$

Moreover, for each $\ell \in \mathbb{N}$ with $\ell \leq k$,

$$d^{(\ell)}(\gamma \circ A) = d^{(\ell)}\gamma \circ \prod_{j=1}^{\ell+1} A. \tag{†}$$

Proof. This is proved by induction on ℓ. The chain rule (Proposition A.1.11) ensures $\gamma \circ A \in \mathcal{C}^k(U, Z)$ and

$$d(\gamma \circ A)(x; h) = d\gamma(A(x); dA(x; h)) = d\gamma(A(x); A(h))$$

for $x \in X$ and $h \in X$, hence (†) is satisfied for $\ell = 1$.

If we assume that (†) holds for an arbitrary $\ell \in \mathbb{N}$, we conclude that for $x \in X$ and $h_1, \ldots, h_\ell, h_{\ell+1} \in X$,

$$\begin{aligned}
&d^{(\ell+1)}(\gamma \circ A)(x; h_1, \ldots, h_\ell, h_{\ell+1}) \\
&\quad = \lim_{t \to 0} \frac{d^{(\ell)}(\gamma \circ A)(x + th_{\ell+1}; h_1, \ldots, h_\ell) - d^{(\ell)}(\gamma \circ A)(x; h_1, \ldots, h_\ell)}{t} \\
&\quad = \lim_{t \to 0} \frac{d^{(\ell)}\gamma(A(x + th_{\ell+1}); A \cdot h_1, \ldots, A \cdot h_\ell) - d^{(\ell)}\gamma(A(x); A \cdot h_1, \ldots, A \cdot h_\ell)}{t} \\
&\quad = \lim_{t \to 0} \frac{1}{t} \int_0^1 d^{(\ell+1)}\gamma(A(x) + stA(h_{\ell+1}); A \cdot h_1, \ldots, A \cdot h_\ell, tA \cdot h_{\ell+1})\, ds \\
&\quad = d^{(\ell+1)}\gamma(A(x); A \cdot h_1, \ldots, A \cdot h_\ell, A \cdot h_{\ell+1})
\end{aligned}$$

so (†) holds for $\ell + 1$ as well. ■

We give a specialization of Proposition A.1.8.

Proposition A.1.18 (Differentiability of parameter-dependent integrals). *Let P be an open subset of a locally convex space, $I \subseteq \mathbb{R}$ a proper interval, $a, b \in I$ and $k \in \overline{\mathbb{N}}$. Further, let $f : P \times I \to X$ be a $\mathcal{C}^k$-map such that the weak integral*

$$\int_a^b f(p, t)\, dt =: g(p)$$

exists for all $p \in P$. Then the map $g : P \to X$ is $\mathcal{C}^k$.

A.1.2.1. *Analytic maps.* Complex analytic maps will be defined as maps which can be locally approximated by polynomials. Real analytic maps are maps that have a *complexification.*

Polynomials and symmetric multilinear maps. For the definition of complex analytic maps we need to define polynomials.

Definition A.1.19. Let $k \in \mathbb{N}$. A *homogeneous polynomial of degree* k from X to Y is a map for which there exists a k-linear map $\beta : X^k \to Y$ such that

$$p(x) = \beta(\underbrace{x, \ldots, x}_{k})$$

for all $x \in X$. In particular, a homogeneous polynomial of degree 0 is a constant map.

A *polynomial of degree* $\leq k$ is a sum of homogeneous polynomials of degree $\leq k$.

There is a bijection between the set of homogeneous polynomials and that of symmetric multilinear maps. In this article, we just need that one can reconstruct a symmetric multilinear map from its homogeneous polynomial.

PROPOSITION A.1.20 (Polarization formula). *Let $\beta : X^k \to Y$ be a symmetric k-linear map, $p : X \to Y : x \mapsto \beta(x, \ldots, x)$ its homogeneous polynomial and $x_0 \in X$. Then*

$$\beta(x_1, \ldots, x_k) = \frac{1}{k!} \sum_{\varepsilon_1, \ldots, \varepsilon_k = 0}^{1} (-1)^{k-(\varepsilon_1 + \cdots + \varepsilon_k)} p(x_0 + \varepsilon_1 x_1 + \cdots + \varepsilon_k x_k)$$

for all $x_1, \ldots, x_k \in X$.

Complex analytic maps. Now we can define complex analytic maps.

DEFINITION A.1.21 (Complex analytic maps). Let X, Y be complex locally convex topological vector spaces and $U \subseteq X$ an open nonempty set. A map $f : U \to Y$ is called *complex analytic* if it is continuous and, for each $x \in U$ there exists a sequence $(p_k)_{k \in \mathbb{N}}$ of continuous homogeneous polynomials $p_k : X \to Y$ of degree k such that

$$f(x + v) = \sum_{k=0}^{\infty} p_k(v)$$

for all v in some zero neighborhood V such that $x + V \subseteq U$.

DEFINITION A.1.22. Let X, Y be complex locally convex topological vector spaces and $U \subseteq X$ an open nonempty set. A map $f : U \to Y$ is called *Gateaux analytic* if its restriction to each affine line is complex analytic; that is, for each $x \in U$ and $v \in X$ the map

$$Z \to Y : z \mapsto f(x + zv)$$

which is defined on the open set $Z := \{z \in \mathbb{C} : x + zv \in U\}$ is complex analytic.

THEOREM A.1.23. *Let X, Y be complex locally convex topological vector spaces and $U \subseteq X$ an open nonempty set. Then for a map $f : U \to Y$ the following assertions are equivalent:*

(a) *f is $\mathcal{C}^\infty_\mathbb{C}$.*
(b) *f is complex analytic.*
(c) *f is continuous and Gateaux analytic.*

We state a few results concerning analytic curves. These share many properties with holomorphic functions. Using Theorem A.1.23, we see that some of these properties carry over to general analytic functions.

DEFINITION A.1.24. Let Y be a complex locally convex topological vector space and $U \subseteq \mathbb{C}$ an open nonempty set. A continuous map $f : U \to Y$ is called a *$\mathcal{C}^0_\mathbb{C}$-curve.* A $\mathcal{C}^0_\mathbb{C}$-curve $f : U \to Y$ is called a *$\mathcal{C}^1_\mathbb{C}$-curve* if for all $z \in U$ the limit

$$f^{(1)}(z) := \lim_{w \to 0} \frac{f(z + w) - f(z)}{w}$$

exists and the curve $f^{(1)} : U \to X$ is a $\mathcal{C}^0_\mathbb{C}$-curve.

Inductively, for $k \in \mathbb{N}$ a curve f is called a *$\mathcal{C}^k_\mathbb{C}$-curve* if it is a $\mathcal{C}^1_\mathbb{C}$-curve and $f^{(1)}$ is a $\mathcal{C}^{k-1}_\mathbb{C}$-curve. In this case, we define $f^{(k)} := (f^{(1)})^{(k-1)}$.

If f is a $\mathcal{C}^k_\mathbb{C}$-curve for all $k \in \mathbb{N}$, f is called a *$\mathcal{C}^\infty_\mathbb{C}$-curve.*

Lemma A.1.25 (Cauchy integral formula). *Let Y be a complex locally convex topological vector space, $U \subseteq \mathbb{C}$ an open nonempty set and $f : U \to Y$ a map. Then*

$$f \text{ is a } \mathcal{C}^k_{\mathbb{C}}\text{-curve} \;\Leftrightarrow\; f \in \mathcal{C}^k_{\mathbb{C}}(U, Y)$$

and furthermore

$$d^{(k)} f(x; h_1, \dots, h_k) = h_1 \cdot \dots \cdot h_k \cdot f^{(k)}(x).$$

A $\mathcal{C}^\infty_{\mathbb{C}}$-curve is complex analytic, and for each $x \in U$, $k \in \mathbb{N}_0$ and $r > 0$ with $\overline{B}_r(x) \subseteq U$ the Cauchy integral formula

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{|\zeta - x| = r} \frac{f(\zeta)}{(\zeta - z)^{k+1}}\, d\zeta$$

holds, where $z \in B_r(x)$.

The Cauchy integral formula implies the Cauchy estimates.

Corollary A.1.26. *Let Y be a complex locally convex topological vector space, $U \subseteq \mathbb{C}$ an open nonempty set, $f : U \to Y$ a complex analytic map, $x \in U$, $r > 0$ such that $\overline{B}_r(x) \subseteq U$ and p a continuous seminorm on Y. Then for each $z \in B_{r/2}(x)$ and $k \in \mathbb{N}$,*

$$\|f^{(k)}(z)\|_p \leq \frac{k!}{(3r/2)^k} \sup_{|\zeta - x| = r} \|f(\zeta)\|_p.$$

Real analytic maps

Definition A.1.27 (Real analytic maps). Let X, Y be real locally convex topological vector spaces and $U \subseteq X$ an open nonempty set. Let $X_{\mathbb{C}}$ resp. $Y_{\mathbb{C}}$ denote the complexifications of X resp. Y. A map $f : U \to Y$ is called *real analytic* if there is an extension $\widetilde{f} : V \to Y_{\mathbb{C}}$ of f to an open neighborhood V of U in $X_{\mathbb{C}}$ that is complex analytic. Such a map $\widetilde{f}$ will be referred to as a *complexification* of f.

A.1.2.2. *Lipschitz continuous maps.* We discuss Lipschitz continuous maps between locally convex spaces.

Definition A.1.28. Let X be a locally convex space and $p : X \to \mathbb{R}$ a continuous seminorm. We denote the Hausdorff space $X/p^{-1}(0)$ by X_p and the quotient map by $\pi_p : X \to X_p$. More generally, for any subset $A \subseteq X$ we set $A_p := \pi_p(A)$.

Further, we let $\mathcal{N}(X)$ denote the set of continuous seminorms on X.

Let $p \in \mathcal{N}(X)$. We call $U \subseteq X$ *open with respect to p* if for each $x \in U$ there exists $r > 0$ such that $\{y \in X : \|y - x\|_p < r\} \subseteq U$.

Remark A.1.29. For any locally convex space X and each $p \in \mathcal{N}(X)$, the norm induced by p on X_p will also be denoted by p. Note that this leads to the identity $p = \pi_p \circ p$, in particular p is a norm and generates the topology on X_p. No confusion will arise.

Lemma A.1.30. *Let X, Y and Z be locally convex spaces, $V \subseteq Y$ an open nonempty set, $k \in \overline{\mathbb{N}}$, $\gamma : V \to Z$ a map and $A \in \mathrm{L}(X, Y)$ surjective such that*

$$\gamma \circ A \in \mathcal{C}^k(U, Z),$$

where $U := A^{-1}(V)$. Then all directional derivatives of γ up to order k exist and satisfy

$$d^{(\ell)}\gamma \circ \prod_{i=1}^{\ell+1} A = d^{(\ell)}(\gamma \circ A) \quad \text{for all } \ell \in \mathbb{N} \text{ with } \ell \leq k.$$

Proof. This is proved by induction on ℓ.

$\ell = 0$: This is obvious.

