Contents

Abstract

Let $\mathbf{P}$ be a poset on the set $[m] \times [n]$, which is given as the disjoint sum of posets on 'columns' of $[m] \times [n]$, and let $\check{\mathbf{P}}$ be the dual poset of $\mathbf{P}$. Then $\mathbf{P}$ is called a generalized Niederreiter–Rosenbloom–Tsfasman poset (gNRTp) if all further posets on columns are weak order posets of the 'same type'. Let G (resp. $\check{G}$) be the group of all linear automorphisms of the space $\mathbb{F}_q^{m\times n}$ preserving the $\mathbf{P}$-weight (resp. $\check{\mathbf{P}}$-weight). We define two partitions of $\mathbb{F}_q^{m\times n}$, one consisting of '$\mathbf{P}$-orbits' and the other of '$\check{\mathbf{P}}$-orbits'. If $\mathbf{P}$ is a gNRTp, then they are respectively the orbits under the action of G on $\mathbb{F}_q^{m\times n}$ and of $\check{G}$ on $\mathbb{F}_q^{m\times n}$. Then, under the assumption that $\mathbf{P}$ is not an antichain, we show that (1) $\mathbf{P}$ is a gNRTp iff (2) the $\mathbf{P}$-orbit distribution of C uniquely determines the $\check{\mathbf{P}}$-orbit distribution of $C^{\perp}$ for every linear code C in $\mathbb{F}_q^{m\times n}$ iff (3) G acts transitively on each $\mathbf{P}$-orbit iff (4) $\mathbb{F}_q^{m\times n}$ together with the classes given by '(u, v) belongs to a class iff $u - v$ belongs to a $\mathbf{P}$-orbit' is a symmetric association scheme. Furthermore, a general method of constructing symmetric association schemes is introduced. When $\mathbf{P}$ is a gNRTp, using this, four association schemes are constructed. Some of their parameters are computed and MacWilliams-type identities for linear codes are derived. Also, we report on the recent developments in the theory of poset codes in the Appendix.

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1. Introduction

The poset codes were introduced in [4] by Brualdi et al. in connection with Niederreiter's problem in [27] and have received considerable attention in recent years (cf. [10], [12], [16–18], [20], [21]). The reader is referred to the Appendix for the recent developments in the theory of poset codes. The Niederreiter–Rosenbloom–Tsfasman weight (metric), introduced in [29], [31] and further developed, for example in [8], [28], [32], is a good example of a poset weight which corresponds to the poset consisting of finite disjoint sums of chains of equal finite lengths.

Let $\mathbb{F}_q$ be the finite field with q elements, and let $n, m_1, \ldots, m_t$ be positive integers with $m = m_1 + \cdots + m_t$. Assume that $\mathbf{P} = \mathbf{P}_1 \dot{+} \cdots \dot{+} \mathbf{P}_n$ is the poset given by the disjoint sum of posets $\mathbf{P}_j$ whose underlying set and order relation are given by

$$
\begin{aligned}
&[m] \times \{j\} = m_1 \mathbf{1}^{(j)} \cup \cdots \cup m_t \mathbf{1}^{(j)}, \\
&m_r \mathbf{1}^{(j)} = \{(m_1 + \cdots + m_{r-1} + 1, j), \ldots, (m_1 + \cdots + m_{r-1} + m_r, j)\}, \\
&a <_{\mathbf{P}_j} b \;\Leftrightarrow\; a \in m_r \mathbf{1}^{(j)},\, b \in m_s \mathbf{1}^{(j)} \text{ for some } r, s \text{ with } 1 \le r < s \le t
\end{aligned}
$$

(cf. (2.4)). Such a $\mathbf{P}$ will be called a generalized Niederreiter–Rosenbloom–Tsfasman poset for the reason to be explained below.

Let $w_{\mathbf{P}}$ be the corresponding $\mathbf{P}$-weight. Namely, for u in the space $\mathbb{F}_q^{m \times n}$ of all $m \times n$ matrices over $\mathbb{F}_q$, $w_{\mathbf{P}}(u)$ is defined to be the cardinality of the smallest ideal containing the support of u (cf. (2.1)). Then $w_{\mathbf{P}}$ is nothing but the Niederreiter–Rosenbloom–Tsfasman weight for $m_1 = \cdots = m_t = 1$ (cf. [8], [23], [31], [32]), while it is the $\mathbf{P}$-weight corresponding to the weak order poset for $n = 1$ (cf. [16–18], [20], [21]).

Assume for the moment that $\mathbf{P}$ is any poset on $[m]$. Then it has been shown that (1) $\mathbf{P}$ is a weak order poset on $[m]$ $\Leftrightarrow$ (2) $(\mathbf{P}, \check{\mathbf{P}})$ is a weak dual MacWilliams pair (wdMp) $\Leftrightarrow$ (3) the group $\mathrm{Aut}(\mathbb{F}_q^m, w_{\mathbf{P}})$ acts transitively on each $\mathbf{P}$-sphere $S_{\mathbf{P}}(i) = \{x \in \mathbb{F}_q^m \mid w_{\mathbf{P}}(x) = i\}$ $(0 \le i \le m)$ $\Leftrightarrow$ (4) $\mathcal{X} = (\mathbb{F}_q^m, \mathcal{R} = \{R_i\}_{i=0}^m)$, with $(x, y) \in R_i \Leftrightarrow x - y \in S_{\mathbf{P}}(i)$, is a symmetric association scheme. Here $\check{\mathbf{P}}$ is the dual poset of $\mathbf{P}$; $(\mathbf{P}, \check{\mathbf{P}})$ is called a wdMp if the $\mathbf{P}$-weight distribution (enumerator) of C uniquely determines the $\check{\mathbf{P}}$-weight distribution (enumerator) of $C^{\perp}$, for every linear code C in $\mathbb{F}_q^m$; $\mathrm{Aut}(\mathbb{F}_q^m, w_{\mathbf{P}})$ is the group of all linear automorphisms of $\mathbb{F}_q^m$ preserving $w_{\mathbf{P}}$; and here $S_{\mathbf{P}}(i)$'s are defined for any $\mathbf{P}$, but they are the orbits under the action of $\mathrm{Aut}(\mathbb{F}_q^m, w_{\mathbf{P}})$ on $\mathbb{F}_q^m$ if $\mathbf{P}$ is the weak order poset. (1)$\Rightarrow$(2) is shown independently in [16] and [21], (2)$\Rightarrow$(1) in [21], (1)$\Rightarrow$(3) in [16], (3)$\Rightarrow$(1) in [17], and (1)$\Leftrightarrow$(4) in [22].

The main purpose of this paper is to generalize the above results to the case of generalized Niederreiter–Rosenbloom–Tsfasman posets. Our main result is the following theorem.

THEOREM. *Let* $\mathbf{P} = \mathbf{P}_1 \dotplus \cdots \dotplus \mathbf{P}_n$ *be the poset given by the disjoint sum of posets* $\mathbf{P}_j$ *on the underlying set* $[m] \times \{j\}$, *for* $j = 1, \ldots, n$. *Assume further that* $\mathbf{P}$ *is not an antichain. Then the following are equivalent:*

(1) $\mathbf{P}$ *is a generalized Niederreiter–Rosenbloom–Tsfasman poset.*
(2) $(\mathbf{P}, \check{\mathbf{P}})$ *is a weak dual orbit pair.*
(3) *The group* $G = \mathrm{Aut}(\mathbb{F}_q^{m\times n}, w_{\mathbf{P}})$ *acts transitively on each set* $\mathcal{O}_{\mathbf{P}}(\beta) \subseteq \mathbb{F}_q^{m\times n}$ $(\beta \in I_{m,n}/S_n)$.
(4) $\mathcal{X}_{\mathbf{P}} = (\mathbb{F}_q^{m\times n}, \mathcal{R}_{\mathbf{P}} = \{R_{\mathbf{P},\beta}\}_{\beta \in I_{m,n}/S_n})$, *with* $(x, y) \in R_{\mathbf{P},\beta} \Leftrightarrow x - y \in \mathcal{O}_{\mathbf{P}}(\beta)$ $(\beta \in I_{m,n}/S_n)$, *is a symmetric association scheme.*

Here $\check{\mathbf{P}}$ is the dual poset of $\mathbf{P}$ just as before, $(\mathbf{P}, \check{\mathbf{P}})$ is a *weak dual orbit pair* if the $\mathbf{P}$-orbit distribution $\{|\mathcal{O}_{\mathbf{P}}(\beta) \cap C|\}_{\beta \in I_{m,n}/S_n}$ of C uniquely determines the $\check{\mathbf{P}}$-orbit distribution $\{|\mathcal{O}_{\check{\mathbf{P}}}(\beta) \cap C^{\perp}|\}_{\beta \in I_{m,n}/S_n}$ of $C^{\perp}$ for any linear code $C \subseteq \mathbb{F}_q^{m\times n}$, and $\{\mathcal{O}_{\mathbf{P}}(\beta)\}_{\beta \in I_{m,n}/S_n}$, $\{\mathcal{O}_{\check{\mathbf{P}}}(\beta)\}_{\beta \in I_{m,n}/S_n}$ are certain partitions of $\mathbb{F}_q^{m\times n}$; they are respectively the orbits under the action of G and $\check{G} = \mathrm{Aut}(\mathbb{F}_q^{m\times n}, w_{\check{\mathbf{P}}})$ on $\mathbb{F}_q^{m\times n}$, if $\mathbf{P}$ is a generalized Niederreiter–Rosenbloom–Tsfasman poset.

The proofs of (1)⇔(3), (1)⇔(4), (2)⇒(1), (1)⇒(2) will each occupy one chapter. The proofs for (2)⇒(1), (3)⇒(1), and (4)⇒(1) are very similar to one another in spirit, while (1)⇒(3) is trivial. (1)⇒(4) is a special case of Theorem 3.1, which says that, for any poset $\mathbf{P}$ on $[N]$ and a subgroup H of $\mathrm{Aut}(\mathbb{F}_q^N, w_{\mathbf{P}})$ containing the involution $\iota \in \mathrm{Aut}(\mathbb{F}_q^N)$ given by $u \mapsto -u$, and the orbits $\{\mathcal{O}_{H,i}\}_{i=0}^d$ under the action of H on $\mathbb{F}_q^N$, with $\mathcal{O}_{H,0} = \{0\}$, $\mathcal{X}_H = (\mathbb{F}_q^N, \mathcal{R}_H = \{R_{H,i}\}_{i=0}^d)$, with $(x, y) \in R_{H,i} \Leftrightarrow x - y \in \mathcal{O}_{H,i}$, is a symmetric association scheme. This gives a general method of constructing association schemes, even though its proof is easy.

The proof (1)⇒(2) requires a longer explanation. We will show that $G = \mathrm{Aut}(\mathbb{F}_q^{m\times n}, w_{\mathbf{P}}) = QP^n \rtimes_{\psi} S_n$ (cf. (4.17)) when $\mathbf{P} = \mathbf{P}_1 \dotplus \cdots \dotplus \mathbf{P}_n$ is the generalized Niederreiter–Rosenbloom–Tsfasman poset with $t > 1$ (i.e., $\mathbf{P}$ is not an antichain). From the three naturally arising subgroups H, K, L (cf. (4.18)) of G, and G itself, we will construct the corresponding association schemes $\mathcal{X}_H$, $\mathcal{X}_K$, $\mathcal{X}_L$ and $\mathcal{X}_{\mathbf{P}}$. To show (1)⇒(2), we only need to consider $\mathcal{X}_{\mathbf{P}}$, which coincides with the ordered Hamming scheme [25] for $m_1 = \cdots = m_t = 1$. Borrowing some idea from [25], we compute the usual parameters of the association schemes such as the adjacency matrices, the irreducible idempotents, q-numbers, and p-numbers. Further, we will show that $(\mathbb{F}_q^{m\times n}, \mathcal{R}_{\check{\mathbf{P}}} = \{R_{\check{\mathbf{P}},\beta}\}_{\beta \in I_{m,n}/S_n})$ is isomorphic to $\mathcal{X}_{\mathbf{P}}^*$, where the former is the scheme corresponding to $\check{G}$ and the latter is the dual scheme to $\mathcal{X}_{\mathbf{P}}$. Similar results also hold for the subgroups H, K, L of G. These results rest on Theorem 3.3, which gives some equivalent conditions for such an isomorphism to exist. Delsarte's well-known result (cf. (3.12), (3.13)) can now be invoked in order to get MacWilliams-type identities for linear codes in the association schemes $\mathcal{X}_{\mathbf{P}}$, $\mathcal{X}_H$, $\mathcal{X}_K$, and $\mathcal{X}_L$, and thus, in particular, the proof for (1)⇒(2) will be completed.

2. Preliminaries

Let $\mathbb{F}_q$ be the finite field with q elements, and let $\mathbf{P} = ([N], \leq)$ be a poset on the underlying set

$$[N] = \{1, \ldots, N\}$$

of coordinate positions of vectors in $\mathbb{F}_q^N$. For any such poset $\mathbf{P}$, the $\mathbf{P}$-*weight* $w_{\mathbf{P}}(u)$ of $u \in \mathbb{F}_q^N$ is defined in [4] to be

$$w_{\mathbf{P}}(u) = |\langle \mathrm{Supp}(u) \rangle|,$$

where

$$\langle \mathrm{Supp}(u) \rangle = \{j \in [N] \mid j \leq i \text{ for some } i \in \mathrm{Supp}(u)\} \tag{2.1}$$

is the smallest ideal (a subset I of $[N]$ is an *ideal* if $a \in I$, $b \leq a \Rightarrow b \in I$) containing $\mathrm{Supp}(u)$, with

$$\mathrm{Supp}(u) = \{i \in [N] \mid u_i \neq 0\}.$$

Then one shows that $w_{\mathbf{P}}(u) \geq 0$ with equality if and only if $u = 0$, $w_{\mathbf{P}}(u) = w_{\mathbf{P}}(-u)$, and $w_{\mathbf{P}}(u+v) \leq w_{\mathbf{P}}(u) + w_{\mathbf{P}}(v)$, for all $u, v \in \mathbb{F}_q^N$, i.e., $d_{\mathbf{P}}(u, v) = w_{\mathbf{P}}(u - v)$ is a metric, called the $\mathbf{P}$-*metric.*

A linear code C of length N over $\mathbb{F}_q$ equipped with $w_{\mathbf{P}}$ is called a *linear* $\mathbf{P}$-*code* (or a linear code on $\mathbf{P}$) of length N over $\mathbb{F}_q$. In particular, the space $\mathbb{F}_q^N$ equipped with $w_{\mathbf{P}}$, denoted by $(\mathbb{F}_q^N, w_{\mathbf{P}})$, is called a $\mathbf{P}$-*weight space*. If $\mathbf{P}$ is an antichain, then $w_{\mathbf{P}} = w$ is the *Hamming weight* and $(\mathbb{F}_q^N, w)$ is the *Hamming space*. By a *linear automorphism of* $\mathbb{F}_q^N$ *preserving* $w_{\mathbf{P}}$ we mean a nonsingular linear map ϕ of $\mathbb{F}_q^N$ satisfying $w_{\mathbf{P}}(\phi u) = w_{\mathbf{P}}(u)$, for all $u \in \mathbb{F}_q^N$. The group of all such automorphisms will be denoted by $\mathrm{Aut}(\mathbb{F}_q^N, w_{\mathbf{P}})$.

Let $n, m_1, \ldots, m_t$ $(t \geq 1)$ be positive integers with $m = m_1 + \cdots + m_t$. For each j $(j \in [n])$, let $\mathbf{P}_j$ be the poset whose underlying set and order relation are given by

$$\begin{aligned} &[m] \times \{j\} = m_1\mathbf{1}^{(j)} \cup \cdots \cup m_t\mathbf{1}^{(j)}, \\ &m_r\mathbf{1}^{(j)} = \{(m_1 + \cdots + m_{r-1} + 1, j), \ldots, (m_1 + \cdots + m_{r-1} + m_r, j)\}, \\ &a <_{p_j} b \;\Leftrightarrow\; a \in m_r\mathbf{1}^{(j)},\, b \in m_s\mathbf{1}^{(j)} \text{ for some } r, s \text{ with } 1 \leq r < s \leq t. \end{aligned} \tag{2.2}$$

Also, let $\mathbf{P}_0$ be the poset whose underlying set and order relation are given by

$$\begin{aligned} &[m] = m_1\mathbf{1} \cup \cdots \cup m_t\mathbf{1}, \\ &m_r\mathbf{1} = \{m_1 + \cdots + m_{r-1} + 1, \ldots, m_1 + \cdots + m_{r-1} + m_r\}, \\ &a <_{\mathbf{P}_0} b \;\Leftrightarrow\; a \in m_r\mathbf{1}, b \in m_s\mathbf{1} \text{ for some } r, s \text{ with } 1 \leq r < s \leq t. \end{aligned} \tag{2.3}$$

Then $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_n$ are all isomorphic to one another under the obvious maps.

