Contents

Abstract

Problems involving cracks are of particular importance in structural mechanics, and gave rise to many interesting mathematical techniques to treat them. The difficulties stem from the singularities of domains, which yield lower regularity of solutions. Of particular interest are techniques which allow us to identify cracks and defects from the mechanical properties. Long before advent of mathematical modeling in structural mechanics, defects were identified by the fact that they changed the sound of a piece of material when struck. These techniques have been refined over the years. This volume gives a compilation of recent mathematical methods used in the solution of problems involving cracks, in particular problems of shape optimization. It is based on a collection of recent papers in this area and reflects the work of many authors, namely Gilles Frémiot (Nancy), Werner Horn (Northridge), Jiří Jarušek (Prague), Alexander Khludnev (Novosibirsk), Antoine Laurain (Graz), Murali Rao (Gainesville), Jan Sokołowski (Nancy) and Carol Ann Shubin (Northridge).

We review the techniques which can be used for numerical analysis and shape optimization of problems with cracks and of the associated variational inequalities. The mathematical results include sensitivity analysis of variational inequalities, based on the concept of conical differential introduced by Mignot. We complete results on conical differentiability obtained for obstacle problems, by results derived for cracks with non-penetration condition and parabolic variational inequalities. Numerical methods for some problems are given as an illustration. From the point of view of applied mathematics numerical analysis is a necessary ingredient of applicability of the models proposed. We also extend the result on conical differentiability to the case of some evolution variational inequalities. The same mathematical model can be represented in different ways, like primal, dual or mixed formulations for an elliptic problem. We use such possibilities for models with cracks.

For the shape sensitivity analysis, in Chapters 1 to 3 we give a thorough introduction to the use of first and second order shape derivatives and their application to problems involving cracks. In Chapter 1, for the convenience of the reader, we provide classical results on shape sensitivity analysis in smooth domains. In Chapter 2, the results on the first order Eulerian semi-derivative in domains with cracks are presented. Of particular interest is the so-called structure theorem for the shape derivative. In Chapter 3, the results on the Fréchet derivative in domains with cracks are presented as well, for first and second order derivatives, using a technique different from that in Chapter 2.

In Chapter 4, we extend those ideas to Banach spaces, and give some applications of this extended theory. The polyhedricity of convex sets is considered in the spirit of [91], [113], in the most general setting. These abstract results can be applied to sensitivity analysis of crack problems with non-linear boundary conditions. The results obtained use non-linear potential theory and are interesting on their own.

In Chapter 5, several techniques for the study of cracked domains with non-penetration conditions on the crack faces in elastic bodies are presented. The classical crack theory in elasticity is characterized by linear boundary conditions which do not correspond to the physical reality since the crack faces can penetrate each other in this model. In this chapter, non-penetration conditions on the crack faces are considered, which leads to a non-linear problem. The model is presented and the shape sensitivity analysis is performed.

Chapter 6 is devoted to the newly developed smooth domain method for cracks. In that chapter the problem on a domain with a crack is transformed into a new problem on a smooth domain. This approach is useful for numerical methods. In [13] this formulation is used combined with mixed finite elements, and some error estimates are derived for the finite element approximation of variational inequalities with non-linear condition on the crack faces. We give applications of this method to some classical problems.

Finally, in Chapter 7 we study integro-differential equations arising from bridged crack models. This is a classical technique, but we introduce a few modern approaches to it for completeness sake.

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1. Introduction: Shape derivatives in smooth domains

In Chapters 2 and 3, the speed method as described in [126] is extended to domains with cracks (cuts in two dimensions). The Hadamard structure theorem for differentiable shape functionals is given for domains with cuts. In particular, the shape derivatives of energy functionals can be used in crack's propagation analysis in solids within the Griffiths criterion.

In the present work, geometrical singularities associated with cracks and free boundary problems of obstacle type are analysed. In general, shape sensitivity analysis can be performed for such problems but the first variation of the shape functional, e.g. of the energy functional, is non-smooth with respect to the direction of the domain perturbations.

In the case of a crack, the singularities at the tips of a crack play a particular role. There are terms associated with the singularities, and this leads in particular to the Griffiths criterion for crack propagation. Such a criterion is derived even for the non-penetration condition imposed on the crack faces, but in that case the singularities are not known in any explicit way.

In the case of variational inequalities, the first variation of solutions with respect to parameters depends on the cone of admissible directions. For problems with the admissible convex set which is polyhedric, first order sensitivity analysis of solutions to variational inequalities can be performed and the so-called conical differentiability of solutions with respect to parameter perturbations is obtained.

For all models of the work, shape sensitivity analysis can be used to obtain the first and second order optimality conditions and to develop the associated numerical methods for solution of shape optimization problems.

As an introduction to Chapters 2 and 3, we provide in this chapter the standard shape sensitivity calculations for the energy functional of the Laplacian in the smooth case. The technique presented can be applied to more general operators and functionals.

1.1. The speed method. The first order shape sensitivity analysis yields shape gradients and leads to gradient type numerical methods for solution of related shape optimization problems. The specificity of problems with cracks in this field can be explained as follows. The general theory of shape optimization applies with some additional terms associated with singularities of the solution at the crack tips. These terms are identified, even for problems where the form of the singularity is unknown. In this case the coefficient of the singularity is given by a path independent functional. Here we present the classical shape sensitivity analysis which is described in [126], where the so-called speed method is proposed and the shape derivatives for broad classes of shape functionals are obtained. We refer the reader to that monograph for the proofs of the results in the smooth case.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\Gamma = \partial\Omega$ (C^2 regularity) and f be a smooth function, e.g. $f \in C^2(\mathbb{R};\mathbb{R})$. We consider the following problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases} \tag{1.1}$$

The variational, or weak, formulation, of the problem (1.1) is given by

$$\int_\Omega \langle \nabla u, \nabla w \rangle_{\mathbb{R}^2}\, dy = \int_\Omega f w\, dy, \qquad \forall w \in H_0^1(\Omega), \tag{1.2}$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^2}$ is the scalar product in $\mathbb{R}^2$. Solving this variational equation is equivalent to minimization of the functional $\pi(\Omega; \cdot)$ which represents the potential energy associated with (1.1),

$$\pi(\Omega;\varphi) = \frac{1}{2}\int_\Omega \|\nabla\varphi\|^2_{\mathbb{R}^2}\, dy - \int_\Omega f\varphi\, dy, \quad \forall \varphi \in H_0^1(\Omega),$$

where $\|\cdot\|_{\mathbb{R}^2}$ is the euclidian norm in $\mathbb{R}^2$. We know that the variational equation (1.2) has a unique solution $u = u_\Omega \in H_0^1(\Omega)$. The energy functional relative to the domain Ω is given by the formula

$$J(\Omega) = \pi(\Omega; u_\Omega) = \frac{1}{2}\int_\Omega \|\nabla u_\Omega\|^2_{\mathbb{R}^2}\, dy - \int_\Omega f u_\Omega\, dy = \inf_{\varphi \in H_0^1(\Omega)} \pi(\Omega;\varphi).$$

Let $V \in W^{1,\infty}(\mathbb{R}^2;\mathbb{R}^2) \cap C^1(\mathbb{R}^2;\mathbb{R}^2)$ be a vector field and $T_t(V)$ ($t \geq 0$) be the map defined by

$$T_t(V) : \mathbb{R}^2 \to \mathbb{R}^2, \qquad X \mapsto x(t),$$

where $x(\cdot)$ is the solution of the ordinary differential equation

$$\begin{cases} \dfrac{dx}{dt}(t) = V(x(t)), \quad t > 0, \\ x(0) = X. \end{cases}$$

We denote by $\Omega_t = \Omega_t(V)$ the image of the domain Ω under the map $T_t(V)$, i.e. $\Omega_t = T_t(V)(\Omega)$, and in the same way the boundary of Ω_t is obtained by $\Gamma_t = \Gamma_t(V) = T_t(V)(\Gamma)$. The coordinates of a given point which belongs to the domains $\Omega = \Omega_0$, Ω_t ($t > 0$), are respectively denoted by $y = (y_1, y_2) \in \Omega$, $x = (x_1, x_2) \in \Omega_t$; moreover, we know that the map $T_t(V)$ is bijective. As previously, there exists a unique function $u^t \in H_0^1(\Omega_t)$ which solves the variational equality

$$\int_{\Omega_t} \langle \nabla u^t, \nabla v \rangle_{\mathbb{R}^2}\, dx = \int_{\Omega_t} f v\, dx, \quad \forall v \in H_0^1(\Omega_t). \tag{1.3}$$

This weak solution u^t is also obtained by minimizing the potential energy associated with (1.3),

$$\pi(\Omega_t;\psi) = \frac{1}{2}\int_{\Omega_t} \|\nabla\psi\|^2_{\mathbb{R}^2}\, dx - \int_{\Omega_t} f\psi\, dx, \quad \forall \psi \in H_0^1(\Omega_t).$$

The energy functional for the domain Ω_t is given by the formula

$$J(\Omega_t) = \pi(\Omega_t; u^t) = \frac{1}{2}\int_{\Omega_t} \|\nabla u^t\|^2_{\mathbb{R}^2}\,dx - \int_{\Omega_t} f u^t\,dx = \inf_{\psi\in H^1_0(\Omega_t)} \pi(\Omega_t;\psi).$$

1.2. The first order shape derivative. First, let us recall some results about the *first order shape derivative*, or *Eulerian semi-derivative*, of a functional J. This shape derivative in direction V is defined by the limit

$$\lim_{t\to 0}\frac{J(\Omega_t)-J(\Omega)}{t},$$

if it exists of course. In our case, for the energy functional, we derive the first shape derivative by differentiation of the volume integral

$$\int_\Omega F_\Omega\,dy,$$

where F_Ω is a function depending on Ω. Indeed, we have [126]

$$\frac{d}{dt}\Big(\int_{\Omega_t} F_{\Omega_t}\,dx\Big)\Big|_{t=0} = \int_\Omega F'_\Omega\,dy + \int_\Gamma F_\Omega \langle V,\nu\rangle_{\mathbb{R}^2}\,d\Gamma(y),$$

where F'_Ω denotes the shape derivative of F_Ω in direction V and ν is the exterior normal vector to Γ. Moreover, according to the structure theorem [126], there exists a distribution $g_{\partial\Omega}\in\mathcal{D}'_1(\Omega)$, supported by Γ, such that

$$\frac{d}{dt}\Big(\int_{\Omega_t} F_{\Omega_t}\,dx\Big)\Big|_{t=0} = \langle g_{\partial\Omega}, \langle V,\nu\rangle_{\mathbb{R}^2}\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes de duality bracket. In our case, we obtain, by integration by parts,

$$g_{\partial\Omega} = -\frac{1}{2}\Big(\frac{\partial u_\Omega}{\partial\nu}\Big)^2 \in L^1(\partial\Omega).$$

Consequently, the Eulerian semi-derivative of the functional J in direction V is given by

$$dJ(\Omega;V) = -\frac{1}{2}\int_\Gamma \Big(\frac{\partial u_\Omega}{\partial\nu}\Big)^2 \langle V,\nu\rangle_{\mathbb{R}^2}\,d\Gamma(y).$$

For (1.3) the shape derivative $u'(v_\nu)$ satisfies

$$\begin{cases} -\Delta u' = 0 & \text{in } \Omega,\\ u' = -v_\nu \dfrac{\partial u}{\partial\nu} & \text{on } \Gamma,\end{cases}$$

where $v_\nu = \langle V,\nu\rangle$.

1.3. The second order shape derivative. The second order shape sensitivity analysis can be applied to derive sufficient optimality conditions for shape optimization problems, and to construct a Newton type method of numerical solution in shape optimization. To this end, the symmetric part of the shape Hessian can be used. We identify the shape Hessian in the non-smooth case, and describe its structure for a model problem. However, the technique proposed is general and can be used for problems with cracks in two or three spatial dimensions.

Now, we consider another vector field $W \in W^{1,\infty}(\mathbb{R}^2;\mathbb{R}^2) \cap C^1(\mathbb{R}^2;\mathbb{R}^2)$, and for all s positive, we define $\Omega_s = \Omega_s(W) = T_s(W)(\Omega)$ and $\Gamma_s = \Gamma_s(W) = T_s(W)(\Gamma)$. The *second order shape derivative* of the functional J in directions V and W is defined by

$$d^2J(\Omega; V, W) = \frac{d}{ds}(dJ(\Omega_s; V))\bigg|_{s=0},$$

with

$$dJ(\Omega_s; V) = -\frac{1}{2}\int_{\Gamma_s}\left(\frac{\partial u_{\Omega_s}}{\partial \nu_s}\right)^2 \langle V, \nu_s\rangle_{\mathbb{R}^2}\, d\Gamma_s(x).$$

In order to obtain the second order shape derivative, we can use a result of [29]. Indeed, if $f \in C^1(\mathbb{R}^2;\mathbb{R})$, we have the following formula for the second order shape derivative of the surface integral:

$$\begin{aligned}\frac{d}{ds}\left(\int_{\Gamma_s} f\langle V, \nu_s\rangle_{\mathbb{R}^2}\, d\Gamma_s(x)\right)\bigg|_{s=0} = \int_\Gamma \Big[& f(\langle D^2b\cdot w_\Gamma, v_\Gamma\rangle_{\mathbb{R}^2} - \langle v_\Gamma, \nabla_\Gamma w_\nu\rangle_{\mathbb{R}^2} - \langle w_\Gamma, \nabla_\Gamma v_\nu\rangle_{\mathbb{R}^2}) \\ & + \left(\frac{\partial f}{\partial \nu} + Hf\right) v_\nu w_\nu + f\langle DV\cdot W, \nu\rangle_{\mathbb{R}^2}\Big]\, d\Gamma(y),\end{aligned}$$

where H denotes the mean curvature of Γ, b is the oriented distance function relative to Γ, and f is the restriction to Γ_s of a given function defined in $\mathbb{R}^2$. For the shape functional we need the formula for the function $f = f(s,x)$; in that case the formula becomes

$$\begin{aligned}\frac{d}{ds}\left(\int_{\Gamma_s} f\langle V, \nu_s\rangle_{\mathbb{R}^2}\, d\Gamma_s(x)\right)\bigg|_{s=0} = \int_\Gamma \Big[& f(\langle D^2b\cdot w_\Gamma, v_\Gamma\rangle_{\mathbb{R}^2} - \langle v_\Gamma, \nabla_\Gamma w_\nu\rangle_{\mathbb{R}^2} - \langle w_\Gamma, \nabla_\Gamma v_\nu\rangle_{\mathbb{R}^2}) \\ & + \frac{\partial f}{\partial s} v_\nu + \left(\frac{\partial f}{\partial \nu} + Hf\right) v_\nu w_\nu + f\langle DV\cdot W, \nu\rangle_{\mathbb{R}^2}\Big]\, d\Gamma(y).\end{aligned}$$

So we need to determine the displacement derivative

$$\frac{\delta f}{\delta s} = \frac{\partial f}{\partial s} + \frac{\partial f}{\partial \nu} w_\nu$$

for the density of the gradient of the energy functional. We have, for $s = 0$,

$$\frac{\partial f}{\partial s} = \frac{\partial}{\partial s}\left(-\frac{1}{2}\left|\frac{\partial u_\Omega}{\partial \nu_s}\right|^2\right) = -\frac{\partial u}{\partial \nu}\,\frac{\partial u'(w_\nu)}{\partial \nu}$$

since $\|\nu_s\| = 1$ hence $\nu^\top \nu' = 0$. This leads to the following formula for the energy functional:

$$\begin{aligned}d^2J(\Omega; V, W) = -\frac{1}{2}\int_\Gamma \bigg[& \left|\frac{\partial u}{\partial \nu}\right|^2 (\langle D^2b\cdot w_\Gamma, v_\Gamma\rangle_{\mathbb{R}^2} - \langle v_\Gamma, \nabla_\Gamma w_\nu\rangle_{\mathbb{R}^2} - \langle w_\Gamma, \nabla_\Gamma v_\nu\rangle_{\mathbb{R}^2}) \\ & + 2\frac{\partial u}{\partial \nu}\,\frac{\partial u'(w_\nu)}{\partial \nu} v_\nu + \left(\frac{\partial}{\partial \nu}\left|\frac{\partial u}{\partial \nu}\right|^2 + H\left|\frac{\partial u}{\partial \nu}\right|^2\right) v_\nu w_\nu \\ & + \left|\frac{\partial u}{\partial \nu}\right|^2 \langle DV\cdot W, \nu\rangle_{\mathbb{R}^2}\bigg]\, d\Gamma(y).\end{aligned}$$

If the last term is omitted in the above formula, then we obtain the symmetric part of the second order shape derivative which leads e.g. to the Newton method in shape optimization.

2. Shape sensitivity analysis for cracks

2.1. The structure theorem. The structure theorem for differentiable shape functionals is given in [126] in the case of smooth domains. The identification of shape gradients becomes easy provided it is established that the functional in question is differentiable. This part of the shape sensitivity analysis is usually performed by using the so-called material derivatives of solutions to PDE's. The second part of the shape sensitivity analysis, important for numerical methods, is based on the structure theorem of shape gradients for specific classes of shape functionals. We present such a structure theorem for problems with cracks. We consider the Gateaux differentiability of shape functionals. The result is given explicitly, provided the singularities of solutions at crack tips are known, or implicitly in the case of unilateral conditions on the crack, since the singularity of the solution in that case is not known in any precise manner.

2.1.1. Introduction. Shape sensitivity analysis of boundary value problems defined in domains with cracks is important for applications; we refer the reader to the review paper [23] for some applications in fracture mechanics and a list of references. In the simplest case such results are derived in [35]; we also refer to [49] and [15]. Since the structure theorem was not established at the time, the direct approach is used in [35] which requires in the case of the energy functional the existence of the shape derivative of solutions to the elliptic equations defined in the domain with cracks. The same approach is in fact used in [34] in the case of unilateral conditions on the crack faces, but in that case the shape differentiability result does not seem to be known in the literature. We refer the reader to [126] for related results on shape differentiability of solutions to variational inequalities in smooth domains.

The problem with unilateral conditions on crack faces is considered in [72] for a scalar equation and in [73] for an elasticity system, where the so-called Rice–Cherepanov formula is derived. It is shown in [72], [73] that the result on shape differentiability of solutions to variational inequalities is not required for the proof of the differentiability of the energy functional with respect to the crack's length for problems with unilateral conditions prescribed on the crack faces.

2.1.2. The structure theorem in 3D. The structure theorem is important for applications in shape optimization, because it allows one to obtain the shape differentiability of a specific shape functional by means of simple verification of hypotheses, usually in the fixed domain setting, by an application of the material derivative method [126].

Let $D \subset \mathbb{R}^3$ be a bounded domain with smooth boundary Γ, and Σ be a part of a smooth surface S. We assume that $\overline{\Sigma}$, the closure of Σ in S, is contained in the domain D. Therefore, we consider the domain $\Omega = D \setminus \overline{\Sigma}$ with crack Σ. Let us denote by $\delta\overline{\Sigma}$ the boundary of Σ in S. Assume that J is a domain functional which is shape differentiable at Ω. We use a velocity field V to construct a family of domains $\Omega_t = T_t(V)(\Omega)$ using the technique described in [126]. Without losing generality, we can consider the problem with autonomous vector fields. We have the following result on the structure of the Eulerian semi-derivative $dJ(\Omega; V)$.

Theorem 2.1 (Structure theorem). *Let k be a non-negative integer. Assume that the mapping $\mathcal{D}^k(D;\mathbb{R}^3) \ni V \mapsto dJ(\Omega;V) \in \mathbb{R}$ is linear and continuous. Then there exist two linear forms ϕ and ψ which are continuous on $C^k(\overline{\Sigma})$ and $C^k(\delta\overline{\Sigma})$ respectively such that for all vector fields $V \in \mathcal{D}^k(D;\mathbb{R}^3)$, we have*

$$dJ(\Omega;V) = \phi(\langle V, n\rangle_{\mathbb{R}^3}) + \psi(\langle V, \nu\rangle_{\mathbb{R}^3}),$$

where $\langle\cdot,\cdot\rangle_{\mathbb{R}^3}$ denotes the scalar product in $\mathbb{R}^3$, n is the normal vector to $\overline{\Sigma}$ in $\mathbb{R}^3$ and ν is the normal vector to $\delta\overline{\Sigma}$ in S.

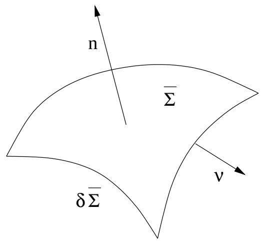

Fig. 2.1. The curved crack Σ in 3D

Proof. To simplify the problem and without loss of generality, we may assume (otherwise we can use an appropriate change of variables) that Σ, $\overline{\Sigma}$ and $\delta\overline{\Sigma}$ are given by

$$\begin{aligned}
\Sigma &= \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 < 1,\ x_3 = 0\},\\
\overline{\Sigma} &= \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 \le 1,\ x_3 = 0\},\\
\delta\overline{\Sigma} &= \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = 1,\ x_3 = 0\}.
\end{aligned}$$

The proof of the structure theorem is based on *Nagumo's theorem* [6] (or on the double viability conditions [28], [29]), and so we need the form, for any $x \in \overline{\Sigma}$, of the tangent set

$$T_{\overline{\Sigma}}(x) = \left\{ v \in \mathbb{R}^3 \,\middle|\, \liminf_{h\to 0^+} \frac{d_{\overline{\Sigma}}(x+hv)}{h} = 0 \right\}.$$

We have to distinguish two cases: $x \in \Sigma$ and $x \in \delta\overline{\Sigma}$.

- *First case*: $x = (x_1,x_2,x_3) \in \Sigma$, i.e. $x_1^2 + x_2^2 < 1$, $x_3 = 0$. In this case, the normal $n(x)$ to $\overline{\Sigma}$ at $x \in \Sigma$ is well defined and $T_{\overline{\Sigma}}(x) = \mathcal{T}_x(\overline{\Sigma})$, where $\mathcal{T}_x(\overline{\Sigma})$ denotes the tangent space to $\overline{\Sigma}$ at x; hence $V(x) \in T_{\overline{\Sigma}}(x)$ if and only if $\langle V(x), n(x)\rangle_{\mathbb{R}^3} = 0$.
- *Second case*: $x = (x_1,x_2,x_3) \in \delta\overline{\Sigma}$, i.e. $x_1^2 + x_2^2 = 1$, $x_3 = 0$. By definition, we have

$$T_{\overline{\Sigma}}(x) = \left\{ v = (v_1,v_2,v_3) \in \mathbb{R}^3 \,\middle|\, \liminf_{h\to 0^+} \frac{d_{\overline{\Sigma}}(x+hv)}{h} = 0 \right\}.$$

  It is not difficult to see that the distance $d_{\overline{\Sigma}}(x+hv)$ (between the point $x+hv$ and the crack $\overline{\Sigma}$) is given by the formula

$$\begin{cases} h|v_3| & \text{if } x_1v_1 + x_2v_2 \le 0,\\ [h^2v_3^2 + (\sqrt{(x_1+hv_1)^2 + (x_2+hv_2)^2} - 1)^2]^{1/2} & \text{if } x_1v_1 + x_2v_2 \ge 0, \end{cases}$$

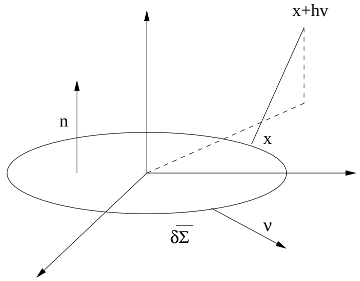

Fig. 2.2. Evaluation of $T_{\overline{\Sigma}}(x)$

thus

$$\liminf_{h\to 0^+} \frac{d_{\overline{\Sigma}} t(x+hv)}{h} = \begin{cases} |v_3| & \text{if } x_1v_1 + x_2v_2 \le 0, \\ [v_3^2 + (x_1v_1+x_2v_2)^2]^{1/2} & \text{if } x_1v_1 + x_2v_2 \ge 0, \end{cases}$$

and consequently the tangent set is given by

$$T_{\overline{\Sigma}}(x) = \{v = (v_1, v_2, v_3) \in \mathbb{R}^3 \mid v_3 = 0 \text{ and } x_1v_1 + x_2v_2 \le 0\}.$$

For any $x \in \Sigma$, $T_{\overline{\Sigma}}(x)$ is a vector space, thus $-T_{\overline{\Sigma}}(x) = T_{\overline{\Sigma}}(x)$. On the other hand, if $x \in \delta\overline{\Sigma}$, we have

$$\begin{aligned} \{T_{\overline{\Sigma}}(x)\} \cap \{-T_{\overline{\Sigma}}(x)\} &= \{v = (v_1, v_2, v_3) \in \mathbb{R}^3 \mid v_3 = 0 \text{ and } x_1v_1 + x_2v_2 = 0\} \\ &= \{v \in \mathbb{R}^3 \mid \langle v, n(x)\rangle_{\mathbb{R}^3} = \langle v, \nu(x)\rangle_{\mathbb{R}^3} = 0\}, \end{aligned}$$

so according to the double viability condition [28], [29], if the field $V \in \mathcal{D}^k(D;\mathbb{R}^3)$ satisfies

$$\begin{cases} \langle V(x), n(x)\rangle_{\mathbb{R}^3} = 0, & \forall x \in \overline{\Sigma}, \\ \langle V(x), \nu(x)\rangle_{\mathbb{R}^3} = 0, & \forall x \in \delta\overline{\Sigma}, \end{cases} \tag{2.1}$$

then $\overline{\Sigma}$ is globally invariant under the associated transformation $T_t(V)$. The exterior boundary $\Gamma = \partial D$ is also invariant under $T_t(V)$, since the support of the field V is included in D. So the boundary of $\Omega = D \setminus \overline{\Sigma}$, i.e. $\partial\Omega = \Gamma \cup \overline{\Sigma}$, is globally invariant under $T_t(V)$. Consequently, $\Omega_t = T_t(V)(\Omega)$, hence

$$dJ(\Omega; V) = 0$$

for any vector field which satisfies (2.1). It follows that it is natural to consider the set

$$F(\Omega) = \{V \in \mathcal{D}^k(D;\mathbb{R}^3) \mid \langle V, n\rangle_{\mathbb{R}^3} \text{ on } \overline{\Sigma},\ \langle V, \nu\rangle_{\mathbb{R}^3} = 0 \text{ on } \delta\overline{\Sigma}\}. \tag{2.2}$$

According to the hypothesis that the mapping $V \mapsto dJ(\Omega; V)$ is linear and continuous from $\mathcal{D}^k(D;\mathbb{R}^3)$ in $\mathbb{R}$, the set $F(\Omega)$ defined by (2.2) is included in its kernel. Consequently, we have the following lemma.

Lemma 2.2. *The mapping*

$$\Phi : \mathcal{D}^k(D;\mathbb{R}^3)/F(\Omega) \to C^k(\overline{\Sigma}) \times C^k(\delta\overline{\Sigma}), \qquad \{V\} \mapsto (\langle V, n\rangle_{\mathbb{R}^3}, \langle V, \nu\rangle_{\mathbb{R}^3}),$$

is an isomorphism.

Proof. First, we have to verify that the linear mapping $\Phi : \{V\} \mapsto (\langle V, n\rangle_{\mathbb{R}^3}, \langle V, \nu\rangle_{\mathbb{R}^3})$ is well defined. If $\{V\} = \{V'\}$, i.e. $V - V' \in F(\Omega)$, then $\langle V - V', n\rangle_{\mathbb{R}^3} = 0$ on $\overline{\Sigma}$ and $\langle V - V', \nu\rangle_{\mathbb{R}^3} = 0$ on $\delta\overline{\Sigma}$, and so $\langle V, n\rangle_{\mathbb{R}^3} = \langle V', n\rangle_{\mathbb{R}^3}$ on $\overline{\Sigma}$ and $\langle V, \nu\rangle_{\mathbb{R}^3} = \langle V', \nu\rangle_{\mathbb{R}^3}$ on $\delta\overline{\Sigma}$. It follows that Φ is well defined.

Let $\{V\} \in \mathcal{D}^k(D;\mathbb{R}^3)/F(\Omega)$ be such that $\Phi(\{V\}) = 0$, i.e. $\langle V, n\rangle_{\mathbb{R}^3} = 0$ on $\overline{\Sigma}$ and $\langle V, \nu\rangle_{\mathbb{R}^3} = 0$ on $\delta\overline{\Sigma}$, which means that $V \in F(\Omega)$ and $\{V\} = \{0\}$. Consequently, Φ is one-to-one.

Now, let us show that Φ is onto. Let $(v_1, v_2) \in C^k(\overline{\Sigma}) \times C^k(\delta\overline{\Sigma})$. We want to construct a vector field $V \in \mathcal{D}^k(D;\mathbb{R}^3)$ such that $\Phi(\{V\}) = (v_1, v_2)$. Since $v_1 \in C^k(\overline{\Sigma}) \simeq C^k(\overline{B}_{\mathbb{R}^2})$ where $\overline{B}_{\mathbb{R}^2} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \le 1\}$, and by definition of $C^k(\overline{B_{\mathbb{R}^2}})$, there exists $\widetilde{v}_1 \in C^k(\mathbb{R}^2)$ such that $\widetilde{v}_1|_{\overline{B}_{\mathbb{R}^2}} = v_1$. So we define

$$\widetilde{V_3}(x_1, x_2, x_3) = \widetilde{v}_1(x_1, x_2), \quad \forall (x_1, x_2, x_3) \in \mathbb{R}^3,$$

and it is evident that $\widetilde{V_3} \in C^k(\mathbb{R}^3)$. Let $\theta \in \mathcal{D}(D;\mathbb{R}) = C_0^\infty(D;\mathbb{R})$ be such that $\theta \equiv 1$ in a sufficiently small neighborhood of $\overline{\Sigma}$. Set

$$V_3(x_1, x_2, x_3) = \theta(x_1, x_2, x_3)\widetilde{V_3}(x_1, x_2, x_3), \quad \forall (x_1, x_2, x_3) \in \mathbb{R}^3;$$

then $V_3 \in \mathcal{D}^k(D;\mathbb{R})$. Let $\widetilde{\theta} \in \mathcal{D}(D;\mathbb{R})$ be such that $\widetilde{\theta} \equiv 0$ in the vicinity of the origin and $\widetilde{\theta} \equiv 1$ in the neighborhood of $\delta\overline{\Sigma}$. Define

$$V_1(x_1, x_2, x_3) = x_1\widetilde{\theta}(x_1, x_2, x_3)v_2\left(\left(\frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2}}\right)\right),$$

$$V_1(x_1, x_2, x_3) = x_2\widetilde{\theta}(x_1, x_2, x_3)v_2\left(\left(\frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2}}\right)\right),$$

for all $(x_1, x_2, x_3) \in \mathbb{R}^3$. Finally, let V be the vector field with components V_1, V_2, V_3. It is not difficult to see that $V \in \mathcal{D}(D;\mathbb{R}^3)$ satisfies

$$\begin{cases} \langle V, n\rangle_{\mathbb{R}^3} = v_1 & \text{on } \overline{\Sigma}, \\ \langle V, \nu\rangle_{\mathbb{R}^3} = v_2 & \text{on } \delta\overline{\Sigma}, \end{cases}$$

i.e. $\Phi(\{V\}) = (v_1, v_2)$. ∎

In order to complete the proof of the structure theorem in 3D, we need the following lemma.

Lemma 2.3. *There exists a continuous linear mapping $\Psi : C^k(\overline{\Sigma}) \times C^k(\delta\overline{\Sigma}) \to \mathbb{R}$ such that for any vector field $V \in \mathcal{D}^k(D;\mathbb{R}^3)$, we have*

$$dJ(\Omega; V) = \Psi(\langle V, n\rangle_{\mathbb{R}^3}, \langle V, \nu\rangle_{\mathbb{R}^3}).$$

Proof. We define

$$\Psi(\{V\}) = dJ(\Omega; V), \qquad \forall \{V\} \in \mathcal{D}^k(D;\mathbb{R}^3)/F(\Omega).$$

This mapping is well defined. Indeed, if $\{V\} = \{V'\}$, i.e. if $V \in \{V'\}$, then $V - V' \in F(\Omega)$, but $F(\Omega)$ is included in the kernel of $dJ(\Omega;\cdot)$ and so $dJ(\Omega; V - V') = 0$. Moreover, by our assumption, the Eulerian semi-derivative $dJ(\Omega,\cdot)$ is linear and we obtain $dJ(\Omega; V) = dJ(\Omega; V')$.

Using Lemma 2.2 and the relation $\mathcal{D}^k(D;\mathbb{R}^3)/F(\Omega) \simeq C^k(\overline{\Sigma}) \times C^k(\delta\overline{\Sigma})$, we can write

$$\{V\} = (\langle V, n\rangle_{\mathbb{R}^3}, \langle V, \nu\rangle_{\mathbb{R}^3}),$$

thus

$$dJ(\Omega; V) = \Psi(\{V\}) = \Psi(\langle V, n\rangle_{\mathbb{R}^3}, \langle V, \nu\rangle_{\mathbb{R}^3}).$$

Furthermore, $dJ(\Omega, \cdot)$ is linear and continuous, which implies that so is Ψ. ■

Now, we can complete the proof of the structure theorem in 3D. Indeed, according to Lemma 2.3, there exists a linear mapping Ψ which is continuous from $C^k(\overline{\Sigma}) \times C^k(\delta\overline{\Sigma})$ in $\mathbb{R}$, such that

$$dJ(\Omega; V) = \Psi(\langle V, n\rangle_{\mathbb{R}^3}, \langle V, \nu\rangle_{\mathbb{R}^3}), \qquad \forall V \in \mathcal{D}^k(D;\mathbb{R}^3),$$

with $\Psi \in (C^k(\overline{\Sigma}) \times C^k(\delta\overline{\Sigma}))' = (C^k(\overline{\Sigma}))' \times (C^k(\delta\overline{\Sigma}))'$. We can conclude that there exist two linear forms ϕ and ψ which are continuous on $C^k(\overline{\Sigma})$ and $C^k(\delta\overline{\Sigma})$ respectively such that

$$dJ(\Omega; V) = \phi(\langle V, n\rangle_{\mathbb{R}^3}) + \psi(\langle V, \nu\rangle_{\mathbb{R}^3}), \qquad \forall V \in \mathcal{D}^k(D;\mathbb{R}^3),$$

which completes the proof of the structure theorem in 3D. ■

2.1.3. The structure theorem in 2D. Let $D \subset \mathbb{R}^2$ be a bounded domain with smooth boundary Γ, and Σ be a part of a smooth curve. We assume that $\overline{\Sigma} \subset D$. Therefore, we consider the domain $\Omega = D \setminus \overline{\Sigma}$ with crack Σ. Let us denote by A and B the tips of $\overline{\Sigma}$. Assume that J is a domain functional which is shape differentiable at Ω. We refer the reader to [126] for the definition of shape differentiability.

The velocity field V is used to construct a family of domains $\Omega_t = T_t(V)(\Omega)$ using the technique described in [126]. Without losing generality, we can consider the problem with autonomous vector fields. We have the following result on the structure of the Eulerian semi-derivative $dJ(\Omega; V)$.

THEOREM 2.4 (Structure theorem). *Let k be a non-negative integer. Assume that the mapping $\mathcal{D}^k(D;\mathbb{R}^2) \ni V \mapsto dJ(\Omega; V) \in \mathbb{R}$ is linear and continuous. Then there exist two real numbers α_A and α_B, and a linear form ϕ which is continuous on $C^k(\overline{\Sigma})$, such that for all fields $V \in \mathcal{D}^k(D;\mathbb{R}^2)$,*

$$dJ(\Omega; V) = \alpha_A \langle V(A), \tau\rangle_{\mathbb{R}^2} + \alpha_B \langle V(B), \tau\rangle_{\mathbb{R}^2} + \phi(\langle V, n\rangle_{\mathbb{R}^2}),$$

where $\langle \cdot, \cdot\rangle_{\mathbb{R}^2}$ denotes the scalar product in $\mathbb{R}^2$, and τ and n are respectively the tangential and normal vectors on $\overline{\Sigma}$.

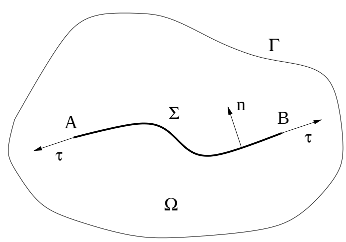

Fig. 2.3. Domain Ω with a curved crack Σ in 2D

Proof. In order to simplify the problem and without loss of generality, we may assume (otherwise we can use an appropriate change of variables) that Σ is given by

$$\Sigma = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < 1,\ x_2 = 0\}.$$

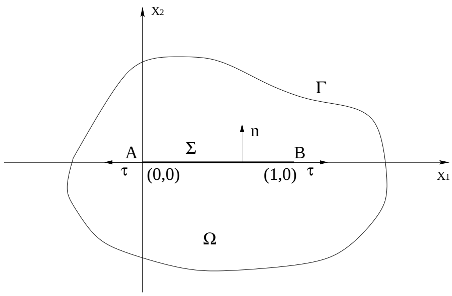

Fig. 2.4. Domain Ω with a rectilinear crack

As for the structure theorem in 3D, the proof of the structure theorem in 2D is based on *Nagumo's theorem* [6] (or on the double viability conditions [28], [29]), and so we need the form, for any $x \in \overline{\Sigma}$, of the tangent set

$$T_{\overline{\Sigma}}(x) = \left\{ v \in \mathbb{R}^2 \,\middle|\, \liminf_{h \to 0^+} \frac{d_{\overline{\Sigma}}(x + hv)}{h} = 0 \right\}.$$

We have to distinguish three cases: $x \in \Sigma$, $x = A$ and $x = B$.

- *First case*: $x = (x_1, x_2) \in \Sigma$, i.e. $0 < x_1 < 1$, $x_2 = 0$. In this case, the normal $n(x)$ to $\overline{\Sigma}$ at $x \in \Sigma$ is well defined and $T_{\overline{\Sigma}}(x) = \mathcal{T}_x(\overline{\Sigma})$, where $\mathcal{T}_x(\overline{\Sigma})$ denotes the tangent space to $\overline{\Sigma}$ at x, and so $V(x) \in T_{\overline{\Sigma}}(x)$ if and only if $\langle V(x), n(x)\rangle_{\mathbb{R}^2} = 0$.
- *Second case*: $x = A = (0, 0)$. By definition, we have

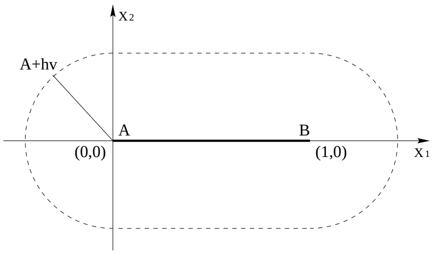

Fig. 2.5. Evaluation of $T_{\overline{\Sigma}}(A)$

$$T_{\overline{\Sigma}}(x) = T_{\overline{\Sigma}}(A) = \left\{ v = (v_1, v_2) \in \mathbb{R}^2 \,\middle|\, \liminf_{h \to 0^+} \frac{d_{\overline{\Sigma}}(A + hv)}{h} = 0 \right\}.$$

It is not difficult to see that the distance $d_{\overline{\Sigma}}(A + hv)$ (between $A + hv$ and the crack $\overline{\Sigma}$) is given by

$$d_{\overline{\Sigma}}(A + hv) = \begin{cases} h\|v\|_{\mathbb{R}^2} & \text{if } v_1 \le 0, \\ h|v_2| & \text{if } v_1 \ge 0, \end{cases}$$

where $\|\cdot\|_{\mathbb{R}^2}$ denotes the Euclidian norm in $\mathbb{R}^2$. Thus, we obtain

$$\liminf_{h\to 0^+} \frac{d_{\overline{\Sigma}}(A+hv)}{h} = \begin{cases} \|v\|_{\mathbb{R}^2} & \text{if } v_1 \le 0, \\ |v_2| & \text{if } v_1 \ge 0, \end{cases}$$

and consequently the tangent set is given by

$$T_{\overline{\Sigma}}((0,0)) = T_{\overline{\Sigma}}(A) = \{v = (v_1, v_2) \in \mathbb{R}^2 \mid v_2 = 0 \text{ and } v_1 \ge 0\}.$$

- *Third case*: $x = B = (1,0)$. Just as for $x = A = (0,0)$, we have

$$T_{\overline{\Sigma}}((1,0)) = T_{\overline{\Sigma}}(B) = \{v = (v_1, v_2) \in \mathbb{R}^2 \mid v_2 = 0 \text{ and } v_1 \le 0\}.$$

For any $x \in \Sigma$, $T_{\overline{\Sigma}}(x)$ is a vector space, thus $-T_{\overline{\Sigma}}(x) = T_{\overline{\Sigma}}(x)$. On the other hand, we have

$$\{T_{\overline{\Sigma}}(A)\} \cap \{-T_{\overline{\Sigma}}(A)\} = \{T_{\overline{\Sigma}}(B)\} \cap \{-T_{\overline{\Sigma}}(B)\} = \{(0,0)\},$$

so according to the double viability condition [28], [29], if the field $V \in \mathcal{D}^k(D;\mathbb{R}^2)$ satisfies

$$\begin{cases} \langle V(x), n(x)\rangle_{\mathbb{R}^2} = 0, & \forall x \in \overline{\Sigma}, \\ \langle V(A), \tau\rangle_{\mathbb{R}^2} = \langle V(B), \tau\rangle_{\mathbb{R}^2} = 0, \end{cases} \tag{2.3}$$

then $\overline{\Sigma}$ is globally invariant under the associated transformation $T_t(V)$. The exterior boundary $\Gamma = \partial D$ is also invariant under $T_t(V)$, since the support of V is included in D. So the boundary of $\Omega = D \setminus \overline{\Sigma}$, i.e. $\partial\Omega = \Gamma \cup \overline{\Sigma}$, is globally invariant under $T_t(V)$. Consequently, $\Omega_t = T_t(V)(\Omega)$, hence

$$dJ(\Omega; V) = 0$$

for any vector field which satisfies (2.3). It follows that it is natural to consider the set

$$F(\Omega) = \{V \in \mathcal{D}^k(D;\mathbb{R}^2) \mid \langle V, n\rangle_{\mathbb{R}^2} \text{ on } \overline{\Sigma},\ \langle V(A), \tau\rangle_{\mathbb{R}^2} = \langle V(B), \tau\rangle_{\mathbb{R}^2} = 0\}. \tag{2.4}$$

According to the hypothesis that the mapping $V \mapsto dJ(\Omega; V)$ is linear and continuous from $\mathcal{D}^k(D;\mathbb{R}^2)$ in $\mathbb{R}$, the set $F(\Omega)$ defined by (2.4) is included in its kernel. Consequently, we have the following lemmas.

Lemma 2.5. *The mapping*

$$\psi:\ \mathcal{D}^k(D;\mathbb{R}^2)/F(\Omega) \to C^k(\overline{\Sigma}) \times \mathbb{R} \times \mathbb{R}, \qquad \{V\} \mapsto (\langle V, n\rangle_{\mathbb{R}^2}, \langle V(A), \tau\rangle_{\mathbb{R}^2}, \langle V(B), \tau\rangle_{\mathbb{R}^2}),$$

is an isomorphism.

Lemma 2.6. *There exists a linear continuous mapping* $\Phi:\ C^k(\overline{\Sigma}) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *such that for any vector field* $V \in \mathcal{D}^k(D;\mathbb{R}^2)$,

$$dJ(\Omega; V) = \Phi(\langle V, n\rangle_{\mathbb{R}^2}, \langle V(A), \tau\rangle_{\mathbb{R}^2}, \langle V(B), \tau\rangle_{\mathbb{R}^2}).$$

The proofs of these two lemmas are the same as in the 3D case. Finally, just as for the 3D case, Lemmas 2.5 and 2.6 lead to the structure theorem in 2D. ∎

2.2. Semi-derivatives of the eigenvalues. Shape sensitivity analysis of eigenvalues is perfomed in [126] for the case of multiple eigenvalues in smooth domains. The shape derivatives of such eigenvalues are in general only directional, so there exists an appropriate subgradient instead of the shape gradient for eigenvalues. In the case of domains with cracks the results include, as we could expect, contributions from the singularities at the crack tips. We provide a complete proof of the result for a model problem. The result can be used in numerical methods of shape optimization, and leads to necessary optimality conditions for specific optimization problems for multiple eigenvalues in domains with geometrical singularities.

2.2.1. Introduction, notations and main result. There are many papers on Eulerian semi-derivatives of eigenvalues in smooth domains; we refer e.g. to Rousselet [119], Zolésio [137], Desaint [32] and Desaint & Zolésio [33]. The non-smooth case is analysed in the present paper for the first time. The case of the first eigenvalue [43] is much simpler compared to the general case of the $(m+1)$th eigenvalue, $m \in \mathbb{N}^*$. We consider the model problem of the eigenvalues of the Laplacian but the method can be applied to the general case of eigenvalues of a second order elliptic operator.

Let $D \subset \mathbb{R}^2$ be a bounded domain with smooth boundary Γ, and set

$$\Sigma_l = \{(y_1, y_2) \in \mathbb{R}^2 \mid 0 < y_1 < l,\, y_2 = 0\}.$$

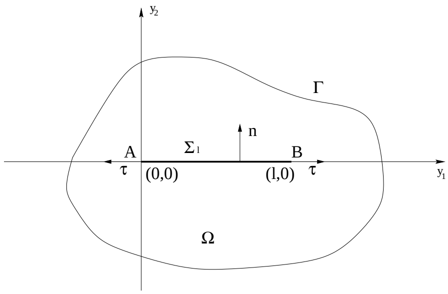

Fig. 2.6. Domain Ω with the crack Σ_l

We assume that $\Sigma_l \subset D$ for $l > 0$ small enough. A and B denote the tips of $\overline{\Sigma}_l$, and n and τ are the normal and tangent vectors to $\overline{\Sigma}_l$ respectively. The domain with crack Σ_l is denoted by $\Omega = D \setminus \overline{\Sigma}_l$. We denote by $\Lambda_i(\Omega)$, $i \geq 1$, the eigenvalues of the Laplacian with the Dirichlet condition on the boundary $\Gamma = \partial D$ and the Neumann condition on the crack's faces $\Sigma_l^{\pm}$. Consequently, there exists an eigenfunction $\varphi \in H^1_\Gamma(\Omega)$, $\varphi \neq 0$, such that

$$\int_\Omega \langle \nabla\varphi, \nabla\psi \rangle_{\mathbb{R}^2}\, dy = \Lambda_i(\Omega) \int_\Omega \varphi\psi\, dy, \qquad \forall \psi \in H^1_\Gamma(\Omega). \tag{2.5}$$

Let us point out that the eigenvalues $\Lambda_i(\Omega)$ are counted without multiplicity. The eigenfunctions $\varphi \in H^1_\Gamma(\Omega)$ in (2.5) constitute a subspace of finite dimension d_i, with $d_1 = 1$ since the first eigenvalue is simple. Moreover, since the Laplacian is a self-adjoint and anti-compact operator, it follows that $\Lambda_i(\Omega) > 0$, $i \geq 1$, the sequence $\{\Lambda_i(\Omega)\}_{i=1}^{+\infty}$ is strictly increasing and $\Lambda_i(\Omega) \to +\infty$ as $i \to +\infty$. Let $F_i(\Omega)$, $i \geq 1$, be the eigenspace corresponding to the eigenvalue $\Lambda_i(\Omega)$,

$$F_i(\Omega) = \{\varphi \in H^1_\Gamma(\Omega) \mid \varphi \text{ satisfies the variational identity (2.5)}\};$$

the dimension of $F_i(\Omega)$ is equal to d_i $(d_1 = 1)$. Therefore, d_i stands for the multiplicity of $\Lambda_i(\Omega)$. In order to take this multiplicity into account, we have to introduce the eigenvalues

λ_i counted with multiplicity, and for $i \geq 2$ we write

$$\Lambda_i = \lambda_{\sum_{k=1}^{i-1} d_k+1} = \cdots = \lambda_{\sum_{k=1}^{i} d_k}.$$

We know that the first eigenvalue $\lambda_1(\Omega)$ is shape differentiable [43]. The purpose of this paper is to find the Eulerian semi-derivatives of the eigenvalues $\lambda_{m+1}(\Omega)$, $m \in \mathbb{N}^*$. In order to find these semi-derivatives, some hypothesis should be verified: such a verification is performed in the proof of the main theorem of this paper.

Let θ_1, $\theta_2 \in C_0^\infty(D) = \mathcal{D}(D)$ be two smooth functions with compact support in D. Then the transformation $T_\delta : (x_1, x_2) \mapsto (y_1, y_2)$ (see [73]) is defined by

$$\begin{cases} y_1 = x_1 - \delta\theta_1(x_1, x_2), \\ y_2 = x_2 - \delta\theta_2(x_1, x_2), \end{cases} \tag{2.6}$$

where $\delta > 0$. Let V denote the vector field with components θ_1, θ_2. The Jacobian of (2.6) equals

$$q_\delta = 1 - \delta \operatorname{div} V + \delta^2 \det(DV),$$

where DV is the Jacobian matrix of the vector field V. For δ small enough, $q_\delta > 0$, so the transformation (2.6) is one-to-one and we write $y = y(x, \delta)$, $x = x(y, \delta)$. Let Ω_δ be the image of Ω under T_δ^{-1}. Since $\theta_1, \theta_2 \in \mathcal{D}(D)$, the exterior boundary Γ is invariant under the transformation (2.6). For given $m \in \mathbb{N}^*$, according to Auchmuty's principle [7], the $(m+1)$th eigenvalue of the Laplacian in the perturbed domain Ω_δ, counted with multiplicity and denoted by $\lambda_{m+1}(\Omega_\delta) = \lambda_{m+1}(\delta)$, is given by the formula

$$-\frac{1}{2\lambda_{m+1}(\delta)} = \max_{\widetilde{\psi} \in (H^1_\Gamma(\Omega_\delta))^m} \ \min_{\substack{\widetilde{\varphi} \in H^1_\Gamma(\Omega_\delta) \\ \int_{\Omega_\delta} \widetilde{\varphi}\widetilde{\psi}_i \, dx = 0 \\ 1 \leq i \leq m}} G_\delta(\widetilde{\varphi}), \tag{2.7}$$

where $\widetilde{\psi}_i$, $1 \leq i \leq m$, are the components of $\widetilde{\psi} \in (H^1_\Gamma(\Omega_\delta))^m$, and

$$G_\delta(\widetilde{\varphi}) = \frac{1}{2} \int_{\Omega_\delta} \|\nabla\widetilde{\varphi}\|^2_{\mathbb{R}^2} \, dx - \sqrt{\int_{\Omega_\delta} \widetilde{\varphi}^2 \, dx}, \qquad \forall \widetilde{\varphi} \in H^1_\Gamma(\Omega_\delta).$$

For simplicity we set

$$\mu_{m+1}(\delta) = -\frac{1}{2\lambda_{m+1}(\delta)}.$$

Indeed, the directional differentiability of $\mu_{m+1}(\delta)$ at $\delta = 0$ is equivalent to the directional differentiability of $\lambda_{m+1}(\delta)$ at $\delta = 0$ since $\lambda_{m+1}(\delta) \neq 0$.

Remark 2.7. According to the proof of Auchmuty's principle, the maximum in (2.7) is attained on the set

$$\widetilde{\mathcal{B}}_{m,\delta} = \Big\{ \widetilde{\psi} \in (H^1_\Gamma(\Omega_\delta))^m \mid (\widetilde{\psi}_i)_{1 \leq i \leq m} \text{ is a family of orthogonal eigenvectors,}$$
$$\widetilde{\psi}_i \text{ associated to } \lambda_i(\delta), \text{ and moreover } \int_{\Omega_\delta} \widetilde{\psi}_i^2 \, dx = 1,\ 1 \leq i \leq m \Big\}.$$

In this notation, $\widetilde{\psi}_i$, $1 \le i \le m$, is an eigenfunction of the Laplacian corresponding to the eigenvalue $\lambda_i(\delta)$ and consequently the following variational equation is satisfied:

$$\int_{\Omega_\delta} \langle \nabla\widetilde{\psi}_i, \nabla\widetilde{\varphi}\rangle_{\mathbb{R}^2}\, dx = \lambda_i(\delta) \int_{\Omega_\delta} \widetilde{\psi}_i \widetilde{\varphi}\, dx, \qquad \forall \widetilde{\varphi} \in H^1_\Gamma(\Omega_\delta).$$

By taking into account Remark 2.7, we can write

$$\mu_{m+1}(\delta) = \max_{\substack{\widetilde{\psi} \in (H^1_\Gamma(\Omega_\delta))^m \\ \int_{\Omega_\delta} \widetilde{\psi}_i^2\, dx = 1 \\ 1 \le i \le m}} \min_{\substack{\widetilde{\varphi} \in H^1_\Gamma(\Omega_\delta) \\ \int_{\Omega_\delta} \widetilde{\varphi}\widetilde{\psi}_i\, dx = 0 \\ 1 \le i \le m}} G_\delta(\widetilde{\varphi}). \tag{2.8}$$

By changing variables in (2.8), in order to transport the problem to the fixed domain Ω, we obtain

$$\mu_{m+1}(\delta) = \max_{\substack{\psi \in (H^1_\Gamma(\Omega))^m \\ \int_\Omega (\psi_i^2/q_\delta)\, dy = 1 \\ 1 \le i \le m}} \min_{\substack{\varphi \in H^1_\Gamma(\Omega) \\ \int_\Omega (\varphi\psi_i/q_\delta)\, dy = 0 \\ 1 \le i \le m}} G(\delta, \varphi), \tag{2.9}$$

where the functional $G(\delta, \cdot)$ is defined by

$$G(\delta, \varphi) = \frac{1}{2} \int_\Omega \frac{\|A_\delta \cdot \nabla\varphi\|^2_{\mathbb{R}^2}}{q_\delta}\, dy - \sqrt{\int_\Omega \frac{\varphi^2}{q_\delta}\, dy}, \qquad \forall \varphi \in H^1_\Gamma(\Omega),$$

with $A_\delta = I - \delta DV^T$.

REMARK 2.8. By Remark 2.7, the maximum in (2.9) is attained on the set

$$\mathcal{B}_{m,\delta} = \{\psi \in (H^1_\Gamma(\Omega))^m \mid \psi_i \text{ satisfies the relations (2.10) and (2.11)}, 1 \le i \le m\},$$

where

$$\int_\Omega \frac{\langle A_\delta \cdot \nabla\psi_i, A_\delta \cdot \nabla\varphi\rangle_{\mathbb{R}^2}}{q_\delta}\, dy = \lambda_i(\delta) \int_\Omega \frac{\psi_i \varphi}{q_\delta}\, dy, \qquad \forall \varphi \in H^1_\Gamma(\Omega),\ 1 \le i \le m, \tag{2.10}$$

$$\int_\Omega \frac{\psi_i \psi_j}{q_\delta}\, dy = \delta_{ij}, \qquad 1 \le i, j \le m, \tag{2.11}$$

and δ_{ij} is the Kronecker symbol.

We use the following notation. For any $\psi = (\psi_1, \ldots, \psi_n) \in (H^1_\Gamma(\Omega))^m$, we define

$$h(\delta, \psi) = \min_{\substack{\varphi \in H^1_\Gamma(\Omega) \\ \int_\Omega (\varphi\psi_i/q_\delta)\, dy = 0 \\ 1 \le i \le m}} G(\delta, \varphi).$$

In view of Remark 2.8, let $\mathcal{K}_{m,\delta}$ be the subset of $\mathcal{B}_{m,\delta}$ given by

$$\mathcal{K}_{m,\delta} = \{\psi \in \mathcal{B}_{m,\delta} \mid h(\delta, \psi) = \max_{\phi \in (H^1_\Gamma(\Omega))^m} h(\delta, \phi)\}.$$

For $\delta_1, \delta_2 \in \mathbb{R}^+$ and $\psi \in (H^1_\Gamma(\Omega))^m$, $\mathcal{L}^{\delta_1}_{\delta_2,\psi}$ denotes the set of minimizers $\varphi \in H^1_\Gamma(\Omega)$ of the functional $G(\delta_1, \cdot)$ under the orthogonality conditions

$$\int_\Omega \frac{\varphi\psi_i}{q_{\delta_2}}\, dy = 0, \qquad 1 \le i \le m.$$

For $\delta \in \mathbb{R}^+$ and $\psi \in (H^1_\Gamma(\Omega))^m$, let

$$\mathcal{H}_{\delta,\psi} = \left\{ \varphi \in H^1_\Gamma(\Omega) \,\middle|\, \int_\Omega \frac{\varphi\psi_i}{q_\delta}\, dy = 0,\ 1 \le i \le m \right\}.$$

Using these notations, we have

$$\mu_{m+1}(\delta) = h(\delta, \psi^\delta) = G(\delta, \varphi^\delta_{\delta,\psi^\delta}),$$

where $\psi^\delta \in \mathcal{K}_{m,\delta}$ and $\varphi^\delta_{\delta,\psi^\delta} \in \mathcal{L}^\delta_{\delta,\psi^\delta}$. Finally, we define, for $i \ge 1$,

$$K_i = \{\varphi \in F_i(\Omega) \mid \exists \widetilde{\varphi} \in F_i(\Omega_\delta),\ \widetilde{\varphi} \circ T_\delta^{-1} \to \varphi \text{ strongly in } H^1_\Gamma(\Omega) \text{ as } \delta \to 0^+\},$$

where $F_i(\Omega_\delta)$ is the eigenspace corresponding to the eigenvalue $\Lambda_i(\Omega_\delta)$ in the perturbed domain Ω_δ. The Eulerian semi-derivatives of the eigenvalues of the Laplacian are found under the assumption of the convergence in the sense of Kuratowski of the sets K_i, $i \ge 1$, with respect to the parameter δ.

THEOREM 2.9. *If $K_i = F_i(\Omega)$, for all $i \ge 1$, which means that any eigenfunction in the domain Ω is a strong limit in $H^1_\Gamma(\Omega)$ of eigenfunctions in the perturbed domain Ω_δ, then all the eigenvalues λ_{m+1}, $m \in \mathbb{N}^*$, have an Eulerian semi-derivative at Ω in direction $V \in \mathcal{D}^1(D, \mathbb{R}^2)$ given by*

$$\begin{aligned} d\lambda_{m+1}(\Omega; V) &= 2\lambda^2_{m+1}(\Omega) \max_{\psi^0 \in \mathcal{K}_{m,0}} \min_{\varphi \in \mathcal{L}^0_{0,\psi^0}} \frac{\partial G}{\partial \delta}(0, \varphi) \\ &= \alpha^{(m+1)}_{A,V} \langle V(A), \tau\rangle_{\mathbb{R}^2} + \alpha^{(m+1)}_{B,V} \langle V(B), \tau\rangle_{\mathbb{R}^2} + \phi^{(m+1)}_V(\langle V, n\rangle_{\mathbb{R}^2}), \end{aligned}$$

where $\alpha^{(m+1)}_{A,V}$, $\alpha^{(m+1)}_{B,V} \in \mathbb{R}$, $\phi^{(m+1)}_V \in (C^1(\overline{\Sigma}_l))'$ and $\frac{\partial G}{\partial \delta}(0,\varphi)$ is given by

$$\frac{\partial G}{\partial \delta}(0, \varphi) = \frac{1}{2} \int_\Omega \|\nabla\varphi\|^2_{\mathbb{R}^2} \operatorname{div} V\, dy - \int_\Omega \langle \nabla\varphi, DV^T \cdot \nabla\varphi\rangle_{\mathbb{R}^2}\, dy - \frac{\int_\Omega \varphi^2 \operatorname{div} V\, dy}{2\|\varphi\|_{L^2(\Omega)}}. \quad (2.12)$$

Moreover, for a vector field V such that $V = (\theta_1, 0)$ where θ_1 has support in D, $B \notin \operatorname{supp}\{\theta_1\}$ and $\theta_1 \equiv -1$ in the vicinity of the origin A, the coefficient $\alpha^{(m+1)}_{A,V}$ takes the form

$$\alpha^{(m+1)}_{A,V} = 2\lambda^2_{m+1}(\Omega) \max_{\psi^0 \in \mathcal{K}_{m,0}} \min_{\varphi \in \mathcal{L}^0_{0,\psi^0}} \frac{\pi c^2_\varphi}{4},$$

where c_φ denotes the coefficient of singularity with respect to A of the function φ. The same form can be obtained for $\alpha^{(m+1)}_{B,V'}$ with $V' = (\theta_2, 0)$, where θ_2 has support in D, $A \notin \operatorname{supp}\{\theta_2\}$ and $\theta_2 \equiv 1$ in the vicinity of B.

REMARK 2.10. If Ω is a smooth domain, integrating by parts in (2.12) leads to the well-known formula

$$d\lambda_{m+1}(\Omega; V) = -\lambda^2_{m+1}(\Omega) \max_{\psi^0 \in \mathcal{K}_{m,0}} \min_{\varphi \in \mathcal{L}^0_{0,\psi^0}} \int_\Gamma \|\nabla\varphi\|^2_{\mathbb{R}^2} \langle V, n\rangle_{\mathbb{R}^2}\, d\sigma(y).$$

The proof of Theorem 2.9 is involved and therefore, for the convenience of the reader, we give an outline which describes the consecutive steps.

2.2.2. Outline of the proof of Theorem 2.9. We start by studying the differentiability in direction V of $\lambda_{m+1}(\delta)$ at $\delta = 0$ which is equivalent to the differentiability of $\mu_{m+1}(\delta)$, because $\lambda_{m+1}(\delta) \neq 0$, and so we have to find the limit of the differential quotient

$$\frac{\mu_{m+1}(\delta) - \mu_{m+1}(0)}{\delta}.$$

We begin by proving the inequalities

$$\frac{G(\delta, \varphi^\delta_{0,\psi^0}) - G(0, \varphi^\delta_{0,\psi^0})}{\delta} \leq \frac{\mu_{m+1}(\delta) - \mu_{m+1}(0)}{\delta} \leq \frac{G(\delta, \varphi^0_{\delta,\psi^\delta}) - G(0, \varphi^0_{\delta,\psi^\delta})}{\delta}$$

for $\delta > 0$, $\psi^0 \in \mathcal{K}_{m,0}$, $\psi^\delta \in \mathcal{K}_{m,\delta}$, $\varphi^\delta_{0,\psi^0} \in \mathcal{L}^\delta_{0,\psi^0}$ and $\varphi^0_{\delta,\psi^\delta} \in \mathcal{L}^0_{\delta,\psi^\delta}$. In view of these inequalities, to complete the proof of Theorem 2.9 it is sufficient to characterize the limits of ψ^δ, $\varphi^\delta_{0,\psi^0}$ and $\varphi^0_{\delta,\psi^\delta}$ as $\delta \to 0^+$.