$\ell \to \ell + 1$: Let $y \in V$ and $h_1, \dots, h_\ell, h_{\ell+1} \in Y$. By the surjectivity of A there exist $x \in U$ and $v_1, \dots, v_\ell, v_{\ell+1} \in X$ with $A \cdot x = y$ and $A \cdot v_i = h_i$ for $i = 1, \dots, \ell, \ell + 1$. Then for all suitable $t \neq 0$,

$$\begin{aligned}
&\lim_{t\to 0} \frac{d^{(\ell)}\gamma(y + th_{\ell+1}; h_1, \dots, h_\ell) - d^{(\ell)}\gamma(y; h_1, \dots, h_\ell)}{t} \\
&\quad = \lim_{t\to 0} \frac{d^{(\ell)}\gamma(A(x + tv_{\ell+1}); A \cdot v_1, \dots, A \cdot v_\ell) - d^{(\ell)}\gamma(A \cdot x; A \cdot v_1, \dots, A \cdot v_\ell)}{t} \\
&\quad = \lim_{t\to 0} \frac{(d^{(\ell)}\gamma \circ \prod_{i=1}^{\ell+1} A)(x + tv_{\ell+1}, v_1, \dots, v_\ell) - (d^{(\ell)}\gamma \circ \prod_{i=1}^{\ell+1} A)(x, v_1, \dots, v_\ell)}{t} \\
&\quad = d^{(\ell+1)}(\gamma \circ A)(x; v_1, \dots, v_\ell, v_{\ell+1}),
\end{aligned}$$

and this completes the proof. ∎

LEMMA A.1.31. *Let X, Y be locally convex spaces, $U \subseteq X$ an open nonempty set, $k \in \overline{\mathbb{N}}$, $\gamma \in \mathcal{C}^{k+1}(U, Y)$ and $\ell \in \mathbb{N}$ with $\ell \leq k$. Then for each $p \in \mathcal{N}(Y)$ and $x_0 \in U$ there exist a seminorm $q \in \mathcal{N}(X)$ and a convex neighborhood $U_{x_0} \subseteq U$ of x with respect to q such that for all $x, y \in U_{x_0}$ and $h_1, \dots, h_\ell \in X$,*

$$\|d^{(\ell)}\gamma(y; h_1, \dots, h_\ell) - d^{(\ell)}\gamma(x; h_1, \dots, h_\ell)\|_p \leq \|y - x\|_q \prod_{i=1}^{\ell} \|h_i\|_q \tag{A.1.31.1}$$

and

$$\|d^{(\ell)}\gamma(x; h_1, \dots, h_\ell)\|_p \leq \prod_{i=1}^{\ell} \|h_i\|_q. \tag{A.1.31.2}$$

Proof. Since $d^{(\ell)}\gamma$ and $d^{(\ell+1)}\gamma$ are continuous at $(x_0, 0, \dots, 0)$ and multilinear in their last ℓ resp. $\ell + 1$ arguments, for each $p \in \mathcal{N}(Y)$ there exist a seminorm $q \in \mathcal{N}(X)$ and an open ball $U_{x_0} := B_q(x_0, r) \subseteq U$ such that

$$1 \geq \sup\{\|d^{(\ell+1)}\gamma(y; h_1, \dots, h_{\ell+1})\|_p : y \in B_q(x_0, r),\ \|h_1\|_q, \dots, \|h_{\ell+1}\|_q \leq 1\}$$

and

$$1 \geq \sup\{\|d^{(\ell)}\gamma(y; h_1, \dots, h_\ell)\|_p : y \in B_q(x_0, r),\ \|h_1\|_q, \dots, \|h_\ell\|_q \leq 1\}.$$

This implies that for each $y \in B_q(x_0, r)$ and $h_1, \dots, h_n \in X$,

$$\|d^{(n)}\gamma(y; h_1, \dots, h_n)\|_p \leq 1 \cdot \prod_{i=1}^{n} \|h_i\|_q, \tag{$\dagger$}$$

where $n \in \{\ell, \ell + 1\}$; this proves (A.1.31.2).

To prove (A.1.31.1), we see that for $x, y \in B_q(x_0, r)$ and $h_1, \dots, h_{\ell+1} \in X$,

$$d^{(\ell)}\gamma(y; h_1, \dots, h_\ell) - d^{(\ell)}\gamma(x; h_1, \dots, h_\ell) = \int_0^1 d^{(\ell+1)}\gamma(ty + (1 - t)x; h_1, \dots, h_\ell, y - x)\, dt.$$

We apply Lemma A.1.7 to the right hand side and get, using (†) with $n = \ell + 1$,

$$\|d^{(\ell)}\gamma(y; h_1, \dots, h_\ell) - d^{(\ell)}\gamma(x; h_1, \dots, h_\ell)\|_p \leq \|h_1\|_q \cdots \|h_\ell\|_q \cdot \|y - x\|_q. \ \blacksquare$$

Definition A.1.32. Let X and Y be locally convex spaces, $U \subseteq X$ an open nonempty set, $k \in \overline{\mathbb{N}}$, $p \in \mathcal{N}(Y)$ and $q \in \mathcal{N}(X)$. We call $\gamma : U \to Y$ *Lipschitz up to order k with respect to p and q* if $\gamma \in \mathcal{C}^k(U, Y)$ and estimates (A.1.31.1) and (A.1.31.2) are satisfied for all $\ell \in \mathbb{N}$ with $\ell \leq k$, $x, y \in U$ and $h_1, \dots, h_\ell \in X$. We write $\mathcal{LC}^k_{q,p}(U, Y)$ for the set of maps that are Lipschitz up to order k with respect to p and q.

Lemma A.1.33. *Let X and Y be locally convex spaces, $U \subseteq X$ an open nonempty set, $k \in \overline{\mathbb{N}}$, $p \in \mathcal{N}(Y)$, $q \in \mathcal{N}(X)$ and $\gamma \in \mathcal{LC}^k_{q,p}(U, Y)$. Then there exists a map $\tilde{\gamma} \in \mathcal{LC}^k_{q,p}(U_q, Y_p)$ that makes the diagram*

$$\begin{tikzcd} U \arrow[r, "\gamma"] \arrow[d, "\pi_q"'] & Y \arrow[d, "\pi_p"] \\ U_q \arrow[r, "\tilde{\gamma}"] & Y_p \end{tikzcd}$$

commutative (*using the notation of Definition* A.1.28).

Proof. Let $\ell \in \mathbb{N}$ with $\ell \leq k$. Since $\gamma \in \mathcal{LC}^k_{q,p}(U, Y)$, the map

$$\pi_p \circ d^{(\ell)}\gamma : (U, q) \times (X, q)^\ell \to Y_p$$

is continuous. Hence by the universal property of the separation there exists a continuous map $\tilde{\gamma}_\ell$ such that the diagram

$$\begin{tikzcd} U \times X^\ell \arrow[rr, "d^{(\ell)}\gamma"] \arrow[dr] \arrow[dd, "\prod_{i=1}^{\ell+1} \pi_q"'] & & Y \arrow[dd, "\pi_p"] \\ & (U, q) \times (X, q)^\ell \arrow[dl, two heads] \arrow[dr] & \\ U_q \times X_q^\ell \arrow[rr, "\tilde{\gamma}_\ell"] & & Y_p \end{tikzcd}$$

commutes, where we denote $\pi_q|_U$ by π_q. The diagram for $\ell = 0$ implies that $\tilde{\gamma} \circ \pi_q = \pi_p \circ \gamma \in \mathcal{C}^k(U, Y_p)$, where $\tilde{\gamma} := \tilde{\gamma}_0$. We proved in Lemma A.1.30 that the ℓth directional derivative of $\tilde{\gamma}$ exists and satisfies the identity

$$d^{(\ell)}\tilde{\gamma} \circ \prod_{i=1}^{\ell+1} \pi_q = d^{(\ell)}(\tilde{\gamma} \circ \pi_q) = d^{(\ell)}(\pi_p \circ \gamma) = \pi_p \circ d^{(\ell)}\gamma = \tilde{\gamma}_\ell \circ \prod_{i=1}^{\ell+1} \pi_q.$$

Since $\prod_{i=1}^{\ell+1} \pi_q$ is surjective, this implies that $d^{(\ell)}\tilde{\gamma} = \tilde{\gamma}_\ell$, so the former is continuous. From this we conclude that $\tilde{\gamma} \in \mathcal{C}^k(U_q, Y_p)$ and that the estimates (A.1.31.1) and (A.1.31.2) are satisfied by $\tilde{\gamma}$. $\blacksquare$

A.2. Fréchet differentiability. For maps between normed spaces, there is the classical notion of *Fréchet differentiability*. This concept relies on the existence of a well-behaved topology on the space of (k-)linear maps between normed spaces.

Spaces of multilinear maps between normed spaces. We provide the details about the norm topology of multilinear operators.

DEFINITION A.2.1. Let X, Y be normed spaces. For each $k \in \mathbb{N}^*$ we define

$$\mathrm{L}^k(X,Y) := \{\Xi : X^k \to Y : \Xi \text{ is } k\text{-linear and continuous}\}.$$

For $k = 1$ we define

$$\mathrm{L}(X,Y) := \mathrm{L}^1(X,Y) \quad \text{and} \quad \mathrm{L}(X) := \mathrm{L}^1(X,X),$$

and furthermore $\mathrm{L}^0(X,Y) := Y$.

The set of multilinear continuous maps can be turned into a normed vector space:

PROPOSITION A.2.2. *Let X, Y be normed spaces and $k \in \mathbb{N}^*$. A k-linear map $\Xi : X^k \to Y$ is continuous iff*

$$\|\Xi\|_{\mathrm{op}} := \sup\{\|\Xi(v_1,\dots,v_k)\| : \|v_1\|,\dots,\|v_k\| \le 1\} < \infty.$$

$\|\Xi\|_{\mathrm{op}}$ *is called the* operator norm *of* Ξ. $\|\cdot\|_{\mathrm{op}}$ *is a norm on* $\mathrm{L}^k(X,Y)$. *The space* $\mathrm{L}^k(X,Y)$*, endowed with this norm, is complete if Y is.*

Proof. The (elementary) proof can be found in [Die60, Chapter V, §7]. ■

LEMMA A.2.3. *Let X, Y be normed spaces and $k \in \mathbb{N}^*$. Then the evaluation map*

$$\mathrm{L}^k(X,Y) \times X^k : (\Xi, v_1,\dots,v_k) \mapsto \Xi(v_1,\dots,v_k)$$

is $(k+1)$-linear and continuous.

Proof. This is trivial. ■

LEMMA A.2.4. *Let X and Y be normed spaces, $k \in \mathbb{N}^*$, $\Xi \in \mathrm{L}^k(X,Y)$ and $h_1,\dots,h_k$, $v_1,\dots,v_k \in X$. Then*

$$\|\Xi(h_1,\dots,h_n) - \Xi(v_1,\dots,v_k)\| \le \sum_{i=1}^{k} \|\Xi(v_1,\dots,v_{i-1},h_i - v_i, h_{i+1},\dots,h_k)\|.$$

Proof. This estimate is derived by an iterated application of the triangle inequality. ■

The following lemma helps to deal with higher derivatives of Fréchet differentiable maps.

LEMMA A.2.5. *Let X, Y be normed spaces and $n,k \in \mathbb{N}^*$. Then the map*

$$\mathcal{E}_{k,n} : \mathrm{L}^k(X,\mathrm{L}^n(X,Y)) \to \mathrm{L}^{k+n}(X,Y),$$
$$\mathcal{E}_{k,n}(\Xi)(h_1,\dots,h_n,v_1,\dots,v_k) := \Xi(v_1,\dots,v_k)(h_1,\dots,h_n),$$

is an isometric isomorphism. In some cases, we will denote $\mathcal{E}_{k,n}$ by $\mathcal{E}^Y_{k,n}$.