Unless otherwise stated, for the rest of this paper, $\mathbf{P} = \mathbf{P}_1 \dotplus \cdots \dotplus \mathbf{P}_n$ *will denote the poset given by the disjoint sum of the posets* $\mathbf{P}_1, \ldots, \mathbf{P}_n$ *defined in* (2.2). In more detail, $\mathbf{P}$ is the poset on the underlying set $[m] \times [n]$ of indices of matrices in $\mathbb{F}_q^{m \times n}$, with the order relation given by

$$a <_{\mathbf{P}} b \;\Leftrightarrow\; a \in m_r\mathbf{1}^{(j)},\, b \in m_s\mathbf{1}^{(j)} \text{ for some } r, s, j \tag{2.4}$$

with $1 \leq r < s \leq t$ and $j \in [n]$.

In accordance with the poset structure of $\mathbf{P}_0$, we write

$$\mathbb{F}_q^m = \mathbb{F}_q^{m_1} \oplus \cdots \oplus \mathbb{F}_q^{m_t}, \quad x = (x_1, \ldots, x_t),\ x_r \in \mathbb{F}_q^{m_r}, \quad \text{for } x \in \mathbb{F}_q^m, \tag{2.5}$$

so that x_r is the rth block of coordinates of x. If, for $x = (x_1, \ldots, x_t) \in \mathbb{F}_q^m$, s is the largest integer with $x_s \neq 0$, then

$$w_{\mathbf{P}_0}(x) = w(x_s) + \sum_{r=1}^{s-1} m_r, \tag{2.6}$$

where $w(x_s)$ is the Hamming weight of x_s (cf. (2.1), (2.3)). The map

$$\tau_j : (\mathbb{F}_q^m, w_{\mathbf{P}_0}) \to (\mathbb{F}_q^{[m]\times\{j\}}, w_{\mathbf{P}_j}) \tag{2.7}$$

given by $(x_1, \ldots, x_m) \mapsto {}^t[x_1 \ \cdots \ x_m] = {}^t[x_{1j} \ \cdots \ x_{mj}]$ is an isomorphism of weight spaces, i.e., τ_j is an isomorphism of vector spaces and $w_{\mathbf{P}_0}(x) = w_{\mathbf{P}_j}(\tau_j x)$ for all $x \in \mathbb{F}_q^m$ (cf. (2.2), (2.3)).

For $u = [u^1 \ \cdots \ u^n] \in \mathbb{F}_q^{m\times n}$, with u^j the jth column of u,

$$w_{\mathbf{P}}(u) = \sum_{j=1}^{n} w_{\mathbf{P}_j}(u^j), \tag{2.8}$$

and we set

$$w_j(u) = w_{\mathbf{P}_j}(u^j) \quad (j = 1, \ldots, n), \tag{2.9}$$

so that

$$w_{\mathbf{P}}(u) = \sum_{j=1}^{n} w_j(u). \tag{2.10}$$

$w_{\mathbf{P}}$ will be called the *generalized Niederreiter–Rosenbloom–Tsfasman weight*.

Whenever it is convenient, we will view $\mathbb{F}_q^{m\times n}$ as

$$\mathbb{F}_q^{m\times n} = \prod_{j=1}^{n} \mathbb{F}_q^{m\times 1}. \tag{2.11}$$

So, for $u^1, \ldots, u^n \in \mathbb{F}_q^{m\times 1}$, $[u^1 \ \cdots \ u^n] = u \in \mathbb{F}_q^{m\times n}$, and we may still write $w_{\mathbf{P}}(u) = \sum_{j=1}^{n} w_j(u)$ with $w_j(u) = w_{\mathbf{P}_j}(u^j)$.

Remark 2.1. (1) If $m_1 = \cdots = m_t = 1$, $w_{\mathbf{P}}$ is the well-known Niederreiter–Rosenbloom–Tsfasman weight (cf. [8], [31], [32]).

(2) If $n = 1$, $w_{\mathbf{P}}$ is the $\mathbf{P}$-weight corresponding to the weak order poset $\mathbf{P}$ (cf. [16–18], [20], [21]).

${}^tA$ indicates the transpose of the matrix A, and $S(Y)$ denotes the group of all permutations of the set Y. Also, we denote by $\check{\mathbf{P}}$ the dual poset of $\mathbf{P}$ having the same underlying set as $\mathbf{P}$ but with the reversed order relation.

3. Construction of association schemes

Here we will first briefly go over some basic facts about association schemes (cf. [2], [3], [6] for details) and then prove Theorem 3.1 about how to construct association schemes associated with subgroups of the full automorphism group. Finally, we will show Theorem 3.3 giving some equivalent conditions for the existence of an isomorphism between the scheme associated with the automorphism group for the dual poset $\check{\mathbf{P}}$ and the dual scheme to that for the poset $\mathbf{P}$.

A pair $(X, \mathcal{R} = \{R_i\}_{i=0}^d)$ consisting of a finite set X and $d+1$ subsets of $X \times X$ is called a *d-class* (*symmetric*) *association scheme* if

$$\begin{aligned} &\text{(i) } \mathcal{R} = \{R_0, R_1, \dots, R_d\} \text{ is a partition of } X \times X, \\ &\text{(ii) } R_0 = \Delta_X = \{(x,x) \mid x \in X\}, \\ &\text{(iii) } R_i = {}^tR_i \text{ for } i = 0, 1, \dots, d, \text{ where } {}^tR_i = \{(x,y) \mid (y,x) \in R_i\}, \\ &\text{(iv) for any } i, j, k \ (0 \le i, j, k \le d), \text{ there are numbers } p_{ij}^k \text{ such that,} \\ &\quad \text{for any } (x,y) \in R_k, \text{ the number of } z \in X \text{ with } (x,z) \in R_i \\ &\quad \text{and } (z,y) \in R_j \text{ equals } p_{ij}^k. \end{aligned} \tag{3.1}$$

Let A_i be the adjacency matrix of R_i, for $i = 0, 1, \dots, d$. Then $A_0, A_1, \dots, A_d$ generate a $(d+1)$-dimensional commutative algebra over the reals under the usual multiplication of matrices, which is called the *Bose–Mesner algebra* of $\mathcal{X} = (X, \mathcal{R})$ and denoted by $\mathcal{A}$. The algebra $\mathcal{A}$ has a basis of irreducible idempotent matrices $E_0, E_1, \dots, E_d$, uniquely determined (up to ordering) by the conditions:

$$\begin{aligned} &\text{(i) } E_k E_l = \delta_{kl} E_k \text{ for } 0 \le k, l \le d, \\ &\text{(ii) } \textstyle\sum_{i=0}^d E_i = I, \\ &\text{(iii) } E_0, E_1, \dots, E_d \text{ are linearly independent over the reals.} \end{aligned} \tag{3.2}$$

These two bases are related by

$$A_j = \sum_{i=0}^d p_{ij} E_i, \qquad E_j = \frac{1}{|X|} \sum_{i=0}^d q_{ij} A_i.$$

p_{ij}'s and q_{ij}'s are respectively called the *p-numbers* and *q-numbers* of $\mathcal{X} = (X, \mathcal{R})$, and (p_{ij}) and (q_{ij}) the *first* and *second eigenmatrices of* $\mathcal{X}$. Further, they satisfy $(p_{ij})(q_{ij}) = (q_{ij})(p_{ij}) = |X| I$.

Now, let $\mathcal{X} = (X, \mathcal{R} = \{R_i\}_{i=0}^d)$ be a *translation association scheme*, i.e., $\mathcal{X}$ is a d-class association scheme, $(X, +)$ is an abelian group, and $(x,y) \in R_i \Rightarrow (x+z, y+z) \in R_i$ for all $z \in X$ and i $(0 \le i \le d)$. Then

$$X_i = \{x \in X \mid (0, x) \in R_i\} \quad (0 \le i \le d) \tag{3.3}$$

gives a partition of X, and

$$(x,y) \in R_i \;\Leftrightarrow\; y - x \in X_i \quad (0 \le i \le d).$$

The *dual association scheme* $\mathcal{X}^* = (X^*, \mathcal{R}^* = \{R_i^*\}_{i=0}^d)$ of $\mathcal{X} = (X, \mathcal{R} = \{R_i\}_{i=0}^d)$ consists of the group X^* of characters on X, together with the $d+1$ classes R_i^* determined by

$$(\chi, \psi) \in R_i^* \;\Leftrightarrow\; \psi \chi^{-1} \in X_i^*. \tag{3.4}$$

Here

$$X_i^* = \{\chi \in X^* \mid E_i \chi = \chi\}, \tag{3.5}$$

where χ is viewed as a column vector with the xth entry given by $\chi(x)$ $(x \in X)$.

Then the idempotents, p-numbers and q-numbers can be expressed in terms of characters:

$$E_j = \frac{1}{|X|} \sum_{\chi \in X_j^*} \chi^t \bar{\chi}, \tag{3.6}$$

$$p_{ij} = \sum_{x \in X_j} \chi(x) \quad \text{for } \chi \in X_i^*, \tag{3.7}$$

$$q_{ij} = \sum_{\chi \in X_j^*} \chi(x) \quad \text{for } x \in X_i \tag{3.8}$$

(cf. (3.3), (3.5)).

Let Y be an *additive code* of the translation association scheme $(X, \mathcal{R})$. This means Y is just a subgroup of $(X, +)$. The dual Y^0 of Y is the additive code of $(X^*, \mathcal{R}^*)$ given by

$$Y^0 = \{\chi \in X^* \mid \chi(x) = 1 \text{ for all } x \in Y\}. \tag{3.9}$$

The *weight distributions* of Y and of Y^0 are respectively defined by

$$(a_i(Y) = |Y \cap X_i|)_{i=0}^d, \tag{3.10}$$

$$(a_i(Y^0) = |Y^0 \cap X_i^*|)_{i=0}^d. \tag{3.11}$$

Then the *generalized MacWilliams identities* of Delsarte say that

$$(a_j(Y^0))_{j=0}^d = \frac{1}{|Y|}(a_i(Y))_{i=0}^d (q_{ij}), \tag{3.12}$$

$$(a_j(Y))_{j=0}^d = \frac{1}{|Y^0|}(a_i(Y^0))_{i=0}^d (p_{ij}). \tag{3.13}$$

Let $\mathbf{P}$ be any poset on $[N]$, and let H be a subgroup of $G = \mathrm{Aut}(\mathbb{F}_q^N, w_{\mathbf{P}})$ containing the involution $\iota \in \mathrm{Aut}(\mathbb{F}_q^N)$ given by

$$\iota(u_1, \ldots, u_N) = (-u_1, \ldots, -u_N). \tag{3.14}$$

Let $\{\mathcal{O}_{H,i}\}_{i=0}^d$ be the orbits under the action of H on $\mathbb{F}_q^N$, with $\mathcal{O}_{H,0} = \{0\}$.

The following theorem provides us with a general method of constructing symmetric association schemes.

THEOREM 3.1. *$\mathcal{X}_H = (\mathbb{F}_q^N, \{R_{H,i}\}_{i=0}^d)$ is a symmetric translation association scheme. Here the classes $R_{H,i}$ are given by*

$$(x, y) \in R_{H,i} \Leftrightarrow x - y \in \mathcal{O}_{H,i} \quad (i = 0, 1, \ldots, d). \tag{3.15}$$

Proof. (i), (ii) of (3.1) are clear. As $\iota \in H$ (cf. (3.14)), $u \in \mathcal{O}_{H,i} \Rightarrow -u \in \mathcal{O}_{H,i}$, and hence we have (iii) of (3.1). So we only need to show that given any k, i, j $(0 \le k, i, j \le d)$ and $x, y, x', y' \in \mathbb{F}_q^N$ with $x - y \in \mathcal{O}_{H,k}, x' - y' \in \mathcal{O}_{H,k}$, we have

$$|\{z \in \mathbb{F}_q^N \mid x - z \in \mathcal{O}_{H,i},\ z - y \in \mathcal{O}_{H,j}\}| = |\{z \in \mathbb{F}_q^N \mid x' - z \in \mathcal{O}_{H,i},\ z - y' \in \mathcal{O}_{H,j}\}|.$$

Put $u = x - y$, $v = x' - y'$. Observe that the map

$$\{z \mid x - z \in \mathcal{O}_{H,i},\ z - y \in \mathcal{O}_{H,j}\} \to \{z \mid u - z \in \mathcal{O}_{H,i},\ z \in \mathcal{O}_{H,j}\} \tag{3.16}$$

given by $z \mapsto z - y$ is a bijection. As this holds with x, y, u respectively replaced by x', y', v, we only need to show that

$$|\{z \mid u - z \in \mathcal{O}_{H,i},\ z \in \mathcal{O}_{H,j}\}| = |\{z \mid v - z \in \mathcal{O}_{H,i},\ z \in \mathcal{O}_{H,j}\}|.$$

As the action of H on each orbit $\mathcal{O}_{H,i}$ is transitive and $u, v \in \mathcal{O}_{H,k}$, there is $\sigma \in H$ such that $\sigma u = v$. Now, the map

$$\{z \mid u - z \in \mathcal{O}_{H,i},\ z \in \mathcal{O}_{H,j}\} \to \{z \mid v - z \in \mathcal{O}_{H,i},\ z \in \mathcal{O}_{H,j}\}$$

given by $z \mapsto \sigma z$ is a bijection. ■

REMARK 3.2. (1) Let H be a subgroup of $G = \mathrm{Aut}(\mathbb{F}_q^N, w_{\mathbf{P}})$. Then we will index by H everything related to H, such as the orbits under the action of H on $\mathbb{F}_q^N$, the association scheme related to H, and parameters of the association scheme like adjacency matrices, irreducible idempotents, p-numbers and q-numbers, etc., but objects related to G itself will be indexed by $\mathbf{P}$.

(2) One could construct, in the same manner as above, association schemes for any subgroups of $\mathrm{Aut}(\mathbb{F}_q^N)$, not just those of $\mathrm{Aut}(\mathbb{F}_q^N, w_{\mathbf{P}})$. However, it seems that the association schemes constructed from subgroups of G are more interesting.

Two d-class association schemes $(X, \{R_{X,i}\}_{i=0}^d)$ and $(Y, \{R_{Y,i}\}_{i=0}^d)$ are *isomorphic* if there exists a bijection $\tau : X \to Y$ such that

$$(x, y) \in R_{X,i} \ \Leftrightarrow\ (\tau(x), \tau(y)) \in R_{Y,i}$$

for $i = 0, 1, \ldots, d$ (after renumbering $R_{Y,1}, \ldots, R_{Y,d}$). In particular, two d-class translation association schemes $(X, \{R_{X,i}\}_{i=0}^d)$ and $(Y, \{R_{Y,i}\}_{i=0}^d)$ are isomorphic if there is an isomorphism $\tau : X \to Y$ of groups such that

$$x \in X_i \ \Leftrightarrow\ \tau(x) \in Y_i \tag{3.17}$$

for $i = 0, 1, \ldots, d$. Here $X_i = \{x \in X \mid (0, x) \in R_{X,i}\}$ and $Y_i = \{y \in Y \mid (0, y) \in R_{Y,i}\}$, for $i = 0, 1, \ldots, d$. In case two d-class association schemes are isomorphic, we may assume that all the parameters of the schemes are the same.