It is not difficult to obtain the existence of a constant $M \in \mathbb{R}^+$ independent of δ such that

$$\|\psi^\delta\|_{(H^1_\Gamma(\Omega))^m} \leq M$$

for $\delta > 0$ small enough. This means that we can assume the weak convergence, for a subsequence, $\psi^\delta \rightharpoonup \overline{\psi}$ in $(H^1_\Gamma(\Omega))^m$ as $\delta \to 0^+$. In the same way, by using the Poincaré inequality and the fact that $\varphi^0_{\delta,\psi^\delta}$ and $\varphi^\delta_{0,\psi^0}$ are minimizers of the functionals $G(0,\cdot)$ and $G(\delta,\cdot)$, respectively, under the orthogonality conditions

$$\int_\Omega \frac{\varphi^0_{\delta,\psi^\delta}\psi^\delta_i}{q_\delta}\, dy = 0, \qquad \int_\Omega \varphi^\delta_{0,\psi^0}\psi^0_i\, dy = 0, \qquad 1 \leq i \leq m,$$

we can show that the norms in $H^1_\Gamma(\Omega)$ of $\varphi^0_{\delta,\psi^\delta}$ and $\varphi^\delta_{0,\psi^0}$ are uniformly bounded with respect to δ for $\delta > 0$ small enough. Consequently,

$$\varphi^0_{\delta,\psi^\delta} \rightharpoonup \overline{\varphi} \quad \text{and} \quad \varphi^\delta_{0,\psi^0} \rightharpoonup \hat{\varphi} \quad \text{weakly in } H^1_\Gamma(\Omega) \text{ as } \delta \to 0^+.$$

The compactness of the imbedding $H^1_\Gamma(\Omega) \hookrightarrow L^2(\Omega)$, the lower semicontinuity of the functional $\varphi \mapsto \int_\Omega \|\nabla\varphi\|^2_{\mathbb{R}^2}\, dy$ for the weak convergence in $H^1_\Gamma(\Omega)$, in view of Poincaré's inequality and the fact that the orthogonality conditions are satisfied for the weak limits for $\delta \to 0^+$, enables us to show that the weak limit of ψ^δ in $(H^1_\Gamma(\Omega))^m$ belongs to $\mathcal{K}_{m,0}$. We set $\overline{\psi} = \overline{\psi}^0$, and we show that $\overline{\varphi}$ and $\hat{\varphi}$ belong to $\mathcal{L}^0_{0,\overline{\psi}^0}$ and $\mathcal{L}^0_{0,\psi^0}$, respectively, where $\overline{\varphi} = \varphi^0_{0,\overline{\psi}^0}$ and $\hat{\varphi} = \varphi^0_{0,\psi^0}$. At this stage, we have shown that

$$\varphi^0_{\delta,\psi^\delta} \rightharpoonup \varphi^0_{0,\overline{\psi}^0} \in \mathcal{L}^0_{0,\overline{\psi}^0} \quad \text{and} \quad \varphi^\delta_{0,\psi^0} \rightharpoonup \varphi^0_{0,\psi^0} \in \mathcal{L}^0_{0,\psi^0} \quad \text{weakly in } H^1_\Gamma(\Omega).$$

Finally, we show that

$$\varphi^0_{\delta,\psi^\delta} \to \varphi^0_{0,\overline{\psi}^0} \in \mathcal{L}^0_{0,\overline{\psi}^0} \quad \text{and} \quad \varphi^\delta_{0,\psi^0} \to \varphi^0_{0,\psi^0} \in \mathcal{L}^0_{0,\psi^0} \quad \text{strongly in } H^1_\Gamma(\Omega).$$

This strong convergence is obtained by using the variational identities, that is, the Euler equations, satisfied by $\varphi^0_{\delta,\psi^\delta}$ and $\varphi^\delta_{0,\psi^0}$. Indeed, these are minimizers of the functionals $G(0,\cdot)$ and $G(\delta,\cdot)$ under appropriate orthogonality conditions. By the Euler equations, it follows that

$$\|\varphi^0_{\delta,\psi^\delta}\|_{H^1_\Gamma(\Omega)} \to \|\varphi^0_{0,\overline{\psi}^0}\|_{H^1_\Gamma(\Omega)} \quad \text{and} \quad \|\varphi^\delta_{0,\psi^0}\|_{H^1_\Gamma(\Omega)} \to \|\varphi^0_{0,\psi^0}\|_{H^1_\Gamma(\Omega)}$$

as $\delta \to 0^+$, which implies the strong convergence of $\varphi^0_{\delta,\psi^\delta}$ and $\varphi^\delta_{0,\psi^0}$ to $\varphi^0_{0,\overline{\psi}^0}$ and φ^0_{0,ψ^0} respectively in $H^1_\Gamma(\Omega)$. Using these convergences, we can pass to the limit as $\delta \to 0^+$ in the inequality

$$\frac{G(\delta,\varphi^\delta_{0,\psi^0}) - G(0,\varphi^\delta_{0,\psi^0})}{\delta} \le \frac{\mu_{m+1}(\delta) - \mu_{m+1}(0)}{\delta} \le \frac{G(\delta,\varphi^0_{\delta,\psi^\delta}) - G(0,\varphi^0_{\delta,\psi^\delta})}{\delta}$$

to obtain estimates from above and from below for the upper and lower limits of the differential quotient. Finally, the convergence in the sense of Kuratowski of $K_i(\Omega_\delta)$ as $\delta \to 0^+$ is used to establish the equality of the inferior and superior limits of the differential quotient, and consequently the Eulerian semi-derivatives of the $(m+1)$th eigenvalue exist.

However, the Eulerian semi-derivatives are not necessarily linear and so we cannot directly apply the structure theorem [44]; nevertheless we can obtain the representation formula

$$d\lambda_{m+1}(\Omega;V) = \alpha^{(m+1)}_{A,V}\langle V(A),\tau\rangle_{\mathbb{R}^2} + \alpha^{(m+1)}_{B,V}\langle V(B),\tau\rangle_{\mathbb{R}^2} + \phi^{(m+1)}_V(\langle V,n\rangle_{\mathbb{R}^2}).$$

2.2.3. Proof of Theorem 2.9. We first prove some preliminary lemmas.

LEMMA 2.11. *We have*

$$G(\delta,\varphi^\delta_{s,\psi}) \le G(\delta,\varphi^\delta_{\delta,\psi^\delta}) \le G(\delta,\varphi_{\delta,\psi^\delta})$$

for $s \in \mathbb{R}^+$, $\psi \in (H^1_\Gamma(\Omega))^m$, $\varphi_{\delta,\psi^\delta} \in \mathcal{H}_{\delta,\psi^\delta}$, $\psi^\delta \in \mathcal{K}_{m,\delta}$ *and* $\varphi^\delta_{s,\psi} \in \mathcal{L}^\delta_{s,\psi}$.

Proof. The second inequality is evident: $\varphi^\delta_{\delta,\psi^\delta}$ minimizes $G(\delta,\cdot)$ subject to the orthogonality condition

$$\int_\Omega \frac{\varphi^\delta_{\delta,\psi^\delta}\psi^\delta_i}{q_\delta}\,dy = 0, \qquad 1 \le i \le m,$$

and $\varphi_{\delta,\psi^\delta} \in \mathcal{H}_{\delta,\psi^\delta}$ satisfies the same orthogonality condition. Let us show the first inequality. For $\psi^\delta \in \mathcal{K}_{m,\delta}$, we have

$$G(\delta,\varphi^\delta_{\delta,\psi^\delta}) = h(\delta,\psi^\delta) \ge h(\delta,\psi), \qquad \forall \psi \in (H^1_\Gamma(\Omega))^m,$$

and replacing $\psi \in (H^1_\Gamma(\Omega))^m$ by $\frac{q_\delta}{q_s}\psi \in (H^1_\Gamma(\Omega))^m$ in the latter inequality, we obtain

$$G(\delta,\varphi^\delta_{\delta,\psi^\delta}) \ge h\left(\delta,\frac{q_\delta}{q_s}\,\psi\right) = \min_{\substack{\varphi \in H^1_\Gamma(\Omega)\\ \int_\Omega (\varphi\psi_i/q_s)\,dy = 0\\ 1 \le i \le m}} G(\delta,\varphi) = G(\delta,\varphi^\delta_{s,\psi}). \ \blacksquare$$

The inequalities in Lemma 2.11 are very important, since we want to find the Eulerian semi-derivatives of the $(m+1)$th eigenvalue, and so it is useful to estimate from above and from below the differential quotient

$$\frac{\mu_{m+1}(\delta) - \mu_{m+1}(0)}{\delta}, \qquad \delta > 0. \tag{2.13}$$

Such estimates are given in the following lemma.

LEMMA 2.12. *If* $\psi^0 \in \mathcal{K}_{m,0}$, $\psi^\delta \in \mathcal{K}_{m,\delta}$, $\varphi^\delta_{0,\psi^0} \in \mathcal{L}^\delta_{0,\psi^0}$, $\varphi^0_{\delta,\psi^\delta} \in \mathcal{L}^0_{\delta,\psi^\delta}$, *then*

$$G(\delta,\varphi^\delta_{0,\psi^0}) - G(0,\varphi^\delta_{0,\psi^0}) \le \mu_{m+1}(\delta) - \mu_{m+1}(0) \le G(\delta,\varphi^0_{\delta,\psi^\delta}) - G(0,\varphi^0_{\delta,\psi^\delta}).$$

Proof. We have, using the notation introduced above,

$$\mu_{m+1}(\delta) - \mu_{m+1}(0) = G(\delta, \varphi^{\delta}_{\delta,\psi^\delta}) - G(0, \varphi^0_{0,\psi^0}).$$

However, the second inequality of Lemma 2.11 leads to

$$G(\delta, \varphi^{\delta}_{\delta,\psi^\delta}) \le G(\delta, \varphi^{0}_{\delta,\psi^\delta}). \tag{2.14}$$

Moreover, by using the first inequality of Lemma 2.11 with $\delta = 0$ we have

$$G(0, \varphi^0_{0,\psi^0}) \ge G(0, \varphi^0_{s,\psi}), \qquad \forall s \in \mathbb{R}^+,\ \forall \psi \in (H^1_\Gamma(\Omega))^m,$$

so for $s = \delta$ and $\psi = \psi^\delta$, we obtain

$$G(0, \varphi^0_{0,\psi^0}) \ge G(0, \varphi^0_{\delta,\psi^\delta}), \tag{2.15}$$

and finally, by combining (2.14) and (2.15) it follows that

$$\mu_{m+1}(\delta) - \mu_{m+1}(0) \le G(\delta, \varphi^{0}_{\delta,\psi^\delta}) - G(0, \varphi^0_{\delta,\psi^\delta}). \tag{2.16}$$

In the same way, we can estimate $\mu_{m+1}(\delta) - \mu_{m+1}(0)$ from below. Indeed, it is evident that

$$G(0, \varphi^0_{0,\psi^0}) \le G(0, \varphi^\delta_{0,\psi^0}), \tag{2.17}$$

so from the first inequality of Lemma 2.11 applied with $s = 0$, $\psi = \psi^0$, it follows that

$$G(\delta, \varphi^\delta_{0,\psi^0}) \le G(\delta, \varphi^\delta_{\delta,\psi^\delta}), \tag{2.18}$$

and finally, using (2.17) and (2.18) we obtain

$$\mu_{m+1}(\delta) - \mu_{m+1}(0) \ge G(\delta, \varphi^\delta_{0,\psi^0}) - G(0, \varphi^\delta_{0,\psi^0}). \ \blacksquare \tag{2.19}$$

The inequalities (2.16) and (2.19) enable us to estimate from below and from above the differential quotient (2.13). It follows that we should characterize the limits as $\delta \to 0^+$ of the functions $\varphi^\delta_{0,\psi^0}$ and $\varphi^0_{\delta,\psi^\delta}$ in order to pass to the limit in the differential quotient (2.13).

LEMMA 2.13. *There exists $\delta_0 > 0$ such that for all $\delta \in [0, \delta_0]$ and for all $i \in \mathbb{N}^*$,*

$$0 < \lambda_i(\delta) \le 27\lambda_i(0).$$

Proof. According to Rayleigh's principle,

$$\lambda_i(\delta) = \min_{E_{i,\delta} \in \mathcal{V}_{i,\delta}} \max_{\substack{v \in E_{i,\delta} \\ v \ne 0}} \frac{\int_{\Omega_\delta} \|\nabla v\|^2_{\mathbb{R}^2}\, dx}{\int_{\Omega_\delta} v^2\, dx},$$

where $\mathcal{V}_{i,\delta}$ denotes the family of subspaces $E_{i,\delta}$ of $H^1_\Gamma(\Omega_\delta)$ such that $\dim \mathcal{V}_{i,\delta} = i$. By a change of variables we can transport the max-min to the fixed domain Ω and obtain

$$\lambda_i(\delta) = \min_{E_i \in \mathcal{V}_i} \max_{\substack{v \in E_i \\ v \ne 0}} \frac{\int_\Omega (\|A_\delta \cdot \nabla v\|^2_{\mathbb{R}^2}/q_\delta)\, dy}{\int_\Omega (v^2/q_\delta)\, dy}, \tag{2.20}$$

where $\mathcal{V}_i$ is the family of subspaces E_i of $H^1_\Gamma(\Omega)$ such that $\dim \mathcal{V}_i = i$. Moreover, by the uniform convergence of q_δ to 1 as $\delta \to 0^+$ on $\overline{\Omega}$, there exists $\delta_1 > 0$ such that $1/2 \le q_\delta \le 3/2$ uniformly on $\overline{\Omega}$ for any $\delta \in [0, \delta_1]$. Moreover, the L^∞-norm of the matrix function A_δ satisfies $\|A_\delta\|_\infty \to 1$ as $\delta \to 0^+$, so there exists $\delta_2 > 0$ such that

$$\|A_\delta\|_\infty \leq 3/2, \qquad \forall \delta \in [0, \delta_2].$$

Consequently, using the above estimates, with $\delta_0 = \min(\delta_1, \delta_2) > 0$ we have

$$\frac{\int_\Omega (\|A_\delta \cdot \nabla v\|^2_{\mathbb{R}^2}/q_\delta)\, dy}{\int_\Omega (v^2/q_\delta)\, dy} \leq 27 \frac{\int_\Omega \|\nabla v\|^2_{\mathbb{R}^2}\, dy}{\int_\Omega v^2\, dy}, \qquad \forall \delta \in [0, \delta_0],$$

and applying the latter inequality to the formula (2.20) for $\lambda_i(\delta)$, we obtain

$$\lambda_i(\delta) \leq 27 \lambda_i(0), \qquad \forall \delta \in [0, \delta_0]. \ \blacksquare$$

LEMMA 2.14. *Let $\phi^\delta \in \mathcal{B}_{m,\delta}$ be such that $\phi^\delta \rightharpoonup \phi$ weakly in $(L^2(\Omega))^m$ as $\delta \to 0^+$. Then for any $\varphi_{0,\phi} \in \mathcal{H}_{0,\phi}$, there exists $\varphi_{\delta,\phi^\delta} \in \mathcal{H}_{\delta,\phi^\delta}$ such that $\varphi_{\delta,\phi^\delta} \to \varphi_{0,\phi}$ strongly in $H^1_\Gamma(\Omega)$ as $\delta \to 0^+$.*

Proof. Let us introduce

$$\varphi_{\delta,\phi^\delta} = q_\delta \varphi_{0,\phi} - \sum_{j=1}^m \phi_j^\delta \langle \varphi_{0,\phi}, \phi_j^\delta - \phi_j \rangle_{L^2(\Omega)}.$$

First, we show that $\varphi_{\delta,\phi^\delta} \in \mathcal{H}_{\delta,\phi^\delta}$. By definition, $\phi^\delta \in \mathcal{B}_{m,\delta}$ and

$$\int_\Omega \frac{\phi_i^\delta \phi_j^\delta}{q_\delta}\, dy = \delta_{ij}, \qquad 1 \leq i, j \leq m,$$

which implies that

$$\int_\Omega \frac{\varphi_{\delta,\phi^\delta} \phi_i^\delta}{q_\delta}\, dy = \int_\Omega \varphi_{0,\phi} \phi_i^\delta\, dy - \langle \varphi_{0,\phi}, \phi_i^\delta - \phi_i \rangle_{L^2(\Omega)} = \langle \varphi_{0,\phi}, \phi_i \rangle_{L^2(\Omega)} = 0$$

for $1 \leq i \leq m$, since $\varphi_{0,\phi} \in \mathcal{H}_{0,\phi}$.

In the second step of the proof we show the strong convergence in $H^1_\Gamma(\Omega)$ of $\varphi_{\delta,\phi^\delta}$ to $\varphi_{0,\phi}$. First, let us note that $q_\delta \varphi_{0,\phi} \to \varphi_{0,\phi}$ converges strongly in $H^1_\Gamma(\Omega)$ as $\delta \to 0^+$. Indeed,

$$\begin{aligned} \|q_\delta \varphi_{0,\phi} - \varphi_{0,\phi}\|^2_{H^1_\Gamma(\Omega)} &= \int_\Omega \|\nabla(q_\delta \varphi_{0,\phi}) - \nabla \varphi_{0,\phi}\|^2_{\mathbb{R}^2}\, dy \\ &= \int_\Omega \|(\nabla q_\delta)\varphi_{0,\phi} + (q_\delta - 1)\nabla \varphi_{0,\phi}\|^2_{\mathbb{R}^2}\, dy, \end{aligned}$$

and by using the identities

$$q_\delta - 1 = -\delta \operatorname{div} V + \delta^2 \det(DV), \qquad \nabla q_\delta = -\delta(\nabla \operatorname{div} V) + \delta^2 \nabla(\det(DV)),$$

the convergences $\|q_\delta - 1\|_{L^\infty(\Omega)} \to 0$ and $\|\nabla q_\delta\|_{(L^\infty(\Omega))^2} \to 0$ for $\delta \to 0^+$, and the estimate

$$\|q_\delta \varphi_{0,\phi} - \varphi_{0,\phi}\|_{H^1_\Gamma(\Omega)} \leq \|\nabla q_\delta\|_{(L^\infty(\Omega))^2} \|\varphi_{0,\phi}\|_{L^2(\Omega)} + \|q_\delta - 1\|_{L^\infty(\Omega)} \|\nabla \varphi_{0,\phi}\|_{L^2(\Omega)},$$

it follows that $\|q_\delta \varphi_{0,\phi} - \varphi_{0,\phi}\|_{H^1_\Gamma(\Omega)} \to 0$ as $\delta \to 0^+$.

Now, let us show that for all j, $1 \leq j \leq m$, $\phi_j^\delta \langle \varphi_{0,\phi}, \phi_j^\delta - \phi_j \rangle_{L^2(\Omega)} \to 0$ strongly in $H^1_\Gamma(\Omega)$ as $\delta \to 0$. By assumptions, $\phi^\delta \in \mathcal{B}_{m,\delta}$ and for $1 \leq i \leq m$,

$$\int_\Omega \frac{\langle A_\delta \cdot \nabla \phi_i^\delta, A_\delta \cdot \nabla \varphi \rangle_{\mathbb{R}^2}}{q_\delta}\, dy = \lambda_i(\delta) \int_\Omega \frac{\phi_i^\delta \varphi}{q_\delta}\, dy, \qquad \forall \varphi \in H^1_\Gamma(\Omega).$$

In particular, by applying this relation with $\varphi = \phi_i^\delta$, we obtain

$$\int_\Omega \frac{\|A_\delta \cdot \nabla \phi_i^\delta\|_{\mathbb{R}^2}^2}{q_\delta}\, dy = \lambda_i(\delta) \int_\Omega \frac{(\phi_i^\delta)^2}{q_\delta}\, dy = \lambda_i(\delta).$$

The estimates of q_δ given above lead to the inequality

$$\|A_\delta \cdot \nabla \phi_i^\delta\|_{L^2(\Omega)} \le \sqrt{\frac{3}{2}\, \lambda_i(\delta)}, \qquad \forall \delta \in [0, \delta_1],$$

which can be rewritten as

$$\|\nabla \phi_i^\delta + \delta B \cdot \nabla \phi_i^\delta\|_{L^2(\Omega)} \le \sqrt{\frac{3}{2}\, \lambda_i(\delta)}, \qquad \forall \delta \in [0, \delta_1],$$

and in view of the inequality $\|B \cdot \nabla \phi_i^\delta\|_{L^2(\Omega)} \le 2\|B\|_\infty \|\nabla \phi_i^\delta\|_{L^2(\Omega)}$ we have the estimate

$$\|\nabla \phi_i^\delta\|_{L^2(\Omega)}(1 - 2\delta\|B\|_\infty) \le \sqrt{\frac{3}{2}\, \lambda_i(\delta)}, \qquad \forall \delta \in [0, \delta_1].$$

Meanwhile there exists $\delta_3 > 0$ such that $1 - 2\delta\|B\|_\infty \ge \frac{1}{2}$ for all $\delta \in [0, \delta_3]$ and so

$$\|\nabla \phi_i^\delta\|_{L^2(\Omega)} \le \sqrt{6\lambda_i(\delta)}, \qquad \forall \delta \in [0, \min(\delta_1, \delta_3)].$$

But according to Lemma 2.13, $0 < \lambda_i(\delta) \le 27\lambda_i(0)$ for all $\delta \in [0, \delta_0]$, and consequently, if we set $\delta_0^* = \min(\delta_0, \delta_3) > 0$, we have

$$\|\nabla \phi_i^\delta\|_{L^2(\Omega)} = \|\phi_i^\delta\|_{H^1_\Gamma(\Omega)} \le 9\sqrt{2}\sqrt{\lambda_i(0)}, \qquad \forall \delta \in [0, \delta_0^*].$$

Having shown this inequality, we can complete the proof of Lemma 2.14. Indeed, for all $\delta \in [0, \delta_0^*]$,

$$\begin{aligned} \left\|\phi_j^\delta \langle \varphi_{0,\phi}, \phi_j^\delta - \phi_j\rangle_{L^2(\Omega)}\right\|_{H^1_\Gamma(\Omega)} &= |\langle \varphi_{0,\phi}, \phi_j^\delta - \phi_j\rangle_{L^2(\Omega)}|\, \|\phi_j^\delta\|_{H^1_\Gamma(\Omega)} \\ &\le 9\sqrt{2}\sqrt{\lambda_j(0)}\,|\langle \varphi_{0,\phi}, \phi_j^\delta - \phi_j\rangle_{L^2(\Omega)}|, \end{aligned}$$

and taking the limit completes the proof, since, by assumptions, $\phi^\delta \to \phi$ weakly in $(L^2(\Omega))^m$ as $\delta \to 0^+$. ∎

LEMMA 2.15. *Let $\phi \in (H^1_\Gamma(\Omega))^m$. Then for all $\varphi_{0,\phi} \in \mathcal{H}_{0,\phi}$, there exists $\varphi_{\delta,\phi} \in \mathcal{H}_{\delta,\phi}$ such that $\varphi_{\delta,\phi} \to \varphi_{0,\phi}$ strongly in $H^1_\Gamma(\Omega)$ as $\delta \to 0^+$.*

Proof. This result is of the same type as Lemma 2.14, but the proof is simpler. Indeed, it is sufficient to set $\varphi_{\delta,\phi} = q_\delta \varphi_{0,\phi}$. We have to show that $\varphi_{\delta,\phi} \in \mathcal{H}_{\delta,\phi}$. Let us note that

$$\int_\Omega \frac{\varphi_{\delta,\phi}\phi_i}{q_\delta}\, dy = \int_\Omega \varphi_{0,\phi}\phi_i\, dy = 0$$

for $1 \le i \le m$, since $\varphi_{0,\phi} \in \mathcal{H}_{0,\phi}$. Moreover, in the same way as in the proof of Lemma 2.14, we establish the strong convergence $\varphi_{\delta,\phi} \to \varphi_{0,\phi}$ in $H^1_\Gamma(\Omega)$ as $\delta \to 0^+$. ∎

Finally, we provide the variational equation satisfied by $\varphi^{\delta_1}_{\delta_2,\psi} \in \mathcal{L}^{\delta_1}_{\delta_2,\psi}$.

LEMMA 2.16. *Let $\delta_1, \delta_2 \in \mathbb{R}^+$ and $\psi \in (H^1_\Gamma(\Omega))^m$. Then any element $\varphi^{\delta_1}_{\delta_2,\psi} \in \mathcal{L}^{\delta_1}_{\delta_2,\psi}$ satisfies the variational identity*

$$\int_\Omega \frac{\langle A_{\delta_1} \cdot \nabla \varphi^{\delta_1}_{\delta_2,\psi}, A_{\delta_1} \cdot \nabla \varphi_{\delta_2,\psi} \rangle_{\mathbb{R}^2}}{q_{\delta_1}}\, dy = \frac{\displaystyle\int_\Omega \frac{\varphi^{\delta_1}_{\delta_2,\psi} \varphi_{\delta_2,\psi}}{q_{\delta_1}}\, dy}{\sqrt{\displaystyle\int_\Omega \frac{{\varphi^{\delta_1}_{\delta_2,\psi}}^2}{q_{\delta_1}}\, dy}}, \qquad \forall \varphi_{\delta_2,\psi} \in \mathcal{H}_{\delta_2,\psi}, \tag{2.21}$$

and in particular,

$$\int_\Omega \frac{\|A_{\delta_1} \cdot \nabla \varphi^{\delta_1}_{\delta_2,\psi}\|^2_{\mathbb{R}^2}}{q_{\delta_1}}\, dy = \sqrt{\int_\Omega \frac{(\varphi^{\delta_1}_{\delta_2,\psi})^2}{q_{\delta_1}}\, dy}. \tag{2.22}$$

Proof. $\varphi^{\delta_1}_{\delta_2,\psi}$ minimizes the functional $G(\delta_1, \cdot)$ subject to the orthogonality condition

$$\int_\Omega \frac{\varphi^{\delta_1}_{\delta_2,\psi} \psi_i}{q_{\delta_2}}\, dy = 0, \qquad 1 \le i \le m,$$

and so $\varphi^{\delta_1}_{\delta_2,\psi}$ is a solution of the Euler equation (2.21). Then (2.22) follows easily by applying (2.21) with $\varphi_{\delta_2,\psi} = \varphi^{\delta_1}_{\delta_2,\psi}$. ■

Now, we can begin the proof of the main result. By Lemma 2.12, if $\psi^0 \in \mathcal{K}_{m,0}$, $\psi^\delta \in \mathcal{K}_{m,\delta}$, $\varphi^\delta_{0,\psi^0} \in \mathcal{L}^\delta_{0,\psi^0}$, $\varphi^0_{\delta,\psi^\delta} \in \mathcal{L}^0_{\delta,\psi^\delta}$, then

$$G(\delta, \varphi^\delta_{0,\psi^0}) - G(0, \varphi^\delta_{0,\psi^0}) \le \mu_{m+1}(\delta) - \mu_{m+1}(0) \le G(\delta, \varphi^0_{\delta,\psi^\delta}) - G(0, \varphi^0_{\delta,\psi^\delta}).$$

Therefore, to find the Eulerian semi-derivatives of the $(m+1)$th eigenvalue of the Laplacian, it is necessary to obtain the limits of $\varphi^0_{\delta,\psi^\delta}$ and $\varphi^\delta_{0,\psi^0}$ as $\delta \to 0^+$. The following theorem gives the strong convergence of the sequence $\varphi^0_{\delta,\psi^\delta}$ in $H^1_\Gamma(\Omega)$, and moreover we precisely identify the limit function.

THEOREM 2.17. $\varphi^0_{\delta,\psi^\delta} \to \varphi^0_{0,\overline{\psi}^0}$ *strongly in $H^1_\Gamma(\Omega)$ as $\delta \to 0^+$ where $\overline{\psi}^0 \in \mathcal{K}_{m,0}$ and $\varphi^0_{0,\overline{\psi}^0} \in \mathcal{L}^0_{0,\overline{\psi}^0}$.*

Proof. According to the proof of Lemma 2.14, there exists a constant $M \in \mathbb{R}^+$ independent of δ such that

$$\|\psi^\delta\|_{(H^1_\Gamma(\Omega))^m} \le M, \qquad \forall \delta \in [0, \delta^*_0]. \tag{2.23}$$

Therefore, we can assume there exists $\overline{\psi} \in (H^1_\Gamma(\Omega))^m$ such that $\psi^\delta \rightharpoonup \overline{\psi}$ weakly in $(H^1_\Gamma(\Omega))^m$ as $\delta \to 0^+$. For the convenience of the reader, the proof of Theorem 2.17 is divided into several steps.

STEP 1. *There exists a constant $C > 0$ such that $\|\varphi^0_{\delta,\psi^\delta}\|_{H^1_\Gamma(\Omega)} \le C$ for all $\delta \in \mathbb{R}^+$.*

We know that $\varphi^0_{\delta,\psi^\delta}$ is a minimizer of the functional $G(0, \cdot)$ subject to the orthogonality condition

$$\int_\Omega \frac{\varphi^0_{\delta,\psi^\delta} \psi^\delta_i}{q_\delta}\, dy = 0, \qquad 1 \le i \le m,$$

and the function which is identically zero satisfies the same orthogonality condition, so $G(0, \varphi^0_{\delta,\psi^\delta}) \leq G(0,0) = 0$, which leads to

$$\frac{1}{2} \|\nabla \varphi^0_{\delta,\psi^\delta}\|^2_{(L^2(\Omega))^2} - \|\varphi^0_{\delta,\psi^\delta}\|_{L^2(\Omega)} \leq 0,$$

and by the Poincaré inequality, there exists a constant $C = C(\Omega) > 0$ such that

$$\|\varphi^0_{\delta,\psi^\delta}\|_{L^2(\Omega)} \leq \frac{C}{2} \|\nabla \varphi^0_{\delta,\psi^\delta}\|_{(L^2(\Omega))^2},$$

thus

$$\|\varphi^0_{\delta,\psi^\delta}\|_{H^1_\Gamma(\Omega)} = \|\nabla \varphi^0_{\delta,\psi^\delta}\|_{(L^2(\Omega))^2} \leq C. \tag{2.24}$$

Therefore, we can assume that $\varphi^0_{\delta,\psi^\delta} \rightharpoonup \overline{\varphi}$ weakly in $H^1_\Gamma(\Omega)$ as $\delta \to 0^+$.

REMARK 2.18. In the same way as in Step 1, we can show that there exists a constant $C = C(\Omega) > 0$ such that $\|\varphi^\delta_{0,\psi^0}\|_{H^1_\Gamma(\Omega)} \leq C$ for $\delta > 0$ small enough, but we cannot claim that the sequence is bounded for any $\delta \in \mathbb{R}^+$, since we use the estimates of A_δ and q_δ which hold only for $\delta > 0$ small enough. Hence, the weak convergence $\varphi^\delta_{0,\psi^0} \rightharpoonup \hat{\varphi}$ in $H^1_\Gamma(\Omega)$ follows for a subsequence as $\delta \to 0^+$.

STEP 2. $\overline{\varphi} \in \mathcal{L}^0_{0,\overline{\psi}}$.

We are going to prove that $\overline{\varphi}$ minimizes the functional $G(0,\cdot)$ and satisfies the orthogonality condition

$$\int_\Omega \overline{\varphi}\overline{\psi}_i \, dy = 0, \qquad 1 \leq i \leq m.$$

We know that $\varphi^0_{\delta,\psi^\delta}$ satisfies the orthogonality condition

$$\int_\Omega \frac{\varphi^0_{\delta,\psi^\delta} \psi^\delta_i}{q_\delta} \, dy = 0, \qquad 1 \leq i \leq m.$$

Moreover, $\varphi^0_{\delta,\psi^\delta} \rightharpoonup \overline{\varphi}$ weakly in $H^1_\Gamma(\Omega)$ as $\delta \to 0^+$ and since the imbedding $H^1_\Gamma(\Omega) \hookrightarrow L^2(\Omega)$ is compact, it follows that $\varphi^0_{\delta,\psi^\delta} \to \overline{\varphi}$ strongly in $L^2(\Omega)$. It is not difficult to see that

$$\int_\Omega \frac{\varphi^0_{\delta,\psi^\delta} \psi^\delta_i}{q_\delta} \, dy \to \int_\Omega \overline{\varphi}\overline{\psi}_i \, dy, \qquad 1 \leq i \leq m,$$

as $\delta \to 0$. Indeed, for $1 \leq i \leq m$, we have

$$\int_\Omega \frac{\varphi^0_{\delta,\psi^\delta} \psi^\delta_i}{q_\delta} \, dy = A_{\delta,i} + B_{\delta,i} + C_{\delta,i} + \int_\Omega \overline{\varphi}\overline{\psi}_i \, dy,$$

where

$$A_{\delta,i} = \int_\Omega \frac{1 - q_\delta}{q_\delta} \varphi^0_{\delta,\psi^\delta} \, \psi^\delta_i \, dy, \quad B_{\delta,i} = \int_\Omega (\varphi^0_{\delta,\psi^\delta} - \overline{\varphi}) \psi^\delta_i \, dy, \quad C_{\delta,i} = \int_\Omega \overline{\varphi}(\psi^\delta_i - \overline{\psi}_i) \, dy.$$

It is not difficult to see that each term $A_{\delta,i}$, $B_{\delta,i}$ and $C_{\delta,i}$ converges to 0 as $\delta \to 0^+$, for any i. Moreover,

$$\begin{aligned}
|A_{\delta,i}| &\leq 2\|1 - q_\delta\|_{L^\infty(\Omega)} \int_\Omega |\varphi^0_{\delta,\psi^\delta} \psi^\delta_i| \, dy, \qquad \forall \delta \in [0, \delta_1], \\
&\leq 2\|1 - q_\delta\|_{L^\infty(\Omega)} \|\varphi^0_{\delta,\psi^\delta}\|_{H^1_\Gamma(\Omega)} \|\psi^\delta\|_{(H^1_\Gamma(\Omega))^n}, \qquad \forall \delta \in [0, \delta_1], \\
&\leq 2MC\|1 - q_\delta\|_{L^\infty(\Omega)}, \qquad \forall \delta \in [0, \delta^*_0],
\end{aligned}$$

in view of (2.23) and (2.24). Finally, the convergence $\|1-q_\delta\|_{L^\infty(\Omega)} \to 0$ as $\delta \to 0^+$ implies the convergence $A_{\delta,i} \to 0$ for $1 \le i \le m$. Furthermore, we have

$$|B_{\delta,i}| \le \|\varphi^0_{\delta,\psi^\delta} - \overline{\varphi}\|_{L^2(\Omega)} \|\psi^\delta\|_{(H^1_\Gamma(\Omega))^m} \le M\|\varphi^0_{\delta,\psi^\delta} - \overline{\varphi}\|_{L^2(\Omega)}, \qquad \forall\delta \in [0,\delta_0^*],$$

and by the strong convergence of $\varphi^0_{\delta,\psi^\delta}$ to $\overline{\varphi}$ in $L^2(\Omega)$ we obtain the convergence of $B_{\delta,i}$ to 0 as $\delta \to 0^+$. Finally, $C_{\delta,i} \to 0$ as $\delta \to 0^+$ since $\psi^\delta \rightharpoonup \overline{\psi}$ weakly in $(H^1_\Gamma(\Omega))^m$. Consequently, for $1 \le i \le m$,

$$0 = \int_\Omega \frac{\varphi^0_{\delta,\psi^\delta}\psi_i^\delta}{q_\delta}\,dy \to \int_\Omega \overline{\varphi}\overline{\psi}_i\,dy$$

as $\delta \to 0^+$, which implies that

$$\int_\Omega \overline{\varphi}\overline{\psi}_i\,dy = 0, \qquad 1 \le i \le m,$$

which means that $\overline{\varphi} \in \mathcal{H}_{0,\overline{\psi}}$.

We have to show that $\overline{\varphi}$ is a minimizer of the functional $G(0,\cdot)$. According to Lemma 2.14, for any $\varphi_{0,\overline{\psi}} \in \mathcal{H}_{0,\overline{\psi}}$, there exists $\varphi_{\delta,\psi^\delta} \in \mathcal{H}_{\delta,\psi^\delta}$ such that $\varphi_{\delta,\psi^\delta} \to \varphi_{0,\overline{\psi}}$ strongly in $H^1_\Gamma(\Omega)$ as $\delta \to 0^+$. Moreover, $\varphi_{\delta,\psi^\delta} \rightharpoonup \overline{\varphi}$ weakly in $H^1_\Gamma(\Omega)$ and the functional $G(0,\cdot)$ is sequentially lower semicontinuous on $H^1_\Gamma(\Omega)$, and consequently

$$G(0,\overline{\varphi}) \le \liminf_{\delta\to 0^+} G(0,\varphi^0_{\delta,\psi^\delta}) \le \liminf_{\delta\to 0^+} G(0,\varphi_{\delta,\psi^\delta}),$$

since $\varphi^0_{\delta,\psi^\delta}$ is a minimizer of $G(0,\cdot)$ subject to the orthogonality condition

$$\int_\Omega \frac{\varphi^0_{\delta,\psi^\delta}\psi_i^\delta}{q_\delta}\,dy = 0, \qquad 1 \le i \le m,$$

and $\varphi_{\delta,\psi^\delta} \in \mathcal{H}_{\delta,\psi^\delta}$. It follows that

$$G(0,\overline{\varphi}) \le \liminf_{\delta\to 0^+} G(0,\varphi_{\delta,\psi^\delta}) = \lim_{\delta\to 0^+} G(0,\varphi_{\delta,\psi^\delta}) = G(0,\varphi_{0,\overline{\psi}}).$$

We have shown that $G(0,\overline{\varphi}) \le G(0,\varphi_{0,\overline{\psi}})$ for any $\varphi_{0,\overline{\psi}} \in \mathcal{H}_{0,\overline{\psi}}$ and $\overline{\varphi} \in \mathcal{H}_{0,\overline{\psi}}$, thus $\overline{\varphi} \in \mathcal{L}^0_{0,\overline{\psi}}$. Using this inequality, we can write $\overline{\varphi} = \varphi^0_{0,\overline{\psi}}$.

The purpose of the following steps is to show that $\overline{\psi} \in \mathcal{K}_{m,0}$. Since $\psi^\delta \in \mathcal{K}_{m,\delta}$, we have

$$h(\delta,\psi^\delta) \ge h(\delta,\psi), \qquad \forall\psi \in (H^1_\Gamma(\Omega))^m,$$

which is equivalent to

$$G(\delta,\varphi^\delta_{\delta,\psi^\delta}) \ge G(\delta,\varphi^\delta_{\delta,\psi})$$

for $\varphi^\delta_{\delta,\psi^\delta} \in \mathcal{L}^\delta_{\delta,\psi^\delta}$ and $\varphi^\delta_{\delta,\psi} \in \mathcal{L}^\delta_{\delta,\psi}$. In order to take the limit as $\delta \to 0^+$ in this inequality, we need to establish the following weak convergences.

Step 3. $\varphi^\delta_{\delta,\psi} \rightharpoonup \varphi^0_{0,\psi} \in \mathcal{L}^0_{0,\psi}$ *weakly in* $H^1_\Gamma(\Omega)$ *as* $\delta \to 0^+$.

The element $\varphi^\delta_{\delta,\psi}$ is a minimizer of $G(\delta,\cdot)$ over $H^1_\Gamma(\Omega)$ subject to the orthogonality condition

$$\int_\Omega \frac{\varphi^\delta_{\delta,\psi}\psi_i}{q_\delta}\,dy = 0, \qquad 1 \le i \le m,$$

and the function which is identically zero satisfies this orthogonality condition, so

$$G(\delta, \varphi^{\delta}_{\delta,\psi}) \le G(\delta, 0) = 0$$

and in the standard way, using the estimates of $\|A_\delta\|_\infty$ and $\|q_\delta\|_{L^\infty(\Omega)}$, the Poincaré inequality, it follows that there exists a constant $C = C(\Omega) > 0$ such that for $\delta > 0$ small enough, we have

$$\|\varphi^{\delta}_{\delta,\psi}\|_{H^1_\Gamma(\Omega)} \le C.$$

Hence there exists $\varphi^* \in H^1_\Gamma(\Omega)$ such that $\varphi^{\delta}_{\delta,\psi} \rightharpoonup \varphi^*$ weakly in $H^1_\Gamma(\Omega)$ and $\varphi^{\delta}_{\delta,\psi} \to \varphi^*$ strongly in $L^2(\Omega)$ as $\delta \to 0^+$, since the imbedding $H^1_\Gamma(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Moreover, in the same way as for the second step, we can take the limit in the orthogonality condition satisfied by $\varphi^{\delta}_{\delta,\psi}$ as $\delta \to 0^+$ and it follows that $\varphi^* \in \mathcal{H}_{0,\psi}$, so we have $\varphi^* = \varphi^*_{0,\psi}$. Now, we show that $\varphi^*_{0,\psi} \in \mathcal{L}^0_{0,\psi}$. Since $\varphi^{\delta}_{\delta,\psi}$ minimizes $G(\delta,\cdot)$ over $H^1_\Gamma(\Omega)$ subject to the orthogonality condition

$$\int_\Omega \frac{\varphi^{\delta}_{\delta,\psi}\psi_i}{q_\delta}\,dy = 0, \qquad 1 \le i \le m,$$

we have

$$G(\delta, \varphi^{\delta}_{\delta,\psi}) \le G(\delta, \varphi_{\delta,\psi}), \qquad \forall \varphi_{\delta,\psi} \in \mathcal{H}_{\delta,\psi}. \tag{2.25}$$

According to Lemma 2.15, for any $\varphi_{0,\psi} \in \mathcal{H}_{0,\psi}$, there exists a sequence $\varphi_{\delta,\psi} \in \mathcal{H}_{\delta,\psi}$ such that $\varphi_{\delta,\psi} \to \varphi_{0,\psi}$ strongly in $H^1_\Gamma(\Omega)$ as $\delta \to 0^+$. Moreover, by the lower sequential semicontinuity of the functional $\varphi \mapsto \int_\Omega \|\nabla\varphi\|^2_{\mathbb{R}^2}\,dy$ for the weak topology of $H^1_\Gamma(\Omega)$, and since $\varphi^{\delta}_{\delta,\psi} \rightharpoonup \varphi^*_{0,\psi}$ weakly in $H^1_\Gamma(\Omega)$ as $\delta \to 0^+$, by taking the lower limit in (2.25), we obtain

$$G(0, \varphi^*_{0,\psi}) \le G(0, \varphi_{0,\psi}), \qquad \forall \varphi_{0,\psi} \in \mathcal{H}_{0,\psi}.$$

Moreover $\varphi^*_{0,\psi} \in \mathcal{H}_{0,\psi}$, so $\varphi^*_{0,\psi} \in \mathcal{L}^0_{0,\psi}$ and we set $\varphi^*_{0,\psi} = \varphi^0_{0,\psi}$.

STEP 4. $\varphi^{\delta}_{\delta,\psi} \to \varphi^0_{0,\psi}$ *strongly in* $H^1_\Gamma(\Omega)$ *as* $\delta \to 0^+$.

$\varphi^{\delta}_{\delta,\psi}$ minimizes the functional $G(\delta,\cdot)$ and satisfies the orthogonality condition

$$\int_\Omega \frac{\varphi^{\delta}_{\delta,\psi}\psi_i}{q_\delta}\,dy = 0, \qquad 1 \le i \le m,$$

and according to Lemma 2.16, we have the relation

$$\int_\Omega \frac{\|A_\delta \cdot \nabla\varphi^{\delta}_{\delta,\psi}\|^2_{\mathbb{R}^2}}{q_\delta}\,dy = \sqrt{\int_\Omega \frac{(\varphi^{\delta}_{\delta,\psi})^2}{q_\delta}\,dy}.$$

Then the strong convergence $\varphi^{\delta}_{\delta,\psi} \to \varphi^0_{0,\psi}$ in $L^2(\Omega)$ as $\delta \to 0^+$ leads to

$$\int_\Omega \frac{\|A_\delta \cdot \nabla\varphi^{\delta}_{\delta,\psi}\|^2_{\mathbb{R}^2}}{q_\delta}\,dy = \sqrt{\int_\Omega \frac{(\varphi^{\delta}_{\delta,\psi})^2}{q_\delta}\,dy} \to \sqrt{\int_\Omega (\varphi^0_{0,\psi})^2\,dy} = \int_\Omega \|\nabla\varphi^0_{0,\psi}\|^2_{\mathbb{R}^2}\,dy,$$

the last inequality being due to Lemma 2.16, since $\varphi^0_{0,\psi}$ minimizes the functional $G(0,\cdot)$ with the orthogonality condition

$$\int_\Omega \varphi^0_{0,\psi}\psi_i\,dy = 0, \qquad 1 \le i \le m.$$

Moreover, it is not difficult to see that, as $\delta \to 0^+$,

$$\int_\Omega \frac{\|A_\delta \cdot \nabla \varphi^\delta_{\delta,\psi}\|^2_{\mathbb{R}^2}}{q_\delta}\, dy - \int_\Omega \|\nabla \varphi^\delta_{\delta,\psi}\|^2_{\mathbb{R}^2}\, dy \to 0.$$

In short, $\varphi^\delta_{\delta,\psi} \rightharpoonup \varphi^0_{0,\psi}$ weakly in $H^1_\Gamma(\Omega)$ as $\delta \to 0^+$, and the convergence of the norms

$$\int_\Omega \|\nabla \varphi^\delta_{\delta,\psi}\|^2_{\mathbb{R}^2}\, dy \to \int_\Omega \|\nabla \varphi^0_{0,\psi}\|^2_{\mathbb{R}^2}\, dy$$

implies the strong convergence $\varphi^\delta_{\delta,\psi} \to \varphi^0_{0,\psi}$ in $H^1_\Gamma(\Omega)$.

STEP 5. $\overline{\psi} \in \mathcal{K}_{m,0}$.

By using the same method as in the third and fourth steps, we can show that $\varphi^\delta_{\delta,\psi^\delta} \to \widetilde{\varphi}^0_{0,\overline{\psi}} \in \mathcal{L}^0_{0,\overline{\psi}}$ strongly in $H^1_\Gamma(\Omega)$ as $\delta \to 0^+$. By taking the limit in the inequality $G(\delta, \varphi^\delta_{\delta,\psi^\delta}) \geq G(\delta, \varphi^\delta_{\delta,\psi})$, we obtain $G(0, \widetilde{\varphi}^0_{0,\overline{\psi}}) \geq G(0, \varphi^0_{0,\psi})$ with $\widetilde{\varphi}^0_{0,\overline{\psi}} \in \mathcal{L}^0_{0,\overline{\psi}}$, $\varphi^0_{0,\psi} \in \mathcal{L}^0_{0,\psi}$ and consequently

$$h(0, \overline{\psi}) \geq h(0, \psi), \qquad \forall \psi \in (H^1_\Gamma(\Omega))^m.$$

It follows that $\overline{\psi} \in \mathcal{K}_{m,0}$ and we write $\overline{\psi} = \overline{\psi}^0$. Now, $\varphi^0_{\delta,\psi^\delta} \rightharpoonup \varphi^0_{0,\overline{\psi}^0}$ weakly in $H^1_\Gamma(\Omega)$ as $\delta \to 0^+$. In fact, in the same way as for the fourth step, we can prove that $\varphi^0_{\delta,\psi^\delta} \to \varphi^0_{0,\overline{\psi}^0}$ strongly in $H^1_\Gamma(\Omega)$. The proof of Theorem 2.17 is complete. ∎

The following theorem gives the limit of $\varphi^\delta_{0,\psi^0}$ as $\delta \to 0^+$.

THEOREM 2.19. *$\varphi^\delta_{0,\psi^0} \to \varphi^0_{0,\psi^0}$ strongly in $H^1_\Gamma(\Omega)$ as $\delta \to 0^+$ with $\varphi^0_{0,\psi^0} \in \mathcal{L}^0_{0,\psi^0}$.*

Proof. The proof is based on the same arguments as for Theorem 2.17, but it is easier since the orthogonality condition satisfied by $\varphi^\delta_{0,\psi^0}$ is independent of δ. ∎

Having established the limits of the functions $\varphi^0_{\delta,\psi^\delta}$ and $\varphi^\delta_{0,\psi^0}$, we can come back to analyzing the differential quotient (2.13). We have previously shown that

$$\frac{G(\delta, \varphi^\delta_{0,\psi^0}) - G(0, \varphi^\delta_{0,\psi^0})}{\delta} \leq \frac{\mu_{m+1}(\delta) - \mu_{m+1}(0)}{\delta} \leq \frac{G(\delta, \varphi^0_{\delta,\psi^\delta}) - G(0, \varphi^0_{\delta,\psi^\delta})}{\delta}$$

for $\delta > 0$. Moreover, by the finite increase theorem, there exists s, $0 \leq s \leq \delta$, such that

$$\frac{G(\delta, \varphi^0_{\delta,\psi^\delta}) - G(0, \varphi^0_{\delta,\psi^\delta})}{\delta} = \frac{\partial G}{\partial \delta}(s, \varphi^0_{\delta,\psi^\delta}) \to \frac{\partial G}{\partial \delta}(0, \varphi^0_{0,\overline{\psi}^0}),$$

since $\varphi^0_{\delta,\psi^\delta} \to \varphi^0_{0,\overline{\psi}^0}$ strongly in $H^1_\Gamma(\Omega)$ as $\delta \to 0^+$. Consequently, we obtain an upper bound

$$\limsup_{\delta \to 0^+} \frac{\mu_{m+1}(\delta) - \mu_{m+1}(0)}{\delta} \leq \frac{\partial G}{\partial \delta}(0, \varphi^0_{0,\overline{\psi}^0}).$$

In the same way,

$$\frac{G(\delta, \varphi^\delta_{0,\psi^0}) - G(0, \varphi^\delta_{0,\psi^0})}{\delta} \to \frac{\partial G}{\partial \delta}(0, \varphi^0_{0,\psi^0}),$$

since $\varphi^\delta_{0,\psi^0} \to \varphi^0_{0,\psi^0}$ strongly in $H^1_\Gamma(\Omega)$ and we have a lower bound

$$\liminf_{\delta \to 0^+} \frac{\mu_{m+1}(\delta) - \mu_{m+1}(0)}{\delta} \geq \frac{\partial G}{\partial \delta}(0, \varphi^0_{0,\psi^0}).$$

To sum up, we have obtained the inequality

$$\frac{\partial G}{\partial \delta}(0, \varphi^0_{0,\psi^0}) \leq \liminf_{\delta \to 0^+} \frac{\mu_{m+1}(\delta) - \mu_{m+1}(0)}{\delta} \leq \limsup_{\delta \to 0^+} \frac{\mu_{m+1}(\delta) - \mu_{m+1}(0)}{\delta} \leq \frac{\partial G}{\partial \delta}(0, \varphi^0_{0,\overline{\psi}^0}).$$

It is important to remark that we do not need the convergence in the sense of Kuratowski up to now. In principle, the functions φ^0_{0,ψ^0} and $\varphi^0_{0,\overline{\psi}^0}$ are different, but owing to the convergence in the sense of Kuratowski, that is, $K_i = F_i(\Omega)$, $i \geq 1$, we can write

$$\frac{\partial G}{\partial \delta}(0, \varphi^0_{0,\psi^0}) \leq \liminf_{\delta \to 0^+} \frac{\mu_{m+1}(\delta) - \mu_{m+1}(0)}{\delta} \leq \limsup_{\delta \to 0^+} \frac{\mu_{m+1}(\delta) - \mu_{m+1}(0)}{\delta} \leq \frac{\partial G}{\partial \delta}(0, \varphi^0_{0,\psi^0})$$

for all $\psi^0 \in \mathcal{K}_{m,0}$ and $\varphi^0_{0,\psi^0} \in \mathcal{L}^0_{0,\psi^0}$. The differential quotient $(\mu_{m+1}(\delta) - \mu_{m+1}(0))/\delta$ has a finite limit as $\delta \to 0^+$, denoted $d\mu_{m+1}(\Omega; V)$ and given by

$$d\mu_{m+1}(\Omega; V) = \max_{\psi^0 \in \mathcal{K}_{m,0}} \min_{\varphi \in \mathcal{L}^0_{0,\psi^0}} \frac{\partial G}{\partial \delta}(0, \varphi).$$

The Eulerian semi-derivative at Ω in direction V of the eigenvalue $\lambda_{m+1}(\Omega)$ is then obtained by the relation

$$d\lambda_{m+1}(\Omega; V) = 2\lambda^2_{m+1}(\Omega) d\mu_{m+1}(\Omega; V).$$

Remark 2.20. Without the convergence in the sense of Kuratowski, we have only an inequality for the inferior and superior limits of the differential quotient.

Step 6. We obtain the representation formula and determine the coefficients $\alpha^{(m+1)}_{A,V}$ for a vector field V such that $V = (\theta_1, 0)$, where θ_1 has support in D, $B \notin \operatorname{supp}\{\theta_1\}$ and $\theta_1 \equiv -1$ in the vicinity of A.

The map $V \mapsto d\lambda_{m+1}(\Omega; V)$ is not necessarily linear and so the structure theorem [44] cannot be directly applied; nevertheless, we can obtain a representation formula for the Eulerian semi-derivatives. Indeed, the semi-derivative of $\lambda_{m+1}(\Omega)$ at Ω in direction $V \in \mathcal{D}^1(D, \mathbb{R}^2)$ is given by

$$d\lambda_{m+1}(\Omega; V) = 2\lambda^2_{m+1}(\Omega) \max_{\psi^0 \in \mathcal{K}_{m,0}} \min_{\varphi \in \mathcal{L}^0_{0,\psi^0}} \frac{\partial G}{\partial \delta}(0, \varphi) = \frac{\partial G}{\partial \delta}(0, \varphi^*_{m+1}),$$

where $\varphi^*_{m+1} = \varphi^*_{m+1}(V)$ depends on the vector field V. But, for a fixed function φ, the map $V \mapsto \frac{\partial G}{\partial \delta}(0, \varphi)$ is linear and continuous with respect to V, and the structure theorem leads to

$$\frac{\partial G}{\partial \delta}(0, \varphi) = \alpha_{A,\varphi} \langle V(A), \tau \rangle_{\mathbb{R}^2} + \alpha_{B,\varphi} \langle V(B), \tau \rangle_{\mathbb{R}^2} + \phi_\varphi(\langle V, n \rangle_{\mathbb{R}^2}),$$

where $\alpha_{A,\varphi}, \alpha_{B,\varphi} \in \mathbb{R}$ and $\phi_\varphi \in (C^1(\overline{\Sigma}))'$. Taking $\varphi = \varphi^*_{m+1}$ yields the representation formula

$$d\lambda_{m+1}(\Omega; V) = \alpha^{(m+1)}_{A,V} \langle V(A), \tau \rangle_{\mathbb{R}^2} + \alpha^{(m+1)}_{B,V} \langle V(B), \tau \rangle_{\mathbb{R}^2} + \phi^{(m+1)}_V(\langle V, n \rangle_{\mathbb{R}^2}).$$

Moreover, let $V = (\theta_1, 0)$ be a vector field such that θ_1 has support in D, $B \notin \operatorname{supp}\{\theta_1\}$ and $\theta_1 \equiv -1$ in the vicinity of the origin A. Let Ω^ε be the subset of Ω defined in polar coordinates by $r > \varepsilon$. By integrating on Ω^ε and taking the limit as $\varepsilon \to 0^+$, we obtain

$$\alpha_{A,V}^{(m+1)} = 2\lambda_{m+1}^2(\Omega) \max_{\psi^0 \in \mathcal{K}_{m,0}} \min_{\varphi \in \mathcal{L}^0_{0,\psi^0}} \frac{\pi c_\varphi^2}{4},$$

where c_φ denotes the coefficient of singularity with respect to A of the function φ. For more details about the method used, we refer the reader to [43]. That completes the proof of Theorem 2.9. ■

2.3. The energy functional for elastic bodies with cracks and unilateral conditions. The singularity at the crack tips for problems with unilateral constraints on the crack faces is not known, in general. Therefore, the classical approach for the shape sensitivity analysis in the case of the perturbation of the position of the tip cannot be applied. We present the method which allows us to obtain a constructive result even in this case. In particular, it allows applying the Griffiths criteria for crack propagation, if applicable, for a specific problem. The proof of our result is elementary; as a result, a path independent integral is obtained for the characterization of the increment of the energy functional for the small perturbation of the position of the crack tip.

2.3.1. Introduction. We consider elasticity equations in a domain having a cut (a crack) with unilateral boundary conditions at the crack faces. The boundary conditions provide a mutual non-penetration between the crack faces, and the problem on the whole is non-linear. Assuming that a general perturbation of the cut is given we find the derivative of the energy functional with respect to the perturbation parameter. It is known that calculation of the material derivative for similar problems meets the difficulty in finding boundary conditions at the crack faces. We use a variational property of the solution, thus avoiding a direct calculation of the material derivative.

There are many results relating to differentiation of the potential energy functional with respect to variable domains (see e.g. [35, 97, 98, 95, 38, 104, 90]). A general theory of calculating material and shape derivatives in linear and non-linear boundary value problems is developed in [126].

Derivatives of energy functionals with respect to the crack length in classical linear elasticity can be found in different ways. It is well known that the classical approach to the crack problem is characterized by equality type boundary conditions at the crack faces [25, 97, 101, 17, 48, 94]. For the analysis of solution dependence on the domain shape for a wide class of elastic problems we refer the reader to [70].

In [72, 67] a technique of finding derivatives of the energy functional with respect to the crack length for unilateral boundary conditions is proposed which can be used for a wide class of unilateral problems. Qualitative properties of solutions (existence, regularity, dependence on parameters etc.) in the crack problem for plates, shells, two- and three-dimensional bodies with unilateral conditions on the crack faces are analyzed in [67] (see also [61, 64, 62, 63]).

2.3.2. Problem formulation. Let $D \subset \mathbb{R}^3$ be a bounded domain with smooth boundary Γ, and $\Xi \subset D$ be a smooth two-dimensional surface. We assume that this surface can be extended up to the outer boundary Γ in such a way that D is divided into subdomains

D_1 and D_2 with Lipschitz boundaries. Assume that this inner surface Ξ is described parametrically by the equations

$$x_i = x_i(y_1, y_2), \qquad i = 1, 2, 3, \tag{2.26}$$

where (y_1, y_2) belongs to the closure of an open bounded connected set $\omega \subset \mathbb{R}^2$ having a smooth boundary γ. We suppose that the rank of the Jacobi matrix $\partial x_i/\partial y_j$ equals 2 at every point $(y_1, y_2) \in \omega \cup \gamma$, and that the map (2.26) is one-to-one. Let $\nu = (\nu_1, \nu_2, \nu_3)$ be a unit normal vector to Ξ, for example

$$\nu = \frac{\dfrac{\partial x}{\partial y_1} \times \dfrac{\partial x}{\partial y_2}}{\left| \dfrac{\partial x}{\partial y_1} \times \dfrac{\partial x}{\partial y_2} \right|}.$$

Set $\Omega = D \setminus \Xi$. In the domain Ω, we consider the following boundary value problem for a function $u = (u_1, u_2, u_3)$:

$$-\sigma_{ij,j} = f_i, \qquad i = 1, 2, 3, \tag{2.27}$$

$$\sigma_{ij} = a_{ijkl}\varepsilon_{kl}, \qquad i, j = 1, 2, 3, \tag{2.28}$$

$$u = 0 \qquad \text{on } \Gamma, \tag{2.29}$$

$$[u]\nu \geq 0,\ \sigma_\nu \leq 0,\ [\sigma_\nu] = 0,\ \sigma_\tau = 0,\ \sigma_\nu[u]\nu = 0 \qquad \text{on } \Xi. \tag{2.30}$$

Here $\varepsilon_{kl} = \varepsilon_{kl}(u) = \frac{1}{2}(u_{k,l} + u_{l,k})$ are the strain tensor components, $u_{k,l} = \partial u_k/\partial x_l$; $\sigma_{ij} = \sigma_{ij}(u)$ denote the stress tensor components,

$$\{\sigma_{ij}\nu_j\}_{i=1}^3 = \sigma_\tau + \sigma_\nu \nu\ , \qquad \sigma_\nu = \sigma_{ij}\nu_j\nu_i.$$

The bracket $[v] = v^+ - v^-$ denotes the jump of v across Ξ, v^+, v^- stand for the values of v on Ξ^+, Ξ^-, respectively, where Ξ^+, Ξ^- are defined for a given choice of positive and negative directions of ν on Ξ. The coefficients a_{ijkl} are assumed to be constant and satisfy the usual conditions of symmetry and positive definiteness, i.e.

$$a_{ijkl} = a_{jikl} = a_{ijlk}, \qquad a_{ijkl}\xi_{kl}\xi_{ij} \geq c|\xi|^2, \qquad c > 0, \qquad \xi_{ij} = \xi_{ji}.$$

The function $f = (f_1, f_2, f_3) \in C^1_{\mathrm{loc}}(\mathbb{R}^3)$ is given. The boundary value problem (2.27)–(2.30) describes an equilibrium state of an elastic body occupying the domain Ω in its non-deformable state, the surface Ξ corresponds to a crack in the body. Conditions (2.30) provide the mutual non-penetration between the crack faces without friction [70]. Considering the problem (2.27)–(2.30) we have in mind its variational formulation. Define

$$K_0 = \{u = (u_1, u_2, u_3) \in H^1(\Omega) \mid u = 0 \text{ on } \Gamma;\ [u]\nu \geq 0 \text{ on } \Xi\}.$$

Then (2.27)–(2.30) corresponds to the following minimization problem:

$$\min_{u \in K_0} \left\{ \frac{1}{2} \int_\Omega a_{ijkl}\varepsilon_{kl}(u)\varepsilon_{ij}(u) - \int_\Omega fu \right\}. \tag{2.31}$$

By the assumptions imposed on Ω, a_{ijkl}, f, the problem (2.31) has a unique solution u satisfying the following variational inequality:

$$u \in K_0 : \qquad \int_\Omega a_{ijkl}\varepsilon_{kl}(u)(\varepsilon_{ij}(\overline{u}) - \varepsilon_{ij}(u)) \geq \int_\Omega f(\overline{u} - u), \qquad \forall \overline{u} \in K_0. \tag{2.32}$$

In this paper, we consider a general perturbation of the boundary value problem (2.27)–(2.30) and find the derivative of the energy functional with respect to the perturbation parameter. Note that the result obtained holds true for other boundary conditions. For example, we may assume that $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\operatorname{meas}\Gamma_1 > 0$, $u = 0$ on Γ_1, $\sigma_{ij}n_j = 0$ on Γ_2. Here $n = (n_1, n_2, n_3)$ is a unit normal vector to Γ.

Let Ω_t be a family of domains such that, for each t, there exists a one-to-one mapping

$$y = \Phi_t(x), \qquad x \in \Omega_t,\ y \in \Omega, \tag{2.33}$$

with positive Jacobian $|\partial\Phi_t/\partial x| > c > 0$, $\Phi_t = (\Phi_t^1, \Phi_t^2, \Phi_t^3)$. We assume that $\Phi_0(x) = x$, $\Phi \in C^2(0, T; W^{2,\infty}_{\text{loc}}(\mathbb{R}^3))$. Let

$$x = x(t, y) = \Phi_t^{-1}(y) \tag{2.34}$$

be the mapping inverse to Φ_t. By fixing y in (2.33) and differentiating (2.33) with respect to t, we have

$$0 = \frac{\partial\Phi_t}{\partial t} + \frac{\partial\Phi_t}{\partial x}\frac{dx(t)}{dt},$$

whence

$$\frac{dx(t)}{dt} = -\left(\frac{\partial\Phi_t}{\partial x}\right)^{-1}\frac{\partial\Phi_t}{\partial t}. \tag{2.35}$$

It is clear that (2.35) can be viewed as a system of ordinary differential equations, thus

$$\frac{dx(t)}{dt} = V(t, x(t)), \tag{2.36}$$

$$x(0) = y, \tag{2.37}$$

where

$$V(t, x(t)) = -\left(\frac{\partial\Phi_t(x(t))}{\partial x}\right)^{-1}\frac{\partial\Phi_t(x(t))}{\partial t},$$

and hence for the solution $x(t)$ of (2.36)–(2.37) we have $x(t) = x(t, y)$. Note, that, by (2.34),

$$\frac{dx(t,y)}{dt} = \frac{\partial\Phi_t^{-1}(y)}{\partial t},$$

hence

$$V(t, x(t)) = \frac{\partial\Phi_t^{-1}(y)}{\partial t}, \qquad x(t) = x(t, y).$$

Let $\Xi_t = \Phi_t^{-1}(\Xi)$ and $\Gamma_t = \Phi_t^{-1}(\Gamma)$. We can assume that Ξ_t has no self-intersections and consider the boundary value problem similar to (2.27)–(2.30) for the domain $\Omega_t = \Phi_t^{-1}(\Omega)$. Namely, in the domain Ω_t we want to find a function $u^t = (u_1^t, u_2^t, u_3^t)$ such that

$$-\sigma_{ij,j}^t = f_i, \qquad i = 1, 2, 3, \tag{2.38}$$

$$\sigma_{ij}^t = a_{ijkl}\varepsilon_{kl}^t, \qquad i, j = 1, 2, 3, \tag{2.39}$$

$$u^t = 0 \qquad \text{on } \Gamma_t, \tag{2.40}$$

$$[u^t]\nu^t \geq 0,\ \sigma_{\nu^t}^t \leq 0,\ [\sigma_{\nu^t}^t] = 0,\ \sigma_{\tau^t}^t = 0,\ \sigma_{\nu^t}^t[u^t]\nu^t = 0 \qquad \text{on } \Xi_t. \tag{2.41}$$

Here ν^t is a normal unit vector to Ξ_t, $\varepsilon^t_{kl}(u^t) = \frac{1}{2}(u^t_{k,l} + u^t_{l,k})$. All the other notations are similar to those of (2.27)–(2.30). In fact, the problem (2.38)–(2.41) can be written in the variational form

$$u^t \in K_t : \quad \int_{\Omega_t} a_{ijkl}\varepsilon_{kl}(u^t)(\varepsilon_{ij}(\overline{u}^t) - \varepsilon_{ij}(u^t)) \geq \int_{\Omega_t} f(\overline{u}^t - u^t), \quad \forall \overline{u}^t \in K_t, \tag{2.42}$$

where

$$K_t = \{u = (u_1, u_2, u_3) \in H^1(\Omega) \mid u = 0 \text{ on } \Gamma_t;\ [u]\nu^t \geq 0 \text{ on } \Xi_t\}.$$

We impose one more condition on the mapping Φ_t. Assume that the condition $v(y) \in K_0$ implies $v^t(x) \in K_t$, $v^t(x) = v(y)$, $x \in \Omega_t$, $y \in \Omega$, $y = \Phi_t(x)$, and conversely, if $v(x) \in K_t$ then $v_t(y) \in K_0$, $v_t(y) = v(x)$, $x = x(t, y)$. Note that this condition is not very restrictive, and it holds in many cases [67].

Let u, u^t be the solutions of the problems (2.32), (2.42), respectively. Consider the energy functionals

$$J(\Omega) = \frac{1}{2}\int_\Omega a_{ijkl}u_{k,l}u_{i,j} - \int_\Omega fu, \qquad J(\Omega_t) = \frac{1}{2}\int_{\Omega_t} a_{ijkl}u^t_{k,l}u^t_{i,j} - \int_{\Omega_t} fu^t.$$

Our purpose is to find the derivative of $J(\Omega_t)$ with respect to the parameter t,

$$\left.\frac{dJ(\Omega_t)}{dt}\right|_{t=0} = \lim_{t\to 0}\frac{J(\Omega_t) - J(\Omega)}{t}.$$

2.3.3. Convergence of solutions. First of all we prove the convergence of u^t to u in a proper sense. Namely, let $u^t(x) = u_t(y)$, $x \in \Omega_t$, $y \in \Omega$, $x = x(t, y)$. Denote by $\|\cdot\|_{1,\Omega}$ the norm in the space $H^1(\Omega)$.

LEMMA 2.21. *We have*

$$\|u_t - u\|_{1,\Omega} \leq ct,$$

where c is a constant independent of t.

