Contents

Abstract

The multilinear Calderón–Zygmund theory is developed in the setting of RD-spaces which are spaces of homogeneous type equipped with measures satisfying a reverse doubling condition. The multiple-weight multilinear Calderón–Zygmund theory in this context is also developed in this work. The bilinear $T1$-theorems for Besov and Triebel–Lizorkin spaces in the full range of exponents are among the main results obtained. Multilinear vector-valued $T1$ type theorems on Lebesgue spaces, Besov spaces, and Triebel–Lizorkin spaces are also proved. Applications include the boundedness of paraproducts and bilinear multiplier operators on products of Besov and Triebel–Lizorkin spaces.

Acknowledgements. Loukas Grafakos is supported by grant DMS 0900946 of the National Science Foundation of the USA. Liguang Liu is supported by the National Natural Science Foundation of China (grant no. 11101425). Diego Maldonado is supported by grant DMS 0901587 of the National Science Foundation of the USA. Dachun Yang is supported by the National Natural Science Foundation of China (grant nos. 11171027 & 11361020) and the Specialized Research Fund for the Doctoral Program of Higher Education of China (grant no. 20120003110003). All authors would like to thank the copy editor, Jerzy Trzeciak, for his valuable remarks which made this article more readable.

2010 *Mathematics Subject Classification*: Primary 42B20, 42B25, 42B35; Secondary 35S50, 42C15, 47G30, 30L99.

Key words and phrases: space of homogeneous type, multilinear Calderón–Zygmund operator, multilinear weighted estimate, paraproduct, bilinear $T1$-theorem, quadratic $T1$ type theorem, bilinear multiplier, Besov space, Triebel–Lizorkin space.

Received 14.11.2012.

1. Introduction

In this work we develop the theory of multilinear analysis related to the Calderón–Zygmund program within the framework of metric spaces. The impetus created by the recent developments in the theory of multilinear operators has naturally led us to consider its extension to the setting of metric spaces. Since the techniques involved in the proofs transcend the algebraic and differential structures of the underlying spaces, it is appropriate to undertake this study in a unified way. This setting is quite general and it includes graphs, fractals, Riemannian manifolds, Carnot–Carathéodory groups, anisotropic structures in $\mathbb{R}^n$, Ahlfors spaces, etc. As a consequence of this work, previously disconnected topics concerning multilinear operators are integrated and streamlined. These topics include bilinear Calderón–Zygmund operators, vector-valued bilinear operators (e.g., square-function-like operators), paraproducts, and Coifman–Meyer multipliers on Lebesgue spaces, Besov spaces, and Triebel–Lizorkin function spaces.

One of the first examples of multilinear operators in Euclidean harmonic analysis are the commutators of Calderón which appear in a series representation of the Cauchy integral along Lipschitz curves. The sharpest possible (endpoint) results for the m-commutators of Calderón were obtained by Calderón himself [15] when $m = 1$, Coifman and Meyer [19] when $m \in \{1, 2\}$ and Duong, Grafakos, and Yan [29] for $m \geq 3$. In particular, the article of Coifman and Meyer [19] not only established delicate estimates for the commutators but also set a solid foundation for a comprehensive study of general multilinear operators; this work, together with [20, 21], has been both fundamental and pioneering in this subject and certainly inspiring in our own work. Another important example of a bilinear operator is the paraproduct of Bony [12], which has been studied extensively and has experienced remarkable development in recent years, in view of its important connections with partial differential equations. Section 8 below is devoted to paraproducts and its introduction contains recent advances in the theory.

Among the main motivations for the multilinear analysis in this work, we mention the m-linear versions of the fractional Leibniz-type rules, that is, inequalities of the type

$$\Big\| D^\alpha\Big(\prod_{j=1}^m f_j\Big)\Big\|_{L^p(\mathbb{R}^n)} \leq C\sum_{j=1}^m \|D^\alpha(f_j)\|_{L^{p_j}(\mathbb{R}^n)} \prod_{\substack{k=1\\k\neq j}}^m \|f_k\|_{L^{p_k}(\mathbb{R}^n)}, \tag{1.1}$$

where C is a positive constant independent of $\{f_j\}_{j=1}^m$, the indices obey the Hölder scaling $1/p = 1/p_1 + \cdots + 1/p_m < 1$ with each p_j in $[1, \infty)$. Indeed, inequalities like (1.1) are based on mapping properties of bilinear Coifman–Meyer multipliers, which in turn follow from paraproduct decompositions and mapping properties for such paraproducts. In [5], Bényi, Maldonado, Nahmod, and Torres proved that paraproducts can be realized as bilinear singular integrals of Calderón–Zygmund type. Consequently, inequalities (1.1) are now best understood via the use of the powerful multilinear Calderón–Zygmund theory that was systematically developed by Grafakos and Torres [53] (see, for example, Grafakos [42] for a survey of these techniques).

In addition, it turns out that there is a rich weighted-norm theory for multilinear operators. In particular, multilinear Calderón–Zygmund operators obey vector-valued and weighted estimates, with respect to certain classes of weights. Very natural classes of multilinear weights surfaced in the work of Lerner, Ombrosi, Pérez, Torres, and Trujillo–González [72]. These weights are intrinsically multilinear and they have brought into fruition a rich weighted theory for multilinear operators analogous to that of the classical A_p weighted theory for linear operators in Euclidean spaces. The metric-space implementation of this class of weights is carried out in the present work.

Besov spaces were originally introduced by Besov [10, 11] as the trace spaces of Sobolev spaces, and were later generalized by Taibleson [96, 97, 98]. These spaces also arise as the real interpolation intermediate spaces of Sobolev spaces. Around 1970, Triebel [100] and Lizorkin [73, 74] started to investigate the scale $F^s_{p,q}$, nowadays known as the Triebel–Lizorkin spaces. The scales of Besov and Triebel–Lizorkin spaces include fundamental function spaces such as Lebesgue spaces, Sobolev spaces, Hardy spaces, and the space BMO of functions with bounded mean oscillation. We refer to Frazier and Jawerth [33] for a survey of the theory of Besov and Triebel–Lizorkin spaces and to Triebel's books [101, 102, 103] for a more comprehensive study. Over the last few decades, Besov and Triebel–Lizorkin spaces have consistently appeared in prominent parts of the literature and their usefulness has been exposed in different areas of mathematics and physics, such as partial differential equations, potential theory, approximation theory, and fluid dynamics. The complete framework of the classical Besov and Triebel–Lizorkin theory was extended to the context of RD-spaces by Han, Müller and Yang [60, 84]. In this work, we establish the bilinear $T1$-theorems for Besov and Triebel–Lizorkin spaces, in the full range of indices. Moreover, we obtain multilinear vector-valued $T1$ type theorems on Lebesgue spaces, Besov spaces, and Triebel–Lizorkin spaces. As an application, we deduce the boundedness of paraproducts and bilinear multiplier operators on products of Besov and Triebel–Lizorkin spaces.

Some of our results, for example those contained in Sections 6, 7, 9 and Subsection 8.4, are new even in the Euclidean case. In particular, in Section 6 we establish $T1$ theorems for bilinear Calderón–Zygmund operators on Triebel–Lizorkin spaces $\dot{F}^s_{p,q}(\mathcal{X})$ and Besov spaces $\dot{B}^s_{p,q}(\mathcal{X})$ for the full admissible range of s, p, q, successfully answering an open problem posed by Grafakos and Torres [55, p. 85]. Appropriate contextual descriptions as well as references are included at the beginning of each section.

As a whole, our results complement, from the Littlewood–Paley and real-analysis side, the recent advances in analysis on metric spaces related to first-order calculus (e.g. Sobolev functions, see Hajłasz and Koskela [56], Koskela and Saksman [69], Shanmugalingam [93] and the references therein), and the (weighted and unweighted) multilinear theory of potential operators in Grafakos and Kalton [43], Kenig and Stein [68], Moen [83], and the references therein.

Notation. Let $\mathbb{N} := \{1, 2, \dots\}$, $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ and $\mathbb{R}_+ := [0, \infty)$.

For any $p \in [1, \infty]$, we denote by p' the *conjugate index*, that is, $1/p + 1/p' = 1$; if $p = 1$, then $p' = \infty$ and, if $p = \infty$, then $p' = 1$.

For any $a, b \in \mathbb{R}$, let $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$.

For any ball $B \subset \mathcal{X}$ and $\kappa > 0$, denote by κB the ball contained in $\mathcal{X}$ with the same center as B but radius dilated by the factor κ.

Let $C_b(\mathcal{X})$ be the set of all continuous functions on $\mathcal{X}$ with bounded support (that is, contained in a ball of $(\mathcal{X}, d)$).

Let $L_b^\infty(\mathcal{X})$ be the set of all bounded functions on $\mathcal{X}$ with bounded support.

We use $L^1_{\mathrm{loc}}(\mathcal{X})$ to represent the collection of all locally integrable functions on $(\mathcal{X}, d, \mu)$. Moreover, for $q \in (0, \infty)$,

$$L^q_{\mathrm{loc}}(\mathcal{X}) := \{f : \ |f|^q \in L^1_{\mathrm{loc}}(\mathcal{X})\}.$$

Let

$$p(s, \epsilon) := \max\{n/(n + \epsilon),\ n/(n + s + \epsilon)\},$$

where $n \in \mathbb{N}$, $\epsilon > 0$ and $s \in \mathbb{R}$.

For any set E of $\mathcal{X}$, we define

$$d(x, E) := \inf\{d(x, y) : y \in E\}.$$

We use $\|T\|_{\mathcal{X} \to \mathcal{Y}}$ to denote the *operator norm* of $T : \mathcal{X} \to \mathcal{Y}$.

Denote by C a positive constant independent of main parameters involved; it may vary at different occurrences. Constants with subscripts do not change through the whole paper. Occasionally we use $C_{\alpha,\beta,\dots}$ or $C(\alpha, \beta, \dots)$ to indicate that the positive constant C depends only on parameters $\alpha, \beta, \dots$. Denote $f \le Cg$ and $f \ge Cg$ by $f \lesssim g$ and $f \gtrsim g$, respectively. If $f \lesssim g \lesssim f$, we then write $f \sim g$. ■

2. Real analysis on spaces of homogeneous type

Spaces of homogeneous type provide a general framework where the real-variable approach in the study of singular integrals of Calderón and Zygmund can be carried out. It turns out that classical analysis topics such as the Littlewood–Paley theory and function spaces can be introduced and developed in this context without resorting to the differential or algebraic structure of the underlying space.

This section provides necessary notions and results related to the spaces of homogeneous type and the so-called RD-spaces; see [22, 23, 75, 76, 27, 64, 61, 59, 60, 65] and the references therein. The readers who are familiar with this basic knowledge can directly proceed to the next section.

2.1. Spaces of homogeneous type and RD-spaces. Let $\mathcal{X}$ be a set. A function $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ is called a *quasi-metric* if

(i) $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{X}$;
(ii) $d(x, y) = 0$ if and only if $x = y$; and
(iii) there exists a constant $K \in [1, \infty)$ such that $d(x, y) \le K[d(x, z) + d(z, y)]$ for all $x, y, z \in \mathcal{X}$.

In this case we call $(\mathcal{X}, d)$ a *quasi-metric space*. In particular, when $K = 1$, we call d a *metric* and $(\mathcal{X}, d)$ a *metric space*. For all $x \in \mathcal{X}$ and $r > 0$, set

$$B(x, r) := \{y \in \mathcal{X} : d(x, y) < r\}.$$

Next we recall the notions of spaces of homogeneous type in the sense of Coifman and Weiss [22, 23] and of RD-spaces introduced in [59, 60].

Definition 2.1. Let $(\mathcal{X}, d)$ be a metric space and let the balls $\{B(x, r) : r > 0\}$ form a basis of open neighborhoods of the point $x \in \mathcal{X}$. Suppose that μ is a regular Borel measure defined on a σ-algebra which contains all Borel sets induced by the open balls $\{B(x, r) : x \in \mathcal{X},\ r > 0\}$, and that $0 < \mu(B(x, r)) < \infty$ for all $x \in \mathcal{X}$ and $r > 0$. The triple $(\mathcal{X}, d, \mu)$ is called a *space of homogeneous type* if there exists a constant $C_1 \in [1, \infty)$ such that, for all $x \in \mathcal{X}$ and $r > 0$,

$$\mu(B(x, 2r)) \le C_1 \mu(B(x, r)) \quad (\textit{doubling condition}). \tag{2.1}$$

The triple $(\mathcal{X}, d, \mu)$ is called an *RD-space* if it is a space of homogeneous type and there exist constants $\kappa \in (0, \infty)$ and $C_2 \in (0, 1]$ such that, for all $x \in \mathcal{X}$, $0 < r < 2\,\mathrm{diam}(\mathcal{X})$ and $1 \le \lambda < 2\,\mathrm{diam}(\mathcal{X})/r$,

$$C_2 \lambda^\kappa \mu(B(x, r)) \le \mu(B(x, \lambda r)); \tag{2.2}$$

here and in what follows,

$$\operatorname{diam}(\mathcal{X}) := \sup_{x,y\in\mathcal{X}} d(x,y).$$

RD-spaces have become the underlying context in numerous areas of analysis and PDEs; we refer the reader to [22, 23, 59, 60, 84, 70, 71, 39, 104, 106, 105] and references therein.

REMARK 2.2. (i) For a space $\mathcal{X}$ of homogeneous type, by (2.1), there exist $C_3 \in [1,\infty)$ and $n \in (0,\infty)$ such that, for all $x \in \mathcal{X}$, $r > 0$ and $\lambda \geq 1$,

$$\mu(B(x,\lambda r)) \leq C_3\lambda^n \mu(B(x,r)).$$

Indeed, we can choose $C_3 := C_1$ and $n := \log_2 C_1$. In some sense, n measures the "dimension" of $\mathcal{X}$. When $\mathcal{X}$ is an RD-space, we obviously have $n \in [\kappa,\infty)$.

(ii) For a space $(\mathcal{X},d,\mu)$ of homogeneous type, if $\mu(\mathcal{X}) < \infty$, then there exists a positive constant R_0 such that $\mathcal{X} = B(x,R_0)$ for all $x \in \mathcal{X}$; see Nakai and Yabuta [86, Lemma 5.1]. It follows that $\mu(\mathcal{X}) < \infty$ if and only if $\operatorname{diam}(\mathcal{X}) < \infty$.

(iii) $\mathcal{X}$ is an RD-space if and only if $\mathcal{X}$ is a space of homogeneous type with the additional property that there is a constant $a_0 > 1$ such that for all $x \in \mathcal{X}$ and $0 < r < \operatorname{diam}(\mathcal{X})/a_0$, $B(x,a_0 r) \setminus B(x,r) \neq \emptyset$; see, for instance, [60, Remark 1.1], [104] and [86]. Consequently, any connected space of homogeneous type is an RD-space.

(iv) For any space $(\mathcal{X},d,\mu)$ of homogeneous type, the set

$$\operatorname{Atom}(\mathcal{X},d,\mu) := \{x \in \mathcal{X} : \mu(\{x\}) > 0\}$$

is countable and, for every $x \in \operatorname{Atom}(\mathcal{X},d,\mu)$, there exists $r > 0$ such that $B(x,r) = \{x\}$; see Macías and Segovia [75, Theorem 1]. From (iii) or (2.2), it follows that any RD-space $(\mathcal{X},d,\mu)$ is non-atomic, i.e., $\mu(\{x\}) = 0$ for all $x \in \mathcal{X}$.

(v) Throughout this paper, we *always assume* that $(\mathcal{X},d,\mu)$ is an RD-space and $\mu(\mathcal{X}) = \infty$, unless it is clearly stated that $(\mathcal{X},d,\mu)$ is a space of homogeneous type.

REMARK 2.3. For any quasi-metric space $(\mathcal{X},d)$, Macías and Segovia [75, Theorem 2] proved that there exists an equivalent quasi-metric ρ such that all balls corresponding to ρ are open in the topology induced by ρ, and there exist constants $C > 0$ and $\theta \in (0,1)$ such that, for all $x,y,z \in \mathcal{X}$,

$$|\rho(x,z) - \rho(y,z)| \leq C[\rho(x,y)]^\theta [\rho(x,z) + \rho(y,z)]^{1-\theta}.$$

If the metric d in Definition 2.1 is replaced by ρ, then all results in this paper have corresponding analogues on $(\mathcal{X},\rho,\mu)$. In order to simplify the presentation, in this work we *always assume* that d is a metric and the balls corresponding to d are open sets in the topology induced by d.

Set

$$V_\delta(x) := \mu(B(x,\delta)) \quad \text{and} \quad V(x,y) := \mu(B(x,d(x,y)))$$

for all $x,y \in \mathcal{X}$ and $\delta > 0$. It follows from (2.1) that $V(x,y) \sim V(y,x)$. Here we present some estimates regarding spaces of homogeneous type; see, for example, [60, Lemma 2.1] or [59].

LEMMA 2.4. *Let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type, $r > 0$, $\delta > 0$, $\alpha > 0$, $\eta \geq 0$ and $\gamma \in (0, 1)$.*

(a) *For all $x, y \in \mathcal{X}$,*

$$\begin{aligned}\mu(B(x, r + d(x, y))) &\sim \mu(B(y, r + d(x, y))) \sim V_r(x) + V(x, y)\\ &\sim V_r(y) + V(x, y) \sim V_r(x) + V_r(y) + V(x, y),\end{aligned}$$

with implicit constants depending only on C_1.

(b) *If $x, x', x_1 \in \mathcal{X}$ satisfy $d(x, x') \leq \gamma(r + d(x, x_1))$, then*

$$r + d(x, x_1) \sim r + d(x', x_1) \quad \text{and} \quad \mu(B(x, r + d(x, x_1))) \sim \mu(B(x', r + d(x', x_1))),$$

with implicit constants depending only on γ and C_1.

(c) *There exists a positive constant C, depending only on C_1 and α, such that, for all $x \in \mathcal{X}$,*

$$\int_{d(x,y)\leq\delta} \frac{d(x, y)^\alpha}{V(x, y)}\, d\mu(y) \leq C\delta^\alpha \quad \text{and} \quad \int_{d(x,y)\geq\delta} \frac{1}{V(x, y)} \frac{\delta^\alpha}{d(x, y)^\alpha}\, d\mu(y) \leq C.$$

(d) *If $\alpha > \eta \geq 0$, then there exists a positive constant C, depending only on C_1, α and η, such that, for all $x \in \mathcal{X}$,*

$$\int_{\mathcal{X}} \frac{1}{V_\delta(x) + V(x, y)} \left[\frac{\delta}{\delta + d(x, y)}\right]^\alpha d(x, y)^\eta\, d\mu(y) \leq C\delta^\eta.$$

2.2. Dyadic cubes, covering lemmas, and the Calderón–Zygmund decomposition. Throughout this subsection we *always assume* that $(\mathcal{X}, d, \mu)$ is a space of homogeneous type.

Recall that in $\mathbb{R}^n$ the dyadic cubes are defined, for all $k \in \mathbb{Z}$ and $\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{Z}^n$, as

$$Q_{k,\ell} := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : 2^{-k}\ell_i \leq x_i < 2^{-k}(\ell_i + 1),\ \forall i \in \{1, \ldots, n\}\}.$$

Most of their properties are retained in the case of abstract spaces of homogeneous type. Indeed, a construction due to Christ (see [16]) allows the following version of the Euclidean dyadic cubes in a general space $\mathcal{X}$ of homogeneous type. It should be remarked that recently Hytönen et. al. [1, 66, 67] constructed a randomized dyadic structure by only assuming that the underlying metric space is geometrically doubling.

LEMMA 2.5. *Let $\mathcal{X}$ be a space of homogeneous type. Then there exists a collection $\mathcal{Q} = \{Q_\alpha^k \subset \mathcal{X} : k \in \mathbb{Z}, \alpha \in I_k\}$ of open subsets, where I_k is some index set, and constants $\delta \in (0, 1)$ and $C_5, C_6 > 0$ such that*

(i) *for each fixed $k \in \mathbb{Z}$, $\mu(\mathcal{X} \setminus \bigcup_\alpha Q_\alpha^k) = 0$ and $Q_\alpha^k \cap Q_\beta^k = \emptyset$ if $\alpha \neq \beta$;*
(ii) *for any α, β, k, ℓ with $\ell \geq k$, either $Q_\beta^\ell \subset Q_\alpha^k$ or $Q_\beta^\ell \cap Q_\alpha^k = \emptyset$;*
(iii) *for each (k, α) and each $\ell < k$, there is a unique β such that $Q_\alpha^k \subset Q_\beta^\ell$;*
(iv) $\mathrm{diam}(Q_\alpha^k) \leq C_5\delta^k$ *and each Q_α^k contains some ball $B(z_\alpha^k, C_6\delta^k)$, where $z_\alpha^k \in \mathcal{X}$.*

One can think of Q_α^k as being a dyadic cube centered at z_α^k with diameter roughly δ^k. In what follows, for simplicity, we *always assume* that $\delta = 1/2$; see, for example, Han and Sawyer [61, pp. 96–98] on how to remove this restriction.

Regarding covering lemmas, we begin with the so-called basic covering lemma (see, for instance, Heinonen [64, p. 2]) on metric spaces, which is particularly useful.

Lemma 2.6. *Every family $\mathcal{F}$ of balls of uniformly bounded diameter in a metric space $(\mathcal{X}, d)$ contains a subfamily $\mathcal{G}$ of pairwise disjoint balls such that*

$$\bigcup_{B\in\mathcal{F}} B \subset \bigcup_{B\in\mathcal{G}} 5B.$$

Moreover, every ball B from $\mathcal{F}$ meets a ball from $\mathcal{G}$ with radius at least half that of B.

Two geometric facts about spaces of homogeneous type, the Vitali–Wiener type covering lemma and Whitney type covering lemma, play fundamental roles in establishing the Calderón–Zygmund theory on $(\mathcal{X}, d, \mu)$; see [22, 23] as well as [76].

Lemma 2.7 (Vitali–Wiener type covering lemma). *Let $E \subset \mathcal{X}$ be a bounded set (that is, contained in a ball). Consider any covering of E of the form $\{B(x, r_x) : x \in E\}$. Then there exists a sequence of points $x_j \in E$ such that $\{B(x_j, r_{x_j})\}_j$ are pairwise disjoint and $\{B(x_j, C_0 r_{x_j})\}_j$ is a covering of E. Here C_0 depends only on the doubling and quasi-triangle constants.*

We remark that, when Ω is an open bounded set, the following Lemma 2.8 was proved in [22, pp. 70–71] and [23, Theorem 3.2]. The current version was claimed in [22, p. 70] without a proof; see also [76, p. 277] for another variant, namely that Ω is assumed to be an open set of finite measure strictly contained in $\mathcal{X}$. In fact, a detailed proof of Lemma 2.8 can be given by borrowing some ideas from [94, pp. 15–16]; see also [47].

Lemma 2.8 (Whitney type covering lemma). *Let Ω be an open proper subset of $\mathcal{X}$. For $x \in \mathcal{X}$ define $d(x) := \operatorname{dist}(x, \Omega^{\complement})$. For any given $c \geq 1$, let $r(x) := d(x)/(2c)$. Then there exist a positive number M, which depends only on c and C_1 but not on Ω, and a sequence $\{x_k\}_k$ such that, if we denote $r(x_k)$ by r_k, then*

- (i) *$\{B(x_k, r_k/4)\}_k$ are pairwise disjoint and $\bigcup_k B(x_k, r_k) = \Omega$;*
- (ii) *for every given k, $B(x_k, cr_k) \subset \Omega$;*
- (iii) *for every given k, $x \in B(x_k, cr_k)$ implies that $cr_k < d(x) < 3cr_k$;*
- (iv) *for every given k, there exists a $y_k \notin \Omega$ such that $d(x_k, y_k) < 3cr_k$;*
- (v) *for every given k, the number of balls $B(x_i, cr_i)$ intersecting the ball $B(x_k, cr_k)$ is at most M.*

For any $f \in L^1_{\mathrm{loc}}(\mathcal{X})$, the *Hardy–Littlewood maximal function* $\mathcal{M}f$ is defined by

$$\mathcal{M}f(x) := \sup_{B\ni x} \frac{1}{\mu(B)} \int_B |f(y)|\, d\mu(y), \qquad \forall x \in \mathcal{X}, \tag{2.3}$$

where the supremum is taken over all balls $B \subset \mathcal{X}$ containing x. It is easy to see that the function $\mathcal{M}f$ is lower semi-continuous (hence μ-measurable) for every $f \in L^1_{\mathrm{loc}}(\mathcal{X})$. Using the Vitali–Wiener type covering lemma, Coifman and Weiss [22] proved that $\mathcal{M}$ is bounded from $L^1(\mathcal{X})$ to $L^{1,\infty}(\mathcal{X})$ and bounded on $L^p(\mathcal{X})$ for all $p \in (1, \infty]$. Also, by an argument similar to Grafakos [40, Exercise 2.1.13], we know that $\mathcal{M}$ is bounded on $L^{p,\infty}(\mathcal{X})$ for $p \in (1, \infty)$. The operator norms

$$\|\mathcal{M}\|_{L^1(\mathcal{X})\to L^{1,\infty}(\mathcal{X})}, \quad \|\mathcal{M}\|_{L^p(\mathcal{X})\to L^p(\mathcal{X})} \quad \text{and} \quad \|\mathcal{M}\|_{L^{p,\infty}(\mathcal{X})\to L^{p,\infty}(\mathcal{X})}$$

all depend only on C_1 and p.

Denote by $C_b(\mathcal{X})$ the space of all continuous functions on $\mathcal{X}$ with bounded supports (that is, contained in a ball of $(\mathcal{X}, d)$). As in Definition 2.1 we are assuming that μ is a regular Borel measure on the metric space $(\mathcal{X}, d)$, which means that μ has the outer and inner regularity (see Heinonen [64]), so $C_b(\mathcal{X})$ is dense in $L^p(\mathcal{X})$ for all $p \in [1,\infty)$. This, combined with the weak type $(1,1)$ boundedness of $\mathcal{M}$ and a standard argument (see, for instance, [64, pp. 12–13]), implies the differentiation theorem for integrals: for all $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ and almost every $x \in \mathcal{X}$,

$$\lim_{B\ni x,\, \mu(B)\to 0} \frac{1}{\mu(B)} \int_B f(y)\, d\mu(y) = f(x)$$

and

$$\lim_{B\ni x,\, \mu(B)\to 0} \frac{1}{\mu(B)} \int_B |f(y) - f(x)|\, d\mu(y) = 0.$$

A consequence of the current Whitney covering lemma and the differentiation theorem for integrals on $(\mathcal{X}, d, \mu)$ as well as the weak-$(1,1)$ boundedness of $\mathcal{M}$ is the celebrated Calderón–Zygmund decomposition process for integrable functions; see Coifman and Weiss [22, 23].

LEMMA 2.9. *Let $f \in L^1(\mathcal{X})$. Then, for every $\lambda > \|f\|_{L^1(\mathcal{X})}/\mu(\mathcal{X})$, there exists a sequence of balls, $\{B_k\}_{k\in I}$, where I is some index set, such that*

(i) *$\{\frac{1}{4}B_k\}_{k\in I}$ are pairwise disjoint;*
(ii) *$|f(x)| \le C\lambda$ for almost every $x \in \mathcal{X} \setminus \bigcup_{k\in I} B_k$;*
(iii) *for any $k \in I$,*
$$\frac{1}{\mu(B_k)} \int_{B_k} |f|\, d\mu > C\lambda;$$
(iv)
$$\sum_{k\in I} \mu(B_k) \le C\|f\|_{L^1(\mathcal{X})}/\lambda,$$

where the positive constant C depends only on C_1.

LEMMA 2.10. *Let $f \in L^1(\mathcal{X})$. For every $\lambda > \|f\|_{L^1(\mathcal{X})}/\mu(\mathcal{X})$, let $\{B_k\}_{k\in I}$ be the sequence of balls provided by Lemma 2.9. Then there exist functions g and $\{b_k\}_{k\in I}$ such that*

(i) *$f = g + \sum_{k\in I} b_k$;*
(ii) *$\|g\|_{L^\infty(\mathcal{X})} \le C\lambda$;*
(iii) *for every $k \in I$,*
$$\int_{\mathcal{X}} b_k\, d\mu = 0;$$
(iv) *for every $k \in I$ we have $\operatorname{supp} b_k \subset B_k$;*
(v) *$\|g\|_{L^1(\mathcal{X})} \le C\|f\|_{L^1(\mathcal{X})}$ and $\sum_{k\in I} \|b_k\|_{L^1(\mathcal{X})} \le C\|f\|_{L^1(\mathcal{X})}$, where C is a positive constant depending only on C_1.*

For any $\eta \in (0,1]$, let $C^\eta(\mathcal{X})$ be the set of all functions $f : \mathcal{X} \to \mathbb{C}$ such that

$$\|f\|_{\dot{C}^\eta(\mathcal{X})} := \sup_{x\neq y} \frac{|f(x) - f(y)|}{d(x,y)^\eta} < \infty.$$

Denote by $\operatorname{supp} f$ the closure of the set $\{x \in \mathcal{X} : f(x) \neq 0\}$ in $\mathcal{X}$. Define

$$C_b^\eta(\mathcal{X}) := \{f \in C^\eta(\mathcal{X}) : f \text{ has bounded support}\}.$$

Then $C_b^\eta(\mathcal{X}) \subset L^\infty(\mathcal{X})$ and the norm on $C_b^\eta(\mathcal{X})$ is given by

$$\|f\|_{C^\eta(\mathcal{X})} := \|f\|_{L^\infty(\mathcal{X})} + \|f\|_{\dot{C}^\eta(\mathcal{X})}.$$

In what follows, we endow $C_b^\eta(\mathcal{X})$ with the strict inductive limit topology (see [76, p. 273]) arising from the sequence of spaces $(C_b^\eta(B_n), \|\cdot\|_{C^\eta(\mathcal{X})})$, where $\{B_n\}_{n\in\mathbb{Z}}$ is any given increasing sequence of balls with the same center such that $\mathcal{X} = \bigcup_{n\in\mathbb{N}} B_n$ and

$$C_b^\eta(B_n) := \{f \in C_b^\eta(\mathcal{X}) : \operatorname{supp} f \subset B_n\}.$$

Denote by $(C_b^\eta(\mathcal{X}))'$ the *dual space* of $C_b^\eta(\mathcal{X})$, that is, the collection of all continuous linear functionals on $C_b^\eta(\mathcal{X})$. The space $(C_b^\eta(\mathcal{X}))'$ is endowed with the weak*-topology.

For functions in $C_b^\eta(\mathcal{X})$, we have a more elaborate version of the Calderón–Zygmund decomposition lemma when $\mu(\mathcal{X}) = \infty$.

LEMMA 2.11. *Let $\mu(\mathcal{X}) = \infty$, $\eta \in (0,1]$ and $f \in C_b^\eta(\mathcal{X})$. For any $\lambda > 0$, f has the decomposition*

$$f = g + \sum_{k\in I} b_k,$$

where g, $\{b_k\}_{k\in I}$ and $\sum_{k\in I} b_k$ are functions in $C_b^\eta(\mathcal{X})$ satisfying (i)–(v) *of Lemma* 2.10.

Proof. Suppose that $f \in C_b^\eta(\mathcal{X})$ with $\eta \in (0,1]$ and $\lambda > 0$. Consider the level set

$$\Omega := \{x \in \mathcal{X} : \mathcal{M}f(x) > \lambda\}.$$

Without loss of generality, we may assume that $\operatorname{supp} f \subset B(y_0, R)$, where $y_0 \in \mathcal{X}$ and $R > 0$. Moreover, since $\mu(\mathcal{X}) = \infty$, we can choose R sufficiently large so that f, λ and $B(y_0, R)$ satisfy the assumptions of [23, p. 625, Lemma (3.9)]. From this, we deduce that Ω is contained in some ball. Then we cover Ω by using balls $\{B_k\}_{k\in I} := \{B(x_k, r_k)\}_{k\in I}$ which satisfy (i)–(v) of Lemma 2.8. Take a radial function $h \in C_c^\infty(\mathbb{R})$ such that $0 \le h \le 1$, $h(t) = 1$ when $|t| \le 1$, and $h(t) = 0$ when $|t| \ge 2$. For every B_k, define

$$\phi_k(x) := h\left(\frac{d(x, x_k)}{r_k}\right), \qquad \forall x \in \mathcal{X}.$$

It is easy to show that every ϕ_k is in $C_b^1(\mathcal{X})$ and $\phi_k(x) = 1$ when $x \in B_k$. Moreover, if we take the constant c in Lemma 2.8 sufficiently large (for example, $c > 2$), then we have $\operatorname{supp} \phi_k \subset 2B_k \subset \Omega$ and $\{2B_k\}_{k\in I}$ has the bounded overlapping property. If we let

$$\Phi_k := \frac{\phi_k}{\sum_j \phi_j}, \qquad k \in I,$$

then $\{\Phi_k\}_{k\in I}$ forms a partition of unity of Ω with every Φ_k in $C_b^1(\mathcal{X})$. Now let

$$b_k := f\Phi_k - \frac{\int_{\mathcal{X}} f\Phi_k \, d\mu}{\int_{\mathcal{X}} \Phi_k \, d\mu}\Phi_k, \qquad k \in I,$$

and

$$g := f - \sum_{k\in I} b_k.$$

Then it is a standard procedure to show that g and $\{b_k\}_{k\in I}$ satisfy (i)–(v) of Lemma 2.10.

As $f, \Phi_k \in C_b^\eta(\mathcal{X})$, we see that every b_k is in $C_b^\eta(\mathcal{X})$ and $\operatorname{supp} b_k \subset 2B_k$. Since $\operatorname{supp} f$ and Ω are both bounded sets, $\sum_{k\in I} b_k$ and g have bounded supports. The finite overlap property of $\{2B_k\}_{k\in I}$ implies that $\sum_{k\in I} b_k(x)$ has only finitely many terms for any fixed $x \in \mathcal{X}$. From this and the fact that each b_k is in $C_b^\eta(\mathcal{X})$, it follows that $\sum_{k\in I} b_k$ is in $C_b^\eta(\mathcal{X})$, hence so is g. ■

2.3. Space of test functions. We recall the notion of the space of test functions on the RD-space $(\mathcal{X}, d, \mu)$ used in [59, 60].

Definition 2.12. Let $x_1 \in \mathcal{X}$, $r \in (0,\infty)$, $\beta \in (0,1]$ and $\gamma \in (0,\infty)$. A function φ on $\mathcal{X}$ is called a *test function of type* (x_1, r, β, γ) if there exists a positive constant C such that

(i) for all $x \in \mathcal{X}$,

$$|\varphi(x)| \le C \frac{1}{V_r(x_1) + V_r(x) + V(x_1, x)} \left[\frac{r}{r + d(x_1, x)}\right]^\gamma;$$

(ii) for all $x, y \in \mathcal{X}$ satisfying $d(x,y) \le [r + d(x_1, x)]/2$,

$$|\varphi(x) - \varphi(y)| \le C \left[\frac{d(x,y)}{r + d(x_1,x)}\right]^\beta \frac{1}{V_r(x_1) + V_r(x) + V(x_1, x)} \left[\frac{r}{r + d(x_1, x)}\right]^\gamma.$$

Denote by $\mathcal{G}(x_1, r, \beta, \gamma)$ the set of all test functions of type (x_1, r, β, γ). If $\varphi \in \mathcal{G}(x_1, r, \beta, \gamma)$, its norm is defined by

$$\|\varphi\|_{\mathcal{G}(x_1,r,\beta,\gamma)} := \inf\{C : \text{(i) and (ii) hold}\}.$$

The space $\mathcal{G}(x_1, r, \beta, \gamma)$ is called the *space of test functions*. Set

$$\mathring{\mathcal{G}}(x_1, r, \beta, \gamma) := \left\{\varphi \in \mathcal{G}(x_1, r, \beta, \gamma) : \int_{\mathcal{X}} \varphi(x)\, d\mu(x) = 0\right\}.$$

The space $\mathring{\mathcal{G}}(x_1, r, \beta, \gamma)$ is called the *space of test functions with mean zero*.

It should be remarked that the prototype of such test functions on $\mathbb{R}^n$ first appeared in the work of Meyer [80], where our Definition 2.12(ii) is replaced by

$$|\varphi(x) - \varphi(y)| \le C \left(\frac{|x - x'|}{r}\right)^\beta \left[\left(\frac{r}{r + |x - x_1|}\right)^\gamma + \left(\frac{r}{r + |y - x_1|}\right)^\gamma\right]. \tag{2.4}$$

Instead of imposing the condition that (2.4) holds for all $x, y \in \mathbb{R}^n$, Han [57] only required (2.4) for the points x, y satisfying $|x - y| \le (r + |x - x_1|)/2$. The above definitions of $\mathcal{G}(x_1, r, \beta, \gamma)$ and $\mathring{\mathcal{G}}(x_1, r, \beta, \gamma)$ for general RD-spaces were first introduced in [59, 60].

Following [60], fix $x_1 \in \mathcal{X}$ and let

$$\mathcal{G}(\beta, \gamma) := \mathcal{G}(x_1, 1, \beta, \gamma).$$

It is easy to see that, for any $x_2 \in \mathcal{X}$ and $r > 0$, we have $\mathcal{G}(x_2, r, \beta, \gamma) = \mathcal{G}(\beta, \gamma)$ with equivalent norms (but with constants depending on x_1, x_2 and r). The space $\mathcal{G}(\beta, \gamma)$ is a Banach space.

For any given $\epsilon \in (0,1]$, let $\mathcal{G}_0^\epsilon(\beta, \gamma)$ be the completion of the space $\mathcal{G}(\epsilon, \epsilon)$ in $\mathcal{G}(\beta, \gamma)$ when $\beta, \gamma \in (0, \epsilon]$. Then $\varphi \in \mathcal{G}_0^\epsilon(\beta, \gamma)$ if and only if $\varphi \in \mathcal{G}(\beta, \gamma)$ and there exists $\{\phi_j\}_{j\in\mathbb{N}} \subset \mathcal{G}(\epsilon, \epsilon)$ such that $\lim_{j\to\infty} \|\varphi - \phi_j\|_{\mathcal{G}(\beta,\gamma)} = 0$. If $\varphi \in \mathcal{G}_0^\epsilon(\beta, \gamma)$, we define

$$\|\varphi\|_{\mathcal{G}_0^\epsilon(\beta,\gamma)} := \|\varphi\|_{\mathcal{G}(\beta,\gamma)}.$$

For the above chosen $\{\phi_j\}_{j\in\mathbb{N}}$, we have

$$\|\varphi\|_{\mathcal{G}_0^\epsilon(\beta,\gamma)} = \lim_{j\to\infty} \|\phi_j\|_{\mathcal{G}(\beta,\gamma)}.$$

Similarly, the space $\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)$ is defined to be the completion of $\mathring{\mathcal{G}}(\epsilon,\epsilon)$ in $\mathring{\mathcal{G}}(\beta,\gamma)$ when $\beta,\gamma \in (0,\epsilon]$ and, for any $\varphi \in \mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)$, we define $\|\varphi\|_{\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)} := \|\varphi\|_{\mathcal{G}(\beta,\gamma)}$. Both $\mathcal{G}_0^\epsilon(\beta,\gamma)$ and $\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)$ are Banach spaces.

Denote by $(\mathcal{G}_0^\epsilon(\beta,\gamma))'$ and $(\mathring{\mathcal{G}}_{b,0}^\epsilon(\beta,\gamma))'$, respectively, the sets of all bounded linear functionals on $\mathcal{G}_0^\epsilon(\beta,\gamma)$ and $\mathring{\mathcal{G}}_{b,0}^\epsilon(\beta,\gamma)$. Define $\langle f,\varphi\rangle$ to be the natural pairing of elements $f \in (\mathcal{G}_0^\epsilon(\beta,\gamma))'$ and $\varphi \in \mathcal{G}_0^\epsilon(\beta,\gamma)$, or $f \in (\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))'$ and $\varphi \in \mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)$. The space $\mathcal{G}_0^\epsilon(\beta,\gamma)$ plays the same role as the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ and the space of all infinitely differentiable compactly supported functions on $\mathbb{R}^n$.

Obviously, any function $f \in C_b^\eta(\mathcal{X})$ with $\eta \in (0,1]$ is a test function of type (x_0,r,η,γ) for all $x_0 \in \mathcal{X}$, $r>0$ and $\gamma>0$; moreover, there exists a positive constant C, depending only on C_1, $\operatorname{supp} f$, x_1, β and γ, such that

$$\|f\|_{\mathcal{G}(x_0,r,\eta,\gamma)} \le C\|f\|_{C^\eta(\mathcal{X})}.$$

Conversely, if $f \in \mathcal{G}(x_1,1,\beta,\gamma)$ for some $\beta \in (0,1]$ and $\gamma>0$, then $f \in C^\beta(\mathcal{X})$. Moreover, by the size condition on f (see Definition 2.12(i)), we see that

$$\|f\|_{L^\infty(\mathcal{X})} \le \frac{1}{V_1(x_1)}\|f\|_{\mathcal{G}(x_1,1,\beta,\gamma)}$$

and, by Definition 2.12(ii) when $d(x,y)\le 1/2$ and Definition 2.12(i) when $d(x,y)>1/2$,

$$\begin{aligned}\|f\|_{\dot{C}^\beta(\mathcal{X})} &= \max\left\{\sup_{x\neq y,\, d(x,y)\le 1/2} \frac{|f(x)-f(y)|}{d(x,y)^\beta},\quad \sup_{x\neq y,\, d(x,y)>1/2}\frac{|f(x)-f(y)|}{d(x,y)^\beta}\right\}\\ &\le \frac{2^{\beta+1}}{V_1(x_1)}\|f\|_{\mathcal{G}(x_1,1,\beta,\gamma)}.\end{aligned}$$

2.4. Approximations of the identity. Approximations of the identity on Ahlfors 1-regular metric measure spaces $(\mathcal{X},d,\mu)$ satisfying $\mu(\mathcal{X})=\infty$ and $\mu(\{x\})=0$ for all $x\in\mathcal{X}$ first appeared in David, Journé and Semmes [26, Lemma 2.2] and Han [58] (see also [57, 61]). Also based on the ideas in [26], the corresponding versions in the context of RD-spaces were proved in [60, Definition 2.2]. The following definition is from [59, 60].

Definition 2.13. Let $\epsilon_1 \in (0,1]$, $\epsilon_2>0$, and $\epsilon_3>0$. A sequence $\{S_k\}_{k\in\mathbb{Z}}$ of bounded linear integral operators on $L^2(\mathcal{X})$ is called an *approximation of the identity of order* $(\epsilon_1,\epsilon_2,\epsilon_3)$ (for short, $(\epsilon_1,\epsilon_2,\epsilon_3)$-ATI) if there exists a positive constant C such that, for all $k\in\mathbb{Z}$ and $x,x',y,y'\in\mathcal{X}$, the integral kernel $S_k(x,y)$ of S_k is a measurable function, from $\mathcal{X}\times\mathcal{X}$ into $\mathbb{C}$, satisfying

(i) $|S_k(x,y)| \le C\frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)+V(x,y)}\,\frac{2^{-k\epsilon_2}}{[2^{-k}+d(x,y)]^{\epsilon_2}}$;

(ii) for all $d(x,x')\le[2^{-k}+d(x,y)]/2$,

$$\begin{aligned}&|S_k(x,y)-S_k(x',y)|\\ &\qquad\le C\frac{d(x,x')^{\epsilon_1}}{[2^{-k}+d(x,y)]^{\epsilon_1}}\,\frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)+V(x,y)}\,\frac{2^{-k\epsilon_2}}{[2^{-k}+d(x,y)]^{\epsilon_2}};\end{aligned}$$

(iii) S_k satisfies (ii) with x and y interchanged;

(iv) for $d(x,x') \le [2^{-k}+d(x,y)]/3$ and $d(y,y') \le [2^{-k}+d(x,y)]/3$,

$$\begin{aligned}|[S_k(x,y)-S_k(x,y')]-[S_k(x',y)-S_k(x',y')]|\\ \le C\frac{d(x,x')^{\epsilon_1}}{[2^{-k}+d(x,y)]^{\epsilon_1}}\frac{d(y,y')^{\epsilon_1}}{[2^{-k}+d(x,y)]^{\epsilon_1}}\\ \times\frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)+V(x,y)}\frac{2^{-k\epsilon_3}}{[2^{-k}+d(x,y)]^{\epsilon_3}};\end{aligned}$$

(v) $\int_{\mathcal{X}} S_k(x,w)\,d\mu(w) = 1 = \int_{\mathcal{X}} S_k(w,y)\,d\mu(w)$.

Remark 2.14. (i) If $\{S_k\}_{k\in\mathbb{Z}}$ is an $(\epsilon_1,\epsilon_2,\epsilon_3)$-ATI with bounded support, that is, there exists a positive constant C such that $S_k(x,y)=0$ whenever $d(x,y)\ge C2^{-k}$, then $\{S_k\}_{k\in\mathbb{Z}}$ is an $(\epsilon_1,\epsilon_2',\epsilon_3')$-ATI for all $\epsilon_2'>0$ and $\epsilon_3'>0$. Such a sequence of operators, $\{S_k\}_{k\in\mathbb{Z}}$, is called an *approximation of the identity of order* ϵ_1 *with bounded support* (for short, ϵ_1-ATI with bounded support). The existence of 1-ATI with bounded support was shown in [60, Theorem 2.6] by using the ideas of David, Journé and Semmes [26].

(ii) Let $\{S_k\}_{k\in\mathbb{Z}}$ be an ϵ_1-ATI with bounded support. For any $\eta\in(0,\epsilon_1]$, there exists a positive constant C such that, for all $x,x',y\in\mathcal{X}$ and all $k\in\mathbb{Z}$,

$$|S_k(x,y)-S_k(x',y)| \le C\left[\frac{d(x,x')}{2^{-k}}\right]^{\eta}\frac{1}{V_{2^{-k}}(y)}. \tag{2.5}$$

Indeed, (2.5) follows from the regular condition of S_k if $d(x,x')\le 2^{-k-1}$ and the size condition of S_k if $d(x,x')>2^{-k-1}$. Combining (2.5) and the size condition of S_k, we see that $S_k(\cdot,y)\in C_b^{\eta}(\mathcal{X})$ for all $k\in\mathbb{Z}$ and $y\in\mathcal{X}$. The same holds true for $S_k(y,\cdot)$.

Classical examples of operators satisfying Definition 2.13 for the special case $\mathcal{X}=\mathbb{R}^n$ can be built as follows. Let $\mathbf{F}_{\rm bass}$ ($\mathbf{F}$ stands for *filter*) be the collection of non-negative radial functions $\varphi\in\mathcal{S}(\mathbb{R}^n)$ such that $\operatorname{supp}\widehat{\varphi}\subset\{\xi\in\mathbb{R}^n:|\xi|\le 2\}$ and $\widehat{\varphi}(\xi)=1$ if $|\xi|\le 1$, where $\widehat{\varphi}$ represents the *Fourier transform* of φ. Let $\mathbf{F}_{\rm band}$ be the collection of non-negative radial functions $\psi\in\mathcal{S}(\mathbb{R}^n)$ such that $\operatorname{supp}\widehat{\psi}\subset\{\xi\in\mathbb{R}^n:1/2\le|\xi|\le 4\}$ and $\widehat{\psi}(\xi)=1$ if $1\le|\xi|\le 2$. Given $\varphi\in\mathbf{F}_{\rm bass}$, define S_j by

$$S_j(f)(x) := \int_{\mathbb{R}^n} S_j(x,y)f(y)\,dy, \qquad \forall f\in\mathcal{S}(\mathbb{R}^n) \text{ and } x\in\mathbb{R}^n,$$

where $j\in\mathbb{Z}$ and $S_j(x,y):=2^{jn}\varphi(2^j(x-y))$. Thus, $S_j(f)(x)=\varphi_j*f(x)$, where we used the following convention: Given a function g and $j\in\mathbb{Z}$, we define the dilations g_j as $g_j(x):=2^{jn}g(2^jx)$. Also, set $D_j(x,y):=\psi_j(x-y)$, where $\psi_j:=\varphi_{j+1}-\varphi_j$ and ψ is such that

$$\widehat{\psi}(\xi)=\widehat{\varphi}(\xi/2)-\widehat{\varphi}(\xi), \qquad \forall\xi\in\mathbb{R}^n.$$

Notice that $\psi\in\mathbf{F}_{\rm band}$. The operators S_j and D_j are the basic tools to develop the Littlewood–Paley theory.

Finally, we summarize some properties concerning the size condition of such approximations of the identity as follows (see [60, p. 16, Proposition 2.7]).

LEMMA 2.15. *Suppose that a sequence* $\{S_k\}_{k\in\mathbb{Z}}$ *of functions defined on* $\mathcal{X}\times\mathcal{X}$ *and taking values in* $\mathbb{C}$ *satisfies Definition* 2.13. *Then:*

(i) *there exists a positive constant* C *such that, for all* $k\in\mathbb{Z}$ *and* $x,y\in\mathcal{X}$,

$$\int_{\mathcal{X}} |S_k(x,z)|\,d\mu(z)\le C \quad \text{and} \quad \int_{\mathcal{X}} |S_k(z,y)|\,d\mu(z)\le C;$$

(ii) *for all* $p\in[1,\infty]$, *there exists a positive constant* C_p *such that, for all* $f\in L^p(\mathcal{X})$,

$$\|S_k(f)\|_{L^p(\mathcal{X})}\le C_p\|f\|_{L^p(\mathcal{X})};$$

(iii) *for all* $p\in[1,\infty)$ *and* $f\in L^p(\mathcal{X})$,

$$\lim_{k\to\infty}\|S_k f-f\|_{L^p(\mathcal{X})}=0;$$

(iv) *there exists a positive constant* C *such that, for all* $k\in\mathbb{Z}$, $f\in L^1_{\mathrm{loc}}(\mathcal{X})$ *and* $x\in\mathcal{X}$,

$$|S_k(f)(x)|\le C\mathcal{M}f(x).$$

2.5. Singular integrals on spaces of homogeneous type. In this subsection we follow the pioneer work of Coifman and Weiss [22, 23], and more recent results provided by Han, Müller and Yang [59, 60], to present the analogs of the boundedness for singular integrals on some classical function spaces on $\mathbb{R}^n$ for spaces of homogeneous type. The following theorem is due to Coifman–Weiss [22, Theorem 2.4].

LEMMA 2.16. *Let* $T: C_b^\eta(\mathcal{X})\to(C_b^\eta(\mathcal{X}))'$ *be a continuous linear operator such that, for all* $f\in C_b^\eta(\mathcal{X})$ *and* $x\in\mathcal{X}$ *away from* $\operatorname{supp} f$,

$$T(f)(x)=\int_{\mathcal{X}} K(x,y)f(y)\,d\mu(y),$$

where the kernel K *satisfies Hörmander's condition*

$$\sup_{y,y'\in\mathcal{X}}\int_{d(y,y')\le d(x,y)/2}[|K(x,y)-K(x,y')|+|K(y,x)-K(y',x)|]\,d\mu(x)\le C_K<\infty$$

for some positive constant C_K. *If* T *is bounded on* $L^p(\mathcal{X})$ *for some* $p\in(1,\infty)$, *then*

(i) T *can be extended to a bounded linear operator on* $L^q(\mathcal{X})$ *for all* $q\in(1,\infty)$;
(ii) T *can be extended to a bounded linear operator from* $L^1(\mathcal{X})$ *to* $L^{1,\infty}(\mathcal{X})$.

The norm of T *in* (i) *or* (ii) *is at most a positive constant multiple of*

$$C_K+\|T\|_{L^p(\mathcal{X})\to L^p(\mathcal{X})}.$$

The $T1$-theorem gives necessary and sufficient conditions for the continuity of singular integral operators in $L^2(\mathcal{X})$. The first instance of such a theorem, in the Euclidean setting, was proved by David and Journé [25]. The theorem also extends to RD-spaces [59, 60].

DEFINITION 2.17. Let $\delta\in(0,1]$. A continuous complex-valued function $K(x,y)$ on

$$\Omega:=\{(x,y)\in\mathcal{X}\times\mathcal{X}: x\ne y\}$$

is called a *Calderón–Zygmund kernel of type* δ if there is a positive constant C_K such that, for all $(x,y),(x',y)\in\Omega$,

$$|K(x,y)|\le\frac{C_K}{V(x,y)}$$

and, when $d(x,x')\le d(x,y)/2$,

$$|K(x,y)-K(x',y)|+|K(y,x)-K(y,x')|\le C_K\left[\frac{d(x,x')}{d(x,y)}\right]^{\delta}\frac{1}{V(x,y)}.$$

In this case, write $K\in \mathrm{Ker}(C_K,\delta)$.

Definition 2.18. Let $\eta\in(0,1]$. A *Calderón–Zygmund singular integral operator* is a continuous operator $T: C_b^\eta(\mathcal{X})\to (C_b^\eta(\mathcal{X}))'$ such that, for all $f\in C_b^\eta(\mathcal{X})$ and $x\notin \operatorname{supp} f$,

$$T(f)(x)=\int_{\mathcal{X}}K(x,y)f(y)\,d\mu(y),$$

where the kernel $K\in\mathrm{Ker}(C_K,\delta)$ for some $C_K>0$ and $\delta\in(0,1]$. The transpose T^* of T is defined by

$$\langle T^*f,g\rangle=\langle Tg,f\rangle$$

for all $f,g\in C_b^\eta(\mathcal{X})$. The kernel K^* of T^* is related to the one of T by $K^*(x,y)=K(y,x)$ for all $x,y\in\mathcal{X}$.

Definition 2.19. Given $\eta\in(0,1]$, $x\in\mathcal{X}$ and $r>0$, a function φ on $\mathcal{X}$ is called a *normalized bump function* for the ball $B(x,r)$ if

(i) $\varphi\in C_b^\eta(\mathcal{X})$ and $\operatorname{supp}\varphi\subset B(x,r)$;
(ii) $\|\varphi\|_{L^\infty(\mathcal{X})}\le 1$;
(iii) $|\varphi(z)-\varphi(y)|\le r^{-\eta}d(z,y)^\eta$ for all $z,y\in\mathcal{X}$.

Denote by $\mathcal{T}(\eta,x,r)$ the collection of all normalized bump functions for the ball $B(x,r)$.

Definition 2.20. Let $0<\eta\le\theta$. A singular operator $T: C_b^\eta(\mathcal{X})\to(C_b^\eta(\mathcal{X}))'$ is said to have the *weak boundedness property* (for short, WBP(η)) if there exists a positive constant C such that, for all $x\in\mathcal{X}$, $r>0$ and $\varphi,\psi\in\mathcal{T}(\eta,x,r)$,

$$|\langle T\varphi,\psi\rangle|\le C\mu(B(x,r)). \tag{2.6}$$

The smallest possible constant C in (2.6) is denoted by $\|T\|_{\mathrm{WBP}(\eta)}$.

The following BMO-type spaces on spaces of homogeneous type $(\mathcal{X},d,\mu)$ were introduced by Coifman and Weiss [23].

Definition 2.21. Let $q\in[1,\infty)$. A function $f\in L^q_{\mathrm{loc}}(\mathcal{X})$ is said to be in the space $\mathrm{BMO}_q(\mathcal{X})$ if

$$\|f\|_{\mathrm{BMO}_q(\mathcal{X})}:=\left\{\sup_{B\subset\mathcal{X}}\frac{1}{\mu(B)}\int_B|f(x)-f_B|^q\,d\mu(x)\right\}^{1/q}<\infty,$$

where

$$f_B:=\frac{1}{\mu(B)}\int_B f(y)\,d\mu(y).$$

Remark 2.22. (i) If $f_1,f_2\in\mathrm{BMO}_q(\mathcal{X})$ and f_1-f_2 is a constant, then we regard f_1 and f_2 as the same element in $\mathrm{BMO}_q(\mathcal{X})$.

(ii) If $q=1$, we write $\mathrm{BMO}(\mathcal{X})$ instead of $\mathrm{BMO}_1(\mathcal{X})$ for simplicity.

(iii) For any given $q\in(1,\infty)$, the two spaces $\mathrm{BMO}_q(\mathcal{X})$ and $\mathrm{BMO}(\mathcal{X})$ coincide with equivalent norms; see [23, pp. 593–594].

In the following $T1$-theorem on spaces of homogeneous type, "(i)$\Leftrightarrow$(ii)" is due to [60, Theorem 5.56], and the proof of [60, Theorem 5.57] implies that "(i)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(ii)".

THEOREM 2.23. *Let $\epsilon \in (0,1]$, $\eta \in (0,\epsilon)$ and T be a continuous linear operator from $C_b^\eta(\mathcal{X})$ to $(C_b^\eta(\mathcal{X}))'$ as in Definition 2.18 associated with a kernel $K \in \mathrm{Ker}(C_K,\epsilon)$ for some $C_K > 0$. Then the following statements are equivalent:*

(i) *T extends to a bounded linear operator on $L^2(\mathcal{X})$;*

(ii) *$T(1) \in \mathrm{BMO}(\mathcal{X})$, $T^*(1) \in \mathrm{BMO}(\mathcal{X})$ and $T \in \mathrm{WBP}(\eta)$;*

(iii) *there exists a positive constant C such that, for all $x \in \mathcal{X}$, $R > 0$ and $\phi \in \mathcal{T}(\eta,x,R)$,*

$$\|T(\phi)\|_{L^2(\mathcal{X})} + \|T^*(\phi)\|_{L^2(\mathcal{X})} < C\mu(B(x,R));$$

(iv) *for all $x \in \mathcal{X}$, $R > 0$ and $\phi \in \mathcal{T}(\eta,x,R)$,*

$$\|T(\phi)\|_{\mathrm{BMO}(\mathcal{X})} + \|T^*(\phi)\|_{\mathrm{BMO}(\mathcal{X})} < \infty.$$

3. Multilinear Calderón–Zygmund theory

This section is entirely devoted to the extension of the multilinear Calderón–Zygmund theory in the Euclidean case, as developed by Grafakos and Torres in [53], to the context of RD-spaces $(\mathcal{X}, d, \mu)$. This theory stems from the work of Coifman and Meyer [19, 20, 21, 82]; see also Kenig and Stein [68].

3.1. Multilinear Calderón–Zygmund operators. Motivated by [53] we study the following multilinear singular integrals on $(\mathcal{X}, d, \mu)$.

Definition 3.1. Given $m \in \mathbb{N}$, set

$$\Omega_m := \mathcal{X}^{m+1} \setminus \{(y_0, y_1, \ldots, y_m) : y_0 = y_1 = \cdots = y_m\}.$$

Suppose that $K : \Omega_m \to \mathbb{C}$ is locally integrable. The function K is called a *Calderón–Zygmund kernel* if there exist constants $C_K \in (0, \infty)$ and $\delta \in (0, 1]$ such that, for all $(y_0, y_1, \ldots, y_m) \in \Omega_m$,

$$|K(y_0, y_1, \ldots, y_m)| \le C_K \frac{1}{[\sum_{k=1}^m V(y_0, y_k)]^m} \tag{3.1}$$

and that, for all $k \in \{0, 1, \ldots, m\}$,

$$\begin{aligned} &|K(y_0, y_1, \ldots, y_k, \ldots, y_m) - K(y_0, y_1, \ldots, y_k', \ldots y_m)| \\ &\qquad\le C_K \left[\frac{d(y_k, y_k')}{\max_{0\le k\le m} d(y_0, y_k)}\right]^\delta \frac{1}{[\sum_{k=1}^m V(y_0, y_k)]^m} \end{aligned} \tag{3.2}$$

whenever $d(y_k, y_k') \le \max_{0\le k\le m} d(y_0, y_k)/2$. In this case, write $K \in \mathrm{Ker}(m, C_K, \delta)$.

Definition 3.2. Let $\eta \in (0, 1]$. An *m-linear Calderón–Zygmund operator* is a continuous operator

$$T : \overbrace{C_b^\eta(\mathcal{X}) \times \cdots \times C_b^\eta(\mathcal{X})}^{m \text{ times}} \to (C_b^\eta(\mathcal{X}))'$$

such that, for all $f_1, \ldots, f_m \in C_b^\eta(\mathcal{X})$ and $x \notin \bigcap_{i=1}^m \operatorname{supp} f_i$,

$$T(f_1, \ldots, f_m)(x) = \int_{\mathcal{X}^m} K(x, y_1, \ldots, y_m) \prod_{i=1}^m f_i(y_i)\, d\mu(y_1) \cdots d\mu(y_m), \tag{3.3}$$

where the kernel K is in $\mathrm{Ker}(m, C_K, \delta)$ for some $C_K > 0$ and $\delta \in (0, 1]$. As an m-linear operator, T has m formal transposes. The jth transpose T^{*j} of T is defined by

$$\langle T^{*j}(f_1, \ldots, f_m), g\rangle = \langle T(f_1, \ldots, f_{j-1}, g, f_{j+1}, \ldots, f_m), f_j\rangle$$

for all $f_1, \ldots, f_m, g$ in $C_b^\eta(\mathcal{X})$. The kernel K^{*j} of T^{*j} is related to the one of T by

$$K^{*j}(x, y_1, \ldots, y_{j-1}, y_j, y_{j+1}, \ldots, y_m) = K(y_j, y_1, \ldots, y_{j-1}, x, y_{j+1}, \ldots, y_m).$$

To maintain uniform notation, we may occasionally denote T by T^{*0} and K by K^{*0}.

3.2. Multilinear weak-type estimates. We use the Calderón–Zygmund decomposition to obtain the endpoint weak-type boundedness for multilinear operators; see [53] when $(\mathcal{X}, d, \mu)$ is the Euclidean space.

THEOREM 3.3. *Let T be an m-linear Calderón–Zygmund operator associated with a kernel $K \in \mathrm{Ker}(m, C_K, \delta)$. Assume that, for some $1 \le q_1, q_2, \ldots, q_m \le \infty$ and some $0 < q < \infty$ with $\sum_{j=1}^m 1/q_j = 1/q$, T maps $L^{q_1}(\mathcal{X}) \times \cdots \times L^{q_m}(\mathcal{X}) \to L^{q,\infty}(\mathcal{X})$. Then T can be extended to a bounded m-linear operator from the m-fold product $L^1(\mathcal{X}) \times \cdots \times L^1(\mathcal{X})$ to $L^{1/m,\infty}(\mathcal{X})$ and*

$$\|T\|_{L^1(\mathcal{X}) \times \cdots \times L^1(\mathcal{X}) \to L^{1/m,\infty}(\mathcal{X})} \le C[C_K + \|T\|_{L^{q_1}(\mathcal{X}) \times \cdots \times L^{q_m}(\mathcal{X}) \to L^{q,\infty}(\mathcal{X})}]$$

for some positive constant C that depends only on C_1, C_2, δ and m.

To show Theorem 3.3, we first establish the following lemma.

LEMMA 3.4. *For any $\delta > 0$, there exists positive constant C, depending only on C_1, δ and m, such that, for all $i \in \{1, \ldots, m\}$ and all $x, y_k \in \mathcal{X}$ with $k \ne i$,*

$$\int_{\mathcal{X}} \left[\frac{1}{\max_{1 \le k \le m} d(x, y_k)}\right]^\delta \frac{1}{\sum_{k=1}^m V(x, y_k)}\, d\mu(y_i) \le C \left[\frac{1}{\max_{1 \le k \le m,\, k \ne i} d(x, y_k)}\right]^\delta.$$

Proof. Let

$$a := \max_{1 \le k \le m,\, k \ne i} d(x, y_k).$$

By (a) and (b) of Lemma 2.4 and (2.1), we see that, when $d(x, y_i) < 2a$,

$$\sum_{k=1}^m V(x, y_k) \sim \mu(B(x, a)),$$

which further implies that

$$\begin{aligned}
\int_{d(x,y_i)<2a} &\left[\frac{1}{\max_{1 \le k \le m} d(x, y_k)}\right]^\delta \frac{1}{\sum_{k=1}^m V(x, y_k)}\, d\mu(y_i) \\
&\lesssim \left[\frac{1}{\max_{1 \le k \le m,\, k \ne i} d(x, y_k)}\right]^\delta \frac{\mu(B(x, 2a))}{\mu(B(x, a))} \\
&\lesssim \left[\frac{1}{\max_{1 \le k \le m,\, k \ne i} d(x, y_k)}\right]^\delta.
\end{aligned}$$

On the other hand, when $d(x, y_i) \ge 2a$, we have

$$\max_{1 \le k \le m} d(x, y_k) = d(x, y_i) \quad \text{and} \quad \sum_{k=1}^m V(x, y_k) \sim V(x, y_i);$$

consequently, again using (2.1) implies that

$$\begin{aligned}
\int_{d(x,y_i)\geq 2a} &\left[\frac{1}{\max_{1\leq k\leq m} d(x,y_k)}\right]^{\delta} \frac{1}{\sum_{k=1}^m V(x,y_k)}\,d\mu(y_i)\\
&\sim \sum_{\ell=1}^{\infty} \int_{2^\ell a\leq d(x,y_i)<2^{\ell+1}a} \left[\frac{1}{d(x,y_i)}\right]^{\delta} \frac{1}{V(x,y_i)}\,d\mu(y_m)\\
&\lesssim \sum_{\ell=1}^{\infty} \left(\frac{1}{2^\ell a}\right)^{\delta} \frac{\mu(B(x,2^{\ell+1}a))}{\mu(B(x,2^\ell a))}\\
&\lesssim \left[\frac{1}{\max_{1\leq k\leq m,\,k\neq i} d(x,y_k)}\right]^{\delta}. \qquad \blacksquare
\end{aligned}$$

Applying Lemmas 2.11 and 3.4, we now show Theorem 3.3.