Recall from [16] that the orbits under the action of $\mathrm{Aut}(\mathbb{F}_q^m, w_{\mathbf{P}_0})$ on $\mathbb{F}_q^m$ are the $\mathbf{P}_0$-spheres

$$S_{\mathbf{P}_0}(i) = \{u \in \mathbb{F}_q^m \mid w_{\mathbf{P}_0}(u) = i\} \quad (i = 0, 1, \ldots, m).$$

Let $\mathcal{X}_{\mathbf{P}_0} = (\mathbb{F}_q^m, \mathcal{R}_{\mathbf{P}_0} = \{R_{\mathbf{P}_0,i}\}_{i=0}^m)$ be the association scheme constructed as in Theorem 3.1, i.e., $(x, y) \in R_{\mathbf{P}_0,i} \Leftrightarrow x - y \in S_{\mathbf{P}_0}(i) \Leftrightarrow w_{\mathbf{P}_0}(x - y) = i$. Let $\mathcal{X}_{\mathbf{P}_j} = (\mathbb{F}_q^{[m]\times\{j\}}, \{R_{\mathbf{P}_j,i}\}_{i=0}^m)$ $(j \in [n])$ be the analogously constructed association schemes. Then $\mathcal{X}_{\mathbf{P}_0} = (\mathbb{F}_q^m, \mathcal{R}_{\mathbf{P}_0}) \to \mathcal{X}_{\mathbf{P}_j} = (\mathbb{F}_q^{[m]\times\{j\}}, \mathcal{R}_{\mathbf{P}_j})$ given by τ_j in (2.7) on the underlying sets is an isomorphism of schemes. So we may assume that all the parameters for the schemes $\mathcal{X}_{\mathbf{P}_0}, \mathcal{X}_{\mathbf{P}_1}, \ldots, \mathcal{X}_{\mathbf{P}_n}$ are the same.

Let λ be a fixed nontrivial additive character of $\mathbb{F}_q$. Then the group of characters of $\mathbb{F}_q^N$ is given by $\{\lambda_x \mid x \in \mathbb{F}_q^N\}$, where $\lambda_x(y) = \lambda(x \cdot y)$, with $x \cdot y$ the usual inner product of x and y in $\mathbb{F}_q^N$.

THEOREM 3.3. *Let $\mathbf{P}$ be any poset on $[N]$, H a subgroup of $G = \mathrm{Aut}(\mathbb{F}_q^N, w_{\mathbf{P}})$ containing the involution $\iota \in \mathrm{Aut}(\mathbb{F}_q^N)$* (cf. (3.14))*, and let $\{\mathcal{O}_{H,i}\}_{i=0}^d$ be the orbits under the action of H on $\mathbb{F}_q^N$, with $\mathcal{O}_{H,0} = \{0\}$. Also, let $\check{H}$ be a subgroup of $\check{G} = \mathrm{Aut}(\mathbb{F}_q^N, w_{\check{\mathbf{P}}})$ containing the involution ι, and let $\{\mathcal{O}_{\check{H},i}\}_{i=0}^d$ be the orbits under the action of $\check{H}$ on $\mathbb{F}_q^N$, with $\mathcal{O}_{\check{H},0} = \{0\}$. Then the following are equivalent:*

(1) $(X = \mathbb{F}_q^N, \mathcal{R}_{\check{H}} = \{R_{\check{H},i}\}_{i=0}^d) \to (X^*, \mathcal{R}_H^* = \{R_{H,i}^*\}_{i=0}^d)$ *given by $x \mapsto \lambda_x$ is an isomorphism of translation association schemes.*

(2) *We have*

$$x \in \mathcal{O}_{\check{H},j} \;\Leftrightarrow\; \lambda_x \in X_{H,j}^* \quad \text{(cf. (3.15)) } \textit{for } j = 0, 1, \ldots, d. \tag{3.18}$$

(3) *The elements*

$$E_{H,j} = \frac{1}{|X|} \sum_{x \in \mathcal{O}_{\check{H},j}} \lambda_x {}^t\bar{\lambda}_x \qquad (j = 0, 1, \ldots, d)$$

are the irreducible idempotents for the association scheme $\mathcal{X}_H = (X = \mathbb{F}_q^N, \mathcal{R}_H = \{R_{H,i}\}_{i=0}^d)$.

(4) *$f_j : X \to \mathbb{C}$ given by*

$$f_j(a) = \sum_{x \in \mathcal{O}_{\check{H},j}} \lambda(a \cdot x)$$

is constant on each $\mathcal{O}_{H,i}$ $(i = 0, 1, \ldots, d)$ for all $j = 0, 1, \ldots, d$.

Proof. (1)$\Rightarrow$(4). As (1)$\Leftrightarrow$(2) by definition, (3.18) is satisfied. Then, from (3.8),

$$q_{H,ij} = \sum_{x \in \mathcal{O}_{\check{H},j}} \lambda(a \cdot x) \qquad \text{for } a \in \mathcal{O}_{H,i}$$

is the q-number of $\mathcal{X}_H$, and hence is constant on each $\mathcal{O}_{H,i}$.

(4)$\Rightarrow$(3). Let $\sum_{x \in \mathcal{O}_{\check{H},j}} \lambda(a \cdot x) = \bar{q}_{H,ij}$ for $a \in \mathcal{O}_{H,i}$. Then

$$\frac{1}{|X|} \sum_{x \in \mathcal{O}_{\check{H},j}} \lambda_x {}^t\bar{\lambda}_x = \frac{1}{|X|} \sum_{j=0}^{d} \bar{q}_{H,ij} A_{H,i}, \tag{3.19}$$

and hence (3.19) belongs to the Bose–Mesner algebra of the association scheme $\mathcal{X}_H$. Set

$$\bar{E}_{H,j} = \frac{1}{|X|} \sum_{x \in \mathcal{O}_{\check{H},j}} \lambda_x {}^t\bar{\lambda}_x, \qquad j = 0, 1, \ldots, d.$$

We need to show conditions (i)–(iii) of (3.2). One can see that

$$(\bar{E}_{H,i} \bar{E}_{H,j})_{ab} = \frac{1}{|X|^2} \sum_{\substack{x \in \mathcal{O}_{\check{H},i} \\ y \in \mathcal{O}_{\check{H},j}}} \lambda_x(a) \bar{\lambda}_y(b) {}^t\bar{\lambda}_x \lambda_y. \tag{3.20}$$

Note here that

$${}^t\bar{\lambda}_x \lambda_y = \sum_{z \in \mathbb{F}_q^N} \lambda(z \cdot (y - x)) = \begin{cases} 0, & x \neq y, \\ |X| = q^N, & x = y. \end{cases} \tag{3.21}$$

Thus (3.20) equals

$$\begin{cases} 0, & \text{if } i \neq j, \\ \dfrac{1}{|X|} \displaystyle\sum_{x \in \mathcal{O}_{\check{H},i}} \lambda(x \cdot (a-b)) = (\bar{E}_{H,i})_{ab}, & \text{if } i = j. \end{cases}$$

This shows condition (i) of (3.2). Next, (ii) is easy to see. Finally, assume that $\sum_{i=0}^{d} \alpha_i \bar{E}_{H,i} = 0$ ($\alpha_i \in \mathbb{R}$). Then $\alpha_i \bar{E}_i = 0$ for all i, by (i). So it is enough to see that $\bar{E}_{H,i} \neq 0$ for all i. In view of (3.21),

$$\bar{E}_{H,j} \lambda_x = \begin{cases} \lambda_x, & x \in \mathcal{O}_{\check{H},j}, \\ 0, & x \notin \mathcal{O}_{\check{H},j}, \end{cases} \tag{3.22}$$

and hence, in particular, $\bar{E}_{H,j} \neq 0$.

(3)⇒(2). By (3.22) above,

$$x \in \mathcal{O}_{\check{H},j} \;\Leftrightarrow\; \bar{E}_{H,j}\lambda_x = \lambda_x \;\Leftrightarrow\; \lambda_x \in X^*_{H,j}.$$

(2)⇒(1). This follows from the definition, as noted above. ∎

4. Automorphism group and orbits

Let $e_{ij} \in \mathbb{F}_q^{m \times n}$ be the matrix with 1 at position (i,j) and 0 elsewhere, so that $\{e_{ij} \mid i \in [m],\, j \in [n]\}$ is a basis for $\mathbb{F}_q^{m\times n}$.

Remark 4.1. (1) For $i \in [m_1 + \cdots + m_l] \setminus [m_1 + \cdots + m_{l-1}]$, $j \in [n]$, we have $w_j(e_{ij}) = m_1 + \cdots + m_{l-1} + 1$, and $w_{j'}(e_{ij}) = 0$ for $j' \neq j$. So $w_{\mathbf{P}}(e_{ij}) = m_1 + \cdots + m_{l-1} + 1$.

(2) Let $u \in \mathbb{F}_q^{m\times n}$, $j \in [n]$. Then $w_j(u) \leq m_1 + \cdots + m_l \Leftrightarrow \mathrm{Supp}(u) \cap ([m] \times \{j\}) \subseteq [m_1 + \cdots + m_l] \times \{j\}$. In particular, $w_j(u) = 0 \Leftrightarrow \mathrm{Supp}(u) \cap ([m] \times \{j\}) = \emptyset$.

(3) For $l \in [t]$, put

$$H_l = \langle e_{ij} \mid i \in [m_1 + \cdots + m_l],\, j \in [n] \rangle. \tag{4.1}$$

Then, in view of (2) above, we have

$$u \in H_l \;\Leftrightarrow\; w_j(u) \leq m_1 + \cdots + m_l \text{ for all } j. \tag{4.2}$$

Lemma 4.2. *Let $\phi \in \mathrm{Aut}(\mathbb{F}_q^{m\times n}, w_{\mathbf{P}})$. Then $\phi(H_l) \subseteq H_l$ (cf. (4.1)), and $\phi|_{H_l} : H_l \to H_l$ is a linear automorphism of H_l ($l \in [t]$).*

Proof. Assume, on the contrary, that $\phi(H_l) \nsubseteq H_l$ for some l. Then there are $i \in [m_1+\cdots+m_l]$, $j \in [n]$ such that $\phi(e_{ij}) \notin H_l$. Then, by (4.2), $w_j(\phi(e_{ij})) \geq m_1+\cdots+m_l+1$ for some j, so that $w_{\mathbf{P}}(\phi(e_{ij})) \geq m_1 + \cdots + m_l + 1$. However, by Remark 4.1(1), $w_{\mathbf{P}}(e_{ij}) \leq m_1 + \cdots + m_{l-1} + 1$. This is a contradiction. ∎

Lemma 4.3. *Let $i \in [m_1+\cdots+m_l] \setminus [m_1+\cdots+m_{l-1}]$, $j \in [n]$, and $\phi \in \mathrm{Aut}(\mathbb{F}_q^{m\times n}, w_{\mathbf{P}})$. Then*

$$w_k(\phi(e_{ij})) = m_1 + \cdots + m_{l-1} + 1 \quad \textit{for some } k \in [n],$$
$$w_{k'}(\phi(e_{ij})) = 0 \quad \textit{for all } k' \in [n] \textit{ with } k' \neq k.$$

Proof. By Remark 4.1, $w_k(\phi(e_{ij})) \le m_1 + \cdots + m_{l-1} + 1$ for all k. Suppose that $w_k(\phi(e_{ij})) \le m_1 + \cdots + m_{l-1}$ for all k. Then, by (4.2), $\phi(e_{ij}) \in H_{l-1}$. So, by Lemma 4.2, there is $u \in H_{l-1}$ such that $\phi(u) = \phi(e_{ij})$. Now, $\phi(u - e_{ij}) = 0$, but $w_{\mathbf{P}}(u - e_{ij}) \ge m_1 + \cdots + m_{l-1} + 1$, which is a contradiction. ■

COROLLARY 4.4. *Let $\phi \in \mathrm{Aut}(\mathbb{F}_q^{m\times n}, w_{\mathbf{P}})$, $i \in [m_1+\cdots+m_l] \setminus [m_1+\cdots+m_{l-1}]$, $j \in [n]$. Then there exist $i' \in [m_1 + \cdots + m_l] \setminus [m_1 + \cdots + m_{l-1}]$ and $j' \in [n]$ such that*

$$\phi(e_{ij}) = a_i^j e_{i'j'} + \sum_{k=1}^{m_1+\cdots+m_{l-1}} (\tilde{\tau}_j)_{ki} e_{kj'} \tag{4.3}$$

for some $a_i^j \in \mathbb{F}_q^\times$, $(\tilde{\tau}_j)_{ki} \in \mathbb{F}_q$.

Unless otherwise stated, from now on we will assume that $t > 1$. This means that $\mathbf{P}$ is not an antichain.

LEMMA 4.5. *Let $\phi \in \mathrm{Aut}(\mathbb{F}_q^{m\times n}, w_{\mathbf{P}})$. Then there exist $\tilde{\rho}_l^j \in S([m_1 + \cdots + m_l] \setminus [m_1 + \cdots + m_{l-1}])$ $(l \in [t], j \in [n])$ and $\sigma \in S_n$ such that, for all $i \in [m_1 + \cdots + m_l] \setminus [m_1 + \cdots + m_{l-1}]$ and $j \in [n]$,*

$$\phi(e_{ij}) = a_i^j e_{\tilde{\rho}_l^j(i)\sigma(j)} + \sum_{k=1}^{m_1+\cdots+m_{l-1}} (\tilde{\tau}_j)_{ki} e_{k\sigma(j)} \tag{4.4}$$

for some $a_i^j \in \mathbb{F}_q^\times$ and $(\tilde{\tau}_j)_{ki} \in \mathbb{F}_q$.

Proof. We first show the assertion for $l = 1$. By Corollary 4.4, $\phi(e_{1j}) = a_1^j e_{1'j'}$ for some $1' \in [m_1]$, $j' \in [n]$, and $a_1^j \in \mathbb{F}_q^\times$. We claim that, for all $i \in [m_1]$, $\phi(e_{ij}) = a_i^j e_{i'j'}$ for some $a_i^j \in \mathbb{F}_q^\times$, $i' \in [m_1]$. The point here is that j' depends only on j and not on i. Suppose that, for some $i \in [m_1]$, $\phi(e_{ij}) = a_i^j e_{i'j''}$ with some $j'' \in [n]$, $j'' \ne j'$. Then we consider $u = e_{1j} + e_{ij} + e_{m_1+1,j}$. Note here that the assumption $t > 1$ is used and $w_{\mathbf{P}}(u) = m_1 + 1$. By (4.3), we have

$$\phi(u) = a_1^j a_{1'j'} + a_i^j e_{i'j''} + a_{m_1+1}^j e_{(m_1+1)'\tilde{j}} + \sum_{k=1}^{m_1} (\tilde{\tau}_j)_{k,(m_1+1)} e_{k\tilde{j}}$$

for some $(m_1 + 1)' \in [m_1 + m_2] \setminus [m_1]$ and $\tilde{j} \in [n]$. If $\tilde{j} \ne j'$ and $\tilde{j} \ne j''$, then $w_{\mathbf{P}}(\phi(u)) = m_1 + 3$. On the other hand, if $\tilde{j} = j'$ or $\tilde{j} = j''$, then $w_{\mathbf{P}}(\phi(u)) = m_1 + 2$. In any event, this is impossible.

So far we have shown that, for $i \in [m_1]$, $j \in [n]$, there are $i' \in [m_1]$ and $\sigma(j) \in [n]$ such that $\phi(e_{ij}) = a_i^j e_{i'\sigma(j)}$. Now, we will show that $\sigma \in S_n$ and the assignment $i \mapsto i'$, denoted by $\tilde{\rho}_1^j$, is a permutation in $S([m_1]) = S_{m_1}$. Assume that

$$\phi(e_{1j}) = a_1^j e_{1'(j)\sigma(j)}, \quad \phi(e_{1k}) = a_1^k e_{1'(k)\sigma(k)}$$

with $j \ne k$ and $\sigma(j) = \sigma(k)$. Here $1'(j), 1'(k) \in [m_1]$ depend respectively on j and k. Then

$$\phi^{-1}(a_1^j e_{1'(j)\sigma(j)}) = e_{1j}, \quad \phi^{-1}(a_1^k e_{1'(k)\sigma(k)}) = e_{1k}.$$

Certainly, $1'(j) \ne 1'(k)$, and, to derive a contradiction, one can consider the vector $u = a_1^j e_{1'(j)\sigma(j)} + a_1^k e_{1'(k)\sigma(k)} + e_{m_1+1,\sigma(j)}$, just as at the beginning of this proof. Thus

$\sigma \in S_n$. Let j be fixed, and assume that

$$\phi(e_{ij}) = a_i^j e_{\tilde{\rho}_1^j(i)\sigma(j)}, \qquad \phi(e_{kj}) = a_k^j e_{\tilde{\rho}_1^j(k)\sigma(j)}$$

with $i \neq k$ and $\tilde{\rho}_1^j(i) = \tilde{\rho}_1^j(k)$. Then $w_{\mathbf{P}}(a_k^j e_{ij} - a_i^j e_{kj}) = 2$, but $\phi(a_k^j e_{ij} - a_i^j e_{kj}) = a_k^j \phi(e_{ij}) - a_i^j \phi(e_{kj}) = 0$, a contradiction.