Proof. The functions u, u^t satisfy the variational inequalities

$$u \in K_0 : \quad \int_\Omega a_{ijkl}u_{k,l}(\overline{u}_{i,j} - u_{i,j}) \geq \int_\Omega f(\overline{u} - u), \quad \forall \overline{u} \in K_0, \tag{2.43}$$

$$u^t \in K_t : \quad \int_{\Omega_t} a_{ijkl}u^t_{k,l}(\overline{u}^t_{i,j} - u^t_{i,j}) \geq \int_{\Omega_t} f(\overline{u}^t - u^t), \quad \forall \overline{u}^t \in K_t. \tag{2.44}$$

Define $f_t(y) = f(x(t, y))$, $q_t(y) = |\partial\Phi_t^{-1}(y)/\partial y|$. We have

$$u^t_{k,l}(x) = u_{tk,p}(y)\Phi^p_{t,l}(x), \qquad k, l = 1, 2, 3.$$

Consequently, the inequality (2.44) can be rewritten in the form

$$u_t \in K_0 : \quad \int_\Omega a_{ijkl}u_{tk,p}\Phi^p_{t,l}(\overline{u}_{ti,s}\Phi^s_{t,j} - u_{ti,s}\Phi^s_{t,j})q_t$$
$$\geq \int_\Omega f_t(\overline{u}_t - u_t)q_t, \quad \forall \overline{u}_t \in K_0. \tag{2.45}$$

It is important to note that $\overline{u}_t \in K_0$ due to the assumption imposed on Φ_t. Here we used the one-to-one mapping between K_t and K_0. In the inequality (2.45), we have

$\Phi^p_{t,l} = \Phi^p_{t,l}(x(t,y))$. Denote by δ^p_l the Kronecker symbol. Then

$$\Phi^p_{t,l}(x(t,y)) = \Phi^p_{0,l}(x(t,y)) + \frac{\partial \Phi^p_{\xi,l}(x(t,y))}{\partial \xi} t, \qquad \xi \in (0,t).$$

Since $\Phi^p_{0,l} = \delta^p_l$, $p, l = 1,2,3$, these equalities can be rewritten as

$$\Phi^p_{t,l}(x(t,y)) = \delta^p_l + \Phi^{pl}_\xi(x(t,y))t, \tag{2.46}$$

where $\xi = \xi(t,p,l)$, and we have denoted $\partial \Phi^p_{\xi,l}/\partial\xi$ by Φ^{pl}_ξ, which satisfies

$$\|\Phi^{pl}_\xi\|_{L^\infty(\Omega)} \le c \quad \text{uniformly in } \xi \in (0,T). \tag{2.47}$$

Moreover, $q_t(y) = q_0(y) + \frac{\partial q_\xi(y)}{\partial \xi} t$, $\xi \in (0,t)$, and $q_0(y) = 1$. Denote $\partial q_\xi(y)/\partial\xi$ by $\overline{q}_\xi(y)$, which gives

$$q_t(y) = 1 + \overline{q}_\xi(y)t, \tag{2.48}$$

with the uniform (in ξ) estimate $\|\overline{q}_\xi\|_{L^\infty(\Omega)} \le c$. By (2.46), the inequality (2.45) can be written in the form

$$\int_\Omega a_{ijkl} u_{tk,p}(\delta^p_l + t\Phi^{pl}_\xi)[\overline{u}_{ti,s}(\delta^s_j + t\Phi^{sj}_\xi) - u_{ti,s}(\delta^s_j + t\Phi^{sj}_\xi)]q_t \ge \int_\Omega f_t(\overline{u}_t - u_t)q_t. \tag{2.49}$$

Now substitute $\overline{u} = u_t, \overline{u}_t = u$ in (2.43), (2.49), respectively. By (2.48), this yields

$$\int_\Omega a_{ijkl} u_{k,l}(u_{ti,j} - u_{i,j}) \ge \int_\Omega f(u_t - u), \tag{2.50}$$

$$\begin{aligned}\int_\Omega a_{ijkl} u_{tk,l}(u_{i,j} - u_{ti,j})(1 + \overline{q}_\xi t) &+ t\int_\Omega a_{ijkl} u_{tk,p}\Phi^{pl}_\xi(u_{i,j} - u_{ti,j})q_t \\ &+ t\int_\Omega a_{ijkl} u_{tk,l}(u_{i,s}\Phi^{sj}_\xi - u_{ti,s}\Phi^{sj}_\xi)q_t \\ &+ t^2\int_\Omega a_{ijkl} u_{tk,p}\Phi^{pl}_\xi(u_{i,s}\Phi^{sj}_\xi - u_{ti,s}\Phi^{sj}_\xi)q_t \ge \int_\Omega f_t(u - u_t)q_t.\end{aligned} \tag{2.51}$$

Summing (2.50) and (2.51) we obtain

$$\begin{aligned}\int_\Omega a_{ijkl}(u_{k,l} - u_{tk,l})(u_{i,j} - u_{ti,j}) &\le t\int_\Omega \overline{q}_\xi a_{ijkl} u_{tk,l}(u_{i,j} - u_{ti,j}) \\ &+ t\int_\Omega [a_{ijkl} u_{tk,p}\Phi^{pl}_\xi(u_{i,j} - u_{ti,j}) + a_{ijkl} u_{tk,l}(u_{i,s}\Phi^{sj}_\xi - u_{ti,s}\Phi^{sj}_\xi)]q_t \\ &+ t^2\int_\Omega a_{ijkl} u_{tk,p}\Phi^{pl}_\xi(u_{i,s}\Phi^{sj}_\xi - u_{ti,s}\Phi^{sj}_\xi)q_t - \int_\Omega f(u_t - u) + \int_\Omega f_t(u_t - u)q_t.\end{aligned} \tag{2.52}$$

Taking $\overline{u}_t = 0$ in (2.49) we derive the uniform (in $t \in (0,T)$) estimate

$$\|u_t\|_{1,\Omega} \le c.$$

Consequently, by (2.47), (2.48), from (2.52) it follows that

$$\int_\Omega a_{ijkl}(u_{i,j} - u_{ti,j})(u_{k,l} - u_{tk,l}) \le ct^2 + \int_\Omega |u - u_t|\,|f - f_t(1 + \overline{q}_\xi t)|, \tag{2.53}$$

where the constant c is independent of $t \in (0,T)$. Since

$$f_k(y) - f_{kt}(y)(1 + \overline{q}_\xi t) = f_k(y) - \left[f_k(y) + \frac{\partial f_k}{\partial x_i}\frac{dx_i(\xi_1)}{d\xi_1}(1 + \overline{q}_\xi t)t\right], \qquad \xi_1 \in (0,t), k = 1,2,3,$$

the inequality (2.53) implies $\|u-u_t\|_{1,\Omega}^2 \le ct^2$, which completes the proof of Lemma 2.21.

2.3.4. Main result. To find the derivative of the energy functional, we shall use the variational property of the solution. Introduce first the notations

$$\Pi(\Omega_t;\varphi) = \frac{1}{2}\int_{\Omega_t} a_{ijkl}\varphi_{k,l}\varphi_{i,j}\,dx - \int_{\Omega_t} f\varphi\,dx,$$

$$\Pi_t(\Omega;\varphi) = \frac{1}{2}\int_{\Omega} a_{ijkl}\varphi_{k,p}\Phi^p_{t,l}\varphi_{i,s}\Phi^s_{t,j}q_t\,dy - \int_{\Omega} f_t\varphi q_t\,dy.$$

Since we have the one-to-one mapping between K_t and K_0, the following equality holds:

$$\min_{\varphi\in K_0} \Pi_t(\Omega;\varphi) = \min_{\varphi\in K_t} \Pi(\Omega_t;\varphi).$$

Note also that

$$J(\Omega) = \Pi(\Omega;u), \qquad J(\Omega_t) = \Pi(\Omega_t;u^t),$$

where u, u^t are the solutions of (2.32) and (2.42), respectively. Consequently,

$$\frac{J(\Omega_t)-J(\Omega)}{t} = \frac{\Pi(\Omega_t;u^t)-\Pi(\Omega;u)}{t} = \frac{\Pi_t(\Omega;u_t)-\Pi(\Omega;u)}{t} \le \frac{\Pi_t(\Omega;u)-\Pi(\Omega;u)}{t}.$$

This implies

$$\limsup_{t\to 0} \frac{J(\Omega_t)-J(\Omega)}{t} \le \limsup_{t\to 0} \frac{\Pi_t(\Omega;u)-\Pi(\Omega;u)}{t}. \tag{2.54}$$

On the other hand,

$$\limsup_{t\to 0} \frac{J(\Omega_t)-J(\Omega)}{t} = \frac{\Pi(\Omega_t;u^t)-\Pi(\Omega;u)}{t} \ge \frac{\Pi_t(\Omega;u_t)-\Pi(\Omega;u_t)}{t},$$

whence

$$\liminf_{t\to 0} \frac{J(\Omega_t)-J(\Omega)}{t} \ge \liminf_{t\to 0} \frac{\Pi_t(\Omega;u_t)-\Pi(\Omega;u_t)}{t}. \tag{2.55}$$

Now we aim to show that the right-hand sides of (2.54), (2.55) coincide, which implies the existence of the limit

$$\lim_{t\to 0} \frac{J(\Omega_t)-J(\Omega)}{t}.$$

Let us find the right-hand side of (2.54). It suffices to find the derivative

$$\frac{d}{dt}\Pi_t(\Omega;u)\bigg|_{t=0} = \frac{d}{dt}\left\{\frac{1}{2}\int_\Omega a_{ijkl}u_{k,p}\Phi^p_{t,l}u_{i,s}\Phi^s_{t,j}q_t - \int_\Omega f_t u q_t\right\}\bigg|_{t=0}. \tag{2.56}$$

By denoting

$$\Lambda(y) = -V(0,y) = \frac{\partial\Phi_t(x(t,y))}{\partial t}\bigg|_{t=0}, \tag{2.57}$$

we have

$$\Phi^p_{t,l}(x(t,y))|_{t=0} = \delta^p_l, \qquad \frac{\partial\Phi^p_{t,l}(x(t,y))}{\partial t}\bigg|_{t=0} = \Lambda^p_{,l}(y)\ , \tag{2.58}$$

and moreover, as $t\to 0$,

$$\Phi^p_{t,l}(x(t,y)) \to \delta^p_l, \qquad \frac{\partial\Phi^p_{t,l}(x(t,y))}{\partial t} \to \Lambda^p_{,l}(y) \quad \text{in } L^\infty(\Omega). \tag{2.59}$$

Indeed,

$$\begin{aligned}\Phi^p_{t,l}(x(t,y)) &= \Phi^p_{t,l}(x(t,y)) - \Phi^p_{0,l}(x(t,y)) + \Phi^p_{0,l}(x(t,y)) \\ &= \frac{\partial\Phi^p_{\xi,l}(x(t,y))}{\partial\xi}t + \Phi^p_{0,l}(x(t,y)), \qquad \xi\in(0,t).\end{aligned}$$

By (2.47),

$$\frac{\partial\Phi^p_{\xi,l}(x(t,y))}{\partial\xi} \text{ is bounded in } L^\infty(\Omega) \text{ uniformly in } \xi, t\in(0,T),$$

which together with the first equality of (2.58) implies the first convergence of (2.59). Similarly, since

$$\frac{\partial^2\Phi^p_{\xi,l}(x(t,y))}{\partial\xi^2} \text{ is bounded in } L^\infty(\Omega) \text{ for all } \xi, t\in(0,T)$$

and $\partial\Phi^p_{0,l}(x)/\partial\xi$ has the Lipschitz property in x, from the equality

$$\begin{aligned}\frac{\partial\Phi^p_{t,l}(x(t,y))}{\partial t} &= \frac{\partial\Phi^p_{t,l}(x(t,y))}{\partial t} - \frac{\partial\Phi^p_{0,l}(x(t,y))}{\partial t} + \frac{\partial\Phi^p_{0,l}(x(t,y))}{\partial t} \\ &= \frac{\partial^2\Phi^p_{\xi,l}(x(t,y))}{\partial\xi^2}t + \frac{\partial\Phi^p_{0,l}(x(t,y))}{\partial t}, \qquad \xi\in(0,t),\end{aligned}$$

we conclude that the second convergence of (2.59) holds.

It is well-known that [100]

$$\frac{\partial q_t(y)}{\partial t} = q_t(y)\operatorname{div} V(t,x(t,y)), \tag{2.60}$$

and by (2.48), (2.57),

$$\left.\frac{\partial q_t(y)}{\partial t}\right|_{t=0} = -\operatorname{div}\Lambda(y).$$

We next obtain

$$\operatorname{div} V(t,x(t,y)) = \operatorname{div} V(0,y) + \frac{d}{d\xi}\operatorname{div} V(\xi,x(\xi,y))t, \qquad \xi\in(0,t),$$

$$\left\|\frac{d}{d\xi}\operatorname{div} V(\xi,x(\xi,y))\right\|_{L^\infty(\Omega)} \le c \quad \text{uniformly in } \xi\in(0,T).$$

Hence, taking into account (2.48), as $t\to 0$,

$$q_t(y)\operatorname{div} V(t,x(t,y)) \to -\operatorname{div}\Lambda(y) \quad \text{in } L^\infty(\Omega).$$

By (2.60), this gives, as $t\to 0$,

$$\frac{\partial q_t}{\partial t} \to -\operatorname{div}\Lambda \quad \text{in } L^\infty(\Omega). \tag{2.61}$$

Also note that (2.58) implies $\Phi^p_{t,ls}(x(t,y))|_{t=0} = 0$, $p,l,s=1,2,3$, so that

$$\left.\frac{d\Phi^p_{t,l}(x(t,y))}{dt}\right|_{t=0} = \left.\frac{\partial\Phi^p_{t,l}(x(t,y))}{\partial t}\right|_{t=0}.$$

Hence, by (2.59), (2.61), we can calculate the right-hand side of (2.56), i.e. the right-hand side of (2.54):

$$\begin{aligned}\frac{d}{dt}\Pi_t(\Omega;u)\Big|_{t=0} &= \frac{1}{2}\int_\Omega (a_{ijkl}u_{k,p}u_{i,s}\Lambda^p_{,l}\delta^s_j + a_{ijkl}u_{k,p}u_{i,s}\Lambda^s_{,j}\delta^p_l) \\ &\quad - \frac{1}{2}\int_\Omega a_{ijkl}u_{k,p}u_{i,s}\delta^p_l\delta^s_j(\operatorname{div}\Lambda) + \int_\Omega u_k(\nabla f_k\Lambda) + \int_\Omega fu\operatorname{div}\Lambda \\ &= \int_\Omega \{\sigma_{kl}u_{k,p}\Lambda^p_{,l} - \frac{1}{2}\sigma_{ij}\varepsilon_{ij}\operatorname{div}\Lambda\} + \int_\Omega u_k\operatorname{div}(f_k\Lambda). \end{aligned} \tag{2.62}$$

Now we find the right-hand side of (2.55). To this end we consider the term

$$\Delta_t = \frac{1}{t}\int_\Omega (u_{tk,p}u_{ti,s}\Phi^p_{t,l}\Phi^s_{t,j}q_t - u_{tk,l}u_{ti,j})$$

for fixed k,l,i,j. It is possible to write Δ_t in the form

$$\begin{aligned}\Delta_t &= \frac{1}{t}\int_\Omega (u_{tk,p}u_{ti,s}\Phi^p_{t,l}\Phi^s_{t,j} - u_{tk,p}\delta^p_l u_{ti,s}\Phi^s_{t,j})q_t + \frac{1}{t}\int_\Omega (u_{tk,p}\delta^p_l u_{ti,s}\Phi^s_{t,j} - u_{tk,l}u_{ti,j}) \\ &\quad + \frac{1}{t}\int_\Omega (u_{tk,p}u_{ti,s}\Phi^p_{t,l}\Phi^s_{t,j} - u_{tk,p}\delta^p_l u_{ti,s}\Phi^s_{t,j}) \\ &\quad + \frac{1}{t}\int_\Omega (u_{tk,p}\delta^p_l u_{ti,s}\Phi^s_{t,j} - u_{tk,p}\delta^p_l u_{ti,s}\delta^s_j)q_t \\ &\quad + \frac{1}{t}\int_\Omega (u_{tk,p}\delta^p_l u_{ti,s}\delta^s_j - u_{tk,p}\Phi^p_{t,l}u_{ti,s}\delta^s_j)q_t \\ &\quad + \frac{1}{t}\int_\Omega (u_{tk,p}\Phi^p_{t,l}u_{ti,s}\delta^s_j q_t - u_{tk,p}\Phi^p_{t,l}u_{ti,s}\delta^s_j) \\ &\quad + \frac{1}{t}\int_\Omega (u_{tk,p}\Phi^p_{t,l}u_{ti,s}\delta^s_j - u_{tk,p}\Phi^p_{t,l}u_{ti,s}\Phi^s_{t,j}). \end{aligned} \tag{2.63}$$

Recall that as $t \to 0$,

$$u_t \to u \quad \text{in } H^1(\Omega), \qquad q_t \to 1 \quad \text{in } L^\infty(\Omega).$$

Since

$$\begin{aligned}\frac{q_t(y) - q_0(y)}{t} &= \frac{\partial q_\xi(y)}{\partial \xi}, & \xi \in (0,t), \\ \frac{\Phi^p_{t,l}(x(t,y)) - \delta^p_l}{t} &= \frac{\partial \Phi^p_{\xi,l}(x(t,y))}{\partial \xi}, & \xi \in (0,t),\end{aligned}$$

by the convergences (2.59), (2.61), we can find the limit of each part of the right-hand side of (2.63) as $t \to 0$, which implies

$$\begin{aligned}\lim_{t\to 0}\Delta_t &= \int_\Omega \{u_{k,p}\Lambda^p_{,l}u_{i,j} + u_{k,l}u_{i,s}\Lambda^s_{,j} + u_{i,j}u_{k,p}\Lambda^p_{,l} + u_{k,l}u_{i,s}\Lambda^s_{,j} \\ &\quad - u_{i,j}u_{k,p}\Lambda^p_{,l} - u_{k,l}u_{i,j}(\operatorname{div}\Lambda) - u_{k,l}u_{i,s}\Lambda^s_{,j}\} \\ &= \int_\Omega \{u_{i,j}u_{k,p}\Lambda^p_{,l} + u_{k,l}u_{i,s}\Lambda^s_{,j} - u_{k,l}u_{i,j}(\operatorname{div}\Lambda)\}. \end{aligned} \tag{2.64}$$

In addition, as $t \to 0$,

$$\frac{f_{kt} - f_k}{t} \to -\nabla f_k \cdot \Lambda \quad \text{in } L^\infty(\Omega), \qquad k = 1, 2, 3,$$

hence

$$\begin{aligned}\lim_{t\to 0} \int_\Omega \frac{f_t q_t u_t - f u_t}{t} &= \lim_{t\to 0} \int_\Omega \frac{(f_t - f) q_t u_t}{t} + \lim_{t\to 0} \int_\Omega \frac{f u_t (q_t - 1)}{t} \\ &= -\int_\Omega u_k (\nabla f_k \Lambda) - \int_\Omega f u \operatorname{div} \Lambda. \end{aligned} \tag{2.65}$$

From (2.64), (2.65) it follows that

$$\begin{aligned}\lim_{t\to 0} \frac{\Pi_t(\Omega; u_t) - \Pi(\Omega; u_t)}{t} &= \int_\Omega \left\{ a_{ijkl} u_{i,j} u_{k,p} \Lambda^p_{,l} - \frac{1}{2} a_{ijkl} u_{i,j} u_{k,l} (\operatorname{div} \Lambda) \right\} \\ &\quad + \int_\Omega u_k (\nabla f_k \Lambda) + \int_\Omega f u \operatorname{div} \Lambda \\ &= \int_\Omega \left\{ \sigma_{kl} u_{k,p} \Lambda^p_{,l} - \frac{1}{2} \sigma_{ij} \varepsilon_{ij} \operatorname{div} \Lambda \right\} + \int_\Omega u_k \operatorname{div}(f_k \Lambda). \end{aligned} \tag{2.66}$$

By (2.62), (2.66), we conclude that the right-hand sides of (2.54), (2.55) coincide and we obtain the following statement.

THEOREM 2.22. *Let the hypotheses concerning Φ_t hold. Then the derivative of the energy functional is given by the formula*

$$\left.\frac{dJ(\Omega_t)}{dt}\right|_{t=0} = \int_\Omega \left\{ \sigma_{kl} u_{k,p} \Lambda^p_{,l} - \frac{1}{2} \sigma_{ij} \varepsilon_{ij} \operatorname{div} \Lambda \right\} + \int_\Omega u_k \operatorname{div}(f_k \Lambda), \tag{2.67}$$

where the vector field Λ is defined by (2.57).

In conclusion note that the formulae similar to (2.67) were obtained for isotropic two- and three-dimensional cracked bodies with conditions (2.30) at the crack faces provided that the perturbation Φ_t of the domain Ω_t describes the crack length change [72, 67].

3. Fréchet differentiability in domains with cracks

In the first chapter, the first and second order shape derivative were computed in the smooth case for the energy functional. In Chapter 2, the structure of the shape derivative was obtained in 2D and 3D, for domains with cracks, for the Eulerian semi-derivative. The Eulerian semi-derivative is only a directional derivative, and in this chapter, the structure of the shape derivative is studied in the framework of Fréchet differentiability, a stronger notion of derivative. This allows us to give the structure theorem for the first order shape derivative and also for the second order shape derivative, which can be used in Newton shape methods. The study is also performed in dimension two or greater. It is necessary to study the 2D case as a special case. More details and examples can be found in [83].

We generalize a method introduced in [96] to obtain the structure of the derivatives in smooth domains. The main idea of this method is to decompose the perturbation field into normal and tangential components by using the implicit function theorem. Before giving this theorem, we will define a class $\mathcal{F}_k(\Omega)$ of cracked domains.

3.1. The structure theorem in dimension $d \geq 3$

3.1.1. The structure theorem. In what follows, U is a bounded domain of $\mathbb{R}^d$, $d \geq 3$, with a smooth boundary. The set U is called the *hold-all* and the perturbations θ defined later leave U globally unchanged. Let also $k \in \mathbb{N}$, $k \geq 1$.

Domains: Define

$$\mathcal{O}_k = \{D \Subset U \mid D \text{ bounded open of class } C^k\}. \tag{3.1}$$

Being of class C^k means that, for every $x \in \partial D$, there exists an open neighbourhood ω_x of x in $\mathbb{R}^2$ and $\xi_x : \omega_x \to \mathbb{R}$ of class C^k such that

$$\nabla \xi_x \neq 0 \quad \text{on } \omega_x, \qquad \xi_x(\omega_x \cap \partial D) = 0,$$
$$\xi_x(\omega_x \cap D) \subset [-\infty, 0), \qquad \xi_x(\omega_x \setminus \bar{D}) \subset (0, +\infty].$$

Notice that $n = \nabla \xi_x / |\nabla \xi_x|$ locally defines the outer unit normal vector to D.

Domains with cracks: Let $D \in \mathcal{O}_k$. The boundary $\partial D = \Sigma$ of D is a closed manifold of dimension $d-1$. Let γ be a closed and connected submanifold of Σ of dimension $d-2$ and of class C^k such that $\Sigma \setminus \gamma$ has two connected components S and S'. Then $\overline{S}$ is called a *crack* and we define the *cracked domain* by $\Omega = U \setminus \overline{S}$.

Let Ω_0 be a cracked domain. Then we have

$$\gamma_0 \subset \Sigma_0 = \partial D_0, \qquad S_0 \cup S_0' = \Sigma_0 \setminus \gamma_0, \qquad \Omega_0 = U \setminus \overline{S}_0,$$

and we denote by n the outer unit normal vector to D. The codimension of γ_0 is two, which means that the complement in $\mathbb{R}^d$ of the tangent set to γ_0 is of dimension two. Then we define the vector ν such that (n, ν) is an orthonormal basis of this space.

For a given domain Ω_0, let us introduce the perturbed domain Ω_θ of Ω_0 by some vector field θ.

Functional framework: Let Θ_k be the space of vector fields from $C^k(\mathbb{R}^d, \mathbb{R}^d)$ which vanish on U^c and whose derivatives are bounded up to order k. Equipped with the usual norm $\|.\|_k$, Θ_k is a Banach space. Define

$$\mathcal{D}_k := \{\theta \in \Theta_k \mid \|\theta\|_k < 1\}.$$

For $\theta \in \mathcal{D}_k$, the map $I + \theta$ is a C^k-diffeomorphism (I is the identity in $\mathbb{R}^d$). For given $\theta \in \mathcal{D}_k$, define

$$D_\theta = \{(I+\theta)(x) \mid x \in D_0\}, \quad \Sigma_\theta = \{(I+\theta)(x) \mid x \in \Sigma_0\},$$
$$\Omega_\theta = \{(I+\theta)(x) \mid x \in \Omega_0\}, \qquad \gamma_\theta = \{(I+\theta)(x) \mid x \in \gamma_0\}.$$

Since $D_0 \in \mathcal{O}_k$, we clearly have $D_\theta \in \mathcal{O}_k$ and Σ_θ is of class C^k. Also, since Ω_0 is a cracked domain with $D_0 \in \mathcal{O}_k$, we define

$$\mathcal{F}_k(\Omega_0) = \{\Omega_\theta \mid \theta \in \mathcal{D}_k\}$$

to be the set of admissible cracked domains. For the sake of simplicity, since Ω_0 is given, we write $\mathcal{F}_k$ instead of $\mathcal{F}_k(\Omega_0)$.

Shape functional: Let $J : \mathcal{F}_k \to \mathbb{R}$ be a given shape functional. We associate to J the functional E defined on $\mathcal{D}_k$ by the equality

$$\forall \theta \in \mathcal{D}_k, \qquad E(\theta) := J(\Omega_\theta).$$

Since Θ_k is a Banach space, the Fréchet derivatives of E can be defined. Denote by $E'(\theta) \in \mathcal{L}(\mathcal{D}_k; \mathbb{R})$ the first-order Fréchet derivative of E with respect to θ and by $E'(\theta)(\xi)$ its value in direction $\xi \in \Theta_k$. In a similar way $E''(\theta) \in \mathcal{L}(\mathcal{D}_k \times \mathcal{D}_k; \mathbb{R})$ is the second-order Fréchet derivative of E and $E''(\theta)(\xi, \eta)$ stands for its value at $\xi, \eta \in \Theta_k$.

Notations: For $x, y \in \mathbb{R}^d$, denote by $\langle x, y\rangle$ the scalar product in $\mathbb{R}^d$. Given a vector field ξ on Σ_0, we will write ξ_n instead of $\langle \xi, n\rangle$ and ξ_ν instead of $\langle \xi, \nu\rangle$. We will also write $\xi_\Sigma = \xi - \xi_n n$ for the component in the tangent space at a point x of the manifold Σ. We then define $\xi_\gamma := \xi_\Sigma - \xi_\nu \nu$, the projection of ξ on the tangent space to the manifold γ. Note that ξ_n and ξ_ν are scalars while ξ_Σ and ξ_γ are vectors in $\mathbb{R}^d$.

We are now able to give the main result of this chapter.

THEOREM 3.1. *Let $k \geq 1$.*

(i) *Let Ω_0 be a cracked domain with $D_0 \in \mathcal{O}_{k+1}$. Assume that E is Fréchet differentiable in Θ_k at 0. Then there exist continuous linear forms $l^1 : C^k(\Sigma_0, \mathbb{R}) \to \mathbb{R}$ and $l^1_\nu : C^k(\gamma_0, \mathbb{R}) \to \mathbb{R}$ such that*

$$E'(0)(\xi) = l^1(\xi_n) + l^1_\nu(\xi_\nu), \qquad \forall \xi \in \Theta_k. \tag{3.2}$$

(ii) *Let Ω_0 be a cracked domain with $D_0 \in \mathcal{O}_{k+2}$. Assume that E is twice Fréchet differentiable at 0 in Θ_k. Then there exist bilinear forms*

$$\begin{aligned} l^2 &: C^k(\Sigma_0, \mathbb{R}) \times C^k(\Sigma_0, \mathbb{R}) \to \mathbb{R}, \\ l^2_\nu &: C^k(\gamma_0, \mathbb{R}) \times C^k(\gamma_0, \mathbb{R}) \to \mathbb{R}, \\ \mathcal{L}^2 &: C^k(\Sigma_0, \mathbb{R}) \times C^k(\gamma_0, \mathbb{R}) \to \mathbb{R}, \end{aligned}$$

and a linear form $l^1_n : C^k(\gamma_0, \mathbb{R}) \to \mathbb{R}$, such that for all vector fields $\xi, \eta \in \Theta_{k+1}$,

$$\begin{aligned} E''(0)(\xi, \eta) = {} & l^2(\xi_n, \eta_n) + l^2_\nu(\xi_\nu, \eta_\nu) + l^1(\Phi''(0)(\xi, \eta)) + l^1_n(\phi''_n(0)(\xi, \eta)) \\ & + l^1_\nu(\phi''_\nu(0)(\xi, \eta)) + \mathcal{L}^2(\xi_n, \eta_\nu) + \mathcal{L}^2(\eta_n, \xi_\nu), \end{aligned} \tag{3.3}$$

with $\Phi''(0)(\xi, \eta)$, $\phi''_n(0)(\xi, \eta)$ and $\phi''_\nu(0)(\xi, \eta)$ given respectively by (3.6), (3.8) *and* (3.9) *below.*

REMARK 3.2. If Ω_0 is a critical shape for E, i.e., $E'(0) = 0$ at 0, then $l^1 \equiv 0$, $l^1_\nu \equiv 0$, and the expression (3.3) simplifies to

$$E''(0)(\xi, \eta) = l^2(\xi_n, \eta_n) + l^2_\nu(\xi_\nu, \eta_\nu) + l^1_n(\phi''_n(0)(\xi, \eta)) + \mathcal{L}^2(\xi_n, \eta_\nu) + \mathcal{L}^2(\eta_n, \xi_\nu).$$

3.1.2. Normal and tangential perturbations. For every $l \in \mathbb{N}$, $1 \leq l \leq k$, we set

$$\begin{aligned} G^{k-l}(\Sigma, \Sigma) &:= \{g \in C^{k-l}(\Sigma, \mathbb{R}^d) \mid g(\Sigma) \subset \Sigma\}, \\ G^{k-l}(\gamma, \gamma) &:= \{g \in C^{k-l}(\gamma, \mathbb{R}^d) \mid g(\gamma) \subset \gamma\}. \end{aligned}$$

Denote by $\Phi(\theta)$ and $H(\theta)$ the normal and tangential perturbations, respectively, on the surface Σ. Denote also by $\phi_n(\theta)$ and $\phi_\nu(\theta)$ the normal perturbations to γ in directions n and ν, and by $h(\theta)$ the tangential perturbations of γ. The implicit function theorem is used to obtain the following lemma, which will be applied in the proof of the structure theorem.

LEMMA 3.3. *Assume that Σ and γ are manifolds of class C^k. For every $1 \le l \le k$, there exists an open neighbourhood $\mathfrak{U}_k$ of 0 in Θ_k and a unique vector of C^l functions*

$$(H, \Phi, h, \phi_n, \phi_\nu) : \mathfrak{U}_k \to G^{k-l}(\Sigma, \Sigma) \times C^{k-l}(\Sigma, \mathbb{R}) \times G^{k-l}(\gamma, \gamma) \times C^{k-l}(\gamma, \mathbb{R}) \times C^{k-l}(\gamma, \mathbb{R})$$

such that $(H, \Phi, h, \phi_n, \phi_\nu)(0) = (I, 0, I, 0, 0)$ and for all $\theta \in \mathfrak{U}_k$,

$$I + \theta = (I + \Phi(\theta)n) \circ H(\theta) \qquad \text{on } \Sigma_0, \tag{3.4}$$

$$H(\theta) = (I + \phi_n(\theta)n + \phi_\nu(\theta)\nu) \circ h(\theta) \qquad \text{on } \gamma_0. \tag{3.5}$$

In addition, for every $\xi, \eta \in \Theta_k$ we get, for $l \ge 1$,

$$\begin{aligned} H'(0)(\xi) &= \xi_\Sigma, \\ \Phi'(0)(\xi) &= \xi_n, \\ h'(0)(\xi) &= \xi_\gamma, \\ \phi_n'(0)(\xi) &= 0, \\ \phi_\nu'(0)(\xi) &= \xi_\nu, \end{aligned}$$

and for $l \ge 2$ the second order derivatives are given by

$$\Phi''(0)(\xi, \eta) = -\langle D\eta\xi_\Sigma, n\rangle - \langle D\xi\eta_\Sigma, n\rangle - \langle Dn\xi_\Sigma, \eta_\Sigma\rangle, \tag{3.6}$$

$$H''(0)(\xi, \eta) = -\langle Dn\eta_\Sigma, \xi_\Sigma\rangle n - \eta_n Dn\xi_\Sigma - \xi_n Dn\eta_\Sigma, \tag{3.7}$$

$$\begin{aligned} h''(0)(\xi, \eta) = {}& \langle \xi_\gamma, Dn\eta_\gamma\rangle n + \langle \xi_\gamma, D\nu\eta_\gamma\rangle\nu - \eta_n Dn\xi_\Sigma - \xi_n Dn\eta_\Sigma \\ &+ \eta_n\langle Dn\xi_\Sigma, \nu\rangle\nu + \xi_n\langle Dn\eta_\Sigma, \nu\rangle\nu - \xi_\nu D\nu\eta_\gamma - \eta_\nu D\nu\xi_\gamma \\ &+ \xi_\nu\langle D\nu\eta_\gamma, n\rangle n + \eta_\nu\langle D\nu\xi_\gamma, n\rangle n, \end{aligned}$$

$$\phi_n''(0)(\xi, \eta) = -\eta_\nu\xi_\nu\langle Dn\nu, \nu\rangle, \tag{3.8}$$

$$\begin{aligned} \phi_\nu''(0)(\xi, \eta) = {}& -\langle \xi_\gamma, D\nu\eta_\gamma\rangle - \langle \nu, D\eta\xi_\gamma\rangle - \langle \nu, D\xi\eta_\gamma\rangle \\ &- \eta_n\langle \nu, Dn\xi_\Sigma\rangle - \xi_n\langle \nu, Dn\eta_\Sigma\rangle - \eta_n\langle n, D\nu\xi_\gamma\rangle - \xi_n\langle n, D\nu\eta_\gamma\rangle. \end{aligned} \tag{3.9}$$

Proof. Since Σ and γ are manifolds of class C^k and of dimension $d-1$ and $d-2$, respectively, there exist $\zeta_0,\ \zeta_1 \in C^k(\mathbb{R}^d, \mathbb{R})$ and open neighbourhoods ω_0 and ω_1 respectively of Σ and γ such that

$$\begin{aligned} \Sigma &= \{x \in \omega_0 \mid \zeta_0(x) = 0\}, && \forall x \in \Sigma,\ \nabla\zeta_0(x) \ne 0, \\ \gamma &= \{x \in \omega_1 \mid \zeta_0(x) = 0,\ \zeta_1(x) = 0\}, && \forall x \in \gamma,\ \nabla\zeta_1(x) \ne 0. \end{aligned}$$

In addition the outer unit normal vector to S is given by

$$\forall x \in \Sigma, \qquad n(x) = \nabla\zeta_0(x)/|\nabla\zeta_0(x)|,$$

and we can choose ζ_0 and ζ_1 so that $(n(x), \nu(x))$ is an orthonormal basis, with $\nu(x)$ defined as

$$\forall x \in \gamma, \qquad \nu(x) = \nabla\zeta_1(x)/|\nabla\zeta_1(x)|.$$

The plane orthogonal to γ at $x \in \gamma$ is then given by the orthonormal basis $(n(x), \nu(x))$. Let us now introduce $Z^l = C^{k-l}(\Sigma, \mathbb{R}^d) \times C^{k-l}(\Sigma, \mathbb{R}) \times C^{k-l}(\gamma, \mathbb{R}^d) \times C^{k-l}(\gamma, \mathbb{R}) \times C^{k-l}(\gamma, \mathbb{R})$ and define $T : \Theta_k \times Z^l \to Z^l$ by

$$(\theta, (H, \Phi, h, \phi_n, \phi_\nu)) \mapsto \begin{pmatrix} I + \theta - (I + \Phi n) \circ H \\ \zeta_0 \circ H \\ H - (I + \phi_n n + \phi_\nu \nu) \circ h \\ \zeta_1 \circ h \\ \zeta_0 \circ h \end{pmatrix}.$$

First of all it can be checked that T is C^l. For all $(H, \Phi, h, \phi_n, \phi_\nu) \in Z^l$, we have

$$\begin{cases} T(0, (I, 0, I, 0, 0)) = (0, 0, 0, 0, 0), \\ D_{(H,\Phi,h,\phi_n,\phi_\nu)} T(0, (I, 0, I, 0, 0))(H, \Phi, h, \phi_n, \phi_\nu) = \begin{pmatrix} -\Phi n - H \\ D\zeta_0 H \\ H - \phi_n n - \phi_\nu \nu - h \\ D\zeta_1 h \\ D\zeta_0 h \end{pmatrix}. \end{cases} \tag{3.10}$$

Note that $D\zeta_0 H = |\nabla \zeta_0| \langle n, H \rangle$, $D\zeta_0 h = |\nabla \zeta_0| \langle n, h \rangle$ and $D\zeta_1 h = |\nabla \zeta_1| \langle n, h \rangle$. In order to prove that $D_{(H,\Phi,h,\phi_n,\phi_\nu)} T(0, (I, 0, I, 0, 0))$ is an isomorphism of Z^l onto Z^l, it must be proved first that for all (A, u, a, w, v), the solution of

$$D_{(H,\Phi,h,\phi_n,\phi_\nu)} T(0, (I, 0, I, 0, 0))(H, \Phi, h, \phi_n, \phi_\nu) = (A, u, a, w, v)$$

is unique. We calculate the scalar product of the first lign of (3.10) with n to find that

$$\Phi = -\langle A, n \rangle - |\nabla \zeta_0|^{-1} u, \qquad H = |\nabla \zeta_0|^{-1} u n - A_\Sigma.$$

Then the scalar products of (3.10) with n and ν are computed and we obtain

$$\begin{aligned} \phi_n &= |\nabla \zeta_0|^{-1} u - |\nabla \zeta_0|^{-1} v - \langle a, n \rangle, \\ \phi_\nu &= -|\nabla \zeta_1|^{-1} w - \langle A, \nu \rangle - \langle a, \nu \rangle, \\ h &= -A_\gamma - a_\gamma + |\nabla \zeta_0|^{-1} v n + |\nabla \zeta_1|^{-1} w \nu. \end{aligned}$$

Thus, $D_{(H,\Phi,h,\phi_n,\phi_\nu)} T(0, (I, 0, I, 0, 0))$ is an isomorphism of Z^l, and the function T satisfies the conditions of the implicit function theorem in the neighbourhood of $(0, (I, 0, I, 0, 0))$. Hence there exists an open neighbourhood $\mathfrak{U}_k \subset \Theta_k$ of 0 and functions

$$(H, \Phi, h, \phi_n, \phi_\nu) : \mathfrak{U}_k \to Z^l$$

of class C^l such that $(H, \Phi, h, \phi_n, \phi_\nu)(0) = (I, 0, I, 0, 0)$ and

$$I + \theta = (I + \Phi(\theta)) \circ H(\theta), \tag{3.11}$$

$$H(\theta) = (I + \phi_n(\theta) n + \phi_\nu(\theta) \nu) \circ h(\theta), \tag{3.12}$$

$$\zeta_0(H(\theta)) = 0, \tag{3.13}$$

$$\zeta_0(h(\theta)) = 0, \tag{3.14}$$

$$\zeta_1(h(\theta)) = 0. \tag{3.15}$$

It is always possible to restrict $\mathcal{U}_k$ so that $H(\theta)$ takes its values in ω_0 and $h(\theta)$ takes its values in $\omega_0 \cap \omega_1$; then (3.13)–(3.15) imply $H(\theta)(x) \in \Sigma$ for all $x \in \Sigma$ and $h(\theta)(x) \in \gamma$ for all $x \in \gamma$. Thus we have shown the first part of the lemma.

Now we can differentiate the five equations (3.11)–(3.15) with respect to θ in the neighbourhood of 0. For all $\xi \in \Theta_k$, we get

$$\xi = H'(\theta)(\xi) + D(\Phi(\theta)n) \circ H(\theta) H'(\theta)(\xi) + (\Phi'(\theta)(\xi)n) \circ H(\theta), \tag{3.16}$$

$$\begin{aligned} H'(\theta)(\xi) = h'(\theta)(\xi) &+ D(\phi_n(\theta)n + \phi_\nu(\theta)\nu) \circ h(\theta) h'(\theta)(\xi) \\ &+ (\phi'_n(\theta)(\xi)n + \phi'_\nu(\theta)(\xi)\nu) \circ h(\theta), \end{aligned} \tag{3.17}$$

$$D\zeta_0(H(\theta))H'(\theta)(\xi) = 0, \tag{3.18}$$

$$D\zeta_0(h(\theta))h'(\theta)(\xi) = 0, \tag{3.19}$$

$$D\zeta_1(h(\theta))h'(\theta)(\xi) = 0. \tag{3.20}$$

In particular, for $\theta = 0$, one obtains

$$\xi = H'(0)(\xi) + D(\Phi(0)n)H'(0)(\xi) + \Phi'(0)(\xi)n, \tag{3.21}$$

$$H'(0)(\xi) = h'(0)(\xi) + D(\phi_n(0)n + \phi_\nu(0)\nu)h'(0)(\xi) + (\phi'_n(0)(\xi)n + \phi'_\nu(0)(\xi)\nu), \tag{3.22}$$

$$D\zeta_0\, H'(0)(\xi) = 0 = |\nabla\zeta_0| \langle n, H'(0)(\xi)\rangle, \tag{3.23}$$

$$D\zeta_0\, h'(0)(\xi) = 0 = |\nabla\zeta_0| \langle n, h'(0)(\xi)\rangle, \tag{3.24}$$

$$D\zeta_1\, h'(0)(\xi) = 0 = |\nabla\zeta_1| \langle \nu, h'(0)(\xi)\rangle. \tag{3.25}$$

Then we multiply (3.21) by n, and in view of (3.23) we get

$$H'(0)(\xi) = \xi_\Sigma, \tag{3.26}$$

$$\Phi'(0)(\xi) = \xi_n. \tag{3.27}$$

Multiplying (3.22) by n and ν, and using equalities (3.24) and (3.25), we get

$$h'(0)(\xi) = \xi_\gamma, \tag{3.28}$$

$$\phi'_n(0)(\xi) = 0, \tag{3.29}$$

$$\phi'_\nu(0)(\xi) = \xi_\nu. \tag{3.30}$$

Now, in order to get the second order shape derivative, we differentiate (3.16)–(3.20) at $\theta = 0$. The following system of equations is derived, for all $\xi, \eta \in \Theta_k$:

$$\begin{aligned} 0 = H''(0)(\xi,\eta) &+ D(\Phi'(0)(\eta)n)H'(0)(\xi) \\ &+ \Phi''(0)(\xi,\eta)n + D(\Phi'(0)(\xi)n)H'(0)(\eta), \end{aligned} \tag{3.31}$$

$$\begin{aligned} H''(0)(\xi,\eta) = h''(0)(\xi,\eta) &+ (\phi''_n(0)(\xi,\eta)n + \phi''_\nu(0)(\xi,\eta)\nu) \\ &+ D(\phi'_n(0)(\xi)n + \phi'_\nu(0)(\xi)\nu)h'(0)(\eta) \\ &+ D(\phi'_n(0)(\eta)n + \phi'_\nu(0)(\eta)\nu)h'(0)(\xi), \end{aligned} \tag{3.32}$$

$$0 = D\zeta_0\, H''(0)(\xi,\eta) + D^2\zeta_0\, H'(0)(\xi)H'(0)(\eta), \tag{3.33}$$

$$0 = D\zeta_0\, h''(0)(\xi,\eta) + D^2\zeta_0\, h'(0)(\xi)h'(0)(\eta), \tag{3.34}$$

$$0 = D\zeta_1\, h''(0)(\xi,\eta) + D^2\zeta_0\, h'(0)(\xi)h'(0)(\eta). \tag{3.35}$$

Nevertheless, since $D\zeta_0\, k = |\nabla\zeta_0|\langle n, k\rangle$ for all $k \in \mathbb{R}^d$, it follows that for all $l \in \mathbb{R}^d$,

$$D^2\zeta_0(k,l) = (D(|\nabla\zeta_0|)\, l)\langle n, k\rangle + |\nabla\zeta_0|\langle Dn\, l, k\rangle$$

so that

$$D^2\zeta_0(\xi_\Sigma, \eta_\Sigma) = |\nabla\zeta_0|\langle Dn\,\eta_\Sigma, \xi_\Sigma\rangle. \tag{3.36}$$

Plugging (3.26)–(3.30) into (3.31)–(3.35) and using (3.36) we get

$$0 = H''(0)(\xi,\eta) + D(\langle\eta, n\rangle n)\xi_\Sigma + \Phi''(0)(\xi,\eta)n + D(\xi_n n)\eta_\Sigma, \tag{3.37}$$

$$\begin{aligned} H''(0)(\xi,\eta) = h''(0)(\xi,\eta) + (\phi_n''(0)(\xi,\eta)n + \phi_\nu''(0)(\xi,\eta)\nu)\\ + D(\xi_\nu\nu)\,\eta_\gamma + D(\langle\eta,\nu\rangle\nu)\,\xi_\gamma, \end{aligned} \tag{3.38}$$

$$0 = \langle\xi_\Sigma, Dn\,\eta_\Sigma\rangle + \langle n, H''(0)(\xi,\eta)\rangle, \tag{3.39}$$

$$0 = \langle\xi_\gamma, Dn\,\eta_\gamma\rangle + \langle n, h''(0)(\xi,\eta)\rangle, \tag{3.40}$$

$$0 = \langle\xi_\gamma, D\nu\,\eta_\gamma\rangle + \langle\nu, h''(0)(\xi,\eta)\rangle. \tag{3.41}$$

Multiplying (3.37) by n and using (3.39) we get

$$\Phi''(0)(\xi,\eta) = \langle\xi_\Sigma, Dn\eta_\Sigma\rangle - \langle D(\eta_n n)\xi_\Sigma, n\rangle - \langle D(\xi_n n)\eta_\Sigma, n\rangle. \tag{3.42}$$

We can simplify (3.42) to

$$\langle D(\eta_n n)\xi_\Sigma, n\rangle = \langle Dn\xi_\Sigma, n\rangle\eta_n + \langle(n\otimes\nabla\eta_n)\xi_\Sigma, n\rangle. \tag{3.43}$$

The first term on the right-hand side of (3.43) is equal to 0 since $\langle n, n\rangle = 1$ implies $\langle Dn\,k, n\rangle = 0$ for all $k \in \mathbb{R}^d$. As a consequence, (3.43) becomes

$$\langle D(\eta_n n)\,\xi_\Sigma, n\rangle = \langle\nabla\eta_n, \xi_\Sigma\rangle \tag{3.44}$$

$$= \langle D\eta\,\xi_\Sigma, n\rangle + \langle Dn\,\xi_\Sigma, \eta\rangle \tag{3.45}$$

$$= \langle D\eta\,\xi_\Sigma, n\rangle + \langle Dn\,\xi_\Sigma, \eta_\Sigma\rangle. \tag{3.46}$$

Then, injecting (3.46) into (3.42) one obtains

$$\Phi''(0)(\xi,\eta) = -\langle D\eta\xi_\Sigma, n\rangle - \langle D\xi\eta_\Sigma, n\rangle - \langle Dn\xi_\Sigma, \eta_\Sigma\rangle. \tag{3.47}$$

Using now equations (3.37) and (3.47) and in view of the decomposition (3.43) we get

$$H''(0)(\xi,\eta) = -\langle Dn\eta_\Sigma, \xi_\Sigma\rangle n - \eta_n Dn\xi_\Sigma - \xi_n Dn\eta_\Sigma. \tag{3.48}$$

Multiplying (3.38) by n and ν and using (3.40) and (3.41) we get

$$\phi_n''(0)(\xi,\eta) = \langle\xi_\gamma, Dn\,\eta_\gamma\rangle - \langle\xi_\Sigma, Dn\,\eta_\Sigma\rangle - \langle n, D(\xi_\nu\nu)\,\eta_\gamma\rangle - \langle n, D(\eta_\nu\nu)\,\xi_\gamma\rangle, \tag{3.49}$$

$$\begin{aligned} \phi_\nu''(0)(\xi,\eta) = \langle\xi_\gamma, D\nu\,\eta_\gamma\rangle - \langle\nu, \eta_n Dn\,\xi_\Sigma\rangle - \langle\nu, \xi_n Dn\,\eta_\Sigma\rangle\\ - \langle\nu, D(\xi_\nu\nu)\,\eta_\gamma\rangle - \langle\nu, D(\eta_\nu\nu)\,\xi_\gamma\rangle. \end{aligned} \tag{3.50}$$

As for (3.43)–(3.46), we can write

$$D(\eta_\nu\nu)\xi_\gamma = \eta_\nu D\nu\xi_\gamma + \langle\nu, D\eta\xi_\gamma\rangle\nu + \langle\eta, D\nu\xi_\gamma\rangle\nu. \tag{3.51}$$

In addition we have the decomposition

$$\langle\xi_\Sigma, Dn\eta_\Sigma\rangle = \langle\xi_\gamma, Dn\eta_\gamma\rangle + \eta_\nu\xi_\nu\langle Dn\,\nu, \nu\rangle + \eta_\nu\langle Dn\nu, \xi_\gamma\rangle + \xi_\nu\langle Dn\eta_\gamma, \nu\rangle. \tag{3.52}$$

Plugging (3.52) and (3.51) into (3.49) one finally obtains

$$\phi_n''(0)(\xi,\eta) = -\eta_\nu\xi_\nu\langle Dn\,\nu, \nu\rangle. \tag{3.53}$$

Now let us deal with the case of $\phi_\nu''(0)(\xi,\eta)$. Inserting (3.51) into (3.50) one obtains

$$\begin{aligned}\phi_\nu''(0)(\xi,\eta) = &-\langle \xi_\gamma, D\nu\,\eta_\gamma\rangle - \langle \nu, D\eta\,\xi_\gamma\rangle - \langle \nu, D\xi\,\eta_\gamma\rangle \\ &- \eta_n\langle \nu, Dn\,\xi_\Sigma\rangle - \xi_n\langle \nu, Dn\,\eta_\Sigma\rangle - \eta_n\langle n, D\nu\,\xi_\gamma\rangle - \xi_n\langle n, D\nu\,\eta_\gamma\rangle. \end{aligned} \tag{3.54}$$

Now let us gather expressions (3.54), (3.53) and (3.48) and insert them in (3.39) to obtain

$$\begin{aligned}h''(0)(\xi,\eta) = &\langle \xi_\gamma, Dn\,\eta_\gamma\rangle n + \langle \xi_\gamma, D\nu\,\eta_\gamma\rangle \nu - \eta_n Dn\,\xi_\Sigma - \xi_n Dn\,\eta_\Sigma \\ &+ \eta_n\langle Dn\,\xi_\Sigma, \nu\rangle\nu + \xi_n\langle Dn\,\eta_\Sigma,\nu\rangle\nu - \xi_\nu D\nu\eta_\gamma - \eta_\nu D\nu\xi_\gamma \\ &+ \xi_\nu\langle D\nu\eta_\gamma, n\rangle n + \eta_\nu\langle D\nu\xi_\gamma, n\rangle n.\end{aligned}$$

Thus the proof of Lemma 3.3 is complete. ∎

3.1.3. Proof of the structure theorem. Our objective is to prove that if θ is close enough to 0 in $\mathcal{D}_k$, there exists a function F such that

$$E(\theta) = F(\Phi(\theta), \phi_n(\theta), \phi_\nu(\theta)) \tag{3.55}$$

where F is a functional on $C^k(\Sigma,\mathbb{R}) \times C^k(\gamma,\mathbb{R}) \times C^k(\gamma,\mathbb{R})$ and the functions Φ, ϕ_n and ϕ_ν are defined in (3.4)–(3.5).

LEMMA 3.4. *Let $h_{1,\varepsilon}$, $h_{2,\varepsilon}$ be in $C^k(\Sigma,\Sigma)$ equipped with the usual norm $\|\cdot\|_k$. Suppose that these functions are close to the identity in the following sense:*

$$\|h_{1,\varepsilon} - I\|_k, \|h_{2,\varepsilon} - I\|_k \to 0, \qquad \varepsilon \to 0. \tag{3.56}$$

Then

$$\exists \varepsilon_0, \forall \varepsilon \le \varepsilon_0, \qquad h_{1,\varepsilon}(\Sigma) = h_{2,\varepsilon}(\Sigma) = \Sigma \text{ and } h_{1,\varepsilon}(\gamma) = h_{2,\varepsilon}(\gamma) \ \Rightarrow\ h_{1,\varepsilon}(S) = h_{2,\varepsilon}(S).$$

Proof. First of all, if ε is close enough to 0, then $h_{1,\varepsilon}$ and $h_{2,\varepsilon}$ are C^k-diffeomorphisms, thanks to (3.56). Thus there exists ε_0 such that for all $\varepsilon \le \varepsilon_0$, $h_{1,\varepsilon}(\Sigma) = h_{2,\varepsilon}(\Sigma) = \Sigma$. Since $S \cup S' = \Sigma \setminus \gamma$, we also have

$$h_{1,\varepsilon}(S) \cup h_{1,\varepsilon}(S') = \Sigma \setminus h_{1,\varepsilon}(\gamma), \tag{3.57}$$

$$h_{2,\varepsilon}(S) \cup h_{2,\varepsilon}(S') = \Sigma \setminus h_{2,\varepsilon}(\gamma), \tag{3.58}$$

because $h_{1,\varepsilon}$ and $h_{2,\varepsilon}$ are C^k-diffeomorphisms. The set $\Sigma \setminus h_{1,\varepsilon}(\gamma)$ has two connected components since $\Sigma \setminus \gamma$ has two connected components. Since the unions of (3.57)–(3.58) are disjoint unions of open sets, $h_{1,\varepsilon}(S)$ and $h_{1,\varepsilon}(S')$ are the sought connected components. The same is true for $h_{2,\varepsilon}$, and as a consequence we have two possible situations:

$$h_{1,\varepsilon}(S) = h_{2,\varepsilon}(S) \quad \text{or} \quad h_{1,\varepsilon}(S) = h_{2,\varepsilon}(S').$$

In the first case the lemma is proven; we will show that the case $h_{1,\varepsilon}(S) = h_{2,\varepsilon}(S')$ is not possible if ε is close enough to zero 0. Otherwise, there exists a sequence (ε_i) which goes to 0 such that $h_{1,\varepsilon_i}(S) \cap h_{2,\varepsilon_i}(S) = \emptyset$. For simplicity, we write ε instead of ε_i, and for $x \in S$ define

$$x_{1,\varepsilon} = h_{1,\varepsilon}(x), \qquad x_{2,\varepsilon} = h_{2,\varepsilon}(x), \qquad h_{1,\varepsilon}(\gamma) = h_{2,\varepsilon}(\gamma) = \gamma_\varepsilon.$$

Since $h_{1,\varepsilon}(S) \cap h_{2,\varepsilon}(S) = \emptyset$, $x_{1,\varepsilon}$ and $x_{2,\varepsilon}$ are separated by γ_ε, which means that for every path α joining $x_{1,\varepsilon}$ to $x_{2,\varepsilon}$, we have $\alpha \cap \gamma_\varepsilon \ne \emptyset$. Let A be the set of smooth curves joining

$x_{1,\varepsilon}$ to $x_{2,\varepsilon}$. Then

$$d(x_{1,\varepsilon}, x_{2,\varepsilon}) = \inf_{\alpha \in A} \mathcal{L}(\alpha)$$

where d is the distance on the manifold Σ and $\mathcal{L}(\alpha)$ is the length of the path α. Hence

$$\forall \delta > 0,\ \exists \varepsilon > 0,\ \exists \alpha_\varepsilon \qquad d(x_{1,\varepsilon}, x_{2,\varepsilon}) \geq \mathcal{L}(\alpha_\varepsilon) - \delta. \tag{3.59}$$

We choose y_ε in $\gamma_\varepsilon \cap \alpha_\varepsilon$ such that

$$\mathcal{L}(\alpha_\varepsilon) = \mathcal{L}(\widehat{x_{1,\varepsilon} y_\varepsilon}) + \mathcal{L}(\widehat{y_\varepsilon x_{2,\varepsilon}}).$$

Taking the limit as $\varepsilon \to 0$ one obtains

$$\lim_{\varepsilon \to 0} \mathcal{L}(\alpha_\varepsilon) \geq 2d(x, \gamma).$$

If one chooses $\delta = d(x, \gamma)$, then (3.59) becomes

$$d(x_{1,\varepsilon}, x_{2,\varepsilon}) \geq \mathcal{L}(\alpha_\varepsilon) - d(x, \gamma)$$

and so

$$\lim_{\varepsilon \to 0} d(x_{1,\varepsilon}, x_{2,\varepsilon}) \geq 2d(x, \gamma) - d(x, \gamma) = d(x, \gamma) > 0,$$

which is impossible since $\lim_{\varepsilon \to 0} d(x_{1,\varepsilon}, x_{2,\varepsilon}) = 0$. ■

Now we apply Lemma 3.3 with k replaced by $k+1$ and $l = 1$. Thus there exists an open neighbourhood $\mathcal{U}_{k+1} \subset \Theta_{k+1}$ of 0 and a vector of C^1 functions

$$(H, \Phi, h, \phi_n, \phi_\nu) : \mathcal{U}_k \to G^{k-l}(\Sigma, \Sigma) \times C^{k-l}(\Sigma, \mathbb{R}) \times G^{k-l}(\gamma, \gamma) \times C^{k-l}(\gamma, \mathbb{R}) \times C^{k-l}(\gamma, \mathbb{R}).$$

We can always restrict $\mathcal{U}_{k+1}$, and thus we can assume that $H(\theta)$ and $h(\theta)$ are bijective from Σ to Σ and from γ to γ, respectively, for all $\theta \in \mathcal{U}_{k+1}$. Using (3.5) it is possible to write

$$\begin{aligned} H(\theta)(\gamma) &= (I + \phi_n(\theta) n + \phi_\nu(\theta)\nu) \circ h(\theta)(\gamma) \\ &= (I + \phi_n(\theta) n + \phi_\nu(\theta)\nu)(\gamma). \end{aligned}$$

Let v be a continuous extension,

$$v : C^k(\gamma, \Sigma) \to C^k(\Sigma, \Sigma),$$

such that the image under v of an element from $C^k(\gamma, \Sigma)$ is close to the identity in the sense of Lemma 3.4. Such an extension exists, and we give an explicit construction of it in the particular case of dimension 2. Then $v(I + \phi_n(\theta)n + \phi_\nu(\theta)\nu) \in C^k(\Sigma, \Sigma)$ can be written as

$$v(I + \phi_n(\theta)n + \phi_\nu(\theta)\nu) = I + w(\phi_n(\theta), \phi_\nu(\theta)))$$

where w is a function from $C^k(\gamma, \mathbb{R}^d)^2$ to $C^k(\Sigma, \mathbb{R}^d)$. Applying Lemma 3.4 we obtain

$$H(\theta)(\gamma) = (I + \phi_n(\theta)n + \phi_\nu(\theta)\nu) \circ h(\theta)(\gamma) \ \Rightarrow\ H(\theta)(S) = (I + w(\phi_n(\theta), \phi_\nu(\theta)))(S).$$

Now let u be a continuous extension,

$$u : C^k(\Sigma, \mathbb{R}^d) \to C^k(\mathbb{R}^d, \mathbb{R}^d).$$

Using (3.4) and (3.5) one obtains

$$
\begin{aligned}
(I+\theta)(S) &= [I+\Phi(\theta)n]\circ[I+w(\phi_n(\theta),\phi_\nu(\theta))](S)\\
&= [I+w(\phi_n(\theta),\phi_\nu(\theta))+\Phi((\theta)n)\circ(I+w(\phi_n(\theta),\phi_\nu(\theta)))](S)\\
&= [I+u[w(\phi_n(\theta),\phi_\nu(\theta))+\Phi((\theta)n)\circ(I+w(\phi_n(\theta),\phi_\nu(\theta)))]](S). \qquad (3.60)
\end{aligned}
$$

Since Ω_θ is defined by $U\setminus\overline{S}_\theta$, we clearly deduce from (3.60) that

$$E(\theta)=E(u[w(\phi_n(\theta),\phi_\nu(\theta))+\Phi((\theta)n)\circ(I+w(\phi_n(\theta),\phi_\nu(\theta)))])=:F(\Phi(\theta),\phi_n(\theta),\phi_\nu(\theta)),$$

which gives (3.55). Now we differentiate (3.55) at $\theta=0$ in direction $\xi\in\theta_{k+1}$. Using Lemma 3.3 and the chain rule we get

$$E'(0)(\xi)=F_\Phi(\Phi'(0)(\xi))+F_{\phi_\nu}(\phi'_\nu(0)(\xi))+F_{\phi_n}(\phi'_n(0)(\xi))=F_\Phi(\xi_n)+F_{\phi_\nu}(\xi_\nu),$$

where F_Φ is the derivative of F with respect to Φ. Since Θ_{k+1} is dense in Θ_k, and F_Φ, F_{ϕ_ν} and F_{ϕ_n} are continuous linear forms, one obtains (3.2) with $l^1=F_\Phi$ and $l^1_\nu=F_{\phi_\nu}$. One also sets $l^1_n=F_{\phi_n}$. In order to prove the second part of Theorem 3.1, one can apply 3.3 with k replaced by $k+2$ and $l=2$. One then obtains (3.3) similarly to (3.2). ■

3.2. The structure theorem in dimension 2

3.2.1. The structure theorem. The 2-dimensional case is a degenerate case of the general framework studied in the previous chapter. Indeed, if we use similar notations, D is a bounded open set of class C^k in $\mathbb{R}^2$, $\Sigma=\partial D$ is a closed curve of class C^k, and γ is a connected manifold of dimension 0. One can see that in order to obtain a crack, γ should be the union of two points in Σ exactly. Therefore we will make this assumption in what follows.

The previous results can actually be adapted without difficulty to the 2-dimensional case. In what follows, the structure theorem and its proof are presented in dimension 2. The results are simpler than in $\mathbb{R}^d$, $d\geq 3$, but the notations are kept as close as possible to those in the general case. We will also refer to the general case when the notions to be introduced or the demonstrations are redundant, the main difference being that γ is the union of two points and thus is not connected.

In what follows, U is a bounded open set of $\mathbb{R}^2$ with a smooth boundary. The set U is called the *hold-all* and the perturbations θ defined later leave U globally unchanged. Let also $k\in\mathbb{N}$, $k\geq 1$. The set $\mathcal{O}_k$ is defined as in (3.1).

Domains with cracks: Let $D\in\mathcal{O}_k$, and write $\Sigma=\partial D$. Let A_1 and A_2 be two distinct points in Σ, and define $\gamma=\{A_1,A_2\}$. Then $\Sigma\setminus\gamma$ has two connected components S and S', $\overline{S}$ is called a *crack* and one defines $\Omega=U\setminus\overline{S}$ to be the *cracked domain*.

Let Ω_0 be a cracked domain. Then we have

$$\gamma_0\subset\Sigma_0=\partial D_0,\qquad S_0\cup S'_0=\Sigma_0\setminus\gamma_0,\qquad \Omega_0=U\setminus\overline{S}_0,$$

and we denote by n the outer unit normal vector to D, and by τ the tangent unit normal vector to Σ. The vector τ corresponds to the vector ν in the general case.

The function space Θ_k, the perturbed domains Ω_θ, the set $\mathcal{F}_k$ of admissible cracked domains and the functional E are defined as in Section 3.1.

Notations: For $x, y \in \mathbb{R}^2$, denote by $\langle x, y\rangle$ the scalar product in $\mathbb{R}^2$. In what follows, for simplicity we use the summation convention: one writes $\alpha_i\beta_i := \sum_{i=1}^2 \alpha_i\beta_i$. Given a vector ξ in $\mathbb{R}^2$, we have $\xi = \xi_n n + \xi_\tau \tau$ where $\xi_n = \langle \xi, n\rangle$ and $\xi_\tau = \langle \xi, \tau\rangle$ stand for the normal and tangential components, respectively. Note that ξ_n and ξ_τ are scalars.

Now we are able to give the main result of this section.