Proof. Obviously $\mathcal{E}_{k,n}$ is linear and injective. Furthermore

$$\begin{aligned}\|\mathcal{E}_{k,n}(\Xi)(h_1,\dots,h_n,v_1,\dots,v_k)\| &= \|\Xi(v_1,\dots,v_k)(h_1,\dots,h_n)\| \\ &\le \|\Xi(v_1,\dots,v_k)\|_{\mathrm{op}} \prod_{i=1}^{n}\|h_i\| \le \|\Xi\|_{\mathrm{op}} \prod_{i=1}^{k}\|v_i\| \prod_{i=1}^{n}\|h_i\|,\end{aligned}$$

and hence

$$\|\mathcal{E}_{k,n}(\Xi)\|_{\mathrm{op}} \le \|\Xi\|_{\mathrm{op}}.$$

On the other hand, for $\|v_1\|, \dots, \|v_k\|, \|h_1\|, \dots, \|h_n\| \le 1$ we have

$$\|\Xi(v_1, \dots, v_k)(h_1, \dots, h_n)\| \le \|\mathcal{E}_{k,n}(\Xi)\|_{\mathrm{op}}.$$

Hence

$$\|\Xi(v_1, \dots, v_k)\|_{\mathrm{op}} \le \|\mathcal{E}_{k,n}(\Xi)\|_{\mathrm{op}},$$

which leads to

$$\|\Xi\|_{\mathrm{op}} \le \|\mathcal{E}_{k,n}(\Xi)\|_{\mathrm{op}},$$

so $\mathcal{E}_{k,n}$ is an isometry. It remains to show that $\mathcal{E}_{k,n}$ is surjective. To this end, for a $M \in \mathrm{L}^{k+n}(X, Y)$ we define the map $\overline{M} \in \mathrm{L}^k(X, \mathrm{L}^n(X, Y))$ by

$$\overline{M}(v_1, \dots, v_k)(h_1, \dots, h_n) := M(h_1, \dots, h_n, v_1, \dots, v_k).$$

Clearly, $\mathcal{E}_{k,n}(\overline{M}) = M$. Since M was arbitrary, $\mathcal{E}_{k,n}$ is surjective. ■

Lemma A.2.6. *Let X, Y and Z be normed spaces and $k \in \mathbb{N}$. Then the map*

$$\mathrm{L}^k(X, Y \times Z) \to \mathrm{L}^k(X, Y) \times \mathrm{L}^k(X, Z) : \Xi \mapsto (\pi_Y \circ \Xi, \pi_Z \circ \Xi), \tag{A.2.6.1}$$

where π_Y and π_Z denote the canonical projections from $Y \times Z$ to Y respectively Z, is an isomorphism of topological vector spaces.

Proof. The map in (A.2.6.1) is linear since its component $\Xi \mapsto \pi_Y \circ \Xi$ and $\Xi \mapsto \pi_Z \circ \Xi$ are. The injectivity of (A.2.6.1) is clear, and the surjectivity can also be shown by an easy computation.

To see that (A.2.6.1) is an isomorphism we denote it by $\mathfrak{i}$ and compute, for $x_1, \dots, x_k \in X$,

$$\big((\pi_{\mathrm{L}^k(X,Y)} \circ \mathfrak{i})(\Xi)(x_1, \dots, x_k), (\pi_{\mathrm{L}^k(X,Z)} \circ \mathfrak{i})(\Xi)(x_1, \dots, x_k)\big) \\ = \big((\pi_Y \circ \Xi)(x_1, \dots, x_k), (\pi_Z \circ \Xi)(x_1, \dots, x_k)\big) = \Xi(x_1, \dots, x_k).$$

From this one can easily derive that $\mathfrak{i}$ and its inverse are continuous since depending on the norm we chose on the products, $\mathfrak{i}$ is an isometry. ■

The calculus. In the following, let X, Y and Z denote normed spaces and U be an open nonempty subset of X. Recall the definition of Fréchet differentiability given in Definition 2.3.1.

We give some examples of Fréchet differentiable maps.

Example A.2.7.

(a) A continuous linear map $A : X \to Y$ is smooth with $DA(x) = A$.
(b) More generally, a continuous k-linear map $b : X_1 \times \cdots \times X_k \to Y$ is smooth with,

$$Db(x_1, \dots, x_k)(h_1, \dots, h_k) = \sum_{i=1}^{k} b(x_1, \dots, x_{i-1}, h_i, x_{i+1}, \dots, x_k).$$

We prove the chain rule and the mean value theorem for Fréchet differentiable maps. Beforehand, we need the following

Lemma A.2.8. *Let X, Y and Z be normed spaces, $U \subseteq X$ an open nonempty set, $k \in \overline{\mathbb{N}}$ and $A : Y \to Z$ a continuous linear map. Then for $\gamma \in \mathcal{FC}^k(U, Y)$,*

$$A \circ \gamma \in \mathcal{FC}^k(U, Z).$$

Proof. We prove this by induction over k. The assertion is obviously true for $k = 0$. If $k = 1$, then $A \circ \gamma$ is $\mathcal{C}^1$ by Proposition A.1.11 with

$$d(A \circ \gamma)(x; \cdot) = dA(\gamma(x); \cdot) \cdot d\gamma(x; \cdot) = A \circ d\gamma(x; \cdot).$$

Since the composition of linear maps is continuous, we conclude that $A \circ \gamma$ is $\mathcal{FC}^1$ with $D(A \circ \gamma) = A \circ D\gamma$.

$k \to k+1$: The map $D\gamma$ is $\mathcal{FC}^k$, hence by the induction hypothesis, so is $A \circ D\gamma = D(A \circ \gamma)$. Hence $A \circ \gamma$ is $\mathcal{FC}^{k+1}$. ∎

Lemma A.2.9. *Let $k \in \overline{\mathbb{N}}$, $\eta \in \mathcal{FC}^k(U, Y)$ and $\gamma \in \mathcal{FC}^k(U, Z)$. Then the map*

$$(\gamma, \eta) : U \to Y \times Z : x \mapsto (\gamma(x), \eta(x))$$

is contained in $\mathcal{FC}^k(U, Y \times Z)$.

Proof. For $k = 0$ the assertion is obviously true. If $k = 1$, we easily calculate that (γ, η) is $\mathcal{C}^1$ with

$$d(\gamma, \eta)(x; h) = (d\gamma(x; h), d\eta(x; h)).$$

Hence

$$d(\gamma, \eta)(x; \cdot) = \mathfrak{i}^{-1}(d\gamma(x; \cdot), d\eta(x; \cdot)),$$

where $\mathfrak{i}$ denotes the isomorphism (A.2.6.1) from Lemma A.2.6. We conclude that (γ, η) is $\mathcal{FC}^1$.

For $k > 1$, the assertion is proved by an easy induction using Lemma A.2.8. ∎

Proposition A.2.10 (Chain Rule). *Let $k \in \overline{\mathbb{N}}$, $\eta \in \mathcal{FC}^k(U, Y)$ and $\gamma \in \mathcal{FC}^k(V, Z)$ such that $\eta(U) \subseteq V$. Then $\gamma \circ \eta \in \mathcal{FC}^k(U, Z)$ and*

$$D(\gamma \circ \eta)(u) = (D\gamma \circ \eta)(u) \cdot D\eta(u) \tag{$*$}$$

for all $u \in U$.

Proof. The proof is by induction on k.

$k = 1$: We apply the chain rule for $\mathcal{C}^1$-maps (Proposition A.1.11) to see that $\gamma \circ \eta$ is $\mathcal{C}^1$, and for $(u, x) \in U \times X$ we have

$$d(\gamma \circ \eta)(u; x) = d\gamma(\eta(u); d\eta(u; x)).$$

From this identity we conclude that $(*)$ holds. Finally we obtain the continuity of $D(\gamma \circ \eta)$ from that of $\cdot$, $D\gamma$, $D\eta$ and η.

$k \to k+1$: By the inductive hypothesis, the maps $D\gamma$ and $D\eta$ are $\mathcal{FC}^k$. We already proved in the case $k = 1$ that $(*)$ holds. By the inductive hypothesis, $D\gamma \circ \eta \in \mathcal{FC}^k$. Since $\cdot$ is smooth (see Example A.2.7), we conclude using Lemma A.2.9 and the inductive hypothesis that $D(\gamma \circ \eta)$ is $\mathcal{FC}^k$. Hence $\gamma \circ \eta$ is $\mathcal{FC}^{k+1}$. ∎

Proposition A.2.11 (Mean Value Theorem). *Let $f \in \mathcal{FC}^1(U, Y)$. Then*

$$f(v) - f(u) = \int_0^1 Df(u + t(v - u)) \cdot (v - u)\, dt$$

for all $v, u \in U$ such that the line segment $\{tu + (1 - t)v : t \in [0, 1]\}$ is contained in U. In particular

$$\|f(v) - f(u)\| \leq \sup_{t \in [0,1]} \|Df(u + t(v - u))\|_{\mathrm{op}} \|v - u\|.$$

Proof. The identity is a reformulation of Proposition A.1.10, hence the estimate is a direct consequence of Lemma A.1.7. ■

The isomorphisms provided by Lemma A.2.5 can be used to characterize Fréchet differentiability of higher order.

REMARK A.2.12. We define inductively

$$L^0_{X,Y} := Y \quad \text{and} \quad L^{k+1}_{X,Y} := \mathrm{L}(X, L^k_{X,Y}).$$

DEFINITION A.2.13 (Higher derivatives). Let $n \in \mathbb{N}$. For each $k \in \mathbb{N}$ with $k \leq n$ we define a linear map

$$D^{(k)} : \mathcal{FC}^n(U,Y) \to \mathcal{FC}^{n-k}(U, \mathrm{L}^k(X,Y))$$

by $D^{(0)} := \mathrm{id}_{\mathcal{FC}^n(U,Y)}$ for $k = 0$, $D^{(1)} := D$ for $k = 1$ and for $1 < k \leq n$ by

$$D^{(k)}\gamma := \mathcal{E}^Y_{k-1,1} \circ \cdots \circ \mathcal{E}^{L^{k-3}_{X,Y}}_{2,1} \circ \mathcal{E}^{L^{k-2}_{X,Y}}_{1,1} \circ (\underbrace{D \circ \cdots \circ D}_{k \text{ times}})(\gamma).$$

Here we used the notation introduced in Remark A.2.12. Note that the image of $D^{(k)}$ is contained in $\mathcal{FC}^{n-k}(U, \mathrm{L}^k(X,Y))$ because $\mathcal{E}^Y_{k-1,1}, \ldots, \mathcal{E}^{L^{k-3}_{X,Y}}_{2,1}, \mathcal{E}^{L^{k-2}_{X,Y}}_{1,1}$ are continuous linear maps and hence smooth (see Example A.2.7); so the chain rule (Proposition A.2.10) gives the result.

We call $D^{(k)}$ the *kth derivative operator*.

The $(k+1)$st derivative of a map γ is closely related to the kth derivative of $D\gamma$:

LEMMA A.2.14. *Let $n \in \overline{\mathbb{N}}^*$, $\gamma \in \mathcal{FC}^n(U,Y)$ and $k \in \mathbb{N}$ with $k < n$. Then*

$$D^{(k+1)}\gamma = \mathcal{E}^Y_{k,1} \circ (D^{(k)}(D\gamma)).$$

Proof. This follows directly from the definition of $D^{(k+1)}\gamma$. ■

A.3. Relation between the differential calculi. We show that the two calculi presented are closely related. First we prove that each $\mathcal{FC}^k$-map is a $\mathcal{C}^k$-map and that the higher differentials are closely related.

LEMMA A.3.1. *Let $k \in \mathbb{N}^*$ and $\gamma \in \mathcal{FC}^k(U,Y)$. Then γ is a $\mathcal{C}^k$-map* (*in the sense of Section* A.1)*, and for each $x \in U$ we have*

$$D^{(k)}\gamma(x) = d^{(k)}\gamma(x;\cdot).$$

Proof. We prove this by induction.