Next, we assume that $l > 1$. By Corollary 4.4, there exist $i' \in [m_1 + \cdots + m_l] \setminus [m_1 + \cdots + m_{l-1}]$ and $j' \in [n]$ such that

$$\phi(e_{ij}) = a_i^j e_{i'j'} + \sum_{k=1}^{m_1+\cdots+m_{l-1}} (\tilde{\tau}_j)_{ki} e_{kj'}$$

for some $a_i^j \in \mathbb{F}_q^\times$ and $(\tilde{\tau}_j)_{ki} \in \mathbb{F}_q$. Then $j' = \sigma(j)$ for the $\sigma \in S_n$ appearing in the $l = 1$ case. Assume, on the contrary, that $j' \neq \sigma(j)$. Then we consider $e_{1j} + e_{ij}$. We have $w_{\mathbf{P}}(e_{1j} + e_{ij}) = m_1 + \cdots + m_{l-1} + 1$, but

$$\phi(e_{1j} + e_{ij}) = a_1^j e_{\tilde{\rho}_1^j(1)\sigma(j)} + a_i^j e_{i'j'} + \sum_{k=1}^{m_1+\cdots+m_{l-1}} (\tilde{\tau}_j)_{ki} e_{kj'}$$

has **P**-weight $m_1 + \cdots + m_{l-1} + 2$, a contradiction. So

$$\phi(e_{ij}) = a_i^j e_{i'\sigma(j)} + \sum_{k=1}^{m_1+\cdots+m_{l-1}} (\tilde{\tau}_j)_{ki} e_{k\sigma(j)}.$$

It only remains to see that $i \mapsto i'$, depending on j and denoted by $\tilde{\rho}_l^j$, is a permutation in $S([m_1 + \cdots + m_l] \setminus [m_1 + \cdots + m_{l-1}])$. This can be proved in the same manner that we used to show $\tilde{\rho}_1^j$ is a permutation. ■

For any $m \times m$ matrix X over a field, let ${}^\rho X$ (resp. X^ρ) denote the matrix obtained from X by permuting the rows (resp. columns) of X according to $\rho \in S_m$. So if $X_1, \ldots, X_m$ (resp. $X^1, \ldots, X^m$) are the rows (resp. columns) of X, then

$${}^\rho X = \begin{bmatrix} X_{\rho^{-1}(1)} \\ \vdots \\ X_{\rho^{-1}(m)} \end{bmatrix}, \qquad X^\rho = [X^{\rho^{-1}(1)} \ \cdots \ X^{\rho^{-1}(m)}].$$

Then we have the following lemma whose proof is elementary.

Lemma 4.6. *Let X, Y be $m \times m$ matrices over a field, and let $\rho, \mu \in S_m$. Then, with $1 = 1_m$, we have the following.*

(1) ${}^\rho X = {}^\rho 1 X$, $X^\rho = X 1^\rho$.
(2) ${}^\mu({}^\rho X) = {}^{\mu\rho} X$, $(X^\rho)^\mu = X^{\mu\rho}$.
(3) ${}^\rho 1 1^\rho = 1$, $({}^\rho 1)^{-1} = 1^\rho = {}^{\rho^{-1}} 1$, $(1^\rho)^{-1} = {}^\rho 1 = 1^{\rho^{-1}}$.
(4) ${}^\rho X^\mu Y = {}^{\rho\mu}(X_\mu Y)$, $X^\rho Y^\mu = (X Y_\rho)^{\mu\rho}$.

Here $X_\mu = {}^{\mu^{-1}} 1 X 1^{\mu^{-1}}$. In addition, for $X = \operatorname{diag}\{x_1, \ldots, x_m\}$, we have $X_\mu = \operatorname{diag}\{x_{\mu(1)}, \ldots, x_{\mu(m)}\}$ and $(X_\mu)_\rho = X_{\mu\rho}$.

We write, for $b = (b_1, \ldots, b_m) \in (\mathbb{F}_q^\times)^m$,

$$\operatorname{diag} b = \operatorname{diag}\{b_1, \ldots, b_m\}.$$

For $\rho_1 \in S_{m_1}, \ldots, \rho_t \in S_{m_t}$, $a_1 \in (\mathbb{F}_q^\times)^{m_1}, \ldots, a_t \in (\mathbb{F}_q^\times)^{m_t}$, let $l_{(\rho_1,a_1),\ldots,(\rho_t,a_t)}$ be the block diagonal matrix given by

$$l_{(\rho_1,a_1),\ldots,(\rho_t,a_t)} = \begin{bmatrix} {}^{\rho_1}(\operatorname{diag} a_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & {}^{\rho_t}(\operatorname{diag} a_t) \end{bmatrix}, \tag{4.5}$$

and let M be the group of all such matrices so that

$$M = M_{m_1,\ldots,m_t} = \left\{ l_{(\rho_1,a_1),\ldots,(\rho_t,a_t)} \;\middle|\; \begin{matrix} a_1 \in (\mathbb{F}_q^\times)^{m_1}, \ldots, a_t \in (\mathbb{F}_q^\times)^{m_t} \\ \rho_1 \in S_{m_1}, \ldots, \rho_t \in S_{m_t} \end{matrix} \right\}. \tag{4.6}$$

One can show that

$$\pi : (S_{m_1} \,{}_{\theta_1}\!\ltimes (\mathbb{F}_q^\times)^{m_1}) \times \cdots \times (S_{m_t} \,{}_{\theta_1}\!\ltimes (\mathbb{F}_q^\times)^{m_t}) \to M \tag{4.7}$$

given by

$$((\rho_1, a_1), \ldots, (\rho_t, a_t)) \mapsto l_{(\rho_1,a_1),\ldots,(\rho_t,a_t)}$$

is an isomorphism. Here $\theta_j : S_{m_j} \to \operatorname{Aut}((\mathbb{F}_q^\times)^{m_j})$ is given by

$$\rho \mapsto (a \mapsto \theta_j(\rho)a = a_\rho),$$

where $a_\rho = (a_{\rho(1)}, \ldots, a_{\rho(m_j)})$ for $a = (a_1, \ldots, a_{m_j})$. Also, let U be the unipotent radical of a maximal parabolic subgroup of $\mathrm{GL}(m, q)$ given by

$$U = U_{m_1,m_2,\ldots,m_t} = \left\{ \begin{bmatrix} 1_{m_1} & * & * & \cdots & * \\ 0 & 1_{m_2} & * & \cdots & * \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1_{m_{t-1}} & * \\ 0 & 0 & \cdots & 0 & 1_{m_t} \end{bmatrix} \right\}. \tag{4.8}$$

Put

$$QP = QP_{m_1,\ldots,m_t} = M \ltimes U, \tag{4.9}$$

$$\check{QP} = \check{QP}_{m_1,\ldots,m_t} = M \ltimes \check{U}, \tag{4.10}$$

where M, U are as in (4.6), (4.8), and $\check{U} = \check{U}_{m_1,m_2,\ldots,m_t}$ is the opposite unipotent radical obtained by transposing all the elements in U. Then, transposing everywhere, we have shown in [16] that

$$\operatorname{Aut}(\mathbb{F}_q^m, w_{\mathbf{P}_0}) \cong QP. \tag{4.11}$$

Here we view the elements in $\mathbb{F}_q^m$ as column vectors.

The group QP^n given by the n-fold direct product of QP acts on $\mathbb{F}_q^{m\times n}$ as

$$\tau u = [\tau_1 u^1 \;\cdots\; \tau_n u^n],$$

where $\tau = (\tau_1, \ldots, \tau_n) \in QP^n$ and $u = [u^1 \;\cdots\; u^n] \in \mathbb{F}_q^{m\times n}$. The map $u \mapsto \tau u$ is in $\operatorname{Aut}(\mathbb{F}_q^{m\times n}, w_{\mathbf{P}})$. In fact, for all j,

$$w_j(\tau u) = w_{\mathbf{P}_j}(\tau_j u^j) = w_{\mathbf{P}_j}(u^j) = w_j(u) \tag{4.12}$$

(cf. (4.10)). Also, note that the map

$$QP^n \to \operatorname{Aut}(\mathbb{F}_q^{m\times n}, w_{\mathbf{P}}) \quad (\tau \mapsto (u \mapsto \tau u))$$

is injective; the map $u \mapsto \tau u$ will simply be denoted by τ, and QP^n will be identified with its image.

The group S_n acts on $\mathbb{F}_q^{m\times n}$ as

$$\sigma u = [u^{\sigma^{-1}(1)} \ \cdots \ u^{\sigma^{-1}(n)}],$$

where $\sigma \in S_n$, $u = [u^1 \ \cdots \ u^n] \in \mathbb{F}_q^{m\times n}$. The map $u \mapsto \sigma u$ is in $\mathrm{Aut}(\mathbb{F}_q^{m\times n}, w_{\mathbf{P}})$. Again,

$$S_n \to \mathrm{Aut}(\mathbb{F}_q^{m\times n}, w_{\mathbf{P}}) \quad (\sigma \mapsto (u \mapsto \sigma u))$$

is injective, the map $u \mapsto \sigma u$ will be denoted by σ, and S_n will be identified with its image.

Let $\phi \in \mathrm{Aut}(\mathbb{F}_q^{m\times n}, w_{\mathbf{P}})$, and assume that the images $\phi(e_{ij})$ are as in (4.4). For each $j \in [n]$, define $\tilde{\tau}_j \in \mathbb{F}_q^{m\times m}$ as follows. It is given by $(\tilde{\tau}_j)_{ki}$ for $l \in [t]$, $i \in [m_1 + \cdots + m_l] \setminus [m_1 + \cdots + m_{l-1}]$, $k \in [m_1 + \cdots + m_{l-1}]$, and 0 elsewhere. So $\tilde{\tau}_j$ is a block strictly upper triangular matrix. Also, for each $j \in [n]$, we put

$$\begin{aligned}
b_1^j &= (a_1^j, \ldots, a_{m_1}^j) \in \mathbb{F}_q^{m_1},\\
b_2^j &= (a_{m_1+1}^j, \ldots, a_{m_1+m_2}^j) \in \mathbb{F}_q^{m_2},\\
&\vdots\\
b_t^j &= (a_{m_1+\cdots+m_{t-1}+1}^j, \ldots, a_{m_1+\cdots+m_{t-1}+m_t}^j) \in \mathbb{F}_q^{m_t},\\
l_j &= l_{(b_1^j,\rho_1^j),\ldots,(b_t^j,\rho_t^j)} \qquad \text{(cf. (4.5))},
\end{aligned} \tag{4.13}$$

where ρ_l^j is the permutation in S_{m_l} uniquely determined by

$$\rho_l^j(i) + m_1 + \cdots + m_{l-1} = \tilde{\rho}_l^j(i + m_1 + \cdots + m_{l-1}) \quad \text{for } i \in [m_l].$$

Finally, for $j \in [n]$ we let

$$\tau_j = l_j + \tilde{\tau}_j \in QP. \tag{4.14}$$

Now, for $u = \sum_{i,j} u_{ij} e_{ij} \in \mathbb{F}_q^{m\times n}$,

$$\begin{aligned}
\phi(u) &= \sum_{i,j} u_{ij}\phi(e_{ij})\\
&= \sum_{j=1}^{n}\sum_{l=1}^{t}\sum_{i=m_1+\cdots+m_{l-1}+1}^{m_1+\cdots+m_{l-1}+m_l} a^j_{(\tilde{\rho}_l^j)^{-1}(i)} u_{(\tilde{\rho}_l^j)^{-1}(i)j} e_{i\sigma(j)} + \sum_{j=1}^{n}\sum_{k=1}^{m}\Big(\sum_{i=1}^{m}(\tilde{\tau}_j)_{ki}u_{ij}\Big)e_{k\sigma(j)}\\
&= \sum_{j=1}^{n}\sum_{k=1}^{m}(l_j u^j)_k e_{k\sigma(j)} + \sum_{j=1}^{n}\sum_{k=1}^{m}(\tilde{\tau}_j u^j)_k e_{k\sigma(j)}\\
&= \sum_{j=1}^{n}\sum_{k=1}^{m}(\tau_j u^j)_k e_{k\sigma(j)} \qquad \text{(cf. (4.14))},
\end{aligned}$$

where l_j is as in (4.13), and $(*)_k$ denotes the kth component of $(*)$. Let

$$\tau = (\tau_{\sigma^{-1}(1)}, \ldots, \tau_{\sigma^{-1}(n)}) \in QP^n. \tag{4.15}$$

Then

$$\begin{aligned}\tau\sigma(u) &= [\tau_{\sigma^{-1}(1)}u^{\sigma^{-1}(1)} \ \cdots \ \tau_{\sigma^{-1}(n)}u^{\sigma^{-1}(n)}] \\ &= \sum_{j=1}^{n}\sum_{k=1}^{m}(\tau_{\sigma^{-1}(j)}u^{\sigma^{-1}(j)})_k e_{kj} = \sum_{j=1}^{n}\sum_{k=1}^{m}(\tau_j u^j)_k e_{k\sigma(j)}.\end{aligned}$$

Thus $\phi = \tau\sigma$ for $\tau = (\tau_{\sigma^{-1}(1)}, \dots, \tau_{\sigma^{-1}(n)}) \in QP^n$ (cf. (4.14–15)) and $\sigma \in S_n$ (cf. (4.4)). So

$$\mathrm{Aut}(\mathbb{F}_q^{m\times n}, w_{\mathbf{P}}) = QP^n \cdot S_n.$$

Also, for $\sigma \in S_n$, $(\tau_1, \dots, \tau_n) \in QP^n$,

$$\sigma(\tau_1, \dots, \tau_n)\sigma^{-1} = (\tau_{\sigma^{-1}(1)}, \dots, \tau_{\sigma^{-1}(n)}),$$

and hence

$$QP^n \lhd \mathrm{Aut}(\mathbb{F}_q^{m\times n}, w_{\mathbf{P}}).$$

Finally, for example, by using (4.12), it can be shown that

$$QP^n \cap S_n = \{\mathrm{id}\}.$$

Thus we have shown that, for $t > 1$,

$$QP^n \rtimes_\psi S_n \to \mathrm{Aut}(\mathbb{F}_q^{m\times n}, w_{\mathbf{P}}) \quad ((\tau, \sigma) \mapsto \tau\sigma)$$

is an isomorphism, with

$$\psi : S_n \to \mathrm{Aut}(QP^n) \quad (\sigma \mapsto (\tau = (\tau_1, \dots, \tau_n) \mapsto (\tau_{\sigma^{-1}(1)}, \dots, \tau_{\sigma^{-1}(n)}))). \tag{4.16}$$

If $t = 1$, $\mathbf{P}$ is just an antichain. Our results so far are summarized in the following theorem.

Theorem 4.7. *Let* $\mathbf{P}$ *be the generalized Niederreiter–Rosenbloom–Tsfasman poset* (cf. (2.4)). *Then*

$$\mathrm{Aut}(\mathbb{F}_q^{m\times n}, w_{\mathbf{P}}) \cong \begin{cases} QP^n \rtimes_\psi S_n & \text{if } t > 1, \\ (\mathbb{F}_q^\times)^{mn} \rtimes_\theta S_{mn} & \text{if } t = 1. \end{cases} \tag{4.17}$$

Here QP *is as in* (4.9) (cf. (4.5–6), (4.8)), ψ *is as in* (4.16), *and* $\theta : S_{mn} \to \mathrm{Aut}((\mathbb{F}_q^\times)^{mn})$ *is given by*

$$\rho \mapsto (a = (a_1, \dots, a_{mn}) \mapsto \theta(\rho)a = a_\rho = (a_{\rho(1)}, \dots, a_{\rho(mn)})).$$

Remark 4.8. (1) The above theorem for $n = 1$ was proved in [16], while that for $m_1 = \cdots = m_t = 1$ was shown in [23].

(2) The two groups $QP^n \rtimes_\psi S_n$ with $t = 1$ and $(\mathbb{F}_q^\times)^{mn} \rtimes_\theta S_{mn}$ are not isomorphic unless $m = 1$ or $n = 1$.