Proof of Theorem 3.3. Let

$$A := \|T\|_{L^{q_1}(\mathcal{X})\times\cdots\times L^{q_m}(\mathcal{X})\to L^{q,\infty}(\mathcal{X})}.$$

Since the space $C_b^\eta(\mathcal{X})$ is dense in $L^1(\mathcal{X})$ for any $\eta\in(0,1]$ (see [60, p. 22, Corollary 2.11]), it suffices to prove the theorem for functions $\{f_j\}_{j=1}^m\subset C_b^\eta(\mathcal{X})$. Fix $\alpha>0$. By homogeneity, without loss of generality, we may assume that $\|f_j\|_{L^1(\mathcal{X})}=1$. For any $\alpha\in(0,\infty)$, let

$$E_\alpha := \{x\in\mathcal{X} : |T(f_1,\dots,f_m)(x)|>\alpha\}.$$

We only need to show that

$$\mu(E_\alpha)\lesssim (C_K+A)^{1/m}\alpha^{-1/m}.$$

Let $\gamma>0$ be a constant to be determined later. For all $j\in\{1,\dots,m\}$, apply the Calderón–Zygmund decomposition (Lemma 2.11) to f_j at height $(\alpha\gamma)^{1/m}$ to obtain good and bad functions g_j and b_j and families of balls, $\{B_{j,k}\}_{k\in I_j, j\in\{1,\dots,m\}}$ with $\{I_j\}_{1\leq j\leq m}$ being index sets, such that $f_j=g_j+b_j$, where $b_j=\sum_{k\in I_j} b_{j,k}$ satisfying, for all $k\in I_j$ and $s\in[1,\infty]$,

(i) $\operatorname{supp} b_{j,k}\subset B_{j,k}$ and $\int_{\mathcal{X}} b_{j,k}(y)\,d\mu(y)=0$;

(ii) $\|b_{j,k}\|_{L^1(\mathcal{X})}\lesssim(\alpha\gamma)^{1/m}\mu(B_{j,k})$;

(iii) $\sum_{k\in I_j}\mu(B_{j,k})\lesssim(\alpha\gamma)^{-1/m}$;

(iv) $\|g_j\|_{L^s(\mathcal{X})}\lesssim(\alpha\gamma)^{(1-1/s)/m}$ and $\|b_j\|_{L^1(\mathcal{X})}\leq\sum_{k\in I_j}\|b_{j,k}\|_{L^1(\mathcal{X})}\lesssim 1.$

Also, notice that g_j and b_j as above are functions in $C_b^\eta(\mathcal{X})$ by Lemma 2.11.

Now let

$$\begin{aligned}
E_1 &= \{x : |T(g_1,g_2,\dots,g_m)(x)|>\alpha/2^m\},\\
E_2 &= \{x : |T(b_1,g_2,\dots,g_m)(x)|>\alpha/2^m\},\\
E_3 &= \{x : |T(g_1,b_2,\dots,g_m)(x)|>\alpha/2^m\},\\
&\ \vdots\\
E_{2^m} &= \{x : |T(b_1,b_2,\dots,b_m)(x)|>\alpha/2^m\}.
\end{aligned}$$

Since

$$\mu(\{x : |T(f_1, \dots, f_m)(x)| > \alpha\}) \le \sum_{r=1}^{2^m} \mu(E_r),$$

we only need to prove that, for all $r \in \{1, \dots, 2^m\}$,

$$\mu(E_r) \lesssim (C_K + A)^{1/m} \alpha^{-1/m}. \tag{3.4}$$

Chebyshev's inequality and the $L^{q_1}(\mathcal{X}) \times \cdots \times L^{q_m}(\mathcal{X}) \to L^{q,\infty}(\mathcal{X})$ boundedness of T give

$$\begin{aligned} \mu(E_1) &\le \left(\frac{2^m A}{\alpha}\right)^q \|g_1\|^q_{L^{q_1}(\mathcal{X})} \cdots \|g_m\|^q_{L^{q_m}(\mathcal{X})} \\ &\lesssim \left(\frac{A}{\alpha}\right)^q \prod_{j=1}^m (\alpha\gamma)^{q/(mq_j')} \sim \left(\frac{A}{\alpha}\right)^q (\alpha\gamma)^{(m-1/q)q/m} \sim A^q \alpha^{-1/m} \gamma^{q-1/m}. \end{aligned}$$

Consider a set E_r as above with $2 \le r \le 2^m$. Suppose that, for some $1 \le \ell \le m$, we have exactly ℓ bad functions appearing in $T(h_1, \dots, h_m)$, where $h_j \in \{b_j, g_j\}$ and assume that the bad functions occur at the entries $j_1, \dots, j_\ell$. The next step is to show that

$$\begin{aligned} \mu(E_r) &= \mu(\{x : |T(h_1, \dots, h_m)(x)| > \alpha/2^m\}) \\ &\lesssim \alpha^{-1/m}[\gamma^{-1/m} + \gamma^{-1/m}(\gamma C_K)^{1/\ell}]. \end{aligned} \tag{3.5}$$

Let $r_{j,k}$ and $c_{j,k}$ be the radius and the center of the ball $B_{j,k}$, respectively. Set

$$(B_{j,k})^{**} := B(c_{j,k}, 5r_{j,k}).$$

Since

$$\mu\Big(\bigcup_{j=1}^m \bigcup_{k \in I_j} (B_{j,k})^{**}\Big) \lesssim (\alpha\gamma)^{-1/m},$$

(3.5) is a consequence of

$$\mu\Big(\Big\{x \notin \bigcup_{j=1}^m \bigcup_{k \in I_j} (B_{j,k})^{**} : |T(h_1, \dots, h_m)(x)| > \alpha/2^m\Big\}\Big) \lesssim (\alpha\gamma)^{-1/m}(\gamma C_K)^{1/\ell}. \tag{3.6}$$

Fix $x \notin \bigcup_{j=1}^m \bigcup_{k \in I_j} (B_{j,k})^{**}$. Then

$$\begin{aligned} |T(h_1, \dots, h_m)(x)| &\le \sum_{k_1 \in I_{j_1}} \cdots \sum_{k_\ell \in I_{j_\ell}} \bigg| \int_{\mathcal{X}^m} K(x, y_1, \dots, y_m) \\ &\qquad \times \prod_{i \notin \{j_1, \dots, j_\ell\}} g_i(y_i) \prod_{i=1}^\ell b_{j_i,k_i}(y_{j_i})\, d\mu(y_1) \cdots d\mu(y_m) \bigg| \\ &=: \sum_{k_1 \in I_{j_1}} \cdots \sum_{k_\ell \in I_{j_\ell}} \mathrm{H}_{k_1, \dots, k_\ell}. \end{aligned}$$

Fix, for the moment, the balls $B^*_{j_1,k_1}, \dots, B^*_{j_\ell,k_\ell}$; without loss of generality, we may suppose that $B^*_{j_1,k_1}$ has the smallest radius among them. Notice that

$$\begin{aligned}
&\left|\int_{B^*_{j_1,k_1}} K(x,y_1,\dots,y_m) b_{j_1,k_1}(y_{j_1})\,d\mu(y_{j_1})\right| \\
&\quad= \left|\int_{B^*_{j_1,k_1}} [K(x,y_1,\dots,y_{j_1},\dots,y_m) - K(x,y_1,\dots,c_{j_1,k_1},\dots,y_m)]\, b_{j_1,k_1}(y_{j_1})\,d\mu(y_{j_1})\right| \\
&\quad\le C_K \int_{B^*_{j_1,k_1}} \left[\frac{d(y_{j_1},c_{j_1,k_1})}{\max_{1\le k\le m} d(x,y_k)}\right]^\delta \frac{1}{[\sum_{k=1}^m V(x,y_k)]^m} |b_{j_1,k_1}(y_{j_1})|\,d\mu(y_{j_1}).
\end{aligned}$$

Integrating with respect to every y_i with $i\notin\{j_1,\dots,j_\ell\}$ and using Lemma 3.4 $m-\ell$ times, we obtain

$$\begin{aligned}
&\int_{\mathcal{X}^{m-\ell}} \left|\int_{B^*_{j_1,k_1}} K(x,y_1,\dots,y_m) b_{j_1,k_1}(y_{j_1})\,d\mu(y_{j_1})\right| \prod_{i\notin\{j_1,\dots,j_\ell\}} d\mu(y_i) \\
&\qquad\le C_K \int_{B^*_{j_1,k_1}} |b_{j_1,k_1}(y_{j_1})| \\
&\qquad\qquad\times \left\{\int_{\mathcal{X}^{m-\ell}} \left[\frac{d(y_{j_1},c_{j_1,k_1})}{\max_{1\le k\le m} d(x,y_k)}\right]^\delta \frac{1}{[\sum_{k=1}^m V(x,y_k)]^m} \prod_{i\notin\{j_1,\dots,j_\ell\}} d\mu(y_i)\right\} d\mu(y_{j_1}) \\
&\qquad\lesssim C_K \int_{B^*_{j_1,k_1}} |b_{j_1,k_1}(y_{j_1})| \left[\frac{d(y_{j_1},c_{j_1,k_1})}{\max_{1\le i\le \ell} d(x,y_{j_i})}\right]^\delta \frac{1}{[\sum_{i=1}^\ell V(x,y_{j_i})]^\ell}\,d\mu(y_{j_1}) \\
&\qquad\lesssim C_K \left[\frac{r_{j_1,k_1}}{\max_{1\le i\le \ell} d(x,c_{j_i,k_i})}\right]^\delta \frac{\|b_{j_1,k_1}\|_{L^1(\mathcal{X})}}{[\sum_{i=1}^\ell V(x,c_{j_i,k_i})]^\ell},
\end{aligned}$$

where, in the last step, we used the fact that $y_{j_i}\in B_{j_i,k_i}$ and $x\notin\bigcup_{j=1}^m\bigcup_{k\in I_j}(B_{j,k})^{**}$ imply that

$$d(x,y_{j_i})\sim d(x,c_{j_i,k_i}) \quad\text{and}\quad V(x,y_{j_i})\sim V(x,c_{j_i,k_i}).$$

By the arithmetic-geometric mean inequality and since we are assuming that r_{j_1,k_1} is the smallest among $\{r_{j_i,k_i}\}_{i=1}^\ell$, we have

$$\left[\frac{r_{j_1,k_1}}{\max_{1\le i\le \ell} d(x,c_{j_i,k_i})}\right]^\delta \frac{1}{[\sum_{i=1}^\ell V(x,c_{j_i,k_i})]^\ell} \le \prod_{i=1}^\ell \left[\frac{r_{j_i,k_i}}{d(x,c_{j_i,k_i})}\right]^{\delta/\ell} \frac{1}{V(x,c_{j_i,k_i})}.$$

Then, by the fact that, for all $i\in\{1,\dots,m\}$,

$$\|g_i\|_{L^\infty(\mathcal{X})}\lesssim(\alpha\gamma)^{1/m},$$

we have

$$\begin{aligned}
\mathrm{H}_{k_1,\dots,k_\ell} &\le \int_{\mathcal{X}^{m-1}} \left|\int_{B^*_{j_1,k_1}} K(x,y_1,\dots,y_m) b_{j_1,k_1}(y_{j_1})\,d\mu(y_{j_1})\right| \\
&\qquad\times \prod_{i\notin\{j_1,\dots,j_\ell\}} |g_i(y_i)| \prod_{i=2}^\ell |b_{j_i,k_i}(y_{j_i})| \prod_{\substack{i\in\{1,\dots,m\}\\ i\ne j_1}} d\mu(y_i) \\
&\lesssim C_K (\alpha\gamma)^{\frac{m-\ell}{m}} \|b_{j_1,k_1}\|_{L^1(\mathcal{X})} \int_{\mathcal{X}^{\ell-1}} \prod_{i=2}^\ell |b_{j_i,k_i}(y_{j_i})| \\
&\qquad\times \prod_{i=1}^\ell \left[\frac{r_{j_i,k_i}}{d(x,c_{j_i,k_i})}\right]^{\delta/\ell} \frac{1}{V(x,c_{j_i,k_i})}\,d\mu(y_{j_2})\,d\mu(y_{j_3})\cdots d\mu(y_{j_\ell})
\end{aligned}$$

$$\lesssim C_K(\alpha\gamma)^{\frac{m-\ell}{m}} \prod_{i=1}^{\ell} \left[\frac{r_{j_i,k_i}}{d(x,c_{j_i,k_i})}\right]^{\delta/\ell} \frac{\|b_{j_i,k_i}\|_{L^1(\mathcal{X})}}{V(x,c_{j_i,k_i})}$$
$$\lesssim C_K(\alpha\gamma)^{\frac{m-\ell}{m}} \prod_{i=1}^{\ell} \left[\frac{r_{j_i,k_i}}{d(x,c_{j_i,k_i})}\right]^{\delta/\ell} \frac{(\alpha\gamma)^{1/m}\mu(B_{j_i,k_i})}{V(x,c_{j_i,k_i})}.$$

We now bound $|T(h_1,\ldots,h_m)(x)|$ as follows: for any $x \notin \bigcup_{j=1}^m \bigcup_{k\in I_j}(B_{j,k})^{**}$,

$$\begin{aligned}|T(h_1,\ldots,h_m)(x)| &\lesssim C_K(\alpha\gamma)^{\frac{m-\ell}{m}} \sum_{k_1\in I_{j_1}} \cdots \sum_{k_\ell\in I_{j_\ell}} \prod_{i=1}^{\ell} \left[\frac{r_{j_i,k_i}}{d(x,c_{j_i,k_i})}\right]^{\delta/\ell} \frac{(\alpha\gamma)^{1/m}\mu(B_{j_i,k_i})}{V(x,c_{j_i,k_i})}\\
&\lesssim C_K\alpha\gamma \prod_{i=1}^{\ell}\left(\sum_{k_i\in I_{j_i}} \left[\frac{r_{j_i,k_i}}{d(x,c_{j_i,k_i})}\right]^{\delta/\ell} \frac{\mu(B_{j_i,k_i})}{V(x,c_{j_i,k_i})}\right)\\
&\lesssim C_K\alpha\gamma \prod_{i=1}^{\ell}\sum_{k_i\in I_{j_i}} [\mathcal{M}(\chi_{B_{j_i,k_i}})(x)]^{1+\delta/(n\ell)},\end{aligned}$$

where, in the last step, the doubling condition (2.1) and $x\notin (B_{j_i,k_i})^{**}$ imply that

$$\begin{aligned}\left[\frac{r_{j_i,k_i}}{d(x,c_{j_i,k_i})}\right]^{\delta/\ell} \frac{\mu(B_{j_i,k_i})}{V(x,c_{j_i,k_i})} &\sim \left[\frac{r_{j_i,k_i}}{d(x,c_{j_i,k_i})+r_{j_i,k_i}}\right]^{\delta/\ell} \frac{\mu(B_{j_i,k_i})}{\mu(B(x,d(x,c_{j_i,k_i})+r_{j_i,k_i}))}\\
&\lesssim \left[\frac{\mu(B_{j_i,k_i})}{\mu(B(x,d(x,c_{j_i,k_i})+r_{j_i,k_i}))}\right]^{1+\delta/(n\ell)}\\
&\lesssim [\mathcal{M}(\chi_{B_{j_i,k_i}})(x)]^{1+\delta/(n\ell)}.\end{aligned}$$

By this, the $L^{1+\delta/(n\ell)}(\mathcal{X})$-boundedness of $\mathcal{M}$ and Hölder's inequality, we conclude that

$$\begin{aligned}&\mu\Big(\Big\{x\notin \bigcup_{j=1}^m\bigcup_{k\in I_j}(B_{j,k})^{**} : |T(h_1,\ldots,h_m)(x)|>\alpha/2^m\Big\}\Big)\\
&\qquad\lesssim \alpha^{-1/\ell}\int_{\mathcal{X}\setminus\bigcup_{j=1}^m\bigcup_{k\in I_j}(B_{j,k})^{**}} |T(h_1,\ldots,h_m)(x)|^{1/\ell}\,d\mu(x)\\
&\qquad\lesssim (C_K\gamma)^{1/\ell}\int_{\mathcal{X}}\Big\{\prod_{i=1}^{\ell}\sum_{k_i\in I_{j_i}}[\mathcal{M}(\chi_{B_{j_i,k_i}})(x)]^{1+\delta/(n\ell)}\Big\}^{1/\ell}\,d\mu(x)\\
&\qquad\lesssim (C_K\gamma)^{1/\ell}\prod_{i=1}^{\ell}\Big\{\sum_{k_i\in I_{j_i}}\int_{\mathcal{X}}[\mathcal{M}(\chi_{B_{j_i,k_i}})(x)]^{1+\delta/(n\ell)}\,d\mu(x)\Big\}^{1/\ell}\\
&\qquad\lesssim (C_K\gamma)^{1/\ell}\prod_{i=1}^{\ell}\Big\{\sum_{k_i\in I_{j_i}}\mu(B_{j_i,k_i})\Big\}^{1/\ell}\\
&\qquad\lesssim (C_K\gamma)^{1/\ell}(\alpha\gamma)^{-1/m}.\end{aligned}$$

This proves (3.6). Selecting $\gamma=(C_K+A)^{-1}$, we see that all the sets E_r satisfy (3.4), which completes the proof of Theorem 3.3. ∎

4. Weighted multilinear Calderón–Zygmund theory

Weighted estimates for multilinear Calderón–Zygmund operators first appear in the article of Grafakos and Torres [53]. Weighted estimates for multilinear commutators via the sharp maximal function were subsequently obtained by Pérez and Torres [91]. One of the main motivations for the results in this section comes from the article by Lerner, Ombrosi, Pérez, Torres, and Trujillo–González [72] where a very natural multiple-weight theory adapted to the multilinear Calderón–Zygmund theory was developed. One should mention the recent work by Bui and Duong [13], in which multiple weighted norm inequalities for multilinear Calderón–Zygmund operators on $\mathbb{R}^n$ were studied, but with kernels satisfying some mild regularity condition which is weaker than the usual Hölder continuity.

If a measure ρ is absolutely continuous with respect to the measure μ, that is, there is a non-negative locally integrable function w such that $d\rho(x) = w(x)d\mu(x)$ for all $x \in \mathcal{X}$, then ρ is called a *weighted measure with respect to* μ and w is called a *weight.* A weight w is said to belong to the *Muckenhoupt class* A_p for $p \in (1, \infty)$ if

$$[w]_{A_p} := \sup_B \left[\frac{1}{\mu(B)} \int_B w(y)\, d\mu(y)\right] \left[\frac{1}{\mu(B)} \int_B w(y)^{1-p'}\, d\mu(y)\right]^{p-1} < \infty,$$

where the supremum is taken over all balls B contained in $\mathcal{X}$. When $p = 1$, a weight w is said to belong to the *Muckenhoupt class* A_1 if

$$[w]_{A_1} := \sup_B \left[\frac{1}{\mu(B)} \int_B w(y)\, d\mu(y)\right] \left[\inf_B w(x)\right]^{-1} < \infty.$$

Set

$$A_\infty := \bigcup_{1 \le p < \infty} A_p.$$

For more details on A_p weights on spaces of homogeneous type, see for instance [95]; for the Muckenhoupt class on $\mathbb{R}^n$, see for example [41, 36].

As part of results in this section we extend the multiple-weight Calderón–Zygmund theory of [72] to the context of spaces of homogeneous type. Multiple-weight norm inequalities for maximal truncated operators of multilinear singular integrals are also obtained.

4.1. Multiple weights. In the context of RD-spaces, motivated by [72], we consider the following multiple weights.

DEFINITION 4.1. For m exponents $p_1, \dots, p_m$, write p for the exponent defined by

$$\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$$

and $\vec{P} := (p_1, \dots, p_m)$.

DEFINITION 4.2. Let $1 \le p_1, \dots, p_m < \infty$ and $p \in (0,\infty)$ be such that $1/p = \sum_{j=1}^m 1/p_j$. Suppose that ν is a weight and $\vec{w} := (w_1, \dots, w_m)$ with every w_j a weight. We say that $(\nu; \vec{w})$ satisfies the $A_{\vec{P}}$ *condition* if

$$\sup_{B \text{ balls}} \left[\frac{1}{\mu(B)} \int_B \nu(x)\, d\mu(x)\right]^{1/p} \prod_{j=1}^m \left[\frac{1}{\mu(B)} \int_B w_j(x)^{1-p_j'}\, d\mu(x)\right]^{1/p_j'} < \infty, \tag{4.1}$$

where, when $p_j = 1$,

$$\left[\frac{1}{\mu(B)} \int_B w_j(x)^{1-p_j'}\, d\mu(x)\right]^{1/p_j'}$$

is understood as $(\inf_B w_j)^{-1}$. The expression on the left-hand side of (4.1) is referred to as the $A_{\vec{P}}$ *constant* of $(\nu; \vec{w})$ and denoted by $[(\nu; \vec{w})]_{A_{\vec{P}}}$.

In particular, if ν is taken to be the weight

$$\nu_{\vec{w}} := \prod_{j=1}^m w_j^{p/p_j}, \tag{4.2}$$

then $(\nu_{\vec{w}}; \vec{w})$ and $[(\nu_{\vec{w}}; \vec{w})]_{A_{\vec{P}}}$ are respectively denoted by $\vec{w}$ and $[\vec{w}]_{A_{\vec{P}}}$.

PROPOSITION 4.3. *Let $1 \le p_1, \dots, p_m < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$ and $\vec{P} = (p_1, \dots, p_m)$. For any given weights $\vec{w} := (w_1, \dots, w_m)$ and $\nu_{\vec{w}}$ as in (4.2), the following hold true:*

(i) *If every w_j is in A_{p_j}, then $\vec{w} \in A_{\vec{P}}$ and*

$$[\vec{w}]_{A_{\vec{P}}} \le \prod_{j=1}^m [w_j]_{A_{p_j}}^{1/p_j}.$$

(ii) *If $\vec{w} \in A_{\vec{P}}$, then $\nu_{\vec{w}} \in A_{mp}$ and $w_j^{1-p_j'} \in A_{mp_j'}$ for all $j \in \{1, \dots, m\}$, where the condition $w_j^{1-p_j'} \in A_{mp_j'}$ in the case $p_j = 1$ is understood as $w_j^{1/m} \in A_1$. Moreover, $[\nu_{\vec{w}}]_{A_{mp}} \le [\vec{w}]_{A_{\vec{P}}}$ and*

$$[w_j^{1-p_j'}]_{A_{mp_j'}} \le [\vec{w}]_{A_{\vec{P}}}^{p_j'} \quad \text{if} \quad p_j > 1 \quad \text{or} \quad [w_j^{1/m}]_{A_1} \le [\vec{w}]_{A_{\vec{P}}}^{1/m} \quad \text{if } p_j = 1.$$

(iii) *If $\nu_{\vec{w}} \in A_{mp}$ and $w_j^{1-p_j'} \in A_{mp_j'}$ for all $j \in \{1, \dots, m\}$, then $\vec{w} \in A_{\vec{P}}$ and*

$$[\vec{w}]_{A_{\vec{P}}} \le [\nu_{\vec{w}}]_{A_{mp}} \Big\{ \prod_{1 \le j \le m,\, p_j > 1} [w_j^{-p_j'/p_j}]_{A_{mp_j'}}^{1/p_j'} \Big\} \Big\{ \prod_{1 \le j \le m,\, p_j = 1} [w_j^{1/m}]_{A_1} \Big\},$$

where the condition $w_j^{1-p_j'} \in A_{mp_j'}$ in the case $p_j = 1$ is understood as $w_j^{1/m} \in A_1$.

Proof. To see (i), if each w_j is in A_{p_j}, then from $1/p = \sum_{j=1}^m 1/p_j$ and Hölder's inequality, we have

$$\begin{aligned}
[\vec{w}]_{A_{\vec{P}}} &= \sup_B \left[\frac{1}{\mu(B)} \int_B \nu_{\vec{w}}(x)\, d\mu(x)\right]^{1/p} \prod_{j=1}^m \left[\frac{1}{\mu(B)} \int_B w_j(x)^{1-p_j'}\, d\mu(x)\right]^{1/p_j'} \\
&\le \sup_B \prod_{j=1}^m \left[\frac{1}{\mu(B)} \int_B w_j(x)\, d\mu(x)\right]^{1/p_j} \left[\frac{1}{\mu(B)} \int_B w_j(x)^{1-p_j'}\, d\mu(x)\right]^{1/p_j'} \\
&\le \prod_{j=1}^m [w_j]_{A_{p_j}}^{1/p_j}.
\end{aligned}$$

The proofs for (ii) and (iii) were indeed given in [72, Theorem 3.6], so we omit the details here. ■

4.2. Weighted estimates for the multi-sublinear maximal function. Given $\vec{f} = (f_1, \dots, f_m)$ with every f_i being a locally integrable function on $\mathcal{X}$, we define the *maximal operator* $\mathcal{M}$ by

$$\mathcal{M}(\vec{f})(x) := \sup_{B \ni x} \prod_{i=1}^{m} \frac{1}{\mu(B)} \int_B |f_i(y_i)| \, d\mu(y_i), \qquad \forall x \in \mathbb{R}^n,$$

where the supremum is taken over all balls $B \subset \mathcal{X}$ containing x.

We use the following notation. For $p \in (0, \infty]$ and weight $w \in A_\infty$, denote by $L^p(w)$ the collection of all functions f satisfying

$$\|f\|_{L^p(w)} := \left[\int_{\mathcal{X}} |f(y)|^p w(y) \, d\mu(y) \right]^{1/p} < \infty.$$

Analogously, we denote by $L^{p,\infty}(w)$ the weak space with norm

$$\|f\|_{L^{p,\infty}(w)} := \sup_{t>0} t[w(\{x \in \mathcal{X} : |f(x)| > t\})]^{1/p},$$

where

$$w(E) := \int_E w(x) \, d\mu(x)$$

for all sets E contained in $\mathcal{X}$.

THEOREM 4.4. *Let $1 \le p_1, \dots, p_m < \infty$ and $1/p = 1/p_1 + \cdots + 1/p_m$. Suppose that ν and all w_j, $j \in \{1, \dots, m\}$, are weights. Then $(\nu; \vec{w}) \in A_{\vec{P}}$ if and only if the inequality*

$$\|\mathcal{M}(\vec{f})\|_{L^{p,\infty}(\nu)} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)} \tag{4.3}$$

holds true for all $\vec{f} = (f_1, \dots, f_m) \in L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$.

Proof. Suppose that (4.3) holds. Let

$$\|\mathcal{M}\| := \|\mathcal{M}\|_{L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m) \to L^{p,\infty}(\nu)}.$$

Then, for any $\vec{f} \in L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$, by the differentiation theorem for integrals on $(\mathcal{X}, d, \mu)$, we see that, for all $\epsilon \in (0, 1)$,

$$\begin{aligned} &\left[\int_B \nu(x) \, d\mu(x) \right]^{1/p} \prod_{j=1}^{m} |f_j|_B \\ &\quad \le \left\{ \nu\Big(x \in B : \mathcal{M}(f_1 \chi_B, \dots, f_m \chi_B)(x) > \epsilon \prod_{j=1}^{m} |f_j|_B \Big) \right\}^{1/p} \prod_{j=1}^{m} |f_j|_B \\ &\quad \le \frac{1}{\epsilon} \|\mathcal{M}\| \prod_{j=1}^{m} \|f_j \chi_B\|_{L^{p_j}(w_j)}, \end{aligned} \tag{4.4}$$

where

$$|f_j|_B := \frac{1}{\mu(B)} \int_B |f_j(y)| \, d\mu(y)$$

for all $j \in \{1, \ldots, m\}$, and ν denotes the measure given by

$$\nu(E) := \int_E \nu(z)\, d\mu(z)$$

for all sets $E \subset \mathcal{X}$.

For $j \in \{1, \ldots, m\}$, we set $f_j := w_j^{1-p_j'}$ if $p_j > 1$ and $f_j := \chi_{S_{j,\eta}}$ if $p_j = 1$, where

$$S_{j,\eta} := \Big\{x \in B : w_j(x) < \eta + \inf_B w_j\Big\},$$

η is a positive sufficiently small constant, and inf is the essential infimum. Then, by (4.4), we see that

$$\begin{aligned}
&\left[\int_B \nu(x)\, d\mu(x)\right]^{1/p} \prod_{\{j:\, p_j=1\}} \frac{\mu(S_{j,\eta})}{\mu(B)} \prod_{\{j:\, p_j>1\}} \frac{1}{\mu(B)} \int_B w_j(x)^{1-p_j'}\, d\mu(x)\\
&\qquad \le \frac{1}{\epsilon}\|\mathcal{M}\| \prod_{\{j:\, p_j=1\}} \int_{S_{j,\eta}} w_j(x)\, d\mu(x) \prod_{\{j:\, p_j>1\}} \left[\int_B w_j(x)^{1-p_j'}\, d\mu(x)\right]^{1/p_j}\\
&\qquad \le \frac{1}{\epsilon}\|\mathcal{M}\| \prod_{\{j:\, p_j=1\}} \mu(S_{j,\eta})\Big(\eta + \inf_B w_j\Big) \prod_{\{j:\, p_j>1\}} \left[\int_B w_j(x)^{1-p_j'}\, d\mu(x)\right]^{1/p_j}.
\end{aligned}$$

Letting $\epsilon \to 1$ and $\eta \to 0$, we then conclude that $(\nu; \vec{w}) \in A_{\vec{P}}$ and

$$[(\nu; \vec{w})]_{A_{\vec{P}}} \le \|\mathcal{M}\|.$$

Now assume that $(\nu; \vec{w}) \in A_{\vec{P}}$, that is, $(\nu; \vec{w})$ satisfies (4.1). Applying Hölder's inequality, we obtain

$$\begin{aligned}
\mathcal{M}(\vec{f})(x) &\le \sup_{B \ni x} \prod_{j=1}^m \left[\frac{1}{\mu(B)} \int_B |f_j(y)|^{p_j} w_j(y)\, d\mu(y)\right]^{1/p_j} \left[\frac{1}{\mu(B)} \int_B w_j(y)^{-p_j'/p_j}\, d\mu(y)\right]^{1/p_j'}\\
&\le [(\nu; \vec{w})]_{A_{\vec{P}}} \prod_{j=1}^m [M_\nu(|f_j|^{p_j} w_j/\nu)(x)]^{1/p_j}.
\end{aligned}$$

Here we remark that, when $p_j = 1$, to obtain the second inequality, we just need to replace

$$\left[\frac{1}{\mu(B)} \int_B w_j(y)^{-p_j'/p_j}\, d\mu(y)\right]^{1/p_j'}$$

by $(\inf_B w_j)^{-1}$. Denote by $\mathcal{M}_\nu$ the *weighted Hardy–Littlewood maximal function*,

$$\mathcal{M}_\nu(f)(x) := \sup_{B \ni x} \frac{1}{\int_B \nu(y)\, d\mu(y)} \int_B |f(y)|\nu(y)\, d\mu(y), \quad \forall x \in \mathcal{X},$$

where the supremum is taken over all balls B of $\mathcal{X}$ containing x. Since ν is an A_∞ weight, it follows that $\nu(x)d\mu(x)$ is a doubling measure (see [95, p. 8, Lemma 12]) and $\mathcal{M}_\nu$ is bounded from $L^1(\nu)$ to $L^{1,\infty}(\nu)$. From this and Hölder's inequality for weak spaces (see [40, p. 10, Exercise 1.1.2]), we deduce that

$$\|\mathcal{M}(\vec{f})\|_{L^{p,\infty}(\nu)} \le [(\nu; \vec{w})]_{A_{\vec{P}}} p^{-1/p} \prod_{j=1}^m p_j^{1/p_j} \|M_\nu(|f_j|^{p_j} w_j/\nu)^{1/p_j}\|_{L^{p_j,\infty}(\nu)}$$

$$
\begin{aligned}
&= [(\nu;\vec{w})]_{A_{\vec{P}}} p^{-1/p} \prod_{j=1}^{m} p_j^{1/p_j} \|M_\nu(|f_j|^{p_j} w_j/\nu)\|_{L^{1,\infty}(\nu)}^{1/p_j} \\
&\le [(\nu;\vec{w})]_{A_{\vec{P}}} \|\mathcal{M}_\nu\|_{L^1(\nu)\to L^{1,\infty}(\nu)} p^{-1/p} \prod_{j=1}^{m} p_j^{1/p_j} \|f_j\|_{L^{p_j}(w_j)}.
\end{aligned}
$$

This concludes the proof of Theorem 4.4. ■

The following theorem shows that the hypothesis $(\nu;\vec{w}) \in A_{\vec{P}}$ is not strong enough to imply the boundedness of $\mathcal{M}$ from $\prod_{j=1}^m L^{p_j}(w_j)$ to $L^p(\nu)$. The proof here is partly motivated by the work of Pérez [88].

Theorem 4.5. *For $1 < p_1, \dots, p_m < \infty$ and $0 < p < \infty$ such that $1/p = \sum_{j=1}^m 1/p_j$, the assumption $(\nu, \vec{w}) \in A_{\vec{P}}$ does not imply that $\mathcal{M}$ is bounded from $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ to $L^p(\nu)$.*

Proof. We prove the conclusion by contradiction. Assume that, for all weights ν and $\{w_j\}_{j=1}^m$ satisfying

$$
\sup_{B\subset\mathcal{X}} \left\{\frac{1}{\mu(B)}\int_B \nu(x)\,d\mu(x)\right\}^{1/p} \prod_{j=1}^m \left\{\frac{1}{\mu(B)}\int_B w_j(x)^{-p_j'/p_j}\,d\mu(x)\right\}^{1/p_j'} =: K < \infty, \tag{4.5}
$$

and for all $f_j \in L^{p_j}(w_j)$ with $j \in \{1,\dots,m\}$, we have

$$
\int_{\mathcal{X}} [\mathcal{M}(\vec{f})(x)]^p \nu(x)\,d\mu(x) \lesssim \prod_{j=1}^m \left\{\int_{\mathcal{X}} |f_j(x)|^{p_j} w_j(x)\,d\mu(x)\right\}^{p/p_j}. \tag{4.6}
$$

Fix $x_0 \in \mathcal{X}$. For any $1 \le j \le m$ and $N \in \mathbb{N}$, set

$$
w_j^N := \chi_{B(x_0,1)} + N\chi_{\mathbb{R}^n\setminus B(x_0,1)} \quad\text{and}\quad \nu_N := \prod_{j=1}^m [\mathcal{M}((w_j^N)^{-p_j'/p_j})]^{-p/p_j'}.
$$

Obviously, all ν_N and w_j^N are weights, and $(\nu_N;\vec{w}_N)$ with $\vec{w}_N := (w_1^N,\dots,w_m^N)$ satisfies (4.5) with constant $K=1$. If we now choose $f_j := \chi_{B(x_0,1)}$, then (4.6) becomes

$$
\int_{\mathcal{X}} \frac{[\mathcal{M}(\chi_{B(x_0,1)})(x)]^{mp}}{\prod_{j=1}^m \sup_{B\subset\mathcal{X},B\ni x}\left\{\frac{1}{\mu(B)}\int_R w_j^N(y)^{-p_j'/p_j}\,d\mu(y)\right\}^{p/p_j'}}\,d\mu(x) \lesssim \mu(B(x_0,1)). \tag{4.7}
$$

Obviously, the left side of (4.7) is not less than

$$
\begin{aligned}
\int_{\mathcal{X}} \frac{[\mathcal{M}(\chi_{B(x_0,1)})(x)]^{mp}}{\mathcal{M}(\chi_{B(x_0,1)})(x)^{\sum_{j=1}^m p/p_j'}}\,d\mu(x) &= \int_{\mathcal{X}} \mathcal{M}(\chi_{B(x_0,1)})(x)\,d\mu(x) \\
&\ge \int_{d(x,x_0)\ge 10} \sup_{B\subset\mathcal{X},\,B\ni x} \frac{\mu(B\cap B(x_0,1))}{\mu(B)}\,d\mu(x) \\
&\ge \int_{d(x,x_0)\ge 10} \frac{\mu(B(x_0,1))}{\mu(B(x_0,2d(x,x_0)))}\,d\mu(x) \\
&= \infty;
\end{aligned}
$$

but the right-hand side of (4.7) is finite. This contradiction completes the proof. ■

To obtain the strong boundedness of $\mathcal{M}$, we assume that $(\nu;\vec{w})$ satisfies some *power bump conditions* as in (4.8) below. For $m=1$ and $\mathcal{X}=\mathbb{R}^n$, this type of power bump conditions appears for the first time in the work of Neugebauer [87] but with an extra power bump in the weight ν. Pérez [88] then removed the power from the weight ν and replaced the power bump in w by a *logarithmic bump* or a more general type of bump (see also [89, 90] and the book [24]). In [45], it was proved that, if $(\nu,\vec{w})$ satisfies a certain power bump condition, which is defined by replacing balls with rectangles of $\mathbb{R}^n$ in (4.8), then the multilinear strong maximal function satisfies the corresponding strong type mutiple weighted estimates. Also, Moen [83] used such bump conditions to study weighted inequalities for multilinear fractional integral operators.

THEOREM 4.6. *Let $1<p_1,\dots,p_m<\infty$ and $1/p=1/p_1+\cdots+1/p_m$. Suppose that ν and all w_j, $j\in\{1,\dots,m\}$, are weights. If, for some $r\in(1,\infty)$,*

$$[(\nu;\vec{w})]_{A_{\vec{P},r}}:=\sup_{\textit{balls } B\subset\mathcal{X}}\left[\frac{1}{\mu(B)}\int_B\nu\,d\mu\right]^{1/p}\prod_{j=1}^m\left[\frac{1}{\mu(B)}\int_B w_j^{-rp_j'/p_j}\,d\mu\right]^{1/(rp_j')} \tag{4.8}$$

is finite, then there exists a positive constant C such that, for all $\vec{f}$ with each $f_j\in L^{p_j}(w_j)$,

$$\|\mathcal{M}(\vec{f})\|_{L^p(\nu)}\le C[(\nu;\vec{w})]_{A_{\vec{P},r}}\prod_{j=1}^m\|f_j\|_{L^{p_j}(w_j)}. \tag{4.9}$$

Proof. For any $N\in\mathbb{N}$ and $x\in\mathcal{X}$, set

$$\mathcal{M}_N(\vec{f})(x):=\sup_{B\ni x,\,r_B\le N}\prod_{j=1}^m\frac{1}{\mu(B)}\int_B|f_j(y)|\,d\mu(y),$$

where, for any ball B of $\mathcal{X}$, we use r_B to denote the radius of B. It suffices to show that there exists a positive constant C, independent of N, such that, for all $\vec{f}$ with each $f_j\in L^{p_j}(w_j)$,

$$\|\mathcal{M}_N(\vec{f})\|_{L^p(\nu)}\lesssim[(\nu;\vec{w})]_{A_{\vec{P},r}}\prod_{j=1}^m\|f_j\|_{L^{p_j}(w_j)}, \tag{4.10}$$

since once (4.10) holds then (4.9) follows by letting $N\to\infty$ and applying the monotone convergence lemma. Moreover, by the density of $L_b^\infty(\mathcal{X})$ in $L^{p_j}(w_j)$ for each $j\in\{1,\dots,m\}$, it suffices to show that (4.10) holds true under the assumption that each $f_j\in L_b^\infty(\mathcal{X})$.

To this end, we assume that every $f_j\in L_b^\infty(\mathcal{X})$. Fix $N\in\mathbb{N}$. For any $k\in\mathbb{Z}$, we set

$$\Omega_k:=\{x\in\mathcal{X}:\mathcal{M}_N(\vec{f})(x)>2^k\}.$$

If $x\in\Omega_k\setminus\Omega_{k+1}$, then there exists a ball $B_x\subset\mathcal{X}$ satisfying $r_{B_x}\le N$, $B_x\ni x$ and

$$2^{k+1}\ge\prod_{j=1}^m\frac{1}{\mu(B_x)}\int_{B_x}|f_j(y)|\,d\mu(y)>2^k.$$

This implies that $B_x\subset\Omega_k\setminus\Omega_{k+1}$, and hence

$$\Omega_k\setminus\Omega_{k+1}=\bigcup_{x\in\Omega_k\setminus\Omega_{k+1}}B_x.$$

Applying the basic covering lemma (see Lemma 2.6) to the family of balls

$$\{B_x : x \in \Omega_k \setminus \Omega_{k+1}\},$$

we obtain the existence of a sequence $\{B^k_\alpha\}_{\alpha\in I_k}$ of pairwise disjoint balls contained in $\Omega_k \setminus \Omega_{k+1}$ such that

$$\Omega_k \setminus \Omega_{k+1} \subset \bigcup_{\alpha\in I_k} 5B^k_\alpha$$

and

$$2^{k+1} \geq \prod_{j=1}^m \frac{1}{\mu(B^k_\alpha)} \int_{B^k_\alpha} |f_j(y)|\, d\mu(y) > 2^k.$$

Notice that balls in $\{B^k_\alpha : k \in \mathbb{Z},\ \alpha \in I_k\}$ are pairwise disjoint. Therefore,

$$\begin{aligned}
\|\mathcal{M}_N(\vec{f})\|^p_{L^p(\nu)} &= \sum_{k\in\mathbb{Z}} \int_{\Omega_k\setminus\Omega_{k+1}} [\mathcal{M}(\vec{f})(x)]^p \nu(x)\, d\mu(x)\\
&\leq \sum_{k\in\mathbb{Z}} 2^{p(k+1)} \nu(\Omega_k \setminus \Omega_{k+1})\\
&\leq 2^p \sum_{k\in\mathbb{Z}} \sum_{\alpha\in I_k} \nu(5B^k_\alpha) \Bigg[\prod_{j=1}^m \frac{1}{\mu(B^k_\alpha)} \int_{B^k_\alpha} |f_j(y)|\, d\mu(y)\Bigg]^p. \qquad (4.11)
\end{aligned}$$

By the bump condition of $(\nu; \vec{w})$, Hölder's inequality and the doubling property of μ, we see that

$$\begin{aligned}
&\nu(5B^k_\alpha)\Bigg[\prod_{j=1}^m \frac{1}{\mu(B^k_\alpha)} \int_{B^k_\alpha} |f_j(y)|\, d\mu(y)\Bigg]^p\\
&\qquad \leq \nu(5B^k_\alpha) \prod_{j=1}^m \Bigg[\frac{1}{\mu(B^k_\alpha)} \int_{B^k_\alpha} |f_j(y)|^{(rp'_j)'} [w_j(y)]^{(rp'_j)'/p_j}\, d\mu(y)\Bigg]^{p/(rp'_j)'}\\
&\qquad\quad \times \Bigg[\frac{1}{\mu(B^k_\alpha)} \int_{B^k_\alpha} [w_j(y)]^{-rp'_j/p_j}\, d\mu(y)\Bigg]^{p/(rp'_j)}\\
&\qquad \lesssim [(\nu;\vec{w})]^p_{A_{\vec{P},r}} \mu(B^k_\alpha) \inf_{x\in B^k_\alpha} \prod_{j=1}^m [\mathcal{M}(|f_j|^{(rp'_j)'} w_j^{(rp'_j)'/p_j})(x)]^{p/(rp'_j)'}.
\end{aligned}$$

Inserting this into (4.11) and using the disjointness of $\{B^k_\alpha\}_{k\in\mathbb{Z},\,\alpha\in I_k}$ and Hölder's inequality, we obtain

$$\begin{aligned}
\|\mathcal{M}_N(\vec{f})\|^p_{L^p(\nu)} &\lesssim [(\nu;\vec{w})]^p_{A_{\vec{P},r}} \sum_{k\in\mathbb{Z}} \sum_{\alpha\in I_k} \int_{B^k_\alpha} \prod_{j=1}^m [\mathcal{M}(|f_j|^{(rp'_j)'} w_j^{(rp'_j)'/p_j})(y)]^{p/(rp'_j)'}\, d\mu(y)\\
&\lesssim [(\nu;\vec{w})]^p_{A_{\vec{P},r}} \int_{\mathcal{X}} \prod_{j=1}^m [\mathcal{M}(|f_j|^{(rp'_j)'} w_j^{(rp'_j)'/p_j})(y)]^{p/(rp'_j)'}\, d\mu(y)\\
&\lesssim [(\nu;\vec{w})]^p_{A_{\vec{P},r}} \prod_{j=1}^m \Bigg\{\int_{\mathcal{X}} [\mathcal{M}(|f_j|^{(rp'_j)'} w_j^{(rp'_j)'/p_j})(y)]^{p_j/(rp'_j)'}\, d\mu(y)\Bigg\}^{p/p_j}\\
&\lesssim [(\nu;\vec{w})]^p_{A_{\vec{P},r}} \prod_{j=1}^m \|f_j\|^p_{L^{p_j}(w_j)},
\end{aligned}$$

where, in the last step, we used $p_j/(rp_j')' > 1$ and the fact that $\mathcal{M}$ is bounded on $L^q(\mathcal{X})$ for all $q \in (1, \infty]$. This proves (4.10) and finishes the proof of (4.9). ■

As a corollary of Theorem 4.6 and Proposition 4.3(ii) the following conclusion holds.

THEOREM 4.7. *Let $\vec{P} := (p_1, \dots, p_m)$ with $1 < p_1, \dots, p_m < \infty$ and $1/p = 1/p_1 + \cdots + 1/p_m$. Then $\vec{w} := (w_1, \dots w_m) \in A_{\vec{P}}$ if and only if there exists a positive constant C such that, for all $\vec{f} := (f_1, \dots, f_m)$ with each $f_j \in L^1_{\mathrm{loc}}(\mathcal{X})$,*

$$\|\mathcal{M}(\vec{f})\|_{L^p(\nu_{\vec{w}})} \le C \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}. \tag{4.12}$$

Proof. Necessity follows from Theorem 4.4, so it remains to prove sufficiency. Assume that $\vec{w} \in A_{\vec{P}}$. By Proposition 4.3(ii), each $w_j^{-p_j'/p_j}$ is in the Muckenhoupt class $A_{mp_j'}$, we thus apply the reverse Hölder inequality, that is, there exists $r_j > 1$ such that, for all $r \in [1, r_j]$ and all balls B,

$$\left[\frac{1}{\mu(B)} \int_B w_j(x)^{-rp_j'/p_j}\, d\mu(x)\right]^{1/r} \lesssim \frac{1}{\mu(B)} \int_B w_j(x)^{-p_j'/p_j}\, d\mu(x).$$

This implies that $(\nu_{\vec{w}}, \vec{w})$ satisfies the bump condition (4.8) with respect to r. Therefore, (4.12) is a consequence of Theorem 4.6. ■

We conclude this subsection by showing that $\prod_{j=1}^m \mathcal{M}(f_j)$ satisfies weighted weak-type estimates by means of mixed weak-type inequalities. By the Calderón–Zygmund decomposition and the Marcinkiewicz interpolation, we proceed as the classical arguments (see [28, p. 37]) and conclude that, for any non-negative measurable function w and $p \in (1, \infty)$, there exists a positive constant C, depending only on p and the doubling constant C_1, such that, for all $f \in L^p(\mathcal{M}(w), d\mu)$,

$$\int_{\mathcal{X}} \mathcal{M}(f)(x)^p\, d\mu(x) \le C \int_{\mathcal{X}} |f(x)|^p \mathcal{M}(w)(x)\, d\mu(x) \tag{4.13}$$

and, for all $f \in L^1(w, d\mu)$,

$$\int_{\{x \in \mathcal{X}:\, \mathcal{M}(f)(x) > \lambda\}} w(x)\, d\mu(x) \le \frac{C}{\lambda} \int_{\mathcal{X}} |f(x)| \mathcal{M}(w)(x)\, d\mu(x), \qquad \forall \lambda > 0.$$

Applying (4.13) and Hölder's inequality, we see that, for $1 < p_1, \dots, p_m < \infty$ and $1/p = 1/p_1 + \cdots + 1/p_m$, there exists a positive constant C such that, for all $\vec{f} = (f_1, \dots, f_m)$ in the m-fold product $L^{p_1}(\mathcal{M}(w_1), d\mu) \times \cdots \times L^{p_m}(\mathcal{M}(w_m), d\mu)$,

$$\Big\| \prod_{j=1}^m \mathcal{M}(f_j) \Big\|_{L^p(\nu_{\vec{w}})} \le C \prod_{j=1}^m \|\mathcal{M}(f_j)\|_{L^{p_j}(w_j)} \le C \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathcal{M}(w_j), d\mu)},$$

where $\nu_{\vec{w}} := \prod_{j=1}^m w_j^{p/p_j}$. If there exists some $p_j = 1$, we have the following weak-type conclusion for the operator $\mathcal{M}$ by means of Theorem 4.4.

COROLLARY 4.8. *Let $1 \le p_1, \dots, p_m < \infty$ and $1/p = 1/p_1 + \cdots + 1/p_m$. Then there exists a constant $C > 0$ such that, for all $(f_1, \dots, f_m) \in L^{p_1}(\mathcal{M}(w_1), d\mu) \times \cdots \times L^{p_m}(\mathcal{M}(w_m), d\mu)$,*

$$\Big\| \prod_{j=1}^m \mathcal{M}(f_j) \Big\|_{L^{p,\infty}(\nu_{\vec{w}})} \le C \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathcal{M}(w_j), d\mu)},$$

where $\nu_{\vec{w}} := \prod_{j=1}^m w_j^{p/p_j}$.

Proof. By Hölder's inequality, we have

$$\begin{aligned}
&\sup_B \left[\frac{1}{\mu(B)}\int_B \nu_{\vec{w}}(x)\,d\mu(x)\right]^{1/p} \prod_{j=1}^m \left[\frac{1}{\mu(B)}\int_B \mathcal{M}(w_j)(x)^{1-p_j'}\,d\mu(x)\right]^{1/p_j'}\\
&\qquad \le \prod_{j=1}^m \left[\frac{1}{\mu(B)}\int_B w_j(x)\,d\mu(x)\right]^{1/p_j} \left[\frac{1}{\mu(B)}\int_B \mathcal{M}(w_j)(x)^{1-p_j'}\,d\mu(x)\right]^{1/p_j'}\\
&\qquad \le \prod_{j=1}^m \left[\inf_{x\in B} \mathcal{M}(w_j)(x)\right]^{1/p_j} \left[\sup_{x\in B} \mathcal{M}(w_j)(x)^{1-p_j'}\right]^{1/p_j'}\\
&\qquad \le 1
\end{aligned}$$

with the usual modification if $p_j = 1$. This, together with Theorem 4.4, implies the desired conclusion. ■

4.3. Weighted estimates for multilinear Calderón–Zygmund operators. In this subsection and the following two subsections, we assume that T is an m-linear operator with a kernel $K \in \mathrm{Ker}(m, C_K, \delta)$ for some $C_K > 0$ and $\delta \in (0,1)$, and that T maps $L^{q_1}(\mathcal{X}) \times \cdots \times L^{q_m}(\mathcal{X})$ to $L^q(\mathcal{X})$ with norm $\|T\|_{L^{q_1}(\mathcal{X})\times\cdots L^{q_m}(\mathcal{X})\to L^q(\mathcal{X})}$, where $1 \le q_1, \dots, q_m < \infty$ and $1/q = 1/q_1 + \cdots + 1/q_m$. Based on Theorem 3.3, T is bounded from the m-fold product space $L^1(\mathcal{X}) \times \cdots \times L^1(\mathcal{X})$ to $L^{1/m,\infty}(\mathcal{X})$.

Our main goal in this subsection is to prove the multiple-weight boundedness of the multilinear Calderón–Zygmund operator T. This indeed follows from a Cotlar-type inequality (see Theorem 4.12 below) and the Fefferman–Stein type inequalities related to sharp maximal functions (see Lemma 4.11 below).

Definition 4.9. For any given locally integrable function f on $(\mathcal{X}, d, \mu)$ and for all $x \in \mathcal{X}$, the *sharp maximal function* $\mathcal{M}^\sharp f(x)$ is defined by

$$\mathcal{M}^\sharp f(x) := \sup_{B\subset\mathcal{X}, B\ni x} \frac{1}{\mu(B)}\int_B |f(y) - f_B|\,d\mu(y);$$

here and in what follows,

$$f_B = \frac{1}{\mu(B)}\int_B f(y)\,d\mu(y).$$

Moreover, for any $\delta \in (0,\infty)$, set

$$\mathcal{M}^\sharp_\delta f(x) = \{\mathcal{M}^\sharp(f^\delta)(x)\}^{1/\delta}, \quad x \in \mathcal{X}.$$

Remark 4.10. Observe that, for $\delta \in (0,\infty)$, by the inequality

$$\min\{1, 2^{\delta-1}\}(a^\delta + b^\delta) \le (a+b)^\delta \le \max\{1, 2^{\delta-1}\}(a^\delta + b^\delta), \qquad a>0,\ b>0,$$

we have

$$\mathcal{M}^\sharp_\delta f(x) \sim \sup_{B\ni x}\inf_{c\in\mathbb{C}} \left[\frac{1}{\mu(B)}\int_B \big||f(y)|^\delta - |c|^\delta\big|\,d\mu(y)\right]^{1/\delta}.$$

The following lemma serves as an analog of the classical Fefferman–Stein inequalities in [31, 18]. It should be remarked that, in the setting of spaces of homogeneous type, Martell [79, Corollary 4.3] proved a kind of Fefferman–Stein inequality for another sharp maximal function defined via certain approximations of the identity of Duong and McIntosh [30]. Our proof here invokes some ideas from [79].

Lemma 4.11. *Let $0 < p_0 \le p < \infty$ and $w \in A_\infty$. Then there exists a positive constant $C := C([w]_{A_\infty}, C_1, p)$ such that, for all $f \in L^1_{\rm loc}(\mathcal{X})$ satisfying $\mathcal{M}f \in L^{p_0,\infty}(w)$,*

(i) *if $p_0 < p$ and $\mu(\mathcal{X}) = \infty$, then*
$$\|\mathcal{M}f\|_{L^p(w)} \le C\|\mathcal{M}^\sharp f\|_{L^p(w)};$$
(ii) *if $p_0 \le p$ and $\mu(\mathcal{X}) = \infty$, then*
$$\|\mathcal{M}f\|_{L^{p,\infty}(w)} \le C\|\mathcal{M}^\sharp f\|_{L^{p,\infty}(w)};$$
(iii) *if $p_0 < p$ and $\mu(\mathcal{X}) < \infty$, then*
$$\|\mathcal{M}f\|_{L^p(w)} \le C[\|f\|_{L^1(\mathcal{X})} + \|\mathcal{M}^\sharp f\|_{L^p(w)}];$$
(iv) *if $p_0 \le p$ and $\mu(\mathcal{X}) < \infty$, then*
$$\|\mathcal{M}f\|_{L^{p,\infty}(w)} \le C[\|f\|_{L^1(\mathcal{X})} + \|\mathcal{M}^\sharp f\|_{L^{p,\infty}(w)}].$$

Proof. Using the Calderón–Zygmund decomposition, we obtain the "good-λ" inequality: for any given $\gamma > 0$, there exist positive constants θ_0 and $\widetilde{C}$, depending only on C_1, γ and w, such that, for all $f \in L^1(\mathcal{X})$ and $\lambda > \|f\|_{L^1(\mathcal{X})}/\mu(\mathcal{X})$,
$$\begin{aligned} w(\{x \in \mathcal{X} : \mathcal{M}f(x) > 2C_1^3\lambda,\ \mathcal{M}^\sharp f(x) \le \gamma\lambda\}) \\ \le \widetilde{C}\gamma^{\theta_0} w(\{x \in \mathcal{X} : \mathcal{M}f(x) > \lambda\}), \end{aligned} \tag{4.14}$$
where we recall that $w(E) = \int_E w(x)\,d\mu(x)$ for any set $E \subset \mathcal{X}$.

To prove (4.14), for every $f \in L^1(\mathcal{X})$ and $\lambda > \|f\|_{L^1(\mathcal{X})}/\mu(\mathcal{X})$, we set
$$\Omega_\lambda := \{x \in \mathcal{X} : \mathcal{M}f(x) > \lambda\}.$$
Then Ω_λ is an open proper set of $\mathcal{X}$ with finite measure. Applying Lemma 2.8 with $\Omega = \Omega_\lambda$ and the constant c therein equal to 10, we obtain sequences $\{x_k\}_k \subset \mathcal{X}$ and $\{r_k\}_k := \{\mathrm{dist}(x_k, \Omega^\complement)/20\}_k$ such that
$$\Omega_\lambda := \bigcup_k B(x_k, r_k),$$
$\{B(x_k, r_k/4)\}_k$ are pairwise disjoint, $B(x_k, 30r_k) \cap \Omega_\lambda \ne \emptyset$ and $\{B(x_k, r_k)\}_k$ has the finite overlap property. This implies that the proof for (4.14) can be reduced to the following estimate:
$$w(\{x \in B(x_k, r_k) : \mathcal{M}f(x) > 2C_1^3\lambda,\ \mathcal{M}^\sharp f(x) \le \gamma\lambda\}) \lesssim \gamma^{\theta_0} w(B(x_k, r_k)), \quad \forall k.$$
To prove this, it suffices to show that, for all k,
$$\mu(\{x \in B(x_k, r_k) : \mathcal{M}f(x) > 2C_1^3\lambda,\ \mathcal{M}^\sharp f(x) \le \gamma\lambda\}) \lesssim \gamma\mu(B(x_k, r_k)). \tag{4.15}$$
Indeed, the hypothesis $w \in A_\infty$ implies the existence of constants $C_0 \ge 1$ and $\theta_0 > 0$ such that
$$\frac{w(E)}{w(B)} \le C_0 \left[\frac{\mu(E)}{\mu(B)}\right]^{\theta_0}$$
for every ball B and every measurable set $E \subset B$.

Set $B_k := B(x_k, r_k)$. We may assume that there exists $x_0 \in B_k$ such that

$$\mathcal{M}^\sharp f(x_0) \le \gamma\lambda;$$

otherwise, (4.15) trivially holds true. We may assume that $B(x_k, 30r_k) \cap \Omega_\lambda \neq \emptyset$ contains a certain point y_k; then

$$|f|_{30B_k} \le \mathcal{M}f(y_k) < \lambda$$

and hence

$$\mathcal{M}(|f|_{32B_k}\chi_{32B_k}) \le |f|_{32B_k} < \lambda.$$

For every $x \in B_k$ satisfying $\mathcal{M}f(x) > 2C_1^3\lambda$, there exists a ball $B \ni x$ with radius r_B such that

$$2C_1^3\lambda < \frac{1}{\mu(B)}\int_B |f(y)|\,d\mu(y) \le C_1^2 \frac{1}{\mu(B(x, 2r_B))}\int_{B(x,2r_B)} |f(y)|\,d\mu(y),$$

which, combined with $\mathcal{M}f(y_k) < \lambda$, gives that $y_k \notin B(x, 2r_B)$. Thus, $2r_B < 31r_k$ and therefore $B(x, 2r_B) \subset 32B_k$; moreover, $\mathcal{M}(|f|\chi_{32B_k})(x) > 2C_1\lambda$ and

$$\mathcal{M}((|f| - |f|_{32B_k})\chi_{32B_k})(x) \ge \mathcal{M}(|f|\chi_{32B_k})(x) - \mathcal{M}(|f|_{32B_k}\chi_{32B_k})(x) > C_1\lambda.$$

Summarizing all these we conclude that the left hand side of (4.15) is bounded by

$$\begin{aligned}
&\mu(\{x \in B(x_k, r_k) : \mathcal{M}((|f| - |f|_{32B_k})\chi_{32B_k})(x) > C_1\lambda\}) \\
&\quad \le \frac{\|\mathcal{M}\|_{L^1(\mathcal{X})\to L^{1,\infty}(\mathcal{X})}}{C_1\lambda}\int_{32B_k} \big||f(y)| - |f|_{32B_k}\big|\,d\mu(y) \lesssim \mu(32B_k)\mathcal{M}^\sharp f(x_0) \lesssim \mu(B_k)\gamma.
\end{aligned}$$

This proves (4.15) and hence (4.14).

Let $A := 2C_1^3$. To prove (i), for each $N \in \mathbb{N}$, we set

$$\mathrm{H}_N := 2^p \int_0^{AN} p\lambda^{p-1} w(\{x \in \mathcal{X} : \mathcal{M}f(x) > 2\lambda\})\,d\lambda.$$

Fix $x_0 \in \mathcal{X}$ and $R > 0$. Write $f = g_R + h_R$, where $g_R := f\chi_{B(x_0,R)}$ and $h_R := f\chi_{B(x_0,R)^\complement}$. Then

$$\begin{aligned}
\mathrm{H}_N &\le 2^p \int_0^{AN} p\lambda^{p-1} w(\{x \in \mathcal{X} : \mathcal{M}g_R(x) > \lambda\})\,d\lambda \\
&\quad + 2^p \int_0^{AN} p\lambda^{p-1} w(\{x \in \mathcal{X} : \mathcal{M}h_R(x) > \lambda\})\,d\lambda \\
&:= \mathrm{H}_{N,R} + \widetilde{\mathrm{H}}_{N,R}.
\end{aligned}$$

We claim that

$$\lim_{R\to\infty} \widetilde{\mathrm{H}}_{N,R} = 0. \tag{4.16}$$

Assume for the moment that (4.16) holds true. Since $p > p_0$ and $\mathcal{M}f \in L^{p_0,\infty}(w)$, it follows that $\mathcal{M}g_R \in L^{p_0,\infty}(w)$ and hence

$$\mathrm{H}_{N,R} \le 2^p \int_0^{AN} p\lambda^{p-1}\lambda^{-p_0}\|\mathcal{M}g_R\|_{L^{p_0,\infty}(w)}^{p_0}\,d\lambda < \infty.$$

By this fact and (4.16), one finds that H_N is finite for large R. Since $g_R \in L^1(\mathcal{X})$ and it has bounded support, we apply (4.14) to obtain

$$\begin{aligned}
\mathrm{H}_{N,R} &= (2A)^p \int_0^N p\lambda^{p-1} w(\{x \in \mathcal{X} : \mathcal{M}g_R(x) > A\lambda\})\, d\lambda \\
&\le (2A)^p \int_0^N p\lambda^{p-1} w(\{x \in \mathcal{X} : \mathcal{M}g_R(x) > A\lambda,\, \mathcal{M}^\sharp g_R(x) \le \gamma\lambda\})\, d\lambda \\
&\quad + (2A)^p \int_0^N p\lambda^{p-1} w(\{x \in \mathcal{X} : \mathcal{M}^\sharp g_R(x) > \gamma\lambda\})\, d\lambda \\
&\le \widetilde{C}(2A)^p \gamma^{\theta_0} \int_0^N p\lambda^{p-1} w(\{x \in \mathcal{X} : \mathcal{M}g_R(x) > \lambda\})\, d\lambda + (2A)^p \|\mathcal{M}^\sharp f\|_{L^p(w)}^p \\
&\le \widetilde{C}(2A)^p \gamma^{\theta_0} \mathrm{H}_N + (2A)^p \|\mathcal{M}^\sharp f\|_{L^p(w)}^p. \qquad (4.17)
\end{aligned}$$

Notice that (4.16) implies that $\mathrm{H}_N = \lim_{R\to\infty} \mathrm{H}_{N,R}$. Thus, in (4.17), by letting $R \to \infty$ and choosing γ small enough such that $\widetilde{C}(2A)^p\gamma^{\theta_0} < 1/2$ we conclude that $\mathrm{H}_N \lesssim \|\mathcal{M}^\sharp f\|_{L^p(w)}^p$. Then letting $N \to \infty$ we obtain (i).