$k = 1$: It follows directly from Definition 2.3.1 that γ is a $\mathcal{C}^1$ map and

$$D^{(1)}\gamma(x) = D\gamma(x) = d\gamma(x;\cdot) = d^{(1)}\gamma(x;\cdot).$$

$k \to k+1$: Let $x \in U$ and $h_1, \ldots, h_{k+1} \in X$. We know from Lemma A.2.14 that

$$\begin{aligned}(D^{(k+1)}\gamma)(x)(h_1,\ldots,h_{k+1}) &= (\mathcal{E}_{k,1} \circ (D^{(k)}D\gamma))(x)(h_1,\ldots,h_{k+1})\\ &= (D^{(k)}D\gamma(x)(h_2,\ldots,h_{k+1})) \cdot h_1.\end{aligned}$$

The inductive hypothesis shows this equals

$$(d^{(k)}D\gamma(x;h_2,\ldots,h_{k+1}))\cdot h_1 = \left(\lim_{t\to 0}\frac{d^{(k-1)}(D\gamma)(x+th_{k+1};h_2,\ldots,h_k)-d^{(k-1)}(D\gamma)(x;h_2,\ldots,h_k)}{t}\right)\cdot h_1.$$

Another application of the inductive hypothesis, together with the continuity of the evaluation of linear maps (Lemma A.2.3) and Lemma A.2.14, gives

$$\begin{aligned}
&= \lim_{t\to 0}\frac{D^{(k-1)}(D\gamma)(x+th_{k+1})(h_2,\ldots,h_k)\cdot h_1 - D^{(k-1)}(D\gamma)(x)(h_2,\ldots,h_k)\cdot h_1}{t}\\
&= \lim_{t\to 0}\frac{(\mathcal{E}_{k-1,1}\circ D^{(k-1)}(D\gamma))(x+th_{k+1})(h_1,\ldots,h_k) - (\mathcal{E}_{k-1,1}\circ D^{(k-1)}(D\gamma))(x)(h_1,\ldots,h_k)}{t}\\
&= \lim_{t\to 0}\frac{D^{(k)}\gamma(x+th_{k+1})(h_1,\ldots,h_k)-D^{(k)}\gamma(x)(h_1,\ldots,h_k)}{t}.
\end{aligned}$$

Another application of the inductive hypothesis finally gives

$$= \lim_{t\to 0}\frac{d^{(k)}\gamma(x+th_{k+1};h_1,\ldots,h_k)-d^{(k)}\gamma(x;h_1,\ldots,h_k)}{t}.$$

Hence $d^{(k+1)}\gamma$ exists and satisfies the identity

$$d^{(k+1)}\gamma(x;h_1,\ldots,h_{k+1}) = D^{(k+1)}\gamma(x)(h_1,\ldots,h_{k+1}).$$

Since $D^{(k+1)}\gamma$ and the evalution of multilinear maps are continuous (see Lemma A.2.3), so is $d^{(k+1)}\gamma$. Proposition 2.2.3 shows that this (and the inductive hypothesis) ensures that γ is a $\mathcal{C}^{k+1}$-map. ■

The preceding can be used to give a characterization of Fréchet differentiable maps.

Proposition A.3.2. *Let $\gamma : U \to Y$ be a continuous map. Then $\gamma \in \mathcal{FC}^k(U,Y)$ iff γ is a $\mathcal{C}^k$-map and the map*

$$U \to \mathrm{L}^\ell(X,Y) : x \mapsto d^{(\ell)}\gamma(x;\cdot) \qquad (*_k)$$

is continuous for each $\ell \in \mathbb{N}$ with $\ell \leq k$.

Proof. For $\gamma \in \mathcal{FC}^k(U,Y)$ we stated in Lemma A.3.1 that $\gamma \in \mathcal{C}^k(U,Y)$ and

$$d^{(\ell)}\gamma(x;\cdot) = D^{(\ell)}\gamma(x)$$

for each $x \in U$ and $\ell \in \mathbb{N}$ with $\ell \leq k$. Since $D^{(\ell)}\gamma$ is continuous by its definition (A.2.13), $(*_k)$ is satisfied.

We have to prove the other direction. This is done by induction on k.

$k = 1$: This follows directly from the definition of $\mathcal{FC}^1(U,Y)$.

$k \to k+1$: We have to show that $\gamma \in \mathcal{FC}^{k+1}(U,Y)$, and this is clearly the case if $D\gamma \in \mathcal{FC}^k(U,\mathrm{L}(X,Y))$. By the inductive hypothesis this is the case if $D\gamma \in \mathcal{C}^k(U,\mathrm{L}(X,Y))$ and it satisfies $(*_k)$. Since $\gamma \in \mathcal{FC}^k(U,Y)$ by the inductive hypothesis and hence $D\gamma \in \mathcal{FC}^{k-1}(U,\mathrm{L}(X,Y))$, we just have to show that $D\gamma$ is $\mathcal{C}^k$ and

$$U \to \mathrm{L}^k(X,\mathrm{L}(X,Y)) : x \mapsto d^{(k)}(D\gamma)(x;\cdot)$$

is continuous. To this end, let $x \in U$, $h, v_1, \ldots, v_{k-1}, v_k \in X$ and $t \in \mathbb{K}$ be such that $\{x + stv_k : s \in [0,1]\} \subseteq U$. We calculate using Lemma A.2.14, the mean value theorem and Lemma A.3.1:

$$\left(\frac{d^{(k-1)}(D\gamma)(x+tv_k; v_1, \ldots, v_{k-1}) - d^{(k-1)}(D\gamma)(x; v_1, \ldots, v_{k-1})}{t}\right) \cdot h$$

$$= \frac{d^{(k)}\gamma(x+tv_k; h, v_1, \ldots, v_{k-1}) - d^{(k)}\gamma(x; h, v_1, \ldots, v_{k-1})}{t}$$

$$= \int_0^1 d^{(k+1)}\gamma(x + stv_k; h, v_1, \ldots, v_{k-1}, v_k)\, ds.$$

Since $x \mapsto d^{(k+1)}\gamma(x; \cdot)$ is continuous by hypothesis, the left hand side converges as $t \to 0$ in the topology of uniform convergence on bounded sets to the linear map

$$h \mapsto d^{(k+1)}\gamma(x; h, v_1, \ldots, v_{k-1}, v_k).$$

Hence $D\gamma$ is $\mathcal{C}^k$ with

$$d^{(k)}(D\gamma)(x; v_1, \ldots, v_{k-1}, v_k) = \mathcal{E}_{k,1}^{-1}(d^{(k+1)}\gamma(x; \cdot))(v_1, \ldots, v_{k-1}, v_k),$$

and since $x \mapsto d^{(k+1)}\gamma(x; \cdot)$ and $\mathcal{E}_{k,1}^{-1}$ are continuous (by hypothesis resp. Lemma A.2.5), so is $x \mapsto d^{(k)}(D\gamma)(x; \cdot)$. ■

Lemma A.3.3. *Let $f : U \to Y$ be a $\mathcal{C}^{k+1}$ map. Then $f \in \mathcal{FC}^k(U, Y)$.*

Proof. We stated in Proposition A.3.2 that f is in $\mathcal{FC}^k(U, Y)$ iff for each $\ell \in \mathbb{N}$ with $\ell \leq k$ the map

$$U \to \mathrm{L}^\ell(X, Y) : x \mapsto d^{(\ell)} f(x; \cdot)$$

is continuous; but this is a direct consequence of Lemma A.1.31. ■

Lemma A.3.4. *Let X and Y be locally convex spaces, $U \subseteq X$ an open nonempty set, $k \in \mathbb{N}$, $\gamma \in \mathcal{C}^{k+1}(U, Y)$, $p \in \mathcal{N}(Y)$ and K a compact subset of U. Then there exists a seminorm $q \in \mathcal{N}(X)$ and an open set V with respect to q such that $K \subseteq V \subseteq U$ and $\tilde{\gamma} \in \mathcal{BC}^k(V_q, Y_p)$.* (For the definition of $\tilde{\gamma}$ see Lemma A.1.33.)

Proof. Using Lemma A.1.31 and standard compactness arguments, we find $q \in \mathcal{N}(X)$ and a neighborhood $\widetilde{V}$ with respect to q of K in U such that estimates (A.1.31.1) and (A.1.31.2) hold for γ on $\widetilde{V}$ and all $\ell \in \mathbb{N}$ with $\ell \leq k$. We proved in Lemma A.1.33 that this implies that $\tilde{\gamma} \in \mathcal{LC}^k_{q,p}(\widetilde{V}_q, Y_p)$, and using Proposition A.3.2 we can conclude that $\tilde{\gamma} \in \mathcal{FC}^k(\widetilde{V}_q, Y_p)$. Further, since $D^{(\ell)}\tilde{\gamma}(K_q)$ is compact for all $\ell \leq k$, there exists a neighborhood V_q of K_q such that $\tilde{\gamma}$ and all its derivatives up to degree k are bounded on V_q. ■

Differential calculus on finite-dimensional spaces. We show that the three definitions of differentiability for maps that are defined on a finite-dimensional space (Fréchet differentiability, Keller's C^k_c theory and continuous partial differentiability) are equivalent.

Definition A.3.5. Let $n, k \in \mathbb{N}^*$ and $\alpha \in \mathbb{N}_0^n$ a multiindex with $|\alpha| = k$. We set

$$I_\alpha := \{(i_1, \ldots, i_k) \in \{1, \ldots, n\}^k : (\forall \ell \in \{1, \ldots, n\})\ \alpha_\ell = |\{j : i_j = \ell\}|\}$$

and use this set to define the continuous k-linear map

$$S_\alpha : (\mathbb{K}^n)^k \to \mathbb{K} : (h_1, \dots, h_k) \mapsto \sum_{(i_1,\dots,i_k)\in I_\alpha} h_{1,i_1} \cdots h_{k,i_k},$$

where $h_j = (h_{j,1}, \dots, h_{j,n})$ for $j = 1, \dots, k$.

PROPOSITION A.3.6. *Let $U \subseteq \mathbb{K}^n$ be open and nonempty and $\gamma : U \to Y$ a map. Then the following conditions are equivalent:*

(a) $\gamma \in \mathcal{FC}^k(U, Y)$.
(b) $\gamma \in \mathcal{C}^k(U, Y)$.
(c) *γ is k-times continuously partially differentiable.*

If one of these conditions is satisfied, then

$$D^{(k)}\gamma(x)(h_1, \dots, h_k) = \sum_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|=k}} S_\alpha(h_1, \dots, h_k) \cdot \partial^\alpha \gamma(x) \tag{A.3.6.1}$$

for all $x \in U$ and $h_1, \dots, h_k \in \mathbb{K}^n$.

Proof. The assertion (a)⇒(b) is a consequence of Lemma A.3.1; and since

$$\frac{\partial^k \gamma}{\partial x_{i_1} \cdots \partial x_{i_k}}(x) = d^{(k)}\gamma(x; e_{i_k}, \dots, e_{i_1})$$

and $d^{(k)}\gamma$ is continuous (Proposition 2.2.3), the implication (b)⇒(c) also holds.

It remains to show that (c)⇒(a). It is well known from calculus that $D_h\gamma = \sum_{i=1}^n h_i \frac{\partial \gamma}{\partial x_i}$. Hence $d^{(\ell)}\gamma(x; h_1, \dots, h_\ell)$ exists and is given by

$$\begin{aligned} d^{(\ell)}\gamma(x; h_1, \dots, h_\ell) &= \sum_{i_1=1,\dots,i_\ell=1}^{n} h_{1,i_1} \cdots h_{\ell,i_\ell} \cdot \frac{\partial^k \gamma}{\partial x_{i_1} \cdots \partial x_{i_\ell}} \\ &= \sum_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|=\ell}} \Big(\sum_{(i_1,\dots,i_\ell)\in I_\alpha} h_{1,i_1} \cdots h_{\ell,i_\ell} \Big) \cdot \partial^\alpha\gamma(x) = \sum_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|=\ell}} S_\alpha(h_1, \dots, h_\ell) \cdot \partial^\alpha\gamma(x). \end{aligned}$$

From this identity we derive the continuity of $x \mapsto d^{(\ell)}\gamma(x; \cdot)$, and we can conclude using Proposition A.3.2 that $\gamma \in \mathcal{FC}^k(U, Y)$ and (A.3.6.1) is satisfied. ■

A.4. Some facts concerning ordinary differential equations. We state some facts about the global solvability of initial value problems and the dependence of solution on parameters.

A.4.1. Maximal solutions of ODEs. In the following, we let $J \subseteq \mathbb{R}$ be a nondegenerate interval and U an open subset of a Banach space X. For a continuous function $f : J \times U \to X$, $x_0 \in U$ and $t_0 \in J$ we consider the initial value problem

$$\gamma'(t) = f(t, \gamma(t)), \qquad \gamma(t_0) = x_0. \tag{A.4.0.2}$$

We state the famous theorem of Picard and Lindelöf:

THEOREM A.4.1. *Let f satisfy a local Lipschitz condition with respect to the second argument, that is, for each $(t_0, x_0) \in J \times U$ there exist a neighborhood W of (t_0, x_0) in*

$J \times U$ and a $K \in \mathbb{R}$ such that for all $(t, x), (t, \tilde{x}) \in W$,

$$\|f(t,x) - f(t,\tilde{x})\| \leq K\|x - \tilde{x}\|.$$

Then for each $(t_0, x_0) \in J \times U$ there exists a neighborhood I of t_0 in J such that the initial value problem (A.4.0.2) *corresponding to t_0 and x_0 has a unique solution that is defined on I.*

It is well-known that the local theorem of Picard and Lindelöf can be used to ensure that there exists a maximal solution.