Let H, K, L be the subgroups of $G = \mathrm{Aut}(\mathbb{F}_q^{m\times n}, w_{\mathbf{P}})$ given as the respective images of

$$\begin{aligned}&(\pi((\mathbb{F}_q^\times)^{m_1} \times \cdots \times (\mathbb{F}_q^\times)^{m_t}) \ltimes U)^n, \\ &QP^n, \\ &(\pi((\mathbb{F}_q^\times)^{m_1} \times \cdots \times (\mathbb{F}_q^\times)^{m_t}) \ltimes U)^n \rtimes_\psi S_n\end{aligned} \tag{4.18}$$

(cf. (4.7), (4.9)) under the isomorphism

$$QP^n \rtimes_\psi S_n \to \mathrm{Aut}(\mathbb{F}_q^{m\times n}, w_{\mathbf{P}}).$$

Also, we let $\check{H}, \check{K}, \check{L}$ be the subgroups of $\check{G} = \mathrm{Aut}(\mathbb{F}_q^{m\times n}, w_{\check{\mathbf{P}}})$ corresponding to H, K, L respectively. In more detail, for $t > 1$ one shows that

$$(\check{QP})^n \rtimes_{\check{\psi}} S_n \to \mathrm{Aut}(\mathbb{F}_q^{m\times n}, w_{\check{\mathbf{P}}}) \qquad ((\tau,\sigma) \mapsto \tau\sigma) \tag{4.19}$$

is an isomorphism, where $\check{\psi} : S_n \to \mathrm{Aut}(\check{QP}^n)$ is the map defined similarly to (4.16) (cf. (4.10)). Then $\check{H}, \check{K}, \check{L}$ are respectively the images of

$$\begin{aligned} &(\pi((\mathbb{F}_q^\times)^{m_1} \times \cdots \times (\mathbb{F}_q^\times)^{m_t}) \ltimes \check{U})^n, \\ &\check{QP}^n, \\ &(\pi((\mathbb{F}_q^\times)^{m_1} \times \cdots \times (\mathbb{F}_q^\times)^{m_t}) \ltimes \check{U})^n \rtimes_\psi S_n \end{aligned} \tag{4.20}$$

(cf. (4.7), (4.10)) under the isomorphism (4.19). Note here that the involution $\iota \in \mathrm{Aut}(\mathbb{F}_q^{m\times n})$ given by $\iota(u_{ij}) = (-u_{ij})$ belongs to $H, K, L, G, \check{H}, \check{K}, \check{L}, \check{G}$.

Let

$$I = I_{m,n} = \{\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n \mid 0 \le \beta_j \le m,\ j \in [n]\}. \tag{4.21}$$

S_n acts on I as

$$\sigma\beta = (\beta_{\sigma 1}, \dots, \beta_{\sigma n}) \tag{4.22}$$

for $\sigma \in S_n$, $\beta = (\beta_1, \dots, \beta_n) \in I$. The set $I_{m,n}/S_n$ of orbits can be identified with

$$\{\beta = (\beta_1, \dots, \beta_n) \mid 0 \le \beta_1 \le \cdots \le \beta_n \le m\}.$$

For $\beta \in I_{m,n}/S_n$, the stabilizer $S_n^{(\beta)}$ of β is

$$S_n^{(\beta)} = \{\sigma \in S_n \mid \sigma\beta = \beta\} \cong S_{n_0} \times S_{n_1} \times \cdots \times S_{n_m},$$

where $n_k = n_k(\beta) = |\{i \in [n] \mid \beta_i = k\}|$ $(0 \le k \le m)$.

For convenience, we put

$$\{0,1\}_{\mathbf{P}}^{m_l} = \begin{cases} \{0,1\}^{m_1} & \text{if } l = 1, \\ \{0,1\}^{m_l} \setminus \{\mathbf{0}^{(l)}\} & \text{if } 1 < l \le t, \end{cases} \tag{4.23}$$

where $\mathbf{0}^{(l)}$ is the all-zero vector in $\{0,1\}^{m_l}$; and

$$\{0,1\}_{\check{\mathbf{P}}}^{m_l} = \begin{cases} \{0,1\}^{m_t} & \text{if } l = t, \\ \{0,1\}^{m_l} \setminus \{\mathbf{0}^{(l)}\} & \text{if } 1 \le l < t. \end{cases} \tag{4.24}$$

Then we put

$$J = J_{m_1,\dots,m_t,n} = \{\lambda = (\lambda^{(l_1)}, \dots, \lambda^{(l_n)}) \mid \lambda^{(l_j)} \in \{0,1\}_{\mathbf{P}}^{m_{l_j}},\ l_j \in [t],\ j \in [n]\}, \tag{4.25}$$

$$\check{J} = \check{J}_{m_1,\dots,m_t,n} = \{\mu = (\mu^{(l_1)}, \dots, \mu^{(l_n)}) \mid \mu^{(l_j)} \in \{0,1\}_{\check{\mathbf{P}}}^{m_{l_j}},\ l_j \in [t],\ j \in [n]\}. \tag{4.26}$$

S_n acts also on J and $\check{J}$ as

$$\sigma\lambda = (\lambda^{(l_{\sigma 1})}, \dots, \lambda^{(l_{\sigma n})}), \qquad \sigma\mu = (\mu^{(l_{\sigma 1})}, \dots, \mu^{(l_{\sigma n})}) \tag{4.27}$$

for $\sigma \in S_n$, $\lambda = (\lambda^{(l_1)}, \dots, \lambda^{(l_n)}) \in J$, $\mu = (\mu^{(l_1)}, \dots, \mu^{(l_n)}) \in \check{J}$. The respective sets of orbits are denoted by J/S_n and $\check{J}/S_n$. The stabilizers of $\lambda \in J$ and of $\mu \in \check{J}$ are respectively denoted by $S_n^{(\lambda)}$ and $S_n^{(\mu)}$.

For $l \in [t]$, $\lambda^{(l)}, \mu^{(l)} \in \{0,1\}^{m_l}$, the fragments $F(\lambda^{(l)})$ and $\check{F}(\mu^{(l)})$ are defined by

$$F(\lambda^{(l)}) = \{{}^t[x_1 \ \cdots \ x_t] \in \mathbb{F}_q^{m\times 1} \mid x_{l+1} = \cdots = x_t = 0,\ \mathrm{Supp}(x_l) = \mathrm{Supp}(\lambda^{(l)})\}, \quad (4.28)$$

$$\check{F}(\mu^{(l)}) = \{{}^t[x_1 \ \cdots \ x_t] \in \mathbb{F}_q^{m\times 1} \mid x_1 = \cdots = x_{l-1} = 0,\ \mathrm{Supp}(x_l) = \mathrm{Supp}(\mu^{(l)})\} \quad (4.29)$$

(cf. (2.5)). Viewing $\mathbb{F}_q^{m\times n}$ as $\mathbb{F}_q^{m\times n} = \prod_{j=1}^{n} \mathbb{F}_q^{m\times 1}$, for $\lambda = (\lambda^{(l_1)}, \ldots, \lambda^{(l_n)}) \in J$ and $\mu = (\mu^{(l_1)}, \ldots, \mu^{(l_n)}) \in \check{J}$ we introduce the sets

$$\mathcal{O}_H(\lambda) = \prod_{j=1}^{n} F(\lambda^{(l_j)}), \qquad \mathcal{O}_{\check{H}}(\mu) = \prod_{j=1}^{n} \check{F}(\mu^{(l_j)}) \quad (4.30)$$

(cf. (4.28–29)). Also, for $\lambda \in J/S_n$ and $\mu \in \check{J}/S_n$, we define

$$\mathcal{O}_L(\lambda) = \bigcup_{\sigma \in S_n/S_n^{(\lambda)}} \mathcal{O}_H(\sigma\lambda), \qquad \mathcal{O}_{\check{L}}(\mu) = \bigcup_{\mu \in S_n/S_n^{(\mu)}} \mathcal{O}_{\check{H}}(\sigma\mu) \quad (4.31)$$

(cf. (4.27)).

For $j \in [n]$ and $0 \le r \le m$, we put

$$S_{\mathbf{P}_j}(r) = \{u \in \mathbb{F}_q^{m\times 1} \mid w_{\mathbf{P}_j}(u) = r\}, \quad (4.32)$$

$$S_{\check{\mathbf{P}}_j}(r) = \{u \in \mathbb{F}_q^{m\times 1} \mid w_{\check{\mathbf{P}}_j}(u) = r\}. \quad (4.33)$$

Then, for $\beta = (\beta_1, \ldots, \beta_n) \in I$ (cf. (4.21)), we let

$$\mathcal{O}_K(\beta) = \prod_{j=1}^{n} S_{\mathbf{P}_j}(\beta_j) = \{u = [u^1 \ \cdots \ u^n] \in \mathbb{F}_q^{m\times n} \mid w_j(u) = w_{\mathbf{P}_j}(u^j) = \beta_j,\ j \in [n]\}, \quad (4.34)$$

$$\mathcal{O}_{\check{K}}(\beta) = \prod_{j=1}^{n} S_{\check{\mathbf{P}}_j}(\beta_j) = \{u = [u^1 \ \cdots \ u^n] \in \mathbb{F}_q^{m\times n} \mid \check{w}_j(u) := w_{\check{\mathbf{P}}_j}(u^j) = \beta_j,\ j \in [n]\} \quad (4.35)$$

(cf. (4.32–33)). So they are products of spheres. Also, for $\beta \in I_{m,n}/S_n$, we put

$$\mathcal{O}_{\mathbf{P}}(\beta) = \bigcup_{\sigma \in S_n/S_n^{(\beta)}} \mathcal{O}_K(\sigma\beta), \qquad \mathcal{O}_{\check{\mathbf{P}}}(\beta) = \bigcup_{\sigma \in S_n/S_n^{(\beta)}} \mathcal{O}_{\check{K}}(\sigma\beta) \quad (4.36)$$

(cf. (4.22)).

Proposition 4.9. *Let* $\mathbf{P}$ *be the generalized Niederreiter–Rosenbloom–Tsfasman poset* (cf. (2.4)) *with* $t > 1$ (*i.e.,* $\mathbf{P}$ *is not an antichain*). *Let* H, K, L *be the subgroups of* $G = \mathrm{Aut}(\mathbb{F}_q^{m\times n}, w_{\mathbf{P}})$ *in* (4.18), *and let* $\check{H}, \check{K}, \check{L}$ *be those of* $\check{G} = \mathrm{Aut}(\mathbb{F}_q^{m\times n}, w_{\check{\mathbf{P}}})$ *in* (4.20). *Then the following hold:*

(1) *The orbits under the action of* H *on* $\mathbb{F}_q^{m\times n}$ *are the* $\mathcal{O}_H(\lambda)$*'s, where* $\lambda \in J$ (cf. (4.30), (4.28), (4.25)). *Those of* $\check{H}$ *on* $\mathbb{F}_q^{m\times n}$ *are the* $\mathcal{O}_{\check{H}}(\mu)$*'s, where* $\mu \in \check{J}$ (cf. (4.30), (4.29), (4.26)).
(2) *The orbits under the action of* L *on* $\mathbb{F}_q^{m\times n}$ *are the* $\mathcal{O}_L(\lambda)$*'s, where* $\lambda \in J/S_n$ (cf. (4.31), (4.27)). *Those of* $\check{L}$ *on* $\mathbb{F}_q^{m\times n}$ *are the* $\mathcal{O}_{\check{L}}(\mu)$*'s, where* $\mu \in \check{J}/S_n$ (cf. (4.31), (4.27)).
(3) *The orbits under the action of* K *on* $\mathbb{F}_q^{m\times n}$ *are the* $\mathcal{O}_K(\beta)$*'s, where* $\beta \in I$ (cf. (4.34), (4.32), (4.21)). *Those of* $\check{K}$ *on* $\mathbb{F}_q^{m\times n}$ *are the* $\mathcal{O}_{\check{K}}(\beta)$*'s, where* $\beta \in I$ (cf. (4.35), (4.33), (4.21)).

(4) *The orbits under the action of G on $\mathbb{F}_q^{m\times n}$ are the $\mathcal{O}_{\mathbf{P}}(\beta)$'s, where $\beta \in I/S_n$* (cf. (4.36), (4.22)). *Those of $\check{G}$ on $\mathbb{F}_q^{m\times n}$ are the $\mathcal{O}_{\check{\mathbf{P}}}(\beta)$'s, where $\beta \in I/S_n$* (cf. (4.36), (4.22)).

5. Equivalent condition I (transitivity of actions)

Here the title of the chapter refers to the equivalence (1)⇔(3) in Theorem 8.7.

Let $\mathbf{P}_j$ be any poset on the underlying set $[m] \times \{j\}$, for $j \in [n]$, and let $\mathbf{P} = \mathbf{P}_1 \dot{+} \cdots \dot{+} \mathbf{P}_n$. For each $j \in [n]$ and $u = [u^1 \ \cdots \ u^n] \in \mathbb{F}_q^{m\times n}$, just as we did when $\mathbf{P}$ is a generalized Niederreiter–Rosenbloom–Tsfasman poset, we define

$$w_j(u) = w_{\mathbf{P}_j}(u^j), \qquad \check{w}_j(u) = w_{\check{\mathbf{P}}_j}(u^j), \tag{5.1}$$

so that $w_{\mathbf{P}} = w_1 + \cdots + w_n$, $w_{\check{\mathbf{P}}} = \check{w}_1 + \cdots + \check{w}_n$.

Let $I = I_{m,n}$ be as in (4.21), and let S_n act on I as in (4.22). Then, for $\beta = (\beta_1, \ldots, \beta_n) \in I$, we define

$$S_{\mathbf{P}}(\beta) = \{u = [u^1 \ \cdots \ u^n] \in \mathbb{F}_q^{m\times n} \mid w_j(u) = \beta_j,\ j \in [n]\}, \tag{5.2}$$

$$S_{\check{\mathbf{P}}}(\beta) = \{u = [u^1 \ \cdots \ u^n] \in \mathbb{F}_q^{m\times n} \mid \check{w}_j(u) = \beta_j,\ j \in [n]\}. \tag{5.3}$$

For each $\beta \in I$, $S_{\mathbf{P}}(\beta)$ and $S_{\check{\mathbf{P}}}(\beta)$ are not empty (see Remark 5.1(3)), and $\{S_{\mathbf{P}}(\beta)\}_{\beta\in I}$, $\{S_{\check{\mathbf{P}}}(\beta)\}_{\beta\in I}$ are partitions of $\mathbb{F}_q^{m\times n}$. Also, for each $\beta \in I/S_n$, we put

$$\mathcal{O}_{\mathbf{P}}(\beta) = \bigcup_{\sigma\in S_n/S_n^{(\beta)}} S_{\mathbf{P}}(\sigma\beta), \qquad \mathcal{O}_{\check{\mathbf{P}}}(\beta) = \bigcup_{\sigma\in S_n/S_n^{(\beta)}} S_{\check{\mathbf{P}}}(\sigma\beta). \tag{5.4}$$

Then obviously $\{\mathcal{O}_{\mathbf{P}}(\beta)\}_{\beta\in I/S_n}$, $\{\mathcal{O}_{\check{\mathbf{P}}}(\beta)\}_{\beta\in I/S_n}$ also form partitions of $\mathbb{F}_q^{m\times n}$. Note here that, when $\mathbf{P}$ is a generalized Niederreiter–Rosenbloom–Tsfasman poset, $S_{\mathbf{P}}(\beta)$'s, $S_{\check{\mathbf{P}}}(\beta)$'s ($\beta \in I$), $\mathcal{O}_{\mathbf{P}}(\beta)$'s, $\mathcal{O}_{\check{\mathbf{P}}}(\beta)$'s ($\beta \in I/S_n$) are respectively the orbits under the action of $K, \check{K}, G = \mathrm{Aut}(\mathbb{F}_q^{m\times n}, w_{\mathbf{P}}), \check{G} = \mathrm{Aut}(\mathbb{F}_q^{m\times n}, w_{\check{\mathbf{P}}})$ (cf. (4.34–36)). However, (5.2), (5.3), and (5.4) are now defined for any poset $\mathbf{P}$ described at the beginning of this chapter. Even so, we will call $\mathcal{O}_{\mathbf{P}}(\beta)$'s the $\mathbf{P}$-orbits and $\mathcal{O}_{\check{\mathbf{P}}}(\beta)$'s the $\check{\mathbf{P}}$-orbits.

Remark 5.1. (1) If $\mathrm{Supp}(u) \subseteq \mathbf{P}_j$ for some j, then $w_{\mathbf{P}}(u) = w_j(u)$ and $w_k(u) = 0$ for all $k \neq j$. Note that $w_k(u) = 0 \Leftrightarrow \mathrm{Supp}(u) \cap \mathbf{P}_k = \emptyset$. So, if $w_k(u) = 0$ for all $k \neq j$, then $\mathrm{Supp}(u) \subseteq \mathbf{P}_j$, and $w_{\mathbf{P}}(u) = w_j(u)$.