THEOREM 3.5. *Let $k \geq 1$.*

1. *Let Ω_0 be a cracked domain with $D_0 \in \mathcal{O}_{k+1}$. Assume that E is Fréchet differentiable in Θ_k at 0. Then there exists a continuous linear form $l^1 : C^k(\Sigma_0, \mathbb{R}) \to \mathbb{R}$ and constants α_i, $i = 1, 2$, such that*
$$E'(0)(\xi) = l^1(\xi_n) + \alpha_i \xi_\tau(A_i), \qquad \forall \xi \in \Theta_k. \tag{3.61}$$
2. *Let Ω_0 be a cracked domain with $D_0 \in \mathcal{O}_{k+2}$. Assume that E is twice Fréchet differentiable at 0 in Θ_k, and denote by $\mathcal{H}$ the mean curvature of Σ_0. Then there exist constants β_i, $i = 1, 2$, constants $\tilde{\alpha}_i$, $i = 1, 2$, and bilinear forms*
$$l^2 : C^k(\Sigma_0, \mathbb{R}) \times C^k(\Sigma_0, \mathbb{R}) \to \mathbb{R}, \qquad \mathcal{L}^2 : C^k(\Sigma_0, \mathbb{R}) \times \mathbb{R}^2 \to \mathbb{R}$$
such that for all vector fields $\xi, \eta \in \Theta_{k+1}$,
$$\begin{aligned} E''(0)(\xi, \eta) = {} & l^2(\xi_n, \eta_n) + \beta_i \xi_\tau(A_i)\eta_\tau(A_i) + \beta_{12}(\xi_\tau(A_1)\eta_\tau(A_2) + \xi_\tau(A_2)\eta_\tau(A_1)) \\ & - l^1(\mathcal{H}\xi_\tau\eta_\tau + \langle n, D\xi\tau\rangle\eta_\tau + \langle n, D\eta\tau\rangle\xi_\tau) \\ & + \alpha_i \mathcal{H}(A_i)(\xi_\tau\eta_n + \xi_n\eta_\tau)(A_i) + \tilde{\alpha}_i\mathcal{H}(A_i)\xi_\tau(A_i)\eta_\tau(A_i) \\ & + \mathcal{L}^2(\xi_n, \eta_\tau(A_1), \eta_\tau(A_2)) + \mathcal{L}^2(\eta_n, \xi_\tau(A_1), \xi_\tau(A_2)). \end{aligned} \tag{3.62}$$

REMARK 3.6. If Ω_0 is a critical shape for E i.e., $E'(0) = 0$ at Ω_0, then $l^1 \equiv 0$ and $\alpha_i = 0$ $i = 1, 2$, thus the expression of the second-order derivative becomes
$$\begin{aligned} E''(0)(\xi, \eta) = {} & l^2(\xi_n, \eta_n) + \beta_i \xi_\tau(A_i)\eta_\tau(A_i) + \beta_{12}(\xi_\tau(A_1)\eta_\tau(A_2) + \xi_\tau(A_2)\eta_\tau(A_1)) \\ & + \tilde{\alpha}_i\mathcal{H}(A_i)\xi_\tau(A_i)\eta_\tau(A_i) \\ & + \mathcal{L}^2(\xi_n, \eta_\tau(A_1), \eta_\tau(A_2)) + \mathcal{L}^2(\eta_n, \xi_\tau(A_1), \xi_\tau(A_2)). \end{aligned}$$

REMARK 3.7. We have used here a relation specific to the 2D case: $\langle Dn\tau, \tau\rangle = \mathcal{H}$, where $\mathcal{H}$ is the mean curvature of the curve Σ_0. This equality links (3.62) and the formula (3.3) in the general case.

3.2.2. Normal and tangential perturbations. For all $l \in \mathbb{N}$ with $1 \leq l \leq k$, we set
$$G^{k-l}(\Sigma, \Sigma) := \{g \in C^{k-l}(\Sigma, \mathbb{R}^2) \mid g(\Sigma) \subset \Sigma\}.$$
Denote by $\Phi(\theta)$ and $H(\theta)$ the normal and tangential perturbations, respectively, on the surface Σ. In contrast to the general case, it is not necessary to introduce the functions $\phi_n(\theta)$, $\phi_\nu(\theta)$ and $h(\theta)$ defined in (3.5). The implicit function theorem is used to obtain the following lemma, which will be used for the proof of the structure theorem.

LEMMA 3.8. *Assume that Σ is a manifold of class C^k. For all $1 \leq l \leq k$, there exists an open neighbourhood $\mathcal{U}_k$ of 0 in Θ_k and a unique vector (H, Φ) of C^l functions*
$$(H, \Phi) : \mathcal{U}_k \to G^{k-l}(\Sigma, \Sigma) \times C^{k-l}(\Sigma, \mathbb{R})$$

such that $(H,\Phi)(0)=(I,0)$ *and for all* $\theta\in\mathcal{U}_k$,

$$I+\theta=(I+\Phi(\theta)n)\circ H(\theta)\qquad \text{on } \Sigma_0. \tag{3.63}$$

In addition, for all $\xi,\eta\in\Theta_k$ *we have, for* $l\geq 1$,

$$H'(0)(\xi)=\xi_\tau\tau,$$
$$\Phi'(0)(\xi)=\xi_n,$$

and for $l\geq 2$ *the second order derivatives are given by*

$$H''(0)(\xi,\eta)=-\mathcal{H}[(\xi_\tau\eta_n+\xi_n\eta_\tau)\tau+\xi_\tau\eta_\tau n], \tag{3.64}$$
$$\Phi''(0)(\xi,\eta)=-\mathcal{H}\xi_\tau\eta_\tau-\langle n,D\xi\tau\rangle\eta_\tau-\langle n,D\eta\tau\rangle\xi_\tau. \tag{3.65}$$

REMARK 3.9. Formulas (3.65) and (3.64) correspond to formulas (3.6) and (3.7) in the general case $d\geq 3$, respectively. These formulae simplify in dimension two due to the relation $\langle Dn\tau,\tau\rangle=\mathcal{H}$.

Proof of Lemma 3.8. The decomposition (3.63) is obtained in much the same way as (3.4) in the general case, i.e. by using the implicit function theorem. Consequently, the proof is not repeated here.

Since Σ_0 is of class C^k, there exists $\zeta\in C^k(\mathbb{R}^2,\mathbb{R})$ such that $\nabla\zeta(x)\neq 0$ for all $x\in\Sigma_0$, and an open neighbourhood ω_0 of Σ_0 in $\mathbb{R}^2$ such that

$$\Sigma_0=\{x\in\omega_0 \mid \zeta(x)=0\}.$$

The oriented distance function to D_0 (see [30] for more details) satisfies this hypothesis. The function ζ can be used in order to define the normal vector n to Σ_0,

$$\forall x\in\Sigma_0,\qquad n(x)=\nabla\zeta(x)/|\nabla\zeta(x)|.$$

Two equations are derived for the maps Φ and H:

$$I+\theta=(I+\Phi(\theta)n)\circ H(\theta), \tag{3.66}$$
$$\zeta(H(\theta))=0. \tag{3.67}$$

The set $\mathcal{U}_k$ can be reduced, thus we can assume that $H(\theta)$ takes values in ω_0. Then equation (3.67) implies that $H(\theta)(\Sigma_0)\subset\Sigma_0$. Differentiating (3.66) and (3.67) with respect to θ in the neighbourhood of 0, it follows that for all $\xi\in\Theta_k$,

$$\xi=H'(\theta)(\xi)+D(\Phi(\theta)n)\circ H(\theta)H'(\theta)(\xi)+(\Phi'(\theta)(\xi)n)\circ H(\theta), \tag{3.68}$$
$$0=D\zeta(H(\theta))H'(\theta)(\xi). \tag{3.69}$$

In particular, for $\theta=0$, in view of $H(0)=I$, we have

$$\xi-H'(0)(\xi)+\Phi'(0)(\xi)n=0, \tag{3.70}$$
$$D\zeta H'(0)(\xi)=0. \tag{3.71}$$

Since $n=\nabla\zeta/|\nabla\zeta|$, multiplying (3.70) by n and using (3.71) one obtains

$$\Phi'(0)(\xi)=\xi_n,\qquad H'(0)(\xi)=\xi_\tau\tau. \tag{3.72}$$

Now, in order to obtain the second order shape derivative, we differentiate (3.68) and (3.69) at $\theta=0$. Using (3.72), the following system of equations is deduced, for all ξ,η

in Θ_k:

$$0 = H''(0)(\xi,\eta) + D(\eta_n n)\xi_\tau\tau + \Phi''(0)(\xi,\eta)n + D(\xi_n n)\eta_\tau\tau, \tag{3.73}$$

$$0 = D^2\zeta(\xi_\tau\tau,\eta_\tau\tau) + \langle|\nabla\zeta|n, H''(0)(\xi,\eta)\rangle. \tag{3.74}$$

The value of $D^2\zeta(\xi_\tau\tau,\eta_\tau\tau)$ can be obtained by differentiating $D\zeta h = \langle|\nabla\zeta|n.h\rangle$. A calculation leads to

$$D^2\zeta(\xi_\tau\tau,\eta_\tau\tau) = |\nabla\zeta|\langle Dn\tau,\tau\rangle\eta_\tau\xi_\tau = |\nabla\zeta|\mathcal{H}\eta_\tau\xi_\tau.$$

Thus, thanks to (3.74),

$$\langle H''(0)(\xi,\eta), n\rangle = -\mathcal{H}\eta_\tau\xi_\tau.$$

In a similar way, using (3.73), we obtain

$$\begin{aligned}\langle H''(0)(\xi,\eta),\tau\rangle &= -\xi_\tau\langle D(\eta_n n)\tau,\tau\rangle - \eta_\tau\langle D(\xi_n n)\tau,\tau\rangle\\ &= -\eta_n\xi_\tau\langle Dn\tau,\tau\rangle - \xi_n\eta_\tau\langle Dn\tau,\tau\rangle = -\mathcal{H}(\eta_n\xi_\tau + \xi_n\eta_\tau).\end{aligned}$$

The equality $\langle D(\eta_n n)\tau,\tau\rangle = \eta_n\langle Dn\tau,\tau\rangle$ follows from

$$D(\eta_n n) = \eta_n Dn + n\otimes\nabla\eta_n.$$

In view of $\langle n,\tau\rangle = 0$, it follows that

$$\langle(n\otimes\nabla\eta_n)\tau,\tau\rangle = \sum_{i,j=1}^{2} n_i\tau_i(\nabla\eta_n)_j\tau_j = 0.$$

Finally, we need to compute $\Phi''(0)(\xi,\eta)$ to check that the result coincides with (3.65). The following calculation is used:

$$\begin{aligned}\langle D(\xi_n n)\eta_\tau\tau, n\rangle &= \langle Dn\eta_\tau\tau, n\rangle\xi_n\eta_\tau + \langle(n\otimes\nabla\xi_n)\tau, n\rangle\eta_\tau = \langle\nabla\xi_n,\tau\rangle\eta_\tau\\ &= \langle D\xi\tau, n\rangle\eta_\tau + \langle Dn\tau,\xi\rangle\eta_\tau = \langle D\xi\tau, n\rangle\eta_\tau + \langle Dn\tau,\tau\rangle\xi_\tau\eta_\tau\end{aligned}$$

and inserting $\langle D(\xi_n n)\eta_\tau\tau, n\rangle = \langle D\xi\tau, n\rangle\eta_\tau + \langle Dn\tau,\tau\rangle\xi_\tau\eta_\tau$ in (3.73) one obtains (3.65). ■

3.2.3. Proof of the structure theorem. First of all we introduce a function K from $\mathcal{D}_k$ to $\Sigma\times\Sigma$:

$$K(\theta) = \begin{pmatrix} H(\theta)(A_1)\\ H(\theta)(A_2)\end{pmatrix}. \tag{3.75}$$

This function plays a role equivalent in dimension two to decomposition (3.5) in the general case. Our objective is to prove that for θ close enough to 0 in $\mathcal{D}_k$, there exists a function F such that

$$E(\theta) = F(\Phi(\theta), K(\theta)) \tag{3.76}$$

where Φ and K are defined respectively in (3.63) and (3.75).

Lemma 3.10. *Let $h_{1,\varepsilon}$ and $h_{2,\varepsilon}$ be functions in $C^k(\Sigma,\Sigma)$ equipped with the usual norm $\|\cdot\|_k$, close to the identity in the following sense:*

$$\|h_{1,\varepsilon} - I\|_k, \|h_{2,\varepsilon} - I\|_k \to 0, \qquad \varepsilon\to 0.$$

Then

$$\exists\varepsilon_0,\quad \forall\varepsilon\le\varepsilon_0,\ h_{1,\varepsilon}(\Sigma) = h_{2,\varepsilon}(\Sigma) = \Sigma \text{ and } h_{1,\varepsilon}(\gamma) = h_{2,\varepsilon}(\gamma) \Rightarrow h_{1,\varepsilon}(S) = h_{2,\varepsilon}(S).$$

The proof of this lemma is similar to the proof of Lemma 3.4. Now we are able to complete the proof of the structure theorem in dimension two.

Lemma 3.8 is applied with k replaced by $k+1$ and $l=1$. Then there exists an open neighbourhood $\mathfrak{U}_{k+1} \subset \Theta_{k+1}$ of 0, and a vector (H,Φ) of C^1 functions

$$(H,\Phi) : \mathfrak{U}_k \to G^{k-l}(\Sigma,\Sigma) \times C^{k-l}(\Sigma,\mathbb{R}).$$

Since we can restrict $\mathfrak{U}_{k+1}$, we can assume that $H(\theta)$ is bijective from Σ to Σ for all $\theta \in \mathfrak{U}_{k+1}$.

Let $v : \Sigma \times \Sigma \to C^k(\Sigma,\Sigma)$ be a continuous extension which satisfies in addition

$$\begin{aligned}&\forall \varepsilon > 0,\ \exists \delta > 0, \forall (B_1,B_2) \in \Sigma \times \Sigma,\\ &\qquad \|(B_1,B_2) - (A_1,A_2)\| \leq \delta \ \Rightarrow\ \|v(B_1,B_2) - I\|_{C^k(\Sigma,\Sigma)} \leq \varepsilon,\end{aligned}$$

where I stands for the identity function in $C^k(\Sigma,\Sigma)$. Such an extension exists, and we can give an explicit construction as follows. The curve Σ can be parameterized by a diffeomorphism φ of class C^k satisfying

$$\varphi([0,1]) = \Sigma, \qquad \varphi(0) = \varphi(1),$$

and we set

$$a_1 := \varphi(A_1), \qquad a_2 := \varphi(A_2).$$

It is always possible to assume, since we can modify the parameterization, that $0 < a_1 < a_2 < 1$. Now we can build a function

$$\tilde{v} : [0,1]^2 \to C^\infty([0,1]^2), \qquad (b_1,b_2) \mapsto \tilde{v}_{b_1,b_2}, \tag{3.77}$$

satisfying

$$\|\tilde{v}_{b_1,b_2} - I\|_{C^\infty([0,1]^2)} \leq 2\max(|b_1 - a_1|, |b_2 - a_2|). \tag{3.78}$$

Indeed, let f be the piecewise affine function defined by

$$f(x) = \begin{cases} \dfrac{b_1}{a_1} x & \text{on } [0,a_1],\\ b_2 \dfrac{x-a_1}{a_2-a_1} + b_1 & \text{on } [a_1,a_2],\\ \dfrac{x-a_2}{1-a_2} + b_2 & \text{on } [a_2,1]. \end{cases}$$

Since $C^\infty([0,1])$ is dense in $C^0([0,1])$, and $\|f - I\| \leq \max(|b_1 - a_1|, |b_2 - a_2|)$, there exists a function $\tilde{v}$ satisfying (3.77) and (3.78). Setting then $v := \tilde{v} \circ \varphi^{-1}$, we obtain the required extension. The function $v(K(\theta)) \in C^k(\Sigma,\Sigma)$ can be written as

$$v(K(\theta)) = I + w(K(\theta))$$

where w is a function from $\Sigma \times \Sigma$ to $C^k(\Sigma,\Sigma)$.

Applying Lemma 3.10, which is possible thanks to (3.78), one obtains the implication

$$H(\theta)(\gamma) = K(\theta)(\gamma) \ \Rightarrow\ H(\theta)(S) = I + w(K(\theta))(S).$$

Now let u be a continuous extension,

$$u : C^k(\Sigma,\mathbb{R}^2) \to C^k(\mathbb{R}^2,\mathbb{R}^2).$$

Using (3.63) one obtains

$$\begin{aligned}(I+\theta)(S) &= [I+\Phi(\theta)n]\circ[I+w(K(\theta))](S)\\ &= [I+w(K(\theta))+\Phi((\theta)n)\circ(I+w(K(\theta))](S)\\ &= [I+u[w(K(\theta))+\Phi((\theta)n)\circ(I+w(K(\theta))]](S).\end{aligned} \tag{3.79}$$

Since Ω_θ is defined by $U\setminus\overline{S}_\theta$, we clearly deduce from (3.79) that

$$E(\theta)=E(u[w(K(\theta))+\Phi((\theta)n)\circ(I+w(K(\theta)))])=:F(\Phi(\theta),K(\theta)),$$

which gives (3.76). Now we differentiate (3.76) at $\theta=0$ in direction $\xi\in\Theta_{k+1}$. Using Lemma 3.8 and the chain rule, one obtains

$$E'(0)(\xi)=F_\Phi(\Phi'(0)(\xi))+F_K(K'(0)(\xi)),$$

where F_Φ is the derivative of F with respect to Φ. Since Θ_{k+1} is dense in Θ_k and F_Φ, F_K are continuous linear forms, one obtains (3.61) with $l^1=F_\Phi$ and

$$F_K=\begin{pmatrix}\alpha_1\langle\cdot,\tau\rangle+\tilde{\alpha}_1\langle\cdot,n\rangle\\ \alpha_2\langle\cdot,\tau\rangle+\tilde{\alpha}_2\langle\cdot,n\rangle\end{pmatrix}.$$

For the proof of the second part of Theorem 3.5, we apply Lemma 3.8 with k replaced by $k+2$ and $l=2$. One then obtains (3.62) in much the same way as we have obtained (3.61):

$$\begin{aligned}E''(0)(\xi,\eta) &= F_{\Phi\Phi}(0)(\Phi'(0)(\xi),\Phi'(0)(\eta))+F_\Phi(0)(\Phi''(0)(\xi,\eta))\\ &\quad+F_{\Phi K}(0)(\Phi'(0)(\xi),K'(0)(\eta))+F_{\Phi K}(0)(\Phi'(0)(\eta),K'(0)(\xi))\\ &\quad+F_{KK}(0)(K'(0)(\xi),K'(0)(\eta))+F_K(0)(K''(0)(\xi,\eta)).\end{aligned}$$

Since Θ_{k+2} is dense in Θ_{k+1}, and from the continuity of the linear forms in this formula, l^2 and $\mathfrak{L}_2$ in (3.62) are given by $l^2=F_{\Phi\Phi}(0)$ and $\mathfrak{L}_2=F_{\Phi K}(0)$. Concerning $F_{KK}(0)$, one notices that the function $K(\theta)$ can be written in the following way:

$$K(\theta)=\begin{pmatrix}K_1(\theta)\\ K_2(\theta)\end{pmatrix}. \tag{3.80}$$

Then we can write

$$\begin{aligned}&F_{KK}(0)(K'(0)(\xi),K'(0)(\eta))\\ &\quad=F_{K_1K_1}(0)(K_1'(0)(\xi),K_1'(0)(\eta))+F_{K_2K_2}(0)(K_2'(0)(\xi),K_2'(0)(\eta))\\ &\qquad+F_{K_2K_1}(0)(K_2'(0)(\xi),K_1'(0)(\eta))+F_{K_1K_2}(0)(K_1'(0)(\xi),K_2'(0)(\eta)),\end{aligned}$$

and the functions $F_{K_1K_1}(0)$, $F_{K_2K_2}(0)$, $F_{K_2K_1}(0)$ and $F_{K_1K_2}(0)$ can each be identified with a second order square matrix. With the notation, for $i=1,2$,

$$F_{K_iK_i}(0)=\begin{pmatrix}\beta_i & \delta_i\\ \lambda_i & \mu_i\end{pmatrix},\qquad F_{K_1K_2}(0)=F_{K_2K_1}(0)=\begin{pmatrix}\beta_{12} & \delta_{12}\\ \lambda_{12} & \mu_{12}\end{pmatrix},$$

we indeed find formula (3.62). ∎

4. Tangent sets in Banach spaces

In variational inequalities the data include some convex set K, and the solution u of the variational inequality is required to belong to this set, i.e., $u \in K$. The definition of the set depends on the problem; we consider in particular sets defined by linear constraints, called unilateral constraints.

In our analysis of variational inequalities, the specific form of the tangent set $T_k(u)$ at $u \in K$, for the convex set K of the problem, plays an important role. Sensitivity analysis requires the explicit form of the tangent set $T_k(u)$ in order to verify the polyhedricity condition at a given solution $u \in K$ of the variational inequality under consideration.

We present an abstract framework for calculation of tangent sets and establishing their properties. Non-linear potential theory is one of the tools used to prove such results. The results of this chapter are interesting on their own, and there are still some open questions we cannot answer, e.g., what is the form of the directional derivative for the metric projection onto the convex set which is not polyhedric, even for unilateral conditions in Sobolev spaces.

4.1. Introduction. We consider the tangent cones and the polyhedricity of convex sets $K = \{v \in \mathbb{B} \mid v \geq 0\}$ in Banach spaces using non-linear potential theory as a tool. In the case of Besov spaces the form of the tangent cones is given in [112]. Besides the form of tangent cones, the so-called polyhedricity of convex sets is of importance for applications in sensitivity analysis [16], [86], [108], [109], [110]. We will give a sufficient condition for the polyhedricity of the set K. For the elementary proof we refer to the proof of Theorem 4.15.

The results on the so-called conical differentiability of the metric projection onto polyhedric convex sets were established by Fulbert Mignot and Alain Haraux in the seventies in the Hilbert space setting. Our results allow us to extend the conical differentiability to more general variational inequalities, in particular for the evolution case—this work is in progress. We provide an example in Sec. 5, for a model static variational inequality with non-penetration condition on the crack faces [72]. We establish the conical differentiability of the solution to the variational inequality with respect to the crack length. Furthermore, the second order directional differentiability of the energy functional with respect to the crack length is proved. Such results seem to be new, and can be extended to the case of linear elasticity with frictionless contact conditions prescribed on the crack faces.

4.2. Notation and preliminaries. Let X be a non-empty set, and let $\mathcal{N}$ be an ideal of subsets of X. More precisely:

(i) If $N \in \mathcal{N}$ and $M \subset N$, then $M \in \mathcal{N}$.
(ii) If $N_k \in \mathcal{N}$ for $k \in \mathbb{N}$, then $\bigcup_{k=1}^{\infty} N_k \in \mathcal{N}$.

We say that a property holds $\mathcal{N}$-almost everywhere ($\mathcal{N}$-a.e.) if it holds outside a set from $\mathcal{N}$.

Let $\mathbb{B}$ be a vector space of $\mathcal{N}$-a.e. defined functions taking values in $[-\infty, +\infty]$ such that for every $u \in \mathbb{B}$, $\{|u| = +\infty\} \in \mathcal{N}$. We assume that there is a seminorm $\|\ \|$ on $\mathbb{B}$

such that $\|u\| = 0$ implies that $u = 0$ $\mathcal{N}$-a.e. As is customary, we identify functions that are equal $\mathcal{N}$-a.e., and denote by the letter $\mathbb{B}$ the vector space of equivalence classes. Then $(\mathbb{B}, \| \ \|)$ becomes a normed space. There is a natural order on $\mathbb{B}$: for $u, v \in \mathbb{B}, u \geq v$ as equivalence classes if $u \geq v$ $\mathcal{N}$-a.e. as functions.

Let $\mathbb{B}^*$ denote the dual space of $\mathbb{B}$ with the dual norm $\| \ \|$. The duality pairing between $\phi \in \mathbb{B}^*$ and $u \in \mathbb{B}$ will be denoted by $\langle \phi, v \rangle$. We will assume that $(\mathbb{B}, \| \ \|)$ satisfies the following hypotheses:

(H1) $\mathbb{B}$ is a reflexive Banach space,
(H2) $\mathbb{B}$ is strictly convex,
(H3) $\mathbb{B}$ is smooth.

Let us recall that a normed space $\mathbb{B}$ is *strictly convex* if for all $u, v \in \mathbb{B}$ such that $u \neq v$ and $\|u\| = \|v\| = 1$, it follows that $\|u + v\| < 2$. Equivalently, every element of $\mathbb{B}^*$ assumes its norm at most at one point in the unit ball of $\mathbb{B}$. The space $\mathbb{B}$ is *smooth* if for every $u \in \mathbb{B}, u \neq 0$, there exists a unique $\phi \in \mathbb{B}^*$ such that $\langle \phi, u \rangle = \|u\|$ and $\|\phi\| = 1$. It is well known that $\mathbb{B}$ is smooth if and only if the norm $\| \ \|$ is Gateaux differentiable on $\mathbb{B} \setminus \{0\}$. Moreover, since $\mathbb{B}$ is reflexive, $\mathbb{B}$ is smooth (respectively strictly convex) if and only if $\mathbb{B}^*$ is strictly convex (respectively smooth). Therefore, by our hypotheses on $\mathbb{B}$, the dual space $\mathbb{B}^*$ is also strictly convex and smooth.

In addition we assume that

(H4) If $(u_n, n \in \mathbb{N})$ is a sequence in $\mathbb{B}$ such that $\lim_{n\to\infty} u_n = u$ in $\mathbb{B}$, then there exists a subsequence $(n_k, k \in \mathbb{N})$ such that $\lim_{n_k\to\infty} u_{n_k} = u$ $\mathcal{N}$-a.e.

We assume that non-negative elements of $\mathbb{B}^*$ are positive measures, and that the following condition holds:

(H5) For every $u \in \mathbb{B}$ there exists an element $v \in \mathbb{B}$ such that
$$u^+ \leq v \quad \text{and} \quad \|v\| \leq \|u\|.$$

A non-negative measure $\mu \in \mathbb{B}^*$ is called a *measure of finite energy*. A subset E of X is called *quasi-null* if $\mu(E) = 0$ for all measures μ of finite energy.

It is known that the collection of quasi-null sets is contained in $\mathcal{N}$ and is in general much smaller than the $\mathcal{N}$-null sets.

If $u \in \mathbb{B}$ and $u \geq 0$ $\mathcal{N}$-a.e. then $u \geq 0$ *quasi-everywhere* (q.e.).

From now on all inequalities and equalities are deemed to hold q.e., i.e., except for a quasi-null set.

4.2.1. Duality mapping. Let us recall some facts about the duality mapping. For completeness and convenience of the reader some proofs are supplied. Let $\omega : [0, \infty) \to [0, \infty)$ be a continuous and strictly increasing function such that $\omega(0) = 0$ and $\lim_{t\to\infty} \omega(t) = +\infty$.

Definition 4.1. A map $\mathcal{J} : \mathbb{B} \to \mathbb{B}^*$ is called an ω-*duality map* if for all $u \in \mathbb{B}$,

(A1) $\langle \mathcal{J}u, u \rangle = \|\mathcal{J}u\| \, \|u\|$,
(A2) $\|\mathcal{J}u\| = \omega(\|u\|)$.

Theorem 4.2. *An ω-duality map exists.*

Proof. By the Hahn–Banach theorem, given an element $u \in \mathbb{B}$, there exists $\widetilde{T}u \in \mathbb{B}^*$ such that

$$\|\widetilde{T}u\| = 1, \qquad \langle \widetilde{T}u, u\rangle = \|u\|.$$

REMARK 4.3. Of course $\|\widetilde{T}u\|$ has to be at least 1, so that $\widetilde{T}u$ is an element in the convex set $\{L \in \mathbb{B}^* \mid Lu = \|u\|\}$, of minimum norm.

To get an ω-duality map simply define $\mathcal{J}u = \omega(\|u\|)\widetilde{T}u$. ∎

THEOREM 4.4. *An ω-duality map with ω strictly increasing is strictly monotone on a strictly convex space, i.e.,*

(A3) $\langle \mathcal{J}u - \mathcal{J}v, u - v\rangle > 0$ *if* $u \neq v$.

Proof. Expanding the left-hand side we have

$$\langle \mathcal{J}u - \mathcal{J}v, u - v\rangle = \langle \mathcal{J}u, u\rangle + \langle \mathcal{J}v, v\rangle - \langle \mathcal{J}u, v\rangle - \langle \mathcal{J}v, u\rangle.$$

Since $\langle \mathcal{J}u, u\rangle = \omega(\|u\|)\|u\|$ etc., $\langle \mathcal{J}u, v\rangle \leq \|\mathcal{J}u\|\,\|v\|$ etc., we get

$$\begin{aligned}\langle \mathcal{J}u - \mathcal{J}v, u - v\rangle &\geq \omega(\|u\|)\|u\| + \omega(\|v\|)\|v\| - \omega(\|u\|)\|v\| - \omega(\|v\|)\|u\| \\ &= (\omega(\|u\|) - \omega(\|v\|))(\|u\| - \|v\|) \geq 0,\end{aligned}$$

and this is strictly positive if ω is strictly $\uparrow$ and $\|u\| \neq \|v\|$. From the above inequality we see that if the left-hand side of (A3) is zero then

$$\langle \mathcal{J}u, v\rangle = \|\mathcal{J}u\|\,\|v\| \quad \text{and} \quad \langle \mathcal{J}v, u\rangle = \|\mathcal{J}v\|\,\|u\|$$

and $\|u\| = \|v\|$. We get

$$\langle \mathcal{J}u, v/\|v\|\rangle = \|\mathcal{J}u\| \quad \text{and of course} \quad \langle \mathcal{J}u, u/\|u\|\rangle = \|\mathcal{J}u\|.$$

Hence by strict convexity $u/\|u\| = v/\|v\|$, and since $\|u\| = \|v\|$ we must have $u = v$. ∎

The following result is due to Asplund [8]: The space $\mathbb{B}$ is smooth if and only if the ω-duality map is single valued. In this case

$$\langle \mathcal{J}u, \varphi\rangle = \frac{d}{dt}\Omega(\|u + t\varphi\|)|_{t=0}, \quad \forall u, \varphi \in \mathbb{B}, \quad \text{where} \quad \Omega(t) = \int_0^t \omega(s)\,ds.$$

In particular, $\mathcal{J}u$ is the unique element of $\mathbb{B}^*$ such that $\langle \mathcal{J}u, u\rangle = \|\mathcal{J}u\|\,\|u\|$ with $\|\mathcal{J}u\| = \omega(\|u\|)$.

4.2.2. Examples covered by our setup

L^p-Potential theory. Let (X, m) be a measure space, G be a kernel on $X \times X$ and $\mathbb{B} = \{Gf \mid f \in L^p(X, m)\}$ with the norm $\|Gf\| = \|f\|_p$. Under very mild hypotheses on G, $\mathbb{B}$ satisfies our conditions. More generally, using $\ell^q(L^p)$ (resp. $L^p(\ell^q)$) spaces instead of L^p-spaces we obtain the Besov space $\mathbb{B}^{p,q}_\alpha$ (respectively the Lizorkin–Triebel space $\mathbb{F}^{p,q}_\alpha$) also on subsets of $\mathbb{B}^n$. For details we refer to [1], [114] and [57].

The weighted Sobolev space $W^{1,p}(\Omega, \mu)$. The spaces $W^{1,p}(\Omega, \mu)$ and $W^{1,p}_0(\Omega, \mu)$ with the norm

$$\|u\|_{1,p,\mu} = \|u\|_{L^p(\mu)} + \|\nabla u\|_{L^p(\mu)}.$$

Here μ is a suitable weight. For details see [51].

Spaces $W_0^{m,2}(B)$. The space $W_0^{m,2}(B)$ where B is an open ball in $\mathbb{R}^N$. For details we refer to [114].

4.3. Non-linear potential theory. Let $K = \mathbb{B}_+$ denote the convex cone of non-negative elements from $\mathbb{B}$,

$$K = \{\varphi \in \mathbb{B} \mid \varphi \geq 0\}.$$

Let $K^* = \mathbb{B}_+^*$ denote the dual cone in $\mathbb{B}^*$,

$$K^* = \{w \in \mathbb{B}^* \mid \langle w, \varphi\rangle \geq 0, \forall\varphi \in K\}.$$

For any closed convex set $\mathcal{C} \subset \mathbb{B}$ there exists a unique element $u_0 \in \mathcal{C}$ which minimizes the norm over $\mathcal{C}$,

$$\|u_0\| = \min\{\|v\| \mid v \in \mathcal{C}\}.$$

The existence follows from the reflexivity of $\mathbb{B}$ using Mazur's lemma, and the uniqueness from the strict convexity of the norm. For the set $\mathcal{C}$ with $\mathcal{C}+K \subset \mathcal{C}$ we have the following result [114].

THEOREM 4.5. *If $\mathcal{C}$ has the property that $v \in \mathcal{C}$, $\varphi \in K$ imply that $v + \varphi \in \mathcal{C}$, then $\mathcal{J}u_0 \in K^*$, where $\|u_0\| = \min\{\|v\| \mid v \in \mathcal{C}\}$ and $\mathcal{J}$ is the ω-duality map.*

Proof. Since $\Omega(t)$ is strictly increasing,

$$\Omega(\|u_0\|) = \min\{\Omega(\|v\|) \mid v \in \mathcal{C}\}.$$

Let $\varphi \geq 0$. Then $u_0 + t\varphi \in \mathcal{C}$ for every $t \geq 0$. Hence $\Omega(\|u_0 + t\varphi\|) \geq \Omega(\|u_0\|)$ for every $t \geq 0$, implying that

$$0 \leq \frac{d}{dt}\Omega(\|u_0 + t\varphi\|)\bigg|_{t=0} = \langle \mathcal{J}u_0, \varphi\rangle.$$

Since this holds for every $\varphi \geq 0$, it follows that $\mathcal{J}u_0 \in K^*$. ∎

DEFINITION 4.6. $u \in \mathbb{B}$ is called a *potential* if $\mathcal{J}u \in K^*$.

THEOREM 4.7. *Let $u \in \mathbb{B}$. The following are equivalent:*

(i) *u is a potential, i.e., $\mathcal{J}u \in K^*$.*
(ii) *For every $v \in \mathbb{B}$ such that $v \geq u$ it follows that $\|v\| \geq \|u\|$.*

Proof. (i)⇒(ii). If $v \in \mathbb{B}$ and $v \geq u$ then $v - u \in K$. Hence

$$\langle \mathcal{J}u, v - u\rangle \geq 0, \quad \text{i.e.} \quad \langle \mathcal{J}u, v\rangle \geq \langle \mathcal{J}u, u\rangle = \|\mathcal{J}u\|\,\|u\|.$$

Since $\mathcal{J}u$ is a linear map this gives $\|v\| \geq \|u\|$.

(ii)⇒(i). If $\varphi \in K$ then $u + t\varphi \geq u$ for all $t > 0$. Hence

$$\|u + t\varphi\| \geq \|u\| \quad \text{so} \quad \Omega(\|u + t\varphi\|) \geq \Omega(\|u\|).$$

It follows that $\langle \mathcal{J}u, \varphi\rangle = \frac{d}{dt}\Omega(\|u + t\varphi\|)|_{t=0} \geq 0$ for all $\varphi \in K$, i.e., $\mathcal{J}u \in K^*$. ∎

4.4. Tangent cones. Let $\mathbb{B}$ be as above and let

$$K = \{v \in \mathbb{B} : v \geq 0\}.$$

Then K is a closed convex set. Each $f \in \mathbb{B}$ has a unique projection $u_0 = P_K f$ onto K characterized by

$$\|u_0 - f\| = \inf_{v \in K} \|v - f\|.$$

We are interested in the differentiability of $t \mapsto P_K(f + th)$. Let $u_t = P_K(f + th)$. Set

$$v_t = \frac{u_t - u_0}{t}.$$

Then

$$u_t = u_0 + tv_t \in K.$$

This motivates the following definition. For $u_0 \in K$ we set

$$C_K(u_0) = \{v \in \mathbb{B} : \exists t > 0,\ u_0 + tv \in K\}, \qquad T_K(u_0) = \overline{C_K(u_0)}.$$

$T_K(u_0)$ is called the *tangent cone* to K at $u_0 \in K$. We see that the derivative q is in $T_K(u_0)$, whenever it exists.

It is thus of interest to determine the form of the tangent cone. But first we have the following result characterizing $P_K f$.

THEOREM 4.8. *Let $f \in \mathbb{B}$ and $\mathcal{C}$ be any closed cone containing K. Then*

$$\mathcal{C} \ni w_0 = P_{\mathcal{C}} f$$

implies

$$w_0 - f \text{ is a potential,} \tag{4.1}$$

$$\text{its measure } \mu = \mathcal{J}(w_0 - f) \text{ satisfies } \int w_0 \, d\mu = 0. \tag{4.2}$$

If $\mathcal{C} = K$, (4.1) and (4.2) are also sufficient. More precisely, if $w_0 \in K$ and $w_0 - f$ is a potential whose measure μ satisfies $\int w_0 \, d\mu = 0$, then $w_0 = P_K f$. In this case, since $w_0 \geq 0$, the measure μ is concentrated on the set $\Xi = \{P_K f = 0\}$.

Proof. Let $w_0 = P_{\mathcal{C}} f$. To show that $w_0 - f$ is a potential we use Theorem 4. So let $\mathbb{B} \ni h \geq w_0 - f$. Then $h + f - w_0 \geq 0$, implying $\mathcal{C} \supset K \ni h + f - w_0$. Since $w_0 \in \mathcal{C}$ and $\mathcal{C}$ is a cone,

$$(h + f - w_0) + w_0 = h + f \in \mathcal{C}.$$

Hence from the definition of w_0,

$$\|h\| = \|(h + f) - f\| \geq \|w_0 - f\|.$$

Thus $w_0 - f$ is a potential. Now for any $g \in \mathcal{C}$,

$$\|w_0 - f + tg\| \geq \|w_0 - f\|,$$

implying

$$\int g \, d\mu = \lim_{t \to 0} [\Omega(\|w_0 - f + tg\|) - \Omega(\|w_0 - f\|)] \geq 0. \tag{4.3}$$

In particular,

$$\int w_0 \, d\mu \geq 0. \tag{4.4}$$

Again, since $\mathcal{C}$ is a cone, $tw_0 \in \mathcal{C}$ for all $t > 0$ implies for $0 < t < 1$

$$\|w_0 - f\| \le \|(1-t)w_0 - f\| = \|w_0 - f - tw_0\|.$$

It follows that we can replace g by $-w_0$ in (4.3) and the right-hand side remains ≥ 0 for $-1 \le t$. Hence

$$-\int w_0 \, d\mu \ge 0.$$

(4.4) together with the last inequality give (4.2).

It remains to prove that for $\mathcal{C} = K$ the conditions on w_0 are sufficient to guarantee that $w_0 = P_K f$. To this end let $w_0 \in K$ satisfy

- *$w_0 - f$ is a potential and its measure ν annihilates w_0: $\int w_0 \, d\nu = 0$.*

Let $\widetilde{w}_0 = P_K f$. From what we have already proved, $\widetilde{w}_0 - f$ is a potential. Let

$$\varphi(t) = \Omega(\|t(\widetilde{w}_0 - f) + (1-t)(w_0 - f)\|).$$

Then φ is convex and because $\widetilde{w}_0 = P_K f$,

$$\varphi(0) = \Omega(\|w_0 - f\|) \ge \Omega(\|\widetilde{w}_0 - f\|) = \varphi(1). \tag{4.5}$$

Thus $\varphi'(0) \le 0$ (because φ' is increasing, so if $\varphi'(0) > 0$, then $\varphi'(t) > 0$ for $t > 0$ implies φ is strictly increasing for $t > 0$, which contradicts (4.5)).

Writing

$$\varphi(t) = \Omega(\|w_0 - f + t(\widetilde{w}_0 - w_0)\|)$$

we find again differentiating

$$0 \ge \varphi'(0) = \int (\widetilde{w}_0 - w_0) \, d\nu = \int \widetilde{w}_0 \, d\nu$$

because $\int w_0 \, d\nu = 0$. Since $\widetilde{w}_0 \ge 0$ and ν is a non-negative measure we conclude $\int \widetilde{w}_0 \, d\nu = 0$, i.e. $\varphi'(0) = 0$. But then (since φ' is increasing) $\varphi' \ge 0$ for $t > 0$ implies $\varphi(1) \ge \varphi(0)$. (4.5) then gives $\|w_0 - f\| = \|\widetilde{w}_0 - f\|$. By uniqueness $w_0 = \widetilde{w}_0$. ∎

The above theorem has an immediate corollary:

COROLLARY 4.9. *Let $f \in \mathbb{B}$ and $u_0 = P_K f$. If $h \in C_K(u_0) \cap [u_0 - f]^\perp$, i.e., if $\int h \, d\mu_0 = 0$, then for all t small enough,*

$$P_K(f + th) = P_K f + th. \tag{4.6}$$

Proof. Since $h \in C_K(P_K f)$, there is t_0 such that

$$P_K f + th \in K, \qquad t \le t_0.$$

Then $P_K f + th - (f + th) = P_K f - f$ is a potential. Its measure μ_0 satisfies

$$\int (P_K f + th) \, d\mu_0 = t \int h \, d\mu_0 = 0$$

because $h \in (u_0 - f)^\perp$. Thus (4.6) follows from Theorem 4.8. ∎

REMARK 4.10. Corollary 4.9 extends to $\overline{C_K(u_0) \cup [u_0 - f]^\perp}$ in the following sense: For $h \in \overline{C_K(u_0) \cup [u_0 - f]^\perp}$,

$$P_K(f + th) = P_K f + th + o(t). \tag{4.7}$$

To establish (4.7) one should prove e.g. that the map $f \mapsto P_K f$ is Lipschitz. This is the case e.g. for the L^p spaces since $P_K f = f^+$ in L^p.

To accommodate more general h we first need to characterize the tangent cone $T_K(u_0)$. This is given in Theorem 4.11 below and has an easy proof.

THEOREM 4.11. *Let $u_0 \in K$ and $\Xi = \{x \mid u_0(x) = 0\}$. Then $T_K(u_0) = \{v \in \mathbb{B} : v \geq 0$ on $\Xi\}$.*

Proof. Since $u_0 = 0$ on Ξ, we see from the definition of $C_K(u_0)$ that all elements in $C_K(u_0)$ are ≥ 0 on Ξ. And taking limits the same is true for elements of $T_K(u_0)$.

Let $w \in \mathbb{B}$ be such that $w \geq 0$ on Ξ and let w_0 be its projection on $T_K(u_0)$. It can be verified that $T_K(u_0)$ is a closed cone. It contains K.

From Theorem 4.8, $w_0 - w$ is a potential whose measure μ annihilates w_0, and

$$\langle \mathcal{J}(w_0 - w), w_0 - w \rangle = \omega(\|w_0 - w\|)\|w_0 - w\| = \int (w_0 - w)\, d\mu = -\int w\, d\mu. \tag{4.8}$$

Recall that $\int h\, d\mu \geq 0$ for all $h \in T_K(u_0)$ (see the proof of Theorem 4.8). Since u_0 and $-u_0$ are in $C_K(u_0) \subset T_K(u_0)$ we see that $\int u_0\, d\mu = 0$, implying μ must be concentrated on Ξ. Since $w \geq 0$ on Ξ, we have therefore

$$\int w\, d\mu \geq 0.$$

Together with (4.8) this gives $\int w\, d\mu = 0$ and hence $w_0 = w$. This completes the proof since $w \in T_K(u_0)$. ∎

Let us now investigate the metric projection onto K in more detail. We establish some notation. Given $f, z \in \mathbb{B}$ write

$$u_0 = P_K f, \qquad \xi_t = \frac{P_K(f + tz) - P_K f}{t},$$

so that

$$P_K(f + tz) = u_0 + t\xi_t.$$

Since the projection P_K is Lipschitz in some cases, we see that in such cases ξ_t is uniformly bounded in norm, which we assume below. Recall that by Theorem 4.8, $u_0 - f$ and $u_0 + t\xi_t - (f + tz)$ are potentials. Denote by μ_0 and μ_t their measures. Then

$$0 = \int u_0\, d\mu_0 = \int (u_0 + t\xi_t)\, d\mu_t. \tag{4.9}$$

With the above notation we have:

THEOREM 4.12. *Every weak limit q of $\{\xi_t\}$ as $t \to 0$ belongs to*

$$q \in T_K(u_0) \cap (u_0 - f)^\perp = T_K(u_0) \cap \mu_0^\perp. \tag{4.10}$$

Proof. Recall that the duality map is monotone,

$$\langle J_x - J_y, x - y \rangle \geq 0. \tag{4.11}$$

For potentials x and y, the maps J_x and J_y are the corresponding measures. Taking $x = u_0 - f$, $y = u_0 + t\xi_t - (f + tz)$ in (4.11) and expanding we get, using (4.9),

$$\begin{aligned} -\int f\,d\mu_0 - \int (f+tz)\,d\mu_t &\geq \int (u_0 + t\xi_t - (f+tz))\,d\mu_0 + \int (u_0 - f)\,d\mu_t \\ &= t\int (\xi_t - z)\,d\mu_0 + \int (u_0 - f)\,d\mu_t - \int f\,d\mu_0. \end{aligned}$$

Simplifying gives

$$-t\int z\,d\mu_t \geq t\int (\xi_t - z)\,d\mu_0 + \int u_0\,d\mu_t \geq t\int (\xi_t - z)\,d\mu_0$$

because μ_t is a positive measure and $u_0 \geq 0$. Canceling t we get

$$-\int z\,d\mu_t \geq \int (\xi_t - z)\,d\mu_0.$$

Duality maps are continuous from strong to weak* topology: as $t \to 0$, $u_0 + t\xi_t - (f + tz)$ converges strongly to $u_0 - f$ so that μ_t converges weak* to μ_0. Therefore, as $t \to 0$, for any weak limit q of ξ_t we have

$$-\int z\,d\mu_0 \geq \int (q - z)\,d\mu_0.$$

Since $q \in T_K(u_0)$ and hence $q \geq 0$ on Ξ, we get (4.10). ■

4.5. Conical differentiability for evolution variational inequalities. A variational inequality with unilateral conditions on the boundary can be seen as a free boundary problem. The evolution of the coincidence set $\{x \in \Sigma_c \mid u(x) = 0\}$ for the condition $v|_{\Sigma_c} \geq 0$ in a parabolic variational inequality is given by a measure μ associated with the variational problem. Variations of the free boundary with respect to perturbations of e.g. the source or the right-hand side of the variational inequality are given by variations of the support of the measure μ. This feature of such problems shows that sensitivity analysis becomes a difficult task. We provide a rigorous analysis of such variations, and introduce the notion of conical differentiability in the parabolic case. Shape sensitivity analysis for the class of evolution inequalities is still to be established.

4.5.1. Tangent sets and measures of finite energy. Let $T > 0$ be a real number. Define $Q = (0,T) \times \Omega$, $\Gamma = \partial\Omega$, $\Sigma = (0,T) \times \Gamma$. Moreover, we assume that there is a disjoint decomposition $\Gamma = \Gamma_c \cup \Gamma_d$ such that both sets have positive $(N-1)$-measures and their relative interiors $\operatorname{ri}\Gamma_c$, $\operatorname{ri}\Gamma_d$ are dense in Γ_c, Γ_d, respectively. We define $\Sigma_z = (0,T) \times \Gamma_z$ for $z = c, d$. Furthermore, let $V = H^{1/2,1}(Q)$ be the anisotropic Sobolev potential space in the sense of Section 4.2 and let us introduce the spaces

$$\mathsf{H} = \{v \in V \mid v|_{\Sigma_d} = 0\} \quad \text{and} \quad \mathring{V} = \{v \in V \mid v|_{\Sigma} = 0\}.$$

Here H is a Hilbert space, indeed a Dirichlet space; let H^* denote the dual space.

Let us recall for the convenience of the reader that a Hilbert space $\mathcal{H}$ of functions defined on Q is called a *Dirichlet space* provided the following three conditions are satisfied:

- $\eta^+, \eta^- \in \mathcal{H}\ \forall \eta \in \mathcal{H}$,
- $((\eta^+, \eta^-)) \leq 0\ \forall \eta \in \mathcal{H}$,
- $\mathcal{H} \cap C_0(Q)$ is dense in $C_0(Q)$,

where $\eta^+ = \max\{0, \eta\}$, $\eta^- = \max\{0, -\eta\}$ and $((\eta^+, \eta^-))$ denotes the scalar product in $\mathcal{H}$.

We refer the reader to [105], [106], [107] for the parabolic case. Our presentation will be self-contained.

Let us write down some results which will be useful later.

Any linear form $L \in \mathsf{H}^*$ such that $L \geq 0$, i.e., $L[u] \geq 0$ if $u \geq 0$ a.e., is given by a unique positive measure μ,

$$L[u] = \langle L, u \rangle = \int u\, d\mu, \qquad u \in \mathsf{H} \cap C(\overline{Q}).$$

Therefore we say a measure μ is of *finite energy* if

$$u \in \mathsf{H} \cap C(\overline{Q}) \ \Rightarrow \ \int |u|\, d\mu \leq C_\mu \|u\|_{\mathsf{H}}.$$

Let

$$\mathfrak{M} = \{\text{set of all (signed) measures of finite energy}\}.$$

A set E will be called *quasi-null* if $\mu(E) = 0$ for all $\mu \in \mathfrak{M}$. Since the Lebesgue measure on Q belongs to $\mathfrak{M}$ we see that quasi-null sets are necessarily of Lebesgue measure zero. The collection of quasi-null sets is much smaller.

The following fact can be easily proved.

PROPOSITION 4.13. *Let $\{u_m\} \subset \mathsf{H} \cap C(\overline{Q})$ be a Cauchy sequence in H. Then there is a subsequence which converges pointwise quasi-everywhere.*

By Proposition 4.13 we see that functions in H have a *value* quasi-everywhere in $\overline{Q}$. Hence we can say $\mathsf{H} \subset L^1(\mu)$ if $\mu \in \mathfrak{M}$.

In the present paper the following convex cone in H is considered:

$$K = \{v \in \mathsf{H} \mid v|_{\Sigma_c} \geq 0\}.$$

Therefore, in order to define quasi-null subsets of Σ_c, the measures from $\mathfrak{M}$ which live on Σ_c are considered. If $u \in \mathsf{H}$ and $u \geq 0$ a.e. with respect to $(N-1)$-dimensional Lebesgue measure on Σ_c then $u \geq 0$ quasi-everywhere (q.e.).

From now on all inequalities and equalities on Σ_c are deemed to hold q.e., i.e., except for a quasi-null set.

Given $u_0 \in K$ we define

$$C_K(u_0) = \{v \in \mathsf{H} \mid \exists t > 0,\ u_0 + tv \in K\}, \qquad T_K(u_0) = \overline{C_K(u_0)}.$$

$T_K(u_0)$ is called the *tangent cone* to K at $u_0 \in K$. For cones with unilateral constraints in function spaces, the tangent cones are determined in [112].

As in the elliptic case [113] we can prove

$$T_K(u_0) = \{v \in H^{1/2,1}(Q) \mid v \geq 0 \text{ on } \Xi,\ v|_{\Sigma_d} = 0\},$$

where

$$\Xi = \{(x,t) \in \Sigma_c \mid u_0(x,t) = 0\}.$$

Let $\Lambda \in \mathfrak{M}$ be a positive measure living on Ξ. Introduce the cone

$$\mathfrak{D}(\Lambda) = \Lambda^{\perp} = \left\{ v \in \mathsf{H} \,\middle|\, \langle \Lambda, v\rangle = \Lambda[v] = \int v \, d\Lambda = 0 \right\}$$

Definition 4.14. Let $u_0 \in K = \{v \in \mathsf{H} \mid v|_{\Sigma_c} \geq 0\}$ be a given element, and $\Lambda \in \mathfrak{M}$ a positive measure living on $\Xi = \{(x,t) \in \Sigma_c \mid u_0(x,t) = 0\}$. The convex set K is called *polyhedric at* $u_0 \in K$ *for* $\Lambda \in \mathfrak{M}$ if

$$T_K(u_0) \cap \mathfrak{D}(\Lambda) = \overline{C_K(u_0) \cap \mathfrak{D}(\Lambda)}. \tag{4.12}$$

If the condition (4.12) is satisfied for all positives measures from $\mathfrak{M}$ living on Ξ, the set K is called *polyhedric at* u_0. If K is polyhedric for all $u_0 \in K$, it is called *polyhedric*, which is the case for the cone K under consideration.

Let us recall that the polyhedricity of the set K at u_0 implies the conical differentiability at u_0 of the metric projection onto K [91].

Polyhedricity of K. We prove the following result due to F. Mignot [91], in a slightly different setting. To be precise, in [91] the convex set $\{v \in H^1(\Omega) \mid v|_{\partial\Omega} \geq 0\}$ is considered.

Actually, if the duality map is a contraction then the proof of polyhedricity is easy.

Theorem 4.15. *For any $u_0 \in K$ and all positive measures $\mu \in \mathfrak{M}$ such that μ lives on $\{(x,t) \in \Sigma_c \mid u_0(x,t) = 0\}$, it follows that*

$$T_K(u_0) \cap \mu^{\perp} = \overline{C_K(u_0) \cap \mu^{\perp}}.$$

Proof. Indeed, let

$$w \in T_K(u_0) \cap \mu^{\perp}.$$

Then $w = 0$ μ-a.e. We construct a sequence in $C_K(u_0) \cap \mu^{\perp}$ which converges strongly to w. Let $C_K(u_0) \ni v_n \to w$. Then $v_n^- \to w^-$, $v_n^+ \to w^+$ and $v_n^+ \wedge w^+ - v_n^- \to w$. Now, if $v \in C_K(u_0)$ then $u_0 + \tau v \geq 0$ for some $\tau > 0$. We claim $v_n^+ \wedge w^+ - v_n^- \in C_K(u_0) \cap \mu^{\perp}$ so the required sequence is of the form $v_n^+ \wedge w^+ - v_n^-$. Indeed, $u_0 + \tau[v_n^+ \wedge w^+ - v_n^-] \geq 0$ so $v_n^+ \wedge w^+ - v_n^- \in C_K(u_0)$ and $\mu[v_n^+ \wedge w^+ - v_n^-] = \mu[v_n^+ \wedge w^+] = 0$ because $\mu[w^+] = 0$. We have taken into account that $v_n \geq 0$ on $\{u_0 = 0\}$ and μ lives on $\{u_0 = 0\}$, therefore $\mu[v_n^-] = 0$. ∎

4.5.2. Conical differentiability. We are interested in directional differentiability of solutions to a parabolic variational inequality with respect to perturbations of the right-hand side. To this end we derive some relations for weak limits.

Denote by A the parabolic operator $\partial/\partial t - \Delta_x$. Then $A : \mathsf{H} \to \mathsf{H}^*$ is defined by

$$\langle Au, v\rangle = \langle \dot{u}, v\rangle + a(u,v),$$

where

$$\langle \dot{u}, v\rangle = \int_\Omega \frac{\partial u}{\partial t} v \, dx \quad \text{and} \quad a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx.$$

We already know [56] that A is an isomorphism $\mathsf{H} \to \mathsf{H}^*$. If u is a solution of

$$\begin{cases} u \in K \ : \ \langle \dot{u}, v-u\rangle_Q + a(u, v-u) \geq \langle f, v-u\rangle_Q \quad \forall v \in K, \\ u(0) = u_0, \end{cases}$$

we see that $Au - f$ is a non-negative element of H^* and hence is given by a measure.

Let now $f_\tau = f + \tau h$ and

$$\Lambda_\tau = Au_\tau - f_\tau \in \mathsf{H}^*.$$

For any τ we have the unique solution u_τ of the variational inequality

$$\begin{cases} u_\tau \in K \ : \ \langle \dot{u}_\tau, v-u_\tau\rangle + a(u_\tau, v-u_\tau) \geq \langle f + \tau h, v-u_\tau\rangle \quad \forall v \in K, \\ u_\tau(0) = \widetilde{u}_0, \end{cases}$$

and the corresponding Λ_τ is a non-negative Radon measure which lives on Σ and integrates functions from the space $H^{1/2,1}(Q)$. Formally, by integration by parts and standard interpretation of variational inequalities [70],

$$\langle \Lambda_\tau, v\rangle = \int_0^T \int_{\Gamma_c} \frac{\partial u_\tau}{\partial \nu}(x,t) v(x,t)\, d\Gamma(x)\, dt$$

and we have

$$\langle \Lambda_\tau, u_\tau\rangle = 0. \tag{4.13}$$

To see this it is enough to take $v = 2u_\tau, v = u_\tau/2$ in the inequality $\langle \Lambda_\tau, v - u_\tau\rangle \geq 0$, which holds for all $v \in K$. Furthermore, $\langle \Lambda_\tau, v\rangle \geq 0$ for any $v \in K$ and, in view of (4.13), Λ_τ lives in the set

$$\Xi_\tau = \{(x,t) \in \Sigma_c \mid u_\tau(x,t) = 0\}.$$

Let us look at the mappings $\tau \mapsto \Lambda_\tau$, $t \mapsto u_t$ more closely. By the Lipschitz continuity we have

$$C|t-s|^2 \geq |\langle \Lambda_t - \Lambda_s, u_t - u_s\rangle| = |-\langle \Lambda_t, u_s\rangle - \langle \Lambda_s, u_t\rangle| = \langle \Lambda_t, u_s\rangle + \langle \Lambda_s, u_t\rangle.$$

Since $u_t, u_s \in K$ and Λ_t, Λ_s are positive measures we see that

$$\langle \Lambda_t, u_s\rangle \geq 0, \qquad \langle \Lambda_s, u_t\rangle \geq 0.$$

Thus for all $t, s \geq 0$,

$$\langle \Lambda_t, u_s\rangle \leq C|t-s|^2. \tag{4.14}$$

Since $t \mapsto \Lambda_t$ and $t \mapsto u_t$ are Lipschitz, we see that

$$\left\{ \frac{u_t - u}{t} \,\middle|\, 0 < t \leq 1 \right\}, \qquad \left\{ \frac{\Lambda_t - \Lambda}{t} \,\middle|\, 0 < t \leq 1 \right\}$$

are relatively weak and weak* compact, respectively.

Let η be any weak* limit of $(\Lambda_t - \Lambda)/t$ as $t \to 0$. Now we show that $\langle \eta, v\rangle \geq 0$ for all $v \in T_K(u) \cap \Lambda^\perp$. To this end we first show $\langle \eta, u\rangle = 0$. We have, in view of (4.14),

$$\langle \eta, u\rangle = \lim_{\tau_i \to 0} \left\langle \frac{\Lambda_{\tau_i} - \Lambda}{\tau_i}, u \right\rangle = \lim_{\tau_i \to 0} \left\langle \frac{\Lambda_{\tau_i}}{\tau_i}, u \right\rangle = 0.$$

Set $u_t = u + tv$, so $u_0 = u$, and by polyhedricity,

$$T_K(u_0) \cap \Lambda^\perp = \overline{C_K(u_0) \cap \Lambda^\perp},$$

so that we only need to show $\eta \geq 0$ on $C_K(u_0) \cap \Lambda^\perp$. Let $v \in C_K(u_0) \cap \Lambda^\perp$. Then for $t > 0$, $u + tv \in K \cap \Lambda^\perp$, and

$$\langle \eta, u + tv \rangle = \lim_{\tau_i \to 0} \left\langle \frac{\Lambda_{\tau_i} - \Lambda}{\tau_i}, u + tv \right\rangle = \lim_{\tau_i \to 0} \left\langle \frac{\Lambda_{\tau_i}}{\tau_i}, u + tv \right\rangle \geq 0$$

because $u + tv \in K$. Thus $\langle \eta, v \rangle \geq 0$ since $\langle \eta, u \rangle = 0$ as was shown before. Thus

$$\eta \geq 0 \quad \text{ on } T_K(u_0) \cap \Lambda^\perp.$$

If q denotes a weak limit of $(u_t - u)/t$ at $t = 0^+$, it follows by definition of the tangent cone $T_K(u)$ that $q \in T_K(u)$. Now we show that

- $q \in S_K(u) = T_K(u) \cap \Lambda^\perp$,
- $\langle \eta, q \rangle = 0$.

Let us observe that $q \in S_K(u)$ follows if we show that

$$\langle \Lambda, q \rangle = 0.$$

We have

$$\langle \Lambda, q \rangle \geq 0$$

since $q \in T_K(u)$ so $q \geq 0$ on Ξ and Λ lives on Ξ (see Section 4.5.1 for details). On the other hand,

$$\left\langle \Lambda_\tau, \frac{u_\tau - u}{\tau} \right\rangle = \left\langle \Lambda, \frac{u_\tau - u}{\tau} \right\rangle + \left\langle \Lambda_\tau - \Lambda, \frac{u_\tau - u}{\tau} \right\rangle \to \langle \Lambda, q \rangle + 0$$

since

$$\left\| \frac{\Lambda_\tau - \Lambda}{\tau} \right\|_{H^*} \leq C.$$

This shows that

$$\lim \left\langle \Lambda_\tau, \frac{u_\tau - u}{\tau} \right\rangle \geq 0.$$

In order to conclude that $\langle \Lambda, q \rangle = 0$, let us note that

$$\left\langle \Lambda_\tau, \frac{u_\tau - u}{\tau} \right\rangle = -\frac{1}{\tau} \langle \Lambda_\tau, u \rangle \leq 0$$

for $\tau \geq 0$ since $u \in K$ is ≥ 0 on Ξ_τ. Therefore,

$$\left\langle \Lambda_\tau, \frac{u_\tau - u}{\tau} \right\rangle \leq 0,$$

which implies that

$$\lim_{\tau \to 0} \left\langle \Lambda_\tau, \frac{u_\tau - u}{\tau} \right\rangle \leq 0.$$

This completes the proof of the required property $\langle \Lambda, q \rangle = 0$, which implies $q \in S_K(u)$.

Now we come to the crucial result, namely $\langle \eta, q \rangle = 0$. Since $\langle Az, z \rangle \geq 0$ we deduce

$$\frac{1}{2}(\langle Au, v \rangle + \langle Av, u \rangle) \leq \langle Au, u \rangle^{1/2} \langle Av, v \rangle^{1/2}. \tag{4.15}$$

Now for any $\theta > 0$, and $h = (f_\theta - f)/\theta$,

$$A(u_\theta - u) = \theta h + \Lambda_\theta - \Lambda. \tag{4.16}$$

Using (4.15) with $u_\tau - u$, $u_t - u$ yields

$$\langle A(u_\tau - u), u_\tau - u\rangle^{1/2}\langle A(u_t - u), u_t - u\rangle^{1/2} \geq \frac{1}{2}\{\langle A(u_\tau - u), u_t - u\rangle + \langle A(u_t - u), u_\tau - u\rangle\}.$$

Using (4.16) we get

$$\langle \tau h + \Lambda_\tau - \Lambda, u_\tau - u\rangle^{1/2}\langle th + \Lambda_t - \Lambda, u_t - u\rangle^{1/2}.$$

Dividing both sides by τt we get

$$\begin{aligned}\left\langle h + \frac{\Lambda_\tau - \Lambda}{\tau}, \frac{u_\tau - u}{\tau}\right\rangle^{1/2} & \left\langle h + \frac{\Lambda_t - \Lambda}{t}, \frac{u_t - u}{t}\right\rangle^{1/2} \\ & \geq \frac{1}{2}\left\{\left\langle h + \frac{\Lambda_\tau - \Lambda}{\tau}, \frac{u_t - u}{t}\right\rangle \left\langle h + \frac{\Lambda_t - \Lambda}{t}, \frac{u_\tau - u}{\tau}\right\rangle\right\}.\end{aligned}$$

Now

$$\langle \Lambda_\theta - \Lambda, u_\theta - u\rangle = -\langle \Lambda_\theta, u\rangle - \langle \Lambda, u_\theta\rangle < 0.$$

Thus

$$\begin{aligned}\left\langle h, \frac{u_\tau - u}{\tau}\right\rangle^{1/2} & \left\langle h, \frac{u_t - u}{t}\right\rangle^{1/2} \\ & \geq \left\langle h + \frac{\Lambda_\tau - \Lambda}{\tau}, \frac{u_\tau - u}{\tau}\right\rangle^{1/2}\left\langle h + \frac{\Lambda_t - \Lambda}{t}, \frac{u_t - u}{t}\right\rangle^{1/2} \\ & \geq \frac{1}{2}\left\{\left\langle h + \frac{\Lambda_\tau - \Lambda}{\tau}, \frac{u_t - u}{t}\right\rangle \left\langle h + \frac{\Lambda_t - \Lambda}{t}, \frac{u_\tau - u}{\tau}\right\rangle\right\}. \qquad (4.17)\end{aligned}$$

Let now τ_i and t_i be sequences tending to zero such that

$$\begin{aligned} & \frac{\Lambda_{\tau_i} - \Lambda}{\tau_i} \overset{w^*}{\rightharpoonup} \eta_1, \qquad \frac{u_{\tau_i} - u}{\tau_i} \overset{w}{\rightharpoonup} q_1, \\ & \left\langle \frac{\Lambda_{\tau_i} - \Lambda}{\tau_i}, \frac{u_{\tau_i} - u}{\tau_i}\right\rangle \to -\alpha \quad (\alpha > 0), \\ & \frac{\Lambda_{t_i} - \Lambda}{t_i} \overset{w^*}{\rightharpoonup} \eta_2, \qquad \frac{u_{t_i} - u}{t_i} \overset{w}{\rightharpoonup} q_2, \\ & \left\langle \frac{\Lambda_{t_i} - \Lambda}{t_i}, \frac{u_{t_i} - u}{t_i}\right\rangle \to -\beta \quad (\beta > 0).\end{aligned}$$

In (4.17) let τ tend to zero along τ_i to get

$$\begin{aligned}\langle h, q_1\rangle^{1/2}\left\langle h, \frac{u_t - u}{t}\right\rangle^{1/2} & \geq (\langle h, q_1\rangle - \alpha))^{1/2}\left\langle h + \frac{\Lambda_t - \Lambda}{t}, \frac{u_t - u}{t}\right\rangle^{1/2} \\ & \geq \frac{1}{2}\left\{\left\langle h + \eta_1, \frac{u_t - u}{t}\right\rangle + \left\langle h + \frac{\Lambda_t - \Lambda}{t}, q_1\right\rangle\right\}.\end{aligned}$$

Now let t tend to zero along t_i to get

$$\begin{aligned}\langle h, q_1\rangle^{1/2}\langle h, q_2\rangle^{1/2} & \geq (\langle h, q_1 - \alpha\rangle)^{1/2}(\langle h, q_2 - \beta\rangle)^{1/2} \\ & \geq \frac{1}{2}\{\langle h + \eta_1, q_2\rangle + \langle h + \eta_2, q_1\rangle\}\end{aligned}$$

$$= \frac{1}{2}\{\langle h, q_1\rangle + \langle h, q_2\rangle + \langle \eta_1, q_2\rangle + \langle \eta_2, q_1\rangle\}$$
$$\geq \frac{1}{2}\{\langle h, q_1\rangle + \langle h, q_2\rangle\} \tag{4.18}$$

(because as shown before, $\eta_i \geq 0$ on $S_K(u)$ and q_i belongs to $S_K(u)$ for $i = 1, 2$)

$$\geq \langle h, q_1\rangle^{1/2}\langle h, q_2\rangle^{1/2}.$$

Thus all the inequalities in (4.18) are equalities. We collect all this as

THEOREM 4.16. *For any two weak limits q_1, q_2 of $(u_\theta - u)/\theta$ and any two weak limits η_1, η_2 of $(\Lambda_\theta - \Lambda)/\theta$ we have*

$$\langle h, q_1\rangle = \langle h, q_2\rangle,$$
$$\langle \eta_1, q_2\rangle = \langle \eta_2, q_1\rangle = 0,$$
$$\lim_{\tau\to 0} \frac{\langle \Lambda_\tau, u\rangle + \langle \Lambda, u_\tau\rangle}{\tau^2} = 0.$$

As an immediate corollary we have:

THEOREM 4.17. *Any weak limit q satisfies the variational inequality*

$$q \in S_K(u) : \langle Aq, v - q\rangle = \langle \dot{q}, v - q\rangle + a(q, v - q) \geq \langle h, v - q\rangle \quad \forall v \in S_K(u), \tag{4.19}$$
$$q(0) = 0. \tag{4.20}$$

In view of the uniqueness of solutions to (4.19)–(4.20), the above theorem shows that the couple η, q is unique.