Proposition A.4.2. *Let f satisfy a local Lipschitz condition with respect to the second argument and let $(t_0, x_0) \in J \times U$. Then there exists an interval $I \subseteq J$ and a function $\phi : I \to U$ that is a maximal solution to* (A.4.0.2)*; that is, if $\gamma : D(\gamma) \to U$ is a solution to* (A.4.0.2) *defined on a connected set, then $D(\gamma) \subseteq I$ and $\gamma = \phi|_{D(\gamma)}$.*

A.4.1.1. *A criterion of global solvability*

Linearly bounded vector fields. One class of ODEs that can be globally solved is that of linear vector fields. This solvability property can be generalized to linearly bounded vector fields.

Definition A.4.3. We call f *linearly bounded* if there exist continuous functions $a, b : J \to \mathbb{R}$ such that

$$\|f(t,x)\| \leq a(t)\|x\| + b(t) \quad \text{ for all } (t,x) \in J \times U.$$

To prove that this condition on f ensures globally defined solutions, we need some lemmas.

Lemma A.4.4. *Let f be a linearly bounded map that satisfies a local Lipschitz condition with respect to the second argument. Let $\phi : I \to U$ be an integral curve of f.*

(a) *If ϕ is bounded, $\overline{I} \subseteq J$ and $\overline{I}$ is compact, then f is bounded on the graph of ϕ.*
(b) *If $\beta := \sup I \neq \sup J$, then ϕ is bounded on $[t_0, \beta[$ for each $t_0 \in J$. The analogous result for $\inf I$ also holds.*

Proof. (a) Let $t \in I$. Then

$$\|f(t,\phi(t))\| \leq a(t)\|\phi(t)\| + b(t)$$

since f is linearly bounded. Because a and b are continuous and defined on $\overline{I}$, they are clearly bounded on I.

(b) For each $t \in [t_0, \beta[$ we have

$$\phi(t) = \phi(t_0) + \int_{t_0}^{t} f(s, \phi(s))\, ds,$$

and from this we deduce, using that f is linearly bounded,

$$\begin{aligned} \|\phi(t)\| &\leq \|\phi(t_0)\| + \left\| \int_{t_0}^{t} f(s,\phi(s))\, ds \right\| \leq \|\phi(t_0)\| + \left| \int_{t_0}^{t} a(s)\|\phi(s)\| + b(s)\, ds \right| \\ &\leq \|a|_{[t_0,\beta]}\|_\infty \left| \int_{t_0}^{t} \|\phi(s)\|\, ds \right| + \|\phi(t_0)\| + \|b\|_{\infty,[t_0,\beta]} |\beta - t_0|. \end{aligned}$$

The assertion is proved by an application of Gronwall's lemma. ∎

LEMMA A.4.5. *Assume that f satisfies a global Lipschitz condition with respect to the second argument. Then f is linearly bounded.*

Proof. Let $(t,x) \in J \times U$ and $x_0 \in U$. Then

$$\begin{aligned}\|f(t,x)\| &\leq \|f(t,x) - f(t,x_0)\| + \|f(t,x_0)\| \\ &\leq L\|x - x_0\| + \|f(t,x_0)\| \leq L\|x\| + L\|x_0\| + \|f(t,x_0)\|.\end{aligned}$$

Defining $a(t) := L$ and $b(t) := L\|x_0\| + \|f(t,x_0)\|$ gives the assertion. ∎

The criterion. We give a sufficient condition for an integral curve to be uniformly continuous. This can be used to extend solutions to larger domains of definition.

LEMMA A.4.6. *Let f satisfy a local Lipschitz condition with respect to the second argument and let $\phi : I \to U$ be an integral curve of f such that f is bounded on the graph of ϕ. Then ϕ is Lipschitz continuous and hence uniformly continuous.*

Proof. Let $t_1, t_2 \in I$. Then

$$\|\phi(t_2) - \phi(t_1)\| = \left\| \int_{t_1}^{t_2} \phi'(s)\, ds \right\| = \left\| \int_{t_1}^{t_2} f(s, \phi(s))\, ds \right\| \leq K|t_2 - t_1|,$$

where $K := \sup_{s \in I} \|f(s, \phi(s))\| < \infty$. ∎

THEOREM A.4.7. *Assume that f satisfies a local Lipschitz condition with respect to the second argument. Let $\phi : I \to U$ be a maximal integral curve of f. Assume further that*

(a) *the image of ϕ is contained in a compact subset of U or*
(b) *f is linearly bounded.*

Then ϕ is a global solution, that is, $I = J$.

Proof. Assume for contradiction that e.g. $\beta := \sup I \neq \sup J$. We choose $t_0 \in I$. In both cases, f is bounded on the graph of $\phi|_{[t_0,\beta[}$: If the image of ϕ is contained in a compact set, we easily see that the graph of $\phi|_{[t_0,\beta[}$ is contained in a compact subset. If f is linearly bounded, we use Lemma A.4.4.

We apply Lemma A.4.6 to see that $\phi|_{[t_0,\beta[}$ is uniformly continuous, and thus has a continuous extension $\widetilde{\phi}$ to $[t_0, \beta]$. We easily calculate that $\widetilde{\phi}$ is a solution to (A.4.0.2) using the integral represention of an ODE. Since $\widetilde{\phi}$ extends ϕ, we get a contradiction to the maximality of ϕ. ∎

A.4.2. Flows and dependence on parameters and initial values. For the sake of full generality, we need a definition.

DEFINITION A.4.8. Let X be a locally convex space. We call $P \subseteq X$ a *locally convex subset with dense interior* if for each $x \in P$, there exists a convex neighborhood $U \subseteq P$ of x and if $P \subseteq \overline{P^\circ}$.

In the following, we let $J \subseteq \mathbb{R}$ be a nondegenerate interval, U an open subset of a Banach space X, P a locally convex subset with dense interior of a locally convex space and $k \in \overline{\mathbb{N}}$ with $k \geq 1$. Further, let f be in $\mathcal{C}^k(J \times U \times P, X)$. We consider the initial value problem

$$\gamma'(t) = f(t, \gamma(t), p), \qquad \gamma(t_0) = x_0, \tag{A.4.8.1}$$

for $t_0 \in J$, $x_0 \in U$ and $p \in P$.

DEFINITION A.4.9. Let $\Omega \subseteq J \times J \times U \times P$. We call a map $\phi : \Omega \to U$ a *flow* for f if for all $t_0 \in J$, $x_0 \in U$ and $p \in P$ the set

$$\Omega_{t_0,x_0,p} := \{t \in J : (t_0, t, x_0, p) \in \Omega\}$$

is connected and the partial map

$$\phi(t_0, \cdot, x_0, p) : \Omega_{t_0,x_0,p} \to U$$

is a solution to (A.4.8.1) corresponding to the initial values t_0, x_0 and p.

A flow is called *maximal* if any other flow is a restriction of it.

REMARK A.4.10. In [Glö06, Theorem 10.3] it was shown that for each $t_0 \in J$, $x_0 \in U$ and $p_0 \in P$ there exist neighborhoods J_0 of t_0, U_0 of x_0 and P_0 of p_0 such that for every $s \in J_0$, $x \in U_0$ and $p \in P_0$ the corresponding initial value problem (A.4.8.1) has a unique solution $\Gamma_{s,x,p} : J_0 \to U$ and the map

$$\Gamma : J_0 \times J_0 \times U_0 \times P_0 \to U : (s, t, x, p) \mapsto \Gamma_{s,x,p}(t)$$

is $\mathcal{C}^k$. Therefore $\mathcal{C}^k$-flows exist.

The following lemma shows that two related flows can be glued together:

LEMMA A.4.11. *Let $I \subseteq J$ be a connected set with nonempty interior and $\gamma : I \to U$ a solution to* (A.4.8.1) *corresponding to $t_\gamma \in J$, $x_\gamma \in U$ and $p_\gamma \in P$. Further let*

$$\phi_0 : J_0 \times I_0 \times U_0 \times P_0 \to U \quad \text{and} \quad \phi_1 : I_1 \times I_1 \times U_1 \times P_1 \to U$$

be $\mathcal{C}^k$-flows for f such that U_1 is open in X and

$$I = I_0 \cup I_1,\ I_0 \cap I_1 \neq \emptyset,\ p_\gamma \in P_0 \cap P_1,\ (t_\gamma, x_\gamma) \in J_0 \times U_0 \text{ and } \gamma(I_1) \subseteq U_1.$$

Then there exist neighborhoods J_γ of t_γ, U_γ of x_γ, P_γ of p_γ and a $\mathcal{C}^k$-flow

$$\phi : J_\gamma \times I \times U_\gamma \times P_\gamma \to U$$

for f.

Proof. We choose $t_1 \in I_0 \cap I_1$. Since ϕ_0 is continuous in $(t_\gamma, t_1, x_\gamma, p_\gamma)$ and

$$\phi_0(t_\gamma, t_1, x_\gamma, p_\gamma) = \gamma(t_1) \in U_1,$$

there exist neighborhoods J_γ of t_γ in J_0, U_γ of x_γ in U_0 and $P_\gamma \subseteq P_0 \cap P_1$ of p_γ such that

$$\phi_0(J_\gamma \times \{t_1\} \times U_\gamma \times P_\gamma) \subseteq U_1.$$

Then the map

$$\phi : J_\gamma \times I \times U_\gamma \times P_\gamma \to U : (t_0, x_0, p, t) \mapsto \begin{cases} \phi_0(t_0, t, x_0, p) & \text{if } t \in I_0, \\ \phi_1(t_1, t, \phi_0(t_0, t_1, x_0, p), p) & \text{if } t \in I_1, \end{cases}$$

is well defined since the curves $\phi_0(t_0, \cdot, x_0, p)$ and $\phi_1(t_1, \cdot, \phi_0(t_0, t_1, x_0, p), p)$ are both solutions to the ODE (A.4.8.1) that coincide in t_1 and hence on $I_0 \cap I_1$. Since both ϕ_0 and ϕ_1 are $\mathcal{C}^k$-flows for f, so is ϕ. ∎

Lemma A.4.12. *Let $I \subseteq J$ be a connected set with nonempty interior, $t_1 \in I$ and $\gamma : I \to U$ a solution to* (A.4.8.1) *corresponding to $t_\gamma \in J$, $x_\gamma \in U$ and $p_\gamma \in P$. Then there exist neighborhoods J_γ of t_γ, U_γ of x_γ, P_γ of p_γ, an interval $\widetilde{I} \subseteq I$ with $t_\gamma, t_1 \in \widetilde{I}$ such that $\widetilde{I}$ is a neighborhood of t_1 in I, and a $\mathcal{C}^k$-flow*

$$\phi : J_\gamma \times \widetilde{I} \times U_\gamma \times P_\gamma \to U$$

for f.