(2) Let us denote, by abuse of notation, the underlying set of the poset Q also by Q. Define $\mathbf{P}_{(i)}$, $i = 0, 1, \ldots,$ inductively as follows:

$$\mathbf{P}_{(0)} = \mathbf{P}, \qquad \mathbf{P}_{(i)} = \mathbf{P}_{(i-1)} \backslash \min \mathbf{P}_{(i-1)} \qquad \text{for } i \geq 1.$$

Here $\min \mathbf{P}_{(j)}$ denotes the set of minimal elements in $\mathbf{P}_{(j)}$. Then

$$\mathbf{P}_{(0)} \supseteq \mathbf{P}_{(1)} \supseteq \mathbf{P}_{(2)} \supseteq \cdots, \qquad \mathbf{P}_{(i)} = \min \mathbf{P}_{(i)} \,\dot{\cup}\, \mathbf{P}_{(i+1)} \qquad (i \geq 0),$$

$$\mathbf{P}_{(i)} = \dot{\bigcup_{j=1}^{n}} \mathbf{P}_{j,(i)} \qquad (i \geq 0), \qquad \min \mathbf{P}_{(i)} = \dot{\bigcup_{j=1}^{n}} \min \mathbf{P}_{j,(i)} \qquad (i \geq 0).$$

(3) Let Q be a poset on $[N]$. Then

$$S_Q(r) = \{u \in \mathbb{F}_q^N \mid w_Q(u) = r\} \neq \emptyset$$

for all r $(0 \le r \le N)$. Indeed, $Q = \dot{\bigcup}_{i=1}^{d} \min Q_{(i)}$, with all $\min Q_{(i)} \neq \emptyset$, for some $d \ge 0$. If we choose $u \in \mathbb{F}_q^N$ with $\mathrm{Supp}(u) = \bigcup_{i=0}^{e-1} \min Q_{(i)} \cup S = I$ for a subset $S \subseteq \min Q_{(e)}$, then I is an ideal of $[N]$, and hence

$$w_Q(u) = \sum_{i=0}^{e-1} |\min Q_{(i)}| + |S| = |\mathrm{Supp}(u)|.$$

THEOREM 5.2. *Let* $\mathbf{P} = \mathbf{P}_1 \dot{+} \cdots \dot{+} \mathbf{P}_n$ *be the poset described at the beginning of this chapter with* $\mathbf{P} \neq \min \mathbf{P}$. *Then* $\mathbf{P}$ *is a generalized Niederreiter–Rosenbloom–Tsfasman poset* (cf. (2.4)) *if and only if* $G = \mathrm{Aut}(\mathbb{F}_q^{m\times n}, w_{\mathbf{P}})$ *acts transitively on each* $\mathcal{O}_{\mathbf{P}}(\beta)$ $(\beta \in I_{m,n}/S_n)$ (cf. (5.2), (5.4)).

Proof. When $\mathbf{P}$ is a generalized Niederreiter–Rosenbloom–Tsfasman poset, $\mathcal{O}_{\mathbf{P}}(\beta)$'s $(\beta \in I_{m,n}/S_n)$ are the orbits under the action of G on $\mathbb{F}_q^{m\times n}$, and so one direction is clear. The other direction requires a lengthy proof.

STEP 1. Here we show that $[\mathrm{Supp}(u) \subseteq \min \mathbf{P}_j,\ \phi \in G] \Rightarrow \mathrm{Supp}(\phi u) \subseteq \min \mathbf{P}_k$ for some k. Note that $w_{\mathbf{P}}(u) = 1 \Leftrightarrow [\mathrm{Supp}(u) \subseteq \min \mathbf{P}, |\mathrm{Supp}(u)| = 1]$. So $[\mathrm{Supp}(u) \subseteq \min \mathbf{P}_j,\ |\mathrm{Supp}(u)| = 1,\ \phi \in G] \Rightarrow \mathrm{Supp}(\phi u) \subseteq \min \mathbf{P}$ (cf. Remark 5.1(2)). Next, we show that $[\mathrm{Supp}(u) \subseteq \min \mathbf{P}_j,\ \phi \in G] \Rightarrow \mathrm{Supp}(\phi u) \subseteq \min \mathbf{P}$. For this, we may assume that $u \neq 0$. Let $\mathrm{Supp}(u) = \{(i_1, j), \ldots, (i_s, j)\} \subseteq \min \mathbf{P}_j$. Then we may write u uniquely as $u = v_1 + \cdots + v_s$, $v_k \in \mathbb{F}_q^{m\times n}$, $\mathrm{Supp}(v_k) = \{(i_k, j)\}$. Now, $\phi u = \phi v_1 + \cdots + \phi v_s$, $\mathrm{Supp}(\phi v_k) \subseteq \min \mathbf{P}$. Thus $\mathrm{Supp}(\phi u) \subseteq \min \mathbf{P}$. Assume now $\mathrm{Supp}(u) \subseteq \min \mathbf{P}_j$ with $|\mathrm{Supp}(u)| = b$ and $\phi \in G$. Then $u \in S_{\mathbf{P}}(0, \ldots, 0, b, 0, \ldots, 0)$ (with b at the jth place), and so $u \in \mathcal{O}_{\mathbf{P}}(\beta) = S_{\mathbf{P}}((b, 0, \ldots, 0)) \cup S_{\mathbf{P}}((0, b, 0, \ldots, 0)) \cup \cdots \cup S_{\mathbf{P}}((0, \ldots, 0, b))$ with $\beta = (0, \ldots, 0, b)$. As G acts transitively on $\mathcal{O}_{\mathbf{P}}(\beta)$ we have $\phi u \in \mathcal{O}_{\mathbf{P}}(\beta)$, and so $\phi u \in S_{\mathbf{P}}((0, \ldots, 0, b, 0, \ldots, 0))$, with b at the kth place for some k. By Remark 5.1(1), $\mathrm{Supp}(\phi u) \subseteq \mathbf{P}_k$. Then $\mathrm{Supp}(\phi u) \subseteq \min \mathbf{P} \cap \mathbf{P}_k = \min \mathbf{P}_k$.

STEP 2. Here we claim that $|\min \mathbf{P}_1| = \cdots = |\min \mathbf{P}_n| = m_1$. For this, we may assume that

$$|\min \mathbf{P}_1| \le \cdots \le |\min \mathbf{P}_n|.$$

Choose $u \in \mathbb{F}_q^{m\times n}$ with $\mathrm{Supp}(u) = \min \mathbf{P}_n$. Let $|\mathrm{Supp}(u)| = b$ (≥ 1). Then $u \in S_{\mathbf{P}}((0, \ldots, 0, b))$. Choose any $v \in S_{\mathbf{P}}((b, 0, \ldots, 0))$. This is possible by Remark 5.1(3). As G acts transitively on $\mathcal{O}_{\mathbf{P}}(\beta)$ $(\beta = (0, \ldots, b))$, and $u, v \in \mathcal{O}_{\mathbf{P}}(\beta)$, there is $\phi \in G$ such that $\phi u = v$. By Step 1, we know that $\mathrm{Supp}(\phi u) = \mathrm{Supp}(v) \subseteq \min \mathbf{P}_k$ for some k. As $\mathrm{Supp}(v) \subseteq \mathbf{P}_1$ by Remark 5.1(1), we must have $\mathrm{Supp}(\phi u) \subseteq \min \mathbf{P}_1$. So $|\min \mathbf{P}_n| = w_{\mathbf{P}}(u) = w_{\mathbf{P}}(\phi u) = |\mathrm{Supp}(\phi u)| \le |\min \mathbf{P}_1|$. This shows our claim.

STEP 3. Here we show that, for each $j \in [n]$, $[(i_1, j) \in \min \mathbf{P}_{j,(1)},\ (i_2, j) \in \min \mathbf{P}_j] \Rightarrow (i_1, j) \ge_{\mathbf{P}_j} (i_2, j)$. It is enough to show that $|\langle (i_1, j) \rangle_{\mathbf{P}_j}| = m_1 + 1$. Assume, on the contrary, that $|\langle (i_1, j) \rangle_{\mathbf{P}_j}| \le m_1$. Then choose $u_1, u_2 \in \mathbb{F}_q^{m\times n}$ such that $\mathrm{Supp}(u_1) = \{(i_1, j)\}$, $\mathrm{Supp}(u_2) \subseteq \min \mathbf{P}_j$, $|\langle \mathrm{Supp}(u_1) \rangle_{\mathbf{P}_j}| = |\langle \mathrm{Supp}(u_2) \rangle_{\mathbf{P}_j}| = b$. Then $u_1, u_2 \in \mathcal{O}_{\mathbf{P}}(\beta)$ with $\beta = (0, \ldots, 0, b)$. So there exists $\phi \in G$ such that $u_1 = \phi u_2$. Then $\{(i_1, j)\} = \mathrm{Supp}(u_1) = \mathrm{Supp}(\phi u_2) \subseteq \min \mathbf{P}_k$ for some k, by Step 1. This is a contradiction, since this implies that $(i_1, j) \in \min \mathbf{P}_{(1)} \cap \min \mathbf{P} \subseteq \mathbf{P}_{(1)} \cap \min \mathbf{P} = \emptyset$.

These steps can be continued. In Step 4, one shows that $[\mathrm{Supp}(u) \subseteq \min \mathbf{P}_{j,(1)},\ \phi \in G] \Rightarrow \mathrm{Supp}(\phi u) \subseteq \min \mathbf{P}_{k,(1)} \cup \min \mathbf{P}_k$ for some k. In Step 5, by using Step 4 one shows that $|\min \mathbf{P}_{1,(1)}| = \cdots = |\min \mathbf{P}_{n,(1)}|$. Again, by exploiting Step 4, in Step 6 one shows that, for each $j \in [n]$, $[(i_1, j) \in \min \mathbf{P}_{j,(2)}$ and $(i_2, j) \in \min \mathbf{P}_{j,(1)}] \Rightarrow (i_1, j) \geq_{\mathbf{P}_j} (i_2, j)$. The details are left to the reader. By continuing in this fashion, we can show that $\mathbf{P}$ is a generalized Niederreiter–Rosenbloom–Tsfasman poset. ∎

6. Equivalent condition II (association scheme)

Here the title of the chapter refers to the equivalence (1)⇔(4) in Theorem 8.7.

THEOREM 6.1. *Let $\mathbf{P} = \mathbf{P}_1 \dotplus \cdots \dotplus \mathbf{P}_n$ be the poset described at the beginning of Chapter 5 with $\mathbf{P} \neq \min \mathbf{P}$. Then $\mathbf{P}$ is a generalized Niederreiter–Rosenbloom–Tsfasman poset* (cf. (2.4)) *if and only if $\mathcal{X}_{\mathbf{P}} = (\mathbb{F}_q^{m\times n}, \{R_{\mathbf{P},\beta}\}_{\beta \in I_{m,n}/S_n})$ is a symmetric association scheme. Here*

$$(u, v) \in R_{\mathbf{P},\beta} \ \Leftrightarrow\ u - v \in \mathcal{O}_{\mathbf{P}}(\beta)$$

(cf. (5.2), (5.4)).

Proof. Assume that $\mathbf{P}$ is a generalized Niederreiter–Rosenbloom–Tsfasman poset. Then the result follows from Proposition 4.9 and Theorem 3.1. The other direction needs a lengthy proof.

STEP 1. Here we show that $|\min \mathbf{P}_1| = \cdots = |\min \mathbf{P}_n| = m_1$. Let $|\min \mathbf{P}_1| = l_1, \ldots, |\min \mathbf{P}_n| = l_n$. Let $1 \leq i < j \leq n$. Choose $u_1, u_2 \in \mathbb{F}_q^{m\times n}$ such that $\mathrm{Supp}(u_1) \subseteq \mathbf{P}_i$, $\mathrm{Supp}(u_2) \subseteq \mathbf{P}_j$, $|\langle \mathrm{Supp}(u_1)\rangle_{\mathbf{P}_i}| = |\langle \mathrm{Supp}(u_2)\rangle_{\mathbf{P}_j}| = m$ (cf. Remark 5.1(3)). Then $u_1 = u_1 - 0$, $u_2 = u_2 - 0 \in \mathcal{O}_{\mathbf{P}}(\beta)$ with $\beta = (0, \ldots, 0, m)$. So we must have

$$|\{v \in \mathbb{F}_q^{m\times n} \mid u_1 - v \in \mathcal{O}_{\mathbf{P}}((0, \ldots, 0, 1)),\ v \in \mathcal{O}_{\mathbf{P}}((0, \ldots, 0, 1, m))\}| \tag{6.1}$$
$$= |\{v \in \mathbb{F}_q^{m\times n} \mid u_2 - v \in \mathcal{O}_{\mathbf{P}}((0, \ldots, 0, 1)),\ v \in \mathcal{O}_{\mathbf{P}}((0, \ldots, 0, 1, m))\}|. \tag{6.2}$$

Observe that $u - v \in \mathcal{O}_{\mathbf{P}}((0, \ldots, 0, 1)) \Leftrightarrow v = u - w$ with $\mathrm{Supp}(w) \subseteq \min \mathbf{P}$, $|\mathrm{Supp}(w)| = 1$. Let $v = u_1 - w_1$ with $\mathrm{Supp}(w_1) \subseteq \min \mathbf{P}$, $|\mathrm{Supp}(w_1)| = 1$, so that $u_1 - v \in \mathcal{O}_{\mathbf{P}}((0, \ldots, 0, 1))$. Now, $v = u_1 - w_1 \in \mathcal{O}_{\mathbf{P}}((0, \ldots, 0, 1, m)) = \bigcup_{\sigma \in S_n/S_n^{(\beta)}} S_{\mathbf{P}}(\sigma\beta)$ with $\beta = (0, \ldots, 0, 1, m) \Leftrightarrow \mathrm{Supp}(w_1) \subseteq \min \mathbf{P} \backslash \min \mathbf{P}_i$ and $|\mathrm{Supp}(w_1)| = 1$. So the cardinality in (6.1) equals $(q - 1)(l_1 + \cdots + l_{i-1} + l_{i+1} + \cdots + l_n)$. Similarly, the cardinality in (6.2) equals $(q - 1)(l_1 + \cdots + l_{j-1} + l_{j+1} + \cdots + l_n)$. This implies $l_i = l_j$. So we have shown what we wanted.

STEP 2. Here we claim that, for each j, $[(i_1, j) \in \min \mathbf{P}_{j,(1)},\ (i_2, j) \in \min \mathbf{P}_j] \Rightarrow (i_1, j) \geq_{\mathbf{P}_j} (i_2, j)$. It is enough to show $|\langle (i_1, j)\rangle_{\mathbf{P}_j}| = m_1 + 1$. Suppose, on the contrary, that $b = |\langle (i_1, j)\rangle_{\mathbf{P}_j}| \leq m_1$. Then $2 \leq b \leq m_1$. Choose $u_1, u_2 \in \mathbb{F}_q^{m\times n}$ such that $\mathrm{Supp}(u_1) = \{(i_1, j)\}$, $\mathrm{Supp}(u_2) \subseteq \min \mathbf{P}_j$, $|\langle \mathrm{Supp}(u_1)\rangle_{\mathbf{P}_j}| = |\langle \mathrm{Supp}(u_2)\rangle_{\mathbf{P}_j}|$. Then $u_1 = u_1 - 0$, $u_2 = u_2 - 0 \in \mathcal{O}_{\mathbf{P}}(\beta)$ with $\beta = (0, \ldots, 0, b)$. So we must have

$$|\{v \in \mathbb{F}_q^{m\times n} \mid u_1 - v \in \mathcal{O}_{\mathbf{P}}((0, \ldots, 0, 1)),\ v \in \mathcal{O}_{\mathbf{P}}((0, \ldots, 0, b + 1))\}| \tag{6.3}$$
$$= |\{v \in \mathbb{F}_q^{m\times n} \mid u_2 - v \in \mathcal{O}_{\mathbf{P}}((0, \ldots, 0, 1)),\ v \in \mathcal{O}_{\mathbf{P}}((0, \ldots, 0, b + 1))\}|. \tag{6.4}$$

Let $v = u_1 - w_1$ with $\mathrm{Supp}(w_1) \subseteq \min \mathbf{P}$, $|\mathrm{Supp}(w_1)| = 1$. So $u_1 - v \in \mathcal{O}_{\mathbf{P}}((0,\dots,0,1))$. Let $\langle (i_1,j)\rangle_{\mathbf{P}_j} = \{(i_1,j),(i_2,j),\dots,(i_b,j)\}$ with $(i_2,j),\dots,(i_b,j) \in \min \mathbf{P}_j$. Then $v = u_1 - w_1 \in \mathcal{O}_{\mathbf{P}}((0,\dots,0,b+1)) \Leftrightarrow \mathrm{Supp}(w_1) \subseteq \min \mathbf{P}_j$ and $\mathrm{Supp}(w_1) \subseteq ([m_1] \setminus \{i_2,\dots,i_b\}) \times \{j\}$. Here we assumed that $\min \mathbf{P}_j = [m_1] \times \{j\}$. Thus the cardinality in (6.3) equals $(m_1 - b + 1)(q-1)$. Let $v = u_2 - w_2$ with $\mathrm{Supp}(w_2) \subseteq \min \mathbf{P}$, $|\mathrm{Supp}(w_2)| = 1$, so that $u_2 - v \in \mathcal{O}_{\mathbf{P}}((0,\dots,0,1))$. Then $v = u_2 - w_2 \in \mathcal{O}_{\mathbf{P}}((0,\dots,0,b+1)) \Leftrightarrow \mathrm{Supp}(w_2) \subseteq \min \mathbf{P}_j$ and $\mathrm{Supp}(w_2) \subseteq [m_1] \times \{j\} \setminus \mathrm{Supp}(u_2)$. So the cardinality in (6.4) equals $(m_1 - b)(q-1)$. This is a contradiction. These steps can be continued. We will briefly sketch the next two steps.