4.6. Applications. We provide applications of the general results we have obtained. We restrict ourselves to the model variational inequality introduced in [72]. The unilateral conditions are prescribed on the crack faces, so the solutions are singular at the crack tips. Our method is general, and therefore can be applied to variational inequalities in elasticity, with frictionless contact conditions prescribed on the crack faces. We also apply the results to an evolution variational inequality in a work in progress.

The analysis of abstract variational inequalities leads to the concept of conical differentiability of solutions. We recall here the result proved in [126] which gives the conical differentiability of solutions to variational inequalities with respect to shape variations. Such a result leads to the first and second order shape differentiability of energy functionals in domains with cracks.

4.6.1. An abstract result. For the convenience of the reader we recall here the abstract result which is a generalization of the implicit function theorem for variational inequalities.

Let $K \subset V$ be a convex and closed subset of a Hilbert space V, and let $\langle\cdot,\cdot\rangle$ denote the duality pairing between V' and V, where V' denotes the dual of V.

We shall consider the following family of variational inequalities depending on a parameter $t \in [0, \delta)$, $\delta > 0$:

$$y_t \in K : \quad a_t(y_t, \varphi - y_t) \geq \langle f_t, \varphi - y_t\rangle \quad \forall \varphi \in K. \tag{4.21}$$

Moreover, let $y_t = \mathcal{P}_t(f_t)$ be a solution to (4.21). Let us note that for $f_t = 0$ and $y_t = \mathcal{P}_t(0)$ we obtain $y' = \Pi'(-\mathcal{A}'y_0)$ in (4.22).

THEOREM 4.18. *Assume that:*

- *the bilinear form* $a_t(\cdot,\cdot) : V \times V \to \mathbb{R}$ *is coercive and uniformly continuous with respect to* $t \in [0,\delta)$*; let* $\mathcal{A}_t \in \mathcal{L}(V;V')$ *be the linear operator defined by* $a_t(\phi,\varphi) = \langle \mathcal{A}_t\phi, \varphi\rangle$ *for* $\phi,\varphi \in V$*; suppose that there exists* $\mathcal{A}' \in \mathcal{L}(V;V')$ *such that*
$$\mathcal{A}_t = \mathcal{A}_0 + t\mathcal{A}' + o(t) \quad \textit{in } \mathcal{L}(V;V');$$
- *for* $t > 0$ *small enough,*
$$f_t = f_0 + tf' + o(t) \quad \textit{in } V',$$
where $f_t, f_0, f' \in V'$*;*
- $K \subset V$ *is convex and closed, and for the solutions to the variational inequality*
$$\Pi f = \mathcal{P}_0(f) \in K : \quad a_0(\Pi f, \varphi - \Pi f) \geq \langle f, \varphi - \Pi f\rangle \quad \forall \varphi \in K$$
the following differential stability result holds:
$$\forall h \in V' : \quad \Pi(f_0 + \varepsilon h) = \Pi f_0 + \varepsilon \Pi' h + o(\varepsilon) \quad \textit{in } V$$
for $\varepsilon > 0$ *small enough, where the mapping* $\Pi' : V' \to V$ *is continuous and positively homogeneous, and* $o(\varepsilon)$ *is uniform with respect to* $h \in V'$ *on compact subsets of* V'*.*

Then the solutions to the variational inequality (4.21) *are right-differentiable with respect to* t *at* $t = 0$*, i.e. for* $t > 0$ *small enough,*
$$y_t = y_0 + ty' + o(t) \quad \textit{in } V,$$
where
$$y' = \Pi'(f' - \mathcal{A}'y_0). \tag{4.22}$$

4.6.2. Unilateral conditions on the crack. Let $D \subset \mathbb{R}^2$ be a bounded domain with smooth boundary Γ, and $\Sigma_{l+\delta}$ be the set $\{(x_1,x_2) \mid 0 < x_1 < l+\delta,\ x_2 = 0\}$. We assume that this set is contained in D for all sufficiently small δ, and some $l > 0$. The domains with cracks $\Sigma_{l+\delta}$, Σ_l are denoted by $\Omega_\delta = D \setminus \overline{\Sigma}_{l+\delta}$, $\Omega = D \setminus \overline{\Sigma}_l$, respectively. We consider an elastic membrane in the reference domain Ω with crack Σ_l of length l and with the unilateral condition prescribed on the crack for the displacement of the membrane [72]. Therefore, in the domain Ω, we consider the following boundary value problem for a function u:
$$-\Delta u = f \quad \text{in } \Omega, \tag{4.23}$$
$$u = 0 \quad \text{on } \Gamma, \tag{4.24}$$
$$[u] \geq 0,\ [u_{y_2}] = 0,\ u_{y_2} \leq 0,\ u_{y_2}[u] = 0 \quad \text{on } \Sigma_l. \tag{4.25}$$

Here $f \in C^1(\bar{D})$ is a given function, $[u] = u^+ - u^-$ is the jump of u across Σ_l. The vector $n = (0,1)$ is orthogonal to Σ_l, and $u^\pm$ denote the traces of u on the crack faces, corresponding to the positive and negative directions of n. In order to define weak solutions to (4.23)–(4.25), we consider the minimization of the functional
$$I(\phi) = \frac{1}{2}\int_\Omega |\nabla\phi|^2 - \int_\Omega f\phi$$

over the set

$$K_0 = \{w \in H^1(\Omega) \mid [w] \geq 0 \text{ on } \Sigma_l;\ w = 0 \text{ on } \Gamma\},$$

of all admissible functions from the Sobolev space

$$H^1_\Gamma(\Omega) = \{v \in H^1(\Omega) \mid v|_\Gamma = 0\}.$$

Therefore, for any value of the parameter $\delta \in [-\delta_0, \delta_0]$, $\delta_0 > 0$, the function u^δ is the solution of the variational inequality

$$u^\delta \in K_\delta : \quad \int_{\Omega_\delta} \langle \nabla u^\delta, \nabla v - \nabla u^\delta \rangle \geq \int_{\Omega_\delta} f(v - u^\delta) \quad \forall v \in K_\delta, \tag{4.26}$$

where

$$K_\delta = \{w \in H^1(\Omega_\delta) \mid [w] \geq 0 \text{ on } \Sigma_{l+\delta};\ w = 0 \text{ on } \Gamma\}.$$

The energy functional for a weak solution of the problem (4.23)–(4.25) is defined by the formula

$$J(\Omega) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega fu, \tag{4.27}$$

and the energy functional for the problem (4.26) is equal to

$$J(\Omega_\delta) = \frac{1}{2}\int_{\Omega_\delta} |\nabla u^\delta|^2 - \int_{\Omega_\delta} fu^\delta.$$

The form of the derivative of the energy functional $J(\Omega_\delta)$ with respect to variations of the crack's length

$$\left.\frac{dJ(\Omega_\delta)}{d\delta}\right|_{\delta=0} = \lim_{\delta\to 0} \frac{J(\Omega_\delta) - J(\Omega)}{\delta} \tag{4.28}$$

is obtained in [72] (see Theorem 4.20 below).

The dependence of the energy functional on the crack length is important in fracture mechanics. The derivative of the functional is often used to formulate fracture criteria. The formulae for derivatives of the energy functional with respect to the crack length are called the *Griffith formulae.* Invariant integrals over curves surrounding the crack tips are usually called the *Rice–Cherepanov integrals.* In the present paper we extend the results of [72] and obtain the second order shape derivative of the energy functional with respect to the crack length.

In order to find the derivative (4.28), the transformation of the domain Ω_δ onto the domain Ω is introduced. The transformation, depending on the function θ, is constructed in the following way.

Let $\theta \in C_0^\infty(D)$ be any function such that $\theta = 1$ in a neighbourhood of the point $x_l = (l, 0)$. To simplify the argument the function θ is assumed to be equal to zero in a neighbourhood of the point $(0, 0)$. Consider the transformation of the independent variables

$$y_1 = x_1 - \delta\theta(x_1, x_2), \qquad y_2 = x_2, \tag{4.29}$$

where $(x_1, x_2) \in \Omega_\delta, (y_1, y_2) \in \Omega$. The Jacobian q_δ of this transformation is equal to

$$\left|\frac{\partial(y_1, y_2)}{\partial(x_1, x_2)}\right| = 1 - \delta\theta_{x_1}.$$

For small δ, the Jacobian q_δ is positive, hence the transformation (4.29) is one-to-one. Therefore, in view of (4.29) we have $y = y(x, \delta), x = x(y, \delta)$.

REMARK 4.19. We shall use the same transformation with θ replaced by ψ, in order to define the second order shape derivatives in the directions θ and ψ. It is then useful to assume that ψ is supported on the set $\theta = 1$, which simplifies the form of the second order derivative. We refer the reader to [124] for the details on the decomposition of the second order shape derivatives.

Let $u^\delta(x)$ be the solution of (4.26), and $u^\delta(x) = u_\delta(y)$, $x = x(y, \delta)$. We have the formulae

$$u^\delta_{x_1} = u_{\delta y_1}(1 - \delta\theta_{x_1}), \qquad u^\delta_{x_2} = u_{\delta y_1}(-\delta\theta_{x_2}) + u_{\delta y_2}.$$

Consequently,

$$\int_{\Omega_\delta} |\nabla u^\delta|^2 \, dx = \int_\Omega \langle A_\delta \nabla u_\delta, \nabla u_\delta \rangle \, dy,$$

where $A_\delta = A_\delta(y)$ is the matrix

$$A_\delta(y) = \frac{1}{1 - \delta\theta_{x_1}} \begin{pmatrix} (1 - \delta\theta_{x_1})^2 + \delta^2\theta^2_{x_2} & -\delta\theta_{x_2} \\ -\delta\theta_{x_2} & 1 \end{pmatrix}, \qquad \theta = \theta(x(y, \delta)).$$

Note that $A_0(y) = E$ is the identity matrix.

It is easy to find the derivative of $A_\delta(y)$ with respect to δ:

$$A'(y) = \left.\frac{dA_\delta(y)}{d\delta}\right|_{\delta=0} = \lim_{\delta\to 0} \frac{A_\delta(y) - A_0(y)}{\delta}.$$

We have

$$A'(y) = \begin{pmatrix} -\theta_{y_1}(y) & -\theta_{y_2}(y) \\ -\theta_{y_2}(y) & \theta_{y_1}(y) \end{pmatrix}, \tag{4.30}$$

and we shall also write $A'(y) = A'(\theta)(y)$ to show the dependence of A' on θ. In the same way $A'(y) = A'(\psi)(y)$ is obtained for the transformation depending on ψ.

By change of variables it follows that

$$\int_{\Omega_\delta} f u^\delta \, dx = \int_\Omega \frac{f(x(y, \delta))u_\delta(y)}{1 - \delta\theta_{x_1}} \, dy.$$

Set

$$f^\delta(y) = \frac{f(x(y, \delta))}{1 - \delta\theta_{x_1}}$$

and find the derivative

$$f'(y) = \left.\frac{df^\delta(y)}{d\delta}\right|_{\delta=0} = \lim_{\delta\to 0} \frac{f^\delta(y) - f^0(y)}{\delta}.$$

Assuming that y, δ are independent variables in (4.29) we have $x = x(y, \delta)$. Differentiation of (4.29) with respect to δ yields

$$0 = \frac{dx_1}{d\delta} - \theta - \delta\theta_{x_1}\frac{dx_1}{d\delta},$$

whence

$$\frac{dx_1}{d\delta} = \frac{\theta}{1 - \delta\theta_{x_1}}, \qquad \frac{dx_2}{d\delta} = 0. \tag{4.31}$$

Consequently, by (4.31),

$$\frac{\partial f(x(y,\delta))}{\partial \delta}\bigg|_{\delta=0} = f_{x_1}\frac{dx_1}{d\delta}\bigg|_{\delta=0} + f_{x_2}\frac{dx_2}{d\delta}\bigg|_{\delta=0} = f_{y_1}\theta. \tag{4.32}$$

Now we are in a position to find the derivative $f'(y)$. Indeed, by (4.32), since $\theta_{x_1} = \theta_{y_1}$,

$$\begin{aligned} f'(y) &= \lim_{\delta\to 0}\left(\frac{f(x(y,\delta))}{1-\delta\theta_{x_1}} - f(y)\right)\frac{1}{\delta} = \lim_{\delta\to 0}\frac{f(x(y,\delta)) - f(y)}{\delta} + \theta_{x_1} f(y)|_{\delta=0} \\ &= f_{y_1}\theta + \theta_{y_1} f = \frac{\partial}{\partial y_1}(\theta f), \end{aligned}$$

i.e. $f'(y) = (\theta f)_{y_1}(y)$. Since $f \in C^1(\bar{\Omega})$ we can see that as $\delta \to 0$,

$$\frac{f^\delta(y) - f^0(y)}{\delta} \to f'(y) \quad \text{in } L^\infty(\Omega).$$

Also, notice that, in addition to (4.30), as $\delta \to 0$,

$$\frac{A_\delta(y) - A_0(y)}{\delta} \to A'(y) \quad \text{in } L^\infty(\Omega).$$

In view of (4.29), let $x = x(y,\delta)$. Then $w^\delta(x) = w_\delta(y)$. The inclusion $w^\delta \in K_\delta$ implies $w_\delta \in K_0$, and conversely, $w_\delta \in K_0$ implies $w^\delta \in K_\delta$. This means that the transformation (4.29) maps K_δ onto K_0, and it is one-to-one. It is easy to see that the solution to the variational inequality is Lipschitz with respect to δ.

Let u^δ be the solution of (4.26), $u^\delta(x) = u_\delta(y)$, and u be the solution of (4.23)–(4.25). Then

$$\|u_\delta - u\|_{H^1(\Omega)} \le C\delta, \qquad \delta \to 0.$$

To underline the dependence of the domain Ω on the crack length l we shall write Ω_l instead of Ω. On the other hand, to underline the dependence of the transformed domain Ω_δ on the function θ when dealing with the first order shape derivative, and on the function ψ in the procedure of derivation of the second order shape derivative, we shall write $\Omega_\delta = \Omega(\theta)$ and $\Omega_\delta = \Omega(\psi)$, respectively.

Let $J(\Omega_l)$ be defined by the formula (4.27). The Griffith formula established in [72] gives the derivative of the energy functional with respect to the crack length for the problem (4.23)–(4.25).

THEOREM 4.20. *The derivative of $J(\Omega_l)$ with respect to l is given by*

$$\frac{dJ(\Omega_l)}{dl} = -\frac{1}{2}\int_\Omega (\theta_{y_1}(u_{y_1}^2 - u_{y_2}^2) + 2\theta_{y_2}u_{y_1}u_{y_2}) - \int_\Omega (\theta f)_{y_1} u.$$

The first derivative is independent of the choice of θ with the required properties. We can obtain the second order shape derivative of the energy functional in directions θ and ψ using the following result which seems to be new and follows by the arguments given in Section 4.1.

THEOREM 4.21. *The set K_0 is polyhedric.*

Theorem 4.18 implies that the function $u_\delta \in K_0$, obtained by transport to the fixed domain Ω of the solution $u^\delta \in K_\delta$ to the variational inequality defined in $\Omega_\delta = \Omega(\psi)$, is right differentiable at $\delta = 0$, i.e. for $\delta > 0$ small enough,

$$u_\delta = u + \delta Q + o(\delta)$$

where $\|o(\delta)\|_{H^1_\Gamma(\Omega)}/\delta \to 0$ as $\delta \downarrow 0$. The directional derivative Q of u_δ in direction θ is given by the unique solution to the variational inequality

$$Q \in S: \quad \int_\Omega \langle \nabla Q, \nabla(v-Q)\rangle_{\mathbb{R}^2} \geq -\int_\Omega \langle A'(\psi)\nabla u, \nabla(v-Q)\rangle_{\mathbb{R}^2} + \int_\Omega f'(\psi)(v-Q) \quad \forall v \in S,$$

where

$$S = \{v \in H^1_\Gamma(\Omega) \mid [v] \geq 0 \text{ on } \Xi,\ v \in \Lambda^\perp\},$$

with

$$\Xi = \{x \in \Sigma_l \mid [u(x)] = 0\} \quad \text{and} \quad \Lambda^\perp = \left\{ v \,\middle|\, \int_\Omega \langle \nabla u, \nabla v\rangle = \int_\Omega fv \right\}.$$

Therefore, we obtain the second order shape derivative of the energy functional $J(\Omega_l)$ in directions θ, ψ.

Theorem 4.22. *The second order directional derivative of the energy functional $J(\Omega_l)$ with respect to the crack length is given by*

$$\begin{aligned}\frac{d^2 J(\Omega_l)}{dl^2} &= \frac{1}{2}\int_\Omega \theta_{y_1}\psi_{y_1}[u^2_{y_1} + u^2_{y_2}] - \int_\Omega [\theta_{y_1}\psi_{y_2}u_{y_1}u_{y_2} - \theta_{y_2}\psi_{y_2}u^2_{y_1}]\\ &\quad - \int_\Omega \theta_{y_1}[u_{y_1}Q_{y_1} - u_{y_2}Q_{y_2}] - \int_\Omega \theta_{y_2}[u_{y_1}Q_{y_2} + u_{y_2}Q_{y_1}]\\ &\quad - \int_\Omega u(\psi(\theta f)_{y_1})_{y_1} - \int_\Omega Q(\psi)(\theta f)_{y_1},\end{aligned}$$

where $Q = Q(\psi)$ solves the variational inequality for $A'(y) = A'(\psi)(y)$ and $f'(y) = (\psi f)_{y_1}(y)$.

Remark 4.23. The same result can be obtained in the case of the elasticity system [73] with frictionless contact conditions on the crack faces, i.e. the convex set is polyhedric and the second order directional differentiability of the energy functional follows by the same argument as above for the scalar equation.

Remark 4.24. Taking ψ with the support in the set $\theta = 1$ we have

$$\begin{aligned}\frac{d^2 J(\Omega_l)}{dl^2} &= -\int_\Omega \theta_{y_1}[u_{y_1}Q_{y_1} - u_{y_2}Q_{y_2}] - \int_\Omega \theta_{y_2}[u_{y_1}Q_{y_2} + u_{y_2}Q_{y_1}]\\ &\quad - \int_\Omega u(\theta(\psi f)_{y_1})_{y_1} - \int_\Omega Q(\psi)(\theta f)_{y_1}.\end{aligned}$$

5. Non-penetration conditions on crack faces in elastic bodies

5.1. Introduction. The chapter is concerned with new recent results related to crack theory in elasticity with possible contact between crack faces. We discuss problem formulations, peculiarities of the problems and possible relations between topics under investigation. It is well known that the classical crack theory in elasticity is characterized by linear boundary conditions which leads to linear boundary value problems. This approach has a clear shortcoming from the mechanical standpoint since opposite crack faces can penetrate each other. We consider non-linear boundary conditions on crack faces, the so-called non-penetration conditions, written in terms of inequalities. From the standpoint of applications these boundary conditions are preferable since they ensure mutual non-penetration between crack faces. As a result, a free boundary problem is obtained, which means that a concrete boundary condition at a given point can be found provided that we have a solution of the problem.

The main attention in this chapter is focused on dependence of solutions of the problem on domain perturbations, and in particular on the crack shape.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary Γ, and $\Gamma_c \subset \Omega$ be a smooth curve without self-intersections, $\Omega_c = \Omega \setminus \overline{\Gamma}_c$ (see Fig. 5.1).

It is assumed that Γ_c can be extended in such a way that this extension crosses Γ at two points, and Ω_c is divided into two subdomains D_1 and D_2 with Lipschitz boundaries ∂D_1, ∂D_2, $\operatorname{meas}(\Gamma \cap \partial D_i) > 0$, $i = 1, 2$. Denote by $\nu = (\nu_1, \nu_2)$ a unit normal vector to Γ_c. We assume that Γ_c does not contain its tip points, i.e. $\Gamma_c = \overline{\Gamma}_c \setminus \partial\Gamma_c$.

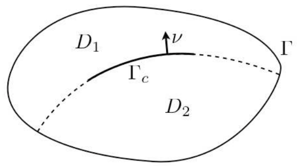

Fig. 5.1. Domain Ω_c

The equilibrium problem for a linear elastic body occupying Ω_c is as follows. In the domain Ω_c we have to find a displacement field $u = (u_1, u_2)$ and stress tensor components $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, such that

$$-\operatorname{div}\sigma = f \quad \text{in } \Omega_c, \tag{5.1}$$

$$\sigma = A\varepsilon(u) \quad \text{in } \Omega_c, \tag{5.2}$$

$$u = 0 \quad \text{on } \Gamma, \tag{5.3}$$

$$[u]\nu \geq 0, \ [\sigma_\nu] = 0, \ \sigma_\nu \cdot [u]\nu = 0 \quad \text{on } \Gamma_c, \tag{5.4}$$

$$\sigma_\nu \leq 0, \ \sigma_\tau = 0 \quad \text{on } \Gamma_c^{\pm}. \tag{5.5}$$

Here $[v] = v^+ - v^-$ is the jump of v on Γ_c, and the signs $\pm$ correspond to the positive and negative crack faces with respect to ν, $f = (f_1, f_2) \in L^2(\Omega_c)$ is a given function,

$$\sigma_\nu = \sigma_{ij}\nu_j\nu_i, \qquad \sigma_\tau = \sigma\nu - \sigma_\nu \cdot \nu, \qquad \sigma_\tau = (\sigma_\tau^1, \sigma_\tau^2), \qquad \sigma\nu = (\sigma_{1j}\nu_j, \sigma_{2j}\nu_j),$$

the strain tensor components are denoted by $\varepsilon_{ij}(u)$,

$$\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \qquad \varepsilon(u) = \{\varepsilon_{ij}(u)\}, \qquad i, j = 1, 2.$$

The elasticity tensor $A = \{a_{ijkl}\}$, $i, j, k, l = 1, 2$, is given and satisfies the usual properties of symmetry and positive definiteness

$$a_{ijkl}\xi_{kl}\xi_{ij} \geq c_0|\xi|^2, \qquad \forall \xi_{ij}, \qquad \xi_{ij} = \xi_{ji}, \qquad c_0 = \text{const},$$

$a_{ijkl} = a_{klij} = a_{jikl}$, $a_{ijkl} \in L^\infty(\Omega)$.

Relations (5.1) are equilibrium equations, and (5.2) is Hooke's law, $u_{i,j} = \partial u_i/\partial u_j$, $(x_1, x_2) \in \Omega_c$. All functions with two lower indices are symmetric in those indices, i.e. $\sigma_{ij} = \sigma_{ji}$ etc. The summation convention over repeated indices is assumed throughout the chapter.

The first condition in (5.4) is called the *non-penetration condition.* It provides a mutual non-penetration between the crack faces $\Gamma_c^\pm$. The second condition of (5.5) provides zero friction on Γ_c. For simplicity we assume a clamping condition (5.3) at the external boundary Γ.

Note that a priori we do not know at which points on Γ_c strict inequalities hold in (5.4), (5.5). Due to this, the problem (5.1)–(5.5) is a free boundary value problem. If we have $\sigma_\nu = 0$ then, together with $\sigma_\tau = 0$, the classical boundary condition $\sigma\nu = 0$ follows which is used in the linear crack theory. On the other hand, due to (5.4), the condition $\sigma_\nu < 0$ implies $[u]\nu = 0$, i.e. we have a contact between the crack faces at a given point. The strict inequality $[u]\nu > 0$ at a given point means that we have no contact between the crack faces.

Hence, the first difficulty in studying the problem (5.1)–(5.5) is concerned with the boundary conditions (5.4)–(5.5). The second one is more general—presence of non-smooth boundaries. We refer the reader to [49], [81] for general results on boundary value problems defined in non-smooth domains.

5.2. Existence of solution. First of all we note that problem (5.1)–(5.5) admits several equivalent formulations. In particular, it corresponds to minimization of an energy functional. To check this, introduce the Sobolev space

$$H^1_\Gamma(\Omega_c) = \{v = (v_1, v_2) \mid v_i \in H^1(\Omega_c),\ v_i = 0 \text{ on } \Gamma,\ i = 1, 2\}$$

and a closed convex set of admissible displacements

$$K = \{v \in H^1_\Gamma(\Omega_c) \mid [v]\nu \geq 0 \text{ a.e. on } \Gamma_c\}. \tag{5.6}$$

In this case, due to the Weierstrass theorem, the problem

$$\min_{v \in K} \left\{ \frac{1}{2} \int_{\Omega_c} \sigma_{ij}(v)\varepsilon_{ij}(v) - \int_{\Omega_c} f_i v_i \right\}$$

has a (unique) solution u satisfying the variational inequality

$$u \in K, \tag{5.7}$$

$$\int_{\Omega_c} \sigma_{ij}(u)\varepsilon_{ij}(v - u) \geq \int_{\Omega_c} f_i(v_i - u_i), \qquad \forall v \in K, \tag{5.8}$$

where $\sigma_{ij}(u) = \sigma_{ij}$ are defined from (5.2).

The formulations (5.1)–(5.5) and (5.7)–(5.8) are equivalent: any smooth solution of (5.1)–(5.5) satisfies (5.7)–(5.8) and conversely.

Below we provide two more equivalent formulations for the problem (5.1)–(5.5), the so-called mixed and smooth domain formulations. To this end, we first discuss in what sense the boundary conditions (5.4)–(5.5) are satisfied. Denote by Σ a closed curve without self-intersections of class $C^{1,1}$, which is an extension of Γ_c such that $\Sigma \subset \Omega$, and the domain Ω is divided into two subdomains Ω_1 and Ω_2 (see Fig. 5.2). In this case Σ is the boundary of the domain Ω_1, and the boundary of Ω_2 is $\Sigma \cup \Gamma$.

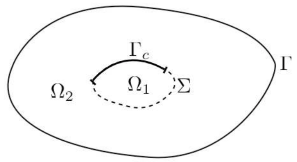

Fig. 5.2. Extension of Γ_c to Σ

Introduce the space $H^{1/2}(\Sigma)$ with the norm

$$\|v\|^2_{H^{\frac{1}{2}}(\Sigma)} = \|v\|^2_{L^2(\Sigma)} + \int_\Sigma \int_\Sigma \frac{|v(x) - v(y)|^2}{|x-y|^2}\, dx\, dy \tag{5.9}$$

and denote by $H^{-1/2}(\Sigma)$ the dual space of $H^{1/2}(\Sigma)$. Also, consider the space

$$H^{1/2}_{00}(\Gamma_c) = \{v \in H^{1/2}(\Gamma_c) \mid v/\sqrt{\rho} \in L^2(\Gamma_c)\}$$

with the norm

$$\|v\|^2_{1/2,00} = \|v\|^2_{1/2} + \int_{\Gamma_c} \rho^{-1} v^2,$$

where $\rho(x) = \mathrm{dist}(x; \partial\Gamma_c)$ and $\|v\|_{1/2}$ is the norm in the space $H^{1/2}(\Gamma_c)$. It is known that functions from $H^{1/2}_{00}(\Gamma_c)$ can be extended to Σ by zero, and moreover this extension belongs to $H^{1/2}(\Sigma)$. More precisely, let v be defined at Γ_c, and $\overline{v}$ be the extension of v by zero, i.e.

$$\overline{v}(x) = \begin{cases} v(x), & x \in \Gamma_c, \\ 0, & x \in \Sigma \setminus \Gamma_c. \end{cases}$$

Then (see [67])

$$v \in H^{1/2}_{00}(\Gamma_c) \quad \text{if and only if} \quad \overline{v} \in H^{1/2}(\Sigma).$$

With the above notations, it is possible to describe in what sense the boundary conditions (5.4)–(5.5) hold. Namely, the condition $\sigma_\nu \le 0$ in (5.5) means that

$$\langle \sigma_\nu, \phi \rangle_{1/2,00} \le 0, \qquad \forall \phi \in H^{1/2}_{00}(\Gamma_c),\ \phi \ge 0 \text{ a.e. on } \Gamma_c,$$

where $\langle \cdot, \cdot \rangle_{1/2,00}$ is the duality pairing between $H^{-1/2}_{00}(\Gamma_c)$ and $H^{1/2}_{00}(\Gamma_c)$. The condition $\sigma_\tau = 0$ in (5.5) means that

$$\langle \sigma_\nu, \phi \rangle_{1/2,00} = 0, \qquad \forall \phi = (\phi_1, \phi_2) \in H^{1/2}_{00}(\Gamma_c).$$

The last condition of (5.4) holds in the following sense:

$$\langle \sigma_\nu, [u]\nu \rangle_{1/2,00} = 0.$$

5.2.1. Mixed formulation of the problem. Now we will give a mixed formulation of the problem (5.1)–(5.5). Introduce the space for stresses

$$H(\operatorname{div}) = \{\sigma = \{\sigma_{ij}\} \mid \sigma \in L^2(\Omega_c), \operatorname{div}\sigma \in L^2(\Omega_c)\}$$

with the norm

$$\|\sigma\|^2_{H(\operatorname{div})} = \|\sigma\|^2_{L^2(\Omega_c)} + \|\operatorname{div}\sigma\|^2_{L^2(\Omega_c)}$$

and the set of admissible stresses

$$H(\operatorname{div};\Gamma_c) = \{\sigma \in H(\operatorname{div}) \mid [\sigma\nu] = 0 \text{ on } \Gamma_c;\ \sigma_\nu \le 0,\ \sigma_\tau = 0 \text{ on } \Gamma_c^\pm\}.$$

We should note at this step that for $\sigma \in H(\operatorname{div})$ the traces $(\sigma\nu)^\pm$ are correctly defined on $\Sigma^\pm$ as elements of $H^{-1/2}(\Sigma)$. The first condition in the definition of $H(\operatorname{div};\Gamma_c)$ is satisfied in the following sense:

$$(\sigma\nu)^+ = (\sigma\nu)^- \quad \text{on } \Sigma$$

for any curve Σ with the prescribed properties (see [67]). The relations $\sigma \le 0$, $\sigma_\tau = 0$ on $\Gamma_c^\pm$ also make sense. The values σ_ν, σ_τ are defined as elements of the space $H_{00}^{-1/2}(\Gamma_c)$.

The mixed formulation of the problem (5.1)–(5.5) is as follows. We have to find a displacement field $u = (u_1, u_2)$ and stress tensor components $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, such that

$$u \in L^2(\Omega_c), \qquad \sigma \in H(\operatorname{div};\Gamma_c), \tag{5.10}$$

$$-\operatorname{div}\sigma = f \qquad \text{in } \Omega_c, \tag{5.11}$$

$$\int_{\Omega_c} C\sigma(\overline{\sigma} - \sigma) + \int_{\Omega_c} u(\operatorname{div}\overline{\sigma} - \operatorname{div}\sigma) \ge 0 \qquad \forall \overline{\sigma} \in H(\operatorname{div};\Gamma_c). \tag{5.12}$$

The tensor C is obtained by inverting Hooke's law (5.2), i.e.

$$C\sigma = \varepsilon(u).$$

It is possible to prove existence of a solution to the problem (5.10)–(5.12) and check that (5.10)–(5.12) is formally equivalent to (5.1)–(5.5) (see [65], [76]). For (5.10)–(5.12) existence can be proved independently of (5.1)–(5.5). On the other hand, the solution exists due to the equivalence, and we already have the solution to the problem (5.1)–(5.5).

5.2.2. Smooth domain formulation. Along with the mixed formulation (5.10)–(5.12) the so-called smooth domain formulation of the problem (5.1)–(5.5) can be provided. In this case the solution of the problem is defined in the smooth domain Ω. To do this, we should notice that the solution of the problem (5.1)–(5.5) satisfies (5.7)–(5.8), thus the condition

$$[\sigma\nu] = 0 \qquad \text{on } \Gamma_c$$

holds, and therefore it can be proved that in the distributional sense

$$-\operatorname{div}\sigma = f \qquad \text{in } \Omega.$$

Hence, the equilibrium equations (5.1) hold in the smooth domain Ω.

Introduce the space for stresses defined in Ω,

$$\mathcal{H}(\operatorname{div}) = \{\sigma = \{\sigma_{ij}\} \mid \sigma, \operatorname{div}\sigma \in L^2(\Omega)\}$$

and the set of admissible stresses

$$\mathcal{H}(\mathrm{div};\Gamma_c) = \{\sigma \in \mathcal{H}(\mathrm{div}) \mid \sigma_\tau = 0,\ \sigma_\nu \le 0 \text{ on } \Gamma_c\}.$$

The norm in the space $\mathcal{H}(\mathrm{div})$ is defined as follows:

$$\|\sigma\|^2_{\mathcal{H}(\mathrm{div})} = \|\sigma\|^2_{L^2(\Omega)} + \|\mathrm{div}\,\sigma\|^2_{L^2(\Omega)}.$$

We see that for $\sigma \in \mathcal{H}(\mathrm{div})$, the boundary conditions $\sigma_\tau = 0$, $\sigma_\nu \le 0$ on Γ_c are correctly defined in the sense $H_{00}^{-1/2}(\Gamma_c)$. Thus, we can provide the smooth domain formulation for the problem (5.1)–(5.5): find a displacement field $u = (u_1, u_2)$ and stress tensor components $\sigma = \{\sigma_{ij}\}$, $i,j = 1,2$, such that

$$u \in L^2(\Omega), \qquad \sigma \in \mathcal{H}(\mathrm{div};\Gamma_c), \tag{5.13}$$

$$-\mathrm{div}\,\sigma = f \quad \text{in } \Omega, \tag{5.14}$$

$$\int_\Omega C\sigma(\overline{\sigma} - \sigma) + \int_\Omega u(\mathrm{div}\,\overline{\sigma} - \mathrm{div}\,\sigma) \ge 0 \qquad \forall \overline{\sigma} \in \mathcal{H}(\mathrm{div};\Gamma_c). \tag{5.15}$$

It is possible to prove existence of a solution to the problem (5.13)–(5.15) (see [65], [76]). Moreover, any smooth solution of (5.1)–(5.5) satisfies (5.13)–(5.15), and conversely, (5.13)–(5.15) implies (5.1)–(5.5). An advantage of the formulation (5.13)–(5.15) is that it is given in the smooth domain. This formulation reminds contact problems with thin obstacle when restrictions are imposed on sets of small dimensions (see [70]).

Numerical aspects of the problems like (5.1)–(5.5) are discussed, for example, in [12], [80].

5.3. Fictitious domain method. In this section we provide a connection between the problem (5.1)–(5.5) and the Signorini contact problem. It turns out that the Signorini problem is a limit problem for a family of problems like (5.1)–(5.5). First we give a formulation of the Signorini problem. Let $\Omega_1 \subset \mathbb{R}^2$ be a bounded domain with smooth boundary Γ_1, $\Gamma_1 = \Gamma_c \cup \Gamma_0$, $\Gamma_c \cap \Gamma_0 = \emptyset$, $\mathrm{meas}\,\Gamma_0 > 0$ (see Fig. 5.3).

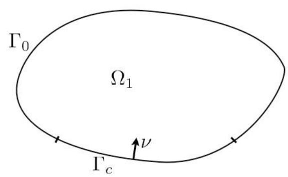

Fig. 5.3. Signorini problem

For simplicity, we assume that Γ_c is a smooth curve (without its tip points). Denote by $\nu = (\nu_1, \nu_2)$ a unit normal inward vector to Γ_c. We have to find a displacement field $u = (u_1, u_2)$ and stress tensor components $\sigma = \{\sigma_{ij}\}$, $i,j = 1,2$, such that

$$-\mathrm{div}\,\sigma = f \qquad \text{in } \Omega_1, \tag{5.16}$$

$$\sigma = A\varepsilon(u) \qquad \text{in } \Omega_1, \tag{5.17}$$

$$u = 0 \qquad \text{on } \Gamma_0, \tag{5.18}$$

$$u\nu \geq 0,\ \sigma_\nu \leq 0,\ \sigma_\tau = 0,\ u\nu \cdot \sigma_\nu = 0 \qquad \text{on } \Gamma_c. \tag{5.19}$$

Here $f = (f_1, f_2) \in L^2_{\text{loc}}(\mathbb{R}^2)$ is a given function, $A = \{a_{ijkl}\}$, $i, j, k, l = 1, 2$, is a given elasticity tensor, $a_{ijkl} \in L^\infty_{\text{loc}}(\mathbb{R}^2)$, with the usual properties of symmetry and positive definiteness.

It is well known (see [40]) that the problem (5.16)–(5.19) has a variational formulation providing existence of solution. Namely, define

$$H^1_{\Gamma_0}(\Omega_1) = \{v = (v_1, v_2) \in H^1(\Omega_1) \mid v_i = 0 \text{ on } \Gamma_0,\ i = 1, 2\}$$

and introduce the set of admissible displacements

$$K_c = \{v = (v_1, v_2) \in H^1_{\Gamma_0}(\Omega_1) \mid v\nu \geq 0 \text{ a.e. on } \Gamma_c\}.$$

In this case the problem (5.16)–(5.19) is equivalent to minimization of the functional

$$\frac{1}{2}\int_{\Omega_1} \sigma_{ij}(v)\varepsilon_{ij}(v) - \int_{\Omega_1} f_i v_i$$

over the set K_c and can be written in the form of the variational inequality

$$u \in K_c, \tag{5.20}$$

$$\int_{\Omega_1} \sigma_{ij}(u)\varepsilon_{ij}(v - u) \geq \int_{\Omega_1} f_i(v_i - u_i) \qquad \forall v \in K_c. \tag{5.21}$$

Here $\sigma_{ij}(u) = \sigma_{ij}$ are defined from Hooke's law (5.17). The variational inequality (5.20)–(5.21) is equivalent to (5.16)–(5.19), and conversely, i.e., any smooth solution of (5.16)–(5.19) satisfies (5.20)–(5.21), and (5.20)–(5.21) implies (5.16)–(5.19). Along with the variational formulation (5.20)–(5.21) the problem (5.16)–(5.19) admits a mixed formulation which is omitted here.

The aim of this section is to prove that the problem (5.16)–(5.19) is a limit problem for a family of problems like (5.1)–(5.5). In what follows we explain this statement.

First of all we extend the domain Ω_1 by adding a domain Ω_2 with smooth boundary Γ_2. The extended domain is denoted by Ω_c, and it has a crack (cut) Γ_c. The boundary of Ω_c is $\Gamma \cup \Gamma_c^\pm$ (see Fig. 5.4). Define $\Sigma_0 = \Gamma_1 \cap \Gamma_2$, $\Sigma = \Sigma_0 \setminus \Gamma$, thus Σ does not contain its tip points.

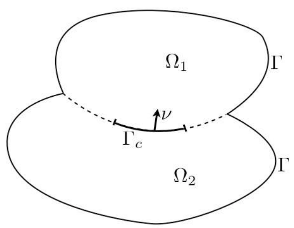

Fig. 5.4. Extended domain Ω_c

We introduce a family of elasticity tensors with a positive parameter λ,

$$a^\lambda_{ijkl} = \begin{cases} a_{ijkl} & \text{in } \Omega_1, \\ \lambda^{-1} a_{ijkl} & \text{in } \Omega_2. \end{cases}$$

Write $A^\lambda = \{a^\lambda_{ijkl}\}$, and in the extended domain Ω_c, consider a family of crack problems. Find a displacement field $u^\lambda = (u^\lambda_1, u^\lambda_2)$ and stress tensor components $\sigma^\lambda = \{\sigma^\lambda_{ij}\}$, $i, j = 1, 2$, such that

$$-\operatorname{div}\sigma^\lambda = f \qquad \text{in } \Omega_c, \tag{5.22}$$

$$\sigma^\lambda = A^\lambda \varepsilon(u^\lambda) \qquad \text{in } \Omega_c, \tag{5.23}$$

$$u^\lambda = 0 \qquad \text{on } \Gamma, \tag{5.24}$$

$$[u^\lambda]\nu \geq 0,\ [\sigma^\lambda_\nu] = 0,\ \sigma^\lambda_\nu \cdot [u]\nu = 0 \qquad \text{on } \Gamma_c, \tag{5.25}$$

$$\sigma^\lambda_\nu \leq 0,\ \sigma^\lambda_\tau = 0 \qquad \text{on } \Gamma^\pm_c. \tag{5.26}$$

As before, $[v] = v^+ - v^-$ is the jump of v through Γ_c, where $\pm$ correspond to the positive and negative crack faces $\Gamma^\pm_c$. All the other notations follow Section 5.1. We see that for any fixed $\lambda > 0$ the problem (5.22)–(5.26) describes an equilibrium state of a linear elastic body with the crack Γ_c where non-penetration conditions are prescribed. Hence, the problem (5.22)–(5.26) is exactly like (5.1)–(5.5), and we are interested in passage to the limit as $\lambda \to 0$. In particular, the problem (5.22)–(5.26) admits a variational formulation. The boundary conditions (5.25)–(5.26) are satisfied in the form as in Section 1. It can be proved (see [52]) that as $\lambda \to 0$,

$$u^\lambda \to u^0 \quad \text{strongly in } H^1_\Gamma(\Omega_c), \tag{5.27}$$

$$\frac{u^\lambda}{\sqrt{\lambda}} \to 0 \quad \text{strongly in } H^1(\Omega_2), \tag{5.28}$$

where $u^0 = u$ on Ω_1, i.e. the restriction of the limit function from (5.27) to Ω_1 coincides with the unique solution of the Signorini problem (5.16)–(5.19). From (5.27)–(5.28) it is seen that the limit function u^0 is zero in Ω_2. On the other hand, there is no limit passage for σ^λ in Ω_2 as $\lambda \to 0$. Thus, the domain Ω_2 can be understood as an undeformable body, and the stresses are not defined in Ω_2. This means that the Signorini problem is, in fact, a crack problem with non-penetration condition between crack faces, where the crack Γ_c is located between the elastic body Ω_1 and the non-deformable (rigid) body Ω_2. It is worth noting that, in fact, we can write the problem (5.22)–(5.26) in the equivalent form in the smooth domain $\Omega_c \cup \overline{\Gamma}_c$ by using the smooth domain formulation (Section 2.2). The details of the fictitious domain method in crack theory can be found in [52], [129].

5.4. Crack on the boundary of rigid inclusion. We can consider a rigid inclusion inside a rigid body. This section is concerned with a crack situated on the boundary of the rigid inclusion.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary Γ, and $\omega \subset \Omega$ be a subdomain with smooth boundary Σ and $\overline{\omega} \subset \Omega$. Assume that Σ is composed of two parts: $\Sigma = \Gamma_c \cup (\Sigma \setminus \Gamma_c)$, $\operatorname{meas}(\Sigma \setminus \Gamma_c) > 0$ (see Fig. 5.5). Write $\Omega_c = \Omega \setminus \overline{\Gamma}_c$. As before, we

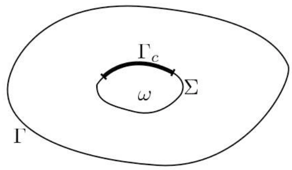

Fig. 5.5. Rigid inclusion ω in an elastic body

denote by $A = \{a_{ijkl}\}$ an elasticity tensor with the usual symmetry and positive definiteness properties, $a_{ijkl} \in L^\infty_{\mathrm{loc}}(\mathbb{R}^2)$. For a positive parameter $\lambda > 0$, introduce the elasticity tensor

$$a^\lambda_{ijkl} = \begin{cases} a_{ijkl} & \text{in } \Omega \setminus \overline{\omega}, \\ \lambda^{-1} a_{ijkl} & \text{in } \omega, \end{cases} \qquad i,j,k,l = 1,2,$$

and consider the problem of finding a displacement field $u^\lambda = (u^\lambda_1, u^\lambda_2)$ and stress tensor components $\sigma^\lambda = \{\sigma^\lambda_{ij}\}$, $i,j = 1,2$, such that

$$-\operatorname{div} \sigma^\lambda = f \qquad \text{in } \Omega_c, \tag{5.29}$$

$$\sigma^\lambda - A^\lambda \varepsilon(u^\lambda) = 0 \qquad \text{in } \Omega_c, \tag{5.30}$$

$$u^\lambda = 0 \qquad \text{on } \Gamma, \tag{5.31}$$

$$[u^\lambda]\nu \geq 0,\ [\sigma^\lambda_\nu] = 0,\ \sigma^\lambda_\nu \cdot [u^\lambda]\nu = 0 \qquad \text{on } \Gamma_c, \tag{5.32}$$

$$\sigma^\lambda_\tau = 0,\ \sigma^\lambda_\nu \leq 0 \qquad \text{on } \Gamma^\pm_c. \tag{5.33}$$

Here $f = (f_1, f_2) \in L^2(\Omega)$ is a given function. We see that for any $\lambda > 0$ the problem (5.29)–(5.33) is like (5.1)–(5.5) describing the equilibrium state for an elastic body with the crack Γ_c. This problem has a variational formulation, mixed formulation and smooth domain formulation. Our aim is to consider the limit case as $\lambda \to 0$. This can be done by analyzing the variational inequality

$$u^\lambda \in K, \tag{5.34}$$

$$\int_{\Omega_c} \sigma^\lambda_{ij}(u^\lambda)\varepsilon_{ij}(v - u^\lambda) \geq \int_{\Omega_c} f_i(v_i - u^\lambda_i) \qquad \forall v \in K. \tag{5.35}$$

Here $\sigma^\lambda_{ij}(u^\lambda) = \sigma^\lambda_{ij}$ are defined from (5.30), and the set K was introduced in (5.6).

We can pass to the limit in (5.34)–(5.35) as $\lambda \to 0$. To this end, we introduce the space of infinitesimal rigid displacements

$$R(\omega) = \{\rho = (\rho_1, \rho_2) \mid \rho(x) = Bx + D,\ x \in \omega\},$$

where

$$B = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}, \qquad D = (d^1, d^2), \qquad b, d^1, d^2 = \text{const}.$$

Consider next the space

$$H^{1,\omega}_\Gamma(\Omega_c) = \{v \in H^1_\Gamma(\Omega_c) \mid v = \rho \text{ on } \omega,\ \rho \in R(\omega)\}$$

and the set of admissible displacements

$$K_\omega = \{v \in H^{1,\omega}_\Gamma(\Omega_c) \mid (v^+ - \rho)\nu \geq 0 \text{ a.e. on } \Gamma_c\}.$$

Here v^+ corresponds to the crack faces Γ_c^+. Now we take $v = 0$, $v = 2u^\lambda$ as test functions in (5.35). This provides the relation

$$\int_{\Omega_c} \sigma^\lambda_{ij}(u^\lambda)\varepsilon_{ij}(u^\lambda) = \int_{\Omega_c} f_i u^\lambda_i,$$

which implies the estimates

$$\|u^\lambda\|_{H^1_\Gamma(\Omega_c)} \leq c_1, \qquad \frac{1}{\lambda}\int_\omega a_{ijkl}\varepsilon_{kl}(u^\lambda)\varepsilon_{ij}(u^\lambda) \leq c_2, \tag{5.36}$$

uniformly in λ, $0 < \lambda < \lambda_0$. Consequently, we can assume that as $\lambda \to 0$,

$$u^\lambda \to u \quad \text{weakly in } H^1_\Gamma(\Omega_c).$$

Moreover, by (5.36),

$$\varepsilon_{ij}(u) = 0 \quad \text{in } \omega,\ i,j = 1,2.$$

This means existence of a function ρ_0 such that

$$u = \rho_0 \quad \text{in } \omega, \qquad \rho_0 \in R(\omega).$$

Since u^λ converges to u weakly in $H^1_\Gamma(\Omega_c)$ and $u^\lambda \in K$, it follows that

$$(u^+ - \rho_0)\nu \geq 0 \text{ on } \Gamma_c.$$

In particular, $u \in K_\omega$. Now we take an arbitrary function $v \in R(\omega)$. In this case, there exists $\rho \in R(\omega)$ such that $v = \rho$ on ω. It is clear that v can be substituted in (5.35) as a test function. Since $A^\lambda = A$ in $\Omega \setminus \overline{\omega}$ we can pass to the limit as $\lambda \to 0$ in (5.34), (5.35), which provides the following variational inequality:

$$u \in K_\omega, \tag{5.37}$$

$$\int_{\Omega\setminus\overline{\omega}} \sigma_{ij}(u)\varepsilon_{ij}(v-u) \geq \int_{\Omega_c} f_i(v_i - u_i) \qquad \forall v \in K_\omega. \tag{5.38}$$

The problem (5.37)–(5.38) describes an equilibrium state of the body occupying the domain Ω_c which has the crack Γ_c and the rigid inclusion ω. The latter means that any possible displacement in ω has the form $\rho(x)$, $x \in \omega$, where $\rho \in R(\omega)$. The problem (5.37)–(5.38) can be written in differential form. This formulation is as follows. In the domain Ω_c, we have to find a displacement field $u = (u_1, u_2)$ with $u = \rho_0$ in ω, and $\rho_0 \in R(\omega)$, and in the domain $\Omega \setminus \overline{\omega}$ we have to find stress tensor components $\sigma = \{\sigma_{ij}\}$, $i,j = 1,2$, such that

$$-\operatorname{div}\sigma = f \qquad \text{in } \Omega \setminus \overline{\omega}, \tag{5.39}$$

$$\sigma - A\varepsilon(u) = 0 \qquad \text{in } \Omega \setminus \overline{\omega}, \tag{5.40}$$

$$u = 0 \qquad \text{on } \Gamma, \tag{5.41}$$

$$(u - \rho_0)\nu \geq 0,\ \sigma_\tau = 0,\ \sigma_\nu \leq 0 \qquad \text{on } \Gamma_c^+, \tag{5.42}$$

$$\sigma_\nu \cdot (u - \rho_0)\nu = 0 \qquad \text{on } \Gamma_c^+, \tag{5.43}$$

$$-\int_\Sigma \sigma\nu \cdot \rho = \int_\omega f_i\rho_i \qquad \forall \rho \in R(\omega). \tag{5.44}$$

The formulations (5.37)–(5.38) and (5.39)–(5.44) are equivalent. This means that any smooth solution of (5.39)–(5.44) satisfies (5.37)–(5.38), and conversely, (5.39)–(5.44) follows from (5.37)–(5.38).

As in the previous sections, it is possible to describe in what sense the boundary conditions (5.42)–(5.44) hold. In particular, the last two conditions of (5.42) are satisfied in the sense of $H_{00}^{-1/2}(\Gamma_c)$. As for (5.43) it holds in the form

$$\langle \sigma_\nu^+, (u-\rho_0)\nu\rangle_{1/2,00,\Gamma_c} = 0.$$

Condition (5.44) holds as follows:

$$-\langle \sigma\nu, \rho\rangle_{1/2,\Sigma} = \int_\omega f_i \rho_i \quad \forall \rho \in R(\omega).$$

To conclude this section, we note that the variational inequality (5.37)–(5.38) is equivalent to minimization of the functional

$$\frac{1}{2}\int_{\Omega\setminus\overline{\omega}} \sigma_{ij}(v)\varepsilon_{ij}(v) - \int_{\Omega_c} f_i v_i$$

over the set K_ω.

5.5. Shape derivatives of energy functionals. In crack theory, the Griffith criterion is widely used to predict a crack propagation. This criterion says that a crack propagates provided that the derivative of the energy functional with respect to the crack length reaches a critical value. In this section we discuss this question for the model (5.1)–(5.5). We also refer to [77] for some developments in the framework of finite strain elasticity and rate-independent model.

A general point of view is that we should consider a perturbed problem with respect to (5.1)–(5.5). In particular, crack length may be perturbed. Perturbation will be characterized by a small parameter t, and $t = 0$ corresponds to the unperturbed problem, i.e. to (5.1)–(5.5). To describe the perturbation properly, we should have a perturbation of the domain Ω_c. This will be done via the velocity method (see [126]). This means that we consider a given velocity field V defined in $\mathbb{R}^2$ and describe a perturbation of Ω_c by solving a Cauchy problem of a system of ODE. Namely, let $V \in W^{1,\infty}(\mathbb{R}^2)^2$ be a given field, $V = (V_1, V_2)$. Consider the Cauchy problem of finding a function $\Phi = (\Phi_1, \Phi_2)$ with

$$\frac{d\Phi}{dt}(t,\cdot) = V(\Phi(t,\cdot)) \quad \text{for } t \neq 0, \qquad \Phi(0,x) = x. \tag{5.45}$$

There exists a unique solution Φ to (5.45) such that

$$\Phi = (\Phi_1, \Phi_2)(t,x) \in C^1([0,t_0]; W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^2)^2), \qquad |t_0| > 0. \tag{5.46}$$

Simultaneously, we can find a solution $\Psi = (\Psi_1, \Psi_2)$ to the Cauchy problem

$$\frac{d\Psi}{dt}(t,\cdot) = -V(\Psi(t,\cdot)) \quad \text{for } t \neq 0, \qquad \Psi(0,y) = y, \tag{5.47}$$

with the same regularity

$$\Psi = (\Psi_1, \Psi_2)(t,y) \in C^1([0,t_0]; W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^2)^2), \qquad |t_0| > 0. \tag{5.48}$$

It can be proved that for any fixed t, the function $\Psi(t,\cdot)$ is inverse to $\Phi(t,\cdot)$, which means (see the proof in [68])

$$y = \Phi(t, \Psi(t,y)), \qquad x \in \Psi(t, \Phi(t,x)), \qquad x, y \in \mathbb{R}^2.$$

Due to this, we have a one-to-one mapping between the domain Ω_c and a perturbed domain Ω_c^t, namely

$$y = \Phi(t,x) : \Omega_c \to \Omega_c^t, \qquad x = \Psi(t,y) : \Omega_c^t \to \Omega_c.$$

Moreover, by (5.46), (5.48), we have the following asymptotic expansions (I denotes the identity operator):

$$\begin{aligned} \Phi(t,x) &= x + tV(x) + r_1(t), \\ \Psi(t,y) &= y - tV(y) + r_2(t), \\ \frac{\partial \Phi(t)}{\partial x} &= I + t\frac{\partial V}{\partial x} + r_3(t), \\ \frac{\partial \Psi(t)}{\partial y} &= I - t\frac{\partial V}{\partial y} + r_4(t), \\ \|r_i(t)\|_{W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^2)^2} &= o(t), \quad i = 1,2, \\ \|r_i(t)\|_{L^{\infty}_{\mathrm{loc}}(\mathbb{R}^2)^{2\times 2}} &= o(t), \quad i = 3,4. \end{aligned} \tag{5.49}$$

Hence, in the domain Ω_c^t it is possible to consider the following boundary value problem (perturbed with respect to (5.1)–(5.5)): Find a displacement field $u^t = (u_1^t, u_2^t)$ and stress tensor components $\sigma^t = \{\sigma_{ij}^t\}$, $i,j = 1,2$, such that

$$-\operatorname{div} \sigma^t = f \quad \text{in } \Omega_c^t, \tag{5.50}$$

$$\sigma^t = A\varepsilon(u^t) \quad \text{in } \Omega_c^t, \tag{5.51}$$

$$u^t = 0 \quad \text{on } \Gamma^t, \tag{5.52}$$

$$[u^t]\nu^t \ge 0,\ [\sigma_{\nu^t}^t] = 0,\ \sigma_{\nu^t}^t \cdot [u^t]\nu^t = 0 \quad \text{on } \Gamma_c^t, \tag{5.53}$$

$$\sigma_{\nu^t}^t \le 0,\ \sigma_{\tau^t}^t = 0 \quad \text{on } \Gamma_c^{t\pm}. \tag{5.54}$$

Here

$$y = \Phi(t,x) : \Gamma \to \Gamma^t,\ \Gamma_c \to \Gamma_c^t,$$

and we assume in this section that $f = (f_1, f_2) \in C^1(\mathbb{R}^2)$ and that $a_{ijkl} = \text{const}$, $i,j,k,l = 1,2$. All the other notations in (5.50)–(5.54) remain those of (5.1)–(5.5), in particular, $\nu^t = (\nu_1^t, \nu_2^t)$ is a unit normal vector to Γ_c^t.

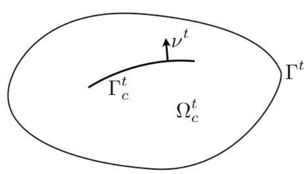

Fig. 5.6. Perturbed domain Ω_c^t

We can provide a variational formulation of the problem (5.50)–(5.54). Indeed, introduce the Sobolev space

$$H^1_{\Gamma^t}(\Omega_c^t) = \{v = (v_1, v_2) \mid v_i \in H^1(\Omega_c^t),\ v_i = 0 \text{ on } \Gamma^t,\ i = 1, 2\}$$

and the set of admissible displacements

$$K^t = \{v \in H^1_{\Gamma^t}(\Omega_c^t) \mid [v]\nu^t \geq 0 \text{ a.e. on } \Gamma_c^t\}.$$

Consider the functional

$$\Pi(\Omega_c^t; v) = \frac{1}{2}\int_{\Omega_c^t} \sigma_{ij}^t(v)\varepsilon_{ij}(v) - \int_{\Omega_c^t} f_i v_i$$

and the minimization problem

$$\min_{v \in K^t} \Pi(\Omega_c^t; v). \tag{5.55}$$

Here $\sigma_{ij}^t(v)$ are defined from Hooke's law similar to (5.51). A solution of the problem (5.55) exists and it satisfies the variational inequality

$$u^t \in K^t, \tag{5.56}$$

$$\int_{\Omega_c^t} \sigma_{ij}^t(u^t)\varepsilon_{ij}(v - u^t) \geq \int_{\Omega_c^t} f_i(v_i - u_i^t) \quad \forall v \in K^t. \tag{5.57}$$

Having found a solution of the problem (5.56)–(5.57) we can define the energy functional

$$\Pi(\Omega_c^t; u^t) = \frac{1}{2}\int_{\Omega_c^t} \sigma_{ij}^t(u^t)\varepsilon_{ij}(u^t) - \int_{\Omega_c^t} f_i u_i^t.$$

Note that for $t = 0$, we have $\Omega_c^0 = \Omega_c$ and $u^0 = u$, where u is the solution of the unperturbed problem (5.7), (5.8). The question is whether it is possible to differentiate the functional $\Pi(\Omega_c^t; u^t)$ with respect to t. We have in mind existence of the following derivative:

$$\left.\frac{d}{dt}\Pi(\Omega_c^t; u^t)\right|_{t=0} = \lim_{t \to 0} \frac{\Pi(\Omega_c^t; u^t) - \Pi(\Omega_c; u)}{t}.$$

The answer is positive in many practical situations. We consider two cases, where the derivative

$$I = \left.\frac{d}{dt}\Pi(\Omega_c^t; u^t)\right|_{t=0} \tag{5.58}$$

exists.

a) Assume that the normal vector ν to Γ_c keeps its value under the mapping $x \mapsto \Phi(t, x)$, i.e. $\nu^t = \nu$. In this case, a formula for I can be obtained (see [69], [79], [84]):

$$I = \frac{1}{2}\int_{\Omega_c} \{\operatorname{div} V \cdot \varepsilon_{ij}(u) - 2E_{ij}(V; u)\}\sigma_{ij}(u) - \int_{\Omega_c} \operatorname{div}(V f_i)u_i, \tag{5.59}$$

where

$$E_{ij}(U; v) = \frac{1}{2}(v_{i,k}U_{k,j} + v_{j,k}U_{k,i}), \qquad U = \{U_{ij}\}, \qquad i, j = 1, 2;$$

Note that the assumption concerning the normal vector ν holds for rectilinear cracks Γ_c and vector fields V tangential to Γ_c (see Fig. 5.7). In this situation, (5.59) can provide a formula for the derivative of the energy functional with respect to the crack length, which is practically needed to use the Griffith criterion. It will be the case when

$V = 1$ in a vicinity of the right crack tip and $\operatorname{supp} V$ lies in a small neighbourhood of this tip (see Fig. 5.7).

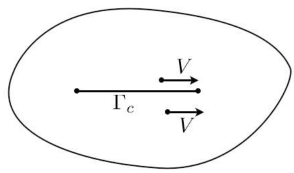

Fig. 5.7. Rectilinear crack Γ_c and tangential field V

b) A formula for the derivative (5.58) can also be derived for curvilinear cracks when the above assumption on the normal vector ν is not satisfied. We provide here the formula (5.58) when the crack Γ_c is described as a graph of a smooth function.

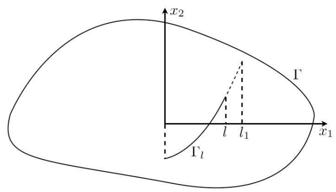

Fig. 5.8. Domain Ω_l with a crack Γ_l

Let $\psi \in H^3(0, l_1)$ be a given function, $l_1 > 0$, and

$$\Sigma = \{(x_1, x_2) \mid x_2 = \psi(x_1),\ 0 < x_1 < l_1\}.$$

Consider a crack $\Gamma_l \subset \Sigma$ as the graph of the function ψ (see Fig. 5.8),

$$\Gamma_l = \{(x_1, x_2) \mid x_2 = \psi(x_1),\ 0 < x_1 < l\}, \qquad 0 < l < l_1.$$

Here l is a parameter that characterizes the length of the projection of the crack Γ_l onto the x_1 axis. Consider a smooth cut-off function θ with support in the vicinity of the crack tip $(l, \psi(l))$; moreover we assume that $\theta = 1$ in a small neighbourhood of $(l, \psi(l))$. We can consider a perturbation of the crack Γ_l along Σ via a small parameter t. Define $\Omega_l = \Omega \setminus \overline{\Gamma}_l$. The perturbed crack Γ_l^t has a tip $(l + t, \psi(l + t))$, and we consider the perturbed domain $\Omega_l^t = \Omega \setminus \overline{\Gamma}_l^t$. It is possible to establish a one-to-one correspondence between Ω_l and Ω_l^t by the formulas

$$\begin{aligned} y_1 &= x_1 + t\theta(x), \\ y_2 &= x_2 + \psi(x_1 + t\theta(x)) - \psi(x_1), \end{aligned} \qquad (x_1, x_2) \in \Omega_l,\ (y_1, y_2) \in \Omega_l^t. \tag{5.60}$$

Transformation (5.60) is equivalent to the following (cf. (5.49)):

$$y = x + tV(x) + r(t, x)$$

with the velocity field

$$V(x) = (\theta(x), \psi'(x_1)\theta(x)). \tag{5.61}$$

In the domain Ω_l^t, we can consider a perturbed problem formulation: find a displacement field $u^t = (u_1^t, u_2^t)$ and stress tensor components $\sigma^t = \{\sigma_{ij}^t\}$, $i, j = 1, 2$, such that

$$-\operatorname{div}\sigma^t = f \quad \text{in } \Omega_l^t, \tag{5.62}$$

$$\sigma^t = A\varepsilon(u^t) \quad \text{in } \Omega_l^t, \tag{5.63}$$

$$u^t = 0 \quad \text{on } \Gamma, \tag{5.64}$$

$$[u^t]\nu^t \geq 0,\ [\sigma_{\nu^t}^t] = 0,\ \sigma_{\nu^t}^t \cdot [u^t]\nu^t = 0 \quad \text{on } \Gamma_l^t, \tag{5.65}$$

$$\sigma_{\nu^t}^t \leq 0,\ \sigma_{\tau^t}^t = 0 \quad \text{on } \Gamma_l^{t\pm}. \tag{5.66}$$

Here $\nu^t = (\nu_1^t, \nu_2^t)$ is a unit normal vector to Γ_l^t. For a solution u^t of (5.62)–(5.66) it is possible to define the energy functional

$$\Pi(\Omega_l^t; u^t) = \frac{1}{2}\int_{\Omega_l^t} \sigma_{ij}^t(u^t)\varepsilon_{ij}(u^t) - \int_{\Omega_l^t} f_i u_i^t$$

and to find the derivative

$$\Pi'(l) = \left.\frac{d\Pi(\Omega_l^t; u^t)}{dt}\right|_{t=0}$$

with the formula (see [115])

$$\begin{aligned}\Pi'(l) = \frac{1}{2}\int_{\Omega_l} &\{\operatorname{div} V \cdot \varepsilon_{ij}(u) - 2E_{ij}(V; u)\}\sigma_{ij}(u) \\ &- \int_{\Omega_l} \operatorname{div}(V f_i)u_i + \int_{\Omega_l} \sigma_{ij}(u)\varepsilon_{ij}(w) - \int_{\Omega_l} f_i w_i,\end{aligned} \tag{5.67}$$

where the vector field V is defined in (5.61) and $w = (0, \theta\psi'' u_1)$ is a given function. Note that the formula (5.67) contains the function θ, but in fact there is no dependence of the right-hand side of (5.67) on θ. In particular, if $\psi'' = 0$, the formula (5.67) reduces to (5.59) with $\Omega_c = \Omega_l$. In this case we have a rectilinear crack and $\nu^t = \nu$. Formula (5.67) defines the derivative of the energy functional with respect to the length of the projection of the crack Γ_l onto the x_1 axis. Hence, the derivative of the energy functional with respect to the length of the curvilinear crack is

$$\Pi'(s) = \Pi'(l)(\psi'(l)^2 + 1)^{-1/2},$$

where

$$s = \int_0^l \sqrt{\psi'(t)^2 + 1}$$

is the length of the crack Γ_l.