Proof. We use [Glö06, Theorem 10.3] to see that for each $s \in I$ there exist neighborhoods J_s of s in J, U_s of $\gamma(s)$ in U, P_s of p_0 in P and a $\mathcal{C}^k$-flow

$$\phi_s : J_s \times J_s \times U_s \times P_s \to U$$

for f; we may assume that $\gamma(J_s) \subseteq U_s$ since γ is continuous and that J_s is open in I. Since I is connected and $\{J_s\}_{s\in I}$ is an open cover of I, there exist finitely many sets $J_{s_1}, \ldots, J_{s_n}$ such that $t_\gamma \in J_{s_1}$, $t_1 \in J_{s_n}$ and $J_{s_m} \cap J_{s_\ell} \neq \emptyset \Leftrightarrow |m - \ell| \leq 1$. Applying Lemma A.4.11 to ϕ_{s_1} and ϕ_{s_2} we find neighborhoods I_1 of t_γ, V_1 of x_γ, P_1 of p_γ and a $\mathcal{C}^k$-flow

$$\phi_1 : I_1 \times (J_{s_1} \cup J_{s_2}) \times V_1 \times P_1 \to U$$

for f. Likewise, ϕ_1 and ϕ_{s_3} lead to ϕ_2, and iterating the argument, we find a $\mathcal{C}^k$-flow

$$\phi_{n-1} : I_{n-1} \times \bigcup_{k=1}^{n} J_{s_k} \times V_{n-1} \times P_{n-1} \to U$$

for f. ∎

Concerning maximal flows, we can state the following

Theorem A.4.13. *For each ODE* (A.4.8.1) *there exists a maximal flow*

$$\phi : J \times J \times U \times P \supseteq \Omega \to U.$$

Ω is an open subset of $J \times J \times U \times P$ and ϕ is a $\mathcal{C}^k$-map.

Proof. The existence of a maximal flow is a direct consequence of the existence of maximal solutions to ODEs without parameters (see Proposition A.4.2). Now let $(t_0, t, x_0, p) \in \Omega$ and $\gamma : I \subseteq J \to U$ the maximal solution corresponding to t_0, x_0 and p. Then $t_0, t \in I$, and according to Lemma A.4.12, there exists a $\mathcal{C}^k$-flow

$$\Gamma : J_\gamma \times \widetilde{I} \times U_\gamma \times P_\gamma \to U$$

for f that is defined on a neighborhood of (t_0, t, x_0, p). Since ϕ is maximal,

$$J_\gamma \times \widetilde{I} \times U_\gamma \times P_\gamma \subseteq \Omega \quad \text{and} \quad \phi|_{J_\gamma \times \widetilde{I} \times U_\gamma \times P_\gamma} = \Gamma.$$

This gives the assertion. ∎

We examine the situation where the initial time is fixed and the initial values depend on the parameters.

Corollary A.4.14. *Let $\alpha : P \to U$ be a $\mathcal{C}^k$-map. Further, let $I \subseteq J$ be a nonempty interval and $t_0 \in I$ such that for every $p \in P$ there exists a solution $\gamma_p : I \to U$ to the* *initial value problem* (A.4.8.1) *corresponding to* p, t_0 *and the initial value* $\alpha(p)$. *Then the map*

$$\Gamma : I \times P \to U : (t,p) \mapsto \gamma_p(t)$$

is $\mathcal{C}^k$.

Proof. We consider a maximal flow $\phi : \Omega \to U$ for f. Since ϕ is maximal,

$$\{t_0\} \times I \times \{(\alpha(p),p) : p \in P\} \subseteq \Omega,$$

and for each $p \in P$,

$$\phi(t_0, \cdot, \alpha(p), p) = \gamma_p.$$

Hence Γ is the composition of ϕ and the $\mathcal{C}^k$-map

$$I \times P \to J \times I \times U \times P : (t,p) \mapsto (t_0, t, \alpha(p), p),$$

and this gives the assertion. ∎

B. Locally convex Lie groups

The goal of this appendix is mainly to fix our conventions and notation concerning manifolds and Lie groups modelled on locally convex spaces. For further information see [Mil84], [Nee06] and [BGN04].

B.1. Locally convex manifolds. Locally convex manifolds are essentially like finite-dimensional ones, replacing the finite-dimensional modelling space by a locally convex space.

Definition B.1.1 (Locally convex manifolds). Let M be a Hausdorff topological space, $k \in \overline{\mathbb{N}}$ and X a locally convex space. A $\mathcal{C}^k$*-atlas* for M is a set $\mathcal{A}$ of homeomorphisms $\phi : U \to V$ from an open subset $U \subseteq M$ onto an open set $V \subseteq X$ whose domains cover M and which are $\mathcal{C}^k$-compatible in the sense that $\phi \circ \psi^{-1}$ is $\mathcal{C}^k$ for all $\phi, \psi \in \mathcal{A}$. A maximal $\mathcal{C}^k$-atlas $\mathcal{A}$ on M is called a *differentiable structure* of class $\mathcal{C}^k$. In this case, the pair $(M, \mathcal{A})$ is called a (*locally convex*) $\mathcal{C}^k$*-manifold modelled on* X.

Direct products of locally convex $\mathcal{C}^k$-manifolds are defined as expected.

Definition B.1.2 (Tangent space and tangent bundle). Let $(M, \mathcal{A})$ be a $\mathcal{C}^k$-manifold modelled on X, where $k \geq 1$. Given $x \in M$, let $\mathcal{A}_x$ be the set of all charts around x (i.e. whose domain contains x). A *tangent vector* of M at x is a family $y = (y_\phi)_{\phi \in \mathcal{A}_x}$ of vectors $y_\phi \in X$ such that $y_\psi = d(\psi \circ \phi^{-1})(\phi(x); y_\phi)$ for all $\phi, \psi \in \mathcal{A}_x$.

The *tangent space* of M at x is the set $\mathbf{T}_x M$ of all tangent vectors of M at x. It has a unique structure of locally convex space such that the map $d\psi|_{\mathbf{T}_x M} : \mathbf{T}_x M \to X : (y_\phi)_{\phi \in \mathcal{A}_x} \mapsto y_\psi$ is an isomorphism of topological vector spaces for any $\psi \in \mathcal{A}_x$.

The *tangent bundle* $\mathbf{T}M$ of M is the union of the (disjoint) tangent spaces $\mathbf{T}_x M$ for all $x \in M$. It admits a unique structure of a $\mathcal{C}^{k-1}$-manifold modelled on $X \times X$ such that $\mathbf{T}\phi := (\phi, d\phi)$ is chart for each $\phi \in \mathcal{A}$. We let $\pi_M : \mathbf{T}M \to M$ be the map taking tangent vectors at x to x for any $x \in M$.

Definition B.1.3. A continuous map $f : M \to N$ between $\mathcal{C}^k$-manifolds is called $\mathcal{C}^k$ if the map $\psi \circ f \circ \phi^{-1}$ is $\mathcal{C}^k$ for all charts ψ of N and ϕ of M.

If $k \geq 1$, then we define the *tangent map* of f as the $\mathcal{C}^{k-1}$-map $\mathbf{T}f : \mathbf{T}M \to \mathbf{T}N$ determined by $d\psi \circ \mathbf{T}f \circ (\mathbf{T}\phi)^{-1} = d(\psi \circ f \circ \phi^{-1})$ for all charts ψ of N and ϕ of M.

Given $x \in M$, we define $\mathbf{T}_x f := \mathbf{T}f|_{\mathbf{T}_x M} : \mathbf{T}_x M \to \mathbf{T}_{f(x)}N$.

Definition B.1.4. Let $k > 0$, M, N and P be $\mathcal{C}^k$-manifolds, and $f : M \times N \to P$ a $\mathcal{C}^k$-map. We define

$$\mathbf{T}_1 f : \mathbf{T}M \times N \to \mathbf{T}P : (v, n) \mapsto \mathbf{T}\Gamma(v, 0_n)$$

and

$$\mathbf{T}_2 f : M \times \mathbf{T}N \to \mathbf{T}P : (m, v) \mapsto \mathbf{T}\Gamma(0_m, v).$$

Definition B.1.5 (Submanifolds). Let M be a $\mathcal{C}^k$-manifold modelled on the locally convex space X and $Y \subseteq X$ be a sequentially closed vector subspace. A *submanifold of M modelled on Y* is a subset $N \subseteq M$ such that for each $x \in N$, there exists a chart $\phi : U \to V$ around x such that $\phi(U \cap N) = V \cap Y$. It is easy to see that a submanifold is also a $\mathcal{C}^k$-manifold.

The following lemma states that submanifolds are initial:

Lemma B.1.6. *Let M be a $\mathcal{C}^k$-manifold and N a submanifold of M. Then the inclusion $\iota : N \to M$ is $\mathcal{C}^k$. Moreover, a map $f : P \to N$ from a $\mathcal{C}^k$-manifold is $\mathcal{C}^k$ iff the map $\iota \circ f : P \to M$ is $\mathcal{C}^k$.*

Definition B.1.7 (Vector fields). A *vector field* on a smooth manifold M is a smooth map $\xi : M \to \mathbf{T}M$ such that $\pi_M \circ \xi = \mathrm{id}_M$. We denote the set of vector fields on M by $\mathfrak{X}(M)$.

A vector field ξ is determined by its local representations $\xi_\phi := d\phi \circ \xi \circ \phi^{-1} : V \to X$ for each chart $\phi : U \to V$ of M. Given vector fields ξ and η on M, there is a unique vector field $[\xi, \eta]$ on M such that $[\xi, \eta]_\phi = d\eta_\phi \circ (\mathrm{id}_V, \xi_\phi) - d\xi_\phi \circ (\mathrm{id}_V, \eta_\phi)$ for all charts $\phi : U \to V$ of M.

Remark B.1.8 (Analytic manifolds). The definition of analytic manifolds and analytic maps between them is literally the same as above, except that "$\mathcal{C}^k$" has to be replaced by "analytic".

B.2. Lie groups

Definition B.2.1 (Lie groups). A (locally convex) *Lie group* is a group G equipped with a smooth manifold structure turning the group operations into smooth maps.

An analytic Lie group is a group G equipped with an analytic manifold structure turning the group operations into analytic maps.

Lemma B.2.2 (Tangent group, action of G group on $\mathbf{T}G$). *Let G be a Lie group with the group multiplication m and the inversion i. Then $\mathbf{T}G$ is a Lie group with the group multiplication*

$$\mathbf{T}m : \mathbf{T}(G \times G) \cong \mathbf{T}G \times \mathbf{T}G \to \mathbf{T}G$$

and the inversion $\mathbf{T}i$*. Identifying* G *with the zero section of* $\mathbf{T}G$*, we obtain a smooth right action*

$$\mathbf{T}G \times G \to \mathbf{T}G : (v, g) \mapsto v.g := \mathbf{T}m(v, 0_g)$$

and a smooth left action

$$G \times \mathbf{T}G \to \mathbf{T}G : (g, v) \mapsto g.v := \mathbf{T}m(0_g, v).$$

Definition B.2.3 (Left invariant vector fields). A vector field V on a Lie group G is called *left invariant* if $g.V(h) = V(gh)$ for all $g, h \in G$. The set $\mathfrak{X}(G)_\ell$ of left invariant vector fields is a Lie algebra under the bracket of vector fields defined above.

Definition B.2.4 (Lie algebra functor). Let G and H be Lie groups. Using the isomorphism $\mathfrak{X}(G)_\ell \to \mathbf{T_1}G : V \mapsto V(\mathbf{1})$ we transport the Lie algebra structure on $\mathfrak{X}(G)_\ell$ to $\mathbf{L}(G) := \mathbf{T_1}G$. If $\phi : G \to H$ is a smooth homomorphism, then the map $\mathbf{L}(\phi) : \mathbf{L}(G) \to \mathbf{L}(H)$ defined as $\mathbf{T}\phi|_{\mathbf{L}(G)}$ is a Lie algebra homomorphism.

B.2.1. Generation of Lie groups. We need the following result concerning the construction of Lie groups from local data (compare [Bou89, Chapter III, §1.9, Proposition 18] for the case of Banach Lie groups; the general proof follows the same pattern).

Lemma B.2.5 (Local description of Lie groups). *Let* G *be a group,* $U \subseteq G$ *a subset which is equipped with a smooth manifold structure, and* $V \subseteq U$ *an open symmetric subset such that* $\mathbf{1} \in V$ *and* $V \cdot V \subseteq U$*. Consider the conditions:*

(a) *The group inversion restricts to a smooth self-map of* V*.*
(b) *The group multiplication restricts to a smooth map* $V \times V \to U$*.*
(c) *For each* $g \in G$*, there exists an open* $\mathbf{1}$*-neighborhood* $W \subseteq U$ *such that* $g \cdot W \cdot g^{-1} \subseteq U$*, and the map*

$$W \to U : w \mapsto g \cdot w \cdot g^{-1}$$

is smooth.