Step 3. Let $|\min \mathbf{P}_{1,(1)}| = l_1, \dots, |\min \mathbf{P}_{n,(1)}| = l_n$. Let $1 \le i < j \le n$, and choose $u_1, u_2 \in \mathbb{F}_q^{m\times n}$ with $\mathrm{Supp}(u_1) \subseteq \mathbf{P}_i$, $\mathrm{Supp}(u_2) \subseteq \mathbf{P}_j$, $|\langle \mathrm{Supp}(u_1)\rangle_{\mathbf{P}_i}| = |\langle \mathrm{Supp}(u_2)\rangle_{\mathbf{P}_j}| = m$. Then $u_1 = u_1 - 0, u_2 = u_2 - 0 \in \mathcal{O}_{\mathbf{P}}(\beta)$ with $\beta = (0,\dots,0,m)$. Then we must have

$$\begin{aligned}&|\{v \in \mathbb{F}_q^{m\times n} \mid u_1 - v \in \mathcal{O}_{\mathbf{P}}((0,\dots,0,m_1+1)),\ v \in \mathcal{O}_{\mathbf{P}}((0,\dots,0,m_1+1,m))\}| \qquad (6.5)\\ &\quad = |\{v \in \mathbb{F}_q^{m\times n} \mid u_2 - v \in \mathcal{O}_{\mathbf{P}}((0,\dots,0,m_1+1)),\ v \in \mathcal{O}_{\mathbf{P}}((0,\dots,0,m_1+1,m))\}|. \qquad (6.6)\end{aligned}$$

Proceeding as in Step 1, one shows the cardinality in (6.5) is $(q-1)(l_1 + \dots + l_{i-1} + l_{i+1} + \dots + l_n)q^{m_1}$, and that in (6.6) is $(q-1)(l_1 + \dots + l_{j-1} + l_{j+1} + \dots + l_n)q^{m_1}$. Thus $l_i = l_j$, and hence $|\min \mathbf{P}_{1,(1)}| = \dots = |\min \mathbf{P}_{n,(1)}| = m_2$.

Step 4. Here we show that, for each j, $[(i_1,j) \in \min \mathbf{P}_{j,(2)},\ (i_2,j) \in \min \mathbf{P}_{j,(1)}] \Rightarrow (i_1,j) \ge_{\mathbf{P}_j} (i_2,j)$. It is enough to show that $|\langle i_1,j\rangle_{\mathbf{P}_j}| = m_1 + m_2 + 1$. Assume, on the contrary, that $|\langle i_1,j\rangle_{\mathbf{P}_j}| \le m_1 + m_2$. Then $|\langle i_1,j\rangle_{\mathbf{P}_j}| = m_1 + b$ with $2 \le b \le m_2$. Choose $u_1, u_2 \in \mathbb{F}_q^{m\times n}$ such that $\mathrm{Supp}(u_1) = \{(i_1,j)\}$, $\mathrm{Supp}(u_2) \subseteq \min \mathbf{P}_{j,(1)}$, $|\mathrm{Supp}(u_2)| = b$. Then $|\langle \mathrm{Supp}(u_1)\rangle_{\mathbf{P}_j}| = m_1 + b = |\langle \mathrm{Supp}(u_2)\rangle_{\mathbf{P}_j}|$, and hence $u_1 = u_1 - 0$, $u_2 = u_2 - 0 \in \mathcal{O}_{\mathbf{P}}(\beta)$ with $\beta = (0,\dots,0,m_1+b)$. So we must have

$$\begin{aligned}&|\{v \in \mathbb{F}_q^{m\times n} \mid u_1 - v \in \mathcal{O}_{\mathbf{P}}((0,\dots,0,m_1+1)),\ v \in \mathcal{O}_{\mathbf{P}}((0,\dots,0,m_1+b+1))\}| \qquad (6.7)\\ &\quad = |\{v \in \mathbb{F}_q^{m\times n} \mid u_2 - v \in \mathcal{O}_{\mathbf{P}}((0,\dots,0,m_1+1)),\ v \in \mathcal{O}_{\mathbf{P}}((0,\dots,0,m_1+b+1))\}|. \qquad (6.8)\end{aligned}$$

Then one shows that the cardinality in (6.7) is $(m_2 - b + 1)(q-1)q^{m_1}$, and that in (6.8) is $(m_2 - b)(q-1)q^{m_1}$. This is a contradiction. ∎

7. A half of equivalent condition III (wdop)

Here the title of the chapter refers to the implication (2)⇒(1) in Theorem 8.7.

Let $\mathbf{P} = \mathbf{P}_1 \dotplus \dots \dotplus \mathbf{P}_n$ be the poset described at the beginning of Chapter 5. Then we put, for any linear code $C \subseteq \mathbb{F}_q^{m\times n}$, and all $\beta \in I_{m,n}/S_n$,

$$a_{\mathbf{P},\beta}(C) = |\mathcal{O}_{\mathbf{P}}(\beta) \cap C|, \quad a_{\check{\mathbf{P}},\beta}(C^\perp) = |\mathcal{O}_{\check{\mathbf{P}}}(\beta) \cap C^\perp|$$

(cf. (5.2), (5.4)). Here $C^\perp$ is with respect to the usual inner product $u \cdot v = \sum u_{ij} v_{ij}$ for $u = (u_{ij}), v = (v_{ij}) \in \mathbb{F}_q^{m\times n}$. The pair $(\mathbf{P}, \check{\mathbf{P}})$ is called a *weak dual orbit pair* (*wdop*) if the $\mathbf{P}$-orbit distribution $\{a_{\mathbf{P},\beta}(C)\}_{\beta \in I_{m,n}/S_n}$ of C uniquely determines the $\check{\mathbf{P}}$-orbit distribution $\{a_{\check{\mathbf{P}},\beta}(C^\perp)\}_{\beta \in I_{m,n}/S_n}$ of $C^\perp$, for every linear code $C \subseteq \mathbb{F}_q^{m\times n}$. For the poset $\mathbf{P}$ as above with $\mathbf{P} \ne \min \mathbf{P}$, in this chapter and the next we will show that $\mathbf{P}$ is a generalized Niederreiter–Rosenbloom–Tsfasman poset if and only if $(\mathbf{P}, \check{\mathbf{P}})$ is a wdop.

REMARK 7.1. For $u \in \mathbb{F}_q^{m\times n}$,

$$u \in \mathcal{O}_{\check{\mathbf{P}}}((m,\dots,m)) \;\Leftrightarrow\; \mathrm{Supp}(u) \supseteq \min \mathbf{P}. \tag{7.1}$$

Indeed,

$$\begin{aligned} u \in \mathcal{O}_{\check{\mathbf{P}}}((m,\dots,m)) = S_{\check{\mathbf{P}}}((m,\dots,m)) \;&\Leftrightarrow w_{\check{\mathbf{P}}_1}(u^1) = m,\dots, w_{\check{\mathbf{P}}_n}(u^n) = m \\ &\Leftrightarrow \mathrm{Supp}(u^1) \supseteq \max \check{\mathbf{P}}_1,\dots,\mathrm{Supp}(u^n) \supseteq \max \check{\mathbf{P}}_n \\ &\Leftrightarrow \mathrm{Supp}(u^1) \supseteq \min \mathbf{P}_1,\dots,\mathrm{Supp}(u^n) \supseteq \min \mathbf{P}_n \\ &\Leftrightarrow \mathrm{Supp}(u) \supseteq \min \mathbf{P}. \end{aligned}$$

Here $\max \check{\mathbf{P}}_j$ denotes the set of maximal elements in $\check{\mathbf{P}}_j$ (cf. Remark 5.1(2)). The idea of the proof of Theorem 7.2 originates from [21].

THEOREM 7.2. *Let* $\mathbf{P} = \mathbf{P}_1 \dot{+} \cdots \dot{+} \mathbf{P}_n$ *be the poset described at the beginning of Chapter* 5 *with* $\mathbf{P} \neq \min \mathbf{P}$. *If* $(\mathbf{P}, \check{\mathbf{P}})$ *is a wdop, then* $\mathbf{P}$ *is a generalized Niederreiter–Rosenbloom–Tsfasman poset* (cf. (2.4)).

Proof. We argue in several steps.

STEP 1. Here we show that, for each $j \in [n]$, $[(i_1,j) \in \min \mathbf{P}_{j,(1)},\, (i_2,j) \in \min \mathbf{P}_j] \Rightarrow (i_1,j) \geq_{\mathbf{P}_j} (i_2,j)$. Assume that $|\min \mathbf{P}_1| = l_1,\dots,|\min \mathbf{P}_n| = l_n$. We claim that, for $u \in \mathbb{F}_q^{m\times n}$ with $\mathrm{Supp}(u) \subseteq \min \mathbf{P}_j$,

$$q^{mn-|\min \mathbf{P}|} \text{ divides } |\{v \in \mathbb{F}_q^{m\times n} \mid u\cdot v = 0,\, v \in \mathcal{O}_{\check{\mathbf{P}}}((m,\dots,m))\}|. \tag{7.2}$$

If $u = 0$, then the cardinality of the set in (7.2) is $(q-1)^{|\min \mathbf{P}|} q^{mn-|\min \mathbf{P}|}$, in view of the observation in (7.1), and we are done. So we assume that $u \neq 0$. Then we may assume that

$$\min \mathbf{P} = \{(1,1),\dots,(l_1,1),(1,2),\dots,(l_2,2),\dots,(1,n),\dots,(l_n,n)\},$$

$u = [u^1 \;\cdots\; u^n]$ with $u^k = 0$ for $k \neq j$, $u^j = {}^t(a_1,\dots,a_{i_0},0,\dots,0)$, where $1 \leq i_0 \leq l_j$, $a_j, \in \mathbb{F}_q^\times$ $(1 \leq j \leq i_0)$. Now, one can see that the cardinality of the set in (7.2) is $|B|(q-1)^{|\min \mathbf{P}|-i_0} q^{mn-|\min \mathbf{P}|}$ (cf. (7.1)), where

$$B = \{(b_1,\dots,b_{i_0}) \in \mathbb{F}_q^{i_0} \mid a_1b_1 + \cdots + a_{i_0}b_{i_0} = 0,\, b_j \neq 0 \text{ for } 1 \leq j \leq i_0\}.$$

This shows our claim. It is enough to see that $|\langle i_1, j\rangle_{\mathbf{P}_j}| = l_j + 1$. Assume, on the contrary, that $b = |\langle i_1, j\rangle_{\mathbf{P}_j}| \leq l_j$. Then we choose $u_1, u_2 \in \mathbb{F}_q^{m\times n}$ such that $\mathrm{Supp}(u_1) = \{(i_1,j)\}$, $\mathrm{Supp}(u_2) \subseteq \min \mathbf{P}_j$, $|\langle \mathrm{Supp}(u_1)\rangle_{\mathbf{P}_j}| = |\langle \mathrm{Supp}(u_2)\rangle_{\mathbf{P}_j}|$. Then $\langle u_1\rangle$ and $\langle u_2\rangle$ have the same $\mathbf{P}$-orbit distribution. Indeed, $a_{\mathbf{P},(0,\dots,0)}(\langle u_1\rangle) = a_{\mathbf{P},(0,\dots,0)}(\langle u_2\rangle) = 1$ and $a_{\mathbf{P},(0,\dots,0,b)}(\langle u_1\rangle) = a_{\mathbf{P},(0,\dots,0,b)}(\langle u_2\rangle) = q-1$. So the two sets

$$\begin{aligned} B_1 = \mathcal{O}_{\check{\mathbf{P}}}((m,\dots,m)) \cap \langle u_1\rangle^\perp &= \{v \in \mathbb{F}_q^{m\times n} \mid v\cdot u_1 = 0,\, v \in \mathcal{O}_{\check{\mathbf{P}}}((m,\dots,m))\} \\ &= \{v \in \mathbb{F}_q^{m\times n} \mid v_{i_1,j} = 0,\, \mathrm{Supp}(v) \supseteq \min \mathbf{P}\}, \\ B_2 = \mathcal{O}_{\check{\mathbf{P}}}((m,\dots,m)) \cap \langle u_2\rangle^\perp &= \{v \in \mathbb{F}_q^{m\times n} \mid v\cdot u_2 = 0,\, v \in \mathcal{O}_{\check{\mathbf{P}}}((m,\dots,m))\} \end{aligned}$$

have the same cardinality. As $(i_1,j) \notin \min \mathbf{P}$, $|B_1| = (q-1)^{|\min \mathbf{P}|} q^{mn-|\min \mathbf{P}|-1}$. As $q^{mn-|\min \mathbf{P}|} \,|\, |B_2|$ by (7.2), and $(q^{mn-|\min \mathbf{P}|}, (q-1)^{|\min \mathbf{P}|}) = 1$, $q^{mn-|\min \mathbf{P}|}$ divides $q^{mn-|\min \mathbf{P}|-1}$, which is a contradiction.