To conclude this section we briefly discuss the existence of so-called invariant integrals in crack theory. It turns out that the formula (5.59) for the derivative of the energy functional can be rewritten as an integral over a closed curve surrounding the crack tip.

Consider the simplest case of a rectilinear crack $\Gamma_c = (0, 1) \times \{0\}$ assuming that $\overline{\Gamma}_c \subset \Omega$ (see Fig. 5.9). Let θ be a smooth cut-off function equal to 1 near the point $(1, 0)$, with $\operatorname{supp}\theta$ in a small neighbourhood of the point $(1, 0)$. Then we can take the vector field $V = (\theta, 0)$ in (5.45), (5.47) which, according to (5.49), corresponds to the following change of independent variables:

$$y_1 = x_1 + t\theta(x) + r_{11}(t), \qquad y_2 = x_2.$$

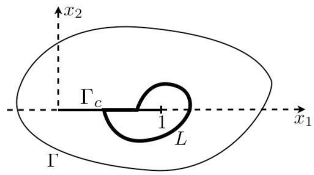

Fig. 5.9. Curve L surrounding a crack tip

In this case the formula (5.59) (or the formula (5.67) in the particular case $\psi = 0$) provides the derivative of the energy functional with respect to the crack length. This formula can be rewritten as an integral over a curve L surrounding the crack tip $(1, 0)$ (see Fig. 5.9, the solid line). Namely, the following formula is valid (see [74], [79]):

$$I = \int_L \left\{ \frac{1}{2}\nu_1 \sigma_{ij}(u)\varepsilon_{ij}(u) - \sigma_{ij}(u)u_{i,1}\nu_j \right\} \tag{5.68}$$

provided that f is equal to zero in a neighbourhood of the point $(1, 0)$. We should underline two important points. First, the formula (5.68) is independent of L, and second, the right-hand side of (5.68) is equal to the derivative of the energy functional with respect to the crack length.

In fact, invariant integrals like (5.68) can be obtained in more complex situations. For example, we can assume that the crack Γ_c is situated on the interface between two media, which means that the elasticity tensor $A = \{a_{ijkl}\}$ is as follows (see Fig. 5.9):

$$a_{ijkl} = \begin{cases} a^1_{ijkl} & \text{for } x_2 > 0, \\ a^2_{ijkl} & \text{for } x_2 < 0. \end{cases}$$

Here $a^1_{ijkl} = \text{const}$, $a^2_{ijkl} = const$, $i, j, k, l = 1, 2$, and $\{a^1_{ijkl}\}$, $\{a^2_{ijkl}\}$ have the usual properties of symmetry and positive definiteness. In this case, formula (5.59) for the derivative of the energy functional holds true provided that V is tangent to Γ_c. This formula provides existence of an invariant integral of the form (5.68). We should remark at this point that when the integral (5.68) is calculated, the values $\sigma_{ij}(u)u_{i,1}\nu_j$ can be taken at Γ_c^+ or at Γ_c^-. This gives the same value of the integral (5.68). This statement holds due to the equality (see [66])

$$[\sigma_{ij}(u)u_{i,1}\nu_j] = 0 \quad \text{on } \Gamma_c.$$

On the other hand, we can analyze the case when the rigidity of the elastic body part $\Omega_c \cap \{x_2 < 0\}$ goes to infinity. Indeed, consider the following elasticity tensor for a positive parameter $\lambda > 0$:

$$a^\lambda_{ijkl} = \begin{cases} a^1_{ijkl} & \text{for } x_2 > 0, \\ \lambda^{-1} a^2_{ijkl} & \text{for } x_2 < 0. \end{cases}$$

Then for any fixed $\lambda > 0$, the solution of the equilibrium problem like (5.1)–(5.5) exists, and a passage to the limit as $\lambda \to 0$ can be made. As already noted in Section 3, in the limit the following contact Signorini problem is obtained: Find a displacement field

$u = (u_1, u_2)$ and stress tensor components $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, such that

$$-\operatorname{div}\sigma = f \qquad \text{in } \Omega_c \cap \{x_2 > 0\}, \tag{5.69}$$

$$\sigma = A\varepsilon(u) \qquad \text{in } \Omega_c \cap \{x_2 > 0\}, \tag{5.70}$$

$$u = 0 \qquad \text{on } \partial(\Omega_c \cap \{x_2 > 0\}) \setminus \Gamma_c, \tag{5.71}$$

$$u\nu \geq 0,\ \sigma_\nu \leq 0,\ \sigma_\tau = 0,\ \sigma_\nu \cdot u\nu = 0 \qquad \text{on } \Gamma_c. \tag{5.72}$$

For the problem (5.69)–(5.72) it is possible to differentiate the energy functional in the direction of the vector field $V = (\theta, 0)$, where the properties of θ are described above. The formula for the derivative has the following form (cf. (5.59)):

$$I = \frac{1}{2}\int_{\Omega_1} \{\operatorname{div} V \cdot \sigma_{ij}(u) - 2E_{ij}(V, u)\}\sigma_{ij}(u) - \int_{\Omega_1} \operatorname{div}(V f_i) u_i. \tag{5.73}$$

Assume that $f = 0$ in a neighbourhood of the point $(1, 0)$. In this case, formula (5.73) can be rewritten in the form of invariant integral

$$I = \int_{L_1} \left\{ \frac{1}{2}\nu_1 \sigma_{ij}(u)\varepsilon_{ij}(u) - \sigma_{ij}(u) u_{i,1}\nu_j \right\}, \tag{5.74}$$

where L_1 is a smooth curve "covering" the point $(1, 0)$ (see Fig. 5.10, solid line). Just as for invariant integrals in crack problems, formula (5.74) is independent of the choice of L_1.

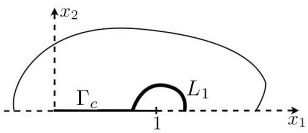

Fig. 5.10. Curve L_1 "covering" a tip of contact set

5.6. Evolution of a kinking crack. The problem of kink is of special interest, because it represents a change of topology from a smooth crack to a non-smooth one. The topology change is the main difficulty of mathematical analysis of cracks with a kink. In this section we apply the shape optimization approach to a two-parameter problem for kinking crack. Namely, we fix a point of kink and find unknown shape parameters of kink angle and crack length, which minimize the total potential energy due to the Griffith approach. This non-linear minimization problem describes evolution of the kinking crack with respect to the time-like loading parameter. In the linear crack theory, the optimization Griffith approach was used in [41].

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary Γ. Assuming that the origin $\mathcal{O} = (0, 0)$ belongs to $\overline{\Omega}$, we consider a given crack $\Gamma_0 \subset \Omega$ with tips at Γ and at the origin, and an unknown part $C_{(r,\phi)}$ of the crack, whose tip is described in polar coordinates as

$$(r\cos\phi, r\sin\phi), \qquad (r, \phi) \in \overline{\omega},$$

where ω is the set of admissible parameters

$$\omega = \{(r, \phi) \mid 0 < r < R(\phi) \text{ for } \phi \in (\phi_0, \phi_1)\}, \qquad [\phi_0, \phi_1] \subset (-\pi, \pi),$$

with a given periodic function $R \in W^{2,\infty}(-\pi, \pi)$.

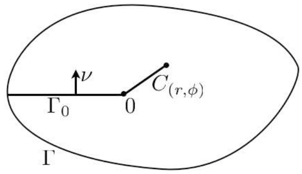

Fig. 5.11. Kinking crack

Admissible kinking cracks are defined as a union $\Gamma_{(r,\phi)} = \Gamma_0 \cup C_{(r,\phi)}$. Denote by $\Omega_{(r,\phi)}$ a domain with a crack $\Gamma_{(r,\phi)}$, i.e. $\Omega_{(r,\phi)} = \Omega \setminus \overline{\Gamma}_{(r,\phi)}$ (see Fig. 5.11). In the domain $\Omega_{(r,\phi)}$ we can consider an equilibrium problem like (5.1)–(5.5). Namely, let ν be a normal vector to $\Gamma_{(r,\phi)}$ and $f = (f_1, f_2) \in C^1(\overline{\Omega})$ be a given function. The problem formulation is as follows. In the domain $\Omega_{(r,\phi)}$ we have to find a displacement vector $u = (u_1, u_2)$ and stress tensor components $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, such that

$$-\operatorname{div}\sigma = f \quad \text{in } \Omega_{(r,\phi)}, \tag{5.75}$$

$$\sigma = A\varepsilon(u) \quad \text{in } \Omega_{(r,\phi)}, \tag{5.76}$$

$$u = 0 \quad \text{on } \Gamma, \tag{5.77}$$

$$[u]\nu \geq 0,\ [\sigma_\nu] = 0,\ \sigma_\nu \cdot [u]\nu = 0 \quad \text{on } \Gamma_{(r,\phi)}, \tag{5.78}$$

$$\sigma_\nu \leq 0,\ \sigma_\tau = 0 \quad \text{on } \Gamma^{\pm}_{(r,\phi)}. \tag{5.79}$$

For any given $(r, \phi) \in \overline{\omega}$, a solution to the problem (5.75)–(5.79) exists in the Sobolev space $H^1_\Gamma(\Omega_{(r,\phi)})$. Hence, for any $(r, \phi) \in \overline{\omega}$ we can define a solution $u^{(r,\phi)}$ and the energy functional

$$\Pi(\Omega_{(r,\phi)}; u^{(r,\phi)}) = \frac{1}{2}\int_{\Omega_{(r,\phi)}} \sigma_{ij}(u^{(r,\phi)})\varepsilon_{ij}(u^{(r,\phi)}) - \int_{\Omega_{(r,\phi)}} f_i u_i^{(r,\phi)},$$

where $\sigma_{ij}(u^{(r,\phi)}) = \sigma_{ij}$ are found from (5.76). Thus, differentiability of the energy functional with respect to (r, ϕ) can be analyzed. These results can be found in [68]. The main difficulty in studying the differentiability is the following. Considering perturbations of the problem (5.75)–(5.79), we have no one-to-one correspondence between sets of admissible displacements for perturbed and unperturbed problems. This requires additional considerations to prove differentiability of $\Pi(\Omega_{(r,\phi)}; u^{(r,\phi)})$ with respect to r, ϕ.

In what follows, we formulate an evolution problem for a kinking crack. Set

$$P(r, \phi) = \Pi(\Omega_{(r,\phi)}; u^{(r,\phi)}).$$

For a time-like loading parameter $t \geq 0$ we consider a family of forces tf in (5.75). Let the length of the crack Γ_0 be equal to $l_0 \geq 0$. Note that if the solution $u^{(r,\phi)}$ corresponds to the force f in (5.75), we obtain a solution $tu^{(r,\phi)}$ for the force tf due to homogeneity of the problem (5.75)–(5.79). Let the initial crack (at $t = 0$) be given as Γ_0. For the loading tf, we look for a propagating crack $\Gamma_{(r(t),\phi^*)} \subset \Omega$ with kink at the origin $\mathcal{O}$ and unknown shape parameters of crack length $l_0 + r(t)$ and kink angle $\phi^* \in [\phi_0, \phi_1]$. To this end, we use a shape optimization approach, which is based on the Griffith hypothesis. Following this hypothesis, we define a function of total potential energy

$$T(r,\phi)(t) = 2\gamma(l_0 + r) + t^2 P(r,\phi), \qquad (r,\phi) \in \overline{\omega}. \tag{5.80}$$

The first term in (5.80) represents the surface energy distributed uniformly at two crack faces with a constant density $\gamma > 0$ (the given material parameter). The second term in (5.80) represents the potential energy which is quadratic in t,

$$P(r,\phi)(t) = \Pi(\Omega_{(r,\phi)}; tu^{(r,\phi)}) = t^2 P(r,\phi).$$

Thus we arrive at the problem of evolution of a kinking crack:

$$r(0) = 0; \tag{5.81}$$

for $t > 0$, find parameters $(r(t), \phi(t)) \in \overline{\omega}$ that

$$\text{minimize} \quad T(r,\phi)(t) \quad \text{over } (r,\phi) \in \overline{\omega}, \tag{5.82}$$

$$\text{subject to} \quad \phi \in \bigcap_{s<t} \{\phi(s)\}. \tag{5.83}$$

The constraint (5.83) allows us to preserve the shape of the kinking crack during its evolution. This means that if the kinking angle ϕ^* is found, its value is preserved during the evolution. Problem (5.81)–(5.83) has a solution (see [68]). It turns out that the radius $r(t)$ during the evolution may be multi-valued, i.e. $r(t) \in [r^-(t), r^+(t)]$, which means a non-stable crack evolution. To conclude this section we refer the reader to the paper [21], where a smooth deviation problem for a crack was analyzed in the framework of linear crack models.

5.7. 3D problems and open questions. Most of the problems discussed in this chapter can be solved in 3D when the crack is represented as a 2D smooth surface. For example, it can be described as

$$x_i = x_i(y_1, y_2), \qquad i = 1, 2, 3,$$

where $(y_1, y_2) \in D$, $D \subset \mathbb{R}^2$ is a bounded domain with smooth boundary, and the mapping $y \mapsto x$ is non-degenerate.

All formulas and statements of Sections 1–5 hold true with suitable specifications. In particular, when discussing the boundary conditions (5.4)–(5.5) we should introduce the Hilbert space $H^{1/2}(\Sigma)$, where Σ is an extension of Γ_c to a closed 2D smooth surface. The norm in $H^{1/2}(\Sigma)$ in this case is defined as follows (cf. (5.9)):

$$\|v\|^2_{H^{1/2}(\Sigma)} = \|v\|^2_{L^2(\Sigma)} + \int_\Sigma \int_\Sigma \frac{|v(x) - v(y)|^2}{|x-y|^3}\, dx\, dy.$$

Mixed and smooth domain formulations in 3D case hold true, as does the fictitious domain method.

Also, we can consider a crack located on the boundary of a rigid inclusion for a 3D elastic body and prove all statements of Section 5. Notice that in 3D the space of infinitesimal rigid inclusions is defined as follows:

$$R(\omega) = \{\rho = (\rho_1, \rho_2, \rho_3) \mid \rho(x) = Bx + D,\ x \in \omega\},$$

where

$$B = \begin{pmatrix} 0 & b_{12} & b_{13} \\ -b_{21} & 0 & b_{23} \\ -b_{13} & -b_{23} & 0 \end{pmatrix}, \qquad D = (d^1,\ d^2,\ d^3),$$

$$b_{ij}, d^i = \text{const}, \qquad i,j = 1,2,3.$$

As for differentiation of energy functionals with respect to a perturbed parameter (Section 6), we have a large variety of perturbations in the 3D case. The simplest ones provide a perturbation of the crack front. For example, let

$$\Gamma_c = \{(x_1, x_2, 0) \mid 0 \le x_1 \le \phi(x_2),\ x_2 \in [-1,1],\ \phi(x_2) > 0\},$$

with a given smooth function ϕ. In this case, the 3D vector field can be

$$V(x) = (\theta(x), 0, 0),$$

where θ is a given smooth function with support in the vicinity of the crack front

$$\{(x_1, x_2, x_3) \mid x_1 = \phi(x_2),\ x_3 = 0,\ x_2 \in [-1,1]\}.$$

This allows us to differentiate the energy functional in the direction of the field V, which implies the formula (5.58) with $i,j = 1,2,3$; see [69], [79].

As in Section 6, in 3D we can consider curvilinear cracks described as the graph of a function

$$x_3 = \psi(x_1, x_2), \qquad (x_1, x_2) \in D,$$

where $D \subset \mathbb{R}^2$ is a bounded domain with smooth boundary. The formulas for derivatives of the energy functional in this case can be found in [116].

As for invariant integrals, in 3D we should integrate over closed 2D surfaces surrounding a crack front (see [66], [74]).

To conclude the chapter, we formulate some open questions:

1. When a crack crosses the external boundary Γ at a zero angle, the problem (5.1)–(5.5) is not solvable in the general case (since Korn's inequality is not valid). Is it possible to overcome this difficulty?
2. Particular invariant integrals in the case of curvilinear cracks are presented in [129]. Is it possible to construct any more?
3. Is it possible to prove uniqueness of solution for the problem (5.80)–(5.82)?

6. Smooth domain method for crack problems

In the so-called smooth domain method of modeling of problems with cracks, the geometrical singularity in the form of a cut in the reference domain is replaced by pointwise constraints in the function space of admissible functions for the model under consideration. This leads, in particular, to efficient numerical methods; we refer to [13] for related results on convergence of numerical methods for the smooth domain method.

6.1. Introduction. A new approach to crack theory for linear elastic bodies with inequality type boundary conditions prescribed on the crack faces was proposed in [76]. The results of this method are summarized in this chapter. This mathematical model allows us to solve the crack problem in a smooth domain. The problem under consideration is characterized by non-linear boundary conditions imposed on non-smooth parts of the boundary [67]. These conditions describe the mutual non-penetration between the crack faces.

It is well known that for a linear elastic body the frictionless contact problem is variational and can be formulated as the minimization of the energy functional over the set of admissible displacements. Such an admissible set contains all displacement fields from a suitable function space, usually a Sobolev space, satisfying the unilateral non-penetration condition on the crack faces. The boundary conditions for stresses on the crack faces follow directly from the variational formulation. In particular, the normal stresses are non-positive and the tangential stresses vanish.

A different setting is proposed for the contact problem, with some inequality type conditions for admissible stress fields on crack faces. For such a setting, the non-penetration conditions for the displacement field follow from the variational formulation and can be derived from the model, i.e., from the equations and the inequalities which form the mathematical model. This is the so called mixed problem formulation. For domains with smooth boundaries and classical boundary conditions mixed problem formulations are analyzed in the book [18]. The peculiarity of the problem analyzed here is that the boundary conditions imposed on non-smooth parts of the boundary are unilateral type relations. It turns out that the setting proposed in this paper is useful for the modeling and analysis of crack problems in smooth domains and results in a smooth domain method for solving the crack models with non-penetration conditions on the boundary. In this case, restrictions imposed on the stress tensor components are considered to be internal restrictions, i.e. to be relations prescribed on given subsets of the smooth domain. In fact, we extend the unknown functions to the crack surface and find the solution in the smooth domain. Note that the problem analyzed in this paper is a free boundary problem. In particular, a specific boundary condition at a given point can be found after the problem is solved. The boundary conditions provide a possibility of contact between crack faces. Notice that the classical crack problem is characterized by equality type boundary conditions on the crack faces; we refer the reader to [25]–[36], [85], [90]–[101]. For the crack theory with a possible contact between crack faces for different constitutive laws, some results can be found in [67]. We should remark that the smooth domain method can be applied to the classical linear crack problems as well as to many other linear and non-linear elliptic boundary value problems.

Throughout this chapter we shall use the following notations for geometrical domains (see Figures 6.1 and 6.2). Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary Γ, and $\Gamma_c \subset \Omega$ be a smooth curve without self-intersections.

We assume that Γ_c can be extended to a closed curve Σ without self-intersections of class $C^{1,1}$ so that $\Sigma \subset \Omega$, and the domain Ω is divided into two subdomains Ω_1, Ω_2. In this case Σ is the boundary of Ω_1, and the boundary of Ω_2 is $\Sigma \cup \Gamma$.

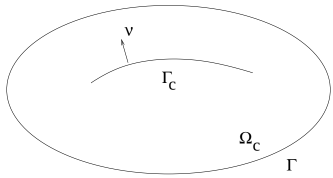

Fig. 6.1. Domain with a crack

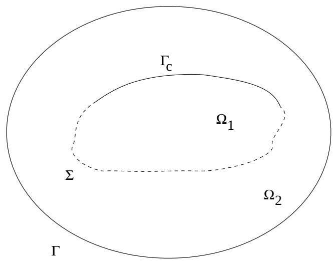

Fig. 6.2. Extension of the crack

Assume that Γ_c does not contain the tip points, i.e., $\Gamma_c = \bar{\Gamma}_c \setminus \partial\Gamma_c$. Denote by $n = (n_1, n_2)$ the unit external normal vector to Γ and by $\nu = (\nu_1, \nu_2)$ a unit normal vector to Σ and therefore to Γ_c. Let $\Omega_c = \Omega \setminus \bar{\Gamma}_c$. In applications Γ_c defines a crack in an elastic body in the reference domain configuration.

To demonstrate the idea of the smooth domain method a simple example for the Poisson equation is discussed. We prescribe the sign of the jump of a displacement on Γ_c for an elastic membrane, i.e. $[u] = u^+ - u^- \geq 0$. The following free boundary problem is considered in Ω_c (see [67], [72]).

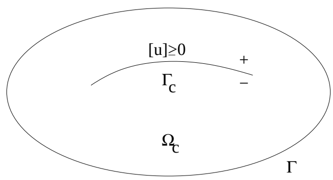

Fig. 6.3. Elastic membrane

Find a function u such that

$$-\Delta u = f \qquad \text{in } \Omega_c, \tag{6.1}$$

$$u = 0 \qquad \text{on } \Gamma, \tag{6.2}$$

$$[u] \geq 0, \ \left[\frac{\partial u}{\partial \nu}\right] = 0, \ [u] \cdot \frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \Gamma_c, \tag{6.3}$$

$$\frac{\partial u}{\partial \nu} \leq 0 \qquad \text{on } \Gamma_c^{\pm}. \tag{6.4}$$

It is clear that there exists a unique weak solution to the problem (6.1)–(6.4) which can be formulated as minimization of the energy functional

$$\frac{1}{2}\int_{\Omega_c}|\nabla v|^2-\int_{\Omega_c}fv$$

over the convex set in the Sobolev space $H^1(\Omega_c)$ with unilateral condition $[v]\geq 0$ on Γ_c and the condition $v=0$ on Γ. For such a problem we can introduce the following smooth domain formulation. In the domain Ω we have to find functions $u, p=(p_1,p_2)$ such that

$$u\in L^2(\Omega),\qquad p\in M, \tag{6.5}$$

$$-\operatorname{div}p=f\qquad\text{in }\Omega, \tag{6.6}$$

$$\int_\Omega p(\bar p-p)+\int_\Omega u(\operatorname{div}\bar p-\operatorname{div}p)\geq 0\qquad\forall\bar p\in M, \tag{6.7}$$

where

$$M=\{p=(p_1,p_2)\in L^2(\Omega)\mid\operatorname{div}p\in L^2(\Omega),\ p\nu\leq 0\text{ on }\Gamma_c\}.$$

The formulations (6.1)–(6.4) and (6.5)–(6.7) are equivalent. The advantage of (6.5)–(6.7) is that the solution is defined in the smooth domain Ω.

PROPOSITION 6.1. *There exists a unique solution to the problem* (6.5)–(6.7).

The proof is similar to the proofs of Theorem 6.8 and Theorem 6.12 below in the more complicated setting of elasticity problems.

6.1.1. Main results. We present two results which are proved in [76]. The smooth domain method is applied to the two-dimensional elasticity and the Kirchhoff plate model. As we can see from Theorem 6.8 and Theorem 6.12 below the variational formulation of the crack contact problem is obtained in the smooth domain Ω. Therefore, from a numerical point of view the discretization is required in the domain Ω, but the restriction imposed on the solution is considered on the curve Γ_c inside Ω. This means that unknown functions are defined in the smooth domain Ω and should satisfy some inequality type constraints. In the last two sections we will present some results for three-dimensional models.

Two-dimensional elasticity. The boundary value problem for frictionless contact on crack faces in two-dimensional elasticity is given in (6.25)–(6.29) below. The unilateral conditions (6.28)–(6.29) are imposed on Γ_c and $\Gamma_c^{\pm}$. The smooth domain formulation of this problem is considered in the smooth domain $\Omega=\Omega_c\cup\bar\Gamma_c$. It takes the following form.

Find $u=(u_1,u_2)$, $\sigma=\{\sigma_{ij}\}$, $i,j=1,2$, such that

$$u\in L^2(\Omega),\qquad\sigma\in N, \tag{6.8}$$

$$-\operatorname{div}\sigma=f\qquad\text{in }\Omega, \tag{6.9}$$

$$(C\sigma,\bar\sigma-\sigma)_\Omega+(u,\operatorname{div}\bar\sigma-\operatorname{div}\sigma)_\Omega\geq 0\qquad\forall\bar\sigma\in N, \tag{6.10}$$

where

$$N=\{\sigma\in H\mid\sigma_\tau=0,\ \sigma_\nu\leq 0\text{ on }\Gamma_c\},\qquad H=\{\sigma=\{\sigma_{ij}\}\mid\sigma,\operatorname{div}\sigma\in L^2(\Omega)\}.$$

Here σ_ν are normal stresses, and σ_τ are tangential forces; $(\cdot,\cdot)_\Omega$ is the scalar product in $L^2(\Omega)$. We prove the following statement.

THEOREM 6.2. *There exists a unique solution to the problem* (6.8)–(6.10).

The proof is given in Section 6.2.

Kirchhoff plate. The boundary value problem for the Kirchhoff plate with an inequality type boundary condition imposed on Γ_c is given in (6.59)–(6.66) below. The smooth domain formulation for this problem is as follows.

We have to find functions u, w, σ, m such that

$$u = (u_1, u_2) \in L^2(\Omega), \qquad w \in L^2(\Omega), \qquad (\sigma, m) \in \mathcal{N}, \tag{6.11}$$

$$-\operatorname{div} \sigma = f \qquad \text{in } \Omega, \tag{6.12}$$

$$-\nabla\nabla m = F \qquad \text{in } \Omega, \tag{6.13}$$

$$(u, \operatorname{div} \bar{\sigma} - \operatorname{div} \sigma)_\Omega + (w, \nabla\nabla \bar{m} - \nabla\nabla m)_\Omega + (C\sigma, \bar{\sigma} - \sigma)_\Omega + (Dm, \bar{m} - m)_\Omega \geq 0 \quad \forall (\bar{\sigma}, \bar{m}) \in \mathcal{N}, \tag{6.14}$$

where

$$\mathcal{N} = \{(\sigma, m) \in \mathcal{H} \mid \sigma_\tau = 0,\ t^\nu(m) = 0,\ |m_\nu| \leq -\sigma_\nu \text{ on } \Gamma_c\},$$
$$\mathcal{H} = \{(\sigma, m) \mid \sigma = \{\sigma_{ij}\}, m = \{m_{ij}\};\ \sigma, \operatorname{div} \sigma \in L^2(\Omega), m, \nabla\nabla m \in L^2(\Omega)\}.$$

Here m_ν are the bending moments, and $t^\nu(m)$ are transverse forces.

THEOREM 6.3. *There exists a unique solution to the problem* (6.11)–(6.14).

The proof is given in Section 6.3.

Note that the case of cracks which come out at $\Gamma = \partial\Omega$ is also treated by the smooth domain formulation. This means that the method is applied to the case when $\bar{\Gamma}_c$ crosses the external boundary Γ (see Remarks 6.9, 6.13).

Three-dimensional case. We study elastoplastic models of Hencky type for bodies $\Omega_c \subset \mathbb{R}^3$ with a crack Γ_c and a smooth boundary Γ. Such models take the form

$$-\sigma_{ij,j} = f_i, \qquad i = 1, 2, 3, \tag{6.15}$$

$$\varepsilon_{ij}(u) = a_{ijkl}\sigma_{kl} + \xi_{ij}, \qquad i, j = 1, 2, 3, \tag{6.16}$$

$$\Phi(\sigma) \leq 0, \quad \xi_{ij}(\bar{\sigma}_{ij} - \sigma_{ij}) \leq 0 \qquad \forall \bar{\sigma},\ \Phi(\bar{\sigma}) \leq 0, \tag{6.17}$$

$$\sigma_{ij} n_j = 0, \qquad i = 1, 2, 3, \text{ on } \Gamma, \tag{6.18}$$

$$\sigma_{ij} \nu_j = 0, \qquad i = 1, 2, 3, \text{ on } \Gamma_c^\pm, \tag{6.19}$$

where $\Phi : \mathbb{R}^6 \to \mathbb{R}$ is a continuous convex function. The main result is

THEOREM 6.4. *The system* (6.15)–(6.19) *has a weak solution.*

This is proved in Section 6.4.

Elastoplasic plates with cracks. The method can also be applied to the two-dimensional problem of an elastoplastic plate with cracks. This problem is described by the following system of equations:

$$-m_{ij,ij} = f, \tag{6.20}$$

$$-w_{,ij} = a_{ijkl} m_{kl} + \xi_{ij}, \qquad i, j = 1, 2, \tag{6.21}$$

$$\Psi(m_{ij}) \le 0,\ \xi_{ij}(\bar m_{ij} - m_{ij}) \le 0 \quad \forall \bar m,\ \Psi(\bar m_{ij}) \le 0, \tag{6.22}$$
$$w = 0,\ m_{ij}n_j n_i = 0 \quad \text{on } \Gamma, \tag{6.23}$$
$$m_{ij}\nu_j\nu_i = 0,\ R_\nu(m_{ij}) = 0 \quad \text{on } \Gamma_c^{\pm}, \tag{6.24}$$

where $\Psi : \mathbb{R}^3 \to \mathbb{R}$ is a convex and continuous function. The main result is:

THEOREM 6.5. *The system* (6.20)–(6.24) *has a weak solution.*

The proof is given in Section 6.5.

6.2. Two-dimensional elasticity. In this section the detailed proof of Theorem 6.2 is given. We start with the variational inequality for frictionless contact on crack faces in two-dimensional elasticity.

6.2.1. Variational formulation. The equilibrium problem for a linear elastic body occupying the domain Ω_c with the interior crack Γ_c can be formulated as follows [67]. We have to find functions $u = (u_1, u_2), \sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, such that

$$-\operatorname{div}\sigma = f \quad \text{in } \Omega_c, \tag{6.25}$$
$$C\sigma - \varepsilon(u) = 0 \quad \text{in } \Omega_c, \tag{6.26}$$
$$u = 0 \quad \text{on } \Gamma, \tag{6.27}$$
$$[u]\nu \ge 0,\ [\sigma_\nu] = 0,\ \sigma_\nu \cdot [u]\nu = 0 \quad \text{on } \Gamma_c, \tag{6.28}$$
$$\sigma_\nu \le 0,\ \sigma_\tau = 0 \quad \text{on } \Gamma_c^{\pm}. \tag{6.29}$$

Here $[u] = u^+ - u^-$ is the jump of the displacement field across Γ_c, and the signs $\pm$ indicate the positive and negative directions of the normal ν; $f = (f_1, f_2) \in L^2(\Omega)$ is a given external force acting on the body, and the following notations are used:

$$\sigma_\nu = \sigma_{ij}\nu_j\nu_i, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \cdot \nu, \quad \sigma_\tau = \{\sigma_\tau^i\}_{i=1}^2, \quad \sigma\nu = \{\sigma_{ij}\nu_j\}_{i=1}^2,$$
$$\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad i, j = 1, 2, \quad \varepsilon(u) = \{\varepsilon_{ij}(u)\}_{i,j=1}^2,$$
$$\{C\sigma\}_{ij} = c_{ijkl}\sigma_{kl}, \quad c_{ijkl} = c_{jikl} = c_{klij}, \quad c_{ijkl} \in L^\infty(\Omega).$$

The tensor C satisfies the ellipticity condition

$$c_{ijkl}\xi_{ji}\xi_{kl} \ge c_0|\xi|^2, \quad \forall \xi_{ji} = \xi_{ij}, \quad c_0 > 0. \tag{6.30}$$

We use the summation convention over repeated indices $i, j, k, l = 1, 2$.

Equations and inequalities (6.28)–(6.29) describe the mutual non-penetration between crack faces without friction. Relation (6.25) is the equilibrium equation, the equation (6.26) is the Hooke constitutive law, and the condition (6.27) corresponds to the fixed displacements on the boundary Γ.

In order to introduce the variational formulation of the problem (6.25)–(6.29) we need the following Sobolev space:

$$H^{1,0}(\Omega_c) = \{v = (v_1, v_2) \mid v_i \in H^1(\Omega_c),\ v_i = 0 \text{ on } \Gamma,\ i = 1, 2\}$$

and a closed convex set of admissible displacements

$$K = \{v \in H^{1,0}(\Omega_c) \mid [v]\nu \ge 0 \text{ a.e. on } \Gamma_c\}. \tag{6.31}$$

In this case we can consider the minimization problem

$$\min_{v \in K} \left\{ \frac{1}{2}(\sigma(v), \varepsilon(v))_{\Omega_c} - (f, v)_{\Omega_c} \right\}$$

which has a unique solution $u \in K$ satisfying the variational inequality

$$(\sigma(u), \varepsilon(v-u))_{\Omega_c} \geq (f, v-u)_{\Omega_c} \quad \forall v \in K. \tag{6.32}$$

Here $(\cdot,\cdot)_{\Omega_c}$ is the scalar product in $L^2(\Omega_c)$, and the stress tensor $\sigma(u) = \sigma$ is found from Hooke's law (6.26). From (6.32) it follows that the equilibrium equation (6.25) is satisfied in the sense of distributions. To verify this it suffices to substitute $v = u \pm \varphi$, $\varphi \in C_0^\infty(\Omega_c)$, in the variational inequality (6.32). It can be shown [67] that for the solution to the variational inequality (6.32) all the boundary conditions (6.28)–(6.29) are satisfied. In the next section we specify the meaning of such conditions.

6.2.2. Mixed formulation. Consider the space of stresses

$$H(\mathrm{div}) = \{\sigma = \{\sigma_{ij}\} \mid \sigma \in L^2(\Omega_c), \mathrm{div}\,\sigma \in L^2(\Omega_c)\}$$

equipped with the norm

$$\|\sigma\|^2_{H(\mathrm{div})} = \|\sigma\|^2_{L^2(\Omega_c)} + \|\mathrm{div}\,\sigma\|^2_{L^2(\Omega_c)}$$

and define the set of admissible stresses

$$H(\mathrm{div};\Gamma_c) = \{\sigma \in H(\mathrm{div}) \mid [\sigma\nu] = 0 \text{ on } \Gamma_c;\ \sigma_\nu \leq 0, \sigma_\tau = 0 \text{ on } \Gamma_c^\pm\}.$$

For simplicity the same notation $L^2(\Omega_c)$ is used for the space of scalar functions and the space $[L^2(\Omega_c)]^2 = L^2(\Omega_c;\mathbb{R}^2)$ of vector functions as well as for the space $[L^2(\Omega_c)]^4$ of tensor-valued functions.

Introduce the space $H^{1/2}(\Sigma)$ with the norm

$$\|\varphi\|^2_{H^{1/2}(\Sigma)} = \|\varphi\|^2_{L^2(\Sigma)} + \int_\Sigma \int_\Sigma \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^2}\, dx\, dy$$

and denote by $H^{-1/2}(\Sigma)$ the space dual to $H^{1/2}(\Sigma)$. Note that for $\sigma \in H(\mathrm{div})$ the traces $(\sigma\nu)^\pm$ can be defined as elements of $H^{-1/2}(\Sigma)$ (see [67], [131]), and the trace operators are continuous from $H(\mathrm{div})$ to $H^{-1/2}(\Sigma)$. Also, it is possible to define $\sigma_\nu^\pm, (\sigma_\tau^i)^\pm \in H^{-1/2}(\Sigma)$, $i = 1, 2$, such that the Green formula holds:

$$(\mathrm{div}\,\sigma, \psi)_{\Omega_1} = -(\sigma, \varepsilon(\psi))_{\Omega_1} + \langle \sigma_\nu^-, \psi_\nu \rangle_{1/2} + \langle \sigma_\tau^-, \psi_\tau \rangle_{1/2} \quad \forall \psi = (\psi_1, \psi_2) \in H^1(\Omega_1),$$

where ν is assumed to be the external normal vector to the boundary $\partial\Omega_1 = \Sigma$, and $\langle \cdot,\cdot \rangle_{1/2}$ is the duality pairing between $H^{-1/2}(\Sigma)$ and $H^{1/2}(\Sigma)$. A similar formula holds for the domain Ω_2 with the external normal vector $-\nu$ to the part Σ of its boundary $\Gamma \cup \Sigma$. The zero jump condition for $\sigma\nu$ in the definition of $H(\mathrm{div};\Gamma_c)$ means

$$\langle (\sigma\nu)^+ - (\sigma\nu)^-, \varphi \rangle_{1/2} = 0 \quad \forall \varphi = (\varphi_1, \varphi_2) \in H^{1/2}(\Sigma).$$

Since $(\sigma\nu)^+$ and $(\sigma\nu)^-$ coincide, it follows that $\sigma_\nu^+ = \sigma_\nu^-$, $(\sigma_\tau^i)^+ = (\sigma_\tau^i)^-$, $i = 1, 2$. Let $\mathrm{supp}\,\varphi$ denote the support of the function φ. The second and the third conditions in the definition of $H(\mathrm{div};\Gamma_c)$ can be written as

$$\langle \sigma_\nu^\pm, \varphi \rangle_{1/2} \leq 0 \quad \forall \varphi \in H^{1/2}(\Sigma), \qquad \varphi \geq 0 \quad \text{a.e. on } \Gamma_c, \qquad \mathrm{supp}\,\varphi \subset \Gamma_c$$

and

$$\langle \sigma_\tau^\pm, \varphi \rangle_{1/2} = 0 \quad \forall \varphi = (\varphi_1, \varphi_2) \in H^{1/2}(\Sigma), \qquad \varphi_i \nu_i = 0 \quad \text{a.e. on } \Gamma_c, \qquad \operatorname{supp} \varphi \subset \Gamma_c,$$

respectively. Thus, the convex cone $H(\mathrm{div}; \Gamma_c)$ is closed in the space $H(\mathrm{div})$. Hence $H(\mathrm{div}; \Gamma_c)$ is weakly closed in $H(\mathrm{div})$.

The above arguments allow us to define function spaces on Γ_c. Recall the definition of the weighted Sobolev space on Γ_c (see e.g. [48] for details):

$$H_{00}^{1/2}(\Gamma_c) = \{\varphi \in H^{1/2}(\Gamma_c) \mid \varphi/\sqrt{\rho} \in L^2(\Gamma_c)\}$$

equipped with the norm

$$\|\varphi\|_{1/2,00}^2 = \|\varphi\|_{1/2}^2 + \int_{\Gamma_c} \rho^{-1} \varphi^2,$$

where $\rho(x) = \mathrm{dist}(x, \partial\Gamma_c)$, and $\|\cdot\|_{1/2}$ is the norm in $H^{1/2}(\Gamma_c)$. It is well known [89] that functions from the space $H_{00}^{1/2}(\Gamma_c)$ can be extended to Σ by zero, and such an extension is an element of $H^{1/2}(\Sigma)$. The extension of φ is denoted by $\bar{\varphi}$, i.e.,

$$\bar{\varphi}(x) = \begin{cases} \varphi(x), & x \in \Gamma_c, \\ 0, & x \in \Sigma \setminus \Gamma_c, \end{cases}$$

and we have $\varphi \in H_{00}^{1/2}(\Gamma_c)$ if and only if $\bar{\varphi} \in H^{1/2}(\Sigma)$.

Let us observe that by the above formulae the elements $\sigma_\nu \in (H_{00}^{1/2}(\Gamma_c))^*$ and $\sigma_\tau^i \in (H_{00}^{1/2}(\Gamma_c))^*$, $i = 1, 2$, can be defined [67]. The inequalities on Γ_c are understood in the sense of duality, i.e. $[\sigma_\nu] = 0, \sigma_\nu \le 0$ in the definition of $H(\mathrm{div}; \Gamma_c)$ mean

$$\langle \sigma_\nu, \varphi \rangle_{1/2,00} \le 0 \qquad \forall \varphi \in H_{00}^{1/2}(\Gamma_c) \text{ such that } \varphi \ge 0 \text{ a.e. on } \Gamma_c;$$

furthermore, the condition $\sigma_\tau = 0$ on $\Gamma_c^\pm$ in the definition of the cone $H(\mathrm{div}; \Gamma_c)$ takes the form

$$\langle \sigma_\tau, \varphi \rangle_{1/2,00} = 0 \qquad \forall \varphi = (\varphi_1, \varphi_2) \in H_{00}^{1/2}(\Gamma_c) \text{ such that } \varphi_i \nu_i = 0 \text{ a.e. on } \Gamma_c.$$

Here $\langle \cdot, \cdot \rangle_{1/2,00}$ is the duality pairing between $(H_{00}^{1/2}(\Gamma_c))^*$ and $H_{00}^{1/2}(\Gamma_c)$.

It is important that in the above formulae the curve Σ is assumed to be arbitrary, but it should be sufficiently smooth. This means that the formulae mentioned are valid for closed curves Σ which are smooth enough. All boundary conditions for σ included in the definition of $H(\mathrm{div}; \Gamma_c)$ are precisely the same as the boundary conditions for the solution $\sigma(u) = \sigma$ of the variational inequality (6.32). Let us note that dependence of the solution on domain variations for classical boundary value problems is analyzed in [126]. For domain variations in free boundary crack problems we refer the reader to [67], [72] (see also [87]).

Now, we are in a position to give the mixed formulation for the problem (6.25)–(6.29). We have to find functions $u = (u_1, u_2)$, $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, such that

$$u \in L^2(\Omega_c), \qquad \sigma \in H(\mathrm{div}; \Gamma_c), \tag{6.33}$$

$$-\mathrm{div}\, \sigma = f \quad \text{in } \Omega_c, \tag{6.34}$$

$$(C\sigma, \bar{\sigma} - \sigma)_{\Omega_c} + (u, \mathrm{div}\, \bar{\sigma} - \mathrm{div}\, \sigma)_{\Omega_c} \ge 0 \qquad \forall \bar{\sigma} \in H(\mathrm{div}; \Gamma_c). \tag{6.35}$$

The boundary value problem (6.25)–(6.29) is formally equivalent to (6.33)–(6.35). Indeed, assuming that the solutions to (6.33)–(6.35) are sufficiently regular, we can derive from (6.35) the Hooke law by taking test functions of the form $\bar{\sigma} = \pm\widetilde{\sigma} + \sigma$, where $\widetilde{\sigma}$ are smooth functions with compact support in Ω_c,

$$C\sigma - \varepsilon(u) = 0 \quad \text{ in } \Omega_c.$$

The boundary conditions

$$[u]\nu \geq 0, \qquad \sigma_\nu \cdot [u]\nu = 0 \quad \text{ on } \Gamma_c \tag{6.36}$$

follow from (6.35) by an application of the Green formula. Thus, all boundary conditions (6.28)–(6.29) are satisfied. On the other hand, by multiplication of (6.26) by $\bar{\sigma} - \sigma$, $\bar{\sigma} \in H(\operatorname{div};\Gamma_c)$, it can be shown that the inequality (6.35) follows from (6.25)–(6.29).

Note that the set $H(\operatorname{div};\Gamma_c)$ includes only the restriction imposed on the stress tensor components. As for the relations (6.36), they are included in the problem (6.33)–(6.35). On the other hand, the set K in the variational inequality (6.32) includes only the restriction imposed on the displacement u, and the equations and inequalities (6.28), (6.29) can be derived from (6.32).

We aim at investigation of the problem (6.33)–(6.35). First, we prove the existence of a solution.

Theorem 6.6. *There exists a solution to the system* (6.33)–(6.35).

Proof. In order to establish a priori estimates for solutions we introduce a function $\sigma^0 \in H(\operatorname{div};\Gamma_c)$ which solves the equilibrium equations

$$-\operatorname{div}\sigma^0 = f \quad \text{ in } \Omega_c.$$

Such a function can be found by solving the variational inequality (6.32) with an arbitrary Hooke's law satisfying the condition (6.30). Let us point out that the existence of a solution to the system (6.33)–(6.35) can in fact be established directly by solving (6.32), but we provide a different argument without any requirement on the solvability of (6.32). The reason to proceed in this way is that later we can use exactly the same arguments in order to analyze the smooth domain formulation for the problem under consideration.

To prove the existence of solutions to (6.33)–(6.35) we introduce the regularized boundary value problem depending on a parameter $\delta > 0$. Then the existence of a solution for the regularized problem is shown and a priori estimates are obtained. The proof is completed by the passage to the limit $\delta \to 0$.

Let us fix $0 < \delta < \delta_0$. The regularized problem takes the form

$$u^\delta \in L^2(\Omega_c), \qquad \sigma^\delta \in H(\operatorname{div};\Gamma_c), \tag{6.37}$$

$$\delta u^\delta - \operatorname{div}\sigma^\delta = f \quad \text{ in } \Omega_c, \tag{6.38}$$

$$(C\sigma^\delta, \bar{\sigma} - \sigma^\delta)_{\Omega_c} + (u^\delta, \operatorname{div}\bar{\sigma} - \operatorname{div}\sigma^\delta)_{\Omega_c} \geq 0 \quad \forall \bar{\sigma} \in H(\operatorname{div};\Gamma_c). \tag{6.39}$$

From (6.38), (6.39) it follows that

$$\delta(u^\delta, u^\delta)_{\Omega_c} - (\operatorname{div}\sigma^\delta, u^\delta)_{\Omega_c} = (f, u^\delta)_{\Omega_c}\ ,$$
$$(C\sigma^\delta, \sigma^0 - \sigma^\delta)_{\Omega_c} + (u^\delta, \operatorname{div}\sigma^0 - \operatorname{div}\sigma^\delta)_{\Omega_c} \geq 0,$$

and the following estimate is obtained:

$$\delta\|u^\delta\|^2_{L^2(\Omega_c)} + \|\sigma^\delta\|^2_{L^2(\Omega_c)} \le c \tag{6.40}$$

with the constant c uniform with respect to δ. Moreover, (6.38) implies that

$$\operatorname{div}\sigma^\delta = \delta u^\delta - f \quad \text{in } \Omega_c.$$

Thus, in view of (6.40), the following uniform estimate is obtained:

$$\|\operatorname{div}\sigma^\delta\|^2_{L^2(\Omega_c)} \le c. \tag{6.41}$$

Let us show that for a given δ there exists a solution to the problem(6.37)–(6.39). Indeed, from (6.38) it follows that $u^\delta = \frac{1}{\delta}(f + \operatorname{div}\sigma^\delta)$. Substituting this value of u^δ in (6.39) we derive the variational inequality

$$(C\sigma^\delta, \bar\sigma - \sigma^\delta)_{\Omega_c} + \frac{1}{\delta}(f + \operatorname{div}\sigma^\delta, \operatorname{div}\bar\sigma - \operatorname{div}\sigma^\delta)_{\Omega_c} \ge 0 \quad \forall\bar\sigma \in H(\operatorname{div};\Gamma_c).$$

It is clear that solving this variational inequality is equivalent to minimization of the functional

$$G(\sigma) = \frac{1}{2}(C\sigma,\sigma)_{\Omega_c} + \frac{1}{2\delta}(\operatorname{div}\sigma, \operatorname{div}\sigma)_{\Omega_c} + \frac{1}{\delta}(f, \operatorname{div}\sigma)_{\Omega_c}$$

over the weakly closed convex set $H(\operatorname{div};\Gamma_c)$. The functional G is coercive and weakly lower semicontinuous on the space $H(\operatorname{div})$, hence the minimization problem has a (unique) solution $\sigma = \sigma^\delta$. Having found σ^δ we define u^δ from (6.38). The solution u^δ, σ^δ satisfies the relations (6.37)–(6.39). Now we perform the passage to the limit in (6.37)–(6.39) as $\delta \to 0$.

From (6.39) it follows that

$$C\sigma^\delta - \varepsilon(u^\delta) = 0 \quad \text{in } \Omega_c$$

in the sense of distributions, i.e., in particular $\varepsilon(u^\delta) \in L^2(\Omega_c)$. Since $u^\delta \in L^2(\Omega_c)$, by an application of the second Korn inequality which holds in the domain Ω_c it follows that $u^\delta \in H^1(\Omega_c)$. On the other hand,

$$u^\delta = 0 \quad \text{on } \Gamma,$$

which can be deduced from (6.39) taking into account that the vector function $\bar\sigma n$ is free on Γ. Hence $u^\delta \in H^{1,0}(\Omega_c)$, and, by the first Korn inequality, the uniform estimate with respect to δ is obtained:

$$\|u^\delta\|_{H^{1,0}(\Omega_c)} \le c.$$

Taking into account (6.40), (6.41), we have the uniform estimate with respect to δ,

$$\|\sigma^\delta\|_{L^2(\Omega_c)} + \|\operatorname{div}\sigma^\delta\|_{L^2(\Omega_c)} \le c.$$

Therefore, there exist elements u, σ such that for $\delta \to 0$ we have the following convergences for subsequences:

$$\begin{aligned} u^\delta &\to u && \text{weakly in } H^{1,0}(\Omega_c) \text{ and strongly in } L^2(\Omega_c),\\ \sigma^\delta &\to \sigma && \text{weakly in } L^2(\Omega_c),\\ \operatorname{div}\sigma^\delta &\to \operatorname{div}\sigma && \text{weakly in } L^2(\Omega_c). \end{aligned}$$

Finally, for $\delta \to 0$ we pass to the limit in (6.38), (6.39), and (6.33)–(6.35) follows, which completes the proof of Theorem 6.6. ∎

Note that the solution to (6.33)–(6.35) is unique. Indeed, if there are two solutions (u^1, σ^1) and (u^2, σ^2) to (6.35), it follows that $\sigma^1 = \sigma^2$. Since $C\sigma^i = \varepsilon(u^i)$, $i = 1, 2$, we have $\varepsilon(u^1 - u^2) = 0$, hence $u^1 = u^2$.

REMARK 6.7. The mixed formulation of the problem (6.25)–(6.29) can be applied to the case when $\bar{\Gamma}_c$ crosses the external boundary Γ, and also to the case when Γ_c is only $C^{0,1}$-regular. The $C^{1,1}$-regularity of the curve Σ was needed to define σ_ν, σ_τ. It is possible to avoid the interpretation of the boundary conditions involved in the set $H(\mathrm{div}; \Gamma_c)$. Indeed, consider a crack Γ_c of $C^{0,1}$-regularity such that $\bar{\Gamma}_c$ crosses the boundary Γ (see Figure 6.4). Assume that the angle between Γ and $\bar{\Gamma}_c$ at the common point x_c is non-zero.

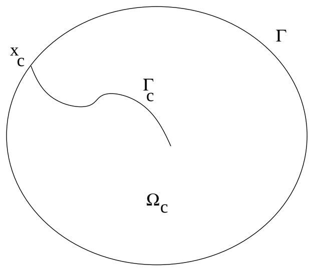

Fig. 6.4. Boundary crack

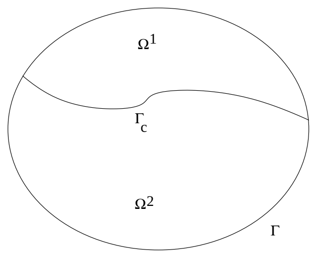

Fig. 6.5. Contact problem

Introduce the set of admissible stresses in the following equivalent form:

$$H(\mathrm{div}; \Gamma_c) = \left\{ \sigma \in H(\mathrm{div}) \;\middle|\; \int_{\Omega_c} (\sigma\varepsilon(\bar{u}) + \bar{u}\,\mathrm{div}\,\sigma) \geq 0 \;\forall \bar{u} \in K \right\},$$

where the set K is defined in (6.31). For such a definition of $H(\mathrm{div}; \Gamma_c)$ we can verify that all boundary conditions for stresses are satisfied provided that the function σ is sufficiently regular. Note that if Γ_c divides Ω into two separate domains Ω^1 and Ω^2, we obtain a contact problem for two elastic bodies occupying the domains Ω^1, Ω^2 with inequality type boundary conditions (6.28)–(6.29) imposed on the common boundary Γ_c (see Figure 6.5).

6.2.3. Smooth domain method. In this section the smooth domain method for the crack problem in two-dimensional elasticity is formulated. The main feature of such a formulation is that the constraints on the stress tenosr are imposed on subsets of the smooth domain Ω, and the unknown functions u, σ are defined in the smooth domain Ω. We extend unknown functions u, σ from the non-smooth domain Ω_c to the smooth domain Ω (cf. [9]). Such an extension reduces in fact to a definition of two fields u, σ on the curve Γ_c. We shall use the same notation u, σ for the extended functions defined on Ω and write the problem (6.25)–(6.29) in the domain Ω. The problem takes the following form.

We have to find functions $u = (u_1, u_2)$, $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, such that

$$-\operatorname{div} \sigma = f \quad \text{in } \Omega, \tag{6.42}$$

$$C\sigma - \varepsilon(u) + p(u)\delta_{\Gamma_c} = 0 \quad \text{in } \Omega, \tag{6.43}$$

$$u = 0 \quad \text{on } \Gamma, \tag{6.44}$$

$$[u]\nu \geq 0,\ \sigma_\nu \leq 0,\ \sigma_\tau = 0,\ \sigma_\nu \cdot [u]\nu = 0 \quad \text{on } \Gamma_c, \tag{6.45}$$

where we write $p(u)_{ij} = \frac{1}{2}([u_i]\nu_j + [u_j]\nu_i)$, and δ_{Γ_c} is the single layer distribution on Γ_c defined by

$$\langle \delta_{\Gamma_c}, \varphi \rangle = \int_{\Gamma_c} \varphi \quad \forall \varphi \in C_0^\infty(\Omega).$$

We have denoted by $\langle T, \varphi \rangle$ the value of a distribution T on the function $\varphi \in C_0^\infty(\Omega)$. Let us point out that solutions to the system (6.25)–(6.29) determined from the variational inequality (6.32) satisfy the jump condition

$$[\sigma\nu] = 0 \quad \text{on } \Gamma_c, \tag{6.46}$$

and therefore equation (6.42) is of the same form as (6.25). Let us verify this statement. It follows from (6.32) that $\sigma = \sigma(u)$ satisfies

$$\sigma \in L^2(\Omega_c), \quad \operatorname{div} \sigma \in L^2(\Omega_c), \quad -\operatorname{div} \sigma = f \quad \text{in } \Omega_c. \tag{6.47}$$

Then in view of (6.46), (6.47) it follows that for any $\varphi \in C_0^\infty(\Omega)$,

$$\begin{aligned} \langle \sigma_{ij,j} + f_i, \varphi \rangle &= -(\sigma_{ij}, \varphi_{,j})_{\Omega_1} - (\sigma_{ij}, \varphi_{,j})_{\Omega_2} + (f_i, \varphi)_\Omega \\ &= \langle [\sigma_{ij}\nu_j], \varphi \rangle_{1/2} + (\sigma_{ij,j} + f_i, \varphi)_{\Omega_1} + (\sigma_{ij,j} + f_i, \varphi)_{\Omega_2} = 0, \quad i = 1, 2, \end{aligned}$$

which proves that equation (6.42) holds in the sense of distributions.

The difference between the systems (6.25)–(6.29) and (6.42)–(6.45) is that now the conditions (6.45) are considered to be internal constraints for the solutions which are imposed on the curve Γ_c located in the interior of the smooth domain Ω. Let us note that the equivalence of the systems (6.25)–(6.29) and (6.42)–(6.45) is straightforward for smooth solutions. We show that this is also the case for weak solutions. We need the following notation for the space of stresses and the convex cone of admissible stresses in the smooth domain Ω:

$$\mathcal{H}(\operatorname{div}) = \{\sigma = \{\sigma_{ij}\} \mid \sigma, \operatorname{div} \sigma \in L^2(\Omega)\},$$
$$\mathcal{H}(\operatorname{div}; \Gamma_c) = \{\sigma \in \mathcal{H}(\operatorname{div}) \mid \sigma_\tau = 0,\ \sigma_\nu \leq 0 \text{ on } \Gamma_c\}.$$

The norm in the space $\mathcal{H}(\operatorname{div})$ is defined by the formula

$$\|\sigma\|^2_{\mathcal{H}(\operatorname{div})} = \|\sigma\|^2_{L^2(\Omega)} + \|\operatorname{div} \sigma\|^2_{L^2(\Omega)}.$$

As indicated before for the cone $H(\operatorname{div}; \Gamma_c)$, also the convex cone $\mathcal{H}(\operatorname{div}; \Gamma_c)$ is closed in the space $\mathcal{H}(\operatorname{div})$ since the conditions $\sigma_\tau = 0$, $\sigma_\nu \leq 0$ on Γ_c are well defined for any $\sigma \in \mathcal{H}(\operatorname{div})$. Indeed, for any curve Σ satisfying the prescribed conditions the functionals σ_ν, σ_τ^i, $i = 1, 2$, are uniquely defined in the space $H^{-1/2}(\Sigma)$. The conditions $\sigma_\tau = 0$, $\sigma_\nu \leq 0$ on Γ_c in the definition of $\mathcal{H}(\operatorname{div}; \Gamma_c)$ are understood in the sense

$$\langle \sigma_\tau, \varphi \rangle_{1/2} = 0 \quad \forall \varphi = (\varphi_1, \varphi_2) \in H^{1/2}(\Sigma),\ \varphi_i\nu_i = 0 \text{ a.e. on } \Gamma_c,\ \operatorname{supp} \varphi \subset \Gamma_c,$$

and

$$\langle \sigma_\nu, \varphi \rangle_{1/2} \le 0 \quad \forall \varphi \in H^{1/2}(\Sigma),\ \varphi \ge 0 \text{ a.e. on } \Gamma_c,\ \operatorname{supp} \varphi \subset \Gamma_c,$$

respectively.

The weak formulation of the system (6.42)–(6.45) takes the form of the following problem in Ω: Find $u = (u_1, u_2)$, $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, such that

$$u \in L^2(\Omega), \qquad \sigma \in \mathcal{H}(\mathrm{div}; \Gamma_c), \tag{6.48}$$

$$-\operatorname{div} \sigma = f \quad \text{in } \Omega, \tag{6.49}$$

$$(C\sigma, \bar\sigma - \sigma)_\Omega + (u, \operatorname{div} \bar\sigma - \operatorname{div} \sigma)_\Omega \ge 0 \quad \forall \bar\sigma \in \mathcal{H}(\mathrm{div}; \Gamma_c). \tag{6.50}$$

Note that (6.50) follows directly from (6.35) since we can replace integration over Ω_c by integration over Ω.

THEOREM 6.8. *There exists a solution to the problem* (6.48)–(6.50).

Proof. The general scheme of the proof remains the same as for Theorem 6.6. First of all, the function σ^0 defined in the proof of Theorem 6.6 can be extended to the domain Ω, the extended function is also denoted by σ^0, $\sigma^0 \in \mathcal{H}(\mathrm{div}; \Gamma_c)$, and furthermore the equilibrium equations are satisfied,

$$-\operatorname{div} \sigma^0 = f \quad \text{in } \Omega.$$

Now, for a positive parameter δ consider the regularized problem

$$u^\delta \in L^2(\Omega), \qquad \sigma^\delta \in \mathcal{H}(\mathrm{div}; \Gamma_c), \tag{6.51}$$

$$\delta u^\delta - \operatorname{div} \sigma^\delta = f \quad \text{in } \Omega, \tag{6.52}$$

$$(C\sigma^\delta, \bar\sigma - \sigma^\delta)_\Omega + (u^\delta, \operatorname{div} \bar\sigma - \operatorname{div} \sigma^\delta)_\Omega \ge 0 \quad \forall \bar\sigma \in \mathcal{H}(\mathrm{div}; \Gamma_c). \tag{6.53}$$

From (6.51)–(6.53) we can obtain the uniform (with respect to δ) estimate

$$\delta \|u\|^2_{L^2(\Omega)} + \|\sigma\|^2_{L^2(\Omega)} + \|\operatorname{div} \sigma\|^2_{L^2(\Omega)} \le c. \tag{6.54}$$

In the same way as in the proof of Theorem 6.6, from (6.52)–(6.53) the following estimate is obtained, uniform with respect to δ:

$$\|u^\delta\|_{H^{1,0}(\Omega_c)} \le c. \tag{6.55}$$

By estimates (6.54), (6.55), we have as $\delta \to 0$ the following convergences, for subsequences,

$$\begin{aligned} u^\delta &\to u && \text{strongly in } L^2(\Omega), \\ \sigma^\delta &\to \sigma && \text{weakly in } L^2(\Omega), \\ \operatorname{div} \sigma^\delta &\to \operatorname{div} \sigma && \text{weakly in } L^2(\Omega). \end{aligned}$$

Consequently, we can pass to the limit as $\delta \to 0$ in (6.51)–(6.53) and obtain (6.48)–(6.50), which completes the proof of Theorem 6.8. ■

The solution to (6.48)–(6.50) is unique.

Formulation of the free boundary crack problem in the form (6.48)–(6.50) is attractive since the domain Ω does not contain non-smooth components of the boundary. Moreover the restrictions imposed on the stress tensor components are given on subsets of Ω. So the formulation (6.48)–(6.50) reminds that of classical contact problems with restrictions imposed on subsets of the domain. A wide class of contact problems with restrictions imposed on subsets of domains can be found in [70].

REMARK 6.9. Similar to the mixed problem formulation (see Remark 6.7) we can consider an equivalent definition of the admissible stresses,

$$\mathcal{H}(\mathrm{div};\Gamma_c) = \left\{ \sigma \in \mathcal{H}(\mathrm{div}) \;\middle|\; \int_{\Omega_c} (\sigma\varepsilon(\bar{u}) + \bar{u}\,\mathrm{div}\,\sigma) \geq 0 \;\forall \bar{u} \in K \right\}.$$

The set K is defined in (6.31). The above definition of $\mathcal{H}(\mathrm{div};\Gamma_c)$ can be applied both for the interior and boundary cracks (see Figures 6.4, 6.5).

REMARK 6.10. Now observe that the classical approach to the two-dimensional crack problem is characterized by the equality type boundary conditions (cf. (6.28), (6.29))

$$\sigma_\nu = \sigma_\tau = 0 \quad \text{on } \Gamma_c^\pm. \tag{6.56}$$

In this case the smooth domain method can be successfully applied to the problem (6.25)–(6.27), (6.56). Indeed, the set of admissible stresses is defined as follows:

$$\mathcal{H}(\mathrm{div};\Gamma_c) = \{\sigma \in \mathcal{H}(\mathrm{div}) \mid \sigma_\nu = 0,\ \sigma_\tau = 0 \text{ on } \Gamma_c\}. \tag{6.57}$$

Instead of (6.50) we obtain the identity

$$(C\sigma, \bar{\sigma})_\Omega + (u, \mathrm{div}\,\bar{\sigma})_\Omega = 0 \qquad \forall \bar{\sigma} \in \mathcal{H}(\mathrm{div};\Gamma_c). \tag{6.58}$$

Hence, the smooth domain method for the classical boundary value crack problem can be formulated in the form (6.48), (6.49), (6.58), where $\mathcal{H}(\mathrm{div};\Gamma_c)$ is defined in (6.57).

6.3. Kirchhoff plate with a crack. In this section we show that the smooth domain method can be applied to equilibrium problems for Kirchhoff plates having cracks with inequality type boundary conditions given at the crack faces. As in two-dimensional elasticity, these boundary conditions describe mutual non-penetration between the crack faces. The problem formulation is as follows [67].