If (a)–(c) *hold, then there exists a unique smooth manifold structure on* G *which makes* G *a Lie group such that* V *is an open submanifold of* G*. If* (a) *and* (b) *hold, then there exists a unique smooth manifold structure on* $\langle V \rangle$ *which makes* $\langle V \rangle$ *a Lie group such that* V *is an open submanifold of* $\langle V \rangle$*.*

B.2.2. Regularity. We recall the notion of regularity (see [Mil84] for further information). To this end, we define left evolutions of smooth curves. As a tool, we use the group multiplication on the tangent bundle $\mathbf{T}G$ of a Lie group G.

Definition B.2.6 (Left logarithmic derivative). Let G be a Lie group, $k \in \mathbb{N}$ and $\eta : [0, 1] \to G$ a $\mathcal{C}^{k+1}$-curve. We define the *left logarithmic derivative* of η as

$$\delta_\ell(\eta) : [0, 1] \to \mathbf{L}(G) : t \mapsto \eta(t)^{-1} \cdot \eta'(t).$$

The curve $\delta_\ell(\eta)$ is obviously $\mathcal{C}^k$.

Definition B.2.7 (Left evolutions). Let G be a Lie group and $\gamma : [0, 1] \to \mathbf{L}(G)$ a smooth curve. A smooth curve $\eta : [0, 1] \to G$ is called a *left evolution* of γ and denoted by $\mathrm{Evol}_G^\ell(\gamma)$ if $\delta_\ell(\eta) = \gamma$ and $\eta(0) = \mathbf{1}$. One can show that if a left evolution exists, it is uniquely determined.

The existence of a left evolution is equivalent to the existence of a solution to a certain initial value problem:

LEMMA B.2.8. *Let G be a Lie group and $\gamma : [0,1] \to \mathbf{L}(G)$ a smooth curve. Then there exists a left evolution* $\mathrm{Evol}^{\ell}(G)\gamma : [0,1] \to G$ *iff the initial value problem*

$$\eta'(t) = \eta(t) \cdot \gamma(t), \qquad \eta(0) = \mathbf{1}, \tag{B.2.8.1}$$

has a solution η. In this case, $\eta = \mathrm{Evol}^{\ell}_G(\gamma)$.

Now we give the definition of regularity:

DEFINITION B.2.9 (Regularity). A Lie group G is called *regular* if for each smooth curve $\gamma : [0,1] \to \mathbf{L}(G)$ there exists a left evolution and the map

$$\mathrm{evol}^{\ell}_G : \mathcal{C}^{\infty}([0,1], \mathbf{L}(G)) \to G : \gamma \mapsto \mathrm{Evol}^{\ell}_G(\gamma)(1)$$

is smooth.

LEMMA B.2.10. *Let G be a Lie group. Suppose there exists a zero neighborhood $\Omega \subseteq \mathcal{C}^{\infty}([0,1], \mathbf{L}(G))$ such that for each $\gamma \in \Omega$ the left evolution $\mathrm{Evol}^{\ell}_G(\gamma)$ exists and the map*

$$\Omega \to G : \gamma \mapsto \mathrm{Evol}^{\ell}_G(\gamma)(1)$$

is smooth. Then G is regular.

REMARK B.2.11. We can define *right logarithmic derivatives* and *right evolutions* in the analogous way. We denote the right logarithmic derivative by δ_{ρ}, the right evolution map by Evol^{ρ} and the endpoint of the right evolution by evol^{ρ}. One can show that a Lie group is left-regular iff it is right-regular. Also the equivalent of Lemma B.2.10 holds. In particular, the initial value problem (B.2.8.1) becomes

$$\eta'(t) = \gamma(t) \cdot \eta(t), \qquad \eta(0) = \mathbf{1}. \tag{B.2.11.1}$$

DEFINITION B.2.12. Let G be a Lie group. A smooth map $\exp_G : \mathbf{L}(G) \to G$ is called an *exponential map* for G if $\mathbf{T}_0 \exp_G = \mathrm{id}_{\mathbf{L}(G)}$ and $\exp_G((s+t)v) = \exp_G(sv) \cdot \exp_G(tv)$ for all $s, t \in \mathbb{R}$ and $v \in \mathbf{L}(G)$.

B.2.3. Group actions

LEMMA B.2.13. *Let G and H be groups and $\alpha : G \times H \to H$ a group action that is a group morphism in its second argument. Further, let $\widetilde{H}$ be a subgroup of H generated by U. Then*

$$\alpha(G \times \widetilde{H}) \subseteq \widetilde{H} \ \Leftrightarrow\ \alpha(G \times U) \subseteq \widetilde{H}.$$

Proof. By our assumption, $\widetilde{H} = \bigcup_{n \in \mathbb{N}} (U \cup U^{-1})^n$. So we calculate

$$\begin{aligned}
\alpha(G \times \widetilde{H}) &= \alpha(G \times \bigcup_{n\in\mathbb{N}} (U \cup U^{-1})^n) = \bigcup_{n\in\mathbb{N}} \alpha(G \times (U \cup U^{-1})^n) \\
&= \bigcup_{n\in\mathbb{N}} \alpha(G \times (U \cup U^{-1}))^n = \bigcup_{n\in\mathbb{N}} (\alpha(G \times U) \cup \alpha(G \times U)^{-1})^n \subseteq \widetilde{H}. \ \blacksquare
\end{aligned}$$

LEMMA B.2.14. *Let G and H be Lie groups and $\alpha : G \times H \to H$ a group action that is a group morphism in its second argument. Then α is smooth iff the following assertions hold:*

(a) *It is smooth on* $U \times V$*, where* U *and* V *are open neighborhoods of the respective units.*
(b) *For each* $h \in H$*, there exists an open unit neighborhood* W *such that the map* $\alpha(\cdot, h) : W \to H$ *is smooth.*
(c) *For each* $g \in G$ *the map* $\alpha(g, \cdot) : H \to H$ *is smooth.*

If U *generates* G*,* (b) *follows from* (a)*. If* V *generates* H*,* (c) *follows from* (a)*.*

Proof. We first show that by our assumptions, α is smooth. To this end, let $(g, h) \in G \times H$. Choose W as in (b). Then $U' := U \cap W \in \mathcal{U}_G(\mathbf{1})$. We show that $\alpha|_{gU' \times Vh}$ is smooth. Since the map $U' \times V \to gU' \times Vh : (u, v) \mapsto (gu, vh)$ is a smooth diffeomorphism, we only need to show that the map

$$U' \times V \to H : (u, v) \mapsto \alpha(gu, hv)$$

is smooth. But

$$\alpha(gu, hv) = \alpha_g(\alpha(u, vh)) = \alpha_g(\alpha(u, v)\alpha(u, h)) = \alpha_g(\alpha(u, v)\alpha^h(u)),$$

where we denote $\alpha(\cdot, h)$ by α^h and $\alpha(g, \cdot)$ by α_g. Since the right hand side is obviously smooth, we are done.

Now we prove the other two assertions. We suppose that (a) holds. We let $S \subseteq H$ be the set of all $h \in H$ such that (b) holds. Then $V \subseteq S$; and since $\alpha^{h^{-1}}(g) = \alpha^h(g)^{-1}$ and $\alpha^{hh'}(g) = \alpha^h(g)\alpha^{h'}(g)$ for all $g \in G$ and $h, h' \in H$, we easily see that S is a subgroup of H. Since V is a generator, $S = H$.

Since U generates G, for each $g \in G$ we find $g_1, \ldots, g_n \in U \cup U^{-1}$ such that

$$\alpha_g = \alpha_{g_n} \circ \cdots \circ \alpha_{g_1}.$$

Further, for $g' \in G$ and $h \in H$, $\alpha_{g'^{-1}}(h) = \alpha_{g'}(h)^{-1}$, so each α_{g_k} is smooth by our assumption. Hence α_g is smooth. ■

Lemma B.2.15. *Let* G *and* H *be Lie groups and* $\omega : G \times H \to H$ *a smooth group action that is a group morphism in its second argument. Then the semidirect product* $H \rtimes_\omega G$ *can be turned into a Lie group that is modelled on* $\mathbf{L}(H) \times \mathbf{L}(G)$*.*

Proof. The semidirect product $H \rtimes_\omega G$ is endowed with the multiplication

$$(H \times G) \times (H \times G) \to H \times G : ((h_1, g_1), (h_2, g_2)) \mapsto (h_1 \cdot \omega(g_1, h_2), g_1 \cdot g_2)$$

and the inversion

$$H \times G \to H \times G : (h, g) \mapsto (\omega(g^{-1}, h^{-1}), g^{-1}),$$

so the smoothness of the group operations follows from that of ω. ■

C. Quasi-inversion in algebras

We give a short introduction to the concept of *quasi-inversion*. It is a useful tool for the treatment of algebras without a unit, where it serves as a replacement for the ordinary inversion. Many of the algebras we treat are without a unit. Unless the contrary is stated, all algebras are assumed associative.

C.1. Definition

Definition C.1.1 (Quasi-inversion). Let A denote a $\mathbb{K}$-algebra with the multiplication $*$. An $x \in A$ is called *quasi-invertible* if there exists a $y \in A$ such that

$$x + y - x * y = y + x - y * x = 0.$$

In this case, we call $QI_A(x) := y$ the *quasi-inverse* of x. The set of all quasi-invertible elements of A is denoted by A^q. The map $A^q \to A^q : x \mapsto QI_A(x)$ is called the *quasi-inversion* of A. Often we will denote QI_A just by QI.

An interesting characterization of quasi-inversion is

Lemma C.1.2. *Let A be a $\mathbb{K}$-algebra with multiplication $*$. Then A, endowed with the operation*

$$A \times A \to A : (x, y) \mapsto x \diamond y := x + y - x * y,$$

is a monoid with the unit 0 *and the unit group A^q. The inversion map is given by QI_A.*

Proof. This is shown by an easy computation. ∎

In unital algebras there is a close relationship between inversion and quasi-inversion.

Lemma C.1.3. *Let A be an algebra with multiplication $*$ and unit e. Then $x \in A$ is quasi-invertible iff $x - e$ is invertible. In this case*

$$QI_A(x) = (x - e)^{-1} + e.$$

Proof. One easily computes that

$$(A, \diamond) \to (A, *) : x \mapsto e - x$$

is an isomorphism of monoids ($\diamond$ was introduced in Lemma C.1.2), and from this we easily deduce the assertion. ∎

C.2. Topological monoids and algebras with continuous quasi-inversion. In this section, we examine algebras that are endowed with a topology. For technical reasons we also examine monoids.

Definition C.2.1. An algebra A is called a *topological algebra* if it is a topological vector space and the multiplication is continuous.

A topological algebra A is called an *algebra with continuous quasi-inversion* if the set A^q is open and the quasi-inversion QI is continuous.

A monoid, endowed with a topology, is called a *topological monoid* if the monoid multiplication is continuous.

A monoid, endowed with a differential structure, is called a *smooth monoid* if the monoid multiplication is smooth.

Remark C.2.2. If A is an algebra with continuous quasi-inversion, then QI is not only continuous, but automatically analytic (see [Glö02a]).

In topological monoids the unit group is open and the inversion continuous if they are so near the unit element:

Lemma C.2.3. *Let M be a topological monoid with unit e and multiplication $*$. Then the unit group $M^\times$ is open iff there exists a neighborhood of e that consists of invertible elements. The inversion map*

$$I : M^\times \to M^\times : x \mapsto x^{-1}$$

is continuous iff it is so at e.

Proof. Let U be a neighborhood of e that consists of invertible elements and $m \in M^\times$. Since the map

$$\ell_m : M \to M : x \mapsto m * x$$

is a homeomorphism, $\ell_m(U)$ is open; and it is clear that $\ell_m(U) \subseteq M^\times$. Hence $M^\times = \bigcup_{m \in M^\times} \ell_m(U)$ is open.