Now it remains to prove (4.16). Since $\mathcal{M}h_R(x) \le \mathcal{M}f(x)$ for all $x \in \mathcal{X}$, $p > p_0$ and $\mathcal{M}(f) \in L^{p_0,\infty}(w)$, we have

$$\int_0^{AN} p\lambda^{p-1} w(\{x \in \mathcal{X} : \mathcal{M}h_R(x) > \lambda\})\, d\lambda < \infty.$$

Thus, (4.16) follows from the dominated convergence theorem and the fact that

$$\lim_{R\to\infty} \mathcal{M}h_R(x) = 0, \qquad \forall x \in \mathcal{X}. \qquad (4.18)$$

To see that (4.18) holds true, observe that, for any fixed $x \in \mathcal{X}$, $\mathcal{M}h_R(x)$ is decreasing as $R \to \infty$. Assume that (4.18) fails, that is, there exists $x_1 \in \mathcal{X}$ such that

$$\lim_{R\to\infty} \mathcal{M}h_R(x_1) = c_0 > 0.$$

For $x \in \mathcal{X}$, set

$$R_x := \max\{d(x_1, x_0),\, d(x, x_0)\} + 1.$$

Then $x, x_1 \in B(x_0, R_x)$. For all $x \in \mathcal{X}$ and $R > 4R_x$,

$$\begin{aligned}
c_0 \le \mathcal{M}h_R(x_1) &= \sup_{B \ni x_1} \frac{1}{\mu(B)} \int_{B\setminus B(x_0,R)} |f(y)|\, d\mu(y) \\
&\le C_1^2 \sup_{r>0} \frac{1}{\mu(B(x_1,r))} \int_{B(x_1,r)\setminus B(x_0,R)} |f(y)|\, d\mu(y) \\
&= C_1^2 \sup_{r>3R_x} \frac{1}{\mu(B(x_1,r))} \int_{B(x_1,r)\setminus B(x_0,R)} |f(y)|\, d\mu(y) \\
&\le C_1^2 \mathcal{M}f(x).
\end{aligned}$$

Thus, with $\lambda_0 := c_0 C_1^{-2}/2$, we have

$$\begin{aligned}
\|\mathcal{M}f\|_{L^{p_0,\infty}(w)}^{p_0} &\ge \lambda_0^{p_0} w(\{x \in \mathcal{X} : \mathcal{M}f(x) > \lambda_0\}) \\
&\ge \lim_{R\to\infty} \lambda_0^{p_0} w(\{x \in \mathcal{X} : R > 4R_x\}) \\
&= \infty.
\end{aligned}$$

This contradiction implies (4.18). Thus, (4.16) holds and we complete the proof of (i).

To prove (ii), for every $N \in \mathbb{N}$, set $A := 2C_1^3$ and

$$\mathrm{J}_N := \sup_{0<\lambda<AN} 2^p \lambda^p w_\mu(\{x \in \mathcal{X} : \mathcal{M}f(x) > 2\lambda\}).$$

Then we argue as in the proof of (i) but using J_N instead of H_N, the details being omitted.

The proofs for (iii) and (iv) are similar to those of (i) and (ii), the presence of an extra term $\|f\|_{L^1(\mathcal{X})}$ is due to the fact that (4.14) holds only for $\lambda > \|f\|_{L^1(\mathcal{X})}/\mu(\mathcal{X})$, the details being omitted. ∎

THEOREM 4.12. *Let $\gamma \in (0, 1/m)$. Then there exists a constant $C := C(\gamma, m, \delta, C_1) > 0$ such that, for all $\vec{f} \in L^{p_1}(\mathcal{X}) \times \cdots \times L^{p_m}(\mathcal{X})$ with $1 \le p_1, \dots, p_m < \infty$,*

$$\mathcal{M}_\gamma^\sharp(T(\vec{f}))(x) \le C(C_K + W)\mathcal{M}\vec{f}(x), \tag{4.19}$$

where

$$W := \|T\|_{L^1(\mathcal{X})\times\cdots\times L^1(\mathcal{X})\to L^{1/m,\infty}(\mathcal{X})}.$$

Proof. Let $\gamma \in (0, 1/m)$. Fix a point $x \in \mathcal{X}$. To obtain (4.19), by Definition 4.9 and Remark 4.10, it suffices to prove that, for any ball B containing x, there exists $c_B \in \mathbb{C}$ such that

$$\left[\frac{1}{\mu(B)} \int_B |T(\vec{f})(z) - c_B|^\gamma \, d\mu(z)\right]^{1/\gamma} \lesssim (C_K + W)\mathcal{M}\vec{f}(x). \tag{4.20}$$

Let z_B and r_B be the center and radius of B, respectively. For $j \in \{1, \dots, m\}$, we set $f_j^0 := f_j \chi_{B^*}$ and $f_j^\infty := f_j - f_j^0$, where $B^* := B(z_B, 5r_B)$. Then

$$\begin{aligned}\prod_{j=1}^m f_j(y_j) = \prod_{j=1}^m [f_j^0(y_j) + f_j^\infty(y_j)] &= \prod_{\alpha_1,\dots,\alpha_m \in \{0,\infty\}} f_1^{\alpha_1}(y_1) \cdots f_m^{\alpha_m}(y_m)\\ &= \prod_{j=1}^m f_j^0(y_j) + \sum\nolimits' f_1^{\alpha_1}(y_1) \cdots f_m^{\alpha_m}(y_m),\end{aligned}$$

where each term of $\sum'$ contains at least one $\alpha_j \neq 0$. Write $\vec{f^0} := (f_1^0, \dots, f_m^0)$ and

$$T(\vec{f}) = T(\vec{f^0}) + \sum\nolimits' T(f_1^{\alpha_1} \cdots f_m^{\alpha_m}). \tag{4.21}$$

Set

$$A := \prod_{j=1}^m \frac{1}{\mu(B)} \|f_j^0\|_{L^1(\mathcal{X})}.$$

Since T is bounded from $L^1(\mathcal{X}) \times \cdots \times L^1(\mathcal{X})$ to $L^{1/m,\infty}(\mathcal{X})$ with norm W, we obtain

$$\begin{aligned}\left[\frac{1}{\mu(B)} \int_B |T(\vec{f^0})(z)|^\gamma \, d\mu(z)\right]^{1/\gamma} &= \left[\frac{1}{\mu(B)} \int_0^{WA} \gamma t^{\gamma-1} \mu(\{x \in B : |T(\vec{f^0})(z)| > t\})\, dt\right.\\ &\qquad \left. + \frac{1}{\mu(B)} \int_{WA}^\infty \gamma t^{\gamma-1} \mu(\{x \in B : |T(\vec{f^0})(z)| > t\})\, dt\right]^{1/\gamma}\\ &\le \left[(WA)^\gamma + W^{1/m} A^{1/m} \int_{WA}^\infty \gamma t^{\gamma-1-1/m}\, dt\right]^{1/\gamma}\\ &\lesssim WA\\ &\lesssim W\mathcal{M}(\vec{f})(x).\end{aligned}$$

Without loss of generality, we may assume that $\alpha_{j_1} = \cdots = \alpha_{j_\ell} = 0$ for some $\{j_1, \ldots, j_\ell\} \subset \{1, \ldots, m\}$ and $0 \le \ell < m$. By convention the set $\{j_1, \ldots, j_\ell\}$ is empty if $\ell = 0$. Recall that $x \in B$. Then for any $z \in B$, by the regularity condition (3.2), we have

$$\begin{aligned}
&|T(f_1^{\alpha_1} \cdots f_m^{\alpha_m})(z) - T(f_1^{\alpha_1} \cdots f_m^{\alpha_m})(x)| \\
&\quad \le C_K \int_{\mathcal{X}^m} \left[\frac{d(z,x)}{\max_{1\le k\le m} d(z,y_k)}\right]^\delta \frac{\prod_{j=1}^m |f_j^{\alpha_j}(y_j)|}{[\sum_{k=1}^m V(z,y_k)]^m}\, d\mu(y_1)\cdots d\mu(y_m) \\
&\quad \le C_K \int_{(B^*)^\ell} |f_{j_1}^0(y_{j_1}) \cdots f_{j_\ell}^0(y_{j_\ell})| \\
&\qquad \times \int_{(\mathcal{X}\setminus B^*)^{m-\ell}} \left[\frac{2r_B}{\max_{1\le k\le m} d(z,y_k)}\right]^\delta \frac{\prod_{j\notin\{j_1,\ldots,j_\ell\}} |f_j(y_j)|}{[\sum_{k=1}^m V(z,y_k)]^m}\, d\mu(y_1)\cdots d\mu(y_m) \\
&\quad \le C_K \sum_{k=1}^\infty \int_{(B^*)^\ell} |f_{j_1}^0(y_{j_1}) \cdots f_{j_\ell}^0(y_{j_\ell})| \int_{(3^kB^*)^{m-\ell}\setminus(3^{k-1}B^*)^{m-\ell}} \\
&\qquad \times \left[\frac{2r_B}{\max_{1\le k\le m} d(z,y_k)}\right]^\delta \frac{\prod_{j\notin\{j_1,\ldots,j_\ell\}} |f_j(y_j)|}{[\sum_{k=1}^m V(z,y_k)]^m}\, d\mu(y_1)\cdots d\mu(y_m) \\
&\quad \lesssim C_K \sum_{k=1}^\infty 3^{-k\delta} \int_{(B^*)^\ell} |f_{j_1}^0(y_{j_1}) \cdots f_{j_\ell}^0(y_{j_\ell})| \\
&\qquad \times \int_{(3^kB^*)^{m-\ell}} \frac{\prod_{j\notin\{j_1,\ldots,j_\ell\}} |f_j(y_j)|}{[\mu(B(z,3^kr_B))]^m}\, d\mu(y_1)\cdots d\mu(y_m) \\
&\quad \lesssim C_K \sum_{k=1}^\infty 3^{-k\delta} \prod_{j=1}^m \frac{1}{\mu(3^kB^*)} \int_{3^kB^*} |f_j(y_j)|\, d\mu(y_j) \\
&\quad \lesssim C_K \mathcal{M}(\vec{f})(x).
\end{aligned}$$

Define c_B in (4.20) to be $c_B := \sum' T(f_1^{\alpha_1} \cdots f_m^{\alpha_m})(x)$. Then from (4.21) it follows that the left hand side of (4.20) is bounded by

$$\begin{aligned}
&2^{1/\gamma-1}\left[\frac{1}{\mu(B)}\int_B |T(\vec{f}^0)(z)|\, d\mu(z)\right]^{1/\gamma} \\
&\qquad + 2^{1/\gamma-1}\left[\frac{1}{\mu(B)}\int_B {\sum}' \, |T(f_1^{\alpha_1} \cdots f_m^{\alpha_m})(z) - T(f_1^{\alpha_1} \cdots f_m^{\alpha_m})(x)|\, d\mu(z)\right]^{1/\gamma} \\
&\qquad \lesssim (W + C_K)\mathcal{M}(\vec{f})(x),
\end{aligned}$$

which completes the proof. ∎

PROPOSITION 4.13. *Let w be an A_∞ weight and $p \in [1/m, \infty)$. Then there exists $C > 0$ such that, for all bounded functions $\vec{f}$ with compact support, if $p > 1/m$, then*

$$\|T(\vec{f})\|_{L^p(w)} \le C\|\mathcal{M}(\vec{f})\|_{L^p(w)}, \tag{4.22}$$

and if $p \ge 1/m$, then

$$\|T(\vec{f})\|_{L^{p,\infty}(w)} \le C\|\mathcal{M}(\vec{f})\|_{L^{p,\infty}(w)}. \tag{4.23}$$

Proof. We only prove (4.22) since similar arguments give the weak-type estimate (4.23). For every $N \in \mathbb{N}$, set $w_N := \min\{w, N\}$. Then $w_N \in A_\infty$, and Fatou's lemma implies that

$$\|T(\vec{f})\|_{L^p(w)} \le \liminf_{j\to\infty} \|T(\vec{f})\|_{L^p(w_N)}.$$

Since each function f_j is in $L_b^\infty(\mathcal{X})$, by assumption,

$$T : L^{q_1}(\mathcal{X}) \times \cdots \times L^{q_m}(\mathcal{X}) \to L^{q,\infty}(\mathcal{X})$$

for some indices $1 \le q_1, \dots, q_m < \infty$ and $0 < q < \infty$ satisfying $1/q = 1/q_1 + \cdots + 1/q_m$, we know that $T(\vec{f}) \in L^{q,\infty}(\mathcal{X})$, and hence $|T(\vec{f})|^\gamma \in L^1_{\mathrm{loc}}(\mathcal{X})$ for all $\gamma \in (0, 1/m)$. Then, using Lemma 4.11(i) and (4.19) we see that, for any fixed $\gamma \in (0, 1/m)$,

$$\begin{aligned}
\|T(\vec{f})\|_{L^p(w_N)} &\le \|[\mathcal{M}(|T(\vec{f})|^\gamma)]^{1/\gamma}\|_{L^p(w_N)} \\
&\lesssim \|\mathcal{M}^\sharp_\gamma(T(\vec{f}))\|_{L^p(w_N)} \\
&\lesssim (C_K + W)\|\mathcal{M}(\vec{f})\|_{L^p(w_N)} \\
&\lesssim (C_K + W)\|\mathcal{M}(\vec{f})\|_{L^p(w)},
\end{aligned}$$

which gives (4.22) by letting $N \to \infty$, provided that we can show that, for some $p_0 \in (0, p)$,

$$\|[\mathcal{M}(|T(\vec{f})|^\gamma)]^{1/\gamma}\|_{L^{p_0,\infty}(w_N)} < \infty. \tag{4.24}$$

To see (4.24), we choose $p_0 = 1/m$. Then, applying $\gamma < 1/m < p$, $\|w_N\|_{L^\infty(\mathcal{X})} \le N$ and the fact $\mathcal{M}$ is bounded on $L^{r,\infty}(\mathcal{X})$ for all $r \in (1, \infty)$ (see [40, Exercise 2.1.13]), we obtain

$$\begin{aligned}
\|[\mathcal{M}(|T(\vec{f})|^\gamma)]^{1/\gamma}\|_{L^{1/m,\infty}(w_N)} &\le N\|\mathcal{M}(|T(\vec{f})|^\gamma)\|^{1/\gamma}_{L^{1/(m\gamma),\infty}(\mathcal{X})} \\
&\le N\|\mathcal{M}\|^{1/\gamma}_{L^{1/(m\gamma),\infty}(\mathcal{X})\to L^{1/(m\gamma),\infty}(\mathcal{X})}\||T(\vec{f})|^\gamma\|^{1/\gamma}_{L^{1/(m\gamma),\infty}(\mathcal{X})} \\
&= N\|\mathcal{M}\|^{1/\gamma}_{L^{1/(m\gamma),\infty}(\mathcal{X})\to L^{1/(m\gamma),\infty}(\mathcal{X})}\|T(\vec{f})\|_{L^{1/m,\infty}(\mathcal{X})},
\end{aligned}$$

which is finite since $T : L^1(\mathcal{X}) \times \cdots \times L^1(\mathcal{X}) \to L^{1/m,\infty}(\mathcal{X})$. Hence, (4.24) holds. ∎

Consequently, this proposition, together with Theorems 4.4 and 4.6, implies the following weighted estimate for multilinear Calderón–Zygmund operators.

Corollary 4.14. *Let $1 \le p_1, \dots, p_m < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$ and $\vec{P} := (p_1, \dots, p_m)$. Suppose that $\nu \in A_\infty$ and $\{w_j\}_{j=1}^m$ are weights. Let $\vec{w} := (w_1, \dots, w_m)$. Then:*

(i) *if $p_j \ge 1$ for all $j \in \{1, \dots, m\}$ and $(\nu; \vec{w})$ satisfies* (4.1)*, then T can be extended to a bounded m-linear operator from $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ to $L^{p,\infty}(\nu)$;*

(ii) *if $p_j > 1$ for all $j \in \{1, \dots, m\}$ and $(\nu; \vec{w})$ satisfies the bump weight condition* (4.8) *for some $r > 1$, then T can be extended to a bounded m-linear operator from $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ to $L^p(\nu)$.*

In either case, the norm of T is bounded by $C(C_K + \|T\|_{L^{q_1}(\mathcal{X})\times\cdots\times L^{q_m}(\mathcal{X})\to L^q(\mathcal{X})})$, where C is a positive constant depending on m, δ, C_1 and $[(\nu; \vec{w})]_{A_{\vec{P}}}$ (or $[(\nu; \vec{w})]_{A_{\vec{P},r}}$).

Likewise, Proposition 4.13, together with Theorems 4.4 and 4.7, implies the following weighted estimate for multilinear Calderón–Zygmund operators.

Corollary 4.15. *Let* $1 \le p_1, \dots, p_m < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$, $\vec{P} := (p_1, \dots, p_m)$, *and* $\vec{w} := (w_1, \dots, w_m) \in A_{\vec{P}}$. *Then:*

(i) T *can be extended to a bounded* m*-linear operator from* $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ *to* $L^p(\nu_{\vec{w}})$ *if all the exponents* p_j *are greater than* 1;
(ii) T *can be extended to a bounded* m*-linear operator from* $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ *to* $L^{p,\infty}(\nu_{\vec{w}})$ *if some of the exponents* p_j *are equal to* 1.

In either case, the norm of T *is bounded by* $C(C_K + \|T\|_{L^{q_1}(\mathcal{X}) \times \cdots \times L^{q_m}(\mathcal{X}) \to L^q(\mathcal{X})})$, *where* C *is a positive constant depending on* m, δ, C_1 *and* $[w]_{A_{\vec{P}}}$.

4.4. Weighted estimates for maximal multilinear singular integrals. Let T be as in the previous subsection. Define the *maximal truncated operator* by

$$T^*(\vec{f})(x) := \sup_{\alpha > 0} |T_\alpha(f_1, \dots, f_m)(x)|, \quad \forall x \in \mathcal{X},$$

where, using the notation $\vec{y} := (y_1, \dots, y_m)$ and $d\mu(\vec{y}) := d\mu(y_1) \cdots d\mu(y_m)$, we set

$$T_\alpha(f_1, \dots, f_m)(x) := \int_{\sum_{j=1}^m d(x, y_j) \ge \alpha} K(x, y_1, \dots, y_m) f_1(y_1) \cdots f_m(y_m)\, d\mu(\vec{y}).$$

Such maximal truncated operators for multilinear integrals on $(\mathbb{R}^n)^m$ were first introduced in [54]. In the Euclidean setting, it was proved in [54] that if multilinear Calderón–Zygmund operators are bounded at one point, say $T : L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ for some $1 \le q_1, \dots, q_m < \infty$ and $1/q = 1/q_1 + \cdots + 1/q_m$, then

$$T^* : L^{p_1}(w) \times \cdots \times L^{p_m}(w) \to L^p(w)$$

for all $1 < p_1, \dots, p_m < \infty$ where $1/p = 1/p_1 + \cdots + 1/p_m$, provided $w \in \bigcap_{1 \le j \le m} A_{p_j}$.

Notice that the size condition of K implies that $T^*(f_1, \dots, f_m)$ is pointwise well-defined when $f_j \in L^{q_j}(\mathcal{X})$ with $q_j \in [1, \infty]$; see [53, p. 1263]. The goal of this subsection is to obtain the following multiple weight norm estimates for T^*.

Theorem 4.16. *Let* $1 \le p_1, \dots, p_m < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$, $\vec{P} := (p_1, \dots, p_m)$, *and* $\vec{w} := (w_1, \dots, w_m) \in A_{\vec{P}}$. *Then:*

(i) T^* *can be extended to a bounded operator from* $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ *to* $L^p(\nu_{\vec{w}})$ *if all the exponents* p_j *are greater than* 1;
(ii) T^* *can be extended to a bounded operator from* $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ *to* $L^{p,\infty}(\nu_{\vec{w}})$ *if some exponent* p_j *is equal to* 1.

In either case, the norm of T^* *is bounded by* $C(C_K + \|T\|_{L^{q_1}(\mathcal{X}) \times \cdots \times L^{q_m}(\mathcal{X}) \to L^q(\mathcal{X})})$, *where* C *is a positive constant depending on* m, δ, C_1 *and* $[w]_{A_{\vec{P}}}$.

Consequently, this result, along with Theorems 4.4 and 4.6, implies the following weighted estimate for multilinear Calderón–Zygmund operators.

Theorem 4.17. *Let* $1 \le p_1, \dots, p_m < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$ *and* $\vec{P} := (p_1, \dots, p_m)$. *Let* $\nu \in A_\infty$ *and* $\{w_j\}_{j=1}^m$ *be weights. Let* $\vec{w} := (w_1, \dots, w_m)$. *Then:*

(i) *if* $p_j \ge 1$ *for all* $j \in \{1, \dots, m\}$ *and* $(\nu; \vec{w})$ *satisfies* (4.1), *then* T^* *can be extended to a bounded operator from* $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ *to* $L^{p,\infty}(\nu)$;

(ii) *if $p_j > 1$ for all $j \in \{1, \dots, m\}$ and $(\nu; \vec{w})$ satisfies the condition (4.8) for some $r > 1$, then T^* can be extended to a bounded operator from $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ to $L^p(\nu)$.*

In either case, the norm of T is bounded by $C(C_K + \|T\|_{L^{q_1}(\mathcal{X}) \times \cdots \times L^{q_m}(\mathcal{X}) \to L^q(\mathcal{X})})$, where C is a positive constant depending on m, δ, C_1 and $[(\nu; \vec{w})]_{A_{\vec{P}}}$ (or $[(\nu; \vec{w})]_{A_{\vec{P},r}}$).

We only prove Theorem 4.16 since the proof for Theorem 4.17 similar. We need the following Cotlar-type inequality.

Proposition 4.18. *For all $\gamma > 0$, there exists a positive constant $C := C(m, C_1, \delta, \gamma)$ such that, for all $\vec{f} := (f_1, \dots, f_m)$ with every $f_j \in L_b^\infty(\mathcal{X})$ and $x \in \mathcal{X}$,*

$$T^*(\vec{f})(x) \le C\{[\mathcal{M}(|T(\vec{f})|^\gamma)(x)]^{1/\gamma} + (C_K + W)\mathcal{M}(\vec{f})(x)\}, \tag{4.25}$$

where

$$W := \|T\|_{L^1(\mathcal{X}) \times \cdots \times L^1(\mathcal{X}) \to L^{1/m,\infty}(\mathcal{X})}.$$

Proof. By Hölder's inequality, it suffices to show that (4.25) holds for $\gamma \in (0, 1/m)$. Fix $\gamma \in (0, 1/m)$ and $x \in \mathcal{X}$. Set

$$S_\alpha(x) := \Big\{\vec{y} \in \mathcal{X}^m : \sup_{1 \le j \le m} d(x, y_j) < \alpha\Big\},$$

$$U_\alpha(x) := \Big\{\vec{y} \in S_\alpha(x) : \sum_{j=1}^m d(x, y_j) \ge \alpha\Big\}.$$

For any $\vec{y} \in U_\alpha(x)$, there exists y_{j_0} such that $d(x, y_{j_0}) \ge \alpha/m$. From this and the doubling property of μ, it follows that

$$\mu(B(x, \alpha)) \le C(m, C_1) \sum_{j=1}^m V(x, y_j).$$

Thus,

$$\begin{aligned}
\sup_{\alpha>0} \bigg| \int_{U_\alpha(x)} & K(x, y_1, \dots, y_m) f_1(y_1) \cdots f_m(y_m)\, d\mu(\vec{y}) \bigg| \\
&\le \sup_{\alpha>0} \int_{U_\alpha(x)} \frac{C_K |f_1(y_1) \cdots f_m(y_m)|}{[\sum_{k=1}^m V(x, y_k)]^m}\, d\mu(\vec{y}) \\
&\lesssim C_K \sup_{\alpha>0} \int_{U_\alpha(x)} \frac{|f_1(y_1) \cdots f_m(y_m)|}{[\mu(B(x, \alpha))]^m}\, d\mu(\vec{y}) \\
&\lesssim C_K \mathcal{M}(\vec{f})(x).
\end{aligned}$$

Therefore, it is enough to prove that (4.25) holds with $T^*(\vec{f})$ replaced by

$$\widetilde{T}^*(\vec{f})(x) := \sup_{\alpha>0} |\widetilde{T}_\alpha(f_1, \dots, f_m)(x)|,$$

where

$$\widetilde{T}_\alpha(f_1, \dots, f_m)(x) := \int_{\vec{y} \notin S_\alpha(x)} K(x, y_1, \dots, y_m) f_1(y_1) \cdots f_m(y_m)\, d\mu(\vec{y}).$$

Fix $\alpha > 0$ and let $B(x, \alpha/2)$ be the ball centered at x and of radius $\alpha/2$. From every $f_j \in L_b^\infty(\mathcal{X})$, together with the boundedness of T from $L^1(\mathcal{X}) \times \cdots \times L^1(\mathcal{X})$ to

$L^{1/m,\infty}(\mathcal{X})$ (see Theorem 3.3), it follows that $T(\vec{f}) \in L^{1/m,\infty}(\mathcal{X})$, and hence it is finite almost everywhere. For $\alpha > 0$, $x \in \mathcal{X}$ and $z \in B(x, \alpha/2)$, we set

$$G_\alpha(\vec{f})(x,z) := \int_{\vec{y} \notin S_\alpha(x)} K(z, y_1, \ldots, y_m) f_1(y_1) \cdots f_m(y_m)\, d\mu(\vec{y}).$$

Observe that, for all $z \in B(x, \alpha/2)$,

$$|\widetilde{T}_\alpha(\vec{f})(x)| \leq |\widetilde{T}_\alpha(\vec{f})(x) - G_\alpha(\vec{f})(x,z)| + |T(\vec{f})(z) - T(\vec{f}_0)(z)|, \tag{4.26}$$

where $\vec{f}_0 := (f_1\chi_{B(x,\alpha)}, \ldots, f_m\chi_{B(x,\alpha)})$.

Applying the regularity condition (3.2), we obtain

$$\begin{aligned}
&|\widetilde{T}_\alpha(\vec{f})(x) - G_\alpha(\vec{f})(x,z)| \\
&\qquad \leq C_K \int_{\vec{y} \notin S_\alpha(x)} \left[\frac{d(x,z)}{\max_{1\leq k\leq m} d(x,y_k)}\right]^\delta \frac{\prod_{j=1}^m |f_j(y_j)|}{[\sum_{k=1}^m V(x,y_k)]^m}\, d\mu(\vec{y}).
\end{aligned} \tag{4.27}$$

Notice that the right hand side of (4.27) can be written as a sum of integrals over sets $R_{j_1,\ldots,j_\ell}$ for some $\{j_1, \ldots, j_\ell\} \subsetneq \{1, \ldots, m\}$ so that, for $\vec{y} := (y_1, \ldots, y_m) \in R_{j_1,\ldots,j_\ell}$, we have $d(x, y_j) < \alpha$ if and only if $j \in \{j_1, \ldots, j_\ell\}$. Set

$$\{k_1, \ldots, k_{m-\ell}\} := \{1, \ldots, m\} \setminus \{j_1, \ldots, j_\ell\}.$$

Then $m - \ell \geq 1$ and

$$\begin{aligned}
&\int_{\vec{y} \in R_{j_1,\ldots,j_\ell}} \left[\frac{d(x,z)}{\max_{1\leq k\leq m} d(x,y_k)}\right]^\delta \frac{\prod_{j=1}^m |f_j(y_j)|}{[\sum_{k=1}^m V(x,y_k)]^m}\, d\mu(\vec{y}) \\
&\quad \leq \alpha^\delta \prod_{j\in\{j_1,\ldots,j_\ell\}} \int_{d(x,y_j)<\alpha} |f_j(y_j)|\, d\mu(y_j) \\
&\qquad \times \int_{(\mathcal{X}\setminus B(x,\alpha))^{m-\ell}} \left[\frac{1}{\max_{1\leq k\leq m} d(x,y_k)}\right]^\delta \frac{\prod_{i=1}^{m-\ell} |f_{k_i}(y_{k_i})|}{[\sum_{k=1}^m V(x,y_k)]^m}\, d\mu(y_{k_1}) \cdots d\mu(y_{k_{m-\ell}}).
\end{aligned}$$

Since

$$\begin{aligned}
&\int_{(\mathcal{X}\setminus B(x,\alpha))^{m-\ell}} \left[\frac{1}{\max_{1\leq k\leq m} d(x,y_k)}\right]^\delta \frac{\prod_{i=1}^{m-\ell} |f_{k_i}(y_{k_i})|}{[\sum_{k=1}^m V(x,y_k)]^m}\, d\mu(y_{k_1}) \cdots d\mu(y_{k_{m-\ell}}) \\
&\quad \lesssim \int_{\{\vec{y}:\, \sum_{i=1}^{m-\ell} d(x,y_{k_i}) \geq \alpha\}} \frac{\prod_{i=1}^{m-\ell} |f_{k_i}(y_{k_i})|}{[\sum_{i=1}^{m-\ell} d(x,y_{k_i})]^\delta [\sum_{i=1}^{m-\ell} V(x,y_{k_i})]^m}\, d\mu(y_{k_1}) \cdots d\mu(y_{k_{m-\ell}}) \\
&\quad \sim \sum_{s=0}^\infty \int_{2^s\alpha \leq \sum_{i=1}^{m-\ell} d(x,y_{k_i}) < 2^{s+1}\alpha} \cdots \\
&\quad \lesssim \sum_{s=0}^\infty \frac{1}{(2^s\alpha)^\delta (\mu(B(x,2^{s+1}\alpha)))^m} \prod_{i=1}^{m-\ell} \int_{B(x,2^{s+1}\alpha)} |f_{k_i}(y_{k_i})|\, d\mu(y_{k_i}),
\end{aligned}$$

we have

$$\begin{aligned}
&\int_{\vec{y} \in R_{j_1,\ldots,j_\ell}} \left[\frac{d(x,z)}{\max_{1\leq k\leq m} d(x,y_k)}\right]^\delta \frac{\prod_{j=1}^m |f_j(y_j)|}{[\sum_{k=1}^m V(x,y_k)]^m}\, d\mu(\vec{y}) \\
&\qquad \lesssim \sum_{s=0}^\infty \frac{1}{(2^s)^\delta [\mu(B(x,2^{s+1}\alpha))]^m} \prod_{j=1}^m \int_{B(x,2^{s+1}\alpha)} |f_j(y_j)|\, d\mu(y_j) \\
&\qquad \lesssim \mathcal{M}(\vec{f})(x).
\end{aligned}$$

Combining this with (4.27) and (4.26), we see that for all $z \in B(x, \alpha/2)$,

$$|\widetilde{T}_\alpha(\vec{f})(x)| \lesssim C_K \mathcal{M}(\vec{f})(x) + |T(\vec{f})(z)| + |T(\vec{f_0})(z)|. \tag{4.28}$$

Raising (4.28) to the power γ, taking integral average over the ball $B := B(x, \alpha/2)$ with respect to the variable z, we obtain

$$|\widetilde{T}_\alpha(\vec{f})(x)|^\gamma \lesssim [C_K \mathcal{M}(\vec{f})(x)]^\gamma + \mathcal{M}(|T(\vec{f})|^\gamma)(x) + \frac{1}{|B|}\int_B |T(\vec{f_0})(z)|^\gamma \, d\mu(z). \tag{4.29}$$

Since

$$\begin{aligned}\int_B |T(\vec{f_0})(z)|^\gamma \, d\mu(z) &= m\gamma \int_0^\infty \lambda^{m\gamma-1} \mu(\{z \in B : |T(\vec{f_0})(z)|^{1/m} > \lambda\}) \, d\lambda \\ &\le m\gamma \int_0^\infty \lambda^{m\gamma-1} \min\left\{\mu(B),\ \lambda^{-1} W^{1/m} \prod_{j=1}^m \|f_j \chi_{B(x,\alpha)}\|_{L^1(\mathcal{X})}^{1/m}\right\} d\lambda \\ &\lesssim \mu(B)^{1-m\gamma} W^\gamma \prod_{j=1}^m \|f_j \chi_{B(x,\alpha)}\|_{L^1(\mathcal{X})}^\gamma,\end{aligned}$$

we then conclude that

$$\left\{\frac{1}{|B|}\int_B |T(\vec{f_0})(z)|^\gamma \, d\mu(z)\right\}^{1/\gamma} \lesssim W \prod_{j=1}^m \frac{\|f_j \chi_{B(x,\alpha)}\|_{L^1(\mathcal{X})}}{|B|} \lesssim W \mathcal{M}(\vec{f})(x).$$

Combining this with (4.29), we obtain (4.25). ∎

Proof of Theorem 4.16. To prove (i), we apply Proposition 4.18 to obtain

$$\|T^*(\vec{f})\|_{L^p(\nu_{\vec{w}})} \lesssim \|[\mathcal{M}(|T(\vec{f})|^\gamma)]^{1/\gamma}\|_{L^p(\nu_{\vec{w}})} + (C_K + W)\|\mathcal{M}(\vec{f})\|_{L^p(\nu_{\vec{w}})}.$$

Theorem 4.7 implies that $\|\mathcal{M}(\vec{f})\|_{L^p(\nu_{\vec{w}})} \lesssim \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}$. By Proposition 4.3(ii) and $\vec{w} \in A_{\vec{P}}$, we have $\nu_{\vec{w}} \in A_{pm}$. If we choose $0 < \gamma < 1/m$, then $\nu_{\vec{w}} \in A_{p/\gamma}$ and

$$\|[\mathcal{M}(|T(\vec{f})|^\gamma)]^{1/\gamma}\|_{L^p(\nu_{\vec{w}})} \le \|\mathcal{M}\|_{L^{p/\gamma}(\nu_{\vec{w}}) \to L^{p/\gamma}(\nu_{\vec{w}})}^{1/\gamma} \|T(\vec{f})\|_{L^p(\nu_{\vec{w}})} \lesssim \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)},$$

where, in the last inequality, we used Corollary 4.15(i). This finishes the proof of Theorem 4.16(i).

Using Proposition 4.18, Theorem 4.4 and Corollary 4.15(ii), together with an argument similar to the proof of (i), we obtain (ii), the further details being omitted. ∎

5. A multilinear $T1$-theorem on Lebesgue spaces

The linear $T1$-theorem was obtained by David and Journé [25]. Multilinear $T1$-theorems have been obtained by Christ and Journé [17] and Grafakos and Torres [53]. In this section we extend the latter to the context of RD-spaces. The multilinear $T1$-theorem provides a criterion for the boundedness of an m-linear Calderón–Zygmund operator on products of Lebesgue spaces.

5.1. Some lemmas on multilinear Calderón–Zygmund operators. The following lemmas for the case $\mathcal{X} = \mathbb{R}^n$ were proved in [53]. The proof of Lemma 5.1 is similar to that of [53, Lemma 1], the details being omitted.

LEMMA 5.1. *Fix $x_0 \in \mathcal{X}$ and $\eta \in (0, 1]$. Assume that $\psi \in C^\infty(\mathbb{R})$ is such that $0 \le \psi \le 1$, $\psi(t) = 1$ when $|t| < 1$, and $\psi(t) = 0$ when $|t| \ge 2$. Define*

$$\psi_k(x) := \psi(2^{-k} d(x, x_0))$$

for all $x \in \mathcal{X}$ and $k \in \mathbb{Z}$. Every m-linear Calderón–Zygmund operator T with a kernel $K \in \operatorname{Ker}(m, C_K, \delta)$ for some $C_K > 0$ and $\delta \in (0, 1]$ as in Definition 3.2 can be extended to the m-fold product space $(C^\eta(\mathcal{X}) \cap L^\infty(\mathcal{X})) \times \cdots \times (C^\eta(\mathcal{X}) \cap L^\infty(\mathcal{X}))$ as an element of $(C_b^\eta(\mathcal{X}))'$ via

$$T(f_1, \dots, f_m)(x) := \lim_{k\to\infty} [T(\psi_k f_1, \dots, \psi_k f_m)(x) + G(\psi_k f_1, \dots, \psi_k f_m)] \tag{5.1}$$

for all $x \in \mathcal{X}$, where

$$G(\psi_k f_1, \dots, \psi_k f_m) \\ := -\int_{\min_{1\le j\le m}\{d(y_j, x_0)\}\ge 1} K(x_0, y_1, \dots, y_m)(\psi_k f_1)(y_1) \cdots (\psi_k f_m)(y_m)\, d\mu(y_1) \cdots d\mu(y_m),$$

and the limit is taken in the weak-topology of $(C_b^\eta(\mathcal{X}))'$.*

REMARK 5.2. For all $f_1, \dots, f_m \in C_b^\eta(\mathcal{X})$ and $x \notin \bigcap_{j=1}^m \operatorname{supp} f_j$, since there exists $k_1 \in \mathbb{Z}$ such that $\operatorname{supp} f_j \subset B(x_0, 2^{k_1})$ for all $j \in \{1, \dots, m\}$, we see that, when $k \ge k_1$,

$$\psi_k(y_j) f_j(y_j) = \psi(2^{-k} d(y_j, x_0)) f_j(y_j) = f_j(y_j)$$

for all $j \in \{1, \dots, m\}$ and $y_j \in \mathcal{X}$, which implies that the value of $T(f_1, \dots, f_m)(x)$ as defined in (5.1) differs from that in Definition 3.2 by the constant

$$-\int_{\min_{1\le j\le m}\{d(y_j, x_0)\}\ge 1} K(x_0, y_1, \dots, y_m) f_1(y_1) \cdots f_m(y_m)\, d\mu(y_1) \cdots d\mu(y_m).$$

However, this causes no difference when $T(f_1, \ldots, f_m)$ is considered as a function in $\mathrm{BMO}(\mathcal{X})$.

LEMMA 5.3. *Let $K \in \mathrm{Ker}(m, C_k, \delta)$ and $f_m \in L^\infty(\mathcal{X})$. Let Ω_{m-1} be as in Definition 3.1. For $(x, y_1, \ldots, y_{m-1}) \in \Omega_{m-1}$ define*

$$K_{f_m}(x, y_1, \ldots, y_{m-1}) := \int_{\mathcal{X}} K(x, y_1, \ldots, y_{m-1}, y_m) f_m(y_m)\, d\mu(y_m). \tag{5.2}$$

Then there exists a positive constant $\widetilde{C}$, depending on $\mathcal{X}$ and m, such that

$$K_{f_m} \in \mathrm{Ker}(m-1, \widetilde{C} C_K \|f_m\|_{L^\infty(\mathcal{X})}, \delta).$$

Proof. Fix $(x, y_1, \ldots, y_{m-1}) \in \Omega_{m-1}$. Without loss of generality, we may assume that

$$d(x, y_1) = \max_{1 \le j \le m-1} d(x, y_j).$$

Then

$$V(x, y_1) \le \sum_{k=1}^{m-1} V(x, y_k) \le (m-1) V(x, y_1). \tag{5.3}$$

First we consider the size estimate of K_{f_m}. Using (3.1) and (5.3), we obtain

$$\begin{aligned}
&|K_{f_m}(x, y_1, \ldots, y_{m-1})| \\
&\qquad \le C_K \|f_m\|_{L^\infty(\mathcal{X})} \int_{\mathcal{X}} \frac{1}{[\sum_{k=1}^m V(x, y_k)]^m}\, d\mu(y_m) \\
&\qquad \lesssim C_K \|f_m\|_{L^\infty(\mathcal{X})} \int_{\mathcal{X}} \frac{1}{[V(x, y_1) + V(x, y_m)]^m}\, d\mu(y_m) \\
&\qquad \sim C_K \|f_m\|_{L^\infty(\mathcal{X})} \sum_{\ell=0}^{\infty} \int_{d(y_m, x) \sim 2^\ell d(x, y_1)} \frac{1}{[V(x, y_1) + V(x, y_m)]^m}\, d\mu(y_m),
\end{aligned}$$

where the notation $d(y_m, x) \sim 2^\ell d(x, y_1)$ means that $d(y_m, x) < 2^\ell d(x, y_1)$ for $\ell = 0$ and $2^{\ell-1} d(x, y_1) \le d(y_m, x) < 2^\ell d(x, y_1)$ for $\ell \ge 1$. Obviously, when $\ell = 0$,

$$\int_{d(y_m, x) < d(x, y_1)} \frac{1}{[V(x, y_1) + V(x, y_m)]^m}\, d\mu(y_m) \le \frac{1}{[V(x, y_1)]^{m-1}}.$$

For each fixed $\ell \ge 1$, using (2.2), we see that

$$\begin{aligned}
&\int_{d(y_m, x) \sim 2^\ell d(x, y_1)} \frac{1}{[V(x, y_1) + V(x, y_m)]^m}\, d\mu(y_m) \\
&\qquad \le \int_{d(y_m, x) \sim 2^\ell d(x, y_1)} \frac{1}{[V(x, y_1) + \mu(B(x, 2^{\ell-1} d(x, y_1)))]^m}\, d\mu(y_m) \\
&\qquad \lesssim \frac{2^{-\ell\kappa(m-1)}}{[V(x, y_1)]^{m-1}}.
\end{aligned}$$

Combining the last two formulae and summing over ℓ, we obtain

$$|K_{f_m}(x, y_1, \ldots, y_{m-1})| \lesssim \frac{C_K \|f_m\|_{L^\infty(\mathcal{X})}}{[V(x, y_1)]^{m-1}} \le \frac{\widetilde{C} C_K \|f_m\|_{L^\infty(\mathcal{X})}}{[\sum_{k=1}^{m-1} V(x, y_k)]^{m-1}}, \tag{5.4}$$

where we used (5.3) in the second inequality. Here and in the remainder of this proof, $\widetilde{C}$ denotes a positive constant depending on $\mathcal{X}$ and m.

Set $y_0 := x$. Suppose that $0 \le j \le m-1$ and

$$d(y_j, y_j') \le \max_{1\le k\le m-1} d(x, y_k)/2.$$

Then $d(y_j, y_j') \le \max_{1\le k\le m} d(x, y_k)/2$. Consequently, applying (3.2) we obtain

$$\begin{aligned}&|K_{f_m}(y_0, y_1, \dots, y_j, \dots, y_{m-1}) - K_{f_m}(y_0, y_1, \dots, y_j', \dots, y_{m-1})|\\ &\qquad \le C_K \|f_m\|_{L^\infty(\mathcal{X})} \left[\frac{d(y_j, y_j')}{\max_{0\le k\le m-1} d(y_0, y_k)}\right]^\delta \int_{\mathcal{X}} \frac{1}{[\sum_{k=1}^m V(y_0, y_k)]^m}\, d\mu(y_m).\end{aligned}$$

Finally, as we proved before,

$$\int_{\mathcal{X}} \frac{1}{[\sum_{k=1}^m V(y_0, y_k)]^m}\, d\mu(y_m) \lesssim \frac{1}{[V(x, y_1)]^{m-1}} \le \frac{\widetilde{C}}{[\sum_{k=1}^{m-1} V(x, y_k)]^{m-1}},$$

which implies that K_{f_m} has the desired smoothness estimate. From this and (5.4), we deduce that

$$K_{f_m} \in \operatorname{Ker}(m-1, \widetilde{C} C_K \|f_m\|_{L^\infty(\mathcal{X})}, \delta). \ ■$$

REMARK 5.4. By symmetry, Lemma 5.3 is true if we use any other variable in K instead of y_m. Given an m-linear operator T and a fixed function $f_j \in C_b^\eta(\mathcal{X})$ for some $j \in \{0, \dots, m\}$, we define the $(m-1)$-linear operator T_{f_j} as

$$T_{f_j}(f_1, \dots, f_{j-1}, f_{j+1}, \dots, f_m) := T(f_1, \dots, f_{j-1}, f_j, f_{j+1}, \dots, f_m).$$

The transposes of T_{f_j} satisfy

$$(T_{f_j})^{*k} = (T^{*k})_{f_j}, \qquad k \in \{1, \dots, j-1\}, \tag{5.5}$$

$$(T_{f_j})^{*k} = (T^{*(k+1)})_{f_j}, \quad k \in \{j, \dots, m-1\}. \tag{5.6}$$

Denote by $L_b^\infty(\mathcal{X})$ the set of all bounded functions on $\mathcal{X}$ with bounded support.

LEMMA 5.5. *Let T be a multilinear operator with kernel $K \in \operatorname{Ker}(m, C_K, \delta)$ which can be extended to a bounded operator $T : L^{p_1}(\mathcal{X}) \times \cdots \times L^{p_m}(\mathcal{X}) \to L^p(\mathcal{X})$ for some indices $1 \le p, p_1, \dots, p_m < \infty$ and $1/p = 1/p_1 + \cdots + 1/p_m$. Given $f_m \in L_b^\infty(\mathcal{X})$, let T_{f_m} be as in* (5.4). *Then T_{f_m} is an $(m-1)$-linear Calderón–Zygmund operator with kernel K_{f_m} given by* (5.2).

Proof. Let $f_1, \dots, f_{m-1} \in C_b^\eta(\mathcal{X})$ and $f_m \in L_b^\infty(\mathcal{X})$. By Lemma 5.3 and Definition 3.2, it is enough to show that, for every $x \notin \bigcap_{\ell=1}^{m-1} \operatorname{supp} f_\ell$,

$$T_{f_m}(f_1, \dots, f_{m-1})(x) = \int_{\mathcal{X}^{m-1}} K_{f_m}(x, y_1, \dots, y_{m-1}) \prod_{\ell=1}^{m-1} f_\ell(y_\ell)\, d\mu(y_\ell). \tag{5.7}$$

To this end, take $h \in C_b^\eta(\mathcal{X})$ such that

$$\operatorname{supp} h \cap \bigcap_{\ell=1}^{m-1} \operatorname{supp} f_\ell = \emptyset. \tag{5.8}$$

By duality and the hypotheses, we obtain

$$\langle T(f_1, \dots, f_{m-1}, f_m), h\rangle = \langle T^{*m}(f_1, \dots, f_{m-1}, h), f_m\rangle,$$

where $T^{*m}(f_1, \ldots, f_{m-1}, h)$ is well defined and belongs to $L^{p'_m}(\mathcal{X})$ (with the convention $1' = \infty$ and $\infty' = 1$). Also, (5.8) implies that $T^{*m}(f_1, \ldots, f_{m-1}, h)$ is given by the absolutely convergent integral

$$z \mapsto \int_{\mathcal{X}^m} K(x, y_1, \ldots, y_{m-1}, z) h(x)\, d\mu(x) \prod_{\ell=1}^{m-1} f_\ell(y_\ell)\, d\mu(y_\ell).$$

By this and (5.8), we know that $\langle T(f_1, \ldots, f_{m-1}, f_m), h\rangle$ is given by the absolutely convergent integral

$$\int_{\mathcal{X}} \bigg[\int_{\mathcal{X}^{m-1}} K_{f_m}(x, y_1, \ldots, y_{m-1}) \prod_{\ell=1}^{m-1} f_\ell(y_\ell)\, d\mu(y_\ell) \bigg] h(x)\, d\mu(x),$$

which implies (5.7). ∎

5.2. BMO-boundedness of multilinear singular integrals

THEOREM 5.6. *Let T be an m-linear operator with a kernel $K \in \mathrm{Ker}(m, C_K, \delta)$ for some $C_K > 0$ and $\delta \in (0,1)$, and T bounded from $L^{q_1}(\mathcal{X}) \times \cdots \times L^{q_m}(\mathcal{X})$ to $L^q(\mathcal{X})$, where $1 < q, q_1, \ldots, q_m < \infty$ and $1/q = 1/q_1 + \cdots + 1/q_m$. Then T extends to a bounded operator from the m-fold product $L^\infty_b(\mathcal{X}) \times \cdots \times L^\infty_b(\mathcal{X})$ to $\mathrm{BMO}(\mathcal{X})$ with norm at most a positive constant multiple of $C_K + \|T\|_{L^{q_1}(\mathcal{X}) \times \cdots \times L^{q_m}(\mathcal{X}) \to L^{q,\infty}(\mathcal{X})}$.*

Proof. Set

$$W := \|T\|_{L^{q_1}(\mathcal{X}) \times \cdots \times L^{q_m}(\mathcal{X}) \to L^{q,\infty}(\mathcal{X})}.$$

We prove the assertion of Theorem 5.6 by induction over m. In the case $m = 1$ the assertion is a known result of linear Calderón–Zygmund theory; see [94, 40]. Next, we assume that the theorem is true for $(m-1)$-linear Calderón–Zygmund operators, and prove that it is valid for the m-linear operators.

To achieve this, we fix a function $f_m \in L^\infty_b(\mathcal{X})$ and define the $(m-1)$-linear operator

$$T_{f_m}(f_1, \ldots, f_{m-1}) := T(f_1, \ldots, f_m).$$

By Lemmas 5.3 and 5.5, T_{f_m} is an $(m-1)$-linear Calderón–Zygmund operator with a kernel $K_{f_m} \in CZK(m-1, \widetilde{C} C_K \|f_m\|_{L^\infty(\mathcal{X})}, \delta)$ as given by (2.17). Notice that the mth transpose T^{*m} of T is a Calderón–Zygmund operator with the following boundedness property:

$$T^{*m} : L^{q_1}(\mathcal{X}) \times \cdots \times L^{q_{m-1}}(\mathcal{X}) \times L^{q'}(\mathcal{X}) \to L^{q'_m}(\mathcal{X})$$

with norm W. Since Corollary 4.15 holds when the multiple weight is $w_1 = \cdots = w_m = 1$, we know that

$$T^{*m} : L^m(\mathcal{X}) \times \cdots \times L^m(\mathcal{X}) \to L^1(\mathcal{X})$$

with norm $C(C_K + W)$ and C as in Corollary 4.15. By duality,

$$T : L^m(\mathcal{X}) \times L^m(\mathcal{X}) \times \cdots \times L^\infty(\mathcal{X}) \to L^{m'}(\mathcal{X})$$

with norm $C(C_K + W)$. It follows that T_{f_m} is bounded from the $(m-1)$-fold product space $L^m(\mathcal{X}) \times \cdots \times L^m(\mathcal{X})$ to $L^{m'}(\mathcal{X})$ with operator norm $C(C_K + W)\|f_m\|_{L^\infty(\mathcal{X})}$. Therefore, the induction hypothesis implies that T_{f_m} is bounded from the $(m-1)$-fold product space

$L_b^\infty(\mathcal{X}) \times \cdots \times L_b^\infty(\mathcal{X})$ to $\mathrm{BMO}(\mathcal{X})$ with operator norm $C(C_K + W)\|f_m\|_{L^\infty(\mathcal{X})}$. Since f_m is an arbitrary $L_b^\infty(\mathcal{X})$ function, the conclusion of the theorem follows. ■

Corollary 5.7. *Let T be as in Theorem 5.6. Then T extends to a bounded operator from the m-fold product $L^\infty(\mathcal{X}) \times \cdots \times L^\infty(\mathcal{X})$ to $\mathrm{BMO}(\mathcal{X})$ with norm at most a positive constant multiple of $C_K + \|T\|_{L^{q_1}(\mathcal{X})\times\cdots\times L^{q_m}(\mathcal{X})\to L^{q,\infty}(\mathcal{X})}$.*

Proof. Fix $x_0 \in \mathcal{X}$ and consider a function $h : \mathbb{R} \to \mathbb{R}$ such that $\operatorname{supp} h \subset (-2, 2)$, $0 \le h \le 1$, and $h(t) = 1$ if $0 \le |t| \le 1$. For $k \in \mathbb{Z}$ and $x \in \mathcal{X}$, define

$$\psi_k(x) := h(2^{-k} d(x, x_0)).$$

For any $k \in \mathbb{Z}$, since T maps the m-fold product space $L^{2m}(\mathcal{X}) \times \cdots \times L^{2m}(\mathcal{X})$ to $L^2(\mathcal{X})$, we know that $T(\psi_k f_1, \ldots, \psi_k f_m) \in L^2(\mathcal{X})$ whenever $f_1, \ldots, f_m \in L^\infty(\mathcal{X})$. Hence, for any given $f_1, \ldots, f_m \in L^\infty(\mathcal{X})$, we define

$$G(\psi_k f_1, \ldots, \psi_k f_m) := -\int_{\min_{1\le j\le m} d(y_j,x_0)>1} K(x_0, y_1, \ldots, y_m) \prod_{j=1}^m \psi_k(y_j) f_j(y_j)\, d\mu(y_j).$$

Then, by Lemma 5.1, T extends to the m-fold product $L^\infty(\mathcal{X}) \times \cdots \times L^\infty(\mathcal{X})$ via

$$T(f_1, \ldots, f_m) = \lim_{k\to\infty} [T(\psi_k f_1, \ldots, \psi_k f_m) + G(\psi_k f_1, \ldots, \psi_k f_m)], \tag{5.9}$$

where the limit exists almost everywhere and defines a locally integrable function. Now, by (5.9) and Theorem 5.6, we have

$$\begin{aligned}
\|T(f_1, \ldots, f_m)\|_{\mathrm{BMO}(\mathcal{X})} &\le \limsup_{k\to\infty} \|T(\psi_k f_1, \ldots, \psi_k f_m) + G(\psi_k f_1, \ldots, \psi_k f_m)\|_{\mathrm{BMO}(\mathcal{X})} \\
&= \limsup_{k\to\infty} \|T(\psi_k f_1, \ldots, \psi_k f_m)\|_{\mathrm{BMO}(\mathcal{X})} \\
&\lesssim (C_K + B) \lim_{k\to\infty} \prod_{j=1}^m \|\psi_k f_j\|_{L^\infty(\mathcal{X})} \\
&\lesssim (C_K + B) \prod_{j=1}^m \|f_j\|_{L^\infty(\mathcal{X})}. \ ■
\end{aligned}$$

5.3. A multilinear $T1$-theorem

Definition 5.8. Let $A > 0$. We say that an m-linear Calderón–Zygmund operator T is *BMO-restrictively bounded with bound A* if there exists $\eta \in (0, 1]$ such that, for all $x_1, \ldots, x_m \in \mathcal{X}$, $R_1, \ldots, R_m \in (0, \infty)$, $\phi_i \in \mathcal{T}(\eta, x_i, R_i)$ and $j \in \{0, \ldots, m\}$,

$$\|T^{*j}(\phi_1, \ldots, \phi_m)\|_{\mathrm{BMO}(\mathcal{X})} \le A.$$

Obviously, any m-linear Calderón–Zygmund operator T as in Theorem 5.6 is BMO-restrictively bounded. Conversely, we have the following multilinear $T1$-type theorem.

Theorem 5.9. *Let T be an m-linear Calderón–Zygmund operator with a kernel K belonging to $\mathrm{Ker}(m, C_K, \delta)$ for some $C_K > 0$ and $\delta \in (0, 1]$. Suppose that T is BMO-restrictively bounded for some positive constant A. Then there exist $1 < q, q_1, \ldots, q_m < \infty$ such that $\sum_{j=1}^m 1/q_j = 1/q$ and T has a bounded extension from $L^{q_1}(\mathcal{X}) \times \cdots \times L^{q_m}(\mathcal{X})$ to $L^q(\mathcal{X})$. Moreover,*

$$\|T\|_{L^{q_1}(\mathcal{X})\times\cdots\times L^{q_m}(\mathcal{X})\to L^q(\mathcal{X})} \le C(A + C_K),$$

where C depends only on C_1, C_2, m and δ.

Proof. We use induction on m. By Theorem 2.23, the assertion is true if $m = 1$. Suppose that the result is valid for $(m-1)$-linear operators. Let T be an m-linear operator, fix $\phi_m \in \mathcal{T}(\eta, x_m, R_m)$, and consider the $(m-1)$-linear operator

$$T_{\phi_m}(f_1, \dots, f_{m-1}) := T(f_1, \dots, f_{m-1}, \phi_m).$$

By (5.5) and (5.6), together with Definition 5.8, we see that T_{ϕ_m} is also BMO-restrictively bounded with constant A. By the induction hypothesis, we know that there exist $1 < q, q_1, \dots, q_{m-1} < \infty$ satisfying $\sum_{j=1}^{m-1} 1/q_j = 1/q$ such that

$$T_{\phi_m} : L^{q_1}(\mathcal{X}) \times \cdots \times L^{q_{m-1}}(\mathcal{X}) \to L^q(\mathcal{X})$$

with norm at most a positive constant multiple of $C_K + A$.

Next, by Lemma 5.5 and the fact that $\phi_m \in \mathcal{T}(\eta, x_m, R_m) \subset L^\infty(\mathcal{X})$, we conclude that T_{ϕ_m} has a Calderón–Zygmund kernel in $\mathrm{Ker}(m-1, C_K, \delta)$. Thus, from Theorem 5.6, it follows that

$$T_{\phi_m} : \overbrace{L_b^\infty(\mathcal{X}) \times \cdots \times L_b^\infty(\mathcal{X})}^{m-1\,\text{times}} \to \mathrm{BMO}(\mathcal{X})$$

with norm less than a positive constant multiple of $C_K + A$. Consequently,

$$\|T(g, \phi_2, \dots, \phi_m)\|_{\mathrm{BMO}(\mathcal{X})} \lesssim (C_K + A)\|g\|_{L^\infty(\mathcal{X})} \quad \text{for all } g \in L_b^\infty(\mathcal{X}). \tag{5.10}$$

Notice that we could repeat this argument when the function g appears in any other entry $2 \le j \le m$. Next, for $1 \le j \le m$ and $g_j \in L_b^\infty(\mathcal{X})$, consider the operators T_{g_j} given by

$$T_{g_j}(f_1, \dots, f_{m-1}) := T(f_1, \dots, f_{j-1}, g_j, f_{j+1}, \dots, f_{m-1}).$$

By (5.10), T_{g_1} satisfies the $(m-1)$-linear BMO-restrictive boundedness with a constant of the form $\widetilde{C}_1 := C(C_K + A)\|g_1\|_{L^\infty(\mathcal{X})}$, where C depends only on C_1, C_2, m, and δ. Analogous conclusions hold for all T_{g_j}, $2 \le j \le m$. Therefore, by the inductive hypothesis we conclude that, for $1 \le j \le m$, the operator T_{g_j} satisfies

$$L^{q_1}(\mathcal{X}) \times \cdots \times L^{q_{j-1}}(\mathcal{X}) \times L^{q_{j+1}}(\mathcal{X}) \cdots \times L^{q_m}(\mathcal{X}) \to L^q(\mathcal{X})$$

for some $1 < q_k := q_k(j)$, $q := q(j) < \infty$ satisfying $\sum_{1 \le k \le m,\, k \ne j} 1/q_k = 1/q$, with bound at most a positive constant multiple of $\widetilde{C}_j := C(C_K + A)\|g_j\|_{L^\infty(\mathcal{X})}$. In other words, T maps

$$L^{q_1}(\mathcal{X}) \times \cdots \times L^{q_{j-1}}(\mathcal{X}) \times L_b^\infty(\mathcal{X}) \times L^{q_{j+1}}(\mathcal{X}) \cdots \times L^{q_m}(\mathcal{X}) \to L^q(\mathcal{X}),$$

with norm bounded by at most a positive constant multiple of $C_K + A$. Now notice that each point of the form $(1/q_1, 1/q_2, \dots, 1/q_m)$, with $1 < q_1, \dots, q_m < \infty$ and $1/q_1 + \cdots + 1/q_m < 1$, lies in the open convex hull of points of the form

$$(1/q_1, \dots, 1/q_{j-1}, \overbrace{\;0\;}^{j\text{th entry}}, 1/q_{j+1}, \dots, 1/q_m), \quad 1 \le j \le m.$$

Based on [43, Theorem 4.6] (see also [50]), T is of strong type for any point in this convex hull with a bound controlled by a positive constant multiple of $C_K + A$, which completes the proof of Theorem 5.9. ∎

6. Bilinear $T1$-theorems on Triebel–Lizorkin and Besov spaces

A powerful method to prove the boundedness of operators on Triebel–Lizorkin or Besov spaces is to show that they map appropriate atoms into molecules. This method goes back to Y. Meyer [81] and it was used by Frazier, Han, Jawerth and Weiss [32] and by Frazier, Torres and Weiss [35] to prove the $T1$-theorem for linear Calderón–Zygmund operators on Triebel–Lizorkin spaces; see also the work of Torres [99]. A systematic treatment of bilinear operators through the use of wavelet decompositions was developed by Grafakos and Torres [51]. Bényi [3] applied such decomposition techniques to obtain a $T1$-theorem for bilinear operators on the space $\dot{F}^0_{p,q}(\mathbb{R}^n)$ for $1 \le p < \infty$ and $1 \le q \le \infty$. Bényi [2] also studied more singular pseudodifferential operators with forbidden symbols on Lipschitz and Besov spaces.

Motivated by these pioneering works, in this section, we prove $T1$-theorems for bilinear Calderón–Zygmund operators on Triebel–Lizorkin spaces $\dot{F}^s_{p,q}(\mathcal{X})$ and Besov spaces $\dot{B}^s_{p,q}(\mathcal{X})$ for full admissible ranges of s, p, q; see Theorems 6.14 and 6.15. This successfully addresses an open problem posed by Grafakos and Torres [55, p. 85].

6.1. Bilinear weak boundedness property. Let $\mathcal{Q}$ denote the collection of Christ's dyadic cubes as in Lemma 2.5. For any $k \in \mathbb{Z}$, let

$$\mathcal{Q}_k := \{Q^k_\alpha : \alpha \in I_k\}.$$

Denote by c_Q the center of a cube $Q \in \mathcal{Q}$.

Definition 6.1. Let $\eta \in (0,1]$. A continuous bilinear operator

$$T : C^\eta_b(\mathcal{X}) \times C^\eta_b(\mathcal{X}) \to (C^\eta_b(\mathcal{X}))'$$

is said to satisfy the *bilinear weak boundedness property* (for short, $T \in \mathbf{BWBP}(\eta)$) if there exists a positive constant C such that, for all $f, g, h \in \mathcal{T}(\eta, x, r)$ and $i \in \{1,2\}$,

$$|\langle T^{*,i}(f,g), h\rangle| \le C\mu(B(x,r)). \tag{6.1}$$

Denote by $\|T\|_{\mathbf{BWBP}(\eta)}$ the smallest C satisfying (6.1).

If $T \in \mathbf{BWBP}(\eta)$ and K is the distribution kernel of T as defined in (3.3), then (6.1) is equivalent to the following:

$$|\langle K, f \otimes g \otimes h\rangle| \le C\mu(B(x,r)), \quad \forall f, g, h \in \mathcal{T}(\eta, x, r); \tag{6.2}$$

here and in what follows,

$$(f \otimes g \otimes h)(x,y,z) = f(x)g(y)h(z), \quad \forall x, y, z \in \mathcal{X}.$$

However, unlike in the Euclidean case, we do not know whether (6.2) holds for general "bump functions" on $\mathcal{X}^3$ or not. In other words, we do not know whether or not (6.2) implies that

$$|\langle K, F\rangle| \le C\mu(B(x,r))$$

whenever $F : \mathcal{X}^3 \to \mathbb{C}$ satisfies, for all $x_0, x_1, x_2 \in \mathcal{X}$,

(i) $\operatorname{supp} F \subset B(x,r) \times B(x,r) \times B(x,r)$ for some $x \in \mathcal{X}$ and $r > 0$;
(ii) $\|F\|_{L^\infty(\mathcal{X}\times\mathcal{X}\times\mathcal{X})} \le 1$;
(iii) $\|F(\cdot, x_1, x_2)\|_{\dot{C}^\eta(\mathcal{X})} \le r^{-\eta}$, $\|F(x_0, \cdot, x_2)\|_{\dot{C}^\eta(\mathcal{X})} \le r^{-\eta}$ and $\|F(x_0, x_1, \cdot)\|_{\dot{C}^\eta(\mathcal{X})} \le r^{-\eta}$.

The following lemma, usually referred to as *Meyer's lemma* (see Meyer [81], also Torres [99] for its linear version), was proved by Bényi [3] for bilinear Calderón–Zygmund operators on $\mathbb{R}^n$. This bilinear Meyer's lemma is crucial for the proof of T mapping atoms into bilinear molecules. We mention that the proof for RD-spaces is much more subtle than that in $\mathbb{R}^n$.

LEMMA 6.2. *Let $\eta \in (0,1]$ and*

$$D^{\complement} := \mathcal{X}^3 \setminus \{(x,x,x) : x \in \mathcal{X}\}.$$

Suppose that $T \in \mathbf{BWBP}(\eta)$ and its kernel K satisfies, for all $(x_0, x_1, x_2) \in D^{\complement}$,

$$|K(x_0, x_1, x_2)| \le C_K \frac{1}{[V(x_0, x_1) + V(x_0, x_2)]^2}.$$

If $f, g, h \in C_b^\eta(\mathcal{X})$ and $f \otimes g \otimes h$ vanishes on the diagonal $\{(x,x,x) : x \in \mathcal{X}\}$, then

$$\langle T(f,g), h\rangle = \int_{D^{\complement}} K(x_0, x_1, x_2) f(x_1) g(x_2) h(x_0)\, dx_0\, dx_1\, dx_2, \tag{6.3}$$

and the integral is absolutely convergent.

Before we prove Lemma 6.2, we first establish the following auxiliary estimate.