In the domain Ω_c, we have to find functions $u = (u_1, u_2), w, \sigma = \{\sigma_{ij}\}, m = \{m_{ij}\}$, $i, j = 1, 2$, such that

$$-\mathrm{div}\,\sigma = f \qquad \text{in } \Omega_c, \tag{6.59}$$

$$-\nabla\nabla m = F \qquad \text{in } \Omega_c, \tag{6.60}$$

$$C\sigma - \varepsilon(u) = 0 \qquad \text{in } \Omega_c, \tag{6.61}$$

$$Dm + \nabla\nabla w = 0 \qquad \text{in } \Omega_c, \tag{6.62}$$

$$u = w = \frac{\partial w}{\partial n} = 0 \qquad \text{on } \Gamma, \tag{6.63}$$

$$[u]\nu \geq \left| \left[\frac{\partial w}{\partial \nu} \right] \right|, \ [\sigma_\nu] = 0, \quad [m_\nu] = 0 \qquad \text{on } \Gamma_c, \tag{6.64}$$

$$|m_\nu| \leq -\sigma_\nu,\ \sigma_\nu \cdot [u]\nu - m_\nu \left[\frac{\partial w}{\partial \nu} \right] = 0 \qquad \text{on } \Gamma_c, \tag{6.65}$$

$$\sigma_\tau = 0,\ t^\nu(m) = 0 \qquad \text{on } \Gamma_c^\pm. \tag{6.66}$$

Here $f = (f_1, f_2)$; $F, f_i \in L^2(\Omega)$ are given functions, $i = 1, 2$,

$$\nabla\nabla m = m_{ij,ij}, \qquad m_\nu = m_{ij}\nu_j\nu_i, \qquad \nabla\nabla w = \{w_{,ij}\}_{i,j=1}^2,$$
$$t^\nu(m) = m_{ij,j}\nu_j + m_{ij,k}\tau_k\tau_j\nu_i, \qquad (\tau_1, \tau_2) = (-\nu_2, \nu_1).$$

We use the same notations as in the previous sections. The tensor C is symmetric and satisfies the condition (6.30). Similar conditions are imposed on the tensor D,

$$\{Dm\}_{ij} = d_{ijkl}m_{kl}, \qquad i, j = 1, 2.$$

Note that (6.59), (6.60) are equilibrium equations; relations (6.61), (6.62) provide the constitutive law. The boundary conditions (6.63) mean that the plate is clamped along the external boundary Γ. Equations and inequalities (6.64)–(6.66) describe mutual non-penetration between the crack faces $\Gamma_c^\pm$. The functions u, w are horizontal and vertical displacements of the mid-surface points of the plate; σ, m are the stress tensor and moment tensor, respectively.

For a variational formulation of the problem (6.59)–(6.66) we need the Sobolev space

$$H^{2,0}(\Omega_c) = \left\{ v \in H^2(\Omega_c) \;\middle|\; v = \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma \right\}.$$

Consider the convex set of admissible displacements,

$$K_c = \left\{ (u, w) \in [H^{1,0}(\Omega_c)]^2 \times H^{2,0}(\Omega_c) \;\middle|\; [u]\nu \geq \left| \left[\frac{\partial w}{\partial \nu} \right] \right| \text{ a.e. on } \Gamma_c \right\}. \tag{6.67}$$

There exists a solution to the following minimization problem:

$$\min_{(u,w)\in K_c} \left\{ \frac{1}{2}(\sigma(u), \varepsilon(u))_{\Omega_c} - \frac{1}{2}(m(w), \nabla\nabla w)_{\Omega_c} - (f, u)_{\Omega_c} - (F, w)_{\Omega_c} \right\},$$

which is equivalent to the variational inequality

$$\begin{aligned} (u, w) \in K_c, \qquad & (\sigma(u), \varepsilon(\bar{u} - u))_{\Omega_c} - (m(w), \nabla\nabla\bar{w} - \nabla\nabla w)_{\Omega_c} \\ & \geq (f, \bar{u} - u)_{\Omega_c} + (F, \bar{w} - w)_{\Omega_c} \quad \forall (\bar{u}, \bar{w}) \in K_c. \end{aligned} \tag{6.68}$$

The set K_c is weakly closed, and the functional being minimized is coercive and weakly lower semicontinuous in the space $[H^{1,0}(\Omega_c)]^2 \times H^{2,0}(\Omega_c)$. Hence, the problem (6.68) is solvable. Furthermore, the solution is unique.

Now, we introduce a mixed formulation of the problem (6.59)–(6.66). Consider the space

$$H(\Omega_c) = \{(\sigma, m) \mid \sigma = \{\sigma_{ij}\},\ m = \{m_{ij}\};\ \sigma, \operatorname{div}\sigma \in L^2(\Omega_c),\ m, \nabla\nabla m \in L^2(\Omega_c)\}$$

equipped with the norm

$$\|(\sigma, m)\|^2_{H(\Omega_c)} = \|\sigma\|^2_{L^2(\Omega_c)} + \|\operatorname{div}\sigma\|^2_{L^2(\Omega_c)} + \|m\|^2_{L^2(\Omega_c)} + \|\nabla\nabla m\|^2_{L^2(\Omega_c)}.$$

We introduce the set of admissible stresses and moments,

$$\begin{aligned} K(\Omega_c) = \{(\sigma, m) \in H(\Omega_c) \mid [\sigma\nu] = [m_\nu] = [t^\nu(m)] = 0 \text{ on } \Gamma_c; \\ |m_\nu| \leq -\sigma_\nu,\ \sigma_\tau = 0,\ t^\nu(m) = 0 \text{ on } \Gamma_c^\pm\}. \end{aligned}$$

Also, consider the space $H^{3/2}(\Sigma)$ with the norm

$$\|\varphi\|^2_{H^{3/2}(\Sigma)} = \|\varphi\|^2_{H^1(\Sigma)} + \int_\Sigma \int_\Sigma \frac{|\nabla\varphi(x) - \nabla\varphi(y)|^2}{|x-y|^2}\,dx\,dy$$

and its dual $H^{-3/2}(\Sigma)$. In the domain Ω_1, we can define traces on the boundary Σ, in particular, $m_\nu^- \in H^{-1/2}(\Sigma)$, $t^\nu(m)^- \in H^{-3/2}(\Sigma)$, and the following Green formula holds [67], [131]:

$$(w, \nabla\nabla m)_{\Omega_1} = (\nabla\nabla w, m)_{\Omega_1} + \langle t^\nu(m)^-, w\rangle_{3/2} - \langle m_\nu^-, \partial w/\partial\nu\rangle_{1/2} \quad \forall w \in H^2(\Omega_1), \tag{6.69}$$

where $\langle\cdot,\cdot\rangle_{3/2}$ stands for the duality pairing between $H^{-3/2}(\Sigma)$ and $H^{3/2}(\Sigma)$. For the domain Ω_2 we can write the Green formula similar to (6.69). In this case the boundary of Ω_2 contains two parts, Σ and Γ. In addition to two-dimensional elasticity, we should explain in what sense the boundary conditions are satisfied in the definition of $K(\Omega_c)$. The zero jump condition for $t^\nu(m)$ means

$$\langle t^\nu(m)^+ - t^\nu(m)^-, \varphi\rangle_{3/2} = 0 \quad \forall\varphi \in H^{3/2}(\Sigma).$$

The condition $t^\nu(m) = 0$ on $\Gamma_c^\pm$ reads

$$\langle t^\nu(m)^\pm, \varphi\rangle_{3/2} = 0 \quad \forall\varphi \in H^{3/2}(\Sigma),\ \operatorname{supp}\varphi \subset \Gamma_c. \tag{6.70}$$

It is seen that the set $K(\Omega_c)$ is convex. By the continuity of the trace operators, the set $K(\Omega_c)$ is closed. Hence $K(\Omega_c)$ is weakly closed.

As in two-dimensional elasticity, in (6.70) we can choose test functions $\bar\varphi$, where $\bar\varphi$ is an extension of φ to Σ by zero, with $\varphi \in H_{00}^{3/2}(\Gamma_c)$. The norm in the space $H_{00}^{3/2}(\Gamma_c)$ is defined by the formula

$$\|\phi\|^2_{H_{00}^{3/2}(\Gamma_c)} = \|\phi\|^2_{H^{3/2}(\Gamma_c)} + \int_{\Gamma_c} \rho^{-1}|\nabla\phi|^2.$$

It is known that $\varphi \in H_{00}^{3/2}(\Gamma_c)$ if and only if $\bar\varphi \in H^{3/2}(\Sigma)$ [89]. Now we are in a position to provide the mixed formulation for the problem (6.59)–(6.66).

We have to find functions $u = (u_1, u_2), w, \sigma = \{\sigma_{ij}\}, m = \{m_{ij}\}$ such that

$$u = (u_1, u_2) \in L^2(\Omega_c), \quad w \in L^2(\Omega_c), \quad (\sigma, m) \in K(\Omega_c), \tag{6.71}$$

$$-\operatorname{div}\sigma = f \quad \text{on } \Omega_c, \tag{6.72}$$

$$-\nabla\nabla m = F \quad \text{on } \Omega_c, \tag{6.73}$$

$$\begin{aligned}(u, \operatorname{div}\bar\sigma - \operatorname{div}\sigma)_{\Omega_c} &+ (w, \nabla\nabla\bar m - \nabla\nabla m)_{\Omega_c} \\ &+ (C\sigma, \bar\sigma - \sigma)_{\Omega_c} + (Dm, \bar m - m)_{\Omega_c} \ge 0 \quad \forall(\bar\sigma, \bar m) \in K(\Omega_c).\end{aligned} \tag{6.74}$$

Inequality (6.74) follows from (6.61)–(6.62). It suffices to multiply these equations by $\bar\sigma - \sigma$, $\bar m - m$, respectively, with $(\bar\sigma, \bar m) \in K(\Omega_c)$. On the other hand, the equations (6.61), (6.62) follow from (6.74). To prove this, it suffices to take in (6.74) the test functions $(\bar\sigma, \bar m) = (\sigma, m) + (\tilde\sigma, \tilde m)$, $(\tilde\sigma, \tilde m) \in C_0^\infty(\Omega_c)$. Moreover, the relations (6.71)–(6.74) contain all boundary conditions (6.63)–(6.66).

The existence of a solution to (6.71)–(6.74) can be shown by the procedure used in the proof of Theorem 6.12 below. The solution is unique.

REMARK 6.11. We can observe that just as in two-dimensional elasticity it is possible to avoid the explicit formulation of the boundary conditions for stresses and moments included in the set $K(\Omega_c)$. Namely, it suffices to introduce the set of admissible stresses and moments by using the "dual" formula

$$K(\Omega_c) = \left\{(\sigma, m) \in H(\Omega_c) \,\middle|\, \int_{\Omega_c} (\sigma\varepsilon(\bar{u}) + \bar{u}\operatorname{div}\sigma) + \int_{\Omega_c} (\bar{w}\nabla\nabla m - m\nabla\nabla\bar{w}) \geq 0 \quad \forall(\bar{u}, \bar{w}) \in K_c\right\},$$

where the set K_c is defined in (6.67). This equivalent definition of the set $K(\Omega_c)$ is suitable also in the case when $\bar{\Gamma}_c$ crosses the external boundary Γ (see Figures 6.4, 6.5). In particular, if Γ_c divides Ω into two separate domains Ω^1, Ω^2, we obtain the contact problem for two elastic plates occupying the domains Ω^1, Ω^2 with contact conditions (6.64)–(6.66) on the common boundary Γ_c.

Now we can formulate the smooth domain method for the problem (6.59)–(6.66). In this case the solution is defined in the smooth domain Ω. In fact, we extend the unknown functions from the domain Ω_c to the domain Ω. To simplify formulae below we use the same notations for the extended functions. The formulation of the problem is as follows.

In the domain Ω, we have to find functions $u = (u_1, u_2), w, \sigma = \{\sigma_{ij}\}, m = \{m_{ij}\}$, $i, j = 1, 2$, such that

$$-\operatorname{div}\sigma = f \quad \text{in } \Omega, \tag{6.75}$$

$$-\nabla\nabla m = F \quad \text{in } \Omega, \tag{6.76}$$

$$C\sigma - \varepsilon(u) + p(u)\delta_{\Gamma_c} = 0 \quad \text{in } \Omega, \tag{6.77}$$

$$Dm + \nabla\nabla w + P(w) = 0 \quad \text{in } \Omega, \tag{6.78}$$

$$u = w = \frac{\partial w}{\partial n} = 0 \quad \text{on } \Gamma, \tag{6.79}$$

$$[u]\nu \geq \left|\left[\frac{\partial w}{\partial \nu}\right]\right|, \ \sigma_\tau = 0, \ t^\nu(m) = 0 \quad \text{on } \Gamma_c, \tag{6.80}$$

$$|m_\nu| \leq -\sigma_\nu, \ \sigma_\nu \cdot [u]\nu - m_\nu \left[\frac{\partial w}{\partial \nu}\right] = 0 \quad \text{on } \Gamma_c. \tag{6.81}$$

Here

$$P(w)_{ij} = -([w]\nu_i\delta_{\Gamma_c})_{,j} - [w_{,i}]\nu_j\delta_{\Gamma_c}.$$

It is very important that the solution of the problem (6.59)–(6.66) determined from the variational inequality (6.68) possesses the properties

$$[\sigma\nu] = 0, \qquad [m_\nu] = 0, \qquad [t^\nu(m)] = 0 \quad \text{on } \Gamma_c. \tag{6.82}$$

This allows us to write the equilibrium equations (6.59), (6.60) in the domain Ω in the same form. Let us verify this statement. The validity of the equation (6.75) in the domain Ω is already shown (see Section 2). So we just check (6.76). From (6.68) it follows

$$m \in L^2(\Omega_c), \qquad \nabla\nabla m \in L^2(\Omega_c), \qquad -\nabla\nabla m = F \quad \text{in } \Omega_c. \tag{6.83}$$

Let m denote the extended function, defined in Ω. Then, by (6.82)–(6.83), for any $\varphi \in C_0^\infty(\Omega)$ we have

$$\begin{aligned}\langle \nabla\nabla m + F, \varphi\rangle &= (m, \nabla\nabla\varphi)_{\Omega_1} + (m, \nabla\nabla\varphi)_{\Omega_2} + (F, \varphi)_\Omega \\ &= (\nabla\nabla m + F, \varphi)_{\Omega_1} + (\nabla\nabla m + F, \varphi)_{\Omega_2} \\ &\quad + \langle [t^\nu(m)], \varphi\rangle_{3/2} - \langle [m_\nu], \partial\varphi/\partial\nu\rangle_{1/2} = 0.\end{aligned}$$

Hence the equilibrium equation (6.76) holds in Ω in the sense of distributions. To give the weak formulation of (6.75)–(6.81) we need additional notations. Consider the space

$$\mathcal{H}(\Omega) = \{(\sigma, m) \mid \sigma = \{\sigma_{ij}\},\ m = \{m_{ij}\};\ \sigma, \operatorname{div}\sigma \in L^2(\Omega),\ m, \nabla\nabla m \in L^2(\Omega)\}$$

equipped with norm

$$\|(\sigma, m)\|^2_{\mathcal{H}(\Omega)} = \|\sigma\|^2_{L^2(\Omega)} + \|\operatorname{div}\sigma\|^2_{L^2(\Omega)} + \|m\|^2_{L^2(\Omega)} + \|\nabla\nabla m\|^2_{L^2(\Omega)}.$$

We introduce the admissible set of stresses and moments

$$\mathcal{K}(\Omega) = \{(\sigma, m) \in \mathcal{H}(\Omega) \mid \sigma_\tau = 0,\ t^\nu(m) = 0,\ |m_\nu| \le -\sigma_\nu \text{ on } \Gamma_c\}.$$

The interpretation of the conditions imposed on σ, m in the definition of $\mathcal{K}(\Omega)$ is simpler than in the case of the non-smooth domain Ω_c since the jumps on Σ of the functions $\sigma\nu$, m_ν, $t^\nu(m)$ are equal to zero by definition. Hence the equalities and inequality are satisfied in the following sense:

$$\begin{gathered}\langle \sigma_\nu \pm m_\nu, \varphi\rangle_{1/2} \le 0 \quad \forall\varphi \in H^{1/2}(\Sigma),\ \varphi \ge 0 \text{ a.e. on } \Gamma_c,\ \operatorname{supp}\varphi \subset \Gamma_c, \\ \langle \sigma_\tau, \varphi\rangle_{1/2} = 0 \quad \forall\varphi = (\varphi_1, \varphi_2) \in H^{1/2}(\Sigma),\ \varphi_i\nu_i = 0 \text{ a.e. on } \Gamma_c,\ \operatorname{supp}\varphi \subset \Gamma_c, \\ \langle t^\nu(m), \varphi\rangle_{3/2} = 0 \quad \forall\varphi \in H^{3/2}(\Sigma),\ \operatorname{supp}\varphi \subset \Gamma_c.\end{gathered}$$

In the weak formulation of the problem (6.75)–(6.81) unknown functions u, w, σ, m are such that

$$u = (u_1, u_2) \in L^2(\Omega), \qquad w \in L^2(\Omega), \qquad (\sigma, m) \in \mathcal{K}(\Omega), \tag{6.84}$$

$$-\operatorname{div}\sigma = f \qquad \text{in } \Omega, \tag{6.85}$$

$$-\nabla\nabla m = F \qquad \text{in } \Omega, \tag{6.86}$$

$$\begin{aligned}&(u, \operatorname{div}\bar\sigma - \operatorname{div}\sigma)_\Omega + (w, \nabla\nabla\bar m - \nabla\nabla m)_\Omega \\ &\qquad + (C\sigma, \bar\sigma - \sigma)_\Omega + (Dm, \bar m - m)_\Omega \ge 0 \quad \forall(\bar\sigma, \bar m) \in \mathcal{K}(\Omega).\end{aligned} \tag{6.87}$$

We can prove the following statement.

Theorem 6.12. *There exists a unique solution to the problem* (6.84)–(6.87).

Proof. The general scheme of the proof is the same as in Theorem 6.8. We introduce functions $(\sigma^0, m^0) \in \mathcal{K}(\Omega)$ satisfying the equations

$$-\operatorname{div}\sigma^0 = f, \quad -\nabla\nabla m^0 = F \quad \text{in } \Omega.$$

The functions (σ^0, m^0) can be obtained by solving the variational inequality (6.68) with arbitrary constitutive laws (6.61)–(6.62) for any given tensors C, D. Of course the tensors C, D should satisfy the required conditions. To prove the existence of a solution a similar regularization procedure is used. For a positive parameter δ the following regularized problem is considered:

$$u^\delta = (u_1^\delta, u_2^\delta) \in L^2(\Omega), \qquad w^\delta \in L^2(\Omega), \qquad (\sigma^\delta, m^\delta) \in \mathcal{K}(\Omega), \tag{6.88}$$

$$\delta u^\delta - \operatorname{div} \sigma^\delta = f \qquad \text{in } \Omega, \tag{6.89}$$

$$\delta w^\delta - \nabla\nabla m^\delta = F \qquad \text{in } \Omega, \tag{6.90}$$

$$\begin{aligned}(C\sigma^\delta, \bar\sigma - \sigma^\delta)_\Omega + (Dm^\delta, \bar m - m^\delta)_\Omega + (u^\delta, \operatorname{div}\bar\sigma - \operatorname{div}\sigma^\delta)_\Omega \\ + (w^\delta, \nabla\nabla\bar m - \nabla\nabla m^\delta)_\Omega \geq 0 \quad \forall (\bar\sigma, \bar m) \in \mathcal{K}(\Omega).\end{aligned} \tag{6.91}$$

Taking $(\bar\sigma, \bar m) = (\sigma^0, m^0)$ in (6.91) and multiplying (6.89), (6.90) by u^δ, w^δ, respectively, we derive the a priori estimate

$$\delta\|u^\delta\|^2_{L^2(\Omega)} + \delta\|w^\delta\|^2_{L^2(\Omega)} + \|\sigma^\delta\|^2_{L^2(\Omega)} + \|m^\delta\|^2_{L^2(\Omega)} \leq c \tag{6.92}$$

where the constant c is uniform with respect to δ. By (6.92), from (6.89), (6.90) we have, uniformly in δ,

$$\|\operatorname{div}\sigma^\delta\|^2_{L^2(\Omega)} + \|\nabla\nabla m^\delta\|^2_{L^2(\Omega)} \leq c. \tag{6.93}$$

Solvability of the problem (6.88)–(6.91) can be obtained by the variational approach. To this end it suffices to substitute the values u^δ, w^δ, taken from (6.89), (6.90), into (6.91). In this way we obtain a variational inequality for $(\sigma^\delta, m^\delta)$ which admits a solution. Let us perform the passage to the limit in (6.89)–(6.91) as $\delta \to 0$. From (6.91) it follows that

$$C\sigma^\delta - \varepsilon(u^\delta) = 0, \qquad Dm^\delta + \nabla\nabla w^\delta = 0 \qquad \text{in } \Omega_c, \tag{6.94}$$

hence $\varepsilon(u^\delta) \in L^2(\Omega_c)$. By the second Korn inequality in Ω_c, since $u^\delta \in L^2(\Omega_c)$, we obtain $u^\delta \in H^1(\Omega_c)$. On the other hand,

$$u^\delta = 0 \qquad \text{on } \Gamma,$$

and consequently $u^\delta = (u_1^\delta, u_2^\delta) \in H^{1,0}(\Omega_c)$. We use the first Korn inequality,

$$\|u_1^\delta\|_{H^{1,0}(\Omega_c)} + \|u_2^\delta\|_{H^{1,0}(\Omega_c)} \leq c\|\varepsilon(u^\delta)\|_{L^2(\Omega_c)},$$

where the constant c depends only on Ω_c. Since the deformations $\varepsilon(u^\delta)$ are bounded in $L^2(\Omega_c)$ uniformly in δ, the following estimate holds:

$$\|u_i^\delta\|_{H^{1,0}(\Omega_c)} \leq c, \qquad i = 1, 2. \tag{6.95}$$

Next, the second equation of (6.94) implies $\nabla\nabla w^\delta \in L^2(\Omega_c)$. Consequently, $w^\delta \in H^2(\Omega_c)$. Taking into account the boundary conditions

$$w^\delta = \frac{\partial w^\delta}{\partial n} = 0 \qquad \text{on } \Gamma$$

it follows that $w^\delta \in H^{2,0}(\Omega_c)$. We can use the inequality

$$\|w^\delta\|_{H^{2,0}(\Omega_c)} \leq c\|\nabla\nabla w^\delta\|_{L^2(\Omega_c)}$$

with the constant c independent of δ, which leads to the uniform estimate with respect to δ,

$$\|w^\delta\|_{H^{2,0}(\Omega_c)} \leq c. \tag{6.96}$$

Hence, by (6.92), (6.93), (6.95), (6.96), we can assume that as $\delta \to 0$,

$$\begin{aligned} u_i^\delta &\to u_i && \text{strongly in } L^2(\Omega),\ i = 1,2, \\ w^\delta &\to w && \text{strongly in } L^2(\Omega), \\ (\sigma^\delta, m^\delta) &\to (\sigma, m) && \text{weakly in } \mathcal{H}(\Omega). \end{aligned}$$

These convergences allow us to pass to the limit in (6.88)–(6.91) as $\delta \to 0$, which implies (6.84)–(6.87).

The solution is unique. Indeed, assume that we have two solutions $(u^1, w^1, \sigma^1, m^1)$ and $(u^2, w^2, \sigma^2, m^2)$. From (6.87) it follows that $\sigma^1 = \sigma^2, m^1 = m^2$. Since

$$C\sigma^i - \varepsilon(u^i) = 0, \qquad Dm^i + \nabla\nabla w^i = 0 \quad \text{in } \Omega_c,\ i = 1,2,$$

we obtain $\varepsilon(u^1 - u^2) = 0,\ \nabla\nabla(w^1 - w^2) = 0$. Consequently, $u^1 = u^2, w^1 = w^2$. ∎

REMARK 6.13. Similar to two-dimensional elasticity we can use a definition of admissible stresses and moments which is suitable both for interior and boundary cracks, namely,

$$\mathcal{K}(\Omega) = \left\{ (\sigma, m) \in \mathcal{H}(\Omega) \;\middle|\; \int_{\Omega_c} (\sigma\varepsilon(\bar{u}) + \bar{u} \operatorname{div} \sigma) + \int_{\Omega_c} (\bar{w}\nabla\nabla m - m\nabla\nabla\bar{w}) \geq 0 \ \forall(\bar{u}, \bar{w}) \in K_c \right\}.$$

In particular, this definition is useful for contact problems (see Figure 6.5).

REMARK 6.14. Consider the classical crack problem for the Kirchhoff plate. In this case instead of (6.64)–(6.66) we have the linear boundary conditions

$$m_\nu = t^\nu(m) = \sigma_\nu = \sigma_\tau = 0 \quad \text{on } \Gamma_c^\pm. \tag{6.97}$$

The smooth domain method proposed in this paper can be applied to the problem (6.59)–(6.63), (6.97). The admissible set of stresses and moments in this linear case is introduced as follows:

$$\mathcal{K}(\Omega) = \{(\sigma, m) \in \mathcal{H}(\Omega) \mid m_\nu = t^\nu(m) = \sigma_\nu = \sigma_\tau = 0 \text{ on } \Gamma_c\}. \tag{6.98}$$

The inequality (6.87) should be replaced by the identity

$$(u, \operatorname{div} \bar{\sigma})_\Omega + (w, \nabla\nabla\bar{m})_\Omega + (C\sigma, \bar{\sigma})_\Omega + (Dm, \bar{m})_\Omega = 0 \quad \forall(\bar{\sigma}, \bar{m}) \in \mathcal{K}(\Omega). \tag{6.99}$$

Hence the smooth domain method in the classical crack problem for plates can be formulated in the form (6.84)–(6.86), (6.99), where the set $\mathcal{K}(\Omega)$ is defined in (6.98).

6.4. Three-dimensional case

6.4.1. Preliminaries. We consider elastoplastic models of the Hencky type for bodies having cracks. The presence of a crack entails that the geometrical domain has a non-smooth component of the boundary. New formulations for these models are proposed. The resulting mathematical models allow us to solve the crack problem in a smooth geometrical domain. The problems under consideration are characterized by linear boundary conditions imposed on non-smooth parts of the boundary. These conditions describe the traction free crack faces. Notice that linear crack problems for elastic bodies have been analyzed in many papers; we refer the reader to [25]–[27], [36], [90]–[101]. For the non-linear crack theory with a possible contact between crack faces for various constitutive laws, the results can be found in [67]. Solvability of elastoplastic problems was analyzed in [19], [20], [31], [58]–[71], [130], [131]. We should remark that the smooth domain method proposed in this paper can be applied to elastic linear crack problems as well as to many other linear and non-linear elliptic boundary value problems. For elastic bodies with non-linear cracks the smooth domain method was developed in [76].

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary Γ, and $\Gamma_c \subset \Omega$ be a smooth orientable two-dimensional surface. We assume that this surface can be extended up to the outer boundary Γ in such a way that Ω is divided into two subdomains with Lipschitz boundaries. We assume that this inner surface Γ_c is described parametrically by the equations

$$x_i = x_i(y_1, y_2), \qquad i = 1, 2, 3,$$

where (y_1, y_2) belong to the closure of a bounded domain $\omega \subset \mathbb{R}^2$ having a smooth boundary γ. We suppose that the rank of the Jacobi matrix $\partial x_i/\partial y_j$ equals 2 at every point $(y_1, y_2) \in \omega \cup \gamma$, and that the map is one-to-one. Let $\nu = (\nu_1, \nu_2, \nu_3)$ be a unit normal vector to Γ_c. Define $\Omega_c = \Omega \setminus \Gamma_c$.

The boundary of Ω_c consists of three components $\Gamma, \Gamma_c^+, \Gamma_c^-$, where $\Gamma_c^\pm$ correspond to the positive and negative directions of the normal ν, respectively. Notice that for all functions on Ω_c to be discussed below, their traces will in general differ on Γ_c^+ and Γ_c^-. We also assume that Γ_c can be extended up to a closed surface Σ without self-intersections of class $C^{0,1}$ so that $\Sigma \subset \Omega$, and the domain Ω is divided into two subdomains Ω_1, Ω_2. In this case Σ is the boundary of the domain Ω_1, and the boundary of Ω_2 is $\Sigma \cup \Gamma$.

We define the Banach space

$$LD(\Omega_c) = \{u = (u_1, u_2, u_3) \mid u_i \in L^1(\Omega_c),\, i = 1, 2, 3,\, \varepsilon_{ij}(u) \in L^1(\Omega_c),\, i, j = 1, 2, 3\}$$

equipped with the norm

$$\|u\|_{LD(\Omega_c)} = \|u\|_{L^1(\Omega_c)} + \sum_{i,j=1}^{3} \|\varepsilon_{ij}(u)\|_{L^1(\Omega_c)}. \tag{6.100}$$

Here $\varepsilon_{ij}(u) = (u_{i,j} + u_{j,i})/2$ are the components of the strain tensor. Consider function spaces whose elements are characterized by the conditions

$$\int_{\Omega_c} u = 0, \qquad \int_{\Omega_c} (u_i x_j - u_j x_i) = 0, \qquad i, j = 1, 2, 3, \qquad u = (u_1, u_2, u_3). \tag{6.101}$$

In particular, we define

$$LD_N(\Omega_c) = \{u \in LD(\Omega_c) \mid u \text{ satisfies (6.101)}\}.$$

Note that the linear space $R(\Omega_c)$ of functions ρ satisfying the conditions $\varepsilon_{ij}(\rho) = 0$ in Ω_c, $i, j = 1, 2, 3$, can be described as $\rho(x) = c + Bx$, $x \in \Omega_c$, where $c = (c_1, c_2, c_3)$ is a constant vector, $B = (b_{ij})$ is a constant matrix with $b_{ij} = -b_{ji}$, for all i, j. In componentwise notation, $\rho_i(x) = c_i + b_{ij} x_j$. One can see that the orthogonal complement of the subspace $R(\Omega_c)$ in $L^2(\Omega_c)$ coincides with the subspace of all functions from $L^2(\Omega_c)$ satisfying (6.101). Therefore we see that if $\rho \in R(\Omega_c)$ satisfies (6.101), then $\rho \equiv 0$.

Since

$$\left| \int_{\Omega_c} u \right| + \sum_{i,j=1}^{3} \left| \int_{\Omega_c} (u_i x_j - u_j x_i) \right|$$

is a seminorm on the space $LD(\Omega_c)$ and a norm on $R(\Omega_c)$, it follows that

$$|u|_{LD(\Omega_c)} = \left|\int_{\Omega_c} u\right| + \sum_{i,j=1}^{3}\left|\int_{\Omega_c}(u_i x_j - u_j x_i)\right| + \sum_{i,j=1}^{3}\|\varepsilon_{ij}(u)\|_{L^1(\Omega_c)}$$

defines a norm on $LD(\Omega_c)$ which is equivalent to the original norm (6.100) (see [131]). Consider next the space of bounded measures $M^1(\Omega_c)$. We know that $M^1(\Omega_c)$ is the dual of the normed space $C_0(\Omega_c)$ of continuous functions with compact support, endowed with the uniform convergence topology (see [47], [131]). Any ball in $M^1(\Omega_c)$ is compact in the weak star topology, and every bounded sequence in $M^1(\Omega_c)$ has a subsequence which is weak* convergent. We recall that by definition a sequence $g_m \in M^1(\Omega_c)$ is weak* convergent to an element $g \in M^1(\Omega_c)$ if

$$g_m(\phi) \to g(\phi), \qquad m \to \infty,$$

for any fixed $\phi \in C_0(\Omega_c)$. Now we can introduce the Banach space of bounded deformations

$$BD(\Omega_c) = \{u = (u_1, u_2, u_3) \mid u_i \in L^1(\Omega_c),\ i = 1,2,3,\ \varepsilon_{ij}(u) \in M^1(\Omega_c),\ i,j = 1,2,3\}$$

equipped with the norm

$$\|u\|_{BD(\Omega_c)} = \|u\|_{L^1(\Omega_c)} + \sum_{i,j=1}^{3}\|\varepsilon_{ij}(u)\|_{M^1(\Omega_c)}.$$

Also, denote by $BD_N(\Omega_c)$ the subspace of $BD(\Omega_c)$ which consists of all elements of $BD(\Omega_c)$ satisfying (6.101). Consider also the space

$$H^1(\Omega_c) = \{u = (u_1, u_2, u_3) \mid u_i \in L^2(\Omega_c),\ i = 1,2,3;\ u_{i,j} \in L^2(\Omega_c),\ i,j = 1,2,3\}$$

with the norm

$$\|u\|_{H^1(\Omega_c)} = \|u\|_0 + \sum_{i,j=1}^{3}\|u_{i,j}\|_0, \tag{6.102}$$

where $\|\cdot\|_0$ is the norm in $L^2(\Omega_c)$. To simplify the notations, we write $H^1(\Omega_c)$ instead of $[H^1(\Omega_c)]^3$. Let

$$H^1_N(\Omega_c) = \{u \in H^1(\Omega_c) \mid u \text{ satisfies } (6.101)\}.$$

We shall use the following norm in $H^1(\Omega_c)$:

$$|u|_{H^1(\Omega_c)} = \left|\int_{\Omega_c} u\right| + \sum_{i,j=1}^{3}\left|\int_{\Omega_c}(u_i x_j - u_j x_i)\right| + \sum_{i,j=1}^{3}\|\varepsilon_{ij}(u)\|_0,$$

which is equivalent to the norm (6.102). It is easy to see that

$$|u|_{H^1_N(\Omega_c)} = \sum_{i,j=1}^{3}\|\varepsilon_{ij}(u)\|_0 \tag{6.103}$$

is a norm on the subspace $H^1_N(\Omega_c)$.

Let us recall the well-known Green formula. Namely, if $\sigma_{ij} \in L^2(\Omega)$, $\sigma_{ij,j} \in L^2(\Omega)$, $i,j = 1,2,3$, then the values $\sigma_{ij}n_j$ can be correctly defined on Γ, and moreover $\sigma_{ij}n_j \in$

$H^{-1/2}(\Gamma)$,

$$-\langle \sigma_{ij,j}, \theta \rangle = \langle \sigma_{ij}, \theta_{,j} \rangle - \langle \sigma_{ij} n_j, \theta \rangle_{1/2,\Gamma}, \qquad \forall \theta \in H^1(\Omega),\ i = 1,2,3. \tag{6.104}$$

Here $n = (n_1, n_2, n_3)$ is the outer normal to the boundary Γ, the brackets $\langle \cdot\,, \cdot \rangle$, $\langle \cdot\,, \cdot \rangle_{1/2,\Gamma}$ denote the scalar product in $L^2(\Omega)$ and the duality pairing between the spaces $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, respectively.

All functions which carry two lower indices are assumed to be symmetric with respect to those indices, i.e. $\sigma_{ij} = \sigma_{ji}$, etc.

6.4.2. Existence of solutions. In this section we prove existence of a solution of the elastoplastic boundary value problem for a body having a crack. The formulation of the elastoplastic problem for a body occupying the domain Ω_c in its undeformed state is as follows. In the domain Ω_c we have to find functions $u = (u_1, u_2, u_3)$, $\sigma = \{\sigma_{ij}\}$, ξ_{ij}, $i, j = 1,2,3$, which satisfy the following equations and inequalities:

$$-\sigma_{ij,j} = f_i, \qquad i = 1,2,3, \tag{6.105}$$

$$\varepsilon_{ij}(u) = a_{ijkl}\sigma_{kl} + \xi_{ij}, \qquad i,j = 1,2,3, \tag{6.106}$$

$$\Phi(\sigma) \le 0, \qquad \xi_{ij}(\bar\sigma_{ij} - \sigma_{ij}) \le 0 \quad \forall \bar\sigma,\ \Phi(\bar\sigma) \le 0, \tag{6.107}$$

$$\sigma_{ij} n_j = 0, \qquad i = 1,2,3, \qquad \text{on } \Gamma, \tag{6.108}$$

$$\sigma_{ij} \nu_j = 0, \qquad i = 1,2,3, \qquad \text{on } \Gamma_c^\pm. \tag{6.109}$$

Here $\Phi : \mathbb{R}^6 \to \mathbb{R}$ is a continuous convex function describing the plastic yield condition. The equations (6.106) provide a decomposition of the strain tensor $\varepsilon_{ij}(u)$ into the sum of an elastic part $a_{ijkl}\sigma_{kl}$ and a plastic part ξ_{ij}, and (6.105) are the equilibrium equations.

We assume that the functions $a_{ijkl}(x)$ possess the property $a_{ijkl} = a_{jikl} = a_{klij}$, and that there exist constants $c_1, c_2 > 0$ such that

$$c_1|\sigma|^2 \le a_{ijkl}\sigma_{kl}\sigma_{ij} \le c_2|\sigma|^2, \qquad \forall \sigma = \{\sigma_{ij}\}. \tag{6.110}$$

This condition allows us to solve the equations $\varepsilon_{ij}(u) = a_{ijkl}\sigma_{kl}$, $i,j = 1,2,3$, with respect to σ_{ij}, and to obtain $\sigma_{ij} = b_{ijkl}\varepsilon_{kl}(u)$, $i,j = 1,2,3$. The functions b_{ijkl} have the same properties as the functions a_{ijkl}. In particular, the inequalities corresponding to (6.110) hold true.

The basic assumption on the function Φ is that the subset $\{\sigma = \{\sigma_{ij}\} \mid \Phi(\sigma) \le 0\}$ of $\mathbb{R}^6$ contains zero as its interior point.

The functions ξ_{ij} can be eliminated from (6.106), (6.107). Indeed, multiply (6.32) by $\bar\sigma_{ij} - \sigma_{ij}$, where $\Phi(\bar\sigma) \le 0$, $\bar\sigma_{ij} n_j = 0$, $i = 1,2,3$, on Γ, and $\bar\sigma_{ij}\nu_j = 0$, $i = 1,2,3$, on $\Gamma_c^\pm$, sum the relations thus obtained over i,j and integrate over Ω_c. By the second inequality (6.33) this yields the relation

$$\int_{\Omega_c} a_{ijkl}\sigma_{kl}(\bar\sigma_{ij} - \sigma_{ij}) + \int_{\Omega_c} u_i(\bar\sigma_{ij,j} - \sigma_{ij,j}) \ge 0,$$

which will be used to define a solution of the problem (6.105)–(6.109).

Introduce two more notations, namely

$$V(\Omega_c) = \{\sigma = \{\sigma_{ij}\} \mid \sigma_{ij} \in L^2(\Omega_c),\ i,j = 1,2,3;\ \sigma_{ij,j} \in L^3(\Omega_c),\ i = 1,2,3;$$
$$\sigma_{ij} n_j = 0,\ i = 1,2,3, \text{ on } \Gamma;\ \sigma_{ij}\nu_j = 0,\ i = 1,2,3, \text{ on } \Gamma_c^\pm\}.$$

The set K of admissible stresses is defined by

$$K = \{\sigma = \{\sigma_{ij}\} \mid \sigma_{ij} \in L^2(\Omega_c),\ i,j = 1,2,3,\ \Phi(\sigma(x)) \le 0 \text{ a.e. in } \Omega_c\}.$$

We moreover assume that there exists a function $\sigma^0 = \{\sigma^0_{ij}\}$ such that $\sigma^0 \in (1+\kappa)^{-1}K$, where $\kappa > 0$ is a constant, and

$$\langle \sigma^0_{ij}, \varepsilon_{ij}(\bar{u})\rangle_c = \langle f, \bar{u}\rangle_c \quad \forall \bar{u} \in H^1_N(\Omega_c). \tag{6.111}$$

Here $\langle \cdot,\cdot\rangle_c$ is the scalar product in $L^2(\Omega_c)$.

Consider next the scalar product in $H^1(\Omega_c)$,

$$(u,v)_c = \langle u,v\rangle_c + \langle \varepsilon_{ij}(u), \varepsilon_{ij}(v)\rangle_c, \quad u,v \in H^1(\Omega_c).$$

Then the space $H^1(\Omega_c)$ can be written as a sum $H^1(\Omega_c) = R(\Omega_c) \oplus H^1_N(\Omega_c)$ of orthogonal subspaces. We assume $\langle f, \rho\rangle_c = 0$ for all $\rho \in R(\Omega_c)$. Consequently, identity (6.111) holds for all $\bar{u} \in H^1(\Omega_c)$. In particular, this means that σ^0 satisfies all boundary conditions on Γ and $\Gamma^\pm_c$ in the definition of $V(\Omega_c)$. Now we can present an existence theorem for problem (6.105)–(6.109).

THEOREM 6.15. *Let $f \in [L^3(\Omega_c)]^3$ be such that $\langle f, \rho\rangle_c = 0$ for all $\rho \in R(\Omega_c)$, and σ^0 have the properties stated above. Then there exist functions $\sigma \in K$ and $u \in BD_N(\Omega_c)$ such that*

$$\langle \sigma_{ij}, \varepsilon_{ij}(\bar{u})\rangle_c = \langle f, \bar{u}\rangle_c \quad \forall \bar{u} \in H^1_N(\Omega_c), \tag{6.112}$$

$$\langle a_{ijkl}\sigma_{kl}, \bar{\sigma}_{ij} - \sigma_{ij}\rangle_c + \langle u_i, \bar{\sigma}_{ij,j} - \sigma_{ij,j}\rangle_c \ge 0 \quad \forall \bar{\sigma} \in K \cap V(\Omega_c). \tag{6.113}$$

Proof. The proof is based on an elliptic regularization of a penalized problem. Let $p(\sigma) = \sigma - \pi\sigma$ be the penalty operator, where $\pi : [L^2(\Omega_c)]^6 \to K$ is the orthogonal projection. Consider the following auxiliary boundary value problem which includes two positive parameters α and δ. In the domain Ω_c we want to find functions $u = (u_1,u_2,u_3)$, $\sigma = \{\sigma_{ij}\}$, $i,j = 1,2,3$, such that

$$-\alpha(b_{ijkl}\varepsilon_{kl}(u))_{,j} - \sigma_{ij,j} = f_i, \quad i = 1,2,3, \tag{6.114}$$

$$a_{ijkl}\sigma_{kl} - \varepsilon_{ij}(u) + \frac{1}{\delta}p(\sigma)_{ij} = 0, \quad i,j = 1,2,3, \tag{6.115}$$

$$\sigma_{ij}n_j + \alpha b_{ijkl}\varepsilon_{kl}(u)n_j = 0, \quad i = 1,2,3, \quad \text{on } \Gamma, \tag{6.116}$$

$$\sigma_{ij}\nu_j + \alpha b_{ijkl}\varepsilon_{kl}(u)\nu_j = 0, \quad i = 1,2,3, \quad \text{on } \Gamma^\pm_c. \tag{6.117}$$

The solvability of the problem (6.114)–(6.117) for fixed parameters α, δ will be proved in the following sense:

$$u \in H^1_N(\Omega_c), \quad \sigma_{ij} \in L^2(\Omega_c), \quad i,j = 1,2,3, \tag{6.118}$$

$$\alpha\langle b_{ijkl}\varepsilon_{kl}(u), \varepsilon_{ij}(\bar{u})\rangle_c + \langle \sigma_{ij}, \varepsilon_{ij}(\bar{u})\rangle_c = \langle f, \bar{u}\rangle_c \quad \forall \bar{u} \in H^1_N(\Omega_c), \tag{6.119}$$

$$a_{ijkl}\sigma_{kl} - \varepsilon_{ij}(u) + \frac{1}{\delta}p(\sigma)_{ij} = 0, \quad i,j = 1,2,3. \tag{6.120}$$

To obtain an a priori estimate of the solution to (6.118)–(6.120) we first substitute $\bar{u} = u$ in (6.119) and multiply (6.120) by $\sigma_{ij} - \sigma^0_{ij}$. This gives the estimate

$$\alpha \sum_{i,j=1}^{3} \|\varepsilon_{ij}(u)\|_0^2 + \|\sigma\|_0^2 \le c.$$

Since $u \in H^1_N(\Omega_c)$ this inequality implies

$$\alpha |u|^2_{H^1_N(\Omega_c)} + \|\sigma\|^2_0 \le c. \tag{6.121}$$

The constant c does not depend on α and δ. The estimate (6.121) allows us to prove the solvability of (6.118)–(6.120) for any fixed parameters α, δ. To verify this, we write the boundary value problem (6.118)–(6.120) in the form

$$B(w) = F, \tag{6.122}$$

with an operator B which maps a Hilbert space V to its dual space V', and where F is a given element of V'. Here we choose $V = H^1_N(\Omega_c) \times [L^2(\Omega_c)]^6$ and define B by the formula

$$B(w)(\bar{w}) = \langle \alpha b_{ijkl}\varepsilon_{kl}(u) + \sigma_{ij}, \varepsilon_{ij}(\bar{u})\rangle_c + \left\langle a_{ijkl}\sigma_{kl} - \varepsilon_{ij}(u) + \frac{1}{\delta}p(\sigma)_{ij}, \bar{\sigma}_{ij} \right\rangle_c,$$

where $w = (u, \sigma), \bar{w} = (\bar{u}, \bar{\sigma})$, and we set $F(\bar{w}) = \langle f, \bar{u}\rangle_c$. The operator B is bounded, monotone and semicontinuous; actually, the computations leading to the estimate (6.121) also provide the coercivity of B in the sense

$$B(w)(w)/\|w\|_V \to \infty, \qquad \|w\|_V \to \infty.$$

Thus, the solvability of the equation (6.122), or, equivalently, of the problem (6.118)–(6.120) follows from classical results (see [88]). The boundary conditions (6.116)–(6.117) are included in the identity (6.119).

In addition to the estimate (6.121) one can prove that the estimate

$$\frac{1}{\delta}\,\|p(\sigma)\|_{L^1(\Omega_c)} \le c$$

holds uniformly in α and δ. We omit the argument since a similar situation appears in the next section when estimating a penalty term in $L^1(\Omega_c)$.

From (6.120) we can derive an additional estimate. Indeed, for any fixed $\delta > 0$ there exists a constant $c(\delta)$ depending on δ such that

$$\sum_{i,j=1}^{3} \|\varepsilon_{ij}(u)\|_0 \le c(\delta)$$

and hence

$$|u|_{H^1_N(\Omega_c)} \le c(\delta). \tag{6.123}$$

Now we can pass to the limit in (6.118)–(6.120) as $\alpha, \delta \to 0$. Denote by $u^{\alpha\delta}, \sigma^{\alpha\delta}$ the solution of (6.118)–(6.120) corresponding to given parameters α, δ. Due to the estimates (6.121) and (6.123), we can choose a subsequence, still denoted by $u^{\alpha\delta}, \sigma^{\alpha\delta}$, such that for $\alpha \to 0$ and any fixed δ,

$$\begin{aligned} u^{\alpha\delta} &\to u^\delta && \text{weakly in } H^1_N(\Omega_c), \\ \sigma^{\alpha\delta}_{ij} &\to \sigma^\delta_{ij} && \text{weakly in } L^2(\Omega_c),\ i,j = 1,2,3. \end{aligned}$$

On passing to the limit as $\alpha \to 0$, the equations (6.118)–(6.120) become

$$u^\delta \in H^1_N(\Omega_c), \quad \sigma^\delta_{ij} \in L^2(\Omega_c), \quad i,j = 1,2,3,$$
$$\langle \sigma^\delta_{ij}, \varepsilon_{ij}(\bar{u})\rangle_c = \langle f, \bar{u}\rangle_c \quad \forall \bar{u} \in H^1_N(\Omega_c), \tag{6.124}$$
$$a_{ijkl}\sigma^\delta_{kl} - \varepsilon_{ij}(u^\delta) + \frac{1}{\delta}\, p(\sigma^\delta)_{ij} = 0, \quad i,j = 1,2,3. \tag{6.125}$$

Consequently, the equations (6.125) imply that

$$\sum_{i,j=1}^{3} \|\varepsilon_{ij}(u^\delta)\|_{L^1(\Omega_c)} \le c$$

and since $u^\delta \in H^1_N(\Omega_c)$, this inequality gives

$$|u^\delta|_{LD_N(\Omega_c)} \le c.$$

The imbeddings $LD(\Omega_c) \subset L^{3/2}(\Omega_c)$, $L^1(\Omega_c) \subset M^1(\Omega_c)$ are continuous in the three-dimensional case, hence

$$\|u^\delta\|_{L^{3/2}(\Omega_c)} \le c, \qquad \sum_{i,j=1}^{3} \|\varepsilon_{ij}(u^\delta)\|_{M^1(\Omega_c)} \le c. \tag{6.126}$$

Due to the estimates (6.121) and (6.126), we can assume that a subsequence u^δ, σ^δ possesses the properties

$$\begin{aligned} u^\delta &\to u && \text{weakly in } L^{3/2}(\Omega_c),\\ \varepsilon_{ij}(u^\delta) &\to \varepsilon_{ij}(u) && \text{weak}^* \text{ in } M^1(\Omega_c),\ i,j = 1,2,3,\\ \sigma^\delta_{ij} &\to \sigma_{ij} && \text{weakly in } L^2(\Omega_c),\ i,j = 1,2,3. \end{aligned}$$

The identity (6.124) easily yields (6.112). We can next derive from (6.124) that the equations

$$-\sigma^\delta_{ij,j} = f_i, \quad i = 1,2,3,$$

hold in the sense of distributions in the domain Ω_c, whence

$$\langle \sigma^\delta_{ij}, \varepsilon_{ij}(u^\delta)\rangle_c = \langle f, u^\delta\rangle_c = -\langle \sigma^\delta_{ij,j}, u^\delta_i\rangle_c. \tag{6.127}$$

Note that

$$\langle \varepsilon_{ij}(u^\delta), \bar{\sigma}_{ij}\rangle_c = -\langle u^\delta_i, \bar{\sigma}_{ij,j}\rangle_c \quad \forall \bar{\sigma} \in V(\Omega_c). \tag{6.128}$$

Let us multiply (6.125) by $\bar{\sigma}_{ij} - \sigma^\delta_{ij}$, where $\bar{\sigma} \in K \cap V(\Omega_c)$. Taking into account (6.127), (6.128) we have

$$\langle a_{ijkl}\sigma^\delta_{kl}, \bar{\sigma}_{ij} - \sigma^\delta_{ij}\rangle_c + \langle u^\delta_i, \bar{\sigma}_{ij,j} - \sigma^\delta_{ij,j}\rangle_c \ge 0 \quad \forall \bar{\sigma} \in K \cap V(\Omega_c). \tag{6.129}$$

The values $\sigma^\delta_{ij,j}$ can be replaced by $-f_i$ in (6.129). This allows us to pass to the limit as $\delta \to 0$, and we arrive at (6.113).

The inclusion $\sigma \in K$ is proved by the standard method. The boundary conditions (6.108)–(6.109) are a consequence of (6.112).

Note that the specific choice of b_{ijkl} as the inverse of the a_{ijkl} for the elliptic regularization appears to be natural, since in the pure elastic case (with $K = [L^2(\Omega)]^6$, respectively

$p(\sigma) \equiv 0$), the boundary conditions (6.116), (6.117) reduce to (6.108), (6.109), respectively. However, the proof of Theorem 6.15 works with any other choice of b_{ijkl} as long as the requirements of symmetry, boundedness and coercivity are met.

6.4.3. Smooth domain method. Since the identity (6.112) holds for all test functions from $H^1(\Omega_c)$ the equilibrium equations (6.105) hold in Ω_c in the sense of distributions,

$$-\sigma_{ij,j} = f_i \quad \text{in } \Omega_c,\ i = 1, 2, 3. \tag{6.130}$$

Also (6.112) implies the zero jump condition,

$$[\sigma_{ij}\nu_j] = 0 \quad \text{on } \Gamma_c,\ i = 1, 2, 3. \tag{6.131}$$

This condition is understood in the following sense:

$$\langle(\sigma_{ij}\nu_j)^+ - (\sigma_{ij}\nu_j)^-, \varphi_i\rangle_{1/2,\Sigma} = 0 \quad \forall\varphi = (\varphi_1, \varphi_2, \varphi_3) \in H^{1/2}(\Sigma),$$

which follows directly from (6.112). By (6.130), (6.131), we can show that

$$-\sigma_{ij,j} = f_i \quad \text{in } \Omega,\ i = 1, 2, 3,$$

in the sense of distributions. Indeed, for any $\varphi \in C_0^\infty(\Omega)$,

$$\begin{aligned}\{\sigma_{ij,j} + f_i, \varphi\} &= -\langle\sigma_{ij}, \varphi_{,j}\rangle_{\Omega_1} - \langle\sigma_{ij}, \varphi_{,j}\rangle_{\Omega_2} + \langle f_i, \varphi\rangle \\ &= \langle[\sigma_{ij}\nu_j], \varphi\rangle_{1/2,\Sigma} + \langle\sigma_{ij,j} + f_i, \varphi\rangle_{\Omega_1} + \langle\sigma_{ij,j} + f_i, \varphi\rangle_{\Omega_2} = 0, \quad i = 1, 2, 3,\end{aligned}$$

where $\{\cdot, \varphi\}$ is the value of a distribution on the element φ, and $\langle\cdot,\cdot\rangle_{\Omega_i}$ are the scalar products in $L^2(\Omega_i)$, respectively. These formulae prove the statement.

Now from Theorem 6.15 we derive existence of solutions to the elastoplastic crack problem (6.105)–(6.109) defined in the smooth geometrical domain Ω. This means that we can solve elastoplastic crack problems in the smooth domain Ω instead of non-smooth domain Ω_c. Set

$$\begin{aligned}V(\Omega) = \{\sigma = \{\sigma_{ij}\} \mid \sigma_{ij} \in L^2(\Omega),\ i, j = 1, 2, 3;\ \sigma_{ij,j} \in L^3(\Omega),\ i = 1, 2, 3; \\ \sigma_{ij}n_j = 0,\ i = 1, 2, 3, \text{ on } \Gamma;\ \sigma_{ij}\nu_j = 0,\ i = 1, 2, 3, \text{ on } \Gamma_c\}.\end{aligned}$$

Notice that the condition $\sigma_{ij}\nu_j = 0$ on Γ_c in the definition of $V(\Omega)$ is imposed on the smooth surface Γ_c located inside the domain Ω.

THEOREM 6.16. *Suppose the assumptions of Theorem* 6.15 *hold. Then there exist functions* u, σ *such that*

$$\sigma \in K \cap V(\Omega), \qquad u \in L^{3/2}(\Omega),$$
$$-\sigma_{ij,j} = f_i \quad \textit{in } \Omega,\ i = 1, 2, 3,$$
$$\langle a_{ijkl}\sigma_{kl}, \bar\sigma_{ij} - \sigma_{ij}\rangle + \langle u_i, \bar\sigma_{ij,j} - \sigma_{ij,j}\rangle \geq 0 \quad \forall\bar\sigma \in K \cap V(\Omega).$$

6.5. Elastoplastic problems for plates with cracks

6.5.1. Existence of solutions—Smooth domain method. We first prove an existence theorem for elastoplasic plates having cracks. Then we formulate the smooth domain approach for this problem.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary Γ, and Γ_c be a smooth curve without self-intersections, $\Gamma_c \subset \Omega$. We assume that Γ_c does not contain its end points, i.e. $\Gamma_c = \bar\Gamma_c \backslash \partial\Gamma_c$. Denote by Ω_c the mid-surface of the plate, $\Omega_c = \Omega\backslash\bar\Gamma_c$. We choose a unit normal vector $\nu = (\nu_1, \nu_2)$ to the curve Γ_c. The curve Γ_c corresponds to the crack in the plate. The crack shape as a surface in $\mathbb{R}^3$ can be described as $x \in \Gamma_c,\ -h \le z \le h$, where $x = (x_1, x_2) \in \Omega$, $2h$ is the thickness of the plate, z is the distance to Ω. The domain Ω_c contains, therefore, three components of the boundary: Γ, Γ_c^+, Γ_c^-. Here $\Gamma_c^\pm$ correspond to the positive and negative directions of the normal ν, respectively. Let $n = (n_1, n_2)$ be the external unit normal vector to Γ.

As in the previous section for the three-dimensional case, we assume that Γ_c can be extended to a closed curve Σ of class $C^{1,1}$, $\Sigma \subset \Omega$, such that Ω_c is divided into two domains Ω_1, Ω_2 with boundaries Σ and $\Sigma \cup \Gamma$, respectively. The normal ν is directed to the domain Ω_2.

The formulation of the elastoplastic problem for the plate having a crack is as follows. In the domain Ω_c we want to find functions w, $m = \{m_{ij}\}$, ξ_{ij}, $i, j = 1, 2$, satisfying the following equations and inequalities:

$$-m_{ij,ij} = f, \tag{6.132}$$

$$-w_{,ij} = a_{ijkl} m_{kl} + \xi_{ij}, \qquad i, j = 1, 2, \tag{6.133}$$

$$\Psi(m_{ij}) \le 0,\ \ \xi_{ij}(\bar{m}_{ij} - m_{ij}) \le 0 \qquad \forall \bar{m},\ \Psi(\bar{m}_{ij}) \le 0, \tag{6.134}$$

$$w = 0,\ \ m_{ij} n_j n_i = 0 \qquad\quad \text{on } \Gamma, \tag{6.135}$$

$$m_{ij} \nu_j \nu_i = 0,\ \ R_\nu(m_{ij}) = 0 \qquad \text{on } \Gamma_c^\pm. \tag{6.136}$$

Here $\Psi : \mathbb{R}^3 \to \mathbb{R}$ is a convex and continuous function describing a plasticity yield condition, $R_\nu(m_{ij}) = m_{ij,j}\nu_i - \frac{\partial}{\partial \tau}[(m_{11} - m_{22})\nu_1\nu_2 + m_{12}(\nu_2^2 - \nu_1^2)]$ are the transverse forces, where $\tau = (-\nu_2, \nu_1)$. The function w describes vertical displacements of the plate, m_{ij} are bending moments; (6.132) is the equilibrium equation, and equations (6.133) give a decomposition of the curvatures $-w_{,ij}$ as the sum of elastic and plastic parts $a_{ijkl}m_{kl}$, ξ_{ij}, respectively. Let $a_{ijkl}(x) = a_{jikl}(x) = a_{klij}(x)$, $i, j, k, l = 1, 2$. There exist two positive constants c_1, c_2 such that

$$c_1 |m|^2 \le a_{ijkl} m_{kl} m_{ij} \le c_2 |m|^2 \qquad \forall m = \{m_{ij}\}.$$

As for the function Ψ, the main assumption is that the set

$$\{m \in \mathbb{R}^3 \mid \Psi(m_{ij}) \le 0\}$$

contains zero as its interior point.

The functions ξ_{ij} can be eliminated from (6.133), (6.134), which gives

$$\Psi(m_{ij}) \le 0, \qquad (a_{ijkl} m_{kl} + w_{,ij})(\bar{m}_{ij} - m_{ij}) \ge 0 \qquad \forall \bar{m},\ \Psi(\bar{m}_{ij}) \le 0.$$

These inequalities will be used to define solutions to the problem (6.132)–(6.136).

Denote by $H^{1,0}(\Omega_c)$ the Sobolev space of functions having the first square integrable derivatives in Ω_c and equal to zero on the external boundary Γ. The space $H^2(\Omega_c)$ contains all functions having derivatives up to the second order square integrable in Ω_c. By $M^1(\Omega_c)$ we denote the space of bounded measures on Ω_c.

Introduce the notation

$$\begin{aligned} U(\Omega_c) = \{m = \{m_{ij}\} \in L^2(\Omega_c) \mid m_{ij,ij} \in L^2(\Omega_c), m_{ij} n_j n_i = 0 \text{ on } \Gamma;& \\ m_{ij}\nu_j\nu_i = R_\nu(m_{ij}) = 0 \text{ on } \Gamma_c^\pm\}.& \end{aligned}$$

In the domain Ω_1, for $m \in U(\Omega_c)$ we can define traces on the boundary Σ, in particular, $m_{ij}\nu_i\nu_j^- \in H^{-1/2}(\Sigma)$, $R_\nu(m_{ij})^- \in H^{-3/2}(\Sigma)$, and the following Green formula holds [67], [131]:

$$\int_{\Omega_1} w m_{ij,ij} = \int_{\Omega_1} w_{,ij} m_{ij} + \langle R_\nu(m_{ij}), w\rangle_{3/2,\Sigma} - \left\langle m_{ij}\nu_j\nu_i, \frac{\partial w}{\partial \nu}\right\rangle_{1/2,\Sigma} \quad \forall w \in H^2(\Omega_1), \tag{6.137}$$

where $\langle\cdot,\cdot\rangle_{i/2,\Sigma}$ stands for the duality pairing between $H^{-i/2}(\Sigma)$ and $H^{i/2}(\Sigma)$, $i = 1, 3$. The same formula is valid for the domain Ω_2 with the boundary $\Sigma \cup \Gamma$.

Define

$$K_c = \{m = \{m_{ij}\} \in L^2(\Omega_c) \mid \Psi(m_{ij}(x)) \leq 0 \text{ a.e. in } \Omega_c\}.$$

Assume that there exists a function $m^0 = \{m^0_{ij}\}$, $(1+\kappa)m^0 \in K_c$, $\kappa = \text{const} > 0$, such that equation (6.132) is satisfied in the following sense:

$$-\langle m^0_{ij}, \bar{w}_{,ij}\rangle_c = \langle f, \bar{w}\rangle_c \quad \forall \bar{w} \in H^2(\Omega_c) \cap H^{1,0}(\Omega_c).$$

The brackets $\langle\,\cdot\,,\,\cdot\,\rangle_c$ denote the scalar product in $L^2(\Omega_c)$.

The theorem of existence of solutions to the problem (6.132)–(6.136) can be formulated as follows.

THEOREM 6.17. *Assume that $f \in L^2(\Omega_c)$ and the above assumption on m^0 holds. Then there exist functions $w, m = \{m_{ij}\}$ such that*

$$\begin{gathered} w \in H^{1,0}(\Omega_c), \qquad w_{,ij} \in M^1(\Omega_c),\ i,j = 1,2, \qquad m \in K_c, \\ -\langle m_{ij}, \bar{w}_{,ij}\rangle_c = \langle f, \bar{w}\rangle_c \quad \forall \bar{w} \in H^2(\Omega_c) \cap H^{1,0}(\Omega_c), \\ \langle a_{ijkl} m_{kl}, \bar{m}_{ij} - m_{ij}\rangle_c + \langle w, \bar{m}_{ij,ij} - m_{ij,ij}\rangle_c \geq 0 \quad \forall \bar{m} \in U(\Omega_c) \cap K_c. \end{gathered} \tag{6.138}$$

Proof. The idea of the proof is to use an elliptic regularization for the penalty equations approximating (6.132)–(6.136). Solutions of the auxiliary problem will depend on two positive parameters ε, δ. The first parameter is responsible for the elliptic regularization and the second one characterizes the penalty approach. More precisely, in the domain Ω_c we want to find functions $w, m = \{m_{ij}\}$ such that

$$\varepsilon w_{,ijij} - m_{ij,ij} = f, \tag{6.139}$$

$$a_{ijkl} m_{kl} + w_{,ij} + \frac{1}{\delta}\, p(m)_{ij} = 0, \qquad i,j = 1,2, \tag{6.140}$$

$$w = 0, \qquad (m_{ij} - \varepsilon w_{,ij}) n_j n_i = 0 \quad \text{on } \Gamma, \tag{6.141}$$

$$(m_{ij} - \varepsilon w_{,ij})\nu_j\nu_i = 0 \quad \text{on } \Gamma_c^\pm, \tag{6.142}$$

$$R_\nu(m_{ij}) - \varepsilon R_\nu(w_{,ij}) = 0 \quad \text{on } \Gamma_c^\pm. \tag{6.143}$$

Here $p(m) = m - \pi(m)$ is the penalty operator, where $\pi : [L^2(\Omega_c)]^3 \to K_c$ is the orthogonal projection operator. Note that p is monotone, continuous and bounded.

We do not show the dependence of the solution to (6.139)–(6.143) on the parameters to simplify the notation. Our aim is first to prove the existence of solution of the problem (6.139)–(6.143) and second to pass to the limit as $\varepsilon \to 0$, $\delta \to 0$.

Let us derive a priori estimates for solutions of (6.139)–(6.143) assuming that the solutions are sufficiently smooth. Multiply (6.139), (6.140) by w, $m_{ij} - m^0_{ij}$, sum and integrate over Ω_c. This gives

$$\varepsilon\langle w_{,ij}, w_{,ij}\rangle_c + \langle a_{ijkl}m_{kl}, m_{ij}\rangle_c + \langle w_{,ij}, m_{ij}\rangle_c + \frac{1}{\delta}\,\langle p(m)_{ij}, m_{ij} - m^0_{ij}\rangle_c$$
$$- \langle m_{ij,ij}, w\rangle_c - \langle w_{,ij}, m^0_{ij}\rangle_c - \langle f, w\rangle_c = \langle a_{ijkl}m_{kl}, m^0_{ij}\rangle_c. \qquad (6.144)$$

Integrate by parts in the fifth and sixth terms of the left-hand side of (6.144) taking into account the boundary conditions (6.141)–(6.143) and the Green formula like (6.137) for the domains Ω_1, Ω_2. The penalty term is non-negative and m^0_{ij} satisfy the equation (6.5.1). Hence the uniform (in ε, δ) estimate follows:

$$\varepsilon\langle w_{,ij}, w_{,ij}\rangle_c + \langle a_{ijkl}m_{kl}, m_{ij}\rangle_c \le c,$$

and consequently

$$\varepsilon\|w\|^2_2 + \|m\|^2_0 \le c. \qquad (6.145)$$

Here $\|\cdot\|_s$ stands for the norm in $H^s(\Omega_c)$. The estimate (6.145) allows us to prove the solvability of the problem (6.139)–(6.143) for the fixed parameters ε, δ in the following sense:

$$w \in H^2(\Omega_c) \cap H^{1,0}(\Omega_c), \qquad m_{ij} \in L^2(\Omega_c), \qquad i, j = 1, 2, \qquad (6.146)$$
$$\varepsilon\langle w_{,ij}, \bar{w}_{,ij}\rangle_c - \langle m_{ij}, \bar{w}_{,ij}\rangle_c = \langle f, \bar{w}\rangle_c \quad \forall \bar{w} \in H^2(\Omega_c) \cap H^{1,0}(\Omega_c), \qquad (6.147)$$
$$\left\langle a_{ijkl}m_{kl} + w_{,ij} + \frac{1}{\delta}\, p(m)_{ij}, \bar{m}_{ij}\right\rangle_c = 0 \quad \forall \bar{m}_{ij} \in L^2(\Omega_c). \qquad (6.148)$$

Now we can pass to the limit as $\varepsilon \to 0$, $\delta \to 0$. Denote the solution of (6.146)–(6.148) by $w^{\varepsilon\delta}, m^{\varepsilon\delta}$. The estimate (6.145) provides the inequality

$$\|m^{\varepsilon\delta}\|_0 \le c.$$

From (6.148) the following equations are obtained:

$$-w^{\varepsilon\delta}_{,ij} = a_{ijkl}m^{\varepsilon\delta}_{kl} + \frac{1}{\delta}\, p(m^{\varepsilon\delta})_{ij}\,, \qquad i, j = 1, 2. \qquad (6.149)$$

Hence, in view of zero boundary conditions for $w^{\varepsilon\delta}$, these equations imply

$$\|w^{\varepsilon\delta}\|_2 \le c(\delta), \qquad (6.150)$$

where the constant $c(\delta)$ depends on δ, in general.