Let I be continuous at e. We show it is so at $x \in M^\times$. For $m \in M^\times$, we have

$$I(m) = m^{-1} = m^{-1} * x * x^{-1} = (x^{-1} * m)^{-1} * x^{-1} = (\rho_{x^{-1}} \circ I \circ \ell_{x^{-1}})(m), \qquad (\dagger)$$

where $\rho_{x^{-1}}$ denotes right multiplication by x^{-1}. Since I is continuous in e and $\ell_{x^{-1}}(x) = e$, we can derive the continuity of I at x from ($\dagger$). ∎

For algebras with continuous multiplication we can deduce

Lemma C.2.4. *Let A be an algebra with continuous multiplication $*$. Then A^q is open if there exists a neighborhood of 0 that consists of invertible elements. The quasi-inversion QI_A is continuous if it is so at 0.*

Proof. Since the map

$$A \times A \to A : (x, y) \mapsto x + y - x * y$$

is continuous, we derive the assertions from Lemmas C.1.2 and C.2.3. ∎

A criterion for quasi-invertibility. We give a criterion that ensures that an element of an algebra is quasi-invertible. It turns out to be quite useful in Banach algebras.

Lemma C.2.5. *Let A be a topological algebra and $x \in A$. If $\sum_{i=1}^\infty x^i$ exists, then x is quasi-invertible with*

$$QI_A(x) = -\sum_{i=1}^\infty x^i.$$

Proof. We just compute that x is quasi-invertible:

$$x + \Big(-\sum_{i=1}^\infty x^i\Big) - x * \Big(-\sum_{i=1}^\infty x^i\Big) = -\sum_{i=2}^\infty x^i + \sum_{i=2}^\infty x^i = 0.$$

The identity $(-\sum_{i=1}^\infty x^i) + x - (-\sum_{i=1}^\infty x^i) * x = 0$ is computed in the same way. So the quasi-invertibility of x follows directly from the definition. ∎

Quasi-inversion in Banach algebras

Lemma C.2.6. *Let A be a Banach algebra. Then $B_1(0) \subseteq A^q$. Moreover, for $x \in B_1(0)$,*

$$QI_A(x) = -\sum_{i=1}^\infty x^i.$$

Proof. For $x \in B_1(0)$ the series $\sum_{i=1}^{\infty} x^i$ exists since it is absolutely convergent and A is complete. So the assertion follows from Lemma C.2.5. ■

LEMMA C.2.7. *Let A be a Banach algebra. Then A^q is open in A and the quasi-inversion QI_A is continuous.*

Proof. This is an immediate consequence of Lemmas C.2.6 and C.2.4 since

$$x \mapsto \sum_{i=1}^{\infty} x^i$$

is analytic (see [Bou67, §3.2.9]) and hence continuous. ■

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Notation

The following list contains the symbols that are used on several occasions, together with a short explanation of their meaning and the page number where the respective symbol is defined. For better overview, the entries are arranged into several categories.

Basic notation

$B_X(x,r)$, $B_r(x)$	Open ball with radius r around x in X	8
$\overline{B}_r(x)$	Closed ball with radius r around x	8
$\mathcal{C}^k(U,Y)$	The set of all k times differentiable functions from U to Y	9
$\mathcal{FC}^k(U,Y)$	The set of all k times Fréchet differentiable functions from U to Y	9
$d^{(k)}f(u;x_1,\dots,x_k)$	kth iterated derivative of f at u in the directions $x_1,\dots,x_k$	9
$D^{(k)}\gamma$	kth Fréchet derivative of γ	9
$\mathbb{D}$	The closed unit disk in $\mathbb{R}$ or $\mathbb{C}$	8
$\operatorname{dist}(A,B)$	Distance between A and B	8
$\mathbb{K}$	$\mathbb{R}$ or $\mathbb{C}$	8
$\overline{\mathbb{N}}$	$\mathbb{N}\cup\{\infty\}=\{\infty,0,1,\dots\}$	8
$\mathbb{N}^*$	$\mathbb{N}\setminus\{0\}$	8
$\overline{\mathbb{R}}$	$\mathbb{R}\cup\{-\infty,\infty\}$	8

Spaces of weighted functions

$\mathcal{BC}^k(U,Y)$	k-times differentiable functions from U to Y with bounded derivatives	10
$\mathcal{BC}^{\partial,k}(U,V)$	functions $\gamma\in\mathcal{BC}^k(U,V)$ such that $\operatorname{dist}(\gamma(U),\partial V)>0$	11
$\mathcal{BC}^k(U,Y)_0$	functions in $\mathcal{BC}^k(U,Y)$ mapping 0 to 0	11
$\mathcal{D}(U,V)$, $\mathcal{C}^\infty_c(U,V)$	compactly supported smooth functions from U to V	11
$\mathcal{C}^k_{\mathcal{W}}(U,Y)$	k-times differentiable functions from U to Y with $\mathcal{W}$-bounded derivatives	10, 30
$\mathcal{C}^k_{\mathcal{W}}(U,V)$	functions in $\mathcal{C}^k_{\mathcal{W}}(U,Y)$ with image in V	11
$\mathcal{C}^{\partial,k}_{\mathcal{W}}(U,V)$	functions $\gamma\in\mathcal{C}^k_{\mathcal{W}}(U,V)$ such that $\operatorname{dist}(\gamma(U),\partial V)>0$	11
$\mathcal{C}^k_{\mathcal{W}}(U,Y)^o$	functions in $\mathcal{C}^k_{\mathcal{W}}(U,Y)$ whose seminorms decay outside of bounded sets	11
$\mathcal{C}^k_{\mathcal{W}}(U,V)^\bullet$	Functions in $\mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet$ with values in V	34
$\mathcal{C}^k_{\mathcal{W}}(U,Y)^\bullet$	Functions in $\mathcal{C}^k_{\mathcal{W}}(U,Y)$ whose seminorms decay outside of compact sets	33

Lie groups and manifolds

$\operatorname{Evol}^\ell_G$	Left evolution	118
$\operatorname{evol}^\ell_G$	Endpoint of the left evolution	119
$\operatorname{Evol}^\rho_G$	Right evolution	119
$\operatorname{evol}^\rho_G$	Endpoint of the right evolution	119
$\exp_G$	Exponential function of the Lie group G	119

$\dot{\omega}$	For a group action ω, a "derivation" at the unital element	63
$\mathbf{L}(\cdot)$	Lie algebra functor	118
$\delta_\ell(\cdot)$	Left logarithmic derivative	118
$\delta_\rho(\cdot)$	Right logarithmic derivative	119
$\mathbf{T}M$	Tangent bundle of M	116
$\mathbf{T}f$	Tangent map	117
$\mathbf{T}_x f$	Restriction of $\mathbf{T}f$ to $\mathbf{T}_x M$ and $\mathbf{T}_{f(x)}N$	117
$\mathbf{T}_1 f$, $\mathbf{T}_2 f$	Partial tangent maps	117
$\mathbf{T}_x M$	Tangent space at $x \in M$	116
$\mathfrak{X}(M)$	The set of vector fields of the manifold M	117

Groups and monoids of functions

$\kappa_{\mathcal{W}}$	Inverse of the canonical chart for $\mathrm{End}_{\mathcal{W}}(X)$ and $\mathrm{Diff}_{\mathcal{W}}(X)$	41
$\mathcal{C}^\ell_{\mathcal{W}}(U,G)$	Lie group of weighted mappings with values in a Banach Lie group	74
$\mathcal{C}^\ell_{\mathcal{W}}(U,G)^\bullet$	Lie group of decaying weighted mappings with values in a Lie group	80
$\mathcal{C}^k_{\mathcal{W}}(U,G)^\bullet_{\mathrm{ex}}$	Lie group normalizing $\mathcal{C}^k_{\mathcal{W}}(U,G)^\bullet$	81
$\mathrm{Diff}(X)$	Diffeomorphisms of the Banach space X	41
$\mathrm{Diff}_c(M)$	Diffeomorphisms of M that are the identity outside some compact set	6
$\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$	Diffeomorphisms of $\mathbb{R}^n$ differing from $\mathrm{id}_{\mathbb{R}^n}$ by a rapidly decreasing $\mathbb{R}^n$-valued map	6
$\mathrm{Diff}_{\mathcal{W}}(X)$	Weighted diffeomorphisms of the Banach space X for weights $\mathcal{W}$	41
$\mathrm{Diff}_{\mathcal{W}}(X)^\circ$	$\mathrm{Diff}_{\mathcal{W}}(X) \cap \mathrm{End}_{\mathcal{W}}(X)^\circ$	55
$\mathrm{Diff}_{\mathcal{W}}(X)_0$	Identity component of $\mathrm{Diff}_{\mathcal{W}}(X)$	63
$\mathrm{End}_{\mathcal{W}}(X)$	Weighted endomorphisms of the Banach space X for weights $\mathcal{W}$	41
$\mathrm{End}_{\mathcal{W}}(X)^\circ$	Functions $\phi \in \mathrm{End}_{\mathcal{W}}(X)$ with $\phi - \mathrm{id}_X \in \mathcal{C}^\infty_{\mathcal{W}}(X,X)^o$	47

Further notation

$\mathrm{L}^k(X,Y)$	k-linear maps between normed spaces, endowed with the operator topology	105
$\mathcal{N}(X)$	Continuous seminorms on X	102
X_p	$X/p^{-1}(0)$ for X and $p \in \mathcal{N}(X)$,	102
π_p	Quotient map $X \to X_p$	102
$\lVert\cdot\rVert_{f,k}$	Supremum of the operator norm of the kth Fréchet derivative multiplied with f	10
$\lVert\gamma\rVert_{p,f,k}$	$\lVert\pi_p \circ \gamma\rVert_{f,k}$	30
$\lVert\cdot\rVert_{\mathrm{op}}$	Operator norm	105
$\lVert T\rVert_{\mathrm{op},p}$	Operator norm with respect to $p \in \mathcal{N}(Y)$	32
QI_A	Quasi-inversion map of the algebra A	121
A^q	Quasi-invertible elements of the algebra A	121

Index

analytic maps, 100
  superposition, *see* superposition with an analytic map

bounded maps, 10
  composition of, 23

centered chart, 70
compactly supported diffeomorphisms, 6
  density in $\mathrm{Diff}_{\mathcal{W}}(X)^{\circ}$, 55
complexification
  good, 27
  of maps, 102
  of power series, 28
composition
  of bounded maps, *see* bounded maps, composition of
  of bounded maps and weighted maps, 25
  of weighted maps und certain subsets of Lie groups, 66

diffeomorphisms, 41
  compactly supported, *see* compactly supported diffeomorphisms
  groups of, 6, 54, 55
    semidirect product with, *see* semidirect product
  weighted, *see* weighted diffeomorphisms

good complexification, *see* complexification, good

mapping groups
  with values in a Banach Lie group, 74
  with values in a locally convex Lie group, 80, 90, 94

quasi-inversion, 120

regularity, 119
  of $\mathcal{C}^k_{\mathcal{W}}(U, G)$, 75
  of $\mathrm{Diff}_{\mathcal{W}}(X)$, 61
  of $\mathrm{Diff}_{\mathcal{W}}(X)^{\circ}$, 62

semidirect product, 120
  of $\mathcal{C}^{\infty}_{\mathcal{W}}(X, G)$ and $\mathrm{Diff}_{\mathcal{W}}(X)$, 78
  of $\mathrm{Diff}_{\mathcal{W}}(X)_0$ and a Lie group acting on X, 63, 67
smooth monoid, 121
smooth normalizer, 81
superposition
  with a bounded map, 23, 25
  with a differentiable map, 40
  with a multilinear map, 18, 19, 35
  with an analytic map, 28

weighted diffeomorphisms, 41
  decreasing, 55
  easier description, 55
weighted maps
  decreasing, 11, 33
  into Banach Lie groups, 74
  into locally convex Lie groups, 80, 81, 90
  into locally convex spaces, 30
  into normed spaces, 10
weights, 10
  condition for completeness, 15