LEMMA 6.3. *Let $\alpha \in (0,\infty)$, $x_0 \in \mathcal{X}$ and $R \in (0,\infty)$. Then*

$$\int_{B(x_0,R)} \int_{B(x_0,R)} \int_{B(x_0,R)} \frac{d(x,x_1)^\alpha + d(x,x_2)^\alpha}{[V(x,x_1) + V(x,x_2)]^2}\, d\mu(x_1)\, d\mu(x_2)\, d\mu(x) < \infty.$$

Proof. Observe that, when $x_1, x_2, x \in B(x_0, R)$, we have

$$d(x,x_1) < 2R \quad \text{and} \quad d(x,x_2) < 2R.$$

With this observation, we may assume that $\alpha < \kappa$; otherwise we may bound

$$d(x,x_1)^\alpha + d(x,x_2)^\alpha$$

by $(2R)^{\alpha-\epsilon}[d(x,x_1)^\epsilon + d(x,x_2)^\epsilon]$ for some $\epsilon \in (0,\kappa)$.

Now assume that $\alpha < \kappa$. In this case it suffices to prove that there exists a positive constant C such that, for all $x \in B(x_0, R)$,

$$\int_{B(x_0,R)} \int_{B(x_0,R)} \frac{d(x,x_1)^\alpha + d(x,x_2)^\alpha}{[V(x,x_1) + V(x,x_2)]^2}\, d\mu(x_1)\, d\mu(x_2) \le C.$$

To this end, by symmetry, we see that it is enough to show that

$$\int_{B(x_0,R)} \int_{d(x,x_2)\ge d(x,x_1)} \frac{d(x,x_1)^\alpha + d(x,x_2)^\alpha}{[V(x,x_1) + V(x,x_2)]^2}\, d\mu(x_2)\, d\mu(x_1) \le C \tag{6.4}$$

for some positive constant C independent of $x \in B(x_0, R)$. To prove (6.4), we observe that

$$\int_{d(x,x_2)\geq d(x,x_1)} \frac{d(x,x_1)^\alpha + d(x,x_2)^\alpha}{[V(x,x_1)+V(x,x_2)]^2}\, d\mu(x_2) \sim \int_{d(x,x_2)\geq d(x,x_1)} \frac{d(x,x_2)^\alpha}{[V(x,x_2)]^2}\, d\mu(x_2),$$

where the right hand side can be estimated as follows:

$$\begin{aligned}
&\int_{d(x,x_2)\geq d(x,x_1)} \frac{d(x,x_2)^\alpha}{[V(x,x_2)]^2}\, d\mu(x_2)\\
&\qquad = \sum_{j=1}^\infty \int_{2^{j-1}d(x,x_1)\leq d(x,x_2)<2^j d(x,x_1)} \frac{d(x,x_2)^\alpha}{[V(x,x_2)]^2}\, d\mu(x_2)\\
&\qquad \sim \sum_{j=1}^\infty \int_{2^{j-1}d(x,x_1)\leq d(x,x_2)<2^j d(x,x_1)} \frac{[2^j d(x,x_1)]^\alpha}{[\mu(B(x,2^j d(x,x_1)))]^2}\, d\mu(x_2)\\
&\qquad \lesssim \sum_{j=1}^\infty \frac{[2^j d(x,x_1)]^\alpha}{\mu(B(x,2^j d(x,x_1)))}.
\end{aligned}$$

Furthermore, from the reverse doubling condition (2.2) and the assumption $\alpha < \kappa$, it follows that

$$\sum_{j=1}^\infty \frac{[2^j d(x,x_1)]^\alpha}{\mu(B(x,2^j d(x,x_1)))} \lesssim \sum_{j=1}^\infty \frac{[2^j d(x,x_1)]^\alpha}{2^{j\kappa}\mu(B(x,d(x,x_1)))} \lesssim \frac{d(x,x_1)^\alpha}{\mu(B(x,d(x,x_1)))}.$$

Thus, we have

$$\int_{d(x,x_2)\geq d(x,x_1)} \frac{d(x,x_1)^\alpha + d(x,x_2)^\alpha}{[V(x,x_1)+V(x,x_2)]^2}\, d\mu(x_2) \lesssim \frac{d(x,x_1)^\alpha}{\mu(B(x,d(x,x_1)))}.$$

Integrating both sides on the ball $B(x_0, R)$ with respect to x_1 and then using Lemma 2.4(c), we see that the right hand side of (6.4) is bounded by a positive constant multiple of

$$\int_{B(x_0,R)} \frac{d(x,x_1)^\alpha}{\mu(B(x,d(x,x_1)))}\, d\mu(x_1) \lesssim R^\alpha,$$

which proves (6.4), and hence completes the proof of Lemma 6.3. ∎

Proof of Lemma 6.2. Let $\{S_k\}_{k\in\mathbb{Z}}$ be a 1-ATI with bounded support and with the additional properties that every $S_k(x,y)$ is non-negative and

$$S_k(x,y) = S_k(y,x)$$

for all $x, y \in \mathcal{X}$ and all $k \in \mathbb{Z}$. Without loss of generality, we may as well assume that $S_k(x,y) = 0$ when $d(x,y) \geq 2^{-k}$. Let $\psi \in C_c^\infty(\mathbb{R})$ be a non-negative radial function such that $0 \leq \psi \leq 1$, $\psi(\xi) = 1$ if $|\xi| \leq 2$, and $\psi(\xi) = 0$ if $|\xi| > 4$. Define

$$\lambda_k(x,y) := \int_{\mathcal{X}} S_k(x,z)\psi(2^k d(z,y))\, d\mu(z), \qquad \forall k \in \mathbb{Z} \text{ and } x, y \in \mathcal{X}.$$

It is easy to see that the functions $\{\lambda_k\}_{k\in\mathbb{Z}}$ enjoy the following properties:

(i) $\lambda_k \in C_b^\eta(\mathcal{X})$;
(ii) $0 \leq \lambda_k(x,x) \leq 1$ for all $x \in \mathcal{X}$;
(iii) $\lambda_k(x,y) = 0$ when $d(x,y) \geq 2^{-k+3}$;

(iv) $\lambda_k(x,y)=1$ when $d(x,y)<2^{-k}$;
(v) $\lambda_k(x,y)=\lambda_k(y,x)$ for all $x,y\in\mathcal{X}$.

Observe that, if $f,g\in C_b^\eta(\mathcal{X})$, then $fg\in C_b^\eta(\mathcal{X})$. So we can write

$$\begin{aligned}\langle T(f,g),h\rangle &= \langle T(f(\cdot)\lambda_k(\cdot,x),\, g(\cdot)\lambda_k(\cdot,x)),\, h(x)\rangle\\ &\quad+\langle T(f(\cdot)\lambda_k(\cdot,x),\, g(\cdot)[1-\lambda_k(\cdot,x)])(x),\, h(x)\rangle\\ &\quad+\langle T(f(\cdot)[1-\lambda_k(\cdot,x)],\, g(\cdot))(x),\, h(x)\rangle. \end{aligned}\tag{6.5}$$

First we consider the second term of (6.5). Since

$$x\notin \operatorname{supp}(f(\cdot)\lambda_k(\cdot,x))\cap\operatorname{supp}(g(\cdot)[1-\lambda_k(\cdot,x)]),$$

by (3.3), we obtain

$$\begin{aligned}&T(f(\cdot)\lambda_k(\cdot,x),\, g(\cdot)[1-\lambda_k(\cdot,x)])(x)\\ &\qquad=\int_{\mathcal{X}^2}K(x,x_1,x_2)f(x_1)\lambda_k(x_1,x)g(x_2)[1-\lambda_k(x_2,x)]\,d\mu(x_1)\,d\mu(x_2).\end{aligned}$$

Without loss of generality, we may assume that f,g,h are supported on some ball $B(x_0,R)$ with $x_0\in\mathcal{X}$ and $R\in(0,\infty)$. Hence,

$$\begin{aligned}&\langle T(f(\cdot)\lambda_k(\cdot,x),\, g(\cdot)[1-\lambda_k(\cdot,x)]),h(x)\rangle\\ &\qquad=\int_{B(x_0,R)^3}K(x,x_1,x_2)\lambda_k(x_1,x)[1-\lambda_k(x_2,x)]\\ &\qquad\qquad\times f(x_1)g(x_2)h(x)\,d\mu(x_1)\,d\mu(x_2)\,d\mu(x)\\ &\qquad=\int_{B(x_0,R)^3}K(x,x_1,x_2)\lambda_k(x_1,x)[1-\lambda_k(x_2,x)]\\ &\qquad\qquad\times [f(x_1)g(x_2)-f(x)g(x)]h(x)\,d\mu(x_1)\,d\mu(x_2)\,d\mu(x),\end{aligned}\tag{6.6}$$

where we used the fact $f(x)g(x)h(x)=0$ for all $x\in\mathcal{X}$. Since $f,g\in C_b^\eta(\mathcal{X})$, we have

$$\begin{aligned}|f(x_1)g(x_2)-f(x)g(x)|&\le|f(x_1)|\,|g(x_2)-g(x)|+|f(x_1)-f(x)|\,|g(x)|\\ &\le\|f\|_{L^\infty(\mathcal{X})}\|g\|_{\dot{C}^\eta(\mathcal{X})}d(x_2,x)^\eta+\|g\|_{L^\infty(\mathcal{X})}\|f\|_{\dot{C}^\eta(\mathcal{X})}d(x_1,x)^\eta,\end{aligned}$$

so the last integrand in (6.6) is bounded by

$$C_K[\|f\|_{L^\infty(\mathcal{X})}\|g\|_{\dot{C}^\eta(\mathcal{X})}+\|g\|_{L^\infty(\mathcal{X})}\|f\|_{\dot{C}^\eta(\mathcal{X})}]\|h\|_{L^\infty(\mathcal{X})}\frac{d(x_1,x)^\eta+d(x_2,x)^\eta}{[V(x,x_1)+V(x,x_2)]^2},$$

which is integrable on the product domain $B(x_0,R)^3$ in view of Lemma 6.3. From this and the Lebesgue dominated convergence theorem, it follows that

$$\begin{aligned}&\lim_{k\to\infty}\langle T(f(\cdot)\lambda_k(\cdot,x),\, g(\cdot)[1-\lambda_k(\cdot,x)]),\, h(x)\rangle\\ &\qquad=\int_{B(x_0,R)^3}\lim_{k\to\infty}\lambda_k(x_1,x)[1-\lambda_k(x_2,x)]K(x,x_1,x_2)\\ &\qquad\qquad\times[f(x_1)g(x_2)-f(x)g(x)]h(x)\,d\mu(x_1)\,d\mu(x_2)\,d\mu(x)=0.\end{aligned}$$

For the third term of (6.5), an argument similar to the above implies that

$$\begin{aligned}
&\lim_{k\to\infty} \langle T(f(\cdot)[1-\lambda_k(\cdot,x)],\, g(\cdot))(x), h(x)\rangle \\
&\quad= \int_{\mathcal{X}^3} \lim_{k\to\infty} [1-\lambda_k(x_2,x)]K(x,x_1,x_2)[f(x_1)g(x_2)-f(x)g(x)]h(x)\,d\mu(x_1)\,d\mu(x_2)\,d\mu(x) \\
&\quad= \int_{D^\complement} K(x,x_1,x_2)f(x_1)g(x_2)h(x)\,d\mu(x_1)\,d\mu(x_2)\,d\mu(x).
\end{aligned}$$

Therefore, to obtain (6.3), it suffices to prove that

$$\lim_{k\to\infty} \langle T(f(\cdot)\lambda_k(\cdot,x),\, g(\cdot)\lambda_k(\cdot,x))(x), h(x)\rangle = 0. \tag{6.7}$$

To this end, we claim that, for any $k\in\mathbb{Z}$ and $x\in\mathcal{X}$,

$$\lambda_k(\cdot,x) = \lim_{J\to\infty} \sum_{Q\in\mathcal{Q}_J} \mu(Q)\, S_k(\cdot,c_Q)\psi(2^k d(c_Q,x)) \tag{6.8}$$

in $C_b^\eta(\mathcal{X})$. Indeed, (6.8) was proved in [60, p. 31, (2.85)]; moreover, for each fixed large number J and $x,y\in\mathcal{X}$, say $J\ge k+10$, the sum

$$\sum_{Q\in\mathcal{Q}_J} \mu(Q)S_k(\cdot,c_Q)\psi(2^k d(c_Q,x))$$

has only finitely many non-zero terms.

For notational convenience, for any $Q\in\mathcal{Q}$ and $k\in\mathbb{Z}$, set

$$\Psi_{k,Q}(\cdot) := \psi(2^k d(c_Q,\cdot)) \quad \text{and} \quad S_{k,Q}(\cdot) := S_k(\cdot,c_Q).$$

By (6.8), we write

$$\begin{aligned}
&\langle T(f(\cdot)\lambda_k(\cdot,x), g(\cdot)\lambda_k(\cdot,x))(x), h(x)\rangle \\
&\qquad= \lim_{J\to\infty} \sum_{Q\in\mathcal{Q}_J}\sum_{P\in\mathcal{Q}_J} \mu(Q)\mu(P)\langle T(f\Psi_{k,Q}, g\Psi_{k,P}), S_{k,Q}S_{k,P}h\rangle.
\end{aligned} \tag{6.9}$$

Taking into account the support condition on $S_{k,Q}S_{k,P}h$, we see that the sums in (6.9) are over all $Q,P\in\mathcal{Q}_J$ satisfying $Q\cap\operatorname{supp} h\neq\emptyset$, $P\cap\operatorname{supp} h\neq\emptyset$ and $d(c_Q,c_P)\le 2^{-k+1}$. Observe that

$$\begin{aligned}
&\langle T(f\Psi_{k,Q}, g\Psi_{k,P}), S_{k,Q}S_{k,P}h\rangle \\
&\qquad\qquad= \langle T([f-f(c_Q)]\Psi_{k,Q}, [g-g(c_P)]\Psi_{k,P}), S_{k,Q}S_{k,P}[h-h(c_Q)]\rangle \\
&\qquad\qquad\quad+ h(c_Q)\langle T([f-f(c_Q)]\Psi_{k,Q}, [g-g(c_P)]\Psi_{k,P}), S_{k,Q}S_{k,P}\rangle \\
&\qquad\qquad\quad+ g(c_P)\langle T([f-f(c_Q)]\Psi_{k,Q}, \Psi_{k,P}), S_{k,Q}S_{k,P}h\rangle \\
&\qquad\qquad\quad+ f(c_Q)\langle T(\Psi_{k,Q}, g\Psi_{k,P}), S_{k,Q}S_{k,P}h\rangle \\
&\qquad\qquad=: \mathrm{Z}_1+\mathrm{Z}_2+\mathrm{Z}_3+\mathrm{Z}_4.
\end{aligned}$$

To estimate Z_1 through Z_4, we apply the hypothesis $T\in\mathbf{BWBP}(\eta)$. Notice that

$$\operatorname{supp}([f-f(c_Q)]\Psi_{k,Q}) \subset B(c_Q, 2^{-k+2})$$

and, for any $x\in\mathcal{X}$,

$$\begin{aligned}
|[f(x)-f(c_Q)]\Psi_{k,Q}(x)| &\le \|f\|_{\dot{C}^\eta(\mathcal{X})} d(x,c_Q)^\eta \psi(2^k d(x,c_Q)) \\
&\le 2^{(-k+2)\eta}\|f\|_{\dot{C}^\eta(\mathcal{X})}.
\end{aligned} \tag{6.10}$$

Now we show that, for all $x,\ x' \in \mathcal{X}$,

$$|[f(x) - f(c_Q)]\Psi_{k,Q}(x) - [f(x') - f(c_Q)]\Psi_{k,Q}(x')| \lesssim d(x,x')^\eta. \tag{6.11}$$

To this end, by the support condition of $\Psi_{k,Q}$, we may as well assume that

$$d(x, c_Q) \le 2^{-k+2} \quad \text{or} \quad d(x', c_Q) \le 2^{-k+2};$$

otherwise the left hand side of (6.11) is equal to 0 and (6.11) holds automatically. By symmetry, it suffices to consider the case $d(x', c_Q) \le 2^{-k+2}$. Then, for all $x,\ x' \in \mathcal{X}$,

$$\begin{aligned}
&|[f(x) - f(c_Q)]\Psi_{k,Q}(x) - [f(x') - f(c_Q)]\Psi_{k,Q}(x')| \\
&\qquad \le |f(x) - f(x')|\,|\Psi_{k,Q}(x)| + |f(x') - f(c_Q)|\,|\Psi_{k,Q}(x) - \Psi_{k,Q}(x')| \\
&\qquad \lesssim \|f\|_{\dot{C}^\eta(\mathcal{X})} d(x,x')^\eta + \|f\|_{\dot{C}^\eta(\mathcal{X})} d(x', c_Q)^\eta [2^k d(x,x')]^\eta \\
&\qquad \lesssim d(x,x')^\eta,
\end{aligned}$$

which proves (6.11). From (6.10) and (6.11), it follows that there exists a positive constant C such that, for all $k \in \mathbb{Z}$, $J \ge k + 10$ and $Q \in \mathcal{Q}_J$,

$$C2^{k\eta}[f - f(c_Q)]\Psi_{k,Q} \in \mathcal{T}(\eta, c_Q, 2^{-k+2}).$$

As we are considering the cubes Q and P such that $d(c_Q, c_P) \le 2^{-k+1}$, an argument similar to that used in the proofs of (6.10) and (6.11) also implies that there exists a positive constant C such that, for all $k \in \mathbb{Z}$, $J \ge k + 10$ and $Q, P \in \mathcal{Q}_J$,

$$\begin{aligned}
C2^{k\eta}[g - g(c_P)]\Psi_{k,P} &\in \mathcal{T}(\eta, c_Q, 2^{-k+2}), \\
CV_{2^{-k}}(c_Q)V_{2^{-k}}(c_P)S_{k,Q}S_{k,P} &\in \mathcal{T}(\eta, c_Q, 2^{-k+2}), \\
C2^{k\eta}V_{2^{-k}}(c_Q)V_{2^{-k}}(c_P)S_{k,Q}S_{k,P}[h - h(c_Q)] &\in \mathcal{T}(\eta, c_Q, 2^{-k+2}).
\end{aligned}$$

From these and the hypothesis $T \in \mathbf{BWBP}(\eta)$, we obtain

$$\begin{aligned}
|\mathrm{Z}_1| &\lesssim 2^{-3k\eta}\mu(B(c_Q, 2^{-k+2}))\frac{1}{V_{2^{-k}}(c_Q)V_{2^{-k}}(c_P)}, \\
|\mathrm{Z}_2| &\lesssim 2^{-2k\eta}\mu(B(c_Q, 2^{-k+2}))\frac{1}{V_{2^{-k}}(c_Q)V_{2^{-k}}(c_P)}
\end{aligned}$$

and

$$\begin{aligned}
|\mathrm{Z}_3| &\le |g(c_P)\langle T([f - f(c_Q)]\Psi_{k,Q}, \Psi_{k,P}), S_{k,Q}S_{k,P}[h - h(c_Q)]\rangle| \\
&\qquad + |g(c_P)h(c_Q)\langle T([f - f(c_Q)]\Psi_{k,Q}, \Psi_{k,P}), S_{k,Q}S_{k,P}\rangle| \\
&\lesssim 2^{-k\eta}\frac{\mu(B(c_Q, 2^{-k+2}))}{V_{2^{-k}}(c_Q)V_{2^{-k}}(c_P)}.
\end{aligned}$$

Using the assumption $f(x)g(x)h(x) = 0$ for all $x \in \mathcal{X}$, we write

$$\begin{aligned}
\mathrm{Z}_4 &= f(c_Q)\langle T(\Psi_{k,Q}, [g - g(c_P)]\Psi_{k,P}), S_{k,Q}S_{k,P}[h - h(c_Q)]\rangle \\
&\qquad + f(c_Q)g(c_P)\langle T(\Psi_{k,Q}, \Psi_{k,P}), S_{k,Q}S_{k,P}[h - h(c_Q)]\rangle \\
&\qquad + f(c_Q)h(c_Q)\langle T(\Psi_{k,Q}, [g - g(c_P)]\Psi_{k,P}), S_{k,Q}S_{k,P}\rangle \\
&\qquad + f(c_Q)h(c_Q)[g(c_P) - g(c_Q)]\langle T(\Psi_{k,Q}, \Psi_{k,P}), S_{k,Q}S_{k,P}\rangle \\
&=: \mathrm{Z}_{4,1} + \mathrm{Z}_{4,2} + \mathrm{Z}_{4,3} + \mathrm{Z}_{4,4}.
\end{aligned}$$

As in the estimates of Z_1 through Z_3, applying the hypothesis $T \in \mathbf{BWBP}(\eta)$ gives

$$|Z_{4,1}| + |Z_{4,2}| + |Z_{4,3}| \lesssim 2^{-k\eta} \mu(B(c_Q, 2^{-k+2})) \frac{1}{V_{2^{-k}}(c_Q) V_{2^{-k}}(c_P)}.$$

From $d(c_Q, c_P) \le 2^{-k+1}$, $g \in C_b^\eta(\mathcal{X})$, $T \in \mathbf{BWBP}(\eta)$ and the facts

$$C\Psi_{k,P}, C\Psi_{k,Q}, CV_{2^{-k}}(c_Q) V_{2^{-k}}(c_P) S_{k,Q} S_{k,P} \in \mathcal{T}(\eta, c_Q, 2^{-k+2}),$$

where C is a positive constant, we also obtain

$$|Z_{4,4}| \lesssim 2^{-k\eta} \mu(B(c_Q, 2^{-k+2})) \frac{1}{V_{2^{-k}}(c_Q) V_{2^{-k}}(c_P)}.$$

Combining the estimates of Z_1 through Z_4, we conclude that

$$\langle T(f\Psi_{k,Q},\, g\Psi_{k,P}), S_{k,Q} S_{k,P} h \rangle \lesssim 2^{-k\eta} \frac{1}{V_{2^{-k}}(c_P)} \lesssim 2^{-k\eta} \frac{1}{V_{2^{-k}}(c_Q)}.$$

Now we insert this into (6.9) and see that, for all $J \ge k + 10$,

$$\begin{aligned} &|\langle T(f\lambda_k(\cdot, x), g\lambda_k(\cdot, x))(x), h(x)\rangle| \\ &\qquad \lesssim 2^{-k\eta} \lim_{J\to\infty} \sum_{\substack{Q \in \mathcal{Q}_J \\ Q \cap \operatorname{supp} h \ne \emptyset}} \sum_{\substack{P \in \mathcal{Q}_J \\ d(c_P, c_Q) < 2^{-k+1}}} \mu(Q)\mu(P) \frac{1}{V_{2^{-k}}(c_Q)} \\ &\qquad \lesssim 2^{-k\eta}. \end{aligned}$$

Letting $k \to \infty$ we obtain (6.7). This concludes the proof of Lemma 6.2. ∎

6.2. Bilinear molecules. Now we elaborate on the concept of bilinear molecules, as introduced by Grafakos and Torres in [51], and stress the importance of mapping properties from atoms to molecules in the boundedness of bilinear operators on Besov and Triebel–Lizorkin spaces. The whole section stems from the seminal work in the linear case by Y. Meyer [81] and David, Journé and Semmes [25, 26].

DEFINITION 6.4. Let $\beta \in (0, 1]$. If $Q \in \mathcal{Q}_k$ for some $k \in \mathbb{Z}$, then a function a_Q is called a *β-atom associated with Q* if the following hold:

(i) $\operatorname{supp} a_Q \subset Q$ and, for all $x \in \mathcal{X}$,

$$|a_Q(x)| \le \mu(Q)^{-1/2};$$

(ii) for all x, $x' \in \mathcal{X}$ such that $d(x, x') \le 2^{-k-1}$,

$$|a_Q(x) - a_Q(x')| \le \mu(Q)^{-1/2} [2^k d(x, x')]^\beta;$$

(iii) $\int_{\mathcal{X}} a_Q(y)\, d\mu(y) = 0$.

Obviously, any β-atom is an element of

$$\{f \in \mathring{\mathcal{G}}(\beta, \gamma) : f \text{ has bounded support}\}$$

for all $\gamma \in (0, \infty)$. The converse is also true modulo a positive constant.

Bilinear molecules on $\mathbb{R}^n$ were first introduced by Grafakos and Torres [51]; these molecules as well as their derivatives decay rapidly at infinity. Here we introduce bilinear molecules on RD-spaces with no requirements on their smoothness.

Definition 6.5. Let $\gamma \in (0,\infty)$ and $\sigma \in (0,\infty)$. If $Q \in \mathcal{Q}_k$ and $P \in \mathcal{Q}_j$ for some $k, j \in \mathbb{Z}$, a function $M_{Q,P}$ is called a *bilinear* (γ,σ)*-molecule associated with* Q *and* P if, for almost every $x \in \mathcal{X}$,

$$|M_{Q,P}(x)| \le 2^{-|k-j|\sigma}\mu(Q)^{1/2}\mu(P)^{1/2}\mathcal{K}(2^{-k};\gamma,x,c_Q)\mathcal{K}(2^{-j};\gamma,x,c_P)$$

and

$$\int_{\mathcal{X}} M_{Q,P}(x)\,d\mu(x) = 0,$$

where, for any given $\epsilon \in (0,\infty)$, $t > 0$ and all x, $y \in \mathcal{X}$,

$$\mathcal{K}(t;\,\epsilon,x,y) := \frac{1}{V_t(x)+V_t(y)+V(x,y)}\left[\frac{t}{t+d(x,y)}\right]^{\epsilon}.$$

Theorem 6.6. *Let* $\eta \in (0,1]$, $\beta \in (0,1]$, $T \in \mathbf{BWBP}(\eta)$ *with a kernel* $K \in \operatorname{Ker}(2, C_K, \delta)$ *for some* $C_K > 0$ *and* $\delta > 0$. *Assume that, for all* $g \in C^\eta_b(\mathcal{X})$,

$$T(g,1) = T^{*,1}(1,g) = 0 \quad \textit{in } (C^\eta_b(\mathcal{X}))'.$$

If a_Q *is a* β*-atom associated with* $Q \in \mathcal{Q}_k$ *for some* $k \in \mathbb{Z}$*, and* a_P *a* β*-atom associated with* $P \in \mathcal{Q}_j$ *for some integer* $j \le k$*, then, for any given*

$$\gamma \in (0,\delta/2] \quad \textit{and} \quad \sigma \in (0,\min\{\delta-\gamma,\beta,\kappa\}],$$

$T(a_Q,a_P)$ *is a constant multiple of a bilinear* (γ,σ)*-molecule associated with* Q *and* P.

Proof. Without loss of generality, we may as well assume that $\beta \le \kappa$, where κ is the reverse doubling exponent in (2.2), since any β-atom is a β'-atom whenever $\beta' \in (0,\beta]$.

We first estimate $|T(a_Q,a_P)(x)|$ for all $x \in \mathcal{X}$ by considering the following four cases:

Case 1: $d(x,c_Q) \ge C_5 2^{-k+10}$ and $d(x,c_P) \ge C_5 2^{-j+10}$. In this case, by (3.3) and

$$\int_{\mathcal{X}} a_Q(y)\,d\mu(y) = 0,$$

we write

$$\begin{aligned}|T(a_Q,a_P)(x)| &= \left|\int_{\mathcal{X}^2} K(x,y_1,y_2)a_Q(y_1)a_P(y_2)\,d\mu(y_1)\,d\mu(y_2)\right| \\ &= \left|\int_{\mathcal{X}^2} [K(x,y_1,y_2)-K(x,c_Q,y_2)]a_Q(y_1)a_P(y_2)\,d\mu(y_1)\,d\mu(y_2)\right|.\end{aligned}$$

Since $\operatorname{supp} a_Q \subset Q$ and $\operatorname{supp} a_P \subset P$, it follows that, if the integrand is non-zero, then

$$d(x,y_1) \sim d(x,c_Q), \quad d(x,y_2) \sim d(x,c_P) \quad \text{and} \quad d(y_1,c_Q) \le C_5 2^{-k} \le d(x,y_1)/2.$$

Applying the regularity condition on K and the size conditions on a_Q and a_P, we conclude that

$$\begin{aligned}|T(a_Q,a_P)(x)| &\lesssim \int_Q\int_P \left[\frac{d(y_1,c_Q)}{\max\{d(x,y_1),d(x,y_2)\}}\right]^{\delta}\frac{\mu(Q)^{-1/2}\mu(P)^{-1/2}}{[V(x,y_1)+V(x,y_2)]^2}\,d\mu(y_1)\,d\mu(y_2) \\ &\lesssim \left[\frac{2^{-k}}{\max\{d(x,c_P),d(x,c_Q)\}}\right]^{\delta}\frac{\mu(Q)^{1/2}\mu(P)^{1/2}}{[\mu(B(x,d(x,c_Q)))+\mu(B(x,d(x,c_P)))]^2} \\ &\lesssim 2^{(j-k)(\delta-\gamma)}\mu(Q)^{1/2}\mu(P)^{1/2}\mathcal{K}(2^{-k};\gamma,x,c_Q)\mathcal{K}(2^{-j};\gamma,x,c_P).\end{aligned}$$

Case 2: $d(x, c_Q) < C_5 2^{-k+10}$ and $d(x, c_P) \geq C_5 2^{-j+10}$. In this case, again by (3.3) and

$$\int_{\mathcal{X}} a_Q(y)\, d\mu(y) = 0,$$

we write

$$|T(a_Q, a_P)(x)| = \left| \int_{\mathcal{X}^2} [K(x, y_1, y_2) - K(x, c_Q, y_2)] a_Q(y_1) a_P(y_2)\, d\mu(y_1)\, d\mu(y_2) \right|.$$

Notice that, for all $y_1 \in Q$ and $y_2 \in P$,

$$d(x, y_2) \sim d(x, c_P) \quad \text{and} \quad d(y_1, c_Q) \leq C_5 2^{-k} \leq C_5 2^{-j} \leq d(x, y_2)/2.$$

Applying the regularity condition on K as well as the size conditions on a_Q and a_P yields

$$\begin{aligned}
|T(a_Q, a_P)(x)| &\lesssim \int_Q \int_P \left[\frac{d(y_1, c_Q)}{\max\{d(x, y_1),\, d(x, y_2)\}} \right]^\delta \frac{\mu(Q)^{-1/2} \mu(P)^{-1/2}}{[V(x, y_1) + V(x, y_2)]^2}\, d\mu(y_1)\, d\mu(y_2) \\
&\lesssim 2^{(j-k)\delta} \left[\frac{2^{-j}}{d(x, c_P)} \right]^\delta \frac{\mu(Q)^{1/2} \mu(P)^{1/2}}{[\mu(B(x, d(x, c_P)))]^2} \\
&\lesssim 2^{(j-k)\delta} \mu(Q)^{1/2} \mu(P)^{1/2} \mathcal{K}(2^{-k}; \gamma, x, c_Q) \mathcal{K}(2^{-j}; \delta, x, c_P),
\end{aligned}$$

where we used the estimates $2^{-k} d(x, c_Q) \lesssim 1$, $j \leq k$ and

$$V_{2^{-k}}(x) + V(x, c_Q) \lesssim \mu(B(x, 2^{-k})) \lesssim \mu(B(x, 2^{-j})) \lesssim \mu(B(x, d(x, c_P))).$$

Case 3: $d(x, c_Q) \geq C_5 2^{-k+10}$ and $d(x, c_P) < C_5 2^{-j+10}$. As in the previous two cases,

$$|T(a_Q, a_P)(x)| = \left| \int_{\mathcal{X}^2} [K(x, y_1, y_2) - K(x, c_Q, y_2)] a_Q(y_1) a_P(y_2)\, d\mu(y_1)\, d\mu(y_2) \right|.$$

Notice that, for all $y_1 \in Q$ and $y_2 \in P$,

$$d(x, y_1) \sim d(x, c_Q) \quad \text{and} \quad d(y_1, c_Q) \leq C_5 2^{-k} \leq d(x, y_1)/2.$$

By the regularity condition on K and the size conditions on a_Q and a_P, we know that

$$\begin{aligned}
|T(a_Q, a_P)(x)| &\lesssim \int_Q \int_P \left[\frac{d(y_1, c_Q)}{\max\{d(x, y_1),\, d(x, y_2)\}} \right]^\delta \frac{\mu(Q)^{-1/2} \mu(P)^{-1/2}}{[V(x, y_1) + V(x, y_2)]^2}\, d\mu(y_1)\, d\mu(y_2) \\
&\lesssim 2^{(j-k)(\delta-\gamma)} \left[\frac{2^{-k}}{d(x, c_Q)} \right]^\gamma \frac{\mu(Q)^{1/2} \mu(P)^{-1/2}}{\mu(B(x, d(x, c_Q)))} \\
&\quad \times \int_P \left[\frac{2^{-j}}{\max\{d(x, c_Q),\, d(x, y_2)\}} \right]^{\delta-\gamma} \frac{1}{V(x, y_2) + V(x, c_Q)}\, d\mu(y_2) \\
&\lesssim 2^{(j-k)(\delta-\gamma)} \mu(Q)^{1/2} \mu(P)^{1/2} \mathcal{K}(2^{-k}; \gamma, x, c_Q) \mathcal{K}(2^{-j}; \gamma, x, c_P),
\end{aligned}$$

where the last inequality is due to the following three estimates:

$$\frac{1}{\mu(P)} \lesssim \mathcal{K}(2^{-j}; \gamma, x, c_P) \quad \text{when } d(x, c_P) \leq C_5 2^{-j+10},$$

$$\left[\frac{2^{-k}}{d(x, c_Q)} \right]^\gamma \frac{1}{\mu(B(x, d(x, c_Q)))} \lesssim \mathcal{K}(2^{-k}; \gamma, x, c_Q) \quad \text{when } d(x, c_Q) > C_5 2^{-j+10},$$

and

$$
\begin{aligned}
&\int_P \left[\frac{2^{-j}}{\max\{d(x,c_Q),\, d(x,y_2)\}}\right]^{\delta-\gamma} \frac{1}{V(x,y_2)+V(x,c_Q)}\, d\mu(y_2)\\
&\quad = \sum_{\ell=0}^{\infty} \int_{d(x,y_2)\sim 2^{\ell-k}} \left[\frac{2^{-j}}{\max\{d(x,c_Q),\, d(x,y_2)\}}\right]^{\delta-\gamma} \frac{1}{V(x,y_2)+V(x,c_Q)}\, d\mu(y_2)\\
&\quad \lesssim \sum_{\ell=0}^{\infty} \int_{d(x,y_2)\sim 2^{\ell-k}} 2^{-\ell(\delta-\gamma)} \frac{1}{\mu(B(x,2^{\ell-k})}\, d\mu(y_2)\\
&\quad \lesssim 1,
\end{aligned}
$$

where the notation $d(x,y_2)\sim 2^{\ell-k}$ means that $2^{\ell-k}\le d(x,y_2)<2^{\ell-k+1}$ when $\ell\in\mathbb{N}$, and $d(x,y_2)<2^{-k}$ when $\ell=0$.

Case 4: $d(x,c_Q)<C_5 2^{-k+10}$ and $d(x,c_P)<C_5 2^{-j+10}$. In this case, it suffices to show that

$$
\|T(a_Q,a_P)\chi_{B(c_Q,C_5 2^{-k+10})\cap B(c_P,C_5 2^{-j+10})}\|_{L^\infty(\mathcal{X})} \lesssim 2^{(j-k)\beta}\mu(Q)^{-1/2}\mu(P)^{-1/2}. \quad (6.12)
$$

Since $(\mathcal{X},d,\mu)$ is assumed to be an RD-space, it follows that $C_b^\gamma(\mathcal{X})$ is dense in $L^1(\mathcal{X})$ (see [60, Corollary 2.11]). Thus, the proof of (6.12) would follow from the estimate

$$
|\langle T(a_Q,a_P),h\rangle| \lesssim 2^{(j-k)\beta}\mu(Q)^{-1/2}\mu(P)^{-1/2} \quad (6.13)
$$

for all functions $h\in C_b^\eta(\mathcal{X})$ with $\|h\|_{L^1(\mathcal{X})}=1$ and

$$
\operatorname{supp} h\subset B(c_Q,\, C_5 2^{-k+10})\cap B(c_P, C_5 2^{-j+10}).
$$

To prove (6.13), by the assumption $T(g,1)=0$ in $(C_b^\eta(\mathcal{X}))'$ for all $g\in C_b^\eta(\mathcal{X})$, we have

$$
\langle T(a_Q,1),\, h a_P\rangle = 0
$$

and hence

$$
\langle T(a_Q,a_P),h\rangle = \langle T(a_Q,[a_P-a_P(x)])(x),\, h(x)\rangle.
$$

Applying Lemma 6.2 and using the size and regularity conditions on a_P, we see that

$$
\begin{aligned}
&|\langle T(a_Q,[a_P-a_P(x)])(x),\, h(x)\rangle|\\
&\quad = \left|\int_{D^{\complement}} K(x,y_1,y_2)a_Q(y_1)[a_P(y_2)-a_P(x)]h(x)\, d\mu(y_1)\, d\mu(y_2)\, d\mu(x)\right|\\
&\quad \lesssim \int_{\substack{(x,y_1,y_2)\in D^{\complement}\\ y_1\in Q}} \frac{\mu(Q)^{-1/2}\mu(P)^{-1/2}}{[V(x,y_1)+V(x,y_2)]^2} \min\left\{\left[\frac{d(x,y_2)}{2^{-j}}\right]^\beta,\, 1\right\}|h(x)|\, d\mu(y_1)\, d\mu(y_2)\, d\mu(x).
\end{aligned}
$$

By the doubling and reverse doubling conditions on μ, we obtain

$$
\begin{aligned}
&\int_Q\int_{\mathcal{X}} \min\left\{\left[\frac{d(x,y_2)}{2^{-j}}\right]^\beta, 1\right\} \frac{1}{[V(x,y_1)+V(x,y_2)]^2}\, d\mu(y_2)\, d\mu(y_1)\\
&\quad \le \sum_{s=0}^{\infty}\sum_{t\in\mathbb{Z}} \int_{\substack{d(x,y_1)\sim C2^{-k-s}\\ d(x,y_2)\sim C2^{-j-t}}} \min\left\{\left[\frac{d(x,y_2)}{2^{-j}}\right]^\beta,\, 1\right\} \frac{1}{[V(x,y_1)+V(x,y_2)]^2}\, d\mu(y_2)\, d\mu(y_1)
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \sum_{s=0}^{\infty}\sum_{t\in\mathbb{Z}} \min\{2^{-t\beta},\, 1\} \frac{\mu(B(x,2^{-s-k}))\,\mu(B(x,2^{-t-j}))}{[\mu(B(x,2^{-s-k}))+\mu(B(x,2^{-t-j}))]^2}\\
&\lesssim \sum_{s=0}^{\infty}\sum_{t+j>s+k} 2^{-t\beta}\frac{\mu(B(x,2^{-t-j}))}{\mu(B(x,2^{-s-k}))} + \sum_{s=0}^{\infty}\sum_{t\le s+k-j} \min\{2^{-t\beta},1\}\frac{\mu(B(x,2^{-s-k}))}{\mu(B(x,2^{-t-j}))}\\
&\lesssim \sum_{s=0}^{\infty}\sum_{t+j>s+k} 2^{-t\beta}\left(\frac{2^{-t-j}}{2^{-s-k}}\right)^{\kappa} + \sum_{s=0}^{\infty}\sum_{t=0}^{s+k-j} 2^{-t\beta}\left(\frac{2^{-s-k}}{2^{-t-j}}\right)^{\kappa} + \sum_{s=0}^{\infty}\sum_{t=-\infty}^{-1}\left(\frac{2^{-s-k}}{2^{-t-j}}\right)^{\kappa}\\
&\lesssim 2^{-(k-j)\beta}
\end{aligned}
$$

(here we used $\beta \le \kappa$), therefore,

$$
\begin{aligned}
|\langle T(a_Q,a_P),h\rangle| &\lesssim 2^{-(k-j)\beta}\mu(Q)^{-1/2}\mu(P)^{-1/2}\int_{\mathcal{X}}|h(x)|\,d\mu(x)\\
&\lesssim 2^{-(k-j)\beta}\mu(Q)^{-1/2}\mu(P)^{-1/2},
\end{aligned}
$$

which proves (6.13) and hence (6.12).

Putting all estimates in the four cases above together, we finally conclude that $T(a_Q,a_P)$ satisfies the size condition of a bilinear (γ,σ)-molecule associated with Q and P modulo a positive constant. Moreover, it is easy to see that

$$
\int_{\mathcal{X}} T(a_Q,a_P)(x)\,d\mu(x) = \langle T(a_Q,a_P),1\rangle = \langle T^{*,1}(1,a_P),a_Q\rangle = 0,
$$

which completes the proof of Theorem 6.6. ■

The following bilinear almost diagonal estimate is a variation of the one in [51, Proposition 3] but it is in the context of RD-spaces; see also [46, Proposition 3.2] and the related work of Bényi and Tzirakis [6].

THEOREM 6.7. *Let ℓ,k,j be integers such that $j\le\ell\le k$, $R\in\mathcal{Q}_\ell$, $Q\in\mathcal{Q}_k$ and $P\in\mathcal{Q}_j$. Then, for all $\gamma\in(0,\infty)$, $\gamma'\in(0,\gamma)$ and $\gamma''\in(0,\gamma)$,*

$$
\begin{aligned}
\mathcal{J} &:= \frac{1}{\mu(R)}\int_{B(c_R,\,C_5 2^{-\ell})}\frac{1}{V_{2^{-k}}(x)+V(x,c_Q)}\left[\frac{1}{1+2^k d(x,c_Q)}\right]^{\gamma}\\
&\qquad\times\frac{1}{V_{2^{-j}}(x)+V(x,c_P)}\left[\frac{1}{1+2^j d(x,c_P)}\right]^{\gamma} d\mu(x)\\
&\le C\mathcal{K}(2^{-\ell};\gamma',c_R,c_Q)\mathcal{K}(2^{-j};\gamma'',c_R,c_P), \qquad (6.14)
\end{aligned}
$$

where C_5 is the constant appearing in Lemma 2.5, and C is a positive constant independent of P, Q, R.

Proof. Let

$$
\begin{aligned}
A_0 &:= \{x\in\mathcal{X} : d(x,c_P) < 2^{t-j}\},\\
A_t &:= \{x\in\mathcal{X} : 2^{t-1-j} \le d(x,c_P) < 2^{t-j}\}, \qquad \forall t\in\mathbb{N}.
\end{aligned}
$$

Let also

$$
\begin{aligned}
W_0 &:= \{x\in\mathcal{X} : d(x,c_Q) < 2^{t-j}\},\\
W_s &:= \{x\in\mathcal{X} : 2^{s-1-k} \le d(x,c_Q) < 2^{s-k}\}, \qquad \forall s\in\mathbb{N}.
\end{aligned}
$$

Then

$$\mathcal{J} \lesssim \sum_{t=0}^{\infty}\sum_{s=0}^{\infty} 2^{-t\gamma}2^{-s\gamma} \frac{1}{\mu(R)} \frac{1}{\mu(B(c_Q, 2^{s-k}))} \frac{1}{\mu(B(c_P, 2^{t-j}))} \int_{B(c_R, C_5 2^{-\ell})\cap A_t \cap W_s} d\mu(x).$$

If $B(c_R, C_5 2^{-\ell}) \cap A_t \cap W_s \neq \emptyset$, then, by $j \le \ell \le k$, we have

$$d(c_R, c_Q) \le C_5 2^{-\ell} + 2^{s-k} \lesssim \max\{2^{-\ell},\, 2^{s-k}\} \quad \text{and} \quad d(c_R, c_P) \le C_5 2^{-\ell} + 2^{t-j} \lesssim 2^{t-j},$$

which implies that, for all $s \ge 0$ and $\gamma' \in (0, \gamma)$,

$$\begin{aligned} \frac{1}{\mu(B(c_Q, 2^{s-k}))} &\lesssim \frac{\mu\big(B(c_Q, 2^{-\ell} + d(c_R, c_Q))\big)}{\mu(B(c_Q, 2^{s-k}))} \left[\frac{2^{-\ell} + d(c_R, c_Q)}{2^{-\ell}}\right]^{\gamma'} \mathcal{K}(2^{-\ell}; \gamma', c_R, c_Q) \\ &\lesssim 2^{s\gamma'} \frac{\mu(B(c_Q, 2^{-\ell} + 2^{s-k}))}{\mu(B(c_Q, 2^{s-k}))} \mathcal{K}(2^{-\ell}; \gamma', c_R, c_Q) \end{aligned} \tag{6.15}$$

and, for all $t \ge 0$ and $\gamma'' \in (0, \gamma)$,

$$\begin{aligned} \frac{1}{\mu(B(c_P, 2^{t-j}))} &\lesssim \frac{\mu\big(B(c_P, 2^{-j} + d(c_R, c_P))\big)}{\mu(B(c_P, 2^{t-j}))} \left[\frac{2^{-j} + d(c_R, c_P)}{2^{-j}}\right]^{\gamma''} \mathcal{K}(2^{-j}; \gamma'', c_R, c_R) \\ &\lesssim 2^{t\gamma''} \mathcal{K}(2^{-\ell}; \gamma'', c_R, c_P). \end{aligned} \tag{6.16}$$

By (6.15), (6.16), together with the facts

$$\mu(B(c_Q, 2^{-\ell} + 2^{s-k})) \sim \mu(B(c_R, 2^{-\ell} + 2^{s-k}))$$

and

$$\mu(B(c_R,\, C_5 2^{-\ell}) \cap A_t \cap W_s) \lesssim \min\{\mu(R),\, \mu(B(c_Q, 2^{s-k}))\},$$

we obtain

$$\begin{aligned} \mathcal{J} &\lesssim \sum_{t=0}^{\infty}\sum_{s=0}^{\infty} 2^{-t(\gamma-\gamma')}2^{-s(\gamma-\gamma'')} \frac{\mu(B(c_Q, 2^{-\ell} + 2^{s-k}))\min\{\mu(R),\, \mu(B(c_Q, 2^{s-k}))\}}{\mu(B(c_Q, 2^{s-k}))\mu(R)} \\ &\quad \times \mathcal{K}(2^{-\ell}; \gamma', c_R, c_Q)\mathcal{K}(2^{-j}; \gamma'', c_R, c_P) \\ &\lesssim \mathcal{K}(2^{-\ell}; \gamma', c_R, c_Q)\mathcal{K}(2^{-j}; \gamma'', c_R, c_P). \end{aligned}$$

Inserting the estimate of $\mathcal{J}$ into (6.17) implies the desired inequality (6.14). This concludes the proof of Theorem 6.7. ∎

As an application of Theorems 6.6 and 6.7, we easily obtain the following conclusion.

THEOREM 6.8. *Let $\eta \in (0, 1]$, $\beta \in (0, 1]$, $T \in \mathbf{BWBP}(\eta)$ with a kernel $K \in \mathrm{Ker}(2, C_K, \delta)$ for some $C_K > 0$ and $\delta > 0$. Assume that, for all $g \in C_b^\eta(\mathcal{X})$,*

$$T(1, g) = T(g, 1) = T^{*,1}(1, g) = 0 \quad \text{in } (C_b^\eta(\mathcal{X}))'.$$

Suppose that ℓ, j and k are integers such that a_P, a_Q and a_R are β-atoms associated with cubes $P \in \mathcal{Q}_j$, $Q \in \mathcal{Q}_k$, and $R \in \mathcal{Q}_\ell$, respectively. Then, for any given

$$\gamma \in (0, \delta/2] \quad \text{and} \quad \sigma \in (0, \min\{\delta - \gamma, \beta, \kappa\}],$$

the following hold: for any given $\gamma', \gamma'' \in (0, \gamma)$,

(i) *when* $j \le \ell \le k$,

$$
\begin{aligned}
&|\langle a_R, T(a_P, a_Q)\rangle| \\
&\qquad \le C 2^{-|k-j|\sigma} \mu(R)^{1/2} \mu(Q)^{1/2} \mu(P)^{1/2} \mathcal{K}(2^{-\ell}; \gamma', c_R, c_Q) \mathcal{K}(2^{-j}; \gamma'', c_R, c_P);
\end{aligned}
$$

(ii) *when* $k \le \ell \le j$,

$$
\begin{aligned}
&|\langle a_R, T(a_P, a_Q)\rangle| \\
&\qquad \le C 2^{-|k-j|\sigma} \mu(R)^{1/2} \mu(Q)^{1/2} \mu(P)^{1/2} \mathcal{K}(2^{-k}; \gamma', c_R, c_Q) \mathcal{K}(2^{-\ell}; \gamma'', c_R, c_P);
\end{aligned}
$$

(iii) *when* $j \le k \le \ell$,

$$
\begin{aligned}
&|\langle a_R, T(a_P, a_Q)\rangle| \\
&\qquad \le C 2^{-|\ell-j|\sigma} \mu(R)^{1/2} \mu(Q)^{1/2} \mu(P)^{1/2} \mathcal{K}(2^{-k}; \gamma', c_Q, c_R) \mathcal{K}(2^{-j}; \gamma'', c_Q, c_P);
\end{aligned}
$$

(iv) *when* $k \le j \le \ell$,

$$
\begin{aligned}
&|\langle a_R, T(a_P, a_Q)\rangle| \\
&\qquad \le C 2^{-|\ell-k|\sigma} \mu(R)^{1/2} \mu(Q)^{1/2} \mu(P)^{1/2} \mathcal{K}(2^{-j}; \gamma', c_P, c_R) \mathcal{K}(2^{-k}; \gamma'', c_P, c_Q);
\end{aligned}
$$

(v) *when* $\ell \le k \le j$,

$$
\begin{aligned}
&|\langle a_R, T(a_P, a_Q)\rangle| \\
&\qquad \le C 2^{-|\ell-j|\sigma} \mu(R)^{1/2} \mu(Q)^{1/2} \mu(P)^{1/2} \mathcal{K}(2^{-\ell}; \gamma', c_Q, c_R) \mathcal{K}(2^{-k}; \gamma'', c_Q, c_P);
\end{aligned}
$$

(vi) *when* $\ell \le j \le k$,

$$
\begin{aligned}
&|\langle a_R, T(a_P, a_Q)\rangle| \\
&\qquad \le C 2^{-|\ell-k|\sigma} \mu(R)^{1/2} \mu(Q)^{1/2} \mu(P)^{1/2} \mathcal{K}(2^{-\ell}; \gamma', c_P, c_R) \mathcal{K}(2^{-j}; \gamma'', c_P, c_Q);
\end{aligned}
$$

here C *is a positive constant independent of* P, Q *and* R.

Proof. Since $T(g,1) = T^{*,1}(1,g) = 0$ in $(C_b^\eta(\mathcal{X}))'$, by Theorems 6.6 and 6.7, we see that, when $j \le \ell \le k$,

$$
\begin{aligned}
&\left| \int_{\mathcal{X}} a_R(x) T(a_P, a_Q)(x)\, d\mu(x) \right| \\
&\quad \lesssim 2^{-|k-j|\sigma} \mu(R)^{1/2} \mu(Q)^{1/2} \mu(P)^{1/2} \frac{1}{\mu(R)} \int_{B(c_R,\, C_5 2^{-\ell})} \frac{1}{V_{2^{-k}}(x) + V(x, c_Q)} \\
&\qquad \times \left[\frac{1}{1 + 2^k d(x, c_Q)} \right]^\gamma \frac{1}{V_{2^{-j}}(x) + V(x, c_P)} \left[\frac{1}{1 + 2^j d(x, c_P)} \right]^\gamma d\mu(x) \\
&\quad \lesssim 2^{-|k-j|\sigma} \mu(R)^{1/2} \mu(Q)^{1/2} \mu(P)^{1/2} \mathcal{K}(2^{-\ell}; \gamma', c_R, c_Q) \mathcal{K}(2^{-j}; \gamma'', c_R, c_P),
\end{aligned}
\tag{6.17}
$$

which proves (i).

By symmetry, (ii) holds true.

To prove (iii) and (iv), we write

$$
\langle a_R, T(a_P, a_Q)\rangle = \langle a_Q, T^{*,2}(a_P, a_R)\rangle.
$$

Since $j \le k \le \ell$ and the operator $S := T^{*,2}$ satisfies

$$
S(g,1) = S^{*,1}(1,g) = 0 \quad \text{in } (C_b^\eta(\mathcal{X}))',
$$

we apply (i) and (ii), respectively, to deduce the estimates (iii) and (iv).

Finally, (v) and (vi) hold true for similar reasons, the details being omitted. This finishes the proof of Theorem 6.8. ∎

6.3. Bilinear $T1$-theorem on Triebel–Lizorkin spaces. Homogeneous Besov and Triebel–Lizorkin spaces on Ahlfors 1-regular metric spaces were first introduced by Han and Sawyer [61] via applying the Calderón reproducing formulae, but only for exponents $p, q \in (1, \infty)$ (for Triebel–Lizorkin spaces $p, q \in [1, \infty)$). An extension to the range of p, q smaller than 1 and near 1 was obtained by Han [57]. For a systematic treatment of the theory of (in)homogeneous Besov and Triebel–Lizorkin spaces on RD-spaces, we refer the reader to the work of Han, Müller and Yang [60]. We recall their definitions.

DEFINITION 6.9. Let $(\mathcal{X}, d, \mu)$ be an RD-space and $\mu(\mathcal{X}) = \infty$. Let $\epsilon_1 \in (0, 1]$, $\epsilon_2 > 0$, $\epsilon_3 > 0$, $\epsilon \in (0, \epsilon_1 \wedge \epsilon_2)$, $|s| < \epsilon$, and $\{S_k\}_{k\in\mathbb{Z}}$ be an $(\epsilon_1, \epsilon_2, \epsilon_3)$-ATI. For $k \in \mathbb{Z}$, let $D_k := S_k - S_{k-1}$. Set also

$$p(s, \epsilon) := \max\{n/(n + \epsilon),\ n/(n + \epsilon + s)\}.$$

(i) Let $p(s, \epsilon) < p \le \infty$ and $0 < q \le \infty$. The *space* $\dot{B}^s_{p,q}(\mathcal{X})$ is defined to be the set of all $f \in (\mathring{\mathcal{G}}^\epsilon_0(\beta, \gamma))'$, for some β, γ satisfying

$$\begin{aligned} &\max\{s, 0, -s + n(1/p - 1)_+\} < \beta < \epsilon, \\ &\max\{s - \kappa/p, n(1/p - 1)_+, -s + n(1/p - 1)_+ - \kappa(1 - 1/p)_+\} < \gamma < \epsilon, \end{aligned} \tag{6.18}$$

such that

$$\|f\|_{\dot{B}^s_{p,q}(\mathcal{X})} := \Big[\sum_{k=-\infty}^{\infty} 2^{ksq} \|D_k(f)\|^q_{L^p(\mathcal{X})}\Big]^{1/q} < \infty$$

with the usual modifications when $p = \infty$ or $q = \infty$.

(ii) Let $p(s, \epsilon) < p < \infty$ and $p(s, \epsilon) < q \le \infty$. The *space* $\dot{F}^s_{p,q}(\mathcal{X})$ is defined to be the set of all $f \in (\mathring{\mathcal{G}}^\epsilon_0(\beta, \gamma))'$, for some β, γ satisfying (6.18), such that

$$\|f\|_{\dot{F}^s_{p,q}(\mathcal{X})} := \Big\|\Big[\sum_{k=-\infty}^{\infty} 2^{ksq} |D_k(f)|^q\Big]^{1/q}\Big\|_{L^p(\mathcal{X})} < \infty,$$

with the usual modification when $q = \infty$. When $p = \infty$ and $p(s, \epsilon) < q \le \infty$, the *space* $\dot{F}^s_{\infty,q}(\mathcal{X})$ is defined to be the set of all $f \in (\mathring{\mathcal{G}}^\epsilon_0(\beta, \gamma))'$, for some β, γ satisfying

$$|s| < \beta < \epsilon, \quad \max\{s,\ 0,\ -s - \kappa\} < \gamma < \epsilon,$$

such that

$$\|f\|_{\dot{F}^s_{\infty,q}(\mathcal{X})} := \sup_{\ell\in\mathbb{Z}} \sup_{\alpha\in I_\ell} \left[\frac{1}{\mu(Q^\ell_\alpha)} \int_{Q^\ell_\alpha} \sum_{k=\ell}^{\infty} 2^{ksq} |D_k(f)(x)|^q \, d\mu(x)\right]^{1/q} < \infty,$$

where the supremum is taken over all dyadic cubes as in Lemma 2.5 with the usual modification when $q = \infty$.

Let

$$\mathcal{Q} := \{Q^k_\alpha \subset \mathcal{X} : k \in \mathbb{Z},\ \alpha \in I_k\}$$

be the collection of all Christ's dyadic cubes as in Lemma 2.5. For $k \in \mathbb{Z}$ and $\tau \in I_k$, we denote by

$$\{Q^{k,\nu}_\tau : \nu \in \{1, \dots, N(k, \tau)\}\}$$

the set of all cubes $Q_{\tau'}^{k+j_0} \subset Q_\tau^k$, where Q_τ^k is the dyadic cube as in Lemma 2.5 and j_0 is a positive integer satisfying

$$2^{-j_0} C_5 < 1/3. \tag{6.19}$$

Denote by $z_\tau^{k,\nu}$ the "center" of $Q_\tau^{k,\nu}$ and by $y_\tau^{k,\nu}$ any point of $Q_\tau^{k,\nu}$.

The Calderón reproducing formulae, due to Calderón [14] in the Euclidean case, are proved to be a powerful tool in the study of Besov and Triebel–Lizorkin spaces on $\mathbb{R}^n$; see [34, 33]. For an extension of these formulae and its applications in spaces of homogeneous type, especially in the context of Ahlfors 1-regular metric measure spaces, see [61]. The following discrete homogeneous Calderón reproducing formula on RD-spaces was proved in [60, Theorem 4.13]; see also [60, Theorems 4.11 and 4.12].

Lemma 6.10. *Let $\epsilon_1 \in (0,1]$, $\epsilon_2 > 0$, $\epsilon_3 > 0$ and $\epsilon \in (0, \epsilon_1 \wedge \epsilon_2)$. Let $\{S_k\}_{k\in\mathbb{Z}}$ be an $(\epsilon_1, \epsilon_2, \epsilon_3)$-ATI. Set $D_k := S_k - S_{k-1}$ for $k \in \mathbb{Z}$. Then, for any fixed j_0 satisfying (6.19) large enough, there exist linear operators, $\{\widetilde{D}_k\}_{k\in\mathbb{Z}}$ and $\{\overline{D}_k\}_{k\in\mathbb{Z}}$, such that, for any fixed $y_\tau^{k,\nu} \in Q_\tau^{k,\nu}$ with $k \in \mathbb{Z}$, $\tau \in I_k$ and $\nu \in \{1, \ldots, N(k,\tau)\}$, and all $f \in (\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))'$ (or $f \in \mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)$) with $\beta, \gamma \in (0,\epsilon)$,*

$$\begin{aligned} f &= \sum_{k\in\mathbb{Z}} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \widetilde{D}_k(\cdot, y_\tau^{k,\nu}) D_k(f)(y_\tau^{k,\nu}) \\ &= \sum_{k\in\mathbb{Z}} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) D_k(\cdot, y_\tau^{k,\nu}) \overline{D}_k(f)(y_\tau^{k,\nu}), \end{aligned}$$

where the series converge in $(\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))'$ (or in $\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)$). Moreover, the kernels of the operators $\{\widetilde{D}_k\}_{k\in\mathbb{Z}}$ satisfy, for all $x, y \in \mathcal{X}$ and $k \in \mathbb{Z}$,

(a) $|\widetilde{D}_k(x,y)| \le C \frac{1}{V_{2^{-k}}(x)+V(x,y)} \left[\frac{2^{-k}}{2^{-k}+d(x,y)}\right]^{\epsilon'}$;

(b) *for $d(x,x') \le [2^{-k} + d(x,y)]/2$,*

$$|\widetilde{D}_k(x,y) - \widetilde{D}_k(x',y)| \le C \left[\frac{d(x,x')}{2^{-k}+d(x,y)}\right]^{\epsilon'} \frac{1}{V_{2^{-k}}(x)+V(x,y)} \left[\frac{2^{-k}}{2^{-k}+d(x,y)}\right]^{\epsilon'};$$

(c) $\int_{\mathcal{X}} \widetilde{D}_k(w,y)\, d\mu(w) = 0 = \int_{\mathcal{X}} \widetilde{D}_k(x,w)\, d\mu(w)$,

where $\epsilon' \in (\epsilon, \epsilon_1 \wedge \epsilon_2)$ and C is a positive constant independent of k, x, x' and y. The kernels of $\{\overline{D}_k\}_{k\in\mathbb{Z}}$ satisfy the above (a), (c) *and*

(b′) *for $d(x,x') \le [2^{-k} + d(x,y)]/2$,*

$$|\overline{D}_k(y,x) - \overline{D}_k(y,x')| \le C \left[\frac{d(x,x')}{2^{-k}+d(x,y)}\right]^{\epsilon'} \frac{1}{V_{2^{-k}}(x)+V(x,y)} \left[\frac{2^{-k}}{2^{-k}+d(x,y)}\right]^{\epsilon'},$$

where ϵ' and C are as in (b).

As a consequence of the Calderón reproducing formulae, in [60, Theorem 7.2, Proposition 5.4] the following *frame characterizations* of Besov and Triebel–Lizorkin spaces were proved.

Lemma 6.11. *With all the notation as in Definition* 6.9 *and Lemma* 6.10, *the following hold with implicit constants independent of f:*

(i) *if* $p(s,\epsilon)<p\le\infty$ *and* $0<q\le\infty$, *then*

$$\begin{aligned}\|f\|_{\dot B^s_{p,q}(\mathcal{X})}&\sim\Big\{\sum_{k\in\mathbb{Z}}2^{ksq}\Big[\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})\inf_{z\in Q_\tau^{k,\nu}}|D_k(f)(z)|^p\Big]^{q/p}\Big\}^{1/q}\\&\sim\Big\{\sum_{k\in\mathbb{Z}}2^{ksq}\Big[\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})|D_k(f)(y_\tau^{k,\nu})|^p\Big]^{q/p}\Big\}^{1/q}\\&\sim\Big\{\sum_{k\in\mathbb{Z}}2^{ksq}\Big[\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})\sup_{z\in Q_\tau^{k,\nu}}|D_k(f)(z)|^p\Big]^{q/p}\Big\}^{1/q};\end{aligned}$$

(ii) *if* $p(s,\epsilon)<p<\infty$ *and* $0<q\le\infty$, *then*

$$\begin{aligned}\|f\|_{\dot F^s_{p,q}(\mathcal{X})}&\sim\Big\|\Big\{\sum_{k\in\mathbb{Z}}\sum_{\nu\in I_k}\sum_{\nu=1}^{N(k,\tau)}2^{ksq}\inf_{z\in Q_\tau^{k,\nu}}|D_k(f)(z)|^q\chi_{Q_\tau^{k,\nu}}\Big\}^{1/q}\Big\|_{L^p(\mathcal{X})}\\&\sim\Big\|\Big\{\sum_{k\in\mathbb{Z}}\sum_{\nu\in I_k}\sum_{\nu=1}^{N(k,\tau)}2^{ksq}|D_k(f)(y_\tau^{k,\nu})|^q\chi_{Q_\tau^{k,\nu}}\Big\}^{1/q}\Big\|_{L^p(\mathcal{X})}\\&\sim\Big\|\Big\{\sum_{k\in\mathbb{Z}}\sum_{\nu\in I_k}\sum_{\nu=1}^{N(k,\tau)}2^{ksq}\sup_{z\in Q_\tau^{k,\nu}}|D_k(f)(z)|^q\chi_{Q_\tau^{k,\nu}}\Big\}^{1/q}\Big\|_{L^p(\mathcal{X})}.\end{aligned}$$

The following technical lemma proved in [60, Lemma 5.3] is very useful when dealing with spaces $\dot B^s_{p,q}(\mathcal{X})$ and $\dot F^s_{p,q}(\mathcal{X})$ for the cases $p<1$ or $q<1$.

Lemma 6.12. *Let* $\epsilon>0$, $k',k\in\mathbb{Z}$, *and* $y_\tau^{k,\nu}$ *be any point in* $Q_\tau^{k,\nu}$ *for* $\tau\in I_k$ *and* $\nu\in\{1,\dots,N(k,\tau)\}$. *If* $r\in(n/(n+\epsilon),1]$, *then there exists a positive constant* C, *depending on* r, *such that, for all* $a_\tau^{k,\nu}\in\mathbb{C}$ *and* $x\in\mathcal{X}$,

$$\begin{aligned}&\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})\frac{1}{V_{2^{-(k'\wedge k)}}(x)+V(x,y_\tau^{k,\nu})}\frac{2^{-(k'\wedge k)\epsilon}}{[2^{-(k'\wedge k)}+d(x,y_\tau^{k,\nu})]^\epsilon}|a_\tau^{k,\nu}|\\&\qquad\le C2^{[(k'\wedge k)-k]n(1-1/r)}\Big[\mathcal{M}\Big(\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}|a_\tau^{k,\nu}|^r\chi_{Q_\tau^{k,\nu}}\Big)(x)\Big]^{1/r},\end{aligned}$$

where C *is also independent of* k, k', τ *and* ν.