By (6.149), (6.150), we choose a subsequence, still denoted by $w^{\varepsilon\delta}, m^{\varepsilon\delta}$, such that for any fixed δ, as $\varepsilon \to 0$,

$$m^{\varepsilon\delta}_{ij} \to m^{\delta}_{ij} \quad \text{weakly in } L^2(\Omega_c),\ i, j = 1, 2,$$
$$w^{\varepsilon\delta} \to w^{\delta} \quad \text{weakly in } H^2(\Omega_c) \cap H^{1,0}(\Omega_c).$$

Passing to the limit as $\varepsilon \to 0$ in (6.146)–(6.148) we have

$$-\langle m^{\delta}_{ij}, \bar{w}_{,ij}\rangle_c = \langle f, \bar{w}\rangle_c \quad \forall \bar{w} \in H^2(\Omega_c) \cap H^{1,0}(\Omega_c), \qquad (6.151)$$
$$\left\langle a_{ijkl}m^{\delta}_{kl} + w^{\delta}_{,ij} + \frac{1}{\delta}\, p(m^{\delta})_{ij}, \bar{m}_{ij}\right\rangle_c = 0 \quad \forall \bar{m}_{ij} \in L^2(\Omega_c). \qquad (6.152)$$

The convergence $p(m^{\varepsilon\delta}) \to p(m^{\delta})$ can be justified by monotonicity arguments.

Let us prove that, uniformly in δ,

$$\sum_{i,j=1}^{2} \|w^{\delta}_{,ij}\|_{L^1(\Omega_c)} \le c. \tag{6.153}$$

First we notice from (6.151), (6.152) that, uniformly in δ,

$$\frac{1}{\delta} \langle p(m^\delta)_{ij}, m^\delta_{ij} - m^0_{ij} \rangle_c \le c. \tag{6.154}$$

Consider the convex functional P on $[L^2(\Omega_c)]^3$,

$$P(m) = \frac{1}{2\delta} \|p(m)\|_0^2.$$

The Gateaux derivative P' of the functional P can be found by the formula $P'(m) = \delta^{-1} p(m)$. Hence, by the convexity of P, we have

$$P(m^0 + q) - P(m^\delta) \ge P'(m^\delta)(m^0 + q - m^\delta), \qquad q = \{q_{ij}\} \in [L^2(\Omega_c)]^3. \tag{6.155}$$

Let $\|q\|_{L^\infty(\Omega_c)} \le \alpha$, where α is chosen to be small enough so that $m^0 + q \in K_c$. Here we use the conditions imposed on m^0 and the set $\{m = \{m_{ij}\} \mid \Psi(m_{ij}) \le 0\}$. Since $P(m^0 + q) = 0$, from (6.155) it follows that

$$\frac{1}{\delta} \langle p(m^\delta), q \rangle_c \le \frac{1}{\delta} \langle p(m^\delta), m^\delta - m^0 \rangle_c.$$

In view of the inequality (6.154) we have

$$\frac{1}{\delta} \langle p(m^\delta), q \rangle_c \le c \quad \forall q, \qquad \|q\|_{L^\infty(\Omega_c)} \le \alpha,$$

which completes the proof of (6.153).

Taking into account the inequality

$$\sum_{i,j=1}^{2} \|w_{,ij}\|_{L^1(\Omega_c)} \ge c \|w\|_{W_1^2(\Omega_c)} \qquad \forall w \in W_1^2(\Omega_c),\ w = 0 \text{ on } \Gamma,$$

with a constant c independent of w we conclude from (6.153) that

$$\|w^\delta\|_{W_1^2(\Omega_c)} \le c. \tag{6.156}$$

Here, $W_1^2(\Omega_c)$ is the Sobolev space of functions having derivatives in Ω_c up to the second order belonging to $L^1(\Omega_c)$.

Now we can use the inequality

$$\|w\|_1 \le c \|w\|_{W_1^2(\Omega_c)} \qquad \forall w \in W_1^2(\Omega_c). \tag{6.157}$$

Hence, from (6.156), (6.157) the boundedness of w^δ follows, i.e.

$$\|w^\delta\|_1 \le c.$$

The imbedding $L^1(\Omega_c) \subset M^1(\Omega_c)$ is continuous, and consequently, by (6.153), from equations (6.152) it is clear that

$$\sum_{i,j=1}^{2} \|w^{\delta}_{,ij}\|_{M^1(\Omega_c)} \le c.$$

As a result we derive the following uniform (in δ) estimate for the solution w^δ, m^δ of the problem (6.151), (6.152):

$$\|m^\delta\|_0 + \|m^\delta_{ij,ij}\|_0 + \|w^\delta\|_1 + \sum_{i,j=1}^{2} \|w^\delta_{,ij}\|_{M^1(\Omega_c)} \le c.$$

Now we can pass to the limit as $\delta \to 0$. Choosing a subsequence w^δ, m^δ we can assume that as $\delta \to 0$,

$$\begin{aligned} m^\delta_{ij} &\to m_{ij} && \text{weakly in } L^2(\Omega_c),\ i,j = 1,2,\\ m^\delta_{ij,ij} &\to m_{ij,ij} && \text{weakly in } L^2(\Omega_c),\\ w^\delta &\to w && \text{weakly in } H^{1,0}(\Omega_c),\\ w^\delta_{,ij} &\to w_{,ij} && \text{weak}^* \text{ in } M^1(\Omega_c),\ i,j = 1,2. \end{aligned}$$

It follows from (6.151) that $-m^\delta_{ij,ij} = f$ in Ω_c in the sense of distributions, whence

$$-m_{ij,ij} = f \quad \text{in } \Omega_c. \tag{6.158}$$

Moreover, from (6.151) we obtain the identity (6.17). Next, by the monotonicity of p, from (6.152) the following inequality is derived:

$$\langle a_{ijkl} m^\delta_{kl}, \bar m_{ij} - m^\delta_{ij}\rangle_c + \langle w^\delta_{,ij}, \bar m_{ij} - m^\delta_{ij}\rangle_c \ge 0 \quad \forall \bar m \in U(\Omega_c) \cap K_c.$$

We see that for $\bar m \in U(\Omega_c)$ the relation

$$\langle w^\delta_{,ij}, \bar m_{ij}\rangle_c = \langle w^\delta, \bar m_{ij,ij}\rangle_c$$

holds. Furthermore, by (6.151), (6.158) the equalities

$$-\langle m^\delta_{ij}, w^\delta_{,ij}\rangle_c = \langle f, w^\delta\rangle_c = -\langle m_{ij,ij}, w^\delta\rangle_c$$

are valid. Consequently, (6.5.1) implies

$$\langle a_{ijkl} m^\delta_{kl}, \bar m_{ij} - m^\delta_{ij}\rangle_c + \langle w^\delta, \bar m_{ij,ij} - m_{ij,ij}\rangle_c \ge 0 \quad \forall \bar m \in U(\Omega_c) \cap K_c.$$

Passing to the limit as $\delta \to 0$ in (6.5.1) we arrive at (6.138).

The inclusion $m \in K_c$ can be proved by standard arguments. The theorem is proved. ∎

Now we aim at formulating an existence theorem for the elastoplastic crack problem (6.132)–(6.136) in the smooth domain Ω.

From (6.17) it follows that

$$-m_{ij,ij} = f \quad \text{in } \Omega_c \tag{6.159}$$

in the sense of distributions. Let us show that equation (6.159) holds also in Ω in the sense of distributions. In fact, identity (6.17) implies the zero jump condition $[m_{ij}\nu_j\nu_i] = 0$, $[R_\nu(m_{ij})] = 0$ on Γ_c in the following sense:

$$\langle m_{ij}\nu_j\nu_i^+ - m_{ij}\nu_j\nu_i^-, \varphi\rangle_{1/2,\Sigma} = 0 \quad \forall \varphi \in H^{1/2}(\Sigma), \tag{6.160}$$

$$\langle R_\nu(m_{ij})^+ - R_\nu(m_{ij})^-, \varphi\rangle_{3/2,\Sigma} = 0 \quad \forall \varphi \in H^{3/2}(\Sigma). \tag{6.161}$$

Hence, by (6.160), (6.161), we have

$$\begin{aligned}\{m_{ij,ij} + f, \varphi\} &= \langle m_{ij}, \varphi_{,ij}\rangle_{\Omega_1} + \langle m_{ij}, \varphi_{,ij}\rangle_{\Omega_2} + \langle f, \varphi\rangle \\ &= \langle m_{ij,ij} + f, \varphi\rangle_{\Omega_1} + \langle m_{ij,ij} + f, \varphi\rangle_{\Omega_2} \\ &\quad + \langle [R_\nu(m_{ij})], \varphi\rangle_{3/2,\Sigma} - \langle [m_{ij}\nu_i\nu_j], \partial\varphi/\partial\nu\rangle_{1/2,\Sigma} = 0,\end{aligned}$$

which proves the statement. Here $\langle\cdot,\cdot\rangle_{\Omega_i}, \langle\cdot,\cdot\rangle$ mean the scalar products in $L^2(\Omega_i), L^2(\Omega)$, respectively. Now we are able to formulate the existence theorem for the elastoplastic crack problem (6.132)–(6.136) in which the functions are defined in the smooth domain Ω. Introduce the space

$$\begin{aligned}U(\Omega) = \{m = \{m_{ij}\} \in L^2(\Omega) \mid m_{ij,ij} \in L^2(\Omega),\ m_{ij}n_jn_i = 0 \text{ on } \Gamma; \\ m_{ij}\nu_j\nu_i = R_\nu(m_{ij}) = 0 \text{ on } \Gamma_c\}.\end{aligned}$$

Note that the conditions $m_{ij}\nu_j\nu_i = R_\nu(m_{ij}) = 0$ on Γ_c in the definition of $U(\Omega)$ are imposed on the smooth curve Γ_c located inside the domain Ω. The following statement holds.

THEOREM 6.18. *Let the conditions of Theorem* 6.17 *be satisfied. Then there exist functions w, $m = \{m_{ij}\}$ such that*

$$\begin{gathered} w \in L^2(\Omega), \qquad m \in U(\Omega) \cup K_c, \\ -m_{ij,ij} = f \qquad \text{in } \Omega, \\ \langle a_{ijkl}m_{kl}, \bar{m}_{ij} - m_{ij}\rangle + \langle w, \bar{m}_{ij,ij} - m_{ij,ij}\rangle \geq 0 \qquad \forall \bar{m} \in U(\Omega) \cap K_c. \end{gathered}$$

7. Bridged crack models and singular integral equations

Finally, we present a model of a bridged crack, with complete mathematical analysis which seems to be new. The mathematical techniques applied to the analysis are very simple, but rather different from the theory used before. Since the model is one-dimensional, the numerical results can be obtained in an elementary way.

7.1. Introduction and derivation of the model. Fully or partially bridged cracks can be modeled using singular integral equations. This is well-documented in the literature and we refer the reader to [45, 46, 53, 54, 78, 135] for examples of derivation of these models and their treatment using classical asymptotic methods. In this chapter we investigate two such models and their mathematical treatment. Both models will contain integro-differential equations with Hilbert transform like kernels.

To derive the model for a bridged crack we consider the elastic half plane $S^- = \{(x, y) \in \mathbb{R}^2 \mid y < 0\}$ with boundary $\partial S^- = \{(x, 0) \mid x \in \mathbb{R}\}$.

The classical results [93, 127] on isotropic elasticity in the half plane S^- state that if the forces (X, Y) are applied on ∂S^- then the displacement field (u, v) in S^- is determined from the following relation due to G. V. Kolosov:

$$2\mu(u_x + iv_x) = \kappa\Phi(z) + \Phi(\overline{z}) - (z - \overline{z})\overline{\Phi}'(\overline{z}) \tag{7.1}$$

with the complex potential derived by Muskhelishvili [93]

$$\Phi(z) = -\frac{1}{2\pi z}(X + iY) + O\left(\frac{1}{z^2}\right)$$

where $z = x + iy \in \mathbb{C}$, $u = u(x, y)$, $v = (x, y)$, $u_x = \partial u/\partial x$, μ, κ are given constants.

In this section we apply the result in the particular case of the point force $(O, \delta(x))$ at the origin, where $\delta(x)$ is the Dirac mass supported at $x = 0$.

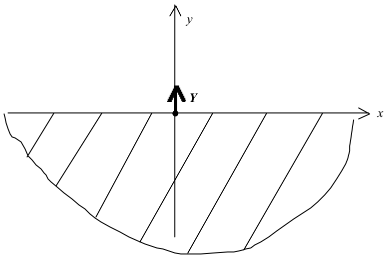

Fig. 7.1. Elastic halfplane

In this case the solution for the unknown displacement v is obtained in closed form as follows.

The potential $\Phi(z)$ is given by

$$\Phi(z) = -\frac{i}{2\pi z}, \qquad \Phi'(z) = \frac{i}{2\pi z^2},$$
$$\Phi(\overline{z}) = -\frac{i}{2\pi \overline{z}}, \qquad \Phi'(\overline{z}) = -\frac{i}{2\pi \overline{z}^2}.$$

Therefore, the equation (7.1) takes the form

$$\begin{aligned} 2\mu\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial x}\right) &= \kappa\Phi(z) + \Phi(\overline{z}) - (z - \overline{z})\overline{\Phi}'(z) \\ &= -\frac{i}{2\pi}\left\{\frac{\overline{z}}{|z|^2}\kappa + \frac{z}{|z|^2} - \frac{(z-\overline{z})z^2}{|z|^4}\right\} \\ &= -\frac{i}{2\pi}\left\{\frac{(\kappa+1)x - i(\kappa-1)y}{|z|^2} + \frac{4xy^2}{|z|^4} - \frac{izy(x^2-y^2)}{|z|^4}\right\}. \end{aligned}$$

The imaginary part is

$$2\mu\frac{\partial v}{\partial x} = -\frac{1}{2\pi}\left\{\frac{(\kappa+1)x}{|z|^2} + \frac{4xy^2}{|z|^4}\right\}, \qquad |z|^2 = x^2 + y^2.$$

On the boundary S^- for $y = 0$,

$$2\mu\frac{\partial v}{\partial x}\bigg|_{y=0} = -\frac{1}{2\pi}\frac{(\kappa+1)x}{x^2} = -\frac{1}{2\pi}(\kappa+1)\frac{1}{x},$$

hence

$$v|_{y=0} = -\frac{\kappa+1}{4\pi\mu}\ln\frac{|x|}{d} + \text{const}$$

where d (in length dimensions) is a constant. From [93] we have the values of κ:

$$\kappa = \begin{cases} 3-4\nu, & \text{plane deformation, } 1<\kappa<3, \\ \dfrac{3-4\nu}{1+\nu}, & \text{mean tension plane, } \frac{5}{3}\kappa<3. \end{cases}$$

Since we are interested in crack modeling, the next step concerns the superposition of the point force applied at the origin and the symmetric distributed force supported on the intervals $(-\infty,-a]$ and $[a,+\infty)$ for some $a>0$.

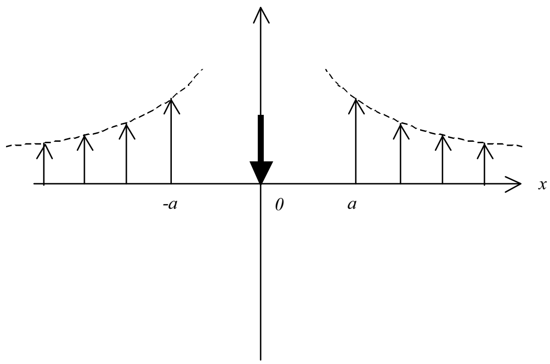

Fig. 7.2. Elastic halfplane loaded on the boundary

In this case, the following solution is obtained for unknown distribution of the force $p(\zeta)$, $\zeta \in (-\infty,-a)\cup(a,+\infty)$:

$$p^-(-\zeta) = p^+(\zeta), \qquad \zeta \in [a,+\infty),$$

$$\int_{\infty}^{-a} p^-(\zeta)\,\ln\frac{|x-\zeta|}{d}\,d\zeta = \int_a^{+\infty} p^-(-t)\ln\frac{|x+t|}{d}\,dt,$$

$$\frac{4\pi\mu}{\kappa+1}\,v\bigg|_{y=0} = P_0\ln\frac{|x|}{d} - \int_a^{+\infty} p(\zeta)\ln\left(\frac{|x-\zeta|}{d}\right)d\zeta - \int_a^{+\infty} p(\zeta)\ln\left(\frac{|x+\zeta|}{d}\right)d\zeta.$$

We can model the problem under consideration here. We have two elastic half planes S^- and S^+ with a thin elastic layer $\{y=0\}$ between S^- and S^+. Assuming symmetry, we can restrict the analysis to the displacement field in S^-, and we impose the relation

$$-p(x) = kv^-|_{y=0}$$

between the reaction $p(x)$, $x\in(-\infty,-a]\cup[a,+\infty)$, and the displacement $v^-|_{y=0}$ on ∂S^-, where k is a coefficient which characterizes the layer $\{y=0\}$. In particular, the Poisson coefficient of the layer is assumed to be zero, $\nu=0$.

This is the limit case, as indicated for example in [93].

We consider the crack problem (see Figure 7.3) described for $|x|>a$ by the equation

$$-\frac{1}{k}\,\frac{4\pi\mu}{\kappa+1}\,p(x) = P_0\ln\frac{|x|}{d} - \int_a^{+\infty} p(\zeta)\left[\ln\frac{|x-\zeta|}{d} + \ln\frac{|x+\zeta|}{d}\right]d\zeta.$$

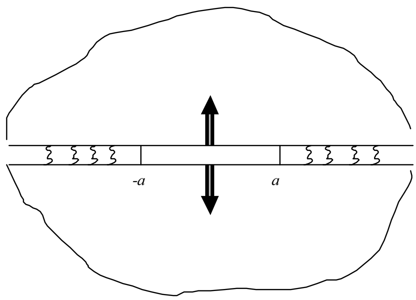

Fig. 7.3. Bridged crack

So by differentiation

$$-A\,\frac{dp(x)}{dx} = \frac{P_0}{x} - \int_a^{+\infty} p(\zeta)\left[\frac{1}{x-\zeta} + \frac{1}{x+\zeta}\right] d\zeta,$$

where

$$P_0 = 2\int_a^{+\infty} p(\zeta)\,d\zeta, \qquad A = \frac{1}{k}\,\frac{4\pi\mu}{\kappa+1}.$$

We will prove that there exists a unique solution p for the above model.

7.2. Mathematical problems

7.2.1. Existence and uniqueness using semigroup methods. We consider the following model with unknown function $p(x)$, $x \in \mathbb{R}$.

Find $p(x)$ such that

$$\begin{cases} \displaystyle\int_{-\infty}^{+\infty} \frac{p(\zeta)}{x-\zeta}\,d\zeta - \frac{P}{x} = A\,\frac{dp}{dx}, & |x| > a, \\ p = 0, & |x| < a, \\ \displaystyle\int_{-\infty}^{+\infty} p\zeta\,d\zeta = P. \end{cases} \tag{7.2}$$

Assume P, A and a are given. This is an integro-differential equation. The integral is a singular integral, the famous Hilbert transform. The integral itself is defined as a principal value:

$$\mathcal{H}f = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{f(\zeta)}{x-\zeta}\,d\zeta = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|x-\zeta|\geq\varepsilon} \frac{f(\zeta)}{x-\zeta}\,d\zeta. \tag{7.3}$$

It is non-trivial that for each $f \in L^1$ the limit exists almost everywhere. It is known that $\mathcal{H} : L^p \to L^p$, $p > 1$ and $\mathcal{H}^2 = -I$. The Hilbert transform is related to the Fourier transform by

$$\widehat{\mathcal{H}f}(\omega) = -i(\operatorname{sign}\omega)\hat{f}(\omega), \qquad f \in L^2.$$

Also the sine and cosine transforms of a function on $[0,\infty)$ are the Hilbert transforms of each other:

$$\int_0^\infty f(x)\cos\omega x\,dx = \mathcal{H}\Big[\int_0^\infty f(x)\sin\omega x\,dx\Big]$$

An alternative method of defining the Hilbert transform is as follows: The function $t \mapsto \log|1 - x/t|$ is in L^q for all $q > 1$, and its L^q norm remains bounded for x in a compact set.

For any $f \in L^p$, the integral

$$\int_{-\infty}^{+\infty} f(t) \log|1 - x/t|\, dt$$

is absolutely convergent. It can be shown that the function

$$x \mapsto \int_{-\infty}^{+\infty} f(t) \log|1 - x/t|\, dt$$

is absolutely continuous. One defines

$$\mathcal{H} f(x) = -\frac{d}{dx}\Big\{ \int_{-\infty}^{+\infty} f(t) \log|1 - x/t|\, dt \Big\}.$$

Now we can reformulate (7.2) as

$$\frac{d}{dx}\Big\{ Ap(x) + \int_{-\infty}^{+\infty} p(t) \log|1 - x/t|\, dt + \Big(\int_{-\infty}^{+\infty} p(t) dt \Big) \log^+ |x/a| \Big\} = 0, \qquad |x| > a.$$

So that apparently we need to solve

$$\begin{aligned} &Ap(x) + \int_{-\infty}^{+\infty} p(t) \log|1 - x/t|\, dt + P \log^+ |x/a| = \text{const}, \qquad |x| > a, \\ &p \equiv 0 \quad \text{ in } |x| < a, \\ &\int_{-\infty}^{+\infty} p(t)\, dt = P. \end{aligned} \tag{7.4}$$

This does not involve any singular integral; however, the function $\log^+ |x/a|$ is not in any L^p, and neither is the integral. We do not know any standard procedure for the solution.

Returning to (7.2) we write it as

$$\begin{aligned} &\mathcal{H} p + f = A \frac{dp}{dx}, \qquad |x| > a, \\ &p = 0, \qquad\qquad\qquad |x| < a. \end{aligned} \tag{7.5}$$

Observe that since p is required to vanish in $(-a, a)$ the values of f cannot be prescribed in this interval. We note that there is yet another connection between the Hilbert transform and d/dx, provided by the Cauchy–Poisson semigroup: For any fixed $p > 1$, define $V_t : L^p \to L^p$ by

$$V_t f = \frac{t}{\pi} \int_{-\infty}^{+\infty} \frac{f(y)}{(x-y)^2 + t^2}\, dy. \tag{7.6}$$

Then V_t is a C_0-semigroup of contractions on L^p:

$$\|V_t\| \le 1, \qquad V_t V_s = V_{t+s}, \qquad \lim_{t \to 0} \| \, |V_t - I|(f) \| = 0.$$

The infinitesimal generator $\mathcal{M}$ of the semigroup is defined by

$$\mathcal{M} f = \lim_{t \to 0} \frac{V_t f - f}{t}$$

whenever the limit exists and the linear space where this limit exists is the domain of $\mathcal{M}$. One can show that this domain is the range of the resolvent:

$$D(\mathcal{M}) = \{R_\lambda f : f \in L^p\} \tag{7.7}$$

where

$$R_\lambda f = \int_0^\infty e^{-\lambda t} V_t f \, dt,$$

and we have

$$\lambda R_\lambda f - \mathcal{M}(R_\lambda f) = f, \qquad f \in L^p. \tag{7.8}$$

For the Cauchy–Poisson semigroup the infinitesimal generator has the following characterization:

$$D(\mathcal{M}) = \{f \in L^p : (\mathcal{H}f)' \in L^p\} \tag{7.9}$$

(see [24]), and for $f \in D(\mathcal{M})$,

$$\mathcal{M}f = -(\mathcal{H}f)'. \tag{7.10}$$

Combining (7.7)–(7.10) we see that for each $f \in L^p$ and each $\lambda > 0$,

$$\lambda R_\lambda f + \frac{d}{dx}(\mathcal{H}(R^\lambda f)) = f. \tag{7.11}$$

Recall that $\mathcal{H}^2 = -I$; writing $v = \mathcal{H}(R_\lambda f)$ we deduce from (7.11) that for each $f \in L^p, v = \mathcal{H}(R_\lambda f)$ solves

$$-\lambda \mathcal{H} v + \frac{d}{dx} v = f,$$

i.e.,

$$\frac{\lambda}{\pi} \int_{-\infty}^{+\infty} \frac{v(y)\, dy}{x - y} + f = \frac{dv}{dx}. \tag{7.12}$$

After some computation we find that if

$$f(x) = \begin{cases} 1/x, & |x| > a, \\ 0, & |x| < a, \end{cases}$$

the solution of (7.12) is given by

$$v(x) = \int_{-\infty}^{+\infty} K(x-y)\, \frac{1}{\pi y} \log \left| \frac{y-a}{y+a} \right| dy$$

where

$$K(x) = \frac{1}{\pi} \int_0^\infty e^{-\lambda t} \, \frac{t}{x^2 + t^2} \, dt = \frac{1}{\pi} \int_0^\infty \frac{\cos x\zeta}{x + \zeta} \, d\zeta.$$

This is not a complete solution of the problem since we require $v = 0$ on $(-a, a)$, where f cannot be prescribed. Hence, we still have to determine f in $(-a, a)$ and the solution v. We formulate the problem:

Let $f_0 \in L^2$. Find f and v so that $f(x) = f_0(x), |x| > a, v(x) = \text{constant in } |x| < a$ and such that

$$\frac{\lambda}{\pi} \int_{-\infty}^{+\infty} \frac{v(y)}{x - y} \, dy + f(x) = \frac{dv}{dx}(x). \tag{7.13}$$

To solve this we proceed as follows: Let v_0 solve (7.13) with $f = f_0$. Then $v_0 = \mathcal{H}(R_\lambda f_0) = R_\lambda(\mathcal{H} f_0)$. Let

$$f_1 = \begin{cases} f_0, & |x| > a, \\ -\lambda \mathcal{H} v_0, & |x| < a, \end{cases} \qquad v_1 = \mathcal{H}(R_\lambda f_1).$$

Then v_1 solves (7.13) with $f = f_1$. If $f_1, \dots, f_n,\ v_1, \dots, v_n$ have been defined we define

$$f_{n+1} = \begin{cases} f_0, & |x| > a, \\ -\lambda \mathcal{H} v_n, & |x| < a, \end{cases} \qquad v_{n+1} = \mathcal{H}(R_\lambda f_{n+1}). \tag{7.14}$$

We have

$$v_{n+1} - v_n = \mathcal{H}[R_\lambda(f_{n+1} - f_n)],$$

so that, $\mathcal{H}$ being an isometry,

$$\|v_{n+1} - v_n\|_2 = \|R_\lambda(f_{n+1} - f_n)\|_2 = \|\lambda R_\lambda[\mathcal{H}(v_n - v_{n-1})]\mathbf{1}_{\{|x|<a\}}\|_2 \tag{7.15}$$

because by definition (7.14) of f_n

$$f_{n+1} - f_n = \begin{cases} 0, & |x| > a, \\ \lambda \mathcal{H}(v_n - v_{n-1}), & |x| < a. \end{cases}$$

To estimate the norm in (7.15) we use the following result.

Proposition 7.1. *Let $0 < \varphi \in L^1$. Let $f \in L^p$ with support in $(-a, a)$. Then $\varphi \star f \in L^p$ and*

$$\|\varphi \star f\|_p \le \alpha \|f\|_p$$

where $\alpha^q = \sup_I \mu(I)$, I is any interval of length $2a$ and μ has density φ.

Proof. This follows by direct computation:

$$\begin{aligned} \int \varphi(x-y) f(y)\, dy &= \int \varphi(x-y) f(y) \mathbf{1}_{\{|y| \le a\}}\, dy \\ &\le \Big(\int \varphi(x-y) f^p(y)\, dy \Big)^{1/p} \Big(\int \varphi(x-y) \mathbf{1}_{\{|y| \le a\}}\, dy \Big)^{1/q} \\ &= \alpha \Big(\int \varphi(x-y) f^p(y)\, dy \Big)^{1/p}. \end{aligned}$$

Now raise both sides to the power p, integrate and use Fubini's Theorem.

Since λR_λ has strictly positive density, the proposition can be used to estimate (7.15): There is $\alpha < 1$ such that

$$\begin{aligned} \|v_{n+1} - v_n\|_2 &\le \alpha \|\mathcal{H}(v_n - v_{n-1}) \mathbf{1}_{\{|x|<a\}}\|_2 \\ &\le \alpha \|\mathcal{H}(v_n - v_{n-1})\|_2 = \alpha \|v_n - v_{n-1}\|_2. \end{aligned}$$

This implies that v_n converge to v in L^2, and $\mathcal{H} v_n$ converges to $\mathcal{H} v$ in L^2. Then f_n converges to f in L^2, where f is defined by

$$f = \begin{cases} f_0, & |x| > a, \\ -\lambda \mathcal{H} v, & |x| < a. \end{cases}$$

Since $v_n = \mathcal{H} R_\lambda f_n$ we see that $v = \mathcal{H} R_\lambda f = R_\lambda \mathcal{H} f$, i.e., v satisfies (7.13).

Using (7.16) in (7.13) we see that $dv/dx = 0$ for $|x| < a$ or v is a constant. Thus (v, f) is a solution of our problem. Note that since $v = R_\lambda \mathcal{H} f$, v is *absolutely continuous.*

Finally, we prove the uniqueness of the solution. Suppose u and v are solutions of the above problem. In other words,

$$\lambda \mathcal{H}_u + f_1 = \frac{d}{dx} u, \qquad \begin{array}{ll} u' = 0, & |x| < a, \\ f_1 = f_0, & |x| > a, \end{array}$$

$$\lambda \mathcal{H}_v + f_2 = \frac{d}{dx} v, \qquad \begin{array}{ll} v' = 0, & |x| < a, \\ f_2 = f_0, & |x| > a. \end{array}$$

Then $w = u - v$ satisfies

$$\begin{cases} \lambda \mathcal{H} w + f = \dfrac{d}{dx} w, \\ w' = 0, \quad |x| < a, \\ f = 0, \quad |x| > a. \end{cases}$$

Since $\frac{d}{dx}\, w = 0$, $|x| < a$, we must have $f = -\lambda \mathcal{H} w$, $|x| < a$.

From what we have said, the unique solution w is given by $w = \mathcal{H} R_\lambda f$. Hence

$$\|w\|_2 = \|R_\lambda f\|_2 = \|R_\lambda[(-\lambda \mathcal{H}_w)\mathbf{1}_{\{|x|<a\}}]\|_2 = \|\lambda R_\lambda(\mathbf{1}_{\{|x|<a\}}\mathcal{H}_w)\|_2 \leq \alpha \|\mathcal{H} w\|_2 = \alpha \|w\|_2$$

where $\alpha < 1$, so we must have $w = 0$. This shows that the solution is unique.

Now suppose the given function f_0 is odd in $|x| > a$, i.e. $f_0(x) = -f_0(-x)$ for $|x| > a$. Extend f_0 to be odd. Now R_λ is a function of $|x|$ so R_λ takes odd functions into odd functions. The same is true of the Hilbert transform. It follows that v_0 is odd. The definition of f_1 shows that f_1 is also odd, etc. All v_n and hence the limit v are odd. But v is constant in $(-a, a)$. So it must be zero.

Thus we have:

Proposition 7.2. *Let $f_0 \in L^2(|x| > a)$ and be odd. Then there exists a unique pair $v, f \in L^2$ such that f is odd, $f = f_0$ in $|x| > a$, $v = 0$ in $|x| < a$ and*

$$\frac{\lambda}{\pi} \int_{-\infty}^{+\infty} \frac{v - (y)}{x - y}\, dy + f = \frac{dy}{dx}. \tag{7.16}$$

7.2.2. Numerical example. Numerical solutions to the model can be computed using Matlab. To this end, the equation

$$-A \frac{dp}{dx}(x) = \frac{P_0}{x} - \int_a^\infty p(\zeta) \left(\frac{1}{x - \zeta} + \frac{1}{x + \zeta} \right) d\zeta, \qquad x > a,$$

is transported to the interval $s \in (0, 1]$ using the change of variable $s = a/x$, $x > a$,

$$\frac{dp}{ds} = \frac{dp}{dx} \left(-\frac{a}{s^2} \right), \quad \text{so} \quad \frac{dp}{dx} = -\frac{s^2}{a} \frac{dp}{dx}.$$

So, the equation becomes

$$A \frac{s^2}{a} \frac{dp}{ds} = \frac{sP_0}{a} - \int_1^0 p(t) \left(\frac{1}{\frac{a}{s} - \frac{a}{t}} + \frac{1}{\frac{a}{s} + \frac{a}{t}} \right) \left(-\frac{a}{t^2} \right) dt = \frac{sP_0}{a} - \int_0^1 \frac{2p(t)}{t^2 - s^2}\, dt,$$

i.e.,

$$As \frac{dp}{ds} + 2a \int_0^1 \frac{p(t)}{t^2 - s^2}\, dt = P_0.$$

This equation is now discretized. We use the standard two-point discretization for the derivative, and a somewhat modified mid-point rule for the singular integral. The sampling points for the integration variable t are chosen to be the midpoints between the sampling points for s. In this way, the singularity does not introduce an added difficulty. However, we cannot evaluate p at these points and replace $p(t_j)$ by the average of the two closest points $(p(s_j) + p(s_{j+1}))/2$.

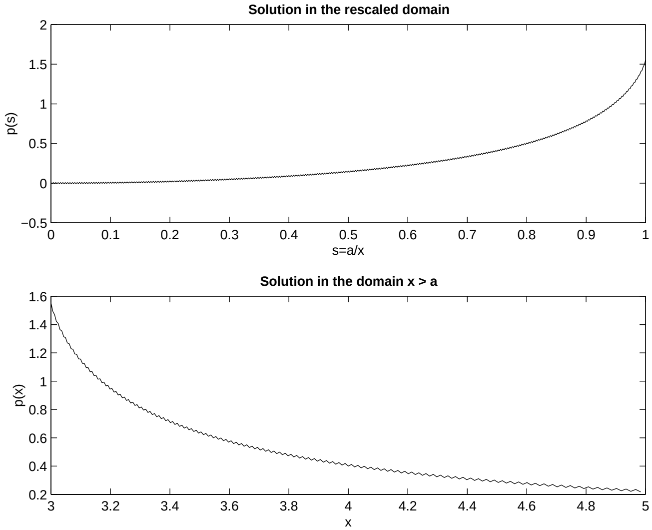

Fig. 7.4. Solution to the integral equation with $a = 3$ and $p = 4$ and 500 subintervals

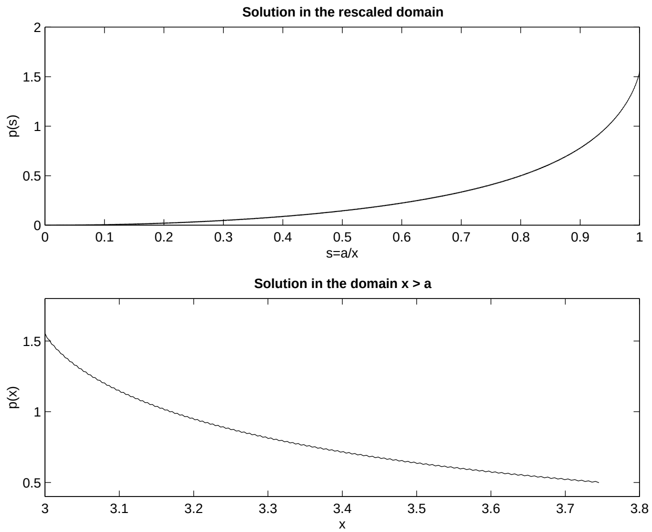

Fig. 7.5. Solution to the integral equation with $a = 3$ and $p = 4$ and 1000 subintervals

Numerical results are shown in the figures. Figures 7.4 and 7.5 contain the solution to the integral equation when $a = 3$ and $p = 4$. The upper graph in both figures shows the solution in the re-scaled finite domain, the lower graph shows the solution in the domain $[a, \infty)$. In Figure 7.4 we used a discretization with 500 subintervals, Figure 7.5 uses a discretization with 1000 subintervals. One can see that the value of $p(x)$ at $x = a$ is independent of the discretization.

7.3. Methods from the theory of pseudo-differential operators

7.3.1. Introduction and statement of the result. In this section we use asymptotic analysis derived from the theory of pseudo-differential operators to examine a bridged crack in $\mathbb{R}^2$ which is located along the x_1 axis. We will present a condensed version of an earlier paper [55]. We assume that the crack is symmetric about $x_1 = 0$ with end points at $x_1 = \pm c$. Its opening is described by a function $U(x_1)$. The crack is fully bridged by fibers which are perpendicular to it (i.e. in the x_2-direction). We assume a linear relationship between the opening of the crack $U(x_1)$ and the bridging force $P(x_1)$, i.e. the force in the x_2-direction at the point x_1,

$$P(x_1) = g(x_1)U(x_1) \quad \text{with} \quad g(x_1) > 0.$$

The balance of forces acting on the crack is given by

$$T(x_1) = \sigma(x_1) + P(x_1), \qquad -c < x_1 < c,$$

where $T(x_1)$ denotes the outside forces and σ denotes the normal stress in the x_2-direction. σ satisfies

$$\sigma = C\mathcal{P}\int_{-c}^{c} \frac{b(y_1)}{x_1 - y_1}\, dy_1,$$

where we understand the integral to be the Cauchy principal value. b denotes the dislocation density along the crack. The constant C depends on the elastic moduli of the material. This dependence is non-trivial and an explicit expression can be found in the paper of M. Hori and S. Nemat-Nasser [53]. This equation is obtained by solving the basic field equations for a transversely isotropic linear elastic solid in plane stress or plane strain. The plane of isotropy is given by x_1, x_3, the axis of symmetry is given by x_2. The complete derivation is carried out in more generality in [53]. The crack-opening displacement $U(x_1)$ is related to the dislocation density b via

$$U(x_1) = -\int_{-c}^{x_1} b(y_1)\, dy_1, \qquad U(\pm c) = 0.$$

We combine these quantities and get

$$C\mathcal{P}\int_{-c}^{c} \frac{b(y_1)}{x_1 - y_1}\, dy_1 - g(x_1)\int_{-c}^{x_1} b(y_1)\, dy_1 = T(x_1).$$

To continue we use the following normalizations:

$$\begin{aligned} x &= x_1/c, \quad & u(x) &= U(x_1)/c, \quad & \hat{b}(x) &= b(x_1),\\ \varepsilon &= C/c, \quad & \lambda(x) &= g(x_1)/C, \quad & -cf(x) &= T(x_1)/C. \end{aligned}$$

We arrive at

$$\varepsilon \mathcal{P} \int_{-1}^{1} \frac{\hat{b}(y)}{x-y}\, dy - \lambda \int_{-1}^{x} \hat{b}(y)\, dy = -f(x).$$

We substitute $\hat{b}(x) = -u'(x)$ to get the following singularly perturbed integral equation:

$$\lambda(x)u(x) - \varepsilon \frac{d}{dx} \mathcal{P} \int_{-1}^{1} \frac{u(y)}{x-y}\, dy = f(x), \qquad -1 < x < 1. \tag{7.17}$$

Equation (7.17) is an example of a strongly singular integral equation. This formulation of the bridged crack problem can be found in [53, 54]. Equation (7.17) also appears in other applications, most notably in the theory of airfoils of finite span where it is known as Prandtl's integro-differential equation [93, 120].

The existence and uniqueness of an L^2-solution to (7.17) is a classical result (see, e.g., [93, 120], and the works cited in those papers). A solution to this equation was formally computed in [45, 54, 135]. In [45] a more general class of strongly singular integral equations with constant coefficient λ is investigated. In [39], Wiener–Hopf techniques are used to analyze the problem. The Wiener–Hopf factorization procedure is appropriate for solving constant coefficient elliptic boundary value problems. This method was generalized by Eskin (see [39]) to solve variable coefficient elliptic boundary value problems using pseudo-differential calculus. Indeed, [39, Chapter 27] gives a general treatment of singularly perturbed elliptic pseudo-differential boundary value problems on manifolds. [39] is formulated in much greater generality than needed to solve (7.17).

7.3.2. Notations and statement of the main results. In this section we will reformulate (7.17) using the notations of pseudo-differential operators. Let $\hat{v}$ denote the usual Fourier transform of v:

$$\hat{v}(\xi) = \int_{-\infty}^{\infty} v(x) e^{i\xi x}\, dx.$$

A pseudo-differential operator $A(x, D)$, where $D = -i\frac{d}{dx}$, is defined by

$$A(x, D)v(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(x, \xi)\hat{v}(\xi) e^{-ix\xi}\, d\xi.$$

The function $A(x, \xi)$ is called the *symbol* of the pseudo-differential operator. To begin, let $l\lambda > 0$ and lf be arbitrary extensions of λ and f in $L^2(\mathbb{R})$, i.e. $lf \in L^2(\mathbb{R})$ and $lf|_{[-1,1]} = f$ and analogously for $l\lambda$. Note that we do not make any further requirements on these extensions. Consider the equation

$$l\lambda(x)u(x) - \varepsilon \frac{d}{dx} \int_{-\infty}^{\infty} \frac{u(y)}{x-y}\, dy = lf(x). \tag{7.18}$$

Taking the Fourier transform of

$$-\varepsilon \frac{d}{dx} \int_{-\infty}^{\infty} \frac{u(y)}{x-y}\, dy,$$

we obtain

$$\varepsilon \sqrt{\pi/2}\, |\xi| \hat{u}(\xi).$$

To simplify notation let $\epsilon = \varepsilon\sqrt{\pi/2}$. The equation (7.18) can therefore be rewritten as a pseudo-differential equation

$$A(x, D, \epsilon)u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(x, \xi, \epsilon)\hat{u}(\xi)e^{-ix\xi}\, d\xi = lf(x), \tag{7.19}$$

with symbol

$$A(x, \xi, \epsilon) = l\lambda(x) + \epsilon|\xi|.$$

Observe that

$$A(x, \xi, \epsilon) > 0, \qquad \forall \xi \in \mathbb{R}.$$

In particular, this means that the equation

$$A(x, \xi, \epsilon) = 0$$

has no real solutions and the corresponding pseudo-differential operator is called elliptic.

Our main concern is the computation of leading order asymptotics of the solution of (7.17). To this end we have the following theorem.

PROPOSITION 7.3. *Let $f \in C^\infty[-1, 1]$ and $\lambda \in C^\infty[-1, 1]$. Then the solution u has an asymptotic expansion $u = u_\epsilon + w_\epsilon$, where the leading term u_ϵ is given by:*

$$u_\epsilon(x) = \begin{cases} \left(\dfrac{1+x}{\varepsilon}\right)^{1/2} \dfrac{f(x)}{\lambda(x)} & \text{as } (1+x)/\epsilon \to 0^+, \\ \dfrac{f(x)}{\lambda(x)} & \text{as } \epsilon \to 0, \\ \left(\dfrac{1-x}{\varepsilon}\right)^{1/2} \dfrac{f(x)}{\lambda(x)} & \text{as } (1-x)/\epsilon \to 0^+, \end{cases} \qquad x \in (-1, 1).$$

Moreover, $w_\epsilon \in C^\infty[-1, 1]$ and satisfies

$$\|w_\epsilon\|_{L^\infty([-1,1])} \le C_\delta \epsilon^{1-\delta}$$

for all $\delta > 0$. Here C_δ only depends on f and δ.

REMARK. Note that this result is independent of the particular extensions of f and λ.

7.3.3. Proof of Proposition 7.3. The equation is a variable coefficient singular Wiener–Hopf-type equation. Constant coefficient equations of this type are commonly solved using the Wiener–Hopf factorization method. In [39] this procedure is extended to symbols with non-constant coefficients. This requires the use of pseudo-differential calculus. Our proof follows this approach.

Step 1: The operator (7.19) is strongly elliptic, i.e. it satisfies

$$\int_{-\infty}^{\infty} A(x, \xi, \epsilon)\hat{u}(\xi)\overline{\hat{u}(\xi)}\, d\xi \ge C\|u\|^2,$$

for a suitable positive constant C. In this equation $\overline{\hat{u}}$ denotes the usual complex conjugate and $\|\cdot\|$ denotes the L^2-norm. It is well known that strongly elliptic equations admit unique solutions (see, e.g., ([39]). Moreover, the solution satisfies

$$\|u\| \le C\|f\|.$$

Indeed, this estimate is not sharp, but follows from

$$\|u\|_{H^{1/2}} \leq \|f\|_{H^{-1/2}},$$

since in our case $f \in L^2$ (cf. [39]).

Step 2: Let

$$A_0(x,\xi) = l\lambda(x)$$

be the unperturbed (reduced) symbol, i.e. the symbol of (7.18) with $\epsilon = 0$. The reduced equation

$$A_0(x,\xi)u_0(x) = l\lambda(x)u_0(x) = lf(x)$$

has a unique solution $u_0 \in L^2(-1,1)$, indeed $u_0(x) = lf(x)/l\lambda(x)$. In $[-1,1]$ this becomes $u_0(x) = f(x)/\lambda(x)$. We can write

$$A(x,\xi,\epsilon) = A_0(x,\xi)A_1(x,\epsilon\xi),$$

where

$$A_1(x,\epsilon\xi) = 1 + \frac{|\epsilon\xi|}{l\lambda(x)}$$

carries all the ϵ-dependence of A. Moreover, $A_1(x,\epsilon\xi)$ converges to 1 as $\epsilon \to 0$.

Using this normalization, we see that $A(x,\xi,\epsilon)$ satisfies the hypotheses of [39, Theorem 27.1, p. 346] (with $A_2 = 0$).

Step 3: To obtain the asymptotic expansions we first expand $A_1(x,\epsilon\xi)$. To shorter notation we will omit the x-dependence of $l\lambda$ and use $\eta = \epsilon\xi$. Observe that

$$A_1(x,\eta) = \frac{1}{l\lambda}(l\lambda + |\eta|) = \frac{1}{l\lambda}\sqrt{l\lambda^2 + \eta^2 + 2l\lambda|\eta|} = \frac{1}{l\lambda}\sqrt{l\lambda^2+\eta^2}\sqrt{1 + \frac{2l\lambda|\eta|}{l\lambda^2+\eta^2}}.$$

By the arithmetic-geometric means inequality we have

$$0 \leq \frac{2l\lambda|\eta|}{l\lambda^2+\eta^2} \leq 1,$$

and therefore the series

$$\sqrt{1+\frac{2l\lambda|\eta|}{l\lambda^2+\eta^2}} = 1 + \frac{1}{2}\frac{2l\lambda|\eta|}{l\lambda^2+\eta^2} - \frac{1}{8}\left(\frac{2l\lambda|\eta|}{l\lambda^2+\eta^2}\right)^2 + \frac{1}{16}\left(\frac{2l\lambda|\eta|}{l\lambda^2+\eta^2}\right)^3 + \cdots$$

converges for all $\eta \in \mathbb{R}$ and all $l\lambda > 0$. Combining this with (6) we obtain

$$\begin{aligned} A_1(x,\eta) = {} & \frac{1}{l\lambda}\sqrt{l\lambda^2+\eta^2} + \frac{1}{2l\lambda}\frac{2l\lambda|\eta|}{\sqrt{l\lambda^2+\eta^2}} - \frac{1}{8l\lambda}\left(\frac{l\lambda|\eta|}{l\lambda^2+\eta^2}\right)^2\sqrt{l\lambda^2+\eta^2} \\ & + \frac{1}{16l\lambda}\left(\frac{2l\lambda|\eta|}{l\lambda^2+\eta^2}\right)^3\sqrt{l\lambda^2+\eta^2} + \cdots . \end{aligned}$$

We can write

$$\begin{aligned} A_1(x,\eta) &= \frac{1}{l\lambda}\sqrt{l\lambda^2+\eta^2} + \frac{1}{2l\lambda}\frac{2l\lambda|\eta|}{\sqrt{l\lambda^2+\eta^2}} + O(|\eta|^{-1/2}) \\ &= A_1^p(x,\eta) + \frac{1}{2l\lambda}\frac{2l\lambda|\eta|}{\sqrt{l\lambda^2+\eta^2}} + O(|\eta|^{-1/2}) = A_1^p(x,\eta) + Q(x,\eta). \end{aligned}$$

Observe that the term $R(x,\eta)$ is of lower order, which has a smoothing effect when taking the inverse Fourier transform. A_1^p is called the principal part of A_1. The function $A_1^p(x,\eta)$ can be analytically continued to a domain in the complex plane which contains the entire real line. To do this we have to choose a branch cut starting at $\eta = +il\lambda$ which stays entirely in the upper half plane, and a second branch cut starting at $\eta = -il\lambda$ which stays entirely in the lower half plane.

Step 4: The principal part $A_1^p(x,\eta)$ as an analytic function of η can be factored into a plus function $A_1^+(x,\eta)$ and a minus function $A_1^-(x,\eta)$ as follows:

$$A_1^p(x,\eta) = \frac{1}{l\lambda}\sqrt{l\lambda^2+\eta^2} = \frac{1}{\sqrt{il\lambda}}\sqrt{\eta+il\lambda}\frac{1}{\sqrt{-il\lambda}}\sqrt{\eta-il\lambda} = A_1^+(x,\eta)A_1^-(x,\eta),$$

where

$$A_1^+(x,\eta) = \frac{1}{\sqrt{il\lambda}}\sqrt{\eta+il\lambda} \quad \text{and} \quad A_1^-(x,\eta) = \frac{1}{\sqrt{-il\lambda}}\sqrt{\eta-il\lambda}.$$

This factorization is consistent with the choice of branches above, i.e. the branch cut for A_1^+ starts at $\eta = -il\lambda$ and stays in the lower half plane, and therefore A_1^+ is analytic in the upper half plane, and analogously for A_1^-.

To continue let $0 < \delta < \epsilon$, and let $\alpha(x)$ be a smooth function with $\alpha(x) = 1$ for $-1 < x < -1+2\delta$ and $\alpha(x) = 0$ for $1-2\delta < x < 1$. Then we get a new factorization for A_1^p via

$$\begin{aligned} A_3^+(x,\eta) &= (A_1^+(x,\eta))^{\alpha(x)}(A_1^-(x,\eta))^{1-\alpha(x)}, \\ A_3^-(x,\eta) &= (A_1^-(x,\eta))^{\alpha(x)}(A_1^+(x,\eta))^{1-\alpha(x)}. \end{aligned} \tag{7.20}$$

Observe that

$$A_1^p(x,\eta) = A_3^+(x,\eta)A_3^-(x,\eta)$$

for all $x \in [-1,1]$ and that

$$\begin{aligned} A_3^+(-1,\eta) &= A_1^+(-1,\eta), \quad & A_3^-(-1,\eta) &= A_1^-(-1,\eta), \\ A_3^+(1,\eta) &= A_1^-(1,\eta), \quad & A_3^-(1,\eta) &= A_1^+(1,\eta). \end{aligned}$$

This process is an extension of the process found in [39].

The purpose of such a factorization is to change the singular perturbation problem, which has the small parameter ϵ in the highest order term, into a regular perturbation problem where ϵ is only in lower order terms.

Now let us consider the extended differential equation

$$A(x,D,\epsilon)u_\epsilon = lf + f_\epsilon^-,$$

where f_ϵ^- is supported outside $[-1,1]$. Following the steps of [39] we write $u_\epsilon = (A_3^+)^{-1}v_\epsilon$ and apply the operator

$$p(A_3^-)^{-1}$$

to the equation. Here p denotes the restriction to $[-1,1]$. We get

$$\begin{aligned} p(A_3^-)^{-1}A(A_3^+)^{-1}v_\epsilon &= p(A_3^-)^{-1}lf, \\ p(A_3^-)^{-1}(A_0A_p^1 + A_0Q)(A_3^+)^{-1}v_\epsilon &= p(A_3^-)^{-1}lf, \\ p(A_0+Q_1)v_\epsilon &= p(A_3^-)^{-1}lf. \end{aligned}$$

In the last expression Q_1 consists of the term $p(A_3^-)^{-1}A_0Q(A_3^+)^{-1}$ and of lower order terms which arise from the rules of composition for pseudo-differential operators. The principal part of $p(A_3^-)^{-1}A_0Q(A_3^+)^{-1}$ is given by

$$p\left(\frac{|\eta|}{l\lambda^2+\eta^2} - \frac{l\lambda}{4}\left(\frac{|\eta|}{l\lambda^2+\eta^2}\right)^2 + \cdots\right).$$

It follows that there exist constants C_s such that

$$\|p(A_3^-)^{-1}A_0Q(A_3^+)^{-1}v_\epsilon\|_{H^{s-1}(\mathbb{R})} \leq C_s\epsilon\|v_\epsilon\|_{H^s(\mathbb{R})} \qquad \forall\epsilon>0,\ \forall s.$$

In particular this statement holds for $s=1$, which gives an L^2-error bound for u. The case $s=2$ gives an L^∞ bound for the higher order terms of the asymptotic expansion. Theorem 18.3 (composition of pseudo-differential operators) of [39] implies a similar estimate for the remaining parts of Q_1 (cf. [39, p. 348, equation (27.54)].

Following the computations of [39, p. 248] the explicit solution u_ϵ is given by

$$u_\epsilon = (A_3^+)^{-1}(I + R_0pA_1^p)^{-1}R_0p(A_3^-)^{-1}lf, \tag{7.21}$$

where I is the identity operator, and R_0 is the inverse of A_0. In our specific case R_0 is just multiplication by $1/l\lambda$.

Step 5: We next obtain an explicit expression near $x=-1$ the expression near $x=1$ follows from an analogous computation. Near $x=-1$, we can use A_1^- and A_1^+ directly. For sufficiently smooth functions f, (7.21) can be rewritten as

$$u_\epsilon = (A_3^+)^{-1}v_\epsilon + u_\epsilon^{(0)},$$

where v_ϵ is the solution to

$$pA_0v_\epsilon = p(A_3^-)^{-1}lf,$$

and $u_\epsilon^{(0)}$ is smooth. Following the steps in [39] we see that

$$v_\epsilon = u_0 + u_\epsilon^{(1)},$$

where $u_0 = f/l\lambda$, the solution to the reduced equation, and $u_\epsilon^{(1)}$ is smooth. Thus we get

$$u_\epsilon = (A_3^+)^{-1}u_0 + u_\epsilon^{(3)}. \tag{7.22}$$

Next observe that near $x=-1$, we can localize the symbol for $(A_3+)^{-1}$ as follows:

$$A_3^+(x,\eta)^{-1} = \frac{\sqrt{i\lambda(-1)}}{\sqrt{\eta+i\lambda}} + B_+(x,\eta) = A_1^+(-1,\eta)^{-1} + B_+(x,\eta)$$

where B_+ is a lower order term. Computing the inverse Fourier transform of $A_1^+(-1,\eta)^{-1}$ we get

$$\begin{aligned} a(x/\epsilon) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sqrt{i\lambda(-1)}}{\sqrt{\eta+i\lambda}}\,e^{ix\xi}\,d\xi = \frac{1}{2\pi\epsilon}\int_{-\infty}^{\infty}\frac{\sqrt{i\lambda(-1)}}{\sqrt{\eta+i\lambda}}\,e^{i\eta x/\epsilon}\,d\eta \\ &= \frac{1}{\sqrt{\pi}\epsilon}\left(\frac{x}{\epsilon}\right)^{-1/2} e^{-\lambda(-1)x/\epsilon}\Theta(x) \end{aligned}$$

where $\Theta(x)$ denotes the usual Heaviside function. Observe that $a(x/\epsilon)\to\delta(x)$ as $\epsilon\to 0$. On the other hand, we may represent the leading term in the pseudo-differential equation

(7.22) as a convolution:

$$u_\epsilon(y) = \int_{-\infty}^{\infty} a(x/\epsilon) u_0(y-x)\, dx + u_\epsilon^{(4)}. \tag{7.23}$$

Observe that $a(x/\epsilon) = 0$ for $x < 0$ and that $u_0(y-x) = 0$ for $y-x < -1$, i.e. for $x > 1+y$. Thus we can replace the limits of integration in (7.23) to get

$$u_\epsilon(y) = \int_0^{1+y} a(x/\epsilon) u_0(y-x)\, dx + u_\epsilon^{(4)}.$$

Finally, we expand $u_0(y-x)$ into a Taylor series around y to get

$$u_0(y-x) = u_0(y) + u_0'(y)x + \cdots .$$

Omitting the higher order terms we obtain

$$\begin{aligned} u_\epsilon(y) &= \int_0^{1+y} a(x/\epsilon) u_0(y)\, dx + u_\epsilon^{(5)} = \int_0^{1+y} \frac{1}{\sqrt{\pi}\epsilon} \left(\frac{x}{\epsilon}\right)^{-1/2} e^{-\lambda(-1)x/\epsilon} u_0(y)\, dx + u_\epsilon^{(5)} \\ &= \int_0^{1+y} \frac{1}{\epsilon} \left(\frac{x}{\epsilon}\right)^{-1/2} u_0(y)\, dx + u_\epsilon^{(6)} = u_0(y) \left(\frac{1+y}{\epsilon}\right)^{1/2} + u_\epsilon^{(6)}. \end{aligned}$$

In this computation we used the fact that $e^{-\lambda(-1)x/\epsilon}$ is rapidly decreasing for $x/\epsilon > 0$ with leading term 1.

In the preceding computations the terms of the form $u_\epsilon^{(k)}$ for $k = 1, \ldots, 6$ all satisfy the estimates given in (27.60–27.80) of [39]. We combine this estimate with the Sobolev imbedding theorem to get

$$\|u_\epsilon^{(k)}\|_{L^\infty([-1,1])} \le C_{\delta,k} \epsilon^{1-\delta},$$

where $C_{\delta,k}$ depends only on f and k. These terms are combined to make up the error term w_ϵ in Theorem 7.3, which therefore satisfies the same estimates.

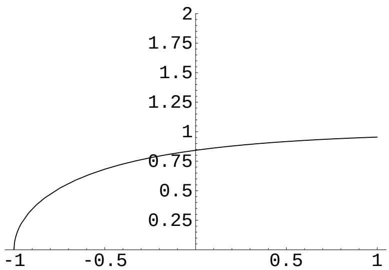

Fig. 7.6. The graph of $S(y,\epsilon)$ for $\epsilon = 1$

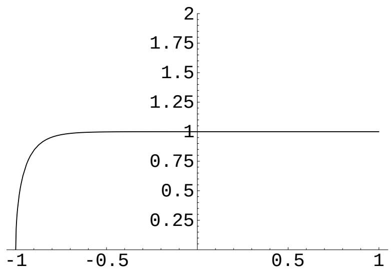

Fig. 7.7. The graph of $S(y,\epsilon)$ for $\epsilon = 0.1$

The graphs of

$$S(y,\epsilon) = \int_0^{1+y} \frac{1}{\epsilon} \left(\frac{x}{\epsilon}\right)^{-1/2} e^{-\lambda(-1)x/\epsilon}\, dx$$

are shown in Figure 7.6 and 7.7. In both cases we used $\lambda(-1) = 1$. In Figure 7.6 we used $\epsilon = 1$ and in Figure 7.7 we used $\epsilon = 0.1$. One can see that this function approaches a step function as $\epsilon \to 0$ in a rather rapid way.

The leading terms of the asymptotic expansion near $x = 1$ are computed in an analogous way. The only difference is that we use $A_1^-(+1, \eta)$. For x in the interior of $(-1, 1)$, u_ϵ converges pointwise to u_0 as $\epsilon \to 0$. This completes the proof of the theorem.

7.3.4. Concluding remarks. 1. There are other ways to obtain the leading order terms for the solution to (1). Indeed, we can explicitly compute the inverse Fourier transform of $(A_1^p(x,\eta))^{(-1)} = \lambda/\sqrt{\lambda^2+\eta^2}$ as follows:

$$\begin{aligned}\tilde{a}(x/\epsilon) &= \frac{1}{2\pi}\int_{\infty}^{\infty} \frac{\lambda}{\sqrt{\lambda^2+\eta^2}} e^{-ix\xi}\, d\xi = \frac{1}{2\pi\epsilon}\,\lambda(x)\int_{-\infty}^{\infty} \frac{e^{-i\frac{x}{\epsilon}\eta}}{\sqrt{\lambda^2+\eta^2}}\, d\eta \\ &= \frac{\lambda(x)}{\epsilon\pi}\, K_0(|x/\epsilon|\lambda(x)).\end{aligned}$$

Here $K_0(\cdot)$ denotes the modified Bessel function (see, for example, [14], for the definition of this function).

As in the previous section, we can write the solution as a convolution. First observe that $u_0(y-x) = 0$ except when $-1 \le x - y \le 1$. Then write $u_0(y-x) = u_0(y) - u_0'(y)x + \cdots$. Using the leading term we get

$$u_\epsilon(y) = u_0(y)\int_{y-1}^{y+1} \frac{\lambda(x)}{\epsilon\pi}\, K_0(|x/\epsilon|\lambda(x))\, dx = u_0(y)S(y,\epsilon).$$

Then we make a change of variables in the integral and let $s = x/\epsilon$ to arrive at

$$\begin{aligned}S(y,\epsilon) &= \int_{(y-1)/\epsilon}^{(y+1)/\epsilon} \frac{\lambda(\epsilon s)}{\pi}\, K_0(|s|\lambda(\epsilon s))\, ds \\ &= \int_{(y-1)/\epsilon}^{(y+1)/\epsilon} \frac{\lambda(0)}{\pi}\, K_0(|s|\lambda(0))\, ds + O(\epsilon) = \tilde{S}(y,\epsilon) + O(\epsilon).\end{aligned}$$

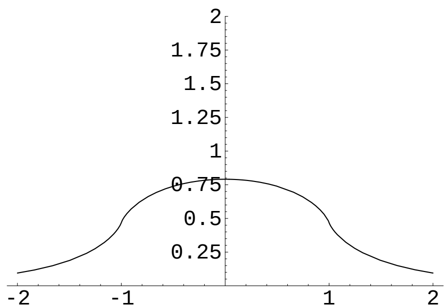

Fig. 7.8. The graph of $\tilde{S}(y,\epsilon)$ for $\epsilon = 1$

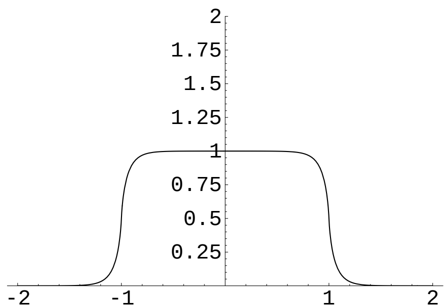

Fig. 7.9. The graph of $\tilde{S}(y,\epsilon)$ for $\epsilon = 0.1$

Using the properties of the modified Bessel function, $\tilde{S}(y,\epsilon)$ can be explicitly computed and its graph is shown in Figures 7.8 and 7.9.

This method appears to be easier and more straightforward. However, the method does not guarantee that the solution is supported in $[-1, 1]$, but only that the solution will decrease rapidly outside this interval. This makes the solutions depend on the specific extensions lf of f and $l\lambda$ of λ. Furthermore, the asymptotics are not correct near the boundary. The factoring of $A_1^p(x,\eta)$ into a $+$ and a $-$ symbol as done in the previous section will guarantee that the support of the solution is contained in $[-1,1]$, and that the leading terms of the asymptotics will be independent of the extension of the solution. The method of this section will yield good results in the interior of $[-1,1]$, but it is not very useful near the endpoints.

2. For constant values of λ asymptotic expansions of the solution to equation (7.17) were obtained by Gautesen [45] and Willis and Nemat-Nasser [135]. These works also include higher order terms. In [45] a Wiener–Hopf factorization is used to obtain a solution up to $O(\epsilon^3 \log \epsilon)$. [135] uses the asymptotic matching methods to obtain solutions correct to order ϵ.

In principle the method introduced in this paper can also be used to find solutions to higher order. One has to use more terms in the Taylor expansion of the symbol $A_1(x,\eta)$. Furthermore, the expression

$$(I + R_0 p A_1^p)^{-1}$$

of (7.21) can be expanded into a Neumann series. One must then collect terms of equal order and can proceed to compute higher order approximations. However, this process is rather arduous.

3. Equation (7.17) could also be considered on the positive real axis instead of a compact interval. Our method can be directly modified for this situation. To do this one has to change the exponent $\alpha(x)$ to a fixed exponent $\alpha = 1$. The whole process becomes easier since only the factorization (7.20) in the form

$$A_3^+(x,\eta) = A_1^+(x,\eta)$$

remains near $x = 0$. This will produce an asymptotic expansion near $x = 0$. Alternately, one could follow the path of [136] and use the change of variable suggested there to transform the equation into a problem on the interval $(0,1)$ and apply the present method to this new problem.

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