The following simple observation is of particular interest for estimates of the Triebel–Lizorkin or Besov norms.

Lemma 6.13. *Let* $p\in(0,\infty)$ *and* $\sigma\in(0,\infty)$. *Then there exists a positive constant* $C_{p,\sigma}$, *depending only on* p *and* σ, *such that, for all non-negative sequences* $\{a_j\}_{j\in\mathbb{N}}$,

$$\Big[\sum_{j\in\mathbb{N}}2^{-j\sigma}a_j\Big]^p\le C_{p,\sigma}\sum_{j\in\mathbb{N}}2^{-j\sigma(p\wedge1)}a_j^p. \tag{6.20}$$

Proof. By the following elementary inequality: for all $\alpha\in(0,1]$ and sequences $\{b_j\}_{j\in\mathbb{N}}$,

$$\Big[\sum_{j\in\mathbb{N}}|b_j|\Big]^\alpha\le\sum_{j\in\mathbb{N}}|b_j|^\alpha, \tag{6.21}$$

we see that (6.20) holds when $p \in (0,1]$. As for the case $p \in (1,\infty)$, applying Hölder's inequality, we know that

$$\Big[\sum_{j\in\mathbb{N}} 2^{-j\sigma} a_j\Big]^p \le \Big[\sum_{j\in\mathbb{N}} 2^{-j\sigma}\Big]^{p/p'} \Big[\sum_{j\in\mathbb{N}} 2^{-j\sigma} a_j^p\Big] \le C_{p,\sigma} \sum_{j\in\mathbb{N}} 2^{-j\sigma} a_j^p. \ \blacksquare$$

THEOREM 6.14. *Let κ be the constant appearing in the reverse doubling condition* (2.2) *and $\epsilon \in (0,1)\cap(0,\kappa]$. Suppose that the bilinear operator T is in* $\mathbf{BWBP}(\eta)$ *for some $\eta \in (0,\epsilon]$ and its kernel K belongs to* $\mathrm{Ker}(2, C_K, \delta)$ *for some $C_K > 0$ and $\delta \ge 2\epsilon$. Assume that, for all $g \in C_b^\eta(\mathcal{X})$,*

$$T(1,g) = T(g,1) = T^{*,1}(1,g) = 0 \quad \text{in } (C_b^\eta(\mathcal{X}))'.$$

For every $j \in \{0,1,2\}$, let $|s_j| < \epsilon$, $p(s_j,\epsilon) < p_j < \infty$ and $p(s_j,\epsilon) < q_j < \infty$ be such that

$$s_0 = s_1 + s_2, \quad \frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad \frac{1}{q_0} = \frac{1}{q_1} + \frac{1}{q_2},$$

and let $\dot F^{s_j}_{p_j,q_j}(\mathcal{X})$ be the Triebel–Lizorkin space as defined in Definition 6.9(ii)*. Then T can be extended to a bounded bilinear operator from $\dot F^{s_1}_{p_1,q_1}(\mathcal{X}) \times \dot F^{s_2}_{p_2,q_2}(\mathcal{X})$ to $\dot F^{s_0}_{p_0,q_0}(\mathcal{X})$.*

Proof. Let ϵ_1 and ϵ_2 be positive real numbers such that $\min\{\epsilon_1,\epsilon_2\} > \epsilon$. By the density of the set

$$\mathring{\mathcal{G}}_b(\epsilon_1,\epsilon_2) := \{f \in \mathring{\mathcal{G}}_0^\epsilon(\epsilon_1,\epsilon_2) : f \text{ has bounded suppoort}\}$$

in $\dot F^{s_j}_{p_j,q_j}(\mathcal{X})$ for $j \in \{1,2\}$ (see [60, Proposition 5.21]), it suffices to show that

$$T : [\dot F^{s_1}_{p_1,q_1}(\mathcal{X}) \cap \mathring{\mathcal{G}}_b(\epsilon_1,\epsilon_2)] \times [\dot F^{s_2}_{p_2,q_2}(\mathcal{X}) \cap \mathring{\mathcal{G}}_b(\epsilon_1,\epsilon_2)] \to \dot F^{s_0}_{p_0,q_0}(\mathcal{X}).$$

Since functions in $\mathring{\mathcal{G}}_b(\epsilon_1,\epsilon_2)$ are indeed atoms, modulo a positive constant, associated to Christ's dyadic cubes in the sense of Definition 6.4, we see that, for all $f,g \in \mathring{\mathcal{G}}_b(\epsilon_1,\epsilon_2)$, it follows from Theorem 6.6 that $T(f,g)$ is a bilinear molecule and hence can be interpreted in the usual way as an element of $(\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))'$, where β and γ satisfy (6.18) with s,p therein replaced by s_0, p_0, respectively. Consequently, it makes sense to write that, for all $\ell \in \mathbb{Z}$ and $x \in \mathcal{X}$,

$$D_\ell(T(f,g))(x) := \langle T(f,g), D_\ell(\cdot,x)\rangle,$$

where $\{S_k\}_{k\in\mathbb{Z}}$ is chosen to be a non-negative and symmetric 1-ATI with bounded support, and $D_k := S_k - S_{k-1}$ for all $k \in \mathbb{Z}$. Then, for all $f,g \in \mathring{\mathcal{G}}_b(\epsilon_1,\epsilon_2)$, by the frame characterization of the Triebel–Lizorkin spaces (see Lemma 6.11), we see that $\|T(f,g)\|_{\dot F^{s_0}_{p_0,q_0}(\mathcal{X})}$ is comparable to

$$\Big\|\Big\{\sum_{\ell\in\mathbb{Z}}\sum_{\tau\in I_\ell}\sum_{\nu=1}^{N(\ell,\tau)} 2^{\ell s_0 q_0}\Big|\inf_{y_\tau^{\ell,\nu}\in Q_\tau^{\ell,\nu}} \langle T(f,g),\, D_\ell(\cdot, y_\tau^{\ell,\nu})\rangle\Big|^{q_0} \chi_{Q_\tau^{\ell,\nu}}\Big\}^{1/q_0}\Big\|_{L^{p_0}(\mathcal{X})}.$$

Denote by $\mathring{C}_b^\eta(\mathcal{X})$ the collection of all $f \in C_b^\eta(\mathcal{X})$ such that $\int_{\mathcal{X}} f(x)\,d\mu(x) = 0$. Since $\epsilon \in (0, \epsilon_1 \wedge \epsilon_2]$ and $\eta \in (0,\epsilon]$, we have

$$(\mathring{\mathcal{G}}_b(\epsilon_1,\epsilon_2), \|\cdot\|_{\mathcal{G}(\epsilon_1,\epsilon_2)}) \hookrightarrow (\mathring{C}_b^\eta(\mathcal{X}), \|\cdot\|_{C_b^\eta(\mathcal{X})}) \hookrightarrow \mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma),$$

where $X \hookrightarrow Y$ means that the space X is continuously embedded into Y. So we apply the Calderón reproducing formulae to $f,g \in \mathring{\mathcal{G}}_b(\epsilon_1,\epsilon_2)$ and obtain, for all $w \in \mathcal{X}$,

$$f(w) = \sum_{k\in\mathbb{Z}}\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')} \mu(Q_{\tau'}^{k,\nu'}) D_k(w, y_{\tau'}^{k,\nu'}) \overline{D}_k(f)(y_{\tau'}^{k,\nu'}),$$

$$g(w)=\sum_{j\in\mathbb{Z}}\sum_{\tau''\in I_j}\sum_{\nu''=1}^{N(j,\tau'')}\mu(Q_{\tau''}^{j,\nu''})D_j(w,y_{\tau''}^{j,\nu''})\overline{D}_j(g)(y_{\tau''}^{j,\nu''}),$$

where both series converge in $\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)$ and hence in $C_b^\eta(\mathcal{X})$. Then, for any $\ell\in\mathbb{Z}$, $\tau\in I_\ell$, $\nu\in\{1,\dots,N(\tau,\ell)\}$ and any $y_\tau^{\ell,\nu}\in Q_\tau^{\ell,\nu}$,

$$\begin{aligned}\langle T(f,g)(\cdot),D_\ell(\cdot,\,y_\tau^{\ell,\nu})\rangle&=\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')}\sum_{\tau''\in I_j}\sum_{\nu''=1}^{N(j,\tau'')}\mu(Q_{\tau'}^{k,\nu'})\mu(Q_{\tau''}^{j,\nu''})\overline{D}_k(f)(y_{\tau'}^{k,\nu'})\\&\quad\times\overline{D}_j(g)(y_{\tau''}^{j,\nu''})\langle T(D_k(\cdot,y_{\tau'}^{k,\nu'}),D_j(\cdot,y_{\tau''}^{j,\nu''}))(x),D_\ell(x,\,y_\tau^{\ell,\nu})\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}&\|T(f,g)\|_{\dot F^{s_0}_{p_0,q_0}(\mathcal{X})}\\&\quad\sim\Bigg\|\Big\{\sum_{\ell\in\mathbb{Z}}\sum_{\nu\in I_\ell}\sum_{\nu=1}^{N(\ell,\tau)}2^{\ell s_0q_0}\\&\qquad\times\inf_{y_\tau^{\ell,\nu}\in Q_\tau^{\ell,\nu}}\Big|\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')}\sum_{\tau''\in I_j}\sum_{\nu''=1}^{N(j,\tau'')}\mu(Q_{\tau'}^{k,\nu'})\mu(Q_{\tau''}^{j,\nu''})\overline{D}_k(f)(y_{\tau'}^{k,\nu'})\\&\qquad\times\overline{D}_j(g)(y_{\tau''}^{j,\nu''})\langle T(D_k(\cdot,y_{\tau'}^{k,\nu'}),\,D_j(\cdot,y_{\tau''}^{j,\nu''}))(x),\,D_\ell(x,\,y_\tau^{\ell,\nu})\rangle\Big|^{q_0}\chi_{Q_\tau^{\ell,\nu}}\Big\}^{1/q_0}\Bigg\|_{L^{p_0}(\mathcal{X})}\\&\quad=:\mathrm{Z}.\end{aligned}$$

If we set

$$\begin{aligned}\mathrm{Z}_{\tau,\nu}^{(\ell,k,j)}&:=\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')}\sum_{\tau''\in I_j}\sum_{\nu''=1}^{N(j,\tau'')}\mu(Q_{\tau'}^{k,\nu'})\mu(Q_{\tau''}^{j,\nu''})|\overline{D}_k(f)(y_{\tau'}^{k,\nu'})||\overline{D}_j(g)(y_{\tau''}^{j,\nu''})|\\&\quad\times|\langle T(D_k(\cdot,y_{\tau'}^{k,\nu'}),\,D_j(\cdot,y_{\tau''}^{j,\nu''}))(x),\,D_\ell(x,\,y_\tau^{\ell,\nu})\rangle|,\end{aligned}$$

then obviously

$$\mathrm{Z}\le\Bigg\|\Big\{\sum_{\ell\in\mathbb{Z}}\sum_{\nu\in I_\ell}\sum_{\nu=1}^{N(\ell,\tau)}2^{\ell s_0q_0}\inf_{y_\tau^{\ell,\nu}\in Q_\tau^{\ell,\nu}}\Big|\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\mathrm{Z}_{\tau,\nu}^{(\ell,k,j)}\Big|^{q_0}\chi_{Q_\tau^{\ell,\nu}}\Big\}^{1/q_0}\Bigg\|_{L^{p_0}(\mathcal{X})}.$$

We estimate Z by splitting the summation $\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}$ into six parts according to the size relationship of k, j and ℓ. More precisely, Z_1 is the part of Z where $j\le\ell\le k$, namely,

$$\mathrm{Z}_1:=\Bigg\|\Big\{\sum_{\ell\in\mathbb{Z}}\sum_{\nu\in I_\ell}\sum_{\nu=1}^{N(\ell,\tau)}2^{\ell s_0q_0}\inf_{y_\tau^{\ell,\nu}\in Q_\tau^{\ell,\nu}}\Big[\sum_{k\ge\ell}\sum_{j\le\ell}\mathrm{Z}_{\tau,\nu}^{(\ell,k,j)}\Big]^{q_0}\chi_{Q_\tau^{\ell,\nu}}\Big\}^{1/q_0}\Bigg\|_{L^{p_0}(\mathcal{X})};$$

similarly, Z_2 is the part of Z where $k\le\ell\le j$, Z_3 the part where $j\le k\le\ell$, Z_4 the part where $k\le j\le\ell$, Z_5 the part where $\ell\le k\le j$, and Z_6 the part where $\ell\le j\le k$. Clearly,

$$\mathrm{Z}\lesssim\sum_{i=1}^{6}\mathrm{Z}_i.$$

So it is enough to show that, for $i\in\{1,\dots,6\}$,

$$\mathrm{Z}_i\lesssim\|f\|_{\dot F^{s_1}_{p_1,q_1}(\mathcal{X})}\|g\|_{\dot F^{s_2}_{p_2,q_2}(\mathcal{X})}.$$

The main tools to be used in the proof below are Theorem 6.8, Lemmas 6.12 and 6.13, and the Fefferman–Stein vector-valued maximal function inequality; for the latter, see [48].

Since $\epsilon \in (0,1)\cap(0,\kappa]\cap(0,\delta/2]$, there exist positive real numbers $\widetilde{\gamma},\gamma,\sigma,r_0,r_1,r_2$ such that, for $i\in\{0,1,2\}$,

$$\begin{cases}\widetilde{\gamma}\in(0,\delta/2],\\ \gamma\in(0,\widetilde{\gamma}),\\ \sigma\in(0,\min\{\delta-\widetilde{\gamma},1,\kappa\}],\\ |s_i|<\min\{\sigma,\epsilon\},\\ \max\left\{\dfrac{n}{n+\gamma},\dfrac{n}{n+\sigma},\dfrac{n}{n+\sigma+s_i}\right\}<r_i<\min\{1,p_i,q_i\}.\end{cases} \tag{6.22}$$

To see the existence of such numbers, we first choose a small positive number η and $\sigma>0$ such that

$$\begin{cases}|s_i|<\epsilon-2\eta<\sigma<\epsilon-\eta<\epsilon,\\ \min\{p_i,q_i\}>\max\left\{\dfrac{n}{n+\epsilon-2\eta},\dfrac{n}{n+\epsilon-2\eta+s_i}\right\}, \qquad i\in\{0,1,2\};\end{cases}$$

it is easy to show that $\widetilde{\gamma}:=\epsilon$ guarantees that

$$\widetilde{\gamma}\in(0,\delta/2] \quad\text{and}\quad \sigma\in(0,\min\{\delta-\widetilde{\gamma},1,\kappa\}];$$

finally, by taking $\gamma:=\epsilon-\eta$, we have

$$\max\left\{\frac{n}{n+\gamma},\frac{n}{n+\sigma},\frac{n}{n+\sigma+s_i}\right\}<\min\{1,p_i,q_i\},$$

so r_i can be taken to be any number in the interval

$$\left(\max\left\{\frac{n}{n+\gamma},\frac{n}{n+\sigma},\frac{n}{n+\sigma+s_i}\right\},\ \min\{1,p_i,q_i\}\right).$$

In the following proof, we fix $\widetilde{\gamma},\gamma,\sigma,r_0,r_1,r_2$ satisfying (6.22).

$\boxed{\textit{Estimate for } \mathrm{Z}_1\textit{: } j\le\ell\le k.}$ By Theorem 6.8(i), we see that, when $j\le\ell\le k$,

$$\begin{aligned}&|\langle T(D_k(\cdot,y_{\tau'}^{k,\nu'}),\,D_j(\cdot,y_{\tau''}^{j,\nu''}))(x),\,D_\ell(x,\,y_\tau^{\ell,\nu})\rangle|\\ &\qquad\lesssim 2^{-(k-j)\sigma}\mathcal{K}(2^{-\ell};\gamma,y_\tau^{\ell,\nu},y_{\tau'}^{k,\nu'})\,\mathcal{K}(2^{-j};\gamma,y_\tau^{\ell,\nu},y_{\tau''}^{j,\nu''}).\end{aligned} \tag{6.23}$$

From (6.23) and Lemma 6.12, it follows that, when $j\le\ell\le k$,

$$\begin{aligned}\mathrm{Z}_{\tau,\nu}^{(\ell,k,j)}&\lesssim 2^{-(k-j)\sigma}\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')}\mu(Q_{\tau'}^{k,\nu'})|\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|\,\mathcal{K}(2^{-\ell};\gamma,y_\tau^{\ell,\nu},y_{\tau'}^{k,\nu'})\\ &\quad\times\sum_{\tau''\in I_j}\sum_{\nu''=1}^{N(j,\tau'')}\mu(Q_{\tau''}^{j,\nu''})|\overline{D}_j(g)(y_{\tau''}^{j,\nu''})|\,\mathcal{K}(2^{-j};\gamma,y_\tau^{\ell,\nu},y_{\tau''}^{j,\nu''})\\ &\lesssim 2^{-(k-j)\sigma}2^{(k-\ell)n(1/r_1-1)}\left\{\mathcal{M}\Big(\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')}|\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{r_1}\chi_{Q_{\tau'}^{k,\nu'}}\Big)(y_\tau^{\ell,\nu})\right\}^{1/r_1}\\ &\quad\times\left\{\mathcal{M}\Big(\sum_{\tau''\in I_j}\sum_{\nu''=1}^{N(j,\tau'')}|\overline{D}_j(g)(y_{\tau''}^{j,\nu''})|^{r_2}\chi_{Q_{\tau''}^{j,\nu''}}\Big)(y_\tau^{\ell,\nu})\right\}^{1/r_2}.\end{aligned}$$

Inserting this into the expression of Z_1, we obtain

$$\begin{aligned}\mathrm{Z}_1 &\lesssim \Bigg\|\Bigg\{\sum_{\ell\in\mathbb{Z}}\sum_{\nu\in I_\ell}\sum_{\nu=1}^{N(\ell,\tau)}\inf_{y_\tau^{\ell,\nu}\in Q_\tau^{\ell,\nu}}\Bigg[\sum_{k\geq\ell}\sum_{j\leq\ell}2^{\ell(s_1+s_2)}2^{-(k-j)\sigma}2^{(k-\ell)n(1/r_1-1)}\\
&\quad\times\Bigg\{\mathcal{M}\Bigg(\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')}|\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{r_1}\chi_{Q_{\tau'}^{k,\nu'}}\Bigg)(y_\tau^{\ell,\nu})\Bigg\}^{1/r_1}\\
&\quad\times\Bigg\{\mathcal{M}\Bigg(\sum_{\tau''\in I_j}\sum_{\nu''=1}^{N(j,\tau'')}|\overline{D}_j(g)(y_{\tau''}^{j,\nu''})|^{r_2}\chi_{Q_{\tau''}^{j,\nu''}}\Bigg)(y_\tau^{\ell,\nu})\Bigg\}^{1/r_2}\Bigg]^{q_0}\chi_{Q_\tau^{\ell,\nu}}\Bigg\}^{1/q_0}\Bigg\|_{L^{p_0}(\mathcal{X})}\\
&\lesssim \Bigg\|\Bigg\{\sum_{\ell\in\mathbb{Z}}\Bigg[\sum_{k\geq\ell}\sum_{j\leq\ell}2^{\ell(s_1+s_2)}2^{-(k-\ell)\sigma}2^{-(\ell-j)\sigma}2^{(k-\ell)n(1/r_1-1)}\\
&\quad\times\Bigg\{\mathcal{M}\Bigg(\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')}|\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{r_1}\chi_{Q_{\tau'}^{k,\nu'}}\Bigg)\Bigg\}^{1/r_1}\\
&\quad\times\Bigg\{\mathcal{M}\Bigg(\sum_{\tau''\in I_j}\sum_{\nu''=1}^{N(j,\tau'')}|\overline{D}_j(g)(y_{\tau''}^{j,\nu''})|^{r_2}\chi_{Q_{\tau''}^{j,\nu''}}\Bigg)\Bigg\}^{1/r_2}\Bigg]^{q_0}\Bigg\}^{1/q_0}\Bigg\|_{L^{p_0}(\mathcal{X})},\end{aligned}$$

where the last inequality was obtained by first removing the infimum and then using the fact that, for all $\ell\in\mathbb{Z}$,

$$\sum_{\tau\in I_\ell}\sum_{\nu=1}^{N(\ell,\tau)}\chi_{Q_\tau^{\ell,\nu}}(x)=1\quad\text{for almost every } x\in\mathcal{X}.$$

Applying Hölder's inequality to the last quantity displayed in the above inequality, we see that

$$\mathrm{Z}_1\lesssim\mathcal{I}\times\mathcal{J},$$

where

$$\begin{aligned}\mathcal{I}&:=\Bigg\|\Bigg\{\sum_{\ell\in\mathbb{Z}}\Bigg[\sum_{k\geq\ell}2^{-(k-\ell)[\sigma+s_1-n(1/r_1-1)]}\\
&\qquad\times\Bigg\{\mathcal{M}\Bigg(\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')}|2^{ks_1}\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{r_1}\chi_{Q_{\tau'}^{k,\nu'}}\Bigg)\Bigg\}^{1/r_1}\Bigg]^{q_1}\Bigg\}^{1/q_1}\Bigg\|_{L^{p_1}(\mathcal{X})},\\
\mathcal{J}&:=\Bigg\|\Bigg\{\sum_{\ell\in\mathbb{Z}}\Bigg[\sum_{j\leq\ell}2^{-(\ell-j)(\sigma-s_2)}\\
&\qquad\times\Bigg\{\mathcal{M}\Bigg(\sum_{\tau''\in I_j}\sum_{\nu''=1}^{N(j,\tau'')}|2^{js_2}\overline{D}_j(g)(y_{\tau''}^{j,\nu''})|^{r_2}\chi_{Q_{\tau''}^{j,\nu''}}\Bigg)\Bigg\}^{1/r_2}\Bigg]^{q_2}\Bigg\}^{1/q_2}\Bigg\|_{L^{p_2}(\mathcal{X})}.\end{aligned}$$

To estimate $\mathcal{I}$, by Lemma 6.13 and the fact that

$$\sum_{k\geq\ell}2^{-(k-\ell)[\sigma+s_1-n(1/r_1-1)]}\lesssim 1$$

(this is because $r_1>n/(n+\sigma+s_1)$), we have

$$\Bigg[\sum_{k\geq\ell}2^{-(k-\ell)[\sigma+s_1-n(1/r_1-1)]}\Bigg\{\mathcal{M}\Bigg(\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')}|2^{ks_1}\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{r_1}\chi_{Q_{\tau'}^{k,\nu'}}\Bigg)(x)\Bigg\}^{1/r_1}\Bigg]^{q_1}$$

$$\lesssim \sum_{k\geq\ell} 2^{-(k-\ell)[\sigma+s_1-n(1/r_1-1)](q_1\wedge 1)} \times \Big[\mathcal{M}\Big(\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')} |2^{ks_1}\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{r_1}\chi_{Q_{\tau'}^{k,\nu'}}\Big)(x)\Big]^{q_1/r_1}.$$

From this and the Fefferman–Stein vector-valued maximal inequality ([48]), we deduce that

$$\begin{aligned}
\mathcal{I} &\lesssim \Bigg\|\Bigg\{\sum_{\ell\in\mathbb{Z}}\sum_{k\geq\ell} 2^{-(k-\ell)[\sigma+s_1-n(1/r_1-1)](q_1\wedge 1)} \\
&\quad\times \Big[\mathcal{M}\Big(\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')} |2^{ks_1}\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{r_1}\chi_{Q_{\tau'}^{k,\nu'}}\Big)\Big]^{q_1/r_1}\Bigg\}^{r_1/q_1}\Bigg\|_{L^{p_1/r_1}(\mathcal{X})}^{1/r_1} \\
&\lesssim \Bigg\|\Bigg\{\sum_{k\in\mathbb{Z}} \Big[\mathcal{M}\Big(\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')} |2^{ks_1}\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{r_1}\chi_{Q_{\tau'}^{k,\nu'}}\Big)\Big]^{q_1/r_1}\Bigg\}^{r_1/q_1}\Bigg\|_{L^{p_1/r_1}(\mathcal{X})}^{1/r_1} \\
&\lesssim \Bigg\|\Bigg\{\sum_{k\in\mathbb{Z}} \Big[\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')} |2^{ks_1}\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{r_1}\chi_{Q_{\tau'}^{k,\nu'}}\Big]^{q_1/r_1}\Bigg\}^{r_1/q_1}\Bigg\|_{L^{p_1/r_1}(\mathcal{X})}^{1/r_1} \\
&\sim \Bigg\|\Bigg\{\sum_{k\in\mathbb{Z}}\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')} 2^{ks_1q_1}|\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{q_1}\chi_{Q_{\tau'}^{k,\nu'}}\Bigg\}^{1/q_1}\Bigg\|_{L^{p_1}(\mathcal{X})} \\
&\lesssim \|f\|_{\dot{F}^{s_1}_{p_1,q_1}(\mathcal{X})}.
\end{aligned}$$

A similar argument gives us

$$\mathcal{J} \lesssim \|g\|_{\dot{F}^{s_2}_{p_2,q_2}(\mathcal{X})}.$$

Combining the estimates of $\mathcal{I}$ and $\mathcal{J}$ implies that

$$\mathrm{Z}_1 \lesssim \|f\|_{\dot{F}^{s_1}_{p_1,q_1}(\mathcal{X})}\|g\|_{\dot{F}^{s_2}_{p_2,q_2}(\mathcal{X})}.$$

$\boxed{\textit{Estimate for } \mathrm{Z}_2\textit{: } k\leq\ell\leq j.}$ By symmetry of f and g, the estimate of Z_2 is similar to that of Z_1. More precisely, when $k\leq\ell\leq j$, applying Theorem 6.8(ii) one deduces that

$$\begin{aligned}
&|\langle T(D_k(\cdot,y_{\tau'}^{k,\nu'}),\, D_j(\cdot,y_{\tau''}^{j,\nu''}))(x),\, D_\ell(x,\, y_\tau^{\ell,\nu})\rangle| \\
&\qquad\lesssim 2^{-(j-k)\sigma}\mathcal{K}(2^{-k};\gamma,y_\tau^{\ell,\nu},y_{\tau'}^{k,\nu'})\,\mathcal{K}(2^{-\ell};\gamma,y_\tau^{\ell,\nu},y_{\tau''}^{j,\nu''}).
\end{aligned}$$

Comparing this inequality with (6.23), as well as the expression of Z_1 with that of Z_2, we see that

$$\begin{aligned}
\mathrm{Z}_2 &:= \Bigg\|\Bigg\{\sum_{\ell\in\mathbb{Z}}\sum_{\nu\in I_\ell}\sum_{\nu=1}^{N(\ell,\tau)} 2^{\ell s_0q_0} \inf_{y_\tau^{\ell,\nu}\in Q_\tau^{\ell,\nu}}\Bigg|\sum_{k\leq\ell}\sum_{j\geq\ell}\mathrm{Z}_{\tau,\nu}^{(\ell,k,j)}\Bigg|^{q_0}\chi_{Q_\tau^{\ell,\nu}}\Bigg\}^{1/q_0}\Bigg\|_{L^{p_0}(\mathcal{X})} \\
&\lesssim \|f\|_{\dot{F}^{s_1}_{p_1,q_1}(\mathcal{X})}\|g\|_{\dot{F}^{s_2}_{p_2,q_2}(\mathcal{X})},
\end{aligned}$$

which was obtained via replacing the estimate of Z_1 by reversing the roles of the terms related to k and those related to j therein, the details being omitted.

$\boxed{\textit{Estimate for } \mathrm{Z}_3\textit{: } j \le k \le \ell.}$ Recall that

$$\mathrm{Z}_3 := \left\| \left\{ \sum_{\ell\in\mathbb{Z}} \sum_{\nu\in I_\ell} \sum_{\nu=1}^{N(\ell,\tau)} 2^{\ell s_0 q_0} \inf_{y_\tau^{\ell,\nu}\in Q_\tau^{\ell,\nu}} \left| \sum_{k\le \ell} \sum_{j\le k} \mathrm{Z}_{\tau,\nu}^{(\ell,k,j)} \right|^{q_0} \chi_{Q_\tau^{\ell,\nu}} \right\}^{1/q_0} \right\|_{L^{p_0}(\mathcal{X})}.$$

From Theorem 6.8(iii) it follows that, when $j \le k \le \ell$,

$$\begin{aligned} &|\langle T(D_k(\cdot, y_{\tau'}^{k,\nu'}),\, D_j(\cdot, y_{\tau''}^{j,\nu''}))(x),\, D_\ell(x, y_\tau^{\ell,\nu})\rangle| \\ &\qquad \lesssim 2^{-(\ell-j)\sigma} \mathcal{K}(2^{-k}; \gamma, y_{\tau'}^{k,\nu'}, y_\tau^{\ell,\nu})\, \mathcal{K}(2^{-j}; \gamma, y_{\tau'}^{k,\nu'}, y_{\tau''}^{j,\nu''}). \end{aligned} \tag{6.24}$$

When $j \le k$, the doubling condition (2.1) implies that, for all z, $y_{\tau'}^{k,\nu'} \in Q_{\tau'}^{k,\nu'}$,

$$\mathcal{K}(2^{-j}; \gamma, y_{\tau'}^{k,\nu'}, y_{\tau''}^{j,\nu''}) \sim \mathcal{K}(2^{-j}; \gamma, z, y_{\tau''}^{j,\nu''}).$$

By this, (6.24) and Lemma 6.12, we deduce that, when $j \le k \le \ell$,

$$\begin{aligned} \mathrm{Z}_{\tau,\nu}^{(\ell,k,j)} &\lesssim 2^{-(\ell-j)\sigma} \sum_{\tau'\in I_k} \sum_{\nu'=1}^{N(k,\tau')} \mu(Q_{\tau'}^{k,\nu'}) \mathcal{K}(2^{-k}; \gamma, y_{\tau'}^{k,\nu'}, y_\tau^{\ell,\nu}) |\overline{D}_k(f)(y_{\tau'}^{k,\nu'})| \\ &\quad \times \inf_{z\in Q_{\tau'}^{k,\nu'}} \sum_{\tau''\in I_j} \sum_{\nu''=1}^{N(j,\tau'')} \mu(Q_{\tau''}^{j,\nu''})\, \mathcal{K}(2^{-j}; \gamma, z, y_{\tau''}^{j,\nu''}) |\overline{D}_j(g)(y_{\tau''}^{j,\nu''})| \\ &\lesssim 2^{-(\ell-j)\sigma} \sum_{\tau'\in I_k} \sum_{\nu'=1}^{N(k,\tau')} \mu(Q_{\tau'}^{k,\nu'}) \mathcal{K}(2^{-k}; \gamma, y_{\tau'}^{k,\nu'}, y_\tau^{\ell,\nu}) |\overline{D}_k(f)(y_{\tau'}^{k,\nu'})| \\ &\quad \times \inf_{z\in Q_{\tau'}^{k,\nu'}} \left[\mathcal{M}\Big(\sum_{\tau''\in I_j} \sum_{\nu''=1}^{N(j,\tau'')} |\overline{D}_j(g)(y_{\tau''}^{j,\nu''})|^{r_2} \chi_{Q_{\tau''}^{j,\nu''}} \Big)(z) \right]^{1/r_2} \\ &\lesssim 2^{-(\ell-j)\sigma} \left\{ \mathcal{M}\Big(\sum_{\tau'\in I_k} \sum_{\nu'=1}^{N(k,\tau')} |\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{r_0} |\mathrm{G}_j|^{r_0} \chi_{Q_{\tau'}^{k,\nu'}} \Big)(y_\tau^{\ell,\nu}) \right\}^{1/r_0}, \end{aligned}$$

where, for all $z \in \mathcal{X}$,

$$\mathrm{G}_j(z) := \left[\mathcal{M}\Big(\sum_{\tau''\in I_j} \sum_{\nu''=1}^{N(j,\tau'')} |\overline{D}_j(g)(y_{\tau''}^{j,\nu''})|^{r_2} \chi_{Q_{\tau''}^{j,\nu''}} \Big)(z) \right]^{1/r_2}.$$

If, for any $z \in \mathcal{X}$, we set

$$\mathrm{F}^{(\ell,k,j)}(z) := \left\{ \mathcal{M}\Big(\sum_{\tau'\in I_k} \sum_{\nu'=1}^{N(k,\tau')} |2^{ks_1} \overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{r_0} |2^{js_2} \mathrm{G}_j|^{r_0} \chi_{Q_{\tau'}^{k,\nu'}} \Big)(z) \right\}^{1/r_0},$$

then

$$\mathrm{Z}_{\tau,\nu}^{(\ell,k,j)} \lesssim 2^{-(\ell-k)\sigma} 2^{-ks_0} 2^{-(k-j)(\sigma-s_2)} \mathrm{F}^{(\ell,k,j)}(y_\tau^{\ell,\nu}).$$

Inserting this into the expression of Z_3, we obtain

$$\begin{aligned} \mathrm{Z}_3 &\lesssim \Bigg\| \Bigg\{ \sum_{\ell\in\mathbb{Z}} \sum_{\nu\in I_\ell} \sum_{\nu=1}^{N(\ell,\tau)} 2^{\ell s_0 q_0} \\ &\quad \times \inf_{y_\tau^{\ell,\nu}\in Q_\tau^{\ell,\nu}} \left| \sum_{k\le\ell} \sum_{j\le k} 2^{-(\ell-k)\sigma} 2^{-ks_0} 2^{-(k-j)(\sigma-s_2)} \mathrm{F}^{(\ell,k,j)}(y_\tau^{\ell,\nu}) \right|^{q_0} \chi_{Q_\tau^{\ell,\nu}} \Bigg\}^{1/q_0} \Bigg\|_{L^{p_0}(\mathcal{X})} \end{aligned}$$

$$\lesssim \Big\| \Big\{ \sum_{\ell\in\mathbb{Z}} \Big| \sum_{k\le \ell} \sum_{j\le k} 2^{-(\ell-k)(\sigma-s_0)} 2^{-(k-j)(\sigma-s_2)} \mathrm{F}^{(\ell,k,j)} \Big|^{q_0} \Big\}^{1/q_0} \Big\|_{L^{p_0}(\mathcal{X})}$$
$$\lesssim \Big\| \Big\{ \sum_{\ell\in\mathbb{Z}} \sum_{k\le \ell} \sum_{j\le k} 2^{-(\ell-k)(\sigma-s_0)(q_0\wedge 1)} 2^{-(k-j)(\sigma-s_2)(q_0\wedge 1)} [\mathrm{F}^{(\ell,k,j)}]^{q_0} \Big\}^{1/q_0} \Big\|_{L^{p_0}(\mathcal{X})},$$

where, in the last step, we used Lemma 6.13. Invoking the definition of $\mathrm{F}^{(\ell,k,j)}$, the above inequality gives us

$$\mathrm{Z}_3 \lesssim \Big\| \Big\{ \sum_{\ell\in\mathbb{Z}} \sum_{k\le \ell} \sum_{j\le k} 2^{-(\ell-k)(\sigma-s_0)(q_0\wedge 1)} 2^{-(k-j)(\sigma-s_2)(q_0\wedge 1)}$$
$$\times \Big[\mathcal{M}\Big(\sum_{\tau'\in I_k} \sum_{\nu'=1}^{N(k,\tau')} |2^{ks_1} \overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{r_0} |2^{js_2} \mathrm{G}_j|^{r_0} \chi_{Q_{\tau'}^{k,\nu'}} \Big) \Big]^{q_0/r_0} \Big\}^{1/q_0} \Big\|_{L^{p_0}(\mathcal{X})}.$$

Since $q_0/r_0 > 1$ and $p_0/r_0 > 1$, by applying the Fefferman–Stein vector-valued maximal function inequality ([48]), we continue the preceding estimate with

$$\mathrm{Z}_3 \lesssim \Big\| \Big\{ \sum_{\ell\in\mathbb{Z}} \sum_{k\le \ell} \sum_{j\le k} 2^{-(\ell-k)(\sigma-s_0)(q_0\wedge 1)} 2^{-(k-j)(\sigma-s_2)(q_0\wedge 1)}$$
$$\times \Big[\sum_{\tau'\in I_k} \sum_{\nu'=1}^{N(k,\tau')} |2^{ks_1} \overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{r_0} |2^{js_2} \mathrm{G}_j|^{r_0} \chi_{Q_{\tau'}^{k,\nu'}} \Big]^{q_0/r_0} \Big\}^{1/q_0} \Big\|_{L^{p_0}(\mathcal{X})}$$
$$\sim \Big\| \Big\{ \sum_{\ell\in\mathbb{Z}} \sum_{k\le \ell} \sum_{j\le k} 2^{-(\ell-k)(\sigma-s_0)(q_0\wedge 1)} 2^{-(k-j)(\sigma-s_2)(q_0\wedge 1)}$$
$$\times [2^{js_2} \mathrm{G}_j]^{q_0} \Big[\sum_{\tau'\in I_k} \sum_{\nu'=1}^{N(k,\tau')} |2^{ks_1} \overline{D}_k(f)(y_{\tau'}^{k,\nu'})| \chi_{Q_{\tau'}^{k,\nu'}} \Big]^{q_0} \Big\}^{1/q_0} \Big\|_{L^{p_0}(\mathcal{X})}.$$

Then Hölder's inequality implies that

$$\mathrm{Z}_3 \lesssim \Big\| \Big\{ \sum_{\ell\in\mathbb{Z}} \sum_{k\le \ell} \sum_{j\le k} 2^{-(\ell-k)(\sigma-s_0)(q_0\wedge 1)} 2^{-(k-j)(\sigma-s_2)(q_0\wedge 1)}$$
$$\times \Big[\sum_{\tau'\in I_k} \sum_{\nu'=1}^{N(k,\tau')} |2^{ks_1} \overline{D}_k(f)(y_{\tau'}^{k,\nu'})| \chi_{Q_{\tau'}^{k,\nu'}} \Big]^{q_0} \Big\}^{1/q_0} \Big\|_{L^{p_0}(\mathcal{X})}$$
$$\times \Big\| \Big\{ \sum_{\ell\in\mathbb{Z}} \sum_{k\le \ell} \sum_{j\le k} 2^{-(\ell-k)(\sigma-s_0)(q_0\wedge 1)} 2^{-(k-j)(\sigma-s_2)(q_0\wedge 1)} [2^{js_2} \mathrm{G}_j]^{q_0} \Big\}^{1/q_0} \Big\|_{L^{p_0}(\mathcal{X})}$$
$$\lesssim \Big\| \Big\{ \sum_{k\in\mathbb{Z}} \Big[\sum_{\tau'\in I_k} \sum_{\nu'=1}^{N(k,\tau')} |2^{ks_1} \overline{D}_k(f)(y_{\tau'}^{k,\nu'})| \chi_{Q_{\tau'}^{k,\nu'}} \Big]^{q_0} \Big\}^{1/q_0} \Big\|_{L^{p_0}(\mathcal{X})}$$
$$\times \Big\| \Big\{ \sum_{j\in\mathbb{Z}} [2^{js_2} \mathrm{G}_j]^{q_0} \Big\}^{1/q_0} \Big\|_{L^{p_0}(\mathcal{X})}$$
$$\lesssim \|f\|_{\dot{F}^{s_1}_{p_1,q_1}(\mathcal{X})} \Big\| \Big\{ \sum_{j\in\mathbb{Z}} [2^{js_2} \mathrm{G}_j]^{q_0} \Big\}^{1/q_0} \Big\|_{L^{p_0}(\mathcal{X})}.$$

Next, the Fefferman–Stein vector-valued maximal function inequality ([48]) and the fact that $r_2 < \min\{p_2, q_2\}$ imply that the second term is dominated by

$$\begin{aligned}
&\Big\|\Big\{\sum_{j\in\mathbb{Z}}[2^{js_2}\mathrm{G}_j]^{q_0}\Big\}^{1/q_0}\Big\|_{L^{p_0}(\mathcal{X})}\\
&\qquad\lesssim \Big\|\Big\{\sum_{j\in\mathbb{Z}}2^{js_2q_2}\Big[\mathcal{M}\Big(\sum_{\tau''\in I_j}\sum_{\nu''=1}^{N(j,\tau'')}|\overline{D}_j(g)(y_{\tau''}^{j,\nu''})|^{r_2}\chi_{Q_{\tau''}^{j,\nu''}}\Big)\Big]^{1/r_2}\Big\}^{1/q_2}\Big\|_{L^{p_2}(\mathcal{X})}\\
&\qquad\lesssim \|g\|_{\dot F^{s_2}_{p_2,q_2}(\mathcal{X})}.
\end{aligned}$$

This proves that

$$\mathrm{Z}_3 \lesssim \|f\|_{\dot F^{s_1}_{p_1,q_1}(\mathcal{X})}\|g\|_{\dot F^{s_2}_{p_2,q_2}(\mathcal{X})}.$$

$\boxed{\textit{Estimate for } \mathrm{Z}_4\textit{: } k \le j \le \ell.}$ The estimate of Z_4 is quite similar to that of Z_3: indeed we just need to repeat the reasoning for Z_3 by reversing the roles of the terms related to k and j therein; we omit the details.

$\boxed{\textit{Estimate for } \mathrm{Z}_5\textit{: } \ell \le k \le j.}$ The estimate of Z_5 is similar to that of Z_3; for the convenience of the reader let us sketch the proof. In this case, applying Theorem 6.8(v), we see that

$$\begin{aligned}
&\big|\big\langle T\big(D_k(\cdot, y_{\tau'}^{k,\nu'}),\, D_j(\cdot, y_{\tau''}^{j,\nu''})\big)(x),\, D_\ell(x,\, y_\tau^{\ell,\nu})\big\rangle\big|\\
&\qquad\lesssim 2^{-(j-\ell)\sigma}\mathcal{K}(2^{-\ell};\gamma, y_{\tau'}^{k,\nu'}, y_\tau^{\ell,\nu})\,\mathcal{K}(2^{-k};\gamma, y_{\tau'}^{k,\nu'}, y_{\tau''}^{j,\nu''}). \qquad (6.25)
\end{aligned}$$

When $k \le j$, the doubling condition (2.1) implies that, for all $z,\ y_{\tau'}^{k,\nu'} \in Q_{\tau'}^{k,\nu'}$,

$$\mathcal{K}(2^{-k};\gamma, y_{\tau'}^{k,\nu'}, y_{\tau''}^{j,\nu''}) \sim \mathcal{K}(2^{-k};\gamma, z, y_{\tau''}^{j,\nu''}).$$

Combining this with (6.24) and Lemma 6.12, we conclude that, when $\ell \le k \le j$,

$$\begin{aligned}
\mathrm{Z}_{\tau,\nu}^{(\ell,k,j)} &\lesssim 2^{-(j-\ell)\sigma}\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')}\mu(Q_{\tau'}^{k,\nu'})\mathcal{K}(2^{-\ell};\gamma, y_{\tau'}^{k,\nu'}, y_\tau^{\ell,\nu})|\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|\\
&\quad\times \inf_{z\in Q_{\tau'}^{k,\nu'}}\sum_{\tau''\in I_j}\sum_{\nu''=1}^{N(j,\tau'')}\mu(Q_{\tau''}^{j,\nu''})\,\mathcal{K}(2^{-k};\gamma, z, y_{\tau''}^{j,\nu''})|\overline{D}_j(g)(y_{\tau''}^{j,\nu''})|\\
&\lesssim 2^{-(j-\ell)\sigma}2^{(j-k)n(1/r_2-1)}\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')}\mu(Q_{\tau'}^{k,\nu'})\mathcal{K}(2^{-k};\gamma, y_{\tau'}^{k,\nu'}, y_\tau^{\ell,\nu})|\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|\\
&\quad\times \inf_{z\in Q_{\tau'}^{k,\nu'}}\Big[\mathcal{M}\Big(\sum_{\tau''\in I_j}\sum_{\nu''=1}^{N(j,\tau'')}|\overline{D}_j(g)(y_{\tau''}^{j,\nu''})|^{r_2}\chi_{Q_{\tau''}^{j,\nu''}}\Big)(z)\Big]^{1/r_2}\\
&\lesssim 2^{-(j-\ell)\sigma}2^{(j-k)n(1/r_2-1)}2^{(k-\ell)n(1/r_0-1)}\\
&\quad\times\Big\{\mathcal{M}\Big(\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')}|\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{r_0}|\mathrm{G}_j|^{r_0}\chi_{Q_{\tau'}^{k,\nu'}}\Big)(y_\tau^{\ell,\nu})\Big\}^{1/r_0},
\end{aligned}$$

where, as in the estimate of Z_3, the function G_j is defined by

$$\mathrm{G}_j(z) := \Big[\mathcal{M}\Big(\sum_{\tau''\in I_j}\sum_{\nu''=1}^{N(j,\tau'')}|\overline{D}_j(g)(y_{\tau''}^{j,\nu''})|^{r_2}\chi_{Q_{\tau''}^{j,\nu''}}\Big)(z)\Big]^{1/r_2},\qquad \forall z\in\mathcal{X}.$$

Inserting this into the expression of Z_5 and noticing that

$$2^{\ell s_0}2^{-(j-\ell)\sigma}2^{(j-k)n(1/r_2-1)}2^{(k-\ell)n(1/r_0-1)}2^{-ks_1}2^{-js_2}$$
$$= 2^{-(k-\ell)[\sigma+s_0-n(1/r_0-1)]}2^{-(j-k)[\sigma+s_2-n(1/r_2-1)]},$$

and also observing that, for any measurable function $H : \mathcal{X} \to [0,\infty)$, all $\ell \in \mathbb{Z}$, and almost every $x \in \mathcal{X}$,

$$\sum_{\nu\in I_\ell}\sum_{\nu=1}^{N(\ell,\tau)}\Big[\inf_{y_\tau^{\ell,\nu}\in Q_\tau^{\ell,\nu}} H(y_\tau^{\ell,\nu})\Big]\chi_{Q_\tau^{\ell,\nu}}(x) \le H(x),$$

we obtain

$$\mathrm{Z}_5 \lesssim \bigg\|\bigg\{\sum_{\ell\in\mathbb{Z}}\sum_{\nu\in I_\ell}\sum_{\nu=1}^{N(\ell,\tau)} 2^{\ell s_0 q_0}\inf_{y_\tau^{\ell,\nu}\in Q_\tau^{\ell,\nu}}\bigg|\sum_{k\ge\ell}\sum_{j\ge k}2^{-(j-\ell)\sigma}2^{(j-k)n(1/r_2-1)}2^{(k-\ell)n(1/r_0-1)}$$
$$\times\bigg\{\mathcal{M}\bigg(\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')}|\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{r_0}|\mathrm{G}_j|^{r_0}\chi_{Q_{\tau'}^{k,\nu'}}\bigg)(y_\tau^{\ell,\nu})\bigg\}^{1/r_0}\bigg|^{q_0}\chi_{Q_\tau^{\ell,\nu}}\bigg\}^{1/q_0}\bigg\|_{L^{p_0}(\mathcal{X})}$$
$$\lesssim \bigg\|\bigg\{\sum_{\ell\in\mathbb{Z}}\bigg|\sum_{k\ge\ell}\sum_{j\ge k}2^{-(k-\ell)[\sigma+s_0-n(1/r_0-1)]}2^{-(j-k)[\sigma+s_2-n(1/r_2-1)]}$$
$$\times\bigg\{\mathcal{M}\bigg(\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')}|2^{ks_1}\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{r_0}|2^{js_2}\mathrm{G}_j|^{r_0}\chi_{Q_{\tau'}^{k,\nu'}}\bigg)\bigg\}^{1/r_0}\bigg|^{q_0}\bigg\}^{1/q_0}\bigg\|_{L^{p_0}(\mathcal{X})}.$$

The choices of σ, r_0 and r_2 imply that $\sigma+s_0-n(1/r_0-1)>0$, $\sigma+s_2-n(1/r_2-1)>0$ and hence

$$\sum_{k\ge\ell}\sum_{j\ge k}2^{-(k-\ell)[\sigma+s_0-n(1/r_0-1)]}2^{-(j-k)[\sigma+s_2-n(1/r_2-1)]}\lesssim 1.$$

By Lemma 6.13, the Fefferman–Stein vector-valued maximal function inequality (see, for example, [48]) and Hölder's inequality, we follow the same procedure as in the estimation of Z_3 to obtain

$$\mathrm{Z}_5 \lesssim \|f\|_{\dot F^{s_1}_{p_1,q_1}(\mathcal{X})}\|g\|_{\dot F^{s_2}_{p_2,q_2}(\mathcal{X})},$$

further details being omitted.

$\boxed{\textit{Estimate for } \mathrm{Z}_6\textit{: } \ell\le j\le k.}$ Analogously to Z_4, to obtain the desired reasoning for Z_6, we just need to repeat the reasoning for Z_5 by reversing the roles of the terms related to k and j; we omit the details.

Summing the estimates of Z_1 through Z_6 completes the proof of Theorem 6.14. ∎

6.4. Bilinear $T1$-theorem on Besov spaces. In this section we prove a bilinear $T1$-theorem on Besov spaces that complements the corresponding results on the Triebel–Lizorkin scale.

THEOREM 6.15. *Let κ be the constant appearing in the reverse doubling condition* (2.2) *and $\epsilon\in(0,1)\cap(0,\kappa]$. Suppose that the bilinear operator T is in* **BWBP**(η) *for some $\eta\in(0,\epsilon]$ and its kernel K belongs to* $\mathrm{Ker}(2, C_K, \delta)$ *for some $C_K>0$ and $\delta\ge 2\epsilon$. Assume that, for all $g\in C_b^\eta(\mathcal{X})$,*

$$T(1,g)=T(g,1)=T^{*,1}(1,g)=0 \quad \textit{in } (C_b^\eta(\mathcal{X}))'.$$

For every $j \in \{0,1,2\}$, let $|s_j| < \epsilon$, $p(s_j,\epsilon) < p_j < \infty$ and $0 < q_j < \infty$ be such that

$$s_0 = s_1 + s_2, \quad \frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{p_2} \quad \textit{and} \quad \frac{1}{q_0} = \frac{1}{q_1} + \frac{1}{q_2},$$

and let $\dot{B}^{s_j}_{p_j,q_j}(\mathcal{X})$ be the Besov space as defined in Definition 6.9(i). Then T can be extended to a bounded bilinear operator from $\dot{B}^{s_1}_{p_1,q_1}(\mathcal{X}) \times \dot{B}^{s_2}_{p_2,q_2}(\mathcal{X})$ to $\dot{B}^{s_0}_{p_0,q_0}(\mathcal{X})$.

Proof. Let ϵ_1 and ϵ_2 be positive real numbers such that $\min\{\epsilon_1,\epsilon_2\} > \epsilon$. Let $\{S_k\}_{k\in\mathbb{Z}}$ be a non-negative and symmetric 1-ATI with bounded support, and $D_k := S_k - S_{k-1}$ for all $k \in \mathbb{Z}$. As in the proof of Theorem 6.14, it suffices to show that, for all $f, g \in \mathring{\mathcal{G}}^{\epsilon}_0(\epsilon_1,\epsilon_2)$ with bounded supports,

$$\|T(f,g)\|_{\dot{B}^{s_0}_{p_0,q_0}(\mathcal{X})} \le C\|f\|_{\dot{B}^{s_1}_{p_1,q_1}(\mathcal{X})}\|g\|_{B^{s_2}_{p_2,q_2}(\mathcal{X})}.$$

By the frame characterization of the Besov spaces (see Lemma 6.11), we write

$$\|T(f,g)\|_{\dot{B}^{s_0}_{p_0,q_0}(\mathcal{X})} \sim \Big\{\sum_{\ell\in\mathbb{Z}} 2^{\ell s_0 q_0}\Big[\sum_{\tau\in I_\ell}\sum_{\nu=1}^{N(\ell,\tau)} \mu(Q^{\ell,\nu}_\tau) \inf_{y^{\ell,\nu}_\tau\in Q^{\ell,\nu}_\tau} |\langle T(f,g)(\cdot),\, D_\ell(\cdot, y^{\ell,\nu}_\tau)\rangle|^{p_0}\Big]^{q_0/p_0}\Big\}^{1/q_0}.$$

Next, applying the Calderón reproducing formula to f and g gives that, for all $w \in \mathcal{X}$,

$$f(w) = \sum_{k\in\mathbb{Z}}\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')} \mu(Q^{k,\nu'}_{\tau'}) D_k(w, y^{k,\nu'}_{\tau'}) \overline{D}_k(f)(y^{k,\nu'}_{\tau'}),$$

$$g(w) = \sum_{j\in\mathbb{Z}}\sum_{\tau''\in I_j}\sum_{\nu''=1}^{N(j,\tau'')} \mu(Q^{j,\nu''}_{\tau''}) D_j(w, y^{j,\nu''}_{\tau''}) \overline{D}_j(g)(y^{j,\nu''}_{\tau''}),$$

where each series converges in $C^\eta_b(\mathcal{X})$. It follows that

$$\begin{aligned}
&\|T(f,g)\|_{\dot{B}^{s_0}_{p_0,q_0}(\mathcal{X})}\\
&\quad\lesssim \Big\{\sum_{\ell\in\mathbb{Z}} 2^{\ell s_0 q_0}\Big[\sum_{\tau\in I_\ell}\sum_{\nu=1}^{N(\ell,\tau)} \mu(Q^{\ell,\nu}_\tau)\\
&\qquad\times \inf_{y^{\ell,\nu}_\tau\in Q^{\ell,\nu}_\tau}\Big|\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')}\sum_{\tau''\in I_j}\sum_{\nu''=1}^{N(j,\tau'')} \mu(Q^{k,\nu'}_{\tau'})\mu(Q^{j,\nu''}_{\tau''})\overline{D}_k(f)(y^{k,\nu'}_{\tau'})\\
&\qquad\times \overline{D}_j(g)(y^{j,\nu''}_{\tau''})\langle T(D_k(\cdot, y^{k,\nu'}_{\tau'}),\, D_j(\cdot, y^{j,\nu''}_{\tau''}))(x),\, D_\ell(x,\, y^{\ell,\nu}_\tau)\rangle\Big|^{p_0}\Big]^{q_0/p_0}\Big\}^{1/q_0}\\
&\quad=: \mathrm{Y}.
\end{aligned}$$

In what follows, we use the notation, for all $k, j, \ell \in \mathbb{Z}$,

$$\begin{aligned}
\mathrm{Y}^{(\ell,k,j)} := &\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')}\sum_{\tau''\in I_j}\sum_{\nu''=1}^{N(j,\tau'')} \mu(Q^{k,\nu'}_{\tau'})\mu(Q^{j,\nu''}_{\tau''})\\
&\times |\overline{D}_k(f)(y^{k,\nu'}_{\tau'})|\,|\overline{D}_j(g)(y^{j,\nu''}_{\tau''})|\,|\langle T(D_k(\cdot, y^{k,\nu'}_{\tau'}),\, D_j(\cdot, y^{j,\nu''}_{\tau''}))(x),\, D_\ell(x,\, y^{\ell,\nu}_\tau)\rangle|.
\end{aligned}$$

With this notation, we have

$$\mathrm{Y} = \Big\{ \sum_{\ell\in\mathbb{Z}} 2^{\ell s_0 q_0} \Big[\sum_{\tau\in I_\ell} \sum_{\nu=1}^{N(\ell,\tau)} \mu(Q_\tau^{\ell,\nu}) \inf_{y_\tau^{\ell,\nu}\in Q_\tau^{\ell,\nu}} \Big| \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \mathrm{Y}^{(\ell,k,j)} \Big|^{p_0} \Big]^{q_0/p_0} \Big\}^{1/q_0}.$$

As in the proof of Theorem 6.14, we estimate Y by splitting the summation $\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}$ into six parts according to the size relationship of k, j, and ℓ. To be precise, Y_1 is the part of Y where $j \le \ell \le k$, Y_2 the part where $k \le \ell \le j$, Y_3 the part where $j \le k \le \ell$, Y_4 the part where $k \le j \le \ell$, Y_5 the part where $\ell \le k \le j$, and Y_6 the part where $\ell \le j \le k$. Clearly,

$$\mathrm{Y} \lesssim \sum_{i=1}^{6} \mathrm{Y}_i.$$

Thus, it is enough to show that, for $i \in \{1, \dots, 6\}$,

$$\mathrm{Y}_i \lesssim \|f\|_{\dot{B}^{s_1}_{p_1,q_1}(\mathcal{X})} \|g\|_{B^{s_2}_{p_2,q_2}(\mathcal{X})}. \tag{6.26}$$

Observe that, by symmetry, the estimates of Y_2, Y_4 and Y_6 are analogous to those of Y_1, Y_3 and Y_5, respectively. So we only prove (6.26) for $j \in \{1, 3, 5\}$.

$\boxed{\textit{Estimate for } \mathrm{Y}_1\textit{: } j \le \ell \le k.}$ Let σ, γ, r_0, r_1 and r_2 be positive numbers satisfying (6.22). Recall that, in the estimate for Z_1 in Theorem 6.14, by Theorem 6.8(i) and Lemma 6.12, we have obtained the following: when $j \le \ell \le k$,

$$\begin{aligned} \mathrm{Y}^{(\ell,k,j)} &\lesssim 2^{-(k-j)\sigma} 2^{(k-\ell)n(1/r_1-1)} \Big[\mathcal{M}\Big(\sum_{\tau'\in I_k} \sum_{\nu'=1}^{N(k,\tau')} |\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{r_1} \chi_{Q_{\tau'}^{k,\nu'}} \Big)(y_\tau^{\ell,\nu}) \Big]^{1/r_1} \\ &\quad \times \Big[\mathcal{M}\Big(\sum_{\tau''\in I_j} \sum_{\nu''=1}^{N(j,\tau'')} |\overline{D}_j(g)(y_{\tau''}^{j,\nu''})|^{r_2} \chi_{Q_{\tau''}^{j,\nu''}} \Big)(y_\tau^{\ell,\nu}) \Big]^{1/r_2}. \end{aligned}$$

From this and Hölder's inequality, it follows that

$$\begin{aligned} \mathrm{Y}_1 &:= \Big\{ \sum_{\ell\in\mathbb{Z}} 2^{\ell s_0 q_0} \Big[\sum_{\tau\in I_\ell} \sum_{\nu=1}^{N(\ell,\tau)} \mu(Q_\tau^{\ell,\nu}) \inf_{y_\tau^{\ell,\nu}\in Q_\tau^{\ell,\nu}} \Big| \sum_{j\le\ell} \sum_{k\ge\ell} \mathrm{Y}^{(\ell,k,j)} \Big|^{p_0} \Big]^{q_0/p_0} \Big\}^{1/q_0} \\ &\lesssim \Big\{ \sum_{\ell\in\mathbb{Z}} 2^{\ell s_0 q_0} \Big\| \sum_{j\le\ell} \sum_{k\ge\ell} 2^{-(k-j)\sigma} 2^{(k-\ell)n(1/r_1-1)} \\ &\quad \times \Big[\mathcal{M}\Big(\sum_{\tau'\in I_k} \sum_{\nu'=1}^{N(k,\tau')} |\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{r_1} \chi_{Q_{\tau'}^{k,\nu'}} \Big) \Big]^{1/r_1} \\ &\quad \times \Big[\mathcal{M}\Big(\sum_{\tau''\in I_j} \sum_{\nu''=1}^{N(j,\tau'')} |\overline{D}_j(g)(y_{\tau''}^{j,\nu''})|^{r_2} \chi_{Q_{\tau''}^{j,\nu''}} \Big) \Big]^{1/r_2} \Big\|_{L^{p_0}(\mathcal{X})}^{q_0} \Big\}^{1/q_0} \\ &\sim \Big\{ \sum_{\ell\in\mathbb{Z}} 2^{\ell s_0 q_0} \Big\| \sum_{j\le\ell} \sum_{k\ge\ell} 2^{-(k-j)\sigma} 2^{(k-\ell)n(1/r_1-1)} \mathrm{F}_k \mathrm{G}_j \Big\|_{L^{p_0}(\mathcal{X})}^{q_0} \Big\}^{1/q_0}, \end{aligned}$$

where

$$\mathrm{F}_k := \Big[\mathcal{M}\Big(\sum_{\tau'\in I_k} \sum_{\nu'=1}^{N(k,\tau')} |\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{r_1} \chi_{Q_{\tau'}^{k,\nu'}} \Big) \Big]^{1/r_1},$$

$$\mathrm{G}_j := \Big[\mathcal{M}\Big(\sum_{\tau''\in I_j}\sum_{\nu''=1}^{N(j,\tau'')} |\overline{D}_j(g)(y_{\tau''}^{j,\nu''})|^{r_2}\chi_{Q_{\tau''}^{j,\nu''}}\Big)\Big]^{1/r_2}.$$

Noticing that $s_0 = s_1 + s_2$ and applying Hölder's inequality, we obtain

$$\mathrm{Y}_1 \lesssim \mathcal{I}\times\mathcal{J},$$

where

$$\mathcal{I} := \Big\{\sum_{\ell\in\mathbb{Z}} 2^{\ell s_1 q_1}\Big\|\sum_{k\geq\ell} 2^{-(k-\ell)[\sigma-n(1/r_1-1)]}\mathrm{F}_k\Big\|_{L^{p_1}(\mathcal{X})}^{q_1}\Big\}^{1/q_1},$$

$$\mathcal{J} := \Big\{\sum_{\ell\in\mathbb{Z}} 2^{\ell s_2 q_2}\Big\|\sum_{j\leq\ell} 2^{-(\ell-j)\sigma}\mathrm{G}_j\Big\|_{L^{p_2}(\mathcal{X})}^{q_2}\Big\}^{1/q_2}.$$

To estimate the term $\mathcal{I}$, for every $\ell\in\mathbb{Z}$, we set

$$\mathcal{I}_\ell := \Big\|\sum_{k\geq\ell} 2^{-(k-\ell)[\sigma+s_1-n(1/r_1-1)]}\mathrm{F}_k\Big\|_{L^{p_1}(\mathcal{X})}.$$

Obviously, $\mathcal{I} = [\sum_{\ell\in\mathbb{Z}}(\mathcal{I}_\ell)^{q_1}]^{1/q_1}$. For each $\ell\in\mathbb{Z}$, we rewrite $\mathcal{I}_\ell$ as

$$\mathcal{I}_\ell = \Big\|\sum_{k\geq\ell} 2^{-(k-\ell)[\sigma+s_1-n(1/r_1-1)]}\mathrm{F}_k^{p_1}\Big\|_{L^1(\mathcal{X})}^{1/p_1}.$$

Invoking the expression for F_k and applying Lemma 6.13 and using

$$\sum_{k\geq\ell} 2^{-(k-\ell)[\sigma-n(1/r_1-1)](p_1\wedge 1)} < \infty,$$

we conclude that

$$\begin{aligned}
\mathcal{I}_\ell &\lesssim \Big\|\sum_{k\geq\ell} 2^{-(k-\ell)[\sigma+s_1-n(1/r_1-1)](p_1\wedge 1)}\\
&\qquad\times\Big[\mathcal{M}\Big(\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')} |2^{ks_1}\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{r_1}\chi_{Q_{\tau'}^{k,\nu'}}\Big)\Big]^{p_1/r_1}\Big\|_{L^1(\mathcal{X})}^{1/p_1}\\
&\sim \sum_{k\geq\ell} 2^{-(k-\ell)[\sigma+s_1-n(1/r_1-1)](p_1\wedge 1)}\\
&\qquad\times\Big\|\mathcal{M}\Big(\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')} |2^{ks_1}\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{r_1}\chi_{Q_{\tau'}^{k,\nu'}}\Big)\Big\|_{L^{p_1/r_1}(\mathcal{X})}^{1/r_1},
\end{aligned}$$

which, combined with the fact that $\mathcal{M}$ is bounded on $L^{p_1/r_1}(\mathcal{X})$ when $p_1/r_1 > 1$, implies

$$\begin{aligned}
\mathcal{I}_\ell &\lesssim \sum_{k\geq\ell} 2^{-(k-\ell)[\sigma+s_1-n(1/r_1-1)](p_1\wedge 1)}\Big\|\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')} |2^{ks_1}\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{r_1}\chi_{Q_{\tau'}^{k,\nu'}}\Big\|_{L^{p_1/r_1}(\mathcal{X})}^{1/r_1}\\
&\sim \sum_{k\geq\ell} 2^{-(k-\ell)[\sigma+s_1-n(1/r_1-1)](p_1\wedge 1)}\Big[\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')} |2^{ks_1}\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{p_1}\mu(Q_{\tau'}^{k,\nu'})\Big]^{1/p_1}.
\end{aligned}$$

Again, applying Lemma 6.13 when $p_1 > 1$ or (6.21) when $p_1 \leq 1$ and the fact that

$$\sum_{k\geq\ell} 2^{-(k-\ell)[\sigma+s_1-n(1/r_1-1)](p_1\wedge 1)(q_1\wedge 1)} < \infty,$$

we know that

$$(\mathcal{I}_\ell)^{q_1} \lesssim \sum_{k\ge\ell} 2^{-(k-\ell)[\sigma+s_1-n(1/r_1-1)](p_1\wedge1)(q_1\wedge1)} \times \Big[\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')} |2^{ks_1}\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{p_1}\mu(Q_{\tau'}^{k,\nu'})\Big]^{q_1/p_1}.$$

Next, we compute $\sum_{\ell\in\mathbb{Z}}(\mathcal{I}_\ell)^{q_1}$ by interchanging the summations in ℓ and k, to obtain

$$\begin{aligned}\mathcal{I} = \Big\{\sum_{\ell\in\mathbb{Z}}(\mathcal{I}_\ell)^{q_1}\Big\}^{1/q_1} &\lesssim \Big\{\sum_{\ell\in\mathbb{Z}}\sum_{k\ge\ell} 2^{-(k-\ell)[\sigma+s_1-n(1/r_1-1)](p_1\wedge1)(q_1\wedge1)} \\ &\qquad\times \Big[\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')} |2^{ks_1}\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{p_1}\mu(Q_{\tau'}^{k,\nu'})\Big]^{q_1/p_1}\Big\}^{1/q_1} \\ &\lesssim \Big\{\sum_{k\in\mathbb{Z}}\Big[\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')} |2^{ks_1}\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{p_1}\mu(Q_{\tau'}^{k,\nu'})\Big]^{q_1/p_1}\Big\}^{1/q_1} \\ &\lesssim \|f\|_{\dot{F}^{s_1}_{p_1,q_1}(\mathcal{X})}.\end{aligned}$$

An argument similar to that used in the estimate of $\mathcal{I}$ gives that

$$\mathcal{J} \lesssim \|g\|_{\dot{F}^{s_2}_{p_2,q_2}(\mathcal{X})}.$$

Combining the estimates of $\mathcal{I}$ and $\mathcal{J}$ implies the desired estimate of Y_1.

$\boxed{\textit{Estimate for } \mathrm{Y}_3\textit{: } j\le k\le\ell.}$ Recall that

$$\mathrm{Y}_3 := \Big\{\sum_{\ell\in\mathbb{Z}} 2^{\ell s_0 q_0}\Big[\sum_{\tau\in I_\ell}\sum_{\nu=1}^{N(\ell,\tau)} \mu(Q_\tau^{\ell,\nu}) \inf_{y_\tau^{\ell,\nu}\in Q_\tau^{\ell,\nu}} \Big|\sum_{k\le\ell}\sum_{j\le k} \mathrm{Y}^{(\ell,k,j)}\Big|^{p_0}\Big]^{q_0/p_0}\Big\}^{1/q_0}.$$

In the estimate of Z_3 in Theorem 6.14, it was proved that, when $j\le k\le\ell$,

$$\mathrm{Y}^{(\ell,k,j)} \lesssim 2^{-(\ell-j)\sigma}\Big[\mathcal{M}\Big(|\mathrm{G}_j|^{r_0}\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')} |\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{r_0}\chi_{Q_{\tau'}^{k,\nu'}}\Big)(y_\tau^{\ell,\nu})\Big]^{1/r_0},$$

where, for all $z\in\mathcal{X}$,

$$\mathrm{G}_j(z) := \Big[\mathcal{M}\Big(\sum_{\tau''\in I_j}\sum_{\nu''=1}^{N(j,\tau'')} |\overline{D}_j(g)(y_{\tau''}^{j,\nu''})|^{r_2}\chi_{Q_{\tau''}^{j,\nu''}}\Big)(z)\Big]^{1/r_2}.$$

If, for any $z\in\mathcal{X}$, we set

$$\mathrm{F}^{(\ell,k,j)}(z) := \Big[\mathcal{M}\Big(|2^{js_2}\mathrm{G}_j|^{r_0}\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')} |2^{ks_1}\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{r_0}\chi_{Q_{\tau'}^{k,\nu'}}\Big)(z)\Big]^{1/r_0},$$

then, by $s_0=s_1+s_2$, we have

$$\mathrm{Y}^{(\ell,k,j)} \lesssim 2^{-(\ell-j)\sigma}2^{-ks_0}2^{(k-j)s_2}\mathrm{F}^{(\ell,k,j)}(y_\tau^{\ell,\nu}).$$

Inserting this estimate into the expression for Y_3 gives that

$$
\begin{aligned}
\mathrm{Y}_3 &\lesssim \Bigg\{\sum_{\ell\in\mathbb{Z}} 2^{\ell s_0 q_0}\Bigg[\sum_{\tau\in I_\ell}\sum_{\nu=1}^{N(\ell,\tau)} \mu(Q_\tau^{\ell,\nu}) \\
&\quad\times \inf_{y_\tau^{\ell,\nu}\in Q_\tau^{\ell,\nu}} \Bigg|\sum_{k\le\ell}\sum_{j\le k} 2^{-(\ell-j)\sigma}2^{-ks_0}2^{(k-j)s_2}\mathrm{F}^{(\ell,k,j)}(y_\tau^{\ell,\nu})\Bigg|^{p_0}\Bigg]^{q_0/p_0}\Bigg\}^{1/q_0} \\
&\lesssim \Bigg\{\sum_{\ell\in\mathbb{Z}} \Bigg\|\sum_{k\le\ell}\sum_{j\le k} 2^{-(\ell-k)(\sigma-s_0)}2^{-(k-j)(\sigma-s_2)}\mathrm{F}^{(\ell,k,j)}\Bigg\|_{L^{p_0}(\mathcal{X})}^{q_0}\Bigg\}^{1/q_0}.
\end{aligned}
$$

For each $\ell\in\mathbb{Z}$, applying Lemma 6.13 and the fact that

$$
\sum_{k\le\ell}\sum_{j\le k} 2^{-(\ell-k)(\sigma-s_0)(p_0\wedge 1)}2^{-(k-j)(\sigma-s_2)(p_0\wedge 1)} \lesssim 1,
$$

we obtain

$$
\begin{aligned}
&\Bigg\|\sum_{k\le\ell}\sum_{j\le k} 2^{-(\ell-k)(\sigma-s_0)}2^{-(k-j)(\sigma-s_2)}\mathrm{F}^{(\ell,k,j)}\Bigg\|_{L^{p_0}(\mathcal{X})} \\
&\quad= \Bigg\|\Bigg\{\sum_{k\le\ell}\sum_{j\le k} 2^{-(\ell-k)(\sigma-s_0)}2^{-(k-j)(\sigma-s_2)}\mathrm{F}^{(\ell,k,j)}\Bigg\}^{p_0}\Bigg\|_{L^1(\mathcal{X})}^{1/p_0} \\
&\quad\lesssim \Bigg\|\sum_{k\le\ell}\sum_{j\le k} 2^{-(\ell-k)(\sigma-s_0)(p_0\wedge 1)}2^{-(k-j)(\sigma-s_2)(p_0\wedge 1)}[\mathrm{F}^{(\ell,k,j)}]^{p_0}\Bigg\|_{L^1(\mathcal{X})}^{1/p_0} \\
&\quad\sim \Bigg[\sum_{k\le\ell}\sum_{j\le k} 2^{-(\ell-k)(\sigma-s_0)(p_0\wedge 1)}2^{-(k-j)(\sigma-s_2)(p_0\wedge 1)}\Bigg\|[\mathrm{F}^{(\ell,k,j)}]^{p_0}\Bigg\|_{L^1(\mathcal{X})}\Bigg]^{1/p_0}. \qquad (6.27)
\end{aligned}
$$

Notice that the $L^{p_0/r_0}(\mathcal{X})$-boundedness of $\mathcal{M}$ and Hölder's inequality imply that

$$
\begin{aligned}
\|[\mathrm{F}^{(\ell,k,j)}]^{p_0}\|_{L^1(\mathcal{X})} &\lesssim \Bigg\||2^{js_2}\mathrm{G}_j|^{r_0}\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')} |2^{ks_1}\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{r_0}\chi_{Q_{\tau'}^{k,\nu'}}\Bigg\|_{L^{p_0/r_0}(\mathcal{X})}^{p_0/r_0} \\
&\sim \Bigg\|2^{js_2}\mathrm{G}_j\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')} |2^{ks_1}\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|\chi_{Q_{\tau'}^{k,\nu'}}\Bigg\|_{L^{p_0}(\mathcal{X})}^{p_0} \\
&\lesssim \Bigg\|\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')} |2^{ks_1}\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|\chi_{Q_{\tau'}^{k,\nu'}}\Bigg\|_{L^{p_1}(\mathcal{X})}^{p_0}\|2^{js_2}\mathrm{G}_j\|_{L^{p_2}(\mathcal{X})}^{p_0}.
\end{aligned}
$$

By this, (6.27), Lemma 6.13, and the fact that

$$
\sum_{k\le\ell}\sum_{j\le k} 2^{-(\ell-k)(\sigma-s_0)}2^{-(k-j)(\sigma-s_2)} \lesssim 1,
$$

we obtain

$$
\begin{aligned}
&\Bigg\|\sum_{k\le\ell}\sum_{j\le k} 2^{-(\ell-k)(\sigma-s_0)}2^{-(k-j)(\sigma-s_2)}\mathrm{F}^{(\ell,k,j)}\Bigg\|_{L^{p_0}(\mathcal{X})}^{q_0} \\
&\qquad\lesssim \Bigg[\sum_{k\le\ell}\sum_{j\le k} 2^{-(\ell-k)(\sigma-s_0)(p_0\wedge 1)}2^{-(k-j)(\sigma-s_2)(p_0\wedge 1)} \\
&\qquad\quad\times \Bigg\|\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')} |2^{ks_1}\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|\chi_{Q_{\tau'}^{k,\nu'}}\Bigg\|_{L^{p_1}(\mathcal{X})}^{p_0}\|2^{js_2}\mathrm{G}_j\|_{L^{p_2}(\mathcal{X})}^{p_0}\Bigg]^{q_0/p_0}
\end{aligned}
$$

$$\lesssim \sum_{k\le \ell}\sum_{j\le k} 2^{-(\ell-k)(\sigma-s_0)(p_0\wedge 1)(\frac{q_0}{p_0}\wedge 1)}2^{-(k-j)(\sigma-s_2)(p_0\wedge 1)(\frac{q_0}{p_0}\wedge 1)}$$
$$\times \Big\|\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')}|2^{ks_1}\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|\chi_{Q_{\tau'}^{k,\nu'}}\Big\|_{L^{p_1}(\mathcal{X})}^{q_0}\|2^{js_2}\mathrm{G}_j\|_{L^{p_2}(\mathcal{X})}^{q_0}.$$

Then, Hölder's inequality further gives that

$$\mathrm{Y}_3 \lesssim \Big\{\sum_{\ell\in\mathbb{Z}}\sum_{k\le \ell}\sum_{j\le k} 2^{-(\ell-k)(\sigma-s_0)(p_0\wedge 1)(\frac{q_0}{p_0}\wedge 1)}2^{-(k-j)(\sigma-s_2)(p_0\wedge 1)(\frac{q_0}{p_0}\wedge 1)}$$
$$\times \Big\|\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')}|2^{ks_1}\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|\chi_{Q_{\tau'}^{k,\nu'}}\Big\|_{L^{p_1}(\mathcal{X})}^{q_1}\Big\}^{1/q_1}$$
$$\times \Big\{\sum_{\ell\in\mathbb{Z}}\sum_{k\le \ell}\sum_{j\le k} 2^{-(\ell-k)(\sigma-s_0)(p_0\wedge 1)(\frac{q_0}{p_0}\wedge 1)}$$
$$\times 2^{-(k-j)(\sigma-s_2)(p_0\wedge 1)(\frac{q_0}{p_0}\wedge 1)}\|2^{js_2}\mathrm{G}_j\|_{L^{p_2}(\mathcal{X})}^{q_2}\Big\}^{1/q_2}$$
$$\lesssim \Big\{\sum_{k\in\mathbb{Z}}\Big\|\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')}|2^{ks_1}\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|\chi_{Q_{\tau'}^{k,\nu'}}\Big\|_{L^{p_1}(\mathcal{X})}^{q_1}\Big\}^{1/q_1}$$
$$\times \Big\{\sum_{j\in\mathbb{Z}}\|2^{js_2}\mathrm{G}_j\|_{L^{p_2}(\mathcal{X})}^{q_2}\Big\}^{1/q_2}.$$

Obviously, the first term is equal to

$$\Big\{\sum_{k\in\mathbb{Z}}\Big[\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')}\mu(Q_{\tau'}^{k,\nu'})|2^{ks_1}\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{p_1}\Big]^{q_1/p_1}\Big\}^{1/q_1}\lesssim \|f\|_{\dot{B}^{s_1}_{p_1,q_1}(\mathcal{X})},$$

and, for the second term, the boundedness of $\mathcal{M}$ on $L^{p_2/r_2}(\mathcal{X})$ implies that

$$\Big\{\sum_{j\in\mathbb{Z}}\|2^{js_2}\mathrm{G}_j\|_{L^{p_2}(\mathcal{X})}^{q_2}\Big\}^{1/q_2}$$
$$\lesssim \Big\{\sum_{j\in\mathbb{Z}}\Big\|2^{js_2}\Big[\mathcal{M}\Big(\sum_{\tau''\in I_j}\sum_{\nu''=1}^{N(j,\tau'')}|\overline{D}_j(g)(y_{\tau''}^{j,\nu''})|^{r_2}\chi_{Q_{\tau''}^{j,\nu''}}\Big)\Big]^{1/r_2}\Big\|_{L^{p_2}(\mathcal{X})}^{q_2}\Big\}^{1/q_2}$$
$$\lesssim \Big\{\sum_{j\in\mathbb{Z}}\Big\|2^{js_2}\sum_{\tau''\in I_j}\sum_{\nu''=1}^{N(j,\tau'')}|\overline{D}_j(g)(y_{\tau''}^{j,\nu''})|\chi_{Q_{\tau''}^{j,\nu''}}\Big\|_{L^{p_2}(\mathcal{X})}^{q_2}\Big\}^{1/q_2}$$
$$\lesssim \|g\|_{\dot{B}^{s_2}_{p_2,q_2}(\mathcal{X})}.$$

Thus,

$$\mathrm{Y}_3 \lesssim \|f\|_{\dot{B}^{s_1}_{p_1,q_1}(\mathcal{X})}\|g\|_{\dot{B}^{s_2}_{p_2,q_2}(\mathcal{X})}.$$

$\boxed{\textit{Estimate for } \mathrm{Y}_5\textit{: } \ell\le k\le j.}$ Recall that, in the estimate of Z_5 in the proof of Theorem 6.14, we proved that, when $\ell\le k\le j$,

$$\mathrm{Y}^{(\ell,k,j)} \lesssim 2^{-(j-\ell)\sigma}2^{(j-k)n(1/r_2-1)}2^{(k-\ell)n(1/r_0-1)}$$
$$\times \Big\{\mathcal{M}\Big(|\mathrm{G}_j|^{r_0}\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')}|\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{r_0}\chi_{Q_{\tau'}^{k,\nu'}}\Big)(y_\tau^{\ell,\nu})\Big\}^{1/r_0},$$

where, for all $z \in \mathcal{X}$,

$$\mathrm{G}_j(z) := \Big[\mathcal{M}\Big(\sum_{\tau''\in I_j}\sum_{\nu''=1}^{N(j,\tau'')} |\overline{D}_j(g)(y_{\tau''}^{j,\nu''})|^{r_2}\chi_{Q_{\tau''}^{j,\nu''}}\Big)(z)\Big]^{1/r_2}.$$

Set, for all $z \in \mathcal{X}$,

$$\mathrm{H}^{(\ell,k,j)}(z) := \Big\{\mathcal{M}\Big(|2^{js_2}\mathrm{G}_j|^{r_0}\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')} |2^{ks_1}\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{r_0}\chi_{Q_{\tau'}^{k,\nu'}}\Big)(z)\Big\}^{1/r_0}.$$

Then

$$\mathrm{Y}^{(\ell,k,j)} \lesssim 2^{-(j-\ell)\sigma}2^{(j-k)n(1/r_2-1)}2^{(k-\ell)n(1/r_0-1)}2^{-ks_0}2^{(k-j)s_2}\mathrm{H}^{(\ell,k,j)}(y_\tau^{\ell,\nu}).$$

Also, notice that

$$\begin{aligned}&2^{\ell s_0}2^{-(j-\ell)\sigma}2^{(j-k)n(1/r_2-1)}2^{(k-\ell)n(1/r_0-1)}2^{-ks_0}2^{(k-j)s_2}\\&\qquad\qquad= 2^{-(k-\ell)[\sigma+s_0-n(1/r_0-1)]}2^{-(j-k)[\sigma+s_2-n(1/r_2-1)]}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{Y}_5 &:= \Big\{\sum_{\ell\in\mathbb{Z}}2^{\ell s_0q_0}\Big[\sum_{\tau\in I_\ell}\sum_{\nu=1}^{N(\ell,\tau)}\mu(Q_\tau^{\ell,\nu})\inf_{y_\tau^{\ell,\nu}\in Q_\tau^{\ell,\nu}}\Big|\sum_{k\ge\ell}\sum_{j\ge k}\mathrm{Y}^{(\ell,k,j)}(y_\tau^{\ell,\nu})\Big|^{p_0}\Big]^{q_0/p_0}\Big\}^{1/q_0}\\
&\lesssim \Big\{\sum_{\ell\in\mathbb{Z}}\Big[\sum_{\tau\in I_\ell}\sum_{\nu=1}^{N(\ell,\tau)}\mu(Q_\tau^{\ell,\nu})\inf_{y_\tau^{\ell,\nu}\in Q_\tau^{\ell,\nu}}\Big|\sum_{k\ge\ell}\sum_{j\ge k}2^{-(k-\ell)[\sigma+s_0-n(1/r_0-1)]}\\
&\qquad\times 2^{-(j-k)[\sigma+s_2-n(1/r_2-1)]}\mathrm{H}^{(\ell,k,j)}(y_\tau^{\ell,\nu})\Big|^{p_0}\Big]^{q_0/p_0}\Big\}^{1/q_0}\\
&\lesssim \Big\{\sum_{\ell\in\mathbb{Z}}\Big\|\sum_{k\ge\ell}\sum_{j\ge k}2^{-(k-\ell)[\sigma+s_0-n(1/r_0-1)]}2^{-(j-k)[\sigma+s_2-n(1/r_2-1)]}\mathrm{H}^{(\ell,k,j)}\Big\|_{L^{p_0}(\mathcal{X})}^{q_0}\Big\}^{1/q_0}.\end{aligned}$$

Applying Lemma 6.13 and the fact that

$$\sum_{k\ge\ell}\sum_{j\ge k}2^{-(k-\ell)[\sigma+s_0-n(1/r_0-1)]}2^{-(j-k)[\sigma+s_2-n(1/r_2-1)]} \lesssim 1,$$

we conclude that, for every $\ell \in \mathbb{Z}$,

$$\begin{aligned}\mathcal{K}_\ell &:= \Big\|\sum_{k\ge\ell}\sum_{j\ge k}2^{-(k-\ell)[\sigma+s_0-n(1/r_0-1)]}2^{-(j-k)[\sigma+s_2-n(1/r_2-1)]}\mathrm{H}^{(\ell,k,j)}\Big\|_{L^{p_0}(\mathcal{X})}^{q_0}\\
&\lesssim \Big\|\sum_{k\ge\ell}\sum_{j\ge k}2^{-(k-\ell)[\sigma+s_0-n(1/r_0-1)](p_0\wedge1)}2^{-(j-k)[\sigma+s_2-n(1/r_2-1)](p_0\wedge1)}[\mathrm{H}^{(\ell,k,j)}]^{p_0}\Big\|_{L^1(\mathcal{X})}^{q_0/p_0}\\
&\sim \Big[\sum_{k\ge\ell}\sum_{j\ge k}2^{-(k-\ell)[\sigma+s_0-n(1/r_0-1)](p_0\wedge1)}\\
&\qquad\times 2^{-(j-k)[\sigma+s_2-n(1/r_2-1)](p_0\wedge1)}\|[\mathrm{H}^{(\ell,k,j)}]^{p_0}\|_{L^1(\mathcal{X})}\Big]^{q_0/p_0}.\end{aligned}$$

The boundedness of $\mathcal{M}$ on $L^{p_0/r_0}(\mathcal{X})$ and Hölder's inequality imply that

$$\begin{aligned}&\|[\mathrm{H}^{(\ell,k,j)}]^{p_0}\|_{L^1(\mathcal{X})}\\
&\qquad= \Big\|\Big\{\mathcal{M}\Big(|2^{js_2}\mathrm{G}_j|^{r_0}\sum_{\tau'\in I_k}\sum_{\nu'=1}^{N(k,\tau')}|2^{ks_1}\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{r_0}\chi_{Q_{\tau'}^{k,\nu'}}\Big)\Big\}^{p_0/r_0}\Big\|_{L^1(\mathcal{X})}\end{aligned}$$

$$\lesssim \left\| |2^{js_2}\mathrm{G}_j|^{p_0} \sum_{\tau'\in I_k} \sum_{\nu'=1}^{N(k,\tau')} |2^{ks_1}\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|^{p_0}\chi_{Q_{\tau'}^{k,\nu'}} \right\|_{L^1(\mathcal{X})}$$

$$\lesssim \left\| \sum_{\tau'\in I_k} \sum_{\nu'=1}^{N(k,\tau')} |2^{ks_1}\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|\chi_{Q_{\tau'}^{k,\nu'}} \right\|_{L^{p_1}(\mathcal{X})}^{p_0/p_1} \|2^{js_2}\mathrm{G}_j\|_{L^{p_2}(\mathcal{X})}^{p_0/p_2}.$$

Combining this with Lemma 6.13, we further see that

$$\mathcal{K}_\ell \lesssim \sum_{k\geq\ell}\sum_{j\geq k} 2^{-(k-\ell)[\sigma+s_0-n(1/r_0-1)](p_0\wedge1)(\frac{q_0}{p_0}\wedge1)} 2^{-(j-k)[\sigma+s_2-n(1/r_2-1)](p_0\wedge1)(\frac{q_0}{p_0}\wedge1)}$$

$$\times \left\| \sum_{\tau'\in I_k} \sum_{\nu'=1}^{N(k,\tau')} |2^{ks_1}\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|\chi_{Q_{\tau'}^{k,\nu'}} \right\|_{L^{p_1}(\mathcal{X})}^{q_0/p_1} \|2^{js_2}\mathrm{G}_j\|_{L^{p_2}(\mathcal{X})}^{q_0/p_2}.$$

From this estimate and Hölder's inequality, we deduce that

$$\begin{aligned}
\mathrm{Y}_5 &\lesssim \left\{\sum_{\ell\in\mathbb{Z}}\mathcal{K}_\ell\right\}^{1/q_0}\\
&\lesssim \left\{\sum_{\ell\in\mathbb{Z}}\sum_{k\geq\ell}\sum_{j\geq k} 2^{-(k-\ell)[\sigma+s_0-n(1/r_0-1)](p_0\wedge1)(\frac{q_0}{p_0}\wedge1)} 2^{-(j-k)[\sigma+s_2-n(1/r_2-1)](p_0\wedge1)(\frac{q_0}{p_0}\wedge1)}\right.\\
&\quad\left.\times \left\| \sum_{\tau'\in I_k} \sum_{\nu'=1}^{N(k,\tau')} |2^{ks_1}\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|\chi_{Q_{\tau'}^{k,\nu'}} \right\|_{L^{p_1}(\mathcal{X})}^{q_1/p_1}\right\}^{1/q_1}\\
&\quad\times \left\{\sum_{\ell\in\mathbb{Z}}\sum_{k\geq\ell}\sum_{j\geq k} 2^{-(k-\ell)[\sigma+s_0-n(1/r_0-1)](p_0\wedge1)(\frac{q_0}{p_0}\wedge1)}\right.\\
&\quad\left.\times\, 2^{-(j-k)[\sigma+s_2-n(1/r_2-1)](p_0\wedge1)(\frac{q_0}{p_0}\wedge1)} \|2^{js_2}\mathrm{G}_j\|_{L^{p_2}(\mathcal{X})}^{q_2/p_2}\right\}^{1/q_2}\\
&\lesssim \left\{\sum_{k\in\mathbb{Z}} \left\| \sum_{\tau'\in I_k} \sum_{\nu'=1}^{N(k,\tau')} |2^{ks_1}\overline{D}_k(f)(y_{\tau'}^{k,\nu'})|\chi_{Q_{\tau'}^{k,\nu'}} \right\|_{L^{p_1}(\mathcal{X})}^{q_1/p_1}\right\}^{1/q_1}\\
&\quad\times \left\{\sum_{j\in\mathbb{Z}} \|2^{js_2}\mathrm{G}_j\|_{L^{p_2}(\mathcal{X})}^{q_2/p_2}\right\}^{1/q_2}\\
&\lesssim \|f\|_{\dot{B}^{s_1}_{p_1,q_1}(\mathcal{X})}\|g\|_{\dot{B}^{s_2}_{p_2,q_2}(\mathcal{X})}.
\end{aligned}$$

This proves the desired estimate for Y_5 and finishes the proof of Theorem 6.15. ■

7. Multilinear vector-valued $T1$ type theorems

Let us recall the Tb-theorem by Semmes [92]. For any given family of functions

$$\theta_k : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$$

such that, for all x, y, $y' \in \mathbb{R}^n$, $k \in \mathbb{Z}$ and some positive numbers A, δ_1 and δ_2,

$$|\theta_k(x,y)| \le \frac{1}{2^{-kn}} \frac{A}{(1+2^k|x-y|)^{n+\delta_2}} \tag{7.1}$$

and

$$|\theta_k(x,y) - \theta_k(x,y')| \le \frac{A}{2^{-kn}} \frac{|y-y'|^{\delta_1}}{2^{-k\delta_1}}, \tag{7.2}$$

we have the following *square function estimate*:

$$\Big\{ \sum_{k\in\mathbb{Z}} \|\Theta_k(f)\|^2_{L^2(\mathbb{R}^n)} \Big\}^{1/2} \le C\|f\|_{L^2(\mathbb{R}^n)} \tag{7.3}$$

with the positive constant C independent of f, provided that, for a certain para-accretive function b,

$$\Theta_k(b) = 0, \quad \forall k \in \mathbb{Z}, \tag{7.4}$$

where Θ_k is the operator associated with the kernel θ_k. By a duality argument, this theorem easily implies, in particular, the celebrated boundedness of the Cauchy integral [92].

The main goal of this section is to investigate the multilinear version of such a quadratic estimate (7.3). Precisely, by assuming certain decay, smoothness and cancelation conditions on the sequence of multilinear operators, $\{\Theta_k\}_{k\in\mathbb{Z}}$, we are interested in the behavior of

$$\Big\{ \sum_{k\in\mathbb{Z}} 2^{ksq} \|\Theta_k(\vec{f})\|^q_{L^p(\mathbb{R}^n)} \Big\}^{1/q} \quad \text{and} \quad \Big\| \Big\{ \sum_{k\in\mathbb{Z}} 2^{ksq} |\Theta_k(\vec{f})|^q \Big\}^{1/q} \Big\|_{L^p(\mathbb{R}^n)}$$

for suitable p, q and s and a vector of functions

$$\vec{f} := (f_1, \ldots, f_m) \in \prod_{j=1}^m \mathcal{Y}_j,$$

with each $\mathcal{Y}_j$ being a Lebesgue or Besov or Triebel–Lizorkin space. A particularly useful tool in the proof of the main results (Theorems 7.6, 7.8 and 7.9) in this section is the Calderón reproducing formula in Lemma 6.10.

7.1. A multilinear off-diagonal estimate. A multilinear version of the family of the operators $\{\Theta_k\}_{k\in\mathbb{Z}}$ appeared in Maldonado [77] when $\mathcal{X}$ is an Ahlfors 1-regular metric space. Here we adopt the following definition.

DEFINITION 7.1. Let $m \in \mathbb{N}$. Suppose that, for any $k \in \mathbb{Z}$,

$$\theta_k : \overbrace{\mathcal{X} \times \cdots \times \mathcal{X}}^{m+1 \text{ times}} \to \mathbb{C},$$

and moreover that there exist constants $A > 0$, $\delta_1 > 0$ and $\delta_2 > 0$ such that, for all $k \in \mathbb{Z}$ and points $x, y_1, \ldots, y_m$ belonging to $\mathcal{X}$,

$$|\theta_k(x, y_1, \ldots, y_m)| \le A \prod_{i=1}^{m} \frac{1}{V_{2^{-k}}(x) + V(x, y_i)} \left[\frac{2^{-k}}{2^{-k} + d(x, y_i)}\right]^{\delta_2} \tag{7.5}$$

and, for all $j_0 \in \{1, \ldots, m\}$ and $y'_{j_0} \in \mathcal{X}$ satisfying $d(y_{j_0}, y'_{j_0}) \le [2^{-j_0} + d(x, y_{j_0})]/2$,

$$\begin{aligned} &|\theta_k(x, y_1, \ldots, y_{j_0}, \ldots, y_m) - \theta_k(x, y_1, \ldots, y'_{j_0}, \ldots y_m)| \\ &\qquad \le A \left[\frac{d(y_{j_0}, y'_{j_0})}{2^{-j_0} + d(x, y_{j_0})}\right]^{\delta_1} \prod_{i=1}^{m} \frac{1}{V_{2^{-k}}(x) + V(x, y_i)} \left[\frac{2^{-k}}{2^{-k} + d(x, y_i)}\right]^{\delta_2}. \end{aligned} \tag{7.6}$$

In this case, write $\{\theta_k\}_{k\in\mathbb{Z}} \in \mathbf{Ker}(m, A, \delta_1, \delta_2)$. Denote by Θ_k the m-linear operator

$$\begin{aligned} &\Theta_k(f_1, \ldots, f_m)(x) \\ &\qquad := \int_{\mathcal{X}^m} \theta_k(x, y_1, \ldots, y_m) f_1(y_1) \cdots f_m(y_m)\, d\mu(y_1) \cdots d\mu(y_m), \quad \forall x \in \mathcal{X}, \end{aligned}$$

which is well defined if $f_i \in \bigcup_{1\le p\le\infty} L^p(\mathcal{X})$ for all $i \in \{1, \ldots, m\}$ in view of (7.5).

REMARK 7.2. For the special case $m = 1$ and $(\mathcal{X}, d, \mu) := (\mathbb{R}^n, |\cdot|, dx)$, conditions (7.5) and (7.6) turn out to be that, for all $x, y \in \mathbb{R}^n$,

$$|\theta_k(x, y)| \le A \frac{2^{-k\delta_2}}{(2^{-k} + |x - y|)^{n+\delta_2}} \tag{7.7}$$

and, when $|y - y'| \le (2^{-k} + |x - y|)/2$,

$$|\theta_k(x, y) - \theta_k(x, y')| \le A \frac{|y - y'|^{\delta_1}}{(2^{-k} + |x - y|)^{\delta_1}} \frac{2^{-k\delta_2}}{(2^{-k} + |x - y|)^{n+\delta_2}}. \tag{7.8}$$

These two conditions are equivalent to (7.1) and (7.2) in the following sense:

(i) (7.7) is exactly (7.1);
(ii) if θ_k satisfies (7.7) and (7.8), then it satisfies (7.2);
(iii) if θ_k satisfies (7.1) and (7.2), then (7.8) holds true but with a new exponent $\delta'_2 \in (0, \delta_2)$.

Obviously (i) holds. To prove (ii), when $|y - y'| \le (2^{-k} + |x - y|)/2$, we see that (7.8) directly implies (7.2) and, when $|y-y'| > (2^{-k}+|x-y|)/2$, we have $|y - y'|^{\delta_2}/2^{-k\delta_2} > 2^{\delta_2}$, which combined with (7.7) implies that

$$|\theta_k(x, y) - \theta_k(x, y')| \le \frac{2}{2^{-kn}} \le \frac{2^{\delta_2+1}}{2^{-kn}} \frac{|y - y'|^{\delta_2}}{2^{-k\delta_2}},$$

we therefore know that (7.2) holds for all x, y, and y'. Finally, (iii) holds if we take the geometric mean between (7.1) and (7.2).

Basic examples of kernels $\{\theta_k\}_{k\in\mathbb{Z}}$ are given by approximations of the identity. For instance, taking $\{S_k\}_{k\in\mathbb{Z}}$ to be an $(\epsilon_1, \epsilon_2, \epsilon_3)$-ATI for some $\epsilon_1 \in (0, 1]$, $\epsilon_2 > 0$, and $\epsilon_3 > 0$, we consider

$$\theta_k(x, y_1, \ldots, y_m) := D_k(x, y_1) S_k(x, y_2) \cdots S_k(x, y_m), \quad \forall x,\, y_1,\, \ldots,\, y_m \in \mathcal{X},$$

where $D_k := S_{k+1} - S_k$ for all $k \in \mathbb{Z}$. It is easy to show that such θ_k satisfies (7.5) and (7.6) with $\delta_1 = \epsilon_1$ and $\delta_2 = \epsilon_2$; moreover, for all $x, y_2, \ldots, y_m \in \mathcal{X}$ and $k \in \mathbb{Z}$,

$$\int_{\mathcal{X}} \theta_k(x, y_1, \ldots, y_m)\, d\mu(y_1) = 0.$$

In what follows, we use the aforementioned notation $\mathcal{K}(t;\, \epsilon, x, y)$: given $\epsilon \in (0, \infty)$, $t > 0$, we let, for all $x, y \in \mathcal{X}$,

$$\mathcal{K}(t;\, \epsilon, x, y) := \frac{1}{V_t(x) + V_t(y) + V(x, y)} \left[\frac{t}{t + d(x, y)}\right]^{\epsilon}.$$

Regarding the family of functions $\{\theta_k\}_{k\in\mathbb{Z}} \in \mathbf{Ker}(m, A, \delta_1, \delta_2)$, the following multilinear version off-diagonal estimates hold true.

Lemma 7.3. *Let $m \in \mathbb{Z}$, $\epsilon_1 \in (0, 1]$, $\epsilon_2 > 0$ and $\{\theta_k\}_{k\in\mathbb{Z}} \in \mathbf{Ker}(m, A, \delta_1, \delta_2)$ for some positive constants A, δ_1 and δ_2. Moreover, assume that each θ_k satisfies the cancelation condition with respect to the y_1-variable, namely, for all $k \in \mathbb{Z}$ and $x, y_2, \ldots, y_m \in \mathcal{X}$,*

$$\int_{\mathcal{X}} \theta_k(x, y_1, \ldots, y_m)\, d\mu(y_1) = 0. \tag{7.9}$$

Suppose that there exists a positive constant B such that the functions $D_j : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$, $j \in \mathbb{Z}$, satisfy, for all $x, x', y \in \mathcal{X}$,

(i) $|D_j(x, y)| \le B \frac{1}{V_{2^{-j}}(x) + V_{2^{-j}}(y) + V(x, y)} \left[\frac{2^{-j}}{2^{-j} + d(x, y)}\right]^{\epsilon_2}$;

(ii) *when $d(x, x') \le [2^{-j} + d(x, y)]/2$,*

$$|D_j(x, y) - D_j(x', y)| \le B \left[\frac{d(x, x')}{2^{-j} + d(x, y)}\right]^{\epsilon_1} \frac{1}{V_{2^{-j}}(x) + V_{2^{-j}}(y) + V(x, y)} \left[\frac{2^{-j}}{2^{-j} + d(x, y)}\right]^{\epsilon_2};$$

(iii) $\int_{\mathcal{X}} D_j(w, y)\, d\mu(w) = 0$.

Then, for any given $\epsilon_1' \in (0, \epsilon_1 \wedge \epsilon_2 \wedge \delta_1 \wedge \delta_2)$, there exists a positive constant C (depending on ϵ_1' and the doubling constant C_1) such that, for all $k, j \in \mathbb{Z}$ and points $x, u, y_2, \ldots, y_m \in \mathcal{X}$,

$$\int_{\mathcal{X}} \theta_k(x, y_1, \ldots, y_m) D_j(y_1, u)\, d\mu(u) \le CAB 2^{-|k-j|\epsilon_1'} \mathcal{K}(2^{-(k\wedge j)};\, \epsilon_2 \wedge \delta_2, x, u) \prod_{i=2}^{m} \mathcal{K}(2^{-k};\, \delta_2, x, y_i).$$

Proof. The reader may easily find that the proof is essentially given in [60, Lemma 3.2] (see also Lemma 8.3 below), so the details are omitted. ∎

Remark 7.4. In Lemma 7.3, instead of assuming that $\{\theta_k\}_{k\in\mathbb{Z}}$ satisfies (7.6) for all $j_0 \in \{1, \ldots, m\}$, it suffices to require that $\{\theta_k\}_{k\in\mathbb{Z}}$ satisfies (7.6) for $j_0 = 1$.

Lemma 7.5. *Let $m \in \mathbb{N}$, $\epsilon_1 \in (0, 1]$, $\epsilon_2 > 0$ and $\{\theta_k\}_{k\in\mathbb{Z}} \in \mathbf{Ker}(m, A, \delta_1, \delta_2)$ for some positive numbers A, δ_1 and δ_2. Moreover, assume that, for all $k \in \mathbb{Z}$, $i \in \{1, \ldots, m\}$ and $x, y_1, \ldots, y_m \in \mathcal{X}$,*

$$\int_{\mathcal{X}} \theta_k(x, y_1, \ldots, y_i, \ldots, y_m)\, d\mu(y_i) = 0. \tag{7.10}$$

Suppose that, for every fixed $i \in \{1, \ldots, m\}$, there exist positive constants $\{B_i\}_{i=1}^m$ such that the functions $\{D_{k_i}\}_{k_i \in \mathbb{Z}}$ satisfy, for all $x, x', y \in \mathcal{X}$ and $k_i \in \mathbb{Z}$,

(i) $|D_{k_i}(x, y)| \le B_i \frac{1}{V_{2^{-k_i}}(x)+V_{2^{-k_i}}(y)+V(x,y)} \left[\frac{2^{-k_i}}{2^{-k_i}+d(x,y)}\right]^{\epsilon_2}$;

(ii) *when* $d(x, x') \le [2^{-k-i} + d(x, y)]/2$,

$$\begin{aligned}|D_{k_i}(x, y) - D_{k_i}(x', y)|\\ \le B_i \left[\frac{d(x, x')}{2^{-k_i} + d(x, y)}\right]^{\epsilon_1} \frac{1}{V_{2^{-k_i}}(x) + V_{2^{-k_i}}(y) + V(x, y)} \left[\frac{2^{-k_i}}{2^{-k_i} + d(x, y)}\right]^{\epsilon_2};\end{aligned}$$

(iii) $\int_{\mathcal{X}} D_{k_i}(w, y)\, d\mu(w) = 0$.

Then, for any given $\sigma \in (0, \epsilon_1 \wedge \epsilon_2 \wedge \delta_1 \wedge \delta_2)$, there exists a positive constant C, depending only on the doubling constant C_1 and σ, such that, for all points $y_1, \ldots, y_m \in \mathcal{X}$,

$$\begin{aligned}&|\Theta_k(D_{k_1}(\cdot, y_1), D_{k_2}(\cdot, y_2), \ldots, D_{k_m}(\cdot, y_m))(x)|\\ &\quad= \left|\int_{\mathcal{X}^m} \theta_k(x, z_1, \ldots, z_m) D_{k_1}(z_1, y_1) \cdots D_{k_m}(z_m, y_m)\, d\mu(z_1) \cdots d\mu(z_m)\right|\\ &\quad\le CA^m \prod_{i=1}^m B_i 2^{-|k-k_i|\sigma} \mathcal{K}(2^{-(k\wedge k_i)};\, \epsilon_2 \wedge \delta_2, x, y_i).\end{aligned}$$

Proof. Fix $k, k_1, \ldots, k_m \in \mathbb{Z}$ and $x, y_1, \ldots, y_m \in \mathcal{X}$. Consider the function

$$\theta_{k,\, y_1}^{(m-1)}(x, z_2, \ldots, z_m) := \int_{\mathcal{X}} \theta_k(x, z_1, \ldots, z_m) D_{k_1}(z_1, y_1)\, d\mu(z_1).$$

For all $j \in \{2, \ldots, m\}$, by (7.10), we have

$$\int_{\mathcal{X}} \theta_{k,\, y_1}^{(m-1)}(x, z_2, \ldots, z_m)\, d\mu(z_j) = 0.$$

Since $\{\theta_k\}_{k\in\mathbb{Z}} \in \mathbf{Ker}(m, A, \delta_1, \delta_2)$ and D_{k_1} satisfies (i) through (iii), we apply Lemma 7.3 to conclude that, for all $z_2, \ldots, z_m \in \mathcal{X}$ and $\sigma \in (0, \epsilon_1 \wedge \epsilon_2 \wedge \delta_1 \wedge \delta_2)$,

$$\begin{aligned}&|\theta_{k,y_1}^{(m-1)}(x, z_2, \ldots, z_m)|\\ &\qquad\lesssim AB_1 2^{-|k-k_1|\sigma} \mathcal{K}(2^{-(k\wedge k_1)}; \epsilon_2 \wedge \delta_2, x, y_1) \prod_{i=2}^m \mathcal{K}(2^{-k}; \delta_2, x, z_i).\end{aligned} \tag{7.11}$$

Now fix $j \in \{2, \ldots, m\}$. We prove that, for all $z_2, \ldots, z_m, z_j' \in \mathcal{X}$ satisfying $d(z_j, z_j') \le [2^{-k} + d(x, z_j)]/2$ and for all $\theta_1 \in (0, 1)$ and $\sigma \in (0, \epsilon_1 \wedge \epsilon_2 \wedge \delta_1 \wedge \delta_2)$,

$$\begin{aligned}&|\theta_{k,\, y_1}^{(m-1)}(x, z_2, \ldots, z_j, \ldots, z_m) - \theta_{k,\, y_1}^{(m-1)}(x, z_2, \ldots, z_j', \ldots, z_m)|\\ &\qquad\lesssim AB_1 2^{-|k-k_1|\sigma(1-\theta_1)} \mathcal{K}(2^{-(k\wedge k_1)};\, \delta_2 \wedge \epsilon_2,\, x, y_1)\\ &\qquad\quad\times \left[\frac{d(z_j, z_j')}{2^{-k} + d(x, z_j)}\right]^{\delta_2\theta_1} \prod_{i=2}^m \mathcal{K}(2^{-k}; \delta_2, x, z_i).\end{aligned} \tag{7.12}$$

Indeed, suppose for the moment that we have proved that, for all $z_2, \ldots, z_m, z_j' \in \mathcal{X}$ satisfying $d(z_j, z_j') \le [2^{-k} + d(x, z_j)]/2$,

$$
\begin{aligned}
&|\theta^{(m-1)}_{k,y_1}(x, z_2, \dots, z_j, \dots, z_m) - \theta^{(m-1)}_{k,y_1}(x, z_2, \dots, z'_j, \dots, z_m)| \\
&\qquad \lesssim AB_1 \mathcal{K}(2^{-(k\wedge k_1)}; \delta_2 \wedge \epsilon_2, x, y_1) \left[\frac{d(z_j, z'_j)}{2^{-k} + d(x, z_j)}\right]^{\delta_2} \prod_{i=2}^{m} \mathcal{K}(2^{-k}; \delta_2, x, z_i); \qquad (7.13)
\end{aligned}
$$

then we obtain (7.12) by taking the geometric mean between (7.13) and using the following estimate: when $d(z_j, z'_j) \le [2^{-k} + d(x, z_j)]/2$,

$$
\begin{aligned}
|\theta^{(m-1)}_{k,\, y_1}&(x, z_2, \dots, z_j, \dots, z_m) - \theta^{(m-1)}_{k,\, y_1}(x, z_2, \dots, z'_j, \dots, z_m)| \\
&\lesssim AB_1 2^{-|k-k_1|\sigma} \mathcal{K}(2^{-(k\wedge k_1)}; \delta_2 \wedge \epsilon_2,\, x, y_1) \prod_{2\le i\le m, i\ne j} \mathcal{K}(2^{-k}; \delta_2,\, x, z_i) \\
&\quad \times [\mathcal{K}(2^{-k}; \delta_2, x, z_j) + \mathcal{K}(2^{-k}; \delta_2,\, x, z'_j)] \\
&\sim AB_1 2^{-|k-k_1|\sigma} \mathcal{K}(2^{-(k\wedge k_1)}; \delta_2 \wedge \epsilon_2,\, x, y_1) \prod_{2\le i\le m} \mathcal{K}(2^{-k}; \delta_2, x, z_i),
\end{aligned}
$$

where we used (7.11) and the fact that, when $d(z_j, z'_j) \le [2^{-k} + d(x, z_j)]/2$,

$$\mathcal{K}(2^{-k};\, \delta_2,\, x, z_j) \sim \mathcal{K}(2^{-k};\, \delta_2,\, x, z'_j).$$

Now we show (7.13) by considering $k \le k_1$ and $k > k_1$ separately.

If $k \le k_1$ and $d(z_j, z'_j) \le [2^{-k} + d(x, z_j)]/2$, then

$$
\begin{aligned}
&|\theta^{(m-1)}_{k,\, y_1}(x, z_2, \dots, z_j, \dots, z_m) - \theta^{(m-1)}_{k,\, y_1}(x, z_2, \dots, z'_j, \dots, z_m)| \\
&\quad = \left| \int_{\mathcal{X}} [\theta_k(x, z_1, \dots, z_j, \dots, z_m) - \theta_k(x, z_1, \dots, z'_j, \dots z_m)] D_{k_1}(z_1, y_1)\, d\mu(z_1) \right| \\
&\quad \lesssim A \left[\frac{d(z_j, z'_j)}{2^{-k} + d(x, z_j)}\right]^{\delta_2} \prod_{i=2}^{m} \mathcal{K}(2^{-k};\, \delta_2,\, x, z_i) \\
&\qquad \times \int_{\mathcal{X}} \mathcal{K}(2^{-k};\, \delta_2, x, z_1) |D_{k_1}(z_1, y_1)|\, d\mu(z_1). \qquad (7.14)
\end{aligned}
$$

To estimate the last integral above, we write

$$
\begin{aligned}
\int_{\mathcal{X}} &\mathcal{K}(2^{-k}; \delta_2, x, z_1) |D_{k_1}(z_1, y_1)|\, d\mu(z_1) \\
&= \int_{d(x,y_1)\le 2[2^{-k}+d(x,z_1)]} \mathcal{K}(2^{-k};\, \delta_2,\, x, z_1) |D_{k_1}(z_1, y_1)|\, d\mu(z_1) \\
&\quad + \int_{d(x,y_1)> 2[2^{-k}+d(x,z_1)]} \cdots \\
&=: \mathcal{J}_1 + \mathcal{J}_2.
\end{aligned}
$$

When $d(x, y_1) \le 2[2^{-k} + d(x, z_1)]$, we have

$$\mathcal{K}(2^{-k};\, \delta_2,\, x, z_1) \lesssim \mathcal{K}(2^{-k};\, \delta_2,\, x, y_1)$$

and hence

$$\mathcal{J}_1 \lesssim \mathcal{K}(2^{-k}; \delta_2, x, y_1) \int_{d(x,y_1)\le 2[2^{-k}+d(x,z_1)]} |D_{k_1}(z_1, y_1)|\, d\mu(z_1) \lesssim B_1 \mathcal{K}(2^{-k}; \delta_2, x, y_1).$$

When $d(x, y_1) > 2[2^{-k} + d(x, z_1)]$, we have

$$\begin{aligned}
d(z_1, y_1) &\geq d(x, y_1) + 2^{-k} - [d(z_1, x) + 2^{-k}] \\
&> d(x, y_1) + 2^{-k} - d(x, y_1)/2 \\
&> [d(x, y_1) + 2^{-k}]/2,
\end{aligned}$$

which, together with the fact $k \leq k_1$ and the size condition of D_{k_1}, implies that

$$\begin{aligned}
|D_{k_1}(z_1, y_1)| &\leq B_1 \frac{1}{V(z_1, y_1)} \left[\frac{1}{1 + 2^{k_1} d(z_1, y_1)}\right]^{\epsilon_2} \\
&\leq B_1 \frac{1}{\mu(B(y_1, 2^{-k} + d(x, y_1)))} \left[\frac{1}{1 + 2^{k} d(x, y_1)}\right]^{\epsilon_2} \\
&\sim B_1 \mathcal{K}(2^{-k}; \epsilon_2, x, y_1)
\end{aligned}$$

and, furthermore,

$$\begin{aligned}
\mathcal{J}_2 &\lesssim B_1 \mathcal{K}(2^{-k}; \epsilon_2, x, y_1) \int_{d(x, y_1) > 2[2^{-k} + d(x, z_1)]} \mathcal{K}(2^{-k}; \delta_2, x, z_1)\, d\mu(z_1) \\
&\lesssim B_1 \mathcal{K}(2^{-k}; \epsilon_2, x, y_1).
\end{aligned}$$

Combining the estimates for $\mathcal{J}_1$ and $\mathcal{J}_2$, we see that

$$\int_{\mathcal{X}} \mathcal{K}(2^{-k};\, \delta_2,\, x, z_1) |D_{k_1}(z_1, y_1)|\, d\mu(z_1) \lesssim B_1 \mathcal{K}(2^{-k}; \epsilon_2 \wedge \delta_2, x, y_1).$$

Inserting this into (7.14), we conclude that, when $k \leq k_1$ and $d(z_j, z_j') \leq [2^{-k} + d(x, z_j)]/2$,

$$\begin{aligned}
&|\theta_{k,\, y_1}^{(m-1)}(x, z_2, \ldots, z_j, \ldots, z_m) - \theta_{k,\, y_1}^{(m-1)}(x, z_2, \ldots, z_j', \ldots, z_m)| \\
&\quad \lesssim A B_1 \left[\frac{d(z_j, z_j')}{2^{-k} + d(x, z_j)}\right]^{\delta_2} \mathcal{K}(2^{-(k \wedge k_1)}; \epsilon_2 \wedge \delta_2, x, y_1) \prod_{i=2}^{m} \mathcal{K}(2^{-k};\, \delta_2,\, x, z_i).
\end{aligned}$$

As for the case $k > k_1$, by (7.6), we see that, when $d(z_j, z_j') \leq [2^{-k} + d(x, z_j)]/2$,

$$\begin{aligned}
&|\theta_{k,\, y_1}^{(m-1)}(x, z_2, \ldots, z_j, \ldots, z_m) - \theta_{k,\, y_1}^{(m-1)}(x, z_2, \ldots, z_j', \ldots, z_m)| \\
&\qquad = \left| \int_{\mathcal{X}} [\theta_k(x, z_1, \ldots, z_j, \ldots, z_m) - \theta_k(x, z_1, \ldots, z_j', \ldots, z_m)] \right. \\
&\qquad\qquad \left. \times [D_{k_1}(z_1, y_1) - D_{k_1}(x, y_1)]\, d\mu(z_1) \right| \\
&\qquad \leq A \left[\frac{d(z_j, z_j')}{2^{-k} + d(x, z_j)}\right]^{\delta_2} \prod_{i=2}^{m} \mathcal{K}(2^{-k};\, \delta_2,\, x, z_i) \\
&\qquad\qquad \times \int_{\mathcal{X}} \mathcal{K}(2^{-k};\, \delta_2,\, x, z_1) |D_{k_1}(z_1, y_1) - D_{k_1}(x, y_1)|\, d\mu(z_1). \qquad (7.15)
\end{aligned}$$

Write

$$\begin{aligned}
&\int_{\mathcal{X}} \mathcal{K}(2^{-k}; \delta_2,\, x, z_1) |D_{k_1}(z_1, y_1) - D_{k_1}(x, y_1)|\, d\mu(z_1) \\
&\qquad \leq \int_{d(z_1, x) \leq [2^{-k_1} + d(x, y_1)]/2} \mathcal{K}(2^{-k}; \delta_2, x, z_1) |D_{k_1}(z_1, y_1) - D_{k_1}(x, y_1)|\, d\mu(z_1)
\end{aligned}$$

$$
\begin{aligned}
&+\int_{d(z_1,x)>[2^{-k_1}+d(x,y_1)]/2} \mathcal{K}(2^{-k};\delta_2,x,z_1)|D_{k_1}(z_1,y_1)|\,d\mu(z_1)\\
&+|D_{k_1}(x,y_1)|\int_{d(z_1,x)>[2^{-k_1}+d(x,y_1)]/2} \mathcal{K}(2^{-k};\delta_2,x,z_1)\,d\mu(z_1)\\
&=: \mathrm{Z}_1+\mathrm{Z}_2+\mathrm{Z}_3.
\end{aligned}
$$

To estimate Z_1, for every z_1 satisfying $d(z_1,x)\le[2^{-k_1}+d(x,y_1)]/2$, we have

$$
|D_{k_1}(z_1,y_1)|\lesssim B_1\mathcal{K}(2^{-k_1};\epsilon_2,z_1,y_1)\sim B_1\mathcal{K}(2^{-k_1};\epsilon_2,x,y_1),
$$

which, together with $|D_{k_1}(x,y_1)|\lesssim B_1\mathcal{K}(2^{-k_1};\epsilon_2,x,y_1)$, implies that

$$
\begin{aligned}
\mathrm{Z}_1&\le\int_{d(z_1,x)\le[2^{-k_1}+d(x,y_1)]/2}\mathcal{K}(2^{-k};\delta_2,\,x,z_1)[|D_{k_1}(z_1,y_1)|+|D_{k_1}(x,y_1)|]\,d\mu(z_1)\\
&\lesssim B_1\mathcal{K}(2^{-k_1};\epsilon_2,\,x,y_1)\int_{d(z_1,x)\le[2^{-k_1}+d(x,y_1)]/2}\mathcal{K}(2^{-k};\delta_2,\,x,z_1)\,d\mu(z_1)\\
&\lesssim B_1\mathcal{K}(2^{-k_1};\epsilon_2,\,x,y_1).
\end{aligned}
$$

If $d(z_1,x)>[2^{-k_1}+d(x,y_1)]/2$, then invoking the fact that $k>k_1$, we obtain

$$
\mathcal{K}(2^{-k};\delta_2,x,z_1)\lesssim\frac{1}{V(x,z_1)}\left[\frac{2^{-k}}{d(x,z_1)}\right]^{\delta_2}\lesssim\mathcal{K}(2^{-k_1};\delta_2,x,y_1)
$$

and hence

$$
\mathrm{Z}_2\lesssim\mathcal{K}(2^{-k_1};\delta_2,x,y_1)\int_{\mathcal{X}}|D_{k_1}(z_1,y_1)|\,d\mu(z_1)\lesssim B_1\mathcal{K}(2^{-k_1};\delta_2,x,y_1).
$$

Also, by the size condition of D_{k_1}, we see that

$$
\mathrm{Z}_3\le|D_{k_1}(x,y_1)|\int_{\mathcal{X}}\mathcal{K}(2^{-k};\delta_2,x,z_1)\,d\mu(z_1)\lesssim|D_{k_1}(x,y_1)|\lesssim\mathcal{K}(2^{-k_1};\epsilon_2,x,y_1).
$$

From the estimates of Z_1, Z_2, Z_3 and (7.15), it follows that (7.13) also holds when $k>k_1$. Thus, we obtain (7.13), and hence (7.12) holds.

Summarizing, all these imply that, for all $\theta_1\in(0,1)$ and $\sigma\in(0,\epsilon_1\wedge\epsilon_2\wedge\delta_1\wedge\delta_2)$,

$$
\{\theta^{(m-1)}_{k,y_1}\}_{k\in\mathbb{Z}}\in\mathbf{Ker}(m-1,A^{(m-1)},\delta_1,\delta_2\theta_1),
$$

where

$$
A^{(m-1)}:=CAB_12^{-|k-k_1|\sigma(1-\theta_1)}\mathcal{K}(2^{-(k\wedge k_1)};\delta_1\wedge\epsilon_1,x,y_1)
$$

and C is a positive constant depending only on σ, θ_1 and C_1.

Likewise, for $z_3,\dots,z_m\in\mathcal{X}$, we define

$$
\theta^{(m-2)}_{k,\,y_1,\,y_2}(z_3,\dots,z_m):=\int_{\mathcal{X}^2}\theta_k(x,z_1,\dots,z_m)D_{k_1}(z_1,y_1)D_{k_2}(z_2,y_2)\,d\mu(z_1)\,d\mu(z_2).
$$

Since

$$
\theta^{(m-2)}_{k,y_1,y_2}(z_3,\dots,z_m)=\int_{\mathcal{X}}\theta^{(m-1)}_{k,\,y_1}(x,z_2,\dots,z_m)D_{k_2}(z_2,y_2)\,d\mu(z_2),
$$

an argument similar to the above gives, for all $\theta_1,\theta_2\in(0,1)$ and $\sigma\in(0,\epsilon_1\wedge\epsilon_2\wedge\delta_1\wedge(\delta_2\theta_1))$,

$$
\{\theta^{(m-2)}_{k,\,y_1,\,y_2}\}_{k\in\mathbb{Z}}\in\mathbf{Ker}(m-2,A^{(m-2)},\delta_1,\delta_2\theta_1\theta_2),
$$

where

$$
\begin{aligned}
A^{(m-2)}:=\,&CA^2B_1B_22^{-|k-k_1|\sigma(1-\theta_1)}2^{-|k-k_2|\sigma(1-\theta_1)(1-\theta_2)}\\
&\times\mathcal{K}(2^{-(k\wedge k_1)};\delta_2\wedge\epsilon_2,x,y_1)\mathcal{K}(2^{-(k\wedge k_2)};\epsilon_2\wedge(\delta_2\theta_1),x,y_2)
\end{aligned}
$$

and C is a positive constant depending only on σ, θ_1, θ_2 and C_1.

Iterating this process $m-1$ times, we see that, for all $\theta_1,\dots,\theta_{m-1}\in(0,1)$ and $\delta\in(0,\epsilon_1\wedge\epsilon_2\wedge\delta_1\wedge(\delta_2\prod_{i=1}^{m-2}\theta_i))$,

$$\begin{aligned}
&\theta^{(1)}_{k,\,y_1,\,y_2,\dots,y_{m-1}}(z_m)\\
&\quad:=\int_{\mathcal{X}^{m-1}}\theta_k(x,z_1,\dots,z_m)D_{k_1}(z_1,y_1)\cdots D_{k_{m-1}}(z_{m-1},y_{m-1})\,d\mu(z_1)\cdots d\mu(z_{m-1})
\end{aligned}$$

belongs to $\mathbf{Ker}(1,A^{(1)},\delta_1,\delta_2\prod_{i=1}^{m-1}\theta_i)$, where

$$A^{(1)}:=CA^{m-1}\prod_{i=1}^{m-1}B_i2^{-|k-k_i|\prod_{1\le\ell\le i}(1-\theta_\ell)}\mathcal{K}(2^{-(k\wedge k_i)};\epsilon_2\wedge(\delta_2\theta_1\cdots\theta_{m-2}),\,x,y_i)$$

and C is a positive constant depending only on σ, θ_1, $\dots$, θ_{m-1} and C_1.

Finally, since

$$\Theta_k(D_{k_1}(\cdot,y_1),\dots,D_{k_m}(\cdot,y_m))(x)=\int_{\mathcal{X}}\theta^{(1)}_{k,y_1,\,y_2,\dots,y_{m-1}}(z_m)D_{k_m}(z_m,y_m)\,d\mu(z_m),$$

we apply (8.5) to find that, for all $\theta_1,\dots,\theta_m\in(0,1)$ and $\delta\in(0,\epsilon_1\wedge\delta_1\wedge(\delta_2\prod_{i=1}^{m-1}\theta_i))$,

$$\begin{aligned}
&|\Theta_k(D_{k_1}(\cdot,y_1),\dots,D_{k_m}(\cdot,y_m))(x)|\\
&\qquad\lesssim A^m\prod_{i=1}^{m}B_i2^{-|k-k_i|\sigma\prod_{1\le\ell\le i}(1-\theta_\ell)}\mathcal{K}(2^{-(k\wedge k_i)};\epsilon_2\wedge(\delta_2\theta_1\cdots\theta_{m-1}),\,x,y_i).
\end{aligned}$$

From this and the arbitrariness of $\theta_i\in(0,1)$, $i\in\{1,\dots,m\}$, we deduce the desired conclusion of Lemma 7.5. ■

7.2. Quadratic $T1$ type theorems on Lebesgue spaces. The main goal of this subsection is to study the multilinear version of the square function estimate on products of Lebesgue and Besov (or Triebel–Lizorkin) spaces.

THEOREM 7.6. *Let $\{\theta_k\}_{k\in\mathbb{Z}}\in\mathbf{Ker}(m,A,\delta_1,\delta_2)$ for some $\delta_1>0$, $\delta_2>0$ and $A>0$. Assume that, for all $k\in\mathbb{Z}$ and $x,y_2,\dots,y_m\in\mathcal{X}$,*

$$\int_{\mathcal{X}}\theta_k(x,y_1,\dots,y_m)\,d\mu(y_1)=0. \tag{7.16}$$

Let $1\le p,p_1,\dots,p_m\le\infty$ be such that $1/p=\sum_{i=1}^m1/p_i$. For $p_1,s\in(-\epsilon,\epsilon)$ with

$$\epsilon\in(0,1\wedge\delta_1\wedge\delta_2)$$

and $q\in(0,\infty]$ as in Definition 6.9, the Besov space $\dot B^s_{p_1,q}(\mathcal{X})$ is defined as a subspace of a distribution space $(\mathring{\mathcal{G}}^\epsilon_0(\beta,\gamma))'$ with β,γ satisfying (6.18). Then there exists a positive constant C such that, for all functions $f_1\in\dot B^s_{p_1,q}(\mathcal{X})$ and $f_i\in L^{p_i}(\mathcal{X})$, $i\in\{2,\dots,m\}$,

$$\Big\{\sum_{k\in\mathbb{Z}}2^{ksq}\|\Theta_k(f_1,\dots,f_m)\|^q_{L^p(\mathcal{X})}\Big\}^{1/q}\le CA\|f_1\|_{\dot B^s_{p_1,q}(\mathcal{X})}\prod_{i=2}^m\|f_i\|_{L^{p_i}(\mathcal{X})}, \tag{7.17}$$

where $C:=C(\delta_1,\delta_2,\epsilon,s,q,p,p_1,\dots,p_m,\mathcal{X})>0$.

A key tool to be used in the proof of Theorem 7.6 is the following continuous homogeneous Calderón reproducing formula; see [60, p. 79, Theorem 3.13].

LEMMA 7.7. *Let $\epsilon_1 \in (0,1]$, $\epsilon_2 > 0$, $\epsilon_3 > 0$ and $\epsilon \in (0, \epsilon_1 \wedge \epsilon_2)$. Let $\{S_k\}_{k\in\mathbb{Z}}$ be an $(\epsilon_1,\epsilon_2,\epsilon_3)$-ATI. Set $D_k := S_k - S_{k-1}$ for $k \in \mathbb{Z}$. Then there exist linear operators $\{\widetilde{D}_k\}_{k\in\mathbb{Z}}$ and $\{\overline{D}_k\}_{k\in\mathbb{Z}}$ such that, for all $f \in (\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))'$ (or $\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)$) with $\beta,\gamma \in (0,\epsilon)$,*

$$f = \sum_{k\in\mathbb{Z}} \widetilde{D}_k D_k(f) = \sum_{k\in\mathbb{Z}} D_k \overline{D}_k(f),$$

where both series converge in $(\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))'$ (or $\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)$). Moreover, the kernels of the operators $\{\widetilde{D}_k\}_{k\in\mathbb{Z}}$ satisfy, for all $x, y \in \mathcal{X}$ and $k \in \mathbb{Z}$,

(a) $|\widetilde{D}_k(x,y)| \le C \frac{1}{V_{2^{-k}}(x)+V(x,y)} \left[\frac{2^{-k}}{2^{-k}+d(x,y)}\right]^{\epsilon'}$,

(b) *if* $d(x,x') \le [2^{-k} + d(x,y)]/2$,

$$|\widetilde{D}_k(x,y) - \widetilde{D}_k(x',y)| \le C\left[\frac{d(x,x')}{2^{-k}+d(x,y)}\right]^{\epsilon'} \frac{1}{V_{2^{-k}}(x)+V(x,y)} \left[\frac{2^{-k}}{2^{-k}+d(x,y)}\right]^{\epsilon'},$$

(c) $\int_{\mathcal{X}} \widetilde{D}_k(w,y)\,d\mu(w) = \int_{\mathcal{X}} \widetilde{D}_k(x,w)\,d\mu(w) = 0$,

where $\epsilon' \in (\epsilon, \epsilon_1 \wedge \epsilon_2)$ and C is a positive constant independent of k, x, x' and y. The kernels of $\{\overline{D}_k\}_{k\in\mathbb{Z}}$ satisfy the above (a), (c) *and, when $d(x,x') \le [2^{-k} + d(x,y)]/2$,*

$$|\overline{D}_k(y,x) - \overline{D}_k(y,x')| \le C\left[\frac{d(x,x')}{2^{-k}+d(x,y)}\right]^{\epsilon'} \frac{1}{V_{2^{-k}}(x)+V(x,y)} \left[\frac{2^{-k}}{2^{-k}+d(x,y)}\right]^{\epsilon'}.$$

Proof of Theorem 7.6. Fix $\epsilon, \beta, \gamma, \delta_1, \delta_2, s, p, q$ and $\{p_i\}_{i=1}^m$ as in Theorem 7.6. Let $\{S_k\}_{k\in\mathbb{Z}}$ be a 1-ATI with bounded support and $D_k := S_k - S_{k-1}$ for all $k \in \mathbb{Z}$. Of course, $\{S_k\}_{k\in\mathbb{Z}}$ is a $(1,1,1)$-ATI. Let $f_1 \in \dot{B}^s_{p_1,q}(\mathcal{X})$. Then the Calderón reproducing formula (see Lemma 7.7) implies that there exist linear operators $\{\widetilde{D}_k\}_{k\in\mathbb{Z}}$ whose kernels satisfy properties (a), (b) and (c) of Lemma 7.7 for any exponent $\epsilon' \in (\epsilon, 1)$ such that

$$f_1 = \sum_{j\in\mathbb{Z}} \widetilde{D}_j D_j(f_1)$$

in $(\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma))'$. For any $x, y_2, \ldots, y_m \in \mathcal{X}$, observe that $\theta_k(x,\cdot,y_2,\ldots,y_m)$ can be viewed as an element of $\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)$. Thus, for all $k \in \mathbb{Z}$,

$$\|2^{ks}\Theta_k(f_1,\ldots,f_m)\|_{L^p(\mathcal{X})} = \left\|\sum_{j\in\mathbb{Z}} 2^{ks}\Theta_k(\widetilde{D}_j D_j f_1, f_2, \ldots, f_m)\right\|_{L^p(\mathcal{X})}. \tag{7.18}$$

Since $\epsilon' > \epsilon$, we fix ϵ_1' such that $\epsilon_1' \in (\epsilon \wedge \delta_1 \wedge \delta_2,\ \epsilon' \wedge \delta_1 \wedge \delta_2)$. Using Lemma 7.3, we see that, for all $k, j \in \mathbb{Z}$ and $x \in \mathcal{X}$,

$$\begin{aligned}\Theta_k(\widetilde{D}_j D_j f_1, f_2, \ldots, f_m)(x)| = &\left|\int_{\mathcal{X}^m}\left[\int_{\mathcal{X}} \theta_k(x,z_1,\ldots,y_m)\widetilde{D}_j(z_1,y_1)\,d\mu(z_1)\right]\right.\\ &\left.\times D_j(f_1)(y_1)\prod_{i=2}^m f_i(y_i)\,d\mu(y_1)\cdots d\mu(y_m)\right|\end{aligned}$$

$$
\begin{aligned}
&\lesssim A2^{-|k-j|\epsilon_1'}\int_{\mathcal{X}^m}\mathcal{K}(2^{-(k\wedge j)};1\wedge\delta_2,x,y_1)\prod_{i=2}^m\mathcal{K}(2^{-k};\delta_2,x,y_i)\\
&\qquad\times|D_j(f_1)(y_1)|\prod_{i=2}^m|f_i(y_i)|\,d\mu(y_1)\cdots d\mu(y_m)\\
&\sim A2^{-|k-j|\epsilon_1'}\left\{\int_{\mathcal{X}}\mathcal{K}(2^{-(k\wedge j)};1\wedge\delta_2,x,y_1)|D_j(f_1)(y_1)|\,d\mu(y_1)\right\}\\
&\qquad\times\prod_{i=2}^m\int_{\mathcal{X}}\mathcal{K}(2^{-k};\delta_2,x,y_i)|f_i(y_i)|\,d\mu(y_i).
\end{aligned}\tag{7.19}
$$

Since $1/p=\sum_{i=1}^m 1/p_i$, we apply Hölder's inequality to obtain

$$
\begin{aligned}
&\left\|\sum_{j\in\mathbb{Z}}2^{ks}\Theta_k(\widetilde{D}_jD_jf_1,f_2,\ldots,f_m)\right\|_{L^p(\mathcal{X})}\\
&\quad\lesssim A\left\|\sum_{j\in\mathbb{Z}}2^{-|k-j|\epsilon_1'+(k-j)s}\int_{\mathcal{X}}\mathcal{K}(2^{-(k\wedge j)};1\wedge\delta_2,\cdot,y_1)|2^{js}D_j(f_1)(y_1)|\,d\mu(y_1)\right\|_{L^{p_1}(\mathcal{X})}\\
&\qquad\times\prod_{i=2}^m\left\|\int_{\mathcal{X}}\mathcal{K}(2^{-k};\delta_2,\cdot,y_i)|f_i(y_i)|\,d\mu(y_i)\right\|_{L^{p_i}(\mathcal{X})}\\
&\quad=:\mathrm{Z}_1\times\prod_{i=2}^m\mathrm{Z}_i.
\end{aligned}
$$

To estimate Z_i for $2\le i\le m$, we use the following fact: for all $\delta>0$, $k\in\mathbb{Z}$ and $g\in L^r(\mathcal{X})$ with $r\in[1,\infty]$,

$$
\left\|\int_{\mathcal{X}}\mathcal{K}(2^{-k};\delta,\cdot,w)|g(w)|\,d\mu(w)\right\|_{L^r(\mathcal{X})}\le C\|g\|_{L^r(\mathcal{X})}\tag{7.20}
$$

for some constant C depending only on ϵ and the doubling constant C_1. To see (7.20), for $r\in(1,\infty)$, we apply Hölder's inequality to deduce that, for all $x\in\mathcal{X}$,

$$
\int_{\mathcal{X}}\mathcal{K}(2^{-k};\delta,x,w)|g(w)|\,d\mu(w)\lesssim\left\{\int_{\mathcal{X}}\mathcal{K}(2^{-k};\delta,x,w)|g(w)|^r\,d\mu(w)\right\}^{1/r},
$$

and then Fubini's theorem further implies that

$$
\begin{aligned}
\left\|\int_{\mathcal{X}}\mathcal{K}(2^{-k};\delta,\cdot,w)|g(w)|\,d\mu(w)\right\|_{L^r(\mathcal{X})}&\lesssim\left\{\int_{\mathcal{X}}\int_{\mathcal{X}}\mathcal{K}(2^{-k};\delta,x,w)|g(w)|^r\,d\mu(w)\,d\mu(x)\right\}^{1/r}\\
&\lesssim\|g\|_{L^r(\mathcal{X})};
\end{aligned}
$$

suitable modifications also yield (7.20) for $r=1$ or ∞. From (7.20), it follows that

$$
\mathrm{Z}_i\lesssim\|f_i\|_{L^{p_i}(\mathcal{X})},\qquad\forall i\in\{2,\ldots,m\}.
$$

Now we turn to the estimate of Z_1. By Hölder's inequality, we obtain

$$
\begin{aligned}
&\int_{\mathcal{X}}\mathcal{K}(2^{-(k\wedge j)};1\wedge\delta_2,x,y_1)|2^{js}D_j(f_1)(y_1)|\,d\mu(y_1)\\
&\qquad\lesssim\left\{\int_{\mathcal{X}}\mathcal{K}(2^{-(k\wedge j)};1\wedge\delta_2,x,y_1)|2^{js}D_j(f_1)(y_1)|^{p_1}\,d\mu(y_1)\right\}^{1/p_1}.
\end{aligned}
$$

Combining this with Lemma 6.13 and Fubini's theorem, we see that

$$
\begin{aligned}
(\mathrm{Z}_1)^q &= \left\| \left[\sum_{j\in\mathbb{Z}} 2^{-|k-j|\epsilon_1'+(k-j)s} \int_{\mathcal{X}} \mathcal{K}(2^{-(k\wedge j)}; 1\wedge\delta_2, \cdot, y_1) |2^{js} D_j(f_1)(y_1)|\, d\mu(y_1) \right]^{p_1} \right\|_{L^1(\mathcal{X})}^{q/p_1} \\
&\lesssim \left\| \sum_{j\in\mathbb{Z}} 2^{-|k-j|(\epsilon_1'-|s|)} \left[\int_{\mathcal{X}} \mathcal{K}(2^{-(k\wedge j)}; 1\wedge\delta_2, \cdot, y_1) |2^{js} D_j(f_1)(y_1)|\, d\mu(y_1) \right]^{p_1} \right\|_{L^1(\mathcal{X})}^{q/p_1} \\
&\lesssim \left\{ \sum_{j\in\mathbb{Z}} 2^{-|k-j|(\epsilon_1'-|s|)} \int_{\mathcal{X}^2} \mathcal{K}(2^{-(k\wedge j)}; 1\wedge\delta_2, x, y_1) |2^{js} D_j(f_1)(y_1)|^{p_1}\, d\mu(y_1)\, d\mu(x) \right\}^{q/p_1} \\
&\lesssim \left\{ \sum_{j\in\mathbb{Z}} 2^{-|k-j|(\epsilon_1'-|s|)} \|2^{js} D_j(f_1)\|_{L^{p_1}(\mathcal{X})}^{p_1} \right\}^{q/p_1} \\
&\lesssim \sum_{j\in\mathbb{Z}} 2^{-|k-j|(\epsilon_1'-|s|)(1\wedge\frac{q}{p_1})} \|2^{js} D_j(f_1)\|_{L^{p_1}(\mathcal{X})}^{q}.
\end{aligned}
$$

Summing the estimates of Z_1 through Z_m, we conclude that

$$
\begin{aligned}
&\left\| \sum_{j\in\mathbb{Z}} 2^{ks} \Theta_k(\widetilde{D}_j D_j f_1, f_2, \ldots, f_m) \right\|_{L^p(\mathcal{X})}^q \\
&\qquad\qquad \lesssim \sum_{j\in\mathbb{Z}} 2^{-|k-j|(\epsilon_1'-|s|)(1\wedge\frac{q}{p_1})} \|2^{js} D_j(f_1)\|_{L^{p_1}(\mathcal{X})}^{q} \prod_{i=2}^m \|f_i\|_{L^{p_i}(\mathcal{X})}^q.
\end{aligned}
$$

This, combined with (7.18), gives us

$$
\begin{aligned}
&\left\{ \sum_{k\in\mathbb{Z}} 2^{ksq} \|\Theta_k(f_1, \ldots, f_m)\|_{L^p(\mathcal{X})}^q \right\}^{1/q} \\
&\qquad \lesssim A \left\{ \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} 2^{-|k-j|(\epsilon_1'-|s|)(1\wedge\frac{q}{p_1})} \|2^{js} D_j(f_1)\|_{L^{p_1}(\mathcal{X})}^{q} \right\}^{1/q} \prod_{i=2}^m \|f_i\|_{L^{p_i}(\mathcal{X})} \\
&\qquad \lesssim A \left\{ \sum_{j\in\mathbb{Z}} \|2^{js} D_j(f_1)\|_{L^{p_1}(\mathcal{X})}^{q} \right\}^{1/q} \prod_{i=2}^m \|f_i\|_{L^{p_i}(\mathcal{X})} \\
&\qquad \lesssim A \|f_1\|_{\dot{B}^s_{p_1,q}(\mathcal{X})} \prod_{i=2}^m \|f_i\|_{L^{p_i}(\mathcal{X})}.
\end{aligned}
$$

Thus, (7.17) holds, and we complete the proof of Theorem 7.6. ∎

THEOREM 7.8. *Let $\{\theta_k\}_{k\in\mathbb{Z}} \in \mathbf{Ker}(m, A, \delta_1, \delta_2)$ for some $\delta_1 > 0$, $\delta_2 > 0$ and $A > 0$, and let every θ_k satisfy* (7.16). *Let p, $\{p_i\}_{i=1}^m$ and q be given numbers such that $1/p = \sum_{i=1}^m 1/p_i$,*

$$1 < q \le \infty, \quad 1 < p_1 < \infty \quad \text{and} \quad 1 < p_i \le \infty \quad \text{for } i \in \{2, \ldots, m\}.$$

For the indices p_1, q and $s \in (-\epsilon, \epsilon)$ with $\epsilon \in (0, 1\wedge\delta_1\wedge\delta_2)$, as in Definition 6.9, *the Triebel–Lizorkin space $\dot{F}^s_{p_1,q}(\mathcal{X})$ is defined as a subspace of a distribution space $(\mathring{\mathcal{G}}^\epsilon_0(\beta,\gamma))'$ with β, γ satisfying* (6.18). *Then there exists a positive constant C such that, for all functions $f_1 \in \dot{F}^s_{p_1,q}(\mathcal{X})$ and $f_i \in L^{p_i}(\mathcal{X})$ with $i \in \{2, \ldots, m\}$,*

$$
\left\| \left\{ \sum_{k\in\mathbb{Z}} 2^{ksq} |\Theta_k(f_1, \ldots, f_m)|^q \right\}^{1/q} \right\|_{L^p(\mathcal{X})} \le CA \|f_1\|_{\dot{F}^s_{p_1,q}(\mathcal{X})} \prod_{i=2}^m \|f_i\|_{L^{p_i}(\mathcal{X})}, \tag{7.21}
$$

where $C := C(\delta_1, \delta_2, \epsilon, s, q, p, p_1, \ldots, p_m, \mathcal{X}) > 0$.

Proof. Let $\{S_k\}_{k\in\mathbb{Z}}$ be a 1-ATI with bounded support, which is also a $(1,1,1)$-ATI, and set $D_k := S_k - S_{k-1}$ for all $k \in \mathbb{Z}$. Let $f_1 \in \dot{F}^s_{p_1,q}(\mathcal{X})$. The Calderón reproducing formula (see Lemma 7.7) implies that there exist linear operators $\{\widetilde{D}_k\}_{k\in\mathbb{Z}}$ with kernels satisfying properties (a)–(c) of Lemma 7.7 for any $\epsilon' \in (\epsilon, 1)$ such that

$$f_1 = \sum_{j\in\mathbb{Z}} \widetilde{D}_j D_j(f_1)$$

in $(\mathcal{G}^\epsilon_0(\beta,\gamma))'$. Hence, we write

$$\begin{aligned}
&\left\|\left\{\sum_{k\in\mathbb{Z}} 2^{ksq}|\Theta_k(f_1,\dots,f_m)|^q\right\}^{1/q}\right\|_{L^p(\mathcal{X})} \\
&\qquad = \left\|\left\{\sum_{k\in\mathbb{Z}} 2^{ksq}\left|\sum_{j\in\mathbb{Z}}\Theta_k(\widetilde{D}_jD_jf_1,f_2,\dots,f_m)\right|^q\right\}^{1/q}\right\|_{L^p(\mathcal{X})}. \qquad (7.22)
\end{aligned}$$

Since $\epsilon' > \epsilon$, we choose ϵ_1' such that $(\epsilon\wedge\delta_1\wedge\delta_2) < \epsilon_1' < (\epsilon'\wedge\delta_1\wedge\delta_2)$. Then $\epsilon_1' > |s|$. For such an ϵ_1', by (7.19), we see that, for all $k, j \in \mathbb{Z}$ and $x \in \mathcal{X}$,

$$\begin{aligned}
&|\Theta_k(\widetilde{D}_jD_jf_1,f_2,\dots,f_m)(x)| \\
&\quad\lesssim A2^{-|k-j|\epsilon_1'}\left\{\int_{\mathcal{X}} \mathcal{K}(2^{-(k\wedge j)};1\wedge\delta_2,x,y_1)|D_j(f_1)(y_1)|\,d\mu(y_1)\right\} \\
&\qquad\times\prod_{i=2}^m\int_{\mathcal{X}}\mathcal{K}(2^{-k};\delta_2,x,y_i)|f_i(y_i)|\,d\mu(y_i) \\
&\quad\lesssim A2^{-|k-j|\epsilon_1'}\mathcal{M}(D_j(f_1))(x)\prod_{i=2}^m\mathcal{M}(f_i)(x), \qquad (7.23)
\end{aligned}$$

where $\mathcal{M}$ is the Hardy–Littlewood maximal operator as in (2.3) and the last inequality of (7.23) is due to Lemma 2.15(iv). Inserting (7.23) into (7.22), we obtain

$$\begin{aligned}
&\left\|\left\{\sum_{k\in\mathbb{Z}} 2^{ksq}|\Theta_k(f_1,\dots,f_m)|^q\right\}^{1/q}\right\|_{L^p(\mathcal{X})} \\
&\qquad\lesssim A\left\|\left\{\sum_{k\in\mathbb{Z}}\left[\sum_{j\in\mathbb{Z}}2^{-|k-j|\epsilon_1'+(k-j)s}\mathcal{M}(2^{js}D_j(f_1))\right]^q\right\}^{1/q}\prod_{i=2}^m\mathcal{M}(f_i)\right\|_{L^p(\mathcal{X})}.
\end{aligned}$$

Notice that $\epsilon_1' > |s|$ and Lemma 6.13 imply that

$$\begin{aligned}
\sum_{k\in\mathbb{Z}}\left[\sum_{j\in\mathbb{Z}}2^{-|k-j|\epsilon_1'+(k-j)s}\mathcal{M}(2^{js}D_j(f_1))\right]^q &\lesssim \sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}2^{-|k-j|(\epsilon_1'-|s|)(q\wedge1)}[\mathcal{M}(2^{js}D_j(f_1))]^q \\
&\lesssim \sum_{j\in\mathbb{Z}}[\mathcal{M}(2^{js}D_j(f_1))]^q.
\end{aligned}$$

From this, Hölder's inequality with exponents $1/p = \sum_{i=1}^m 1/p_i$, the Fefferman–Stein vector-valued maximal function inequality (see, for example, [48]) and the boundedness of $\mathcal{M}$ on $L^{p_i}(\mathcal{X})$ for $i \in \{2,\dots,m\}$, we continue to estimate (7.9):

$$\begin{aligned}
&\left\|\left\{\sum_{k\in\mathbb{Z}} 2^{ksq}|\Theta_k(f_1,\dots,f_m)|^q\right\}^{1/q}\right\|_{L^p(\mathcal{X})} \\
&\qquad\lesssim A\left\|\left\{\sum_{j\in\mathbb{Z}}[\mathcal{M}(2^{js}D_j(f_1))]^q\right\}^{1/q}\prod_{i=2}^m\mathcal{M}(f_i)\right\|_{L^p(\mathcal{X})}
\end{aligned}$$

$$\lesssim A\Big\|\Big\{\sum_{j\in\mathbb{Z}}[\mathcal{M}(2^{js}D_j(f_1))]^q\Big\}^{1/q}\Big\|_{L^{p_1}(\mathcal{X})}\prod_{i=2}^m\|\mathcal{M}(f_i)\|_{L^{p_i}(\mathcal{X})}$$
$$\lesssim A\Big\|\Big\{\sum_{j\in\mathbb{Z}}[2^{js}|D_j(f_1)|]^q\Big\}^{1/q}\Big\|_{L^{p_1}(\mathcal{X})}\prod_{i=2}^m\|f_i\|_{L^{p_i}(\mathcal{X})}$$
$$\sim \|f_1\|_{\dot F^s_{p_1,q}(\mathcal{X})}\prod_{i=2}^m\|f_i\|_{L^{p_i}(\mathcal{X})}.$$

This proves (7.21), and finishes the proof of Theorem 7.8. ∎

7.3. Quadratic $T1$ type theorems on Besov and Triebel–Lizorkin spaces. By applying the off-diagonal estimate for the sequence of multilinear kernels $\{\theta_k\}_{k\in\mathbb{Z}}$ (see Lemma 7.5) and the Calderón reproducing formula, we prove the following main result of this subsection.

THEOREM 7.9. *Let $m\in\mathbb{N}$, and $\{\theta_k\}_{k\in\mathbb{Z}}\in\mathbf{Ker}(m,A,\delta_1,\delta_2)$ for some $\delta_1>0$, $\delta_2>0$ and $A>0$. Assume that, for all $k\in\mathbb{Z}$, $i\in\{1,\dots,m\}$ and x, y_1, …, $y_m\in\mathcal{X}$,*

$$\int_{\mathcal{X}}\theta_k(x,y_1,\dots,y_i,\dots,y_m)\,d\mu(y_i)=0.$$

Let $\epsilon\in(0,1\wedge\delta_1\wedge\delta_2)$ and s, s_1, …, $s_m\in(-\epsilon,\epsilon)$ satisfy $s=\sum_{i=1}^m s_i$. Then:

(i) *if $0<p\le\infty$, $0<q\le\infty$, $p(s_i,\epsilon)<p_i<\infty$ and $0<q_i<\infty$ for all $i\in\{1,\dots,m\}$ are such that $1/p=\sum_{i=1}^m 1/p_i$ and $1/q=\sum_{i=1}^m 1/q_i$, and, as in Definition 6.9, $\dot B^{s_i}_{p_i,q_i}(\mathcal{X})$ is defined as a subspace of $(\mathring{\mathcal{G}}^\epsilon_0(\beta,\gamma))'$ with certain β,γ satisfying*

$$\begin{aligned}&\max\{s_i,\ 0,\ -s_i+n(1/p_i-1)_+\}<\beta<\epsilon\quad\text{and}\\&\max\{s_i-\kappa/p_i,\ n(1/p_i-1)_+,-s_i+n(1/p_i-1)_+-\kappa(1-1/p_i)_+\}<\gamma<\epsilon,\end{aligned}\tag{7.24}$$

then there exists a positive constant C such that, for all $i\in\{1,\dots,m\}$ and $f_i\in\mathring{\mathcal{G}}^\epsilon_0(\beta,\gamma)\subset\dot B^{s_i}_{p_i,q_i}(\mathcal{X})$,

$$\Big\{\sum_{k=-\infty}^{\infty}2^{ksq}\|\Theta_k(f_1,\dots,f_m)\|^q_{L^p(\mathcal{X})}\Big\}^{1/q}\le CA^m\prod_{i=1}^m\|f_i\|_{\dot B^{s_i}_{p_i,q_i}(\mathcal{X})};$$

(ii) *if $0<p<\infty$, $0<q\le\infty$, $p(s_i,\epsilon)<p_i<\infty$ and $p(s_i,\epsilon)<q_i<\infty$ for $i\in\{1,\dots,m\}$ are such that $1/p=\sum_{i=1}^m 1/p_i$ and $1/q=\sum_{i=1}^m 1/q_i$, and, as in Definition 6.9, $\dot F^{s_i}_{p_i,q_i}(\mathcal{X})$ is defined as a subspace of $(\mathring{\mathcal{G}}^\epsilon_0(\beta,\gamma))'$ with certain β,γ satisfying (7.24), then there exists a positive constant C such that, for all $f_i\in\mathring{\mathcal{G}}^\epsilon_0(\beta,\gamma)\subset\dot F^{s_i}_{p_i,q_i}(\mathcal{X})$,*

$$\Big\|\Big\{\sum_{k=-\infty}^{\infty}2^{ksq}|\Theta_k(f_1,\dots,f_m)|^q\Big\}^{1/q}\Big\|_{L^p(\mathcal{X})}\le CA^m\prod_{i=1}^m\|f_i\|_{\dot F^{s_i}_{p_i,q_i}(\mathcal{X})}.\tag{7.25}$$

REMARK 7.10. Using the fact that $\dot F^0_{p,2}(\mathcal{X})$ coincides with the space $L^p(\mathcal{X})$, we see that the conclusions of Theorem 7.9 do not cover those of Theorems 7.6 and 7.8. A result along the lines of Theorem 7.8, in the Euclidean setting, has recently and independently been proved by Hart [62, Theorem 1.4]. Examination indicates that Theorem 1.4 in [62] requires a weaker cancelation condition and stronger regularity compared to Theorem 7.8 in our work. It should also be noticed that an alternative approach to the bilinear $T1$-theorem was also obtained by Hart [63] in the Euclidean setting. Quadratic estimates of the form (7.25) were also studied by Grafakos and Oliveira [49] via multilinear Carleson measure techniques.

Proof of Theorem 7.9. Let $\{S_k\}_{k\in\mathbb{Z}}$ be a 1-ATI with bounded support, which is also a $(1,1,1)$-ATI, and set $D_k := S_k - S_{k-1}$ for all $k\in\mathbb{Z}$. For every $i\in\{1,\ldots,m\}$ and $f_i\in \mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)$, by the Calderón reproducing formula in Lemma 6.10, we know that there exist linear operators $\{\overline{D}_{k_i}\}_{k_i\in\mathbb{Z}}$ such that, for any fixed $y_{\tau_i}^{k_i,\nu_i}\in Q_{\tau_i}^{k_i,\nu_i}$ with $k_i\in\mathbb{Z}$, $\tau_i\in I_{k_i}$ and $\nu_i\in\{1,\ldots,N(k_i,\tau_i)\}$,

$$f_i(\cdot)=\sum_{k_i\in\mathbb{Z}}\sum_{\tau_i\in I_{k_i}}\sum_{\nu=1}^{N(k_i,\tau_i)}\mu(Q_{\tau_i}^{k_i,\nu_i})D_{k_i}(\cdot,y_{\tau_i}^{k_i,\nu_i})\overline{D}_{k_i}(f_i)(y_{\tau_i}^{k_i,\nu_i}), \tag{7.26}$$

where the series converges in $\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)$, and the kernels of the operators $\{\overline{D}_{k_i}\}_{k_i\in\mathbb{Z}}$ satisfy (a), (b$'$) and (c) of Lemma 6.10 for any $\epsilon'\in(\epsilon,1)$. In what follows, for the sake of simplicity, we use $\sum_{(k_i,\tau_i,\nu_i)}$ to denote $\sum_{k_i\in\mathbb{Z}}\sum_{\tau_i\in I_{k_i}}\sum_{\nu=1}^{N(k_i,\tau_i)}$ for all $i\in\{1,\ldots,m\}$.

To show (i), applying (7.26) to each f_i and then using Lemma 7.4, we see that, for any given $\sigma\in(0,1\wedge\delta_1\wedge\delta_2)$ and all $x\in\mathcal{X}$,

$$\begin{aligned}
&|\Theta_k(f_1,\ldots,f_m)(x)|\\
&\quad\le \sum_{(k_1,\tau_1,\nu_1)}\cdots\sum_{(k_m,\tau_m,\nu_m)}\mu(Q_{\tau_1}^{k_1,\nu_1})\cdots\mu(Q_{\tau_m}^{k_m,\nu_m})\\
&\qquad\times|\Theta_k(D_{k_1}(\cdot,y_{\tau_1}^{k_1,\nu_1}),\ldots,D_{k_m}(\cdot,y_{\tau_m}^{k_m,\nu_m}))(x)|\,|\overline{D}_{k_1}(f_1)(y_{\tau_1}^{k_1,\nu_1})\cdots\overline{D}_{k_m}(f_m)(y_{\tau_m}^{k_m,\nu_m})|\\
&\quad\lesssim A^m\prod_{i=1}^m\sum_{(k_i,\tau_i,\nu_i)}2^{-|k-k_i|\sigma}\mu(Q_{\tau_i}^{k_i,\nu_i})\mathcal{K}(2^{-(k\wedge k_i)};1\wedge\delta_2,\,x,y_{\tau_i}^{k_i,\nu_i})|\overline{D}_{k_i}(f_i)(y_{\tau_i}^{k_i,\nu_i})|.
\end{aligned}$$

By Lemma 6.12, the last quantity above is bounded by

$$\begin{aligned}
&A^m\prod_{i=1}^m\sum_{k_i\in\mathbb{Z}}2^{-|k-k_i|\sigma+[k_i-(k\wedge k_i)n(1/r_i-1)]}\\
&\qquad\qquad\times\left\{\mathcal{M}\Big(\sum_{\tau_i\in I_{k_i}}\sum_{\nu=1}^{N(k_i,\tau_i)}|\overline{D}_{k_i}(f_i)(y_{\tau_i}^{k_i,\nu_i})|^{r_i}\chi_{Q_{\tau_i}^{k_i,\nu_i}}\Big)(x)\right\}^{1/r_i},
\end{aligned}$$

whenever $r_i\in(\frac{n}{n+(1\wedge\delta_2)},1]$. In particular, since $\epsilon\in(0,1\wedge\delta_1\wedge\delta_2)$ and

$$\min\{p_i,q_i\}>p(s_i,\epsilon)=\max\left\{\frac{n}{n+\epsilon},\frac{n}{n+s_i+\epsilon}\right\},\qquad\forall i\in\{1,\ldots,m\},$$

there exist $\sigma\in(0,1\wedge\delta_1\wedge\delta_2)$ and $\{r_i\}_{i=1}^m$ such that, for all $i\in\{1,\ldots,m\}$,

$$\frac{n}{n+(1\wedge\delta_2)}<r_i\le 1,\quad r_i<\min\{p_i,q_i,\}\quad\text{and}\quad n(1/r_i-1)<\sigma+s_i. \tag{7.27}$$

Fix such σ and $\{r_i\}_{i=1}^m$. Set

$$\mathrm{F}_i:=\left\{\mathcal{M}\Big(\sum_{\tau_i\in I_{k_i}}\sum_{\nu=1}^{N(k_i,\tau_i)}|\overline{D}_{k_i}(f_i)(y_{\tau_i}^{k_i,\nu_i})|^{r_i}\chi_{Q_{\tau_i}^{k_i,\nu_i}}\Big)\right\}^{1/r_i}.$$

Thus, we have proved that, for σ and $\{r_i\}_{i=1}^m$ as in (7.27), and all $x\in\mathcal{X}$,

$$|\Theta_k(f_1,\ldots,f_m)(x)|\lesssim A^m\prod_{i=1}^m\sum_{k_i\in\mathbb{Z}}2^{-|k-k_i|[\sigma-n(1/r_i-1)]}\mathrm{F}_i(x). \tag{7.28}$$

Since $1/p=\sum_{i=1}^m 1/p_i$, it follows, from (7.28) and Hölder's inequality, that

$$\|\Theta_k(f_1,\ldots,f_m)\|_{L^p(\mathcal{X})}\lesssim A^m\prod_{i=1}^m\Big\|\sum_{k_i\in\mathbb{Z}}2^{-|k-k_i|[\sigma-n(1/r_i-1)]}\mathrm{F}_i\Big\|_{L^{p_i}(\mathcal{X})}.$$

Notice that $s=\sum_{i=1}^m s_i$. Again, using $1/q=\sum_{i=1}^m 1/q_i$ and Hölder's inequality we obtain

$$\begin{aligned}
&\Big\{\sum_{k=-\infty}^{\infty}2^{ksq}\|\Theta_k(f_1,\ldots,f_m)\|_{L^p(\mathcal{X})}^q\Big\}^{1/q}\\
&\quad\lesssim A^m\Big\{\sum_{k=-\infty}^{\infty}\prod_{i=1}^m 2^{ks_iq}\Big\|\sum_{k_i\in\mathbb{Z}}2^{-|k-k_i|[\sigma-n(1/r_i-1)]}\mathrm{F}_i\Big\|_{L^{p_i}(\mathcal{X})}^q\Big\}^{1/q}\\
&\quad\lesssim A^m\prod_{i=1}^m\Big\{\sum_{k=-\infty}^{\infty}2^{ks_iq_i}\Big\|\sum_{k_i\in\mathbb{Z}}2^{-|k-k_i|[\sigma-n(1/r_i-1)]}\mathrm{F}_i\Big\|_{L^{p_i}(\mathcal{X})}^{q_i}\Big\}^{1/q_i}\\
&\quad\lesssim A^m\prod_{i=1}^m\Big\{\sum_{k=-\infty}^{\infty}\Big\|\sum_{k_i\in\mathbb{Z}}2^{-|k-k_i|[\sigma+s_i-n(1/r_i-1)]}2^{k_is_i}\mathrm{F}_i\Big\|_{L^{p_i}(\mathcal{X})}^{q_i}\Big\}^{1/q_i}.
\end{aligned}\tag{7.29}$$

For every $i\in\{1,\ldots,m\}$, by $\sigma+s_i-n(1/r_i-1)>0$, Lemma 6.13 and the boundedness of $\mathcal{M}$ on $L^{p_i/r_i}(\mathcal{X})$, we conclude that

$$\begin{aligned}
&\Big\|\sum_{k_i\in\mathbb{Z}}2^{-|k-k_i|[\sigma+s_i-n(1/r_i-1)]}2^{k_is_i}\mathrm{F}_i\Big\|_{L^{p_i}(\mathcal{X})}^{q_i}\\
&\quad=\Big\{\int_{\mathcal{X}}\Big|\sum_{k_i\in\mathbb{Z}}2^{-|k-k_i|[\sigma+s_i-n(1/r_i-1)]}2^{k_is_i}\mathrm{F}_i(x)\Big|^{p_i}\,d\mu(x)\Big\}^{q_i/p_i}\\
&\quad\lesssim\Big\{\int_{\mathcal{X}}\sum_{k_i\in\mathbb{Z}}2^{-|k-k_i|[\sigma+s_i-n(1/r_i-1)](p_i\wedge1)}|2^{k_is_i}\mathrm{F}_i(x)|^{p_i}\,d\mu(x)\Big\}^{q_i/p_i}\\
&\quad\sim\Big\{\sum_{k_i\in\mathbb{Z}}2^{-|k-k_i|[\sigma+s_i-n(1/r_i-1)](p_i\wedge1)}\\
&\qquad\times\int_{\mathcal{X}}\Big[\mathcal{M}\Big(\sum_{\tau_i\in I_{k_i}}\sum_{\nu=1}^{N(k_i,\tau_i)}|2^{k_i,s_i}\overline{D}_{k_i}(f_i)(y_{\tau_i}^{k_i,\nu_i})|^{r_i}\chi_{Q_{\tau_i}^{k_i,\nu_i}}\Big)(x)\Big]^{p_i/r_i}\,d\mu(x)\Big\}^{q_i/p_i}\\
&\quad\lesssim\Big[\sum_{k_i\in\mathbb{Z}}2^{-|k-k_i|[\sigma+s_i-n(1/r_i-1)](p_i\wedge1)}\\
&\qquad\times\sum_{\tau_i\in I_{k_i}}\sum_{\nu=1}^{N(k_i,\tau_i)}|2^{k_is_i}\overline{D}_{k_i}(f_i)(y_{\tau_i}^{k_i,\nu_i})|^{p_i}\mu(Q_{\tau_i}^{k_i,\nu_i})\Big]^{q_i/p_i}\\
&\quad\lesssim\sum_{k_i\in\mathbb{Z}}2^{-|k-k_i|[\sigma+s_i-n(1/r_i-1)](p_i\wedge1)(q_i/p_i\wedge1)}\\
&\qquad\times\Big[\sum_{\tau_i\in I_{k_i}}\sum_{\nu=1}^{N(k_i,\tau_i)}|2^{k_is_i}\overline{D}_{k_i}(f_i)(y_{\tau_i}^{k_i,\nu_i})|^{p_i}\mu(Q_{\tau_i}^{k_i,\nu_i})\Big]^{q_i/p_i}.
\end{aligned}$$

With this estimate, we continue (7.29) as

$$
\begin{aligned}
&\left\{\sum_{k=-\infty}^{\infty} 2^{ksq}\|\Theta_k(f_1,\ldots,f_m)\|_{L^p(\mathcal{X})}^q\right\}^{1/q}\\
&\qquad\lesssim A^m\prod_{i=1}^m\left\{\sum_{k=-\infty}^{\infty}\sum_{k_i\in\mathbb{Z}} 2^{-|k-k_i|[\sigma+s_i-n(1/r_i-1)](p_i\wedge 1)(q_i/p_i\wedge 1)}\right.\\
&\qquad\qquad\times\left.\left[\sum_{\tau_i\in I_{k_i}}\sum_{\nu=1}^{N(k_i,\tau_i)}|2^{k_is_i}\overline{D}_{k_i}(f_i)(y_{\tau_i}^{k_i,\nu_i})|^{p_i}\mu(Q_{\tau_i}^{k_i,\nu_i})\right]^{q_i/p_i}\right\}^{1/q_i}\\
&\qquad\lesssim A^m\prod_{i=1}^m\left\{\sum_{k_i\in\mathbb{Z}}\left[\sum_{\tau_i\in I_{k_i}}\sum_{\nu=1}^{N(k_i,\tau_i)}|2^{k_is_i}\overline{D}_{k_i}(f_i)(y_{\tau_i}^{k_i,\nu_i})|^{p_i}\mu(Q_{\tau_i}^{k_i,\nu_i})\right]^{q_i/p_i}\right\}^{1/q_i}\\
&\qquad\lesssim A^m\prod_{i=1}^m\|f_i\|_{\dot{B}^{s_i}_{p_i,q_i}(\mathcal{X})}.
\end{aligned}
$$

This concludes the proof of (i).

Next we turn to (ii). For $\{f_i\}_{i=1}^m\subset\mathring{\mathcal{G}}_0^\epsilon(\beta,\gamma)$ as in (i), using the Calderón reproducing formula and Lemma 6.12, we see that (7.28) holds for some σ and a sequence $\{r_i\}_{i=1}^m$ satisfying (7.27). By (7.28) and $s=\sum_{i=1}^m s_i$, we obtain

$$
\begin{aligned}
&\left\|\left\{\sum_{k=-\infty}^{\infty}2^{ksq}|\Theta_k(f_1,\ldots,f_m)|^q\right\}^{1/q}\right\|_{L^p(\mathcal{X})}\\
&\qquad\lesssim A^m\left\|\left\{\sum_{k=-\infty}^{\infty}\left|\prod_{i=1}^m\sum_{k_i\in\mathbb{Z}}2^{ks_i}2^{-|k-k_i|[\sigma-n(1/r_i-1)]}\mathrm{F}_i\right|^q\right\}^{1/q}\right\|_{L^p(\mathcal{X})}\\
&\qquad\lesssim A^m\left\|\left\{\sum_{k=-\infty}^{\infty}\prod_{i=1}^m\left|\sum_{k_i\in\mathbb{Z}}2^{-|k-k_i|[\sigma+s_i-n(1/r_i-1)]}2^{k_is_i}\mathrm{F}_i\right|^q\right\}^{1/q}\right\|_{L^p(\mathcal{X})}.
\end{aligned}\tag{7.30}
$$

Since $1/q=\sum_{i=1}^m 1/q_i$ and $\sigma+s_i-n(1/r_i-1)>0$ for all $i\in\{1,\ldots,m\}$, from Hölder's inequality and Lemma 6.13, it follows that

$$
\begin{aligned}
&\left\{\sum_{k=-\infty}^{\infty}\prod_{i=1}^m\left|\sum_{k_i\in\mathbb{Z}}2^{-|k-k_i|[\sigma+s_i-n(1/r_i-1)]}2^{k_is_i}\mathrm{F}_i\right|^q\right\}^{1/q}\\
&\qquad\qquad\lesssim\prod_{i=1}^m\left\{\sum_{k=-\infty}^{\infty}\left|\sum_{k_i\in\mathbb{Z}}2^{-|k-k_i|[\sigma+s_i-n(1/r_i-1)]}2^{k_is_i}\mathrm{F}_i\right|^{q_i}\right\}^{1/q_i}\\
&\qquad\qquad\lesssim\prod_{i=1}^m\left\{\sum_{k=-\infty}^{\infty}\sum_{k_i\in\mathbb{Z}}2^{-|k-k_i|[\sigma+s_i-n(1/r_i-1)](q_i\wedge 1)}|2^{k_is_i}\mathrm{F}_i|^{q_i}\right\}^{1/q_i}\\
&\qquad\qquad\lesssim\prod_{i=1}^m\left\{\sum_{k_i\in\mathbb{Z}}|2^{k_is_i}\mathrm{F}_i|^{q_i}\right\}^{1/q_i}.
\end{aligned}
$$

Inserting this into (7.30) and applying Hölder's inequality with indices $1/p=\sum_{i=1}^m 1/p_i$, we deduce that

$$\begin{aligned}\left\|\left\{\sum_{k=-\infty}^{\infty}2^{ksq}|\Theta_k(f_1,\dots,f_m)|^q\right\}^{1/q}\right\|_{L^p(\mathcal{X})} &\lesssim A^m\left\|\prod_{i=1}^m\left\{\sum_{k_i\in\mathbb{Z}}|2^{k_is_i}\mathrm{F}_i|^{q_i}\right\}^{1/q_i}\right\|_{L^p(\mathcal{X})}\\ &\lesssim A^m\prod_{i=1}^m\left\|\left\{\sum_{k_i\in\mathbb{Z}}|2^{k_is_i}\mathrm{F}_i|^{q_i}\right\}^{1/q_i}\right\|_{L^{p_i}(\mathcal{X})}. \quad (7.31)\end{aligned}$$

For any $i\in\{1,\dots,m\}$, since $p_i/r_i>1$ and $q_i/r_i>1$, we use the Fefferman–Stein vector-valued maximal function inequality (see, for example, [48]) to find that

$$\begin{aligned}&\left\|\left\{\sum_{k_i\in\mathbb{Z}}|2^{k_is_i}\mathrm{F}_i|^{q_i}\right\}^{1/q_i}\right\|_{L^{p_i}(\mathcal{X})}\\ &\quad=\left\|\left\{\sum_{k_i\in\mathbb{Z}}\left[\mathcal{M}\left(\sum_{\tau_i\in I_{k_i}}\sum_{\nu=1}^{N(k_i,\tau_i)}|2^{k_is_i}\overline{D}_{k_i}(f_i)(y_{\tau_i}^{k_i,\nu_i})|^{r_i}\chi_{Q_{\tau_i}^{k_i,\nu_i}}\right)\right]^{q_i/r_i}\right\}^{1/q_i}\right\|_{L^{p_i}(\mathcal{X})}\\ &\quad\lesssim\left\|\left\{\sum_{k_i\in\mathbb{Z}}\left[\sum_{\tau_i\in I_{k_i}}\sum_{\nu=1}^{N(k_i,\tau_i)}|2^{k_is_i}\overline{D}_{k_i}(f_i)(y_{\tau_i}^{k_i,\nu_i})|^{r_i}\chi_{Q_{\tau_i}^{k_i,\nu_i}}\right]^{q_i/r_i}\right\}^{1/q_i}\right\|_{L^{p_i}(\mathcal{X})}\\ &\quad\sim\left\|\left\{\sum_{k_i\in\mathbb{Z}}\sum_{\tau_i\in I_{k_i}}\sum_{\nu=1}^{N(k_i,\tau_i)}|2^{k_is_i}\overline{D}_{k_i}(f_i)(y_{\tau_i}^{k_i,\nu_i})|^{q_i}\chi_{Q_{\tau_i}^{k_i,\nu_i}}\right\}^{1/q_i}\right\|_{L^{p_i}(\mathcal{X})}\\ &\quad\lesssim\|f_i\|_{\dot{F}^{s_i}_{p_i,q_i}(\mathcal{X})}.\end{aligned}$$

Then, applying (7.31), we conclude that

$$\left\|\left\{\sum_{k=-\infty}^{\infty}2^{ksq}|\Theta_k(f_1,\dots,f_m)|^q\right\}^{1/q}\right\|_{L^p(\mathcal{X})}\lesssim A^m\prod_{i=1}^m\|f_i\|_{\dot{F}^{s_i}_{p_i,q_i}(\mathcal{X})}.$$

This proves (ii) and completes the proof of Theorem 7.9. ∎

8. Paraproducts as bilinear Calderón–Zygmund operators

In this section, we prove that paraproducts on spaces of homogeneous type can be viewed as bilinear Calderón–Zygmund singular integrals of the kind considered in Sections 3 and 4. As applications, such paraproducts have weighted boundedness properties as in Corollaries 4.14 and 4.15. In the special case $\mathcal{X} = \mathbb{R}^n$, these paraproducts go back to the classical ones investigated by Coifman and Meyer [21, 82], Muscalu, Tao and Thiele [85], Grafakos and Kalton [44, Section 8], and Gilbert and Nahmod [37, 38]. More recent developments on paraproducts and their applications can be found in Bernicot's works [7, 8, 9]. Unweighted Euclidean counterparts to the results in this section were obtained by Bényi, Maldonado, Nahmod and Torres [5].

8.1. Paraproducts. Paraproducts were first introduced and systematically studied by J.-M. Bony [12] and they now play a central role in numerous areas of analysis and PDEs; see, for example, [4] for an exposition on the evolution of the concept of paraproduct.

Definition 8.1. Given $\beta, \gamma > 0$ and a dyadic cube $Q \in \mathcal{Q}$, say $Q := Q_\alpha^k$ for some $k \in \mathbb{Z}$ and $\alpha \in I_k$, a function ψ is called a *bump function* adapted to Q if, for all $x, y \in \mathcal{X}$,

$$|\psi(x)| \le \frac{\mu(Q)^{1/2}}{\mu(Q) + V(x, c_Q)} \left[\frac{1}{1 + 2^k d(x, c_Q)}\right]^\gamma \tag{8.1}$$

and, when $d(x, y) \le [2^{-k} + d(x, y)]/2$,

$$|\psi(x) - \psi(y)| \le \frac{\mu(Q)^{1/2}}{\mu(Q) + V(x, c_Q)} \left[\frac{d(x, y)}{2^{-k} + d(x, c_Q)}\right]^\beta \left[\frac{1}{1 + 2^k d(x, c_Q)}\right]^\gamma, \tag{8.2}$$

where c_Q denotes the center of Q, namely, $B(c_Q, C_6 2^{-k}) \subset Q \subset B(c_Q, C_5 2^{-k})$ as in Lemma 2.5.

In this case, we write ψ_Q to indicate that ψ is a bump function adapted to Q. If, in addition, ψ_Q satisfies $\int_{\mathcal{X}} \psi(x)\, d\mu(x) = 0$, then ψ_Q is called a (β, γ)-*smooth molecule* for Q. From now on, β and γ are assumed to be uniform in ψ_Q.

Bump functions are actually test functions as in Definition 2.12. Obviously, if ψ_Q is a bump function adapted to the dyadic cube $Q := Q_\alpha^k$, then $\mu(Q)^{-1/2}\psi_Q$ is a test function of type $(c_Q, 2^{-k}, \beta, \gamma)$, and vice versa. Also, ψ_Q is a molecule if and only if $\mu(Q)^{-1/2}\psi_Q \in \mathring{\mathcal{G}}(c_Q, 2^{-k}, \beta, \gamma)$.

Paraproducts are defined as follows.

Definition 8.2. For any given three families of bump functions, $\{\psi_Q^{(0)}\}_{Q\in\mathcal{Q}}$, $\{\psi_Q^{(1)}\}_{Q\in\mathcal{Q}}$, and $\{\psi_Q^{(2)}\}_{Q\in\mathcal{Q}}$, the *bilinear* (*discrete*) *paraproduct* is defined as follows:

$$\Pi(f,g)(x) := \sum_{Q\in\mathcal{Q}} \mu(Q)^{-1/2}\langle\psi_Q^{(1)}, f\rangle\langle\psi_Q^{(2)}, g\rangle\psi_Q^{(0)}(x), \quad \forall x\in\mathcal{X}. \tag{8.3}$$

In Lemma 8.9 below, we prove that if $\{\psi_Q^{(1)}\}_{Q\in\mathcal{Q}}$ and $\{\psi_Q^{(2)}\}_{Q\in\mathcal{Q}}$ are smooth molecules and $\{\psi_Q^{(0)}\}_{Q\in\mathcal{Q}}$ are bump functions, then Π as in (8.3) is bounded from $L^2(\mathcal{X})\times L^2(\mathcal{X})$ to $L^1(\mathcal{X})$. Notice that the kernel of Π is given by

$$K_\Pi(x_0,x_1,x_2) = \sum_{Q\in\mathcal{Q}} \mu(Q)^{-1/2}\psi_Q^{(0)}(x_0)\psi_Q^{(1)}(x_1)\psi_Q^{(2)}(x_2), \quad \forall x_0,x_1,x_2\in\mathcal{X}.$$

Indeed, K is a bilinear Calderón–Zygmund kernel provided that $\{\psi_Q^{(0)}\}_{Q\in\mathcal{Q}}$, $\{\psi_Q^{(1)}\}_{Q\in\mathcal{Q}}$ and $\{\psi_Q^{(2)}\}_{Q\in\mathcal{Q}}$ are bump functions with *fast decay*; see Lemma 8.10 below. Therefore, weighted estimates of Π on the 2-fold product of Lebesgue spaces follow from the multilinear Calderón–Zygmund theory of Section 4.

8.2. Almost diagonal estimates. For simplicity, we use the following notation: for any given $\epsilon\in(0,\infty)$, $t>0$ and all $x,y\in\mathcal{X}$,

$$\mathcal{K}(t;\epsilon,x,y) := \frac{1}{V_t(x)+V_t(y)+V(x,y)}\left[\frac{t}{t+d(x,y)}\right]^\epsilon.$$

A useful observation is that the doubling property of μ implies that, when $d(x',x)\le a\epsilon$ and $d(y,y')\le a\epsilon$,

$$\mathcal{K}(t;\epsilon,x,y)\sim\mathcal{K}(t;\epsilon,x',y')$$

with implicit constants depending only on a, ϵ and C_1.

The following off-diagonal estimates for any two test functions are proved in [60, Lemmas 3.2 and 3.19].

LEMMA 8.3. *Let $\epsilon_1\in(0,1]$, $\epsilon_2>0$ and $\epsilon_3>0$. Fix $\epsilon_1'\in(0,\epsilon_1\wedge\epsilon_2)$. Assume that there exist positive constants A_1 and A_2 such that, for any $k\in\mathbb{Z}$, the functions $Q_k:\mathcal{X}\times\mathcal{X}\to\mathbb{C}$ and $P_k:\mathcal{X}\times\mathcal{X}\to\mathbb{C}$ satisfy, for all $x,y\in\mathcal{X}$,*

$$|Q_k(x,y)|\le A_1\mathcal{K}(2^{-k};\epsilon_2,x,y), \quad |P_k(x,y)|\le A_2\mathcal{K}(2^{-k};\epsilon_2,x,y)$$

and

$$Q_kP_\ell(x,y) := \int_{\mathcal{X}} Q_k(x,w)P_\ell(w,y)\,d\mu(w).$$

Then there exists a positive constant C (depending on ϵ_1', ϵ_1, ϵ_2, and ϵ_3) and δ (depending on ϵ_1', ϵ_1 and ϵ_2) such that the following hold:

(i) *when $\ell=k$, for all $x,y\in\mathcal{X}$,*

$$|P_kQ_k(x,y)|\le CA_1A_2\mathcal{K}(2^{-k};\epsilon_2,x,y); \tag{8.4}$$

(ii) *when $\ell>k$, if*

$$\int_{\mathcal{X}} P_\ell(x,w)\,d\mu(w)=0, \quad \forall x\in\mathcal{X},$$

and, for all $x,x',y\in\mathcal{X}$ satisfying $d(x,x')\le[2^{-k}+d(x,y)]/2$,

$$|Q_k(x,y)-Q_k(x',y)|\le A_1\left[\frac{d(x,x')}{2^{-k}+d(x,y)}\right]^{\epsilon_2}\mathcal{K}(2^{-k};\epsilon_2,x,y),$$

then, for all $x, y \in \mathcal{X}$,

$$|P_\ell Q_k(x,y)| \le CA_1A_2 2^{-|k-\ell|\epsilon_1'} \mathcal{K}(2^{-(k\wedge\ell)}; \epsilon_2, x, y). \tag{8.5}$$

An extension of (8.4) and (8.5) to an estimate of three test functions is as follows.

LEMMA 8.4. *Let $\gamma > 0$. Then, for any $\gamma' \in (0,\gamma)$ and $\gamma'' \in (0, \gamma-\gamma')$, there exists a positive constant C, depending only on γ, γ', γ'' and C_1, such that, for all $x_0, x_1, x_2 \in \mathcal{X}$,*

$$\int_{\mathcal{X}} \prod_{j=0}^{2} \mathcal{K}(2^{-k}; \gamma, x, x_j)\, d\mu(x) \le C\mathcal{K}(2^{-k}; \gamma', x_0, x_1)\mathcal{K}(2^{-k}; \gamma'', x_2, x_1). \tag{8.6}$$

The proof of Lemma 8.4 is based on Lemma 8.5 below. In the Euclidean case, Lemma 8.5 was proved in [51, Lemma 1]; see [46] for its extension to spaces of homogeneous type.

LEMMA 8.5. *Let $\epsilon \in (0,\infty)$. Then there exists a positive constant C, depending only on ϵ, C_1 and n, such that, for all $x, w \in \mathcal{X}$, $r \in (0,\infty)$ and $R \in (0,\infty)$,*

$$\int_{d(x,y)<R} \mathcal{K}(r; \epsilon, y, w)\, d\mu(y) \le C \max\left\{\left(\frac{R}{r}\right)^{\epsilon}, 1\right\} V_R(x)\mathcal{K}(r; \epsilon, x, w).$$

Proof of Lemma 8.4. We decompose the space $\mathcal{X}$ as follows:

$$\begin{aligned}
&\int_{\mathcal{X}} \prod_{j=0}^{2} \mathcal{K}(2^{-k}; \gamma, x, x_j)\, d\mu(x) \\
&\quad = \sum_{s=0}^{\infty}\sum_{t=0}^{\infty} \int_{\substack{d(x,x_1)\sim 2^{-k+s} \\ d(x,x_2)\sim 2^{-k+t}}} \prod_{j=0}^{2} \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(x_j) + V(x,x_j)} \left[\frac{1}{1+2^k d(x,x_j)}\right]^{\gamma} d\mu(x) \\
&\quad \lesssim \sum_{s=0}^{\infty}\sum_{t=0}^{\infty} 2^{-s\gamma}2^{-t\gamma} \frac{1}{\mu(B(x_1, 2^{-k+s}))} \frac{1}{\mu(B(x_2, 2^{-k+t}))} \\
&\qquad \times \int_{\substack{d(x,x_1)\sim 2^{-k+s} \\ d(x,x_2)\sim 2^{-k+t}}} \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(x_0) + V(x,x_0)} \left[\frac{1}{1+2^k d(x,x_0)}\right]^{\gamma} d\mu(x),
\end{aligned}$$

where the notation $d(x,x_1) \sim 2^{-k+s}$ means $2^{-k+s} \le d(x,x_1) < 2^{-k+s+1}$ if $s \ge 1$ and $d(x,x_1) \le 2^{-k}$ if $s = 0$; likewise for $d(x,x_2) \sim 2^{-k+t}$.

Assume first that $s \le t$. Take $\gamma' \in (0,\gamma)$ and $\gamma'' \in (0, \gamma-\gamma')$. Observe that, if there exists an $x \in \mathcal{X}$ such that $d(x,x_1) \sim 2^{-k+s}$ and $d(x,x_2) \sim 2^{-k+t}$, then

$$d(x_1,x_2) \le d(x_1,x) + d(x,x_2) \le 2^{-k+2}\max\{2^s, 2^t\},$$

and hence

$$\begin{aligned}
\frac{1}{\mu(B(x_2, 2^{-k+t}))} &= \frac{V_{2^{-k}}(x) + V_{2^{-k}}(x_2) + V(x,x_2)}{\mu(B(x_2, 2^{-k+t}))} [1+2^k d(x_2,x_1)]^{\gamma''} \mathcal{K}(2^{-k}; \gamma'', x_2, x_1) \\
&\lesssim \left[\frac{\max\{2^s, 2^t\}}{2^t}\right]^{n} [\max\{2^s, 2^t\}]^{\gamma''} \mathcal{K}(2^{-k}; \gamma'', x_2, x_1).
\end{aligned}$$

Applying Lemma 8.5 with $\epsilon = \gamma'$, we see that

$$\frac{1}{\mu(B(x_1,2^{-k+s}))}\int_{\substack{d(x,x_1)\sim 2^{-k+s}\\ d(x,x_2)\sim 2^{-k+t}}}\frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(x_0)+V(x,x_0)}\left[\frac{1}{1+2^kd(x,x_0)}\right]^{\gamma}d\mu(x)$$
$$\lesssim \max\{2^{s\gamma'},1\}\mathcal{K}(2^{-k};\gamma',x_0,x_1).$$

From the last two estimates, it follows that

$$\begin{aligned}
&\sum_{s=0}^{\infty}\sum_{t\ge s}2^{-s\gamma}2^{-t\gamma}\frac{1}{\mu(B(x_1,2^{-k+s}))}\frac{1}{\mu(B(x_2,2^{-k+t}))}\\
&\qquad\times\int_{\substack{d(x,x_1)\sim 2^{-k+s}\\ d(x,x_2)\sim 2^{-k+t}}}\frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(x_0)+V(x,x_0)}\left[\frac{1}{1+2^kd(x,x_0)}\right]^{\gamma}d\mu(x)\\
&\qquad\lesssim \sum_{s=0}^{\infty}\sum_{t\ge s}2^{-s\gamma}2^{-t\gamma}\max\{2^{s\gamma'},1\}\left[\frac{\max\{2^s,2^t\}}{2^t}\right]^{n}(\max\{2^s,2^t\})^{\gamma''}\\
&\qquad\quad\times\mathcal{K}(2^{-k};\gamma',x_0,x_1)\mathcal{K}(2^{-k};\gamma'',x_2,x_1)\\
&\qquad\lesssim \mathcal{K}(2^{-k};\gamma',x_0,x_1)\mathcal{K}(2^{-k};\gamma'',x_2,x_1),
\end{aligned}\tag{8.7}$$

where the last inequality is due to the fact that, when $\gamma'\in(0,\gamma-n)$ and $\gamma''\in(0,\gamma-n-\gamma')$,

$$\sum_{s=0}^{\infty}\sum_{t\ge s}2^{-s\gamma}2^{-t\gamma}\max\{2^{s\gamma'},1\}\left[\frac{\max\{2^s,2^t\}}{2^t}\right]^{n}[\max\{2^s,2^t\}]^{\gamma''}<\infty.$$

Likewise, a symmetric argument (by reversing the roles of x_1 and x_2) also implies that

$$\begin{aligned}
&\sum_{s=0}^{\infty}\sum_{t\le s}2^{-s\gamma}2^{-t\gamma}\frac{1}{\mu(B(x_1,2^{-k+s}))}\frac{1}{\mu(B(x_2,2^{-k+t}))}\\
&\qquad\times\int_{\substack{d(x,x_1)\sim 2^{-k+s}\\ d(x,x_2)\sim 2^{-k+t}}}\frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(x_0)+V(x,x_0)}\left[\frac{1}{1+2^kd(x,x_0)}\right]^{\gamma}d\mu(x)\\
&\qquad\qquad\lesssim \mathcal{K}(2^{-k};\gamma',x_0,x_1)\,\mathcal{K}(2^{-k};\gamma'',x_2,x_1).
\end{aligned}\tag{8.8}$$

Combining (8.7) and (8.8), we obtain the desired estimate (8.6). ■

Lemma 8.6. *Suppose that $x_0,x_1,x_2\in\mathcal{X}$ are such that*

$$d(x_0,x_1)\ge d(x_1,x_2)\ge d(x_0,x_2).$$

If $\beta'>0$, $\gamma'>n+\beta'$ and $\gamma''>0$, then there exists a positive constant C, independent of k and $\{x_j\}_{j=0}^2$, such that

$$\sum_{k\in\mathbb{Z}}[2^kd(x_0,x_1)]^{\beta'}\mathcal{K}(2^{-k};\gamma',x_0,x_1)\mathcal{K}(2^{-k};\gamma'',x_2,x_1)$$
$$\le C\frac{1}{[V(x_0,x_1)+V(x_0,x_2)+V(x_1,x_2)]^2}.\tag{8.9}$$

Proof. Define the sets

$$\Lambda_1:=\{k\in\mathbb{Z}:2^kd(x_0,x_1)\le 1\},$$
$$\Lambda_2:=\{k\in\mathbb{Z}:2^kd(x_0,x_1)>1>2^kd(x_1,x_2)\},\qquad \Lambda_3:=\mathbb{Z}\setminus\{\Lambda_1,\Lambda_2\}.$$

By the reverse doubling property of μ, we find that

$$\sum_{k\in\Lambda_1}[2^k d(x_0,x_1)]^{\beta'}\mathcal{K}(2^{-k};\gamma',x_0,x_1)\mathcal{K}(2^{-k};\gamma'',x_2,x_1) \lesssim \sum_{k\in\Lambda_1}\frac{1}{V(x_0,x_1)}\frac{1}{V_{2^{-k}}(x_1)} \lesssim \frac{1}{[V(x_0,x_1)]^2}.$$

Since $\gamma'>\beta'$, it follows that

$$\begin{aligned}\sum_{k\in\Lambda_2}&[2^k d(x_0,x_1)]^{\beta'}\mathcal{K}(2^{-k};\gamma',x_0,x_1)\mathcal{K}(2^{-k};\gamma'',x_2,x_1)\\ &\lesssim \sum_{k\in\Lambda_2}[2^k d(x_0,x_1)]^{\beta'-\gamma'}\frac{1}{V(x_0,x_1)}\frac{1}{V_{2^{-k}}(x_1)}\\ &\lesssim \frac{1}{[V(x_0,x_1)]^2}\sum_{\{k\in\mathbb{Z}:\,2^k d(x_0,x_1)>1\}}[2^k d(x_0,x_1)]^{\beta'-\gamma'}\\ &\lesssim \frac{1}{[V(x_0,x_1)]^2}.\end{aligned}$$

For any $k\in\Lambda_3$, we have $1\le 2^k d(x_1,x_2)\le 2^k d(x_0,x_1)$ and hence, by using $\gamma'>n+\beta'$ and the doubling property of μ, we obtain

$$\begin{aligned}\sum_{k\in\Lambda_3}&[2^k d(x_0,x_1)]^{\beta'}\mathcal{K}(2^{-k};\gamma',x_0,x_1)\mathcal{K}(2^{-k};\gamma'',x_2,x_1)\\ &\lesssim \sum_{k\in\Lambda_3}\frac{1}{V(x_0,x_1)}\frac{1}{V_{2^{-k}}(x_1)}[2^k d(x_0,x_1)]^{\beta'-\gamma'}[2^k d(x_1,x_2)]^{-\gamma''}\\ &\lesssim \frac{1}{[V(x_0,x_1)]^2}\sum_{k\in\Lambda_3}\frac{\mu\big(B(x_1,d(x_1,x_0))\big)}{\mu(B(x_1,2^{-k}))}[2^k d(x_0,x_1)]^{\beta'-\gamma'}[2^k d(x_1,x_2)]^{-\gamma''}\\ &\lesssim \frac{1}{[V(x_0,x_1)]^2}\sum_{\{k\in\mathbb{Z}:\,2^k d(x_1,x_2)\ge 1\}}[2^k d(x_0,x_1)]^{\beta'+n-\gamma'}[2^k d(x_1,x_2)]^{-\gamma''}\\ &\lesssim \frac{1}{[V(x_0,x_1)]^2}.\end{aligned}$$

Also, the doubling condition of μ and the assumption $d(x_0,x_1)\ge d(x_1,x_2)\ge d(x_0,x_2)$ imply that

$$V(x_0,x_1)\sim V(x_0,x_1)+V(x_0,x_2)+V(x_1,x_2).$$

Therefore, summarizing all these estimates, we obtain (8.9). ∎

8.3. Paraproducts as bilinear Calderón–Zygmund operators. This subsection is concerned with the weighted boundedness of the following paraproducts. The approach taken here is that these paraproducts can be viewed as bilinear Calderón–Zygmund operators; see Bényi, Maldonado and Nahmod and Torres [5] for the special case $\mathcal{X}=\mathbb{R}^n$.

LEMMA 8.7. *There exists a positive constant N_∞, depending only on C_1, C_5, C_6, γ and β, such that, for all $Q\in\mathcal{Q}$, bump functions ψ_Q adapted to Q and $h\in L^\infty(\mathcal{X})$,*

$$|\langle\psi_Q,h\rangle|\le N_\infty\mu(Q)^{1/2}\|h\|_{L^\infty(\mathcal{X})}.$$

Proof. The size condition of ψ_Q implies that

$$|\langle \psi_Q, h\rangle| \le \|h\|_{L^\infty(\mathcal{X})}\mu(Q)^{1/2}\int_{\mathcal{X}} \frac{1}{\mu(Q)+V(x,c_Q)}\left[\frac{1}{1+2^k d(x,c_Q)}\right]^\gamma d\mu(x).$$

Without loss of generality, we may as well assume that $Q \in I_k$ for some constant $k \in \mathbb{Z}$, that is, $B(c_Q, C_6 2^{-k}) \subset Q \subset B(c_Q, C_5 2^{-k})$. Therefore, by splitting the integral over $\mathcal{X}$ into the annulus $d(x,c_Q) < 2^{-k}$ and $2^{-k+\nu} \le d(x,c_Q) < 2^{-k+\nu+1}$ for $\nu \ge 1$, we apply the doubling condition of μ to conclude that

$$\int_{\mathcal{X}} \frac{1}{\mu(Q)+V(x,c_Q)}\left[\frac{1}{1+2^k d(x,c_Q)}\right]^\gamma d\mu(x) < \infty. \blacksquare$$

LEMMA 8.8. *Given a family $\{\psi_Q\}$ of smooth molecules, there exists a positive constant N_2, depending only on C_1, C_5, C_6, γ and β, such that, for all $f \in L^2(\mathcal{X})$,*

$$\sum_{Q\in\mathcal{Q}} |\langle \psi_Q, f\rangle|^2 \le N_2\|f\|^2_{L^2(\mathcal{X})}. \tag{8.10}$$

Proof. Without loss of generality, we may assume that $\sum_{Q\in\mathcal{Q}} |\langle \psi_Q, f\rangle|^2$ is finite; otherwise we only need to prove that (8.10) holds for a finite number N of terms of the summation and then let $N \to \infty$. Given $f \in L^2(\mathcal{X})$ with $\|f\|_{L^2(\mathcal{X})} = 1$, by Hölder's inequality, we have

$$\sum_{Q\in\mathcal{Q}} |\langle \psi_Q, f\rangle|^2 = \sum_{Q\in\mathcal{Q}} \langle \psi_Q, f\rangle\langle f, \psi_Q\rangle \le \Big\| \sum_{Q\in\mathcal{Q}} \langle f, \psi_Q\rangle \psi_Q\Big\|_{L^2(\mathcal{X})}.$$

Hence

$$\Big[\sum_{Q\in\mathcal{Q}} |\langle \psi_Q, f\rangle|^2\Big]^2 = \sum_{Q,R\in\mathcal{Q}} \langle \psi_Q, f\rangle\langle \psi_Q, \psi_R\rangle\langle f, \psi_R\rangle.$$

Suppose that $Q = Q^k_\alpha$ for some $k \in \mathbb{Z}$ and $\alpha \in I_k$, and $R = R^{k'}_{\alpha'}$ for some $k' \in \mathbb{Z}$ and $\alpha' \in I_{k'}$. Then the decay, the regularity, and the cancelation conditions for the molecules imply that we can apply (8.5) and (8.4) to deduce that

$$\begin{aligned}|\langle \psi_Q, \psi_R\rangle| &\lesssim \mu(Q^k_\alpha)^{1/2}\mu(R^{k'}_{\alpha'})^{1/2}2^{-|k-k'|\epsilon}\mathcal{K}(2^{-(k\wedge j)};\gamma, c_{Q^k_\alpha}, c_{R^j_{\alpha'}})\\ &\sim \mu(Q^k_\alpha)^{1/2}\mu(R^{k'}_{\alpha'})^{1/2}2^{-|k-k'|\epsilon} \inf_{y\in Q^k_\alpha,\, z\in R^{k'}_{\alpha'}} \mathcal{K}(2^{-(k\wedge j)};\gamma, y, z),\end{aligned}$$

where $\epsilon \in (0, \beta\wedge\gamma)$. Thus,

$$\begin{aligned}\Big[\sum_{Q\in\mathcal{Q}} |\langle \psi_Q, f\rangle|^2\Big]^2 &\lesssim \sum_{k\in\mathbb{Z}}\sum_{\alpha\in I_k}\sum_{k'\in\mathbb{Z}}\sum_{\alpha'\in I_{k'}} 2^{-|k-k'|\epsilon}\mu(Q^k_\alpha)\mu(R^{k'}_{\alpha'})|\langle \mu(Q^k_\alpha)^{-1/2}\psi_{Q^k_\alpha}, f\rangle|\\ &\quad\times |\langle f, \mu(R^{k'}_{\alpha'})^{-1/2}\psi_{R^{k'}_{\alpha'}}\rangle| \inf_{y\in Q^k_\alpha,\, z\in R^{k'}_{\alpha'}} \mathcal{K}(2^{-(k\wedge j)};\gamma, y, z).\end{aligned}$$

By symmetry and estimating the factors $|\langle \mu(Q^k_\alpha)^{-1/2}\psi_{Q^k_\alpha}, f\rangle|$ and $|\langle f, \mu(R^{k'}_{\alpha'})^{-1/2}\psi_{R^{k'}_{\alpha'}}\rangle|$ by the bigger one, we continue the above estimation with

$$\begin{aligned}&\Big[\sum_{Q\in\mathcal{Q}} |\langle \psi_Q, f\rangle|^2\Big]^2\\ &\quad\lesssim \sum_{k\in\mathbb{Z}}\sum_{\alpha\in I_k}\sum_{k'\in\mathbb{Z}}\sum_{\alpha'\in I_{k'}} 2^{-|k-k'|\epsilon}|\langle \psi_{Q^k_\alpha}, f\rangle|^2\mu(R^{k'}_{\alpha'}) \inf_{y\in Q^k_\alpha,\, z\in R^{k'}_{\alpha'}} \mathcal{K}(2^{-(k\wedge j)};\gamma, y, z)\end{aligned}$$

$$
\begin{aligned}
&\lesssim \sum_{k\in\mathbb{Z}}\sum_{\alpha\in I_k} |\langle \psi_{Q_\alpha^k}, f\rangle|^2 \sum_{k'\in\mathbb{Z}} 2^{-|k-k'|\epsilon} \inf_{y\in Q_\alpha^k}\int_{\mathcal{X}} \mathcal{K}(2^{-(k\wedge j)};\gamma,y,z)\,d\mu(z)\\
&\lesssim \sum_{k\in\mathbb{Z}}\sum_{\alpha\in I_k} |\langle \psi_{Q_\alpha^k}, f\rangle|^2\\
&\sim \sum_{Q\in\mathcal{Q}} |\langle \psi_Q, f\rangle|^2.
\end{aligned}
$$

Dividing both sides of this inequality by $\sum_{Q\in\mathcal{Q}}|\langle\psi_Q,f\rangle|^2$ yields (8.10). ■

Lemma 8.9. *Let Π be the bilinear paraproduct defined as in* (8.3) *with $\{\psi_Q^{(1)}\}_{Q\in\mathcal{Q}}$ and $\{\psi_Q^{(2)}\}_{Q\in\mathcal{Q}}$ smooth molecules and $\{\psi_Q^{(0)}\}_{Q\in\mathcal{Q}}$ bump functions. Then Π is bounded from $L^2(\mathcal{X})\times L^2(\mathcal{X})$ to $L^1(\mathcal{X})$ with norm at most a positive constant multiple of N_2N_∞.*

Proof. By duality and Lemmas 8.8 and 8.7, given $h\in L^\infty(\mathcal{X})$, we have

$$
\begin{aligned}
|\langle \Pi(f,g),h\rangle| &= \Big|\sum_{Q\in\mathcal{Q}} \mu(Q)^{-1/2}\langle\psi_Q^{(1)},f\rangle\langle\psi_Q^{(2)},g\rangle\langle\psi_Q^{(0)},h\rangle\Big|\\
&\le N_\infty\|h\|_{L^\infty(\mathcal{X})}\sum_{Q\in\mathcal{Q}}|\langle\psi_Q^{(1)},f\rangle|\,|\langle\psi_Q^{(2)},g\rangle|\\
&\le N_\infty\|h\|_{L^\infty(\mathcal{X})}\Big[\sum_{Q\in\mathcal{Q}}|\langle\psi_Q^{(1)},f\rangle|^2\Big]^{1/2}\Big[\sum_{Q\in\mathcal{Q}}|\langle\psi_Q^{(2)},g\rangle|^2\Big]^{1/2}\\
&\le N_2N_\infty\|f\|_{L^2(\mathcal{X})}\|g\|_{L^2(\mathcal{X})}\|h\|_{L^\infty(\mathcal{X})}. \quad ■
\end{aligned}
$$

Lemma 8.10. *Let Π be the bilinear paraproduct defined as in* (8.3) *with $\{\psi_Q^{(0)}\}_{Q\in\mathcal{Q}}$, $\{\psi_Q^{(1)}\}_{Q\in\mathcal{Q}}$ and $\{\psi_Q^{(2)}\}_{Q\in\mathcal{Q}}$ being bump functions satisfying* (8.1) *and* (8.2) *with $\beta>0$ and $\gamma>n$. Then, for any given $\beta'\in(0,\beta\wedge(\gamma-n))$, Π has a Calderón–Zygmund kernel $K_\Pi\in\mathrm{Ker}(2,C_K,\beta')$, where C_K is a positive constant depending only on C_1,C_5,C_6, γ, β, and β'.*

Proof. Notice that the kernel of Π is given by

$$
K_\Pi(x_0,x_1,x_2)=\sum_{Q\in\mathcal{Q}}\mu(Q)^{-1/2}\psi_Q^{(0)}(x_0)\psi_Q^{(1)}(x_1)\psi_Q^{(2)}(x_2).
$$

By Lemma 2.5, we may as well assume that $Q=Q_\alpha^k$ for some $k\in\mathbb{Z}$ and $\alpha\in I_k$. Without loss of generality we may also assume that

$$
d(x_0,x_1)\ge d(x_1,x_2)\ge d(x_0,x_2).
$$

For any $\gamma'\in(0,\gamma-n)$ and $\gamma''\in(0,\gamma-n-\gamma')$, by Lemmas 8.4 and 8.6, there exists a positive constant C, independent of $\{x_j\}_{j=0}^2$, such that, for all $x_0,x_1,x_2\in\mathcal{X}$,

$$
\begin{aligned}
|K_\Pi(x_0,x_1,x_2)| &\le \sum_{Q\in\mathcal{Q}}\mu(Q)^{-1/2}|\psi_Q^{(0)}(x_0)\psi_Q^{(1)}(x_1)\psi_Q^{(2)}(x_2)|\\
&\lesssim \sum_{k\in\mathbb{Z}}\sum_{\alpha\in I_k}\mu(Q_\alpha^k)\prod_{j=0}^{2}\mathcal{K}(2^{-k};\gamma,c_{Q_\alpha^k},x_j)
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \sum_{k\in\mathbb{Z}}\sum_{\alpha\in I_k}\mu(Q_\alpha^k)\inf_{x\in Q_\alpha^k}\prod_{j=0}^{2}\mathcal{K}(2^{-k};\gamma,x,x_j)\\
&\lesssim \sum_{k\in\mathbb{Z}}\int_{\mathcal{X}}\prod_{j=0}^{2}\mathcal{K}(2^{-k};\gamma,x,x_j)\,d\mu(x)\\
&\lesssim \sum_{k\in\mathbb{Z}}\mathcal{K}(2^{-k};\gamma',x_0,x_1)\mathcal{K}(2^{-k};\gamma'',x_2,x_1)\\
&\lesssim \frac{1}{[V(x_0,x_1)+V(x_0,x_2)+V(x_1,x_2)]^2}.
\end{aligned}
$$

Now we prove the regularity of K_Π. Fix $\beta'\in(0,\beta\wedge(\gamma-n))$. Given any $x_1,x_1'\in\mathcal{X}$ with

$$
d(x_1,x_1')\le \max_{0\le i,j\le 2}\{d(x_i,x_j)/2\}=d(x_0,x_1)/2,
$$

we write

$$
\begin{aligned}
&|K_\Pi(x_0,x_1,x_2)-K_\Pi(x_0,x_1',x_2)|\\
&\qquad=\Big|\sum_{k\in\mathbb{Z}}\sum_{\alpha\in I_k}\mu(Q_\alpha^k)^{-1/2}[\psi^{(1)}_{Q_\alpha^k}(x_1)-\psi^{(1)}_{Q_\alpha^k}(x_1')]\psi^{(2)}_{Q_\alpha^k}(x_2)\psi^{(0)}_{Q_\alpha^k}(x_0)\Big|. \qquad (8.11)
\end{aligned}
$$

For every $k\in\mathbb{Z}$, set

$$
I_{k,1}:=\{\alpha\in I_k: d(x_1,x_1')\le(2^{-k}+d(x_1,c_{Q_\alpha^k}))/2\}\quad\text{and}\quad I_{k,2}:=I_k\setminus I_{k,1}.
$$

Observe that, when $\alpha\in I_{k,1}$, we have $d(x_1,x_1')\le(2^{-k}+d(x_1,c_{Q_\alpha^k}))/2$ and hence, by (8.2),

$$
\begin{aligned}
&\mu(Q_\alpha^k)^{-1/2}|\psi^{(1)}_{Q_\alpha^k}(x_1)-\psi^{(1)}_{Q_\alpha^k}(x_1')|\,|\psi^{(2)}_{Q_\alpha^k}(x_2)\psi^{(0)}_{Q_\alpha^k}(x_0)|\\
&\qquad\lesssim \mu(Q_\alpha^k)\left[\frac{d(x_1,x_1')}{2^{-k}+d(x_1,c_{Q_\alpha^k})}\right]^{\beta'}\prod_{j=0}^{2}\mathcal{K}(2^{-k};\gamma,c_{Q_\alpha^k},x_j)\\
&\qquad\lesssim \mu(Q_\alpha^k)\left[\frac{d(x_1,x_1')}{d(x_1,x_0)}\right]^{\beta'}[2^kd(x_0,x_1)]^{\beta'}\prod_{j=0}^{2}\mathcal{K}(2^{-k};\gamma,c_{Q_\alpha^k},x_j).
\end{aligned}
$$

From this, we apply Lemmas 8.4 and 8.6 to see that, for $\gamma'\in(n+\beta',\gamma)$ and $\gamma''\in(0,\gamma)$,

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}}\sum_{\alpha\in I_{k,1}}\mu(Q_\alpha^k)^{-1/2}|[\psi^{(1)}_{Q_\alpha^k}(x_1)-\psi^{(1)}_{Q_\alpha^k}(x_1')]\psi^{(2)}_{Q_\alpha^k}(x_2)\psi^{(0)}_{Q_\alpha^k}(x_0)|\\
&\quad\lesssim \sum_{k\in\mathbb{Z}}\sum_{\alpha\in I_{k,1}}\mu(Q_\alpha^k)\left[\frac{d(x_1,x_1')}{d(x_1,x_0)}\right]^{\beta'}[2^kd(x_0,x_1)]^{\beta'}\prod_{j=0}^{2}\mathcal{K}(2^{-k};\gamma,c_{Q_\alpha^k},x_j)\\
&\quad\lesssim \left[\frac{d(x_1,x_1')}{d(x_1,x_0)}\right]^{\beta'}\sum_{k\in\mathbb{Z}}[2^kd(x_0,x_1)]^{\beta'}\int_{\mathcal{X}}\prod_{j=0}^{2}\mathcal{K}(2^{-k};\gamma,x,x_j)\,d\mu(x)\\
&\quad\lesssim \left[\frac{d(x_1,x_1')}{d(x_1,x_0)}\right]^{\beta'}\sum_{k\in\mathbb{Z}}[2^kd(x_0,x_1)]^{\beta'}\mathcal{K}(2^{-k};\gamma',x_0,x_1)\mathcal{K}(2^{-k};\gamma'',x_2,x_1)\\
&\quad\lesssim \left[\frac{d(x_1,x_1')}{d(x_1,x_0)}\right]^{\beta'}\frac{1}{[V(x_0,x_1)+V(x_0,x_2)+V(x_1,x_2)]^2}. \qquad (8.12)
\end{aligned}
$$

When $\alpha \in I_{k,2}$, we have $1 < \frac{2d(x_1,x_1')}{2^{-k}+d(x_1,c_{Q_\alpha^k})}$, and hence

$$\begin{aligned}
&\mu(Q_\alpha^k)^{-1/2}|\psi_{Q_\alpha^k}^{(1)}(x_1)\psi_{Q_\alpha^k}^{(2)}(x_2)\psi_{Q_\alpha^k}^{(0)}(x_0)| \\
&\qquad\lesssim \mu(Q_\alpha^k)\left[\frac{d(x_1,x_1')}{2^{-k}+d(x_1,c_{Q_\alpha^k})}\right]^{\beta'}\prod_{j=0}^{2}\mathcal{K}(2^{-k};\gamma,c_{Q_\alpha^k},x_j) \\
&\qquad\lesssim \mu(Q_\alpha^k)\left[\frac{d(x_1,x_1')}{d(x_1,x_0)}\right]^{\beta'}[2^k d(x_0,x_1)]^{\beta'}\prod_{j=0}^{2}\mathcal{K}(2^{-k};\gamma,c_{Q_\alpha^k},x_j),
\end{aligned}$$

so an argument similar to that used in (8.12) gives

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}}\sum_{\alpha\in I_{k,2}}\mu(Q_\alpha^k)^{-1/2}|\psi_{Q_\alpha^k}^{(1)}(x_1)\psi_{Q_\alpha^k}^{(2)}(x_2)\psi_{Q_\alpha^k}^{(0)}(x_0)| \\
&\qquad\lesssim \left[\frac{d(x_1,x_1')}{d(x_1,x_0)}\right]^{\beta'}\frac{1}{[V(x_0,x_1)+V(x_0,x_2)+V(x_1,x_2)]^2}. \qquad (8.13)
\end{aligned}$$

The assumption $d(x_1,x_1') \le d(x_0,x_1)/2$ implies that $d(x_0,x_1) \le 2d(x_0,x_1')$ and

$$2d(x_1',x_0) \ge d(x_1,x_2) \ge d(x_0,x_2).$$

Also, when $\alpha \in I_{k,2}$, we have $d(x_1,x_1') > [2^{-k}+d(x_1,c_{Q_\alpha^k})]/3$ and hence

$$\begin{aligned}
&\mu(Q_\alpha^k)^{-1/2}|\psi_{Q_\alpha^k}^{(1)}(x_1')\psi_{Q_\alpha^k}^{(2)}(x_2)\psi_{Q_\alpha^k}^{(0)}(x_0)| \\
&\qquad\lesssim \mu(Q_\alpha^k)\left[\frac{d(x_1,x_1')}{2^{-k}+d(x_1',c_{Q_\alpha^k})}\right]^{\beta'}\mathcal{K}(2^{-k};\gamma,c_{Q_\alpha^k},x_1')\prod_{j=0,2}\mathcal{K}(2^{-k};\gamma,c_{Q_\alpha^k},x_j) \\
&\qquad\lesssim \mu(Q_\alpha^k)\left[\frac{d(x_1,x_1')}{d(x_1,x_0)}\right]^{\beta'}[2^k d(x_0,x_1)]^{\beta'}\mathcal{K}(2^{-k};\gamma,c_{Q_\alpha^k},x_1')\prod_{j=0,2}\mathcal{K}(2^{-k};\gamma,c_{Q_\alpha^k},x_j),
\end{aligned}$$

so an argument similar to that used in (8.12) shows that

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}}\sum_{\alpha\in I_{k,2}}\mu(Q_\alpha^k)^{-1/2}|\psi_{Q_\alpha^k}^{(1)}(x_1)\psi_{Q_\alpha^k}^{(2)}(x_2)\psi_{Q_\alpha^k}^{(0)}(x_0)| \\
&\qquad\lesssim \left[\frac{d(x_1,x_1')}{d(x_1,x_0)}\right]^{\beta'}\frac{1}{[V(x_0,x_1')+V(x_0,x_2)+V(x_1',x_2)]^2} \\
&\qquad\lesssim \left[\frac{d(x_1,x_1')}{d(x_1,x_0)}\right]^{\beta'}\frac{1}{[V(x_0,x_1)+V(x_0,x_2)+V(x_1,x_2)]^2}. \qquad (8.14)
\end{aligned}$$

Inserting the estimates (8.12)–(8.14) into (8.11), we see that K_Π satisfies the regularity condition in the x_1-variable. Similar proofs show the regularity of K_Π in the x_0- and x_2-variables. Thus, K_Π satisfies (3.1) and (3.2) in the definition of a Calderón–Zygmund kernel, which completes the proof of Lemma 8.10. ∎

Consequently, applying Lemmas 8.9 and 8.10 and the theory of weighted multilinear singular integrals in Section 4 (see Corollaries 4.14 and 4.15) as well as Theorem 5.6, we have the following conclusions, the details being omitted.

THEOREM 8.11. *Let* Π *be the bilinear paraproduct defined as in* (8.3), *with* $\{\psi_Q^{(1)}\}_{Q\in\mathcal{Q}}$ *and* $\{\psi_Q^{(2)}\}_{Q\in\mathcal{Q}}$ *being smooth molecules, and* $\{\psi_Q^{(0)}\}_{Q\in\mathcal{Q}}$ *being bump functions, satisfying* (8.1) *and* (8.2) *with* $\beta > 0$ *and* $\gamma > n$. *Let* $1/2 \le p < \infty$ *and* $1 \le p_1, p_2 < \infty$ *be such that* $1/p_1 + 1/p_2 = 1/p$. *Let* $\nu \in A_\infty$, $\{w_j\}_{j=1}^2$ *be weights and* $\vec{w} := (w_1, w_2)$. *Then:*

(i) Π *is a bilinear Calderón–Zygmund singular integral operator;*
(ii) *if* $p_j \ge 1$ *for all* $j \in \{1,2\}$ *and* $(\nu;\vec{w})$ *satisfies* (4.1), *then* Π *is bounded from* $L^{p_1}(w_1) \times L^{p_2}(w_2)$ *to* $L^{p,\infty}(\nu)$;
(iii) *if* $p_j > 1$ *for all* $j \in \{1,2\}$ *and* $(\nu;\vec{w})$ *satisfies the bump weight condition* (4.8) *for some* $r > 1$, *then* Π *is bounded from* $L^{p_1}(w_1) \times L^{p_2}(w_2)$ *to* $L^p(\nu)$.

THEOREM 8.12. *Let* Π *be as in Theorem* 8.11. *For any* $1/2 \le p < \infty$ *and* $1 \le p_1,\, p_2 < \infty$ *such that* $1/p_1 + 1/p_2 = 1/p$, *and* $\vec{w} := (w_1, w_2) \in A_{(p,q)}$ *with* $\nu_{\vec{w}} := w_1^{p/p_1} w_2^{p/p_2}$, *the following hold:*

(i) Π *is bounded from* $L^{p_1}(w_1) \times L^{p_2}(w_2)$ *to* $L^p(\nu_{\vec{w}})$ *if* $p_1, p_2 \in (1,\infty)$;
(ii) Π *is bounded from* $L^{p_1}(w_1) \times L^{p_2}(w_2)$ *to* $L^{p,\infty}(\nu_{\vec{w}})$ *if* $p_1 = 1$ *or* $p_2 = 1$.

8.4. Boundedness of paraproducts on Triebel–Lizorkin and Besov spaces. Recall that paraproducts considered in Subsection 8.3 can be viewed as bilinear Calderón–Zygmund operators. Consequently, Theorems 6.14 and 6.15 imply the boundedness of such paraproducts on products of Triebel–Lizorkin or Besov spaces. We have the following conclusion.

THEOREM 8.13. *Let* Π *be the bilinear paraproduct defined by*

$$\Pi(f,g)(x) := \sum_{Q\in\mathcal{Q}} \mu(Q)^{-1/2} \langle \psi_Q^{(1)}, f\rangle \langle \psi_Q^{(2)}, g\rangle \psi_Q^{(0)}(x), \qquad \forall x \in \mathcal{X},$$

where, for $i \in \{0,1,2\}$, $\{\psi_Q^{(i)}\}_{Q\in\mathcal{Q}}$ *are* (β,γ)*-molecules with* $\beta > 0$ *and* $\gamma > n$. *Let* κ *be the constant appearing in the reverse doubling condition* (2.2) *and let* ϵ *satisfy*

$$0 < \epsilon \le \kappa \quad \text{and} \quad 0 < \epsilon < \min\left\{1, \frac{1}{2}[\beta \wedge (\gamma - n)]\right\}.$$

Then:

(a) *For every* $j \in \{0,1,2\}$, *let* $|s_j| < \epsilon$, $p(s_j,\epsilon) < p_j < \infty$ *and* $p(s_j,\epsilon) < q_j < \infty$ *be such that*

$$s_0 = s_1 + s_2, \quad \frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad \frac{1}{q_0} = \frac{1}{q_1} + \frac{1}{q_2},$$

and let $\dot{F}^{s_j}_{p_j,q_j}(\mathcal{X})$ *be the Triebel–Lizorkin space as defined in Definition* 6.9(ii). *Then* Π *can be extended to a bounded bilinear operator from* $\dot{F}^{s_1}_{p_1,q_1}(\mathcal{X}) \times \dot{F}^{s_2}_{p_2,q_2}(\mathcal{X})$ *to* $\dot{F}^{s_0}_{p_0,q_0}(\mathcal{X})$.
(b) *For every* $j \in \{0,1,2\}$, *let* $|s_j| < \epsilon$, $p(s_j,\epsilon) < p_j < \infty$ *and* $0 < q_j < \infty$ *be such that*

$$s_0 = s_1 + s_2, \quad \frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad \frac{1}{q_0} = \frac{1}{q_1} + \frac{1}{q_2},$$

and let $\dot{B}^{s_j}_{p_j,q_j}(\mathcal{X})$ *be the Besov space as defined in Definition* 6.9(i). *Then* Π *can be extended to a bounded bilinear operator from* $\dot{B}^{s_1}_{p_1,q_1}(\mathcal{X}) \times \dot{B}^{s_2}_{p_2,q_2}(\mathcal{X})$ *to* $\dot{B}^{s_0}_{p_0,q_0}(\mathcal{X})$.

Proof. From Lemma 8.10, it follows that the kernel of Π is of the class $\mathrm{Ker}(2, C, \beta')$ for some positive constant C and $\beta' \in (0, \beta \wedge (\gamma - n))$ as in Definition 3.1. Also, by Lemma 8.9, Π is bounded from $L^2(\mathcal{X}) \times L^2(\mathcal{X})$ to $L^1(\mathcal{X})$, so $\Pi \in \mathbf{BWBP}(\eta)$ for all $\eta \in (0, \epsilon]$. Since every $\psi_Q^{(i)}$ is a molecule, we have $\int_{\mathcal{X}} \psi_Q^{(i)}(x)\, d\mu(x) = 0$ and hence

$$\Pi(1, g) = \Pi(g, 1) = \Pi^{*,1}(1, g) = 0, \qquad \forall g \in (C_b^\eta(\mathcal{X}))'.$$

Applying Theorems 6.14 and 6.15 yields (a) and (b), respectively. ∎

9. Bilinear multiplier operators on Triebel–Lizorkin and Besov spaces

t is well known that the boundedness of pseudodifferential operators associated with some classes of symbols is a typical application of the Calderón–Zygmund theory. The main goal of this section is to apply the previous bilinear $T1$-theorems to obtain the boundedness of bilinear multiplier operators on products of Triebel–Lizorkin and Besov spaces.

9.1. Bilinear multiplier operators. Consider the bilinear multiplier operators of the form

$$T_\sigma(f,g)(x) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sigma(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{ix\cdot(\xi+\eta)}\,d\xi\,d\eta, \qquad \forall x\in\mathbb{R}^n, \tag{9.1}$$

where $\widehat{f}$ denotes the Fourier transform of the function f, that is,

$$\widehat{f}(x) := \int_{\mathbb{R}^n} f(\xi)e^{-i\xi\cdot x}\,d\xi, \qquad \forall x\in\mathbb{R}^n,$$

and the *symbol* σ is an infinitely differentiable function on $(\mathbb{R}^n\times\mathbb{R}^n)\setminus\{(0,0)\}$ with the property that, for any multiindices $\beta\in\mathbb{Z}_+^2$ and $\gamma\in\mathbb{Z}_+^2$, there exists a positive constant $C_{\gamma,\beta}$ such that, for all points $(\xi,\eta)\in(\mathbb{R}^n\times\mathbb{R}^n)\setminus\{(0,0)\}$,

$$|\partial_\xi^\gamma\partial_\eta^\beta\sigma(\xi,\eta)| \le C_{\gamma,\beta}(|\xi|+|\eta|)^{-|\gamma|-|\beta|}. \tag{9.2}$$

These bilinear multipliers are also known as *Coifman–Meyer multipliers* and render the bilinear version of the (linear) Mikhlin multipliers. The basic mapping properties of these operators in Lebesgue spaces have been obtained in [19, 20, 21].

Using the molecular decompositions of homogeneous Besov spaces, Grafakos and Torres in [52, Theorem 3] also proved that T_σ is bounded from $\dot B^{\alpha_1}_{p,p}(\mathbb{R}^n)\times\dot B^{\alpha_2}_{q,q}(\mathbb{R}^n)$ to $\dot B^{\alpha_1+\alpha_2}_{r,r}(\mathbb{R}^n)$ for all $\alpha_1,\alpha_2>0$, $1<p,q,r<\infty$ and $1/p+1/q=1/r$, provided that σ satisfies (9.2) and the following cancelation conditions: for all $\xi\neq 0$,

$$\partial_\xi^\beta\sigma(0,\xi) = \partial_\xi^\beta\sigma(\xi,0) = \partial_\xi^\beta\sigma(\xi,-\xi) = 0 \tag{9.3}$$

for all multiindices β up to a suitable order and all $\xi\in\mathbb{R}^n\setminus\{0\}$. Under the assumption that σ satisfies (9.2) and (9.3) for all $|\beta|\le 1$, Bényi [3, Proposition 3] proved that

$$T_\sigma : \dot F^0_{p_1,q_1}(\mathbb{R}^n)\times\dot F^0_{p_2,q_2}(\mathbb{R}^n)\to\dot F^0_{p_0,q_0}(\mathbb{R}^n)$$

when $1<p_1,\,p_2,\,p_0<\infty$, $1<q_1,\,q_2,\,q_0\le\infty$, $1/p_0=1/p_1+1/p_2$ and $1/q_0=1/q_1+1/q_2$. For $1\le p_1,\,p_2,\,p_0\le\infty$, $1/p_0=1/p_1+1/p_2$, $1\le q\le\infty$ and $s\in\mathbb{R}$, the boundedness of

$$T_\sigma : \dot B^s_{p_1,q}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)\to\dot B^s_{p_0,q}(\mathbb{R}^n)$$

was studied by Maldonado and Naibo [78, Theorem 3.1].

The operator T_σ as in (9.1) is related to bilinear operators in the following way: for all compactly supported Schwartz functions f, g and for all $x \notin \operatorname{supp} f \cap \operatorname{supp} g$,

$$T_\sigma(f,g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K_\sigma(x,y_1,y_2) f(y_1) g(y_2)\, dy_1\, dy_2,$$

where the kernel K_σ is defined by

$$K_\sigma(x,y_1,y_2) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(\xi,\eta) e^{i\xi\cdot(x-y_1)} e^{i\eta\cdot(x-y_2)}\, d\xi\, d\eta, \qquad \forall x,y_1,y_2 \in \mathbb{R}^n.$$

By using $K(x,y_1,y_2) = K^{*,1}(y_1,x,y_2) = K^{*,2}(y_2,y_1,x)$ and the Fourier transform, for all $x \in \mathbb{R}^n$, we obtain

$$T_\sigma^{*,1}(f,g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(-\xi-\eta,\, \eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{ix\cdot(\xi+\eta)}\, d\xi\, d\eta,$$

$$T_\sigma^{*,2}(f,g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(\xi,\, -\xi-\eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{ix\cdot(\xi+\eta)}\, d\xi\, d\eta.$$

In what follows, for convenience, we denote by $T_\sigma^{*,0}$ the operator T_σ.

9.2. Off-diagonal estimates for bilinear multiplier operators. Now we fix a real-valued radial function $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\operatorname{supp} \widehat{\phi} \subset \{\xi \in \mathbb{R}^n : \pi/4 \le |\xi| \le \pi\},$$

$\widehat{\phi}$ is bounded away from zero on the annulus $\{\xi \in \mathbb{R}^n : \pi/2 \le |\xi| \le 3\pi/4\}$ and, for all $\xi \neq 0$,

$$\sum_{\nu\in\mathbb{Z}} |\widehat{\phi}(2^\nu \xi)|^2 = 1. \tag{9.4}$$

For $\nu \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, let $Q_{\nu,k}$ be the dyadic cube

$$Q_{\nu,k} := \{(x_1,\dots,x_n) \in \mathbb{R}^n : k_i \le 2^\nu x_i < k_i + 1 \text{ for all } i \in \{1,\dots,n\}\}.$$

The lower-left corner of $Q_{\nu,k}$ is denoted by $2^{-\nu}k$ and we set

$$\phi_{\nu,k}(x) := 2^{\nu n/2} \phi(2^\nu x - k), \qquad \forall x \in \mathbb{R}^n.$$

Obviously, $\widehat{\phi_{\nu,k}}(x) = 2^{-\nu n/2} 2^{-i2^{-\nu}k\cdot x} \widehat{\phi}(2^{-\nu}x)$ for all $x \in \mathbb{R}^n$ and

$$\operatorname{supp} \widehat{\phi_{\nu,k}} \subset \{\xi \in \mathbb{R}^n : 2^{\nu-2}\pi \le |\xi| \le 2^\nu \pi\}.$$

Also, for any given multiindex γ and $N \in \mathbb{N}$, there exists a positive constant $C_{\gamma,N,n}$, depending only on γ, N, n, such that, for all $x \in \mathbb{R}^n$,

$$|\partial^\gamma \phi_{\nu,k}(x)| \le C_{\gamma,N,n} \frac{2^{\nu n/2} 2^{|\gamma|\nu}}{(1 + 2^\nu |x - 2^{-\nu}k|)^N}.$$

The condition (9.4) implies the following Calderón reproducing formula: if $f \in \mathcal{S}(\mathbb{R}^n)$ (or $(\mathcal{S}(\mathbb{R}^n))'$), then f can be written as

$$f = \sum_{\nu\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \langle f, \phi_{\nu,k}\rangle \phi_{\nu,k}, \tag{9.5}$$

where the series converges in $\mathcal{S}(\mathbb{R}^n)$ (or $(\mathcal{S}(\mathbb{R}^n))'$); see Frazier and Jawerth [33].

Regarding the wavelets $\{\phi_{\nu,k}\}_{\nu\in\mathbb{Z},\, k\in\mathbb{Z}^n}$, we have the following off-diagonal estimates, which are indeed parallel to those of Theorem 6.8.

LEMMA 9.1. *Let T_σ be as in* (9.1) *and let the symbol σ satisfy* (9.2) *and the cancelation condition: for all $\xi \neq 0$,*

$$\sigma(\xi,0)=\sigma(0,\xi)=\sigma(\xi,-\xi)=0. \tag{9.6}$$

Then, for all integers $N>n$, there exists a positive constant $C_{N,n}$ such that, for all $x\in\mathbb{R}^n$, $\nu,\mu\in\mathbb{Z}$ and $k,\ell\in\mathbb{Z}^n$ and all $i\in\{0,1,2\}$,

$$|T_\sigma^{*,i}(\phi_{\nu,k},\phi_{\mu,\ell})(x)|\le C_{N,n}\frac{2^{-|\nu-\mu|}\,2^{\nu n/2}\,2^{\mu n/2}}{(1+2^\nu|x-2^{-\nu}k|)^N\,(1+2^\mu|x-2^{-\mu}\ell|)^N}. \tag{9.7}$$

Moreover, if $\nu\ge\lambda\ge\mu$, then, for all $i\in\{0,1,2\}$,

$$\begin{aligned}&|\langle\phi_{\lambda,m},\,T_\sigma^{*,i}(\phi_{\nu,k},\phi_{\mu,\ell})\rangle|\\&\qquad\le C_{N,n}\frac{2^{-|\nu-\mu|}\,2^{-\nu n/2}\,2^{\lambda n/2}2^{\mu n/2}}{(1+2^\lambda|2^{-\lambda}m-2^{-\nu}k|)^N\,(1+2^\mu|2^{-\lambda}m-2^{-\mu}\ell|)^N}.\end{aligned} \tag{9.8}$$

Proof. Observe that (9.8) follows easily from (9.7) and [51, Proposition 3]. So it suffices to prove (9.7). Without loss of generality, we may as well assume that $\nu\ge\mu$.

First we prove that (9.7) holds when $i=0$. By a simple change of variables and the cancelation condition $\sigma(\xi,0)=0$, we see that, for all $x\in\mathbb{R}^n$,

$$\begin{aligned}&T_\sigma(\phi_{\nu,k},\phi_{\mu,\ell})(x)\\&\quad=2^{\nu n/2}2^{\mu n/2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{i((2^\nu x-k)\xi+(2^\mu x-\ell)\eta)}\sigma(2^\nu\xi,2^\mu\eta)\widehat{\phi}(\xi)\widehat{\phi}(\eta)\,d\xi\,d\eta\\&\quad=2^{\nu n/2}2^{\mu n/2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{i((2^\nu x-k)\xi+(2^\mu x-\ell)\eta)}[\sigma(2^\nu\xi,2^\mu\eta)-\sigma(2^\nu\xi,0)]\widehat{\phi}(\xi)\widehat{\phi}(\eta)\,d\xi\,d\eta,\end{aligned}$$

where the integration indeed takes place for $\pi/4\le|\xi|\le\pi$ and $\pi/4\le|\eta|\le\pi$. If $\triangle_\xi$ denotes the Laplace operator in ξ, then

$$(1-\triangle_\xi)^N e^{i((2^\nu x-k)\xi+(2^\mu x-\ell)\eta)}=(1+|2^\nu x-k|)^N e^{i((2^\nu x-k)\xi+(2^\mu x-\ell)\eta)}.$$

Therefore, integration by parts gives us

$$\begin{aligned}T_\sigma(\phi_{\nu,k},\phi_{\mu,\ell})(x)=&\frac{2^{\nu n/2}2^{\mu n/2}}{(1+|2^\nu x-k|)^N}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{i((2^\nu x-k)\xi+(2^\mu x-\ell)\eta)}\\&\times(1-\triangle_\xi)^N([\sigma(2^\nu\xi,2^\mu\eta)-\sigma(2^\nu\xi,0)]\widehat{\phi}(\xi))\widehat{\phi}(\eta)\,d\xi\,d\eta.\end{aligned} \tag{9.9}$$

Notice that

$$\begin{aligned}&|(1-\triangle_\xi)^N([\sigma(2^\nu\xi,2^\mu\eta)-\sigma(2^\nu\xi,0)]\widehat{\phi}(\xi))|\\&\qquad\lesssim\sum_{\substack{\alpha_1,\,\alpha_2\in\mathbb{Z}_+^n\\|\alpha_1|+|\alpha_2|\le2N}}|\partial_\xi^{\alpha_1}[\sigma(2^\nu\xi,2^\mu\eta)-\sigma(2^\nu\xi,0)]|\,|\partial_\xi^{\alpha_2}\widehat{\phi}(\xi)|.\end{aligned}$$

By the mean value theorem and (9.2), we conclude that, for some $\theta\in(0,1)$,

$$|\partial_\xi^{\alpha_1}[\sigma(2^\nu\xi,2^\mu\eta)-\sigma(2^\nu\xi,0)]|\lesssim\frac{2^{\nu|\alpha_1|}|2^\mu\eta|}{(|2^\nu\xi|+|2^\mu\theta\eta|)^{|\alpha_1|+1}}\lesssim\frac{2^{\nu|\alpha_1|}|2^\mu\eta|}{|2^\nu\xi|^{|\alpha_1|+1}}\lesssim2^{\mu-\nu},$$

where the second inequality is due to the facts $\nu\ge\mu$, $|\xi|\sim1$ and $|\eta|\sim1$. Since $\phi\in\mathcal{S}(\mathbb{R}^n)$, it follows that $|\partial^{\alpha_2}\widehat{\phi}(\xi)| \lesssim 1$. Summarizing these estimates gives that

$$|(1-\triangle_\xi)^N([\sigma(2^\nu\xi,2^\mu\eta)-\sigma(2^\nu\xi,0)]\widehat{\phi}(\xi))| \lesssim 2^{-|\nu-\mu|}.$$

Inserting this estimate into (9.9) further implies that, for all $x\in\mathbb{R}^n$,

$$|T_\sigma(\phi_{\nu,k},\phi_{\mu,\ell})(x)| \lesssim \frac{2^{\nu n/2}2^{\mu n/2}2^{-|\nu-\mu|}}{(1+|2^\nu x-k|)^N}. \tag{9.10}$$

The same computations using integration by parts in the variable η give us that, for all $x\in\mathbb{R}^n$,

$$|T_\sigma(\phi_{\nu,k},\phi_{\mu,\ell})(x)| \lesssim \frac{2^{\nu n/2}2^{\mu n/2}2^{-|\nu-\mu|}}{(1+|2^\mu x-\ell|)^N}. \tag{9.11}$$

Considering the geometric mean between (9.10) and (9.11) shows that (9.7) holds for $i=0$.

To prove that (9.7) holds for $i=1$ or $i=2$, we repeat the above computations but using the expression of $T_\sigma^{*,i}$ and the cancelation condition $\sigma(\xi,-\xi)=0$, the details being omitted. ■

Remark 9.2. (i) The decay coefficient $2^{-|\nu-\mu|}$ in (9.7) and (9.8) comes from the cancelation condition (9.6). Indeed, if instead of (9.6) we assume that $\partial^\gamma\sigma(\xi,0)=\partial^\gamma\sigma(0,\xi)=\partial^\gamma\sigma(-\xi,\xi)=0$ for multiindices γ such that $|\gamma|\le L$, then both (9.7) and (9.8) hold with the decay $2^{-|\nu-\mu|}$ there replaced by $2^{-|\nu-\mu|(L+1)}$.

(ii) From the conclusion that (9.8) holds for $T_\sigma^{*,i}$ with $i\in\{0,1,2\}$, it follows easily that T_σ satisfies all off-diagonal estimates (i)–(vi) listed in Theorem 6.8, but with the indices γ, γ' and σ there being chosen as N, N and 1, respectively.

9.3. Boundedness of bilinear multiplier operators. In analogy with Lemma 6.12, we have the following estimate from [33, p. 147, Lemma A.2 and Remark A.3].

Lemma 9.3. *Let $\nu\in\mathbb{Z}$ and $\nu'\in\mathbb{Z}$. Then, for any given $r\in(0,1]$ and $N>n/r$, there exists a positive constant C, depending on r, n and N, such that, for all $\nu,\nu'\in\mathbb{Z}$, sequences $\{a_{\nu,k}\}_{k\in\mathbb{Z}^n}\subset\mathbb{C}$ and $x\in\mathbb{R}^n$,*

$$\sum_{k\in\mathbb{Z}^n}\frac{2^{-[\nu-(\nu\wedge\nu')]n}}{(1+2^{(\nu\wedge\nu')}|x-2^{-\nu}k|)^N}|a_{\nu,k}| \le C2^{[(\nu\wedge\nu')-\nu]n(1-1/r)}\Big\{\mathcal{M}\Big(\sum_{k\in\mathbb{Z}^n}|a_{\nu,k}|^r\chi_{Q_{\nu,k}}\Big)(x)\Big\}^{1/r},$$

where $\mathcal{M}$ denotes the Hardy–Littlewood maximal function on $\mathbb{R}^n$.

Applying Lemmas 9.1 and 9.3, Remark 9.2(ii), and the Calderón reproducing formula (9.5), we obtain the corresponding $T1$-theorem for the bilinear multiplier operator T_σ. The proof is essentially contained in the proof of Theorems 6.14 and 6.15, the details being omitted here.

Theorem 9.4. *Let T_σ be as in* (9.1) *and the symbol σ satisfy* (9.2) *and the cancelation condition, for all $\xi\neq0$,*

$$\sigma(\xi,0)=\sigma(0,\xi)=\sigma(-\xi,\xi)=0.$$

(a) *For every* $j \in \{0,1,2\}$, *let* $|s_j| < 1$, $0 < p_j < \infty$ *and* $0 < q_j < \infty$ *be such that*

$$s_0 = s_1 + s_2, \quad \frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{p_2} \quad \textit{and} \quad \frac{1}{q_0} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then T_σ *can be extended to a bounded bilinear operator from the product space* $\dot{F}^{s_1}_{p_1,q_1}(\mathbb{R}^n) \times \dot{F}^{s_2}_{p_2,q_2}(\mathbb{R}^n)$ *to* $\dot{F}^{s_0}_{p_0,q_0}(\mathbb{R}^n)$.

(b) *For every* $j \in \{0,1,2\}$, *let* $|s_j| < 1$, $0 < p_j < \infty$ *and* $0 < q_j < \infty$ *be such that*

$$s_0 = s_1 + s_2, \quad \frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{p_2} \quad \textit{and} \quad \frac{1}{q_0} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then T_σ *can be extended to a bounded bilinear operator from the product space* $\dot{B}^{s_1}_{p_1,q_1}(\mathbb{R}^n) \times \dot{B}^{s_2}_{p_2,q_2}(\mathbb{R}^n)$ *to* $\dot{B}^{s_0}_{p_0,q_0}(\mathbb{R}^n)$.

REMARK 9.5. Theorem 9.4 extends the results in [51, 52], where the diagonal cases $p_1 = q_1$, $p_2 = q_2$ and $p_0 = q_0$ were considered.

REMARK 9.6. (i) According to Remark 9.2(i), if in Theorem 9.4 we further assume that, for any multiindices γ,

$$\partial^\gamma \sigma(\xi, 0) = \partial^\gamma \sigma(0, \xi) = \partial^\gamma \sigma(-\xi, \xi) = 0,$$

then, for all integers $N > n$ and $L \in \mathbb{N}$, there exists a positive constant $C_{N,n,L}$ such that, for all $m, k, \ell \in \mathbb{Z}^n$, $\lambda, \nu, \mu \in \mathbb{Z}$ satisfying $\nu \le \lambda \le \mu$, and $i \in \{0,1,2\}$,

$$\begin{aligned} &|\langle \phi_{\lambda,m}, T^{*,i}_\sigma(\phi_{\nu,k}, \phi_{\mu,\ell})\rangle| \\ &\qquad \le C_{N,n,L} \frac{2^{-|\nu-\mu|L} 2^{-\nu n/2} 2^{\lambda n/2} 2^{\mu n/2}}{(1 + 2^\lambda |2^{-\lambda} m - 2^{-\nu} k|)^N (1 + 2^\mu |2^{-\lambda} m - 2^{-\mu} \ell|)^N}. \end{aligned} \tag{9.12}$$

Using (9.12), Lemma 9.3 and following the proofs of Theorems 6.14 and 6.15, we see that (a) and (b) of Theorem 9.4 hold for all $s_0, s_1, s_2 \in \mathbb{R}$ satisfying $s_0 = s_1 + s_2$.

(ii) In principle, if the symbol σ satisfies (9.12), then the operator T_σ satisfies (a) and (b) of Theorem 9.4 for all $s_0, s_1, s_2 \in \mathbb{R}$ satisfying $s_0 = s_1 + s_2$.

References

[1] P. Auscher and T. Hytönen, *Orthonormal bases of regular wavelets in spaces of homogeneous type*, Appl. Comput. Harmon. Anal. 34 (2013), 266–296.

[2] Á. Bényi, *Bilinear pseudodifferential operators with forbidden symbols on Lipschitz and Besov spaces*, J. Math. Anal. Appl. 284 (2003), 97–103.

[3] Á. Bényi, *Bilinear singular integral operators, smooth atoms and molecules*, J. Fourier Anal. Appl. 9 (2003), 301–319.

[4] Á. Bényi, D. Maldonado and V. Naibo, *What is ... a paraproduct?*, Notices Amer. Math. Soc. 57 (2010), 858–860.

[5] Á. Bényi, D. Maldonado, A. Nahmod and R. Torres, *Bilinear paraproducts revisited*, Math. Nachr. 283 (2010), 1257–1276.

[6] Á. Bényi and N. Tzirakis, *Multilinear almost diagonal estimates and applications*, Studia Math. 164 (2004), 75–89.

[7] F. Bernicot, *Uniform estimates for multilinear paraproducts*, Rev. Mat. Iberoamer. 25 (2009), 1055–1088.

[8] F. Bernicot, *A $T(1)$-theorem in relation to a semigroup of operators and applications to new paraproducts*, Trans. Amer. Math. Soc. 364 (2012), 6071–6108.

[9] F. Bernicot, *Fiber-wise Calderón–Zygmund decomposition and application to a bi-dimensional paraproduct*, Illinois J. Math, to appear.

[10] O. V. Besov, *On some families of function spaces. Embedding and extension theorems*, Dokl. Akad. Nauk SSSR 126 (1959), 1163–1165 (in Russian).

[11] O. V. Besov, *Investigation of a family of function spaces in connection with embeddings and extensions*, Trudy Mat. Inst. Steklov. 60 (1961), 42–81 (in Russian); English transl.: Amer. Math. Soc. Transl. Ser. 2 40 (1964), 85–126.

[12] J.-M. Bony, *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires*, Ann. Sci. École Norm. Sup. 14 (1981), 209–246.

[13] T. A. Bui and X. T. Duong, *Weighted norm inequalities for multilinear operators and applications to multilinear Fourier multipliers*, Bull. Sci. Math. 137 (2013), 63–75.

[14] A. P. Calderón, *An atomic decomposition of distributions in parabolic H^p spaces*, Adv. Math. 25 (1977), 216–225.

[15] C. P. Calderón, *On commutators of singular integrals*, Studia Math. 53 (1975), 139–174.

[16] M. Christ, *A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral*, Colloq. Math. 60/61 (1990), 601–628.

[17] M. Christ and J.-L. Journé, *Polynomial growth estimates for multilinear singular integral operators*, Acta Math. 159 (1987), 51–80.

[18] R. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. 51 (1974), 241–250.

[19] R. R. Coifman and Y. Meyer, *On commutators of singular integrals and bilinear singular integrals*, Trans. Amer. Math. Soc. 212 (1975), 315–331.

[20] R. R. Coifman et Y. Meyer, *Commutateurs d'intégrales singulières et opérateurs multilinéaires*, Ann. Inst. Fourier (Grenoble) 28 (1978), no. 3, 177–202.

[21] R. R. Coifman et Y. Meyer, *Au delà des opérateurs pseudo-différentiels*, Astérisque 57 (1978).

[22] R. R. Coifman et G. Weiss, *Analyse Harmonique Non-Commutative sur Certains Espaces Homogènes*, Lecture Notes in Math. 242, Springer, Berlin, 1971.

[23] R. R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. 83 (1977), 569–645.

[24] D. V. Cruz-Uribe, J. M. Martell and C. Pérez, *Weights, Extrapolation and the Theory of Rubio de Francia*, Operator Theory Adv. Appl. 215, Birkhäuser, Basel, 2011.

[25] G. David and J.-L. Journé, *A boundedness criterion for generalized Calderón–Zygmund operators*, Ann. of Math. (2) 120 (1984), 371–397.

[26] G. David, J.-L. Journé et S. Semmes, *Opérateurs de Calderón–Zygmund, fonctions para-accrétives et interpolation*, Rev. Mat. Iberoamer. 1 (1985), 1–56.

[27] D. Deng and Y. Han, *Harmonic Analysis on Spaces of Homogeneous Type*, Lecture Notes in Math. 1966, Springer, Berlin, 2009.

[28] J. Duoandikoetxea, *Fourier Analysis*, Grad. Stud. Math. 29, Amer. Math. Soc., Providence, RI, 2001.

[29] X. T. Duong, L. Grafakos and L. Yan, *Multilinear operators with non-smooth kernels and commutators of singular integrals*, Trans. Amer. Math. Soc. 362 (2010), 2089–2113.

[30] X. T. Duong and A. McIntosh, *Singular integral operators with non-smooth kernels on irregular domains*, Rev. Mat. Iberoamer. 15 (1999), 233–265.

[31] C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. 129 (1972), 137–193.

[32] M. Frazier, Y. Han, B. Jawerth and G. Weiss, *The $T1$-theorem for Triebel–Lizorkin spaces*, in: Harmonic Analysis and Partial Differential Equations (El Escorial, 1987), Lecture Notes in Math. 1384, Springer, Berlin, 1989, 168–181.

[33] M. Frazier and B. Jawerth, *A discrete transform and decompositions of distribution spaces*, J. Funct. Anal. 93 (1990), 34–170.

[34] M. Frazier, B. Jawerth and G. Weiss, *Litttlewood–Paley Theory and the Study of Function Spaces*, CBMS Reg. Conf. Ser. Math. 79, Amer. Math. Soc., Providence, RI, 1991.

[35] M. Frazier, R. H. Torres and G. Weiss, *The boundedness of Calderón–Zygmund operators on the space $\dot{F}^s_{p,q}$*, Rev. Mat. Iberoamer. 4 (1988), 41–72.

[36] J. García-Cuerva and J. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland, Amsterdam, 1985.

[37] J. Gilbert and A. Nahmod, *Bilinear operators with non-smooth symbol. I*, J. Fourier Anal. Appl. 7 (2001), 435–467.

[38] J. Gilbert and A. Nahmod, *L^p-boundedness for time-frequency paraproducts. II*, J. Fourier Anal. Appl. 8 (2002), 109–172.

[39] A. Gogatishvili, P. Koskela and Y. Zhou, *Characterizations of Besov and Triebel–Lizorkin spaces on metric measure spaces*, Forum Math. 25 (2013), 787–819.

[40] L. Grafakos, *Classical Fourier Analysis*, 2nd ed., Grad. Texts in Math. 249, Springer, New York, 2008.

[41] L. Grafakos, *Modern Fourier Analysis*, 2nd ed., Grad. Texts in Math. 250, Springer, New York, 2009.

[42] L. Grafakos, *Multilinear operators in harmonic analysis and partial differential equations*, RIMS Kôkyûroku Bessatsu B33 (2012), 11–27.

[43] L. Grafakos and N. Kalton, *Some remarks on multilinear maps and interpolation*, Math. Ann. 319 (2001), 151–180.

[44] L. Grafakos and N. Kalton, *The Marcinkiewicz multiplier condition for bilinear operators*, Studia Math. 146 (2001), 115–156.

[45] L. Grafakos, L. Liu, C. Pérez and R. H. Torres, *The multilinear strong maximal function*, J. Geom. Anal. 21 (2011), 118–149.

[46] L. Grafakos, L. Liu and D. Yang, *Boundedness of paraproduct operators on RD-spaces*, Sci. China Ser. A 53 (2010), 2097–2114.

[47] L. Grafakos, L. Liu and D. Yang, *Maximal function characterizations of Hardy spaces on RD-spaces and their applications*, Sci. China Ser. A 51 (2008), 2253–2284.

[48] L. Grafakos, L. Liu and D. Yang, *Vector-valued singular integrals and maximal functions on spaces of homogeneous type*, Math. Scand. 104 (2009), 296–310.

[49] L. Grafakos and L. Oliveira, *Carleson measures associated with families of multilinear operators*, Studia Math. 211 (2012), 71–94.

[50] L. Grafakos and T. Tao, *Multilinear interpolation between adjoint operators*, J. Funct. Anal. 199 (2003), 379–385.

[51] L. Grafakos and R. H. Torres, *Discrete decompositions for bilinear operators and almost diagonal conditions*, Trans. Amer. Math. Soc. 354 (2001), 1153–1176.

[52] L. Grafakos and R. H. Torres, *A multilinear Schur test and multiplier operators*, J. Funct. Anal. 187 (2001), 1–24.

[53] L. Grafakos and R. H. Torres, *Multilinear Calderón–Zygmund theory*, Adv. Math. 165 (2002), 124–164.

[54] L. Grafakos and R. H. Torres, *Maximal operator and weighted norm inequalities for multilinear singular integrals*, Indiana Univ. Math. J. 51 (2002), 1261–1276.

[55] L. Grafakos and R. H. Torres, *On multilinear singular intergrals of Calderón–Zygmund type*, Publ. Mat. 2002, Vol. Extra, 57–91.

[56] P. Hajłasz and P. Koskela, *Sobolev met Poincaré*, Mem. Amer. Math. Soc. 145 (2000), no. 688.

[57] Y. Han, *Calderón-type reproducing formula and the Tb theorem*, Rev. Mat. Iberoamer. 10 (1994), 51–91.

[58] Y. Han, *Triebel–Lizorkin spaces on spaces of homogeneous type*, Studia Math. 108 (1994), 247–273.

[59] Y. Han, D. Müller and D. Yang, *Littlewood–Paley characterizations for Hardy spaces on spaces of homogeneous type*, Math. Nachr. 279 (2006), 1505–1537.

[60] Y. Han, D. Müller and D. Yang, *A theory of Besov and Triebel–Lizorkin spaces on metric measure spaces modeled on Carnot–Carathéodory spaces*, Abstr. Appl. Anal. 2008, art. ID 893409, 250 pp.

[61] Y. Han and E. T. Sawyer, *Littlewood–Paley theory on spaces of homogeneous type and the classical function spaces*, Mem. Amer. Math. Soc. 110 (1994), no. 530.

[62] J. Hart, *Bilinear square functions and vector-valued Calderón–Zygmund operators*, J. Fourier Anal. Appl. 18 (2012), 1291–1313.

[63] J. Hart, *A new proof of the bilinear $T(1)$ theorem*, Proc. Amer. Math. Soc., to appear.

[64] J. Heinonen, *Lectures on Analysis on Metric Spaces*, Springer, New York, 2001.

[65] G. Hu, Do. Yang and Da. Yang, *Boundedness of maximal singular integral operators on spaces of homogeneous type and its applications*, J. Math. Soc. Japan 59 (2007), 323–349.

[66] T. Hytönen and A. Kairema, *Systems of dyadic cubes in a doubling metric space*, Colloq. Math. 126 (2012), 1–33.

[67] T. Hytönen and H. Martikainen, *Non-homogeneous Tb theorem and random dyadic cubes on metric measure spaces*, J. Geom. Anal. 22 (2012), 1071–1107.

[68] C. E. Kenig and E. M. Stein, *Multilinear estimates and fractional integration*, Math. Res. Lett. 6 (1999), 1–15, 467.

[69] P. Koskela and E. Saksman, *Pointwise characterizations of Hardy–Sobolev functions*, Math. Res. Lett. 15 (2008), 727–744.

[70] P. Koskela, D. Yang and Y. Zhou, *A characterization of Hajłasz–Sobolev and Triebel–Lizorkin spaces via grand Littlewood–Paley functions*, J. Funct. Anal. 258 (2010), 2637–2661.

[71] P. Koskela, D. Yang and Y. Zhou, *Pointwise characterizations of Besov and Triebel–Lizorkin spaces and quasiconformal mappings*, Adv. Math. 226 (2011), 3579–3621.

[72] A. K. Lerner, S. Ombrosi, C. Pérez, R. H. Torres and R. Trujillo-González, *New maximal functions and multiple weights for the multilinear Calderón–Zygmund theory*, Adv. Math. 220 (2009), 1222–1264.

[73] P. I. Lizorkin, *Operators connected with fractional differentiation, and classes of differentiable functions*, Trudy Mat. Inst. Steklov. 117 (1972), 212–243 (in Russian); English transl.: Proc. Steklov Inst. Math. 117 (1972), 251–286.

[74] P. I. Lizorkin, *Properties of functions in the spaces* $\Lambda^r_{p,\theta}$, Trudy Mat. Inst. Steklov. 131 (1974), 158–181 (in Russian); English transl.: Proc. Steklov Inst. Math. 131 (1974), 165–188.

[75] R. A. Macías and C. Segovia, *Lipschitz functions on spaces of homogeneous type*, Adv. Math. 33 (1979), 257–270.

[76] R. A. Macías and C. Segovia, *A decomposition into atoms of distributions on spaces of homogeneous type*, Adv. Math. 33 (1979), 271–309.

[77] D. Maldonado, *Multilinear singular integrals and quadratic estimates*, PhD thesis, Univ. of Kansas, 2005.

[78] D. Maldonado and V. Naibo, *On the boundedness of bilinear operators on products of Besov and Lebesgue spaces*, J. Math. Anal. Appl. 352 (2009), 591–603.

[79] J. M. Martell, *Sharp maximal functions associated with approximations of the identity in spaces of homogeneous type and applications*, Studia Math. 161 (2004), 113–145.

[80] Y. Meyer, *Real analysis and operator theory*, Math. Rep. McMaster Univ. 123, 1985.

[81] Y. Meyer, *Les nouveaux opérateurs de Calderón–Zygmund*, Astérisque 131 (1985), 237–254.

[82] Y. Meyer and R. R. Coifman, *Wavelets: Calderón–Zygmund and Multilinear Operators*, Cambridge Univ. Press, Cambridge, 1997.

[83] K. Moen, *Weighted inequalities for multilinear fractional integral operators*, Collect. Math. 60 (2009), 213–238.

[84] D. Müller and D. Yang, *A difference characterization of Besov and Triebel–Lizorkin spaces on RD-spaces*, Forum Math. 21 (2009), 259–298.

[85] C. Muscalu, T. Tao and C. Thiele, *Uniform estimates on paraproducts*, J. Anal. Math. 87 (2002), 369–384.

[86] E. Nakai and K. Yabuta, *Pointwise multipliers for functions of weighted bounded mean oscillation on spaces of homogeneous type*, Math. Japon. 46 (1997), 15–28.

[87] C. J. Neugebauer, *Inserting* A_p*-weights*, Proc. Amer. Math. Soc. 87 (1983), 644–648.

[88] C. Pérez, *On sufficient conditions for the boundedness of the Hardy–Littlewood maximal operator between weighted* L^p*-spaces with different weights*, Proc. London Math. Soc. 71 (1995), 135–157.

[89] C. Pérez, *Weighted norm inequalities for singular integral operators*, J. London Math. Soc. 49 (1994), 296–308.

[90] C. Pérez, *Two weighted inequalities for potential and fractional type maximal operators*, Indiana Univ. Math. J. 43 (1994), 663–683.

[91] C. Pérez and R. H. Torres, *Sharp maximal function estimates for multilinear singular integrals*, in: Harmonic Analysis at Mount Holyoke (South Hadley, MA, 2001), Contemp. Math. 320, Amer. Math. Soc., Providence, RI, 2003, 323–331.

[92] S. Semmes, *Square function estimates and the $T(b)$ theorem*, Proc. Amer. Math. Soc. 110 (1990), 721–726.

[93] N. Shanmugalingam, *Newtonian spaces: an extension of Sobolev spaces to metric measure spaces*, Rev. Mat. Iberoamer. 16 (2000), 243–279.

[94] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press, Princeton, NJ, 1993.

[95] J. O. Strömberg and A. Torchinsky, *Weighted Hardy Spaces*, Lecture Notes in Math. 1381, Springer, Berlin, 1989.

[96] M. H. Taibleson, *On the theory of Lipschitz spaces of distributions on Euclidean n-space. I. Principal properties*, J. Math. Mech. 13 (1964), 407–479.

[97] M. H. Taibleson, *On the theory of Lipschitz spaces of distributions on Euclidean n-space. II. Translation invariant operators, duality, and interpolation*, J. Math. Mech. 14 (1965), 821–839.

[98] M. H. Taibleson, *On the theory of Lipschitz spaces of distributions on Euclidean n-space. III. Smoothness and integrability of Fourier transforms, smoothness of convolution kernels*, J. Math. Mech. 15 (1966), 973–981.

[99] R. H. Torres, *Boundedness results for operators with singular kernels on distribution spaces*, Mem. Amer. Math. Soc. 90 (1991), no. 442.

[100] H. Triebel, *Spaces of distributions of Besov type on Euclidean n-space. Duality, interpolation*, Ark. Mat. 11 (1973), 13–64.

[101] H. Triebel, *Theory of Function Spaces*, Monogr. Math. 78, Birkhäuser, Basel, 1983.

[102] H. Triebel, *Theory of Function Spaces II*, Monogr. Math. 84, Birkhäuser, Basel, 1992.

[103] H. Triebel, *Function Spaces and Wavelets on Domains*, EMS Tracts in Math. 7, Eur. Math. Soc., Zürich, 2008.

[104] D. Yang and Y. Zhou, *Radial maximal function characterizations of Hardy spaces on RD-spaces and their applications*, Math. Ann. 346 (2010), 307–333.

[105] D. Yang and Y. Zhou, *New properties of Besov and Triebel–Lizorkin spaces on RD-spaces*, Manuscripta Math. 134 (2011), 59–90.

[106] D. Yang and Y. Zhou, *Localized Hardy spaces H^1 related to admissible functions on RD-spaces and applications to Schrödinger operators*, Trans. Amer. Math. Soc. 363 (2011), 1197–1239.