STEP 2. We show that $|\min \mathbf{P}_1| = \cdots = |\min \mathbf{P}_n|$. For this, we may assume that

$$|\min \mathbf{P}_1| = l_1 \le |\min \mathbf{P}_2| = l_2 \le \cdots \le |\min \mathbf{P}_n| = l_n.$$

Then we claim that $l_1 \ge l_n$. Assume, on the contrary, that $l_1 < l_n$. Choose $u_2 \in \mathbb{F}_q^{m\times n}$ with $\mathrm{Supp}(u_2) = \min \mathbf{P}_n$, and $u_1 \in \mathbb{F}_q^{m\times n}$ such that $\mathrm{Supp}(u_1) \subseteq \mathbf{P}_{1,(1)}$, $|\mathrm{Supp}(u_1)| = l_n - l_1$, $w_{\mathbf{P}_1}(u_1) = l_n$. Note that $l_n - l_1 \le m - l_1$ and such a choice of u_1 is possible in view of Remark 5.1(3) and Step 1. Observe now that $\langle u_1\rangle$ and $\langle u_2\rangle$ have the same $\mathbf{P}$-orbit distribution. Indeed, $a_{\mathbf{P},(0,\dots,0)}(\langle u_1\rangle) = a_{\mathbf{P},(0,\dots,0)}(\langle u_2\rangle) = 1$ and $a_{\mathbf{P},(0,\dots,0,l_n)}(\langle u_1\rangle) = a_{\mathbf{P},(0,\dots,0,l_n)}(\langle u_2\rangle) = q-1$. So the two sets

$$\begin{aligned} B_1 = \mathcal{O}_{\check{\mathbf{P}}}((m,\dots,m)) \cap \langle u_1\rangle^\perp &= \{v \in \mathbb{F}_q^{m\times n} \mid v\cdot u_1 = 0,\ v \in \mathcal{O}_{\check{\mathbf{P}}}((m,\dots,m))\} \\ &= \{v \in \mathbb{F}_q^{m\times n} \mid v\cdot u_1 = 0,\ \mathrm{Supp}(v) \supseteq \min \mathbf{P}\}, \\ B_2 = \mathcal{O}_{\check{\mathbf{P}}}((m,\dots,m)) \cap \langle u_2\rangle^\perp &= \{v \in \mathbb{F}_q^{m\times n} \mid v\cdot u_2 = 0,\ v \in \mathcal{O}_{\check{\mathbf{P}}}((m,\dots,m))\} \end{aligned}$$

have the same cardinality. As $\mathrm{Supp}(u_2) = \min \mathbf{P}_n$, in the same way as in Step 1 one sees that

$$q^{mn-|\min \mathbf{P}|} \mid |B_2|.$$

Write $\mathbb{F}_q^{m\times n}$, with the obvious meaning, as

$$\mathbb{F}_q^{m\times n} = \mathbb{F}_q^{[m]\times[n]-\min \mathbf{P}} \oplus \mathbb{F}_q^{\min \mathbf{P}} \quad (v = (v_1, v_2)).$$

Then

$$B_1 = \{(v_1, v_2) \in \mathbb{F}_q^{m\times n} \mid v_1 \cdot u_1 = 0,\ \mathrm{Supp}(v_2) = \min \mathbf{P}\}.$$

As $\mathrm{Supp}(u_1) \subseteq \mathbf{P}_{(1)} = \mathbf{P}\setminus \min \mathbf{P} = [m]\times[n]\setminus \min \mathbf{P}$, and $\mathrm{Supp}(u_1) \ne \emptyset$, $\mathbb{F}_q^{[m]\times[n]-\min \mathbf{P}} \to \mathbb{F}_q$ $(w \mapsto w\cdot u_1)$ is a nonzero linear functional. Hence $|B_1| = (q-1)^{|\min \mathbf{P}|} q^{mn-|\min \mathbf{P}|-1}$. So we can derive the same contradiction as in Step 1. So $l_1 \ge l_n$, and we obtain $|\min \mathbf{P}_1| = \cdots = |\min \mathbf{P}_n| = m_1$.

STEP 3. Let C_1', C_2' be linear $\mathbf{P}_{(1)}$-codes in $\mathbb{F}_q^{(m-m_1)\times n}$ with the same $\mathbf{P}_{(1)}$-orbit distribution. Then

$$\begin{aligned} C_1 &= \left\{ \begin{bmatrix} u \\ v \end{bmatrix} \in \mathbb{F}_q^{m\times n} \;\middle|\; u \in \mathbb{F}_q^{m_1\times n},\ v \in C_1' \right\}, \\ C_2 &= \left\{ \begin{bmatrix} u \\ v \end{bmatrix} \in \mathbb{F}_q^{m\times n} \;\middle|\; u \in \mathbb{F}_q^{m_1\times n},\ v \in C_2' \right\} \end{aligned}$$

are linear $\mathbf{P}$-codes with the same $\mathbf{P}$-orbit distribution. By Step 1, for each $j \in [n]$, $[(i_1, j) \in \mathbf{P}_{j,(1)},\ (i_2, j) \in \min \mathbf{P}_j] \Rightarrow (i_1, j) \ge_{\mathbf{P}_j} (i_2, j)$. From this observation and Step 2, we indeed have, for $R = (r_1, \dots, r_n) \ne 0$ with $0 \le r_1, \dots, r_n \le m - m_1$,

$$\begin{aligned} a_{\mathbf{P},(m_1+r_1,\dots,m_1+r_n)}(C_1) &= q^{m_1|\mathrm{Supp}(R)|}(q-1)^{m_1(n-|\mathrm{Supp}(R)|)} a_{\mathbf{P}_{(1)},(r_1,\dots,r_n)}(C_1') \\ &= q^{m_1|\mathrm{Supp}(R)|}(q-1)^{m_1(n-|\mathrm{Supp}(R)|)} a_{\mathbf{P}_{(1)},(r_1,\dots,r_n)}(C_2') \\ &= a_{\mathbf{P},(m_1+r_1,\dots,m_1+r_n)}(C_2), \end{aligned}$$

and, for $0 \le r_1, \dots, r_n \le m_1$,

$$a_{\mathbf{P},(r_1,\dots,r_n)}(C_1) = |S_n/S_n^{(R)}| \binom{m_1}{r_1}(q-1)^{r_1} \cdots \binom{m_n}{r_n}(q-1)^{r_n} = a_{\mathbf{P},(r_1,\dots,r_n)}(C_2).$$

So $C_1^{\perp}$, $C_2^{\perp}$ have the same $\check{\mathbf{P}}$-orbit distributions. As $C_i^{\perp} = \left\{ \left[\begin{smallmatrix} 0 \\ v \end{smallmatrix}\right] \in \mathbb{F}_q^{m\times n} \;\middle|\; v \in C_i'^{\perp} \right\}$, for $i = 1, 2, C_1'^{\perp}, C_2'^{\perp}$ have the same $\check{\mathbf{P}}_{(1)}$-orbit distributions. By induction our proof is now complete. ■

8. A half of equivalent condition III (MacWilliams-type identity)

Here the title of the chapter refers to the implication (1)⇒(2) in Theorem 8.7.

The following results were obtained in [22], and the reader is referred to their paper for the details. Recall from [14, Ch. 5, (53)] that the Krawtchouk polynomial $P_k(x; n, q)$, for any positive integer n and prime power q, is defined as

$$P_k(x) = P_k(x; n, q) = \sum_{j=0}^{k} (-1)^j (q-1)^{k-j} \binom{x}{j} \binom{n-x}{k-j} \qquad (k = 0, 1, \ldots, n).$$

THEOREM 8.1 ([22]). *Let $\mathcal{X}_{\mathbf{P}_0} = (X = \mathbb{F}_q^m, \mathcal{R}_{\mathbf{P}_0} = \{R_{\mathbf{P}_0,i}\}_{i=0}^m)$ be the association scheme given by*

$$(x, y) \in R_{\mathbf{P}_0,i} \;\Leftrightarrow\; w_{\mathbf{P}_0}(x-y) = i \ (i = 0, 1, \ldots, m)$$

(cf. the paragraph below (3.17)). *Then*

(1) $(X = \mathbb{F}_q^m, \mathcal{R}_{\check{\mathbf{P}}_0} = \{R_{\check{\mathbf{P}}_0,i}\}_{i=0}^m) \to (X^*, \mathcal{R}^*_{\mathbf{P}_0})$ $(x \mapsto \lambda_x)$ *is an isomorphism of association schemes* (cf. (3.17)).

(2) $E_j = |X|^{-1} \sum_{w_{\check{\mathbf{P}}_0}(x)=j} \lambda_x {}^t\bar{\lambda}_x$ $j = 0, 1, \ldots, m)$ *are the irreducible idempotents for the scheme $\mathcal{X}_{\mathbf{P}_0}$.*

(3) *Let $j = m_1 + \cdots + m_{l-1} + j_0$ $(1 \le l \le t$ and $1 \le j_0 \le m_l$, or $l = 1$ and $j_0 = 0)$. Then the p-numbers are given by*

$$p_{ij} = \begin{cases} 0 & \text{if } i > m_l + \cdots + m_t, \\ q^{m_1+\cdots+m_{l-1}} P_{j_0}(i - (m_{l+1} + \cdots + m_t); m_l, q) & \\ & \text{if } m_{l+1} + \cdots + m_t < i \le m_l + \cdots + m_t, \\ q^{m_1+\cdots+m_{l-1}} \binom{m_l}{j_0} (q-1)^{j_0} & \\ & \text{if } i \le m_{l+1} + \cdots + m_t. \end{cases} \tag{8.1}$$

(4) *Let $j = m_t + \cdots + m_{l+1} + j_0$ $(1 \le l \le t$ and $1 \le j_0 \le m_l$, or $l = t$ and $j_0 = 0)$. Then the q-numbers are given by*

$$q_{ij} = \begin{cases} 0 & \text{if } i > m_1 + \cdots + m_l, \\ q^{m_{l+1}+\cdots+m_t} P_{j_0}(i - (m_1 + \cdots + m_{l-1}); m_l, q) & \\ & \text{if } m_1 + \cdots + m_{l-1} < i \le m_l + \cdots + m_l, \\ q^{m_{l+1}+\cdots+m_t} \binom{m_l}{j_0} (q-1)^{j_0} & \\ & \text{if } i \le m_1 + \cdots + m_{l-1}. \end{cases} \tag{8.2}$$

We obtain the following MacWilliams-type identity from the above theorem and Delsarte's well-known results in (3.12) and (3.13). Previously, this result was obtained independently and in different manner in [16] and [21] by using the Poisson summation formula as the main tool.

COROLLARY 8.2. *Let C be a linear code in $\mathbb{F}_q^m$, $a_{\mathbf{P}_0,i}(C) = |S_{\mathbf{P}_0}(i) \cap C|$, $a_{\check{\mathbf{P}}_0,i}(C^\perp) = |S_{\check{\mathbf{P}}_0}(i) \cap C^\perp|$, for $i = 0, 1, \dots, m$. Then*

$$(a_{\check{\mathbf{P}}_0,j}(C^\perp))_{j=0}^m = \frac{1}{|C|}(a_{\mathbf{P}_0,i}(C))_{i=0}^m (q_{ij}), \qquad (a_{\mathbf{P}_0,j}(C))_{j=0}^m = \frac{1}{|C^\perp|}(a_{\check{\mathbf{P}}_0,i}(C^\perp))_{i=0}^m (p_{ij}),$$

where (p_{ij}) and (q_{ij}) are the matrices given by (8.1) and (8.2), respectively.

For the rest of this chapter, we will assume that $\mathbf{P}$ is a generalized Niederreiter–Rosenbloom–Tsfasman poset and determine some parameters for the association schemes

$$\mathcal{X}_K = (\mathbb{F}_q^{m\times n}, \mathcal{R}_K = \{R_{K,\beta}\}_{\beta\in I_{m,n}}),$$
$$\mathcal{X}_\mathbf{P} = (\mathbb{F}_q^{m\times n}, \mathcal{R}_\mathbf{P} = \{R_{\mathbf{P},\beta}\}_{\beta\in I_{m,n}/S_n}),$$

at the same time. Here $(u, v) \in R_{K,\beta} \Leftrightarrow u - v \in \mathcal{O}_K(\beta)$ (cf. (4.34)), and $(u, v) \in R_{\mathbf{P},\beta} \Leftrightarrow u - v \in \mathcal{O}_\mathbf{P}(\beta)$ (cf. (4.36)).

Let $\mathcal{X}_{\mathbf{P}_0} = (\mathbb{F}_q^m, \mathcal{R}_{\mathbf{P}_0} = \{R_{\mathbf{P}_0,i}\}_{i=0}^m)$, $\mathcal{X}_{\mathbf{P}_j} = (\mathbb{F}_q^{[m]\times\{j\}}, \mathcal{R}_{\mathbf{P}_j} = \{R_{\mathbf{P}_j,i}\}_{i=0}^m)$ be the association schemes described in the paragraph below (3.17). Then, as we mentioned there, via the maps τ_j in (2.7), $\mathcal{X}_{\mathbf{P}_0}, \mathcal{X}_{\mathbf{P}_1}, \dots, \mathcal{X}_{\mathbf{P}_n}$ are all isomorphic, and we may assume that the parameters of them are all the same.

Keeping $\mathbf{P}_j$'s momentarily, assume that $A_{\mathbf{P}_j,i}$ $(i = 0, 1, \dots, m)$ are the adjacency matrices for $\mathcal{X}_{\mathbf{P}_j}$ $(j = 1, \dots, n)$. Then, for $u = [u^1 \ \cdots \ u^n]$, $v = [v^1 \ \cdots \ v^n] \in \mathbb{F}_q^{m\times n}$, and $\beta = (\beta_1, \dots, \beta_n) \in I_{m,n}$,

$$\begin{aligned}(A_{\mathbf{P}_1,\beta_1} \otimes \cdots \otimes A_{\mathbf{P}_n,\beta_n})_{uv} &= (A_{\mathbf{P}_1,\beta_1})_{u^1v^1} \cdots (A_{\mathbf{P}_n,\beta_n})_{u^nv^n} \\ &= \begin{cases} 1 & \text{if } (u^1, v^1) \in R_{\mathbf{P}_1,\beta_1}, \dots, (u^n, v^n) \in R_{\mathbf{P}_n,\beta_n}, \\ 0 & \text{otherwise}\end{cases} \\ &= \begin{cases} 1 & \text{if } w_{\mathbf{P}_1}(u^1 - v^1) = \beta_1, \dots, w_{\mathbf{P}_n}(u^n - v^n) = \beta_n, \\ 0 & \text{otherwise}\end{cases} \\ &= \begin{cases} 1 & \text{if } u - v \in \mathcal{O}_K(\beta), \\ 0 & \text{otherwise.}\end{cases} \qquad (8.3)\end{aligned}$$

So we also have

$$\Big(\sum_{\sigma\in S_n/S_n^{(\beta)}} A_{\mathbf{P}_1,\beta_{\sigma 1}} \otimes \cdots \otimes A_{\mathbf{P}_n,\beta_{\sigma n}}\Big)_{uv} = \begin{cases} 1 & \text{if } u - v \in \mathcal{O}_\mathbf{P}(\beta), \\ 0 & \text{otherwise.}\end{cases} \qquad (8.4)$$

Denote the adjacency matrices for $\mathcal{X}_{\mathbf{P}_0}$ by $A_0, A_1, \dots, A_m$ (suppressing $\mathbf{P}_0$). Then, by the above remark and (8.3–4), the adjacency matrices for $\mathcal{X}_K$ and $\mathcal{X}_\mathbf{P}$ are respectively given by

$$A_{K,\beta} = A_{\beta_1} \otimes \cdots \otimes A_{\beta_n} \quad (\beta = (\beta_1, \dots, \beta_n) \in I_{m,n}), \qquad (8.5)$$

$$\begin{aligned}A_{\mathbf{P},\beta} &= \sum_{\sigma\in S_n/S_n^{(\beta)}} A_{K,\sigma\beta} \\ &= \sum_{\sigma\in S_n/S_n^{(\beta)}} A_{\beta_{\sigma 1}} \otimes \cdots \otimes A_{\beta_{\sigma n}} \quad (\beta = (\beta_1, \dots, \beta_n) \in I_{m,n}/S_n). \qquad (8.6)\end{aligned}$$

Let us denote the irreducible idempotents of $\mathcal{X}_{\mathbf{P}_0}$ by $E_0, E_1, \ldots, E_m$ (again suppressing $\mathbf{P}_0$). Then the irreducible idempotents for $\mathcal{X}_K$ and $\mathcal{X}_{\mathbf{P}}$ are given by

$$E_{K,\beta} = E_{\beta_1} \otimes \cdots \otimes E_{\beta_n} \quad (\beta = (\beta_1, \ldots, \beta_n) \in I_{m,n}), \tag{8.7}$$

$$\begin{aligned} E_{\mathbf{P},\beta} &= \sum_{\sigma \in S_n / S_n^{(\beta)}} E_{K,\sigma\beta} \\ &= \sum_{\sigma \in S_n / S_n^{(\beta)}} E_{\beta_{\sigma 1}} \otimes \cdots \otimes E_{\beta_{\sigma n}} \quad (\beta = (\beta_1, \ldots, \beta_n) \in I_{m,n}/S_n). \end{aligned} \tag{8.8}$$

Let q_{ij} $(i, j = 0, \ldots, m)$ be the q-numbers of $\mathcal{X}_{\mathbf{P}_0}$. Then it is easy to see that

$$E_{K,\beta} = \frac{1}{q^{mn}} \sum_{\alpha = (\alpha_1, \ldots, \alpha_n) \in I} q_{K,\alpha\beta} A_{K,\alpha} \tag{8.9}$$

with

$$q_{K,\alpha\beta} = q_{\alpha_1\beta_1} \cdots q_{\alpha_n\beta_n}, \tag{8.10}$$

and

$$E_{\mathbf{P},\beta} = \frac{1}{q^{mn}} \sum_{\alpha = (\alpha_1, \ldots, \alpha_n) \in I/S_n} q_{\mathbf{P},\alpha\beta} A_{\mathbf{P},\alpha} \tag{8.11}$$

with

$$q_{\mathbf{P},\alpha\beta} = \sum_{\sigma \in S_n / S_n^{(\beta)}} q_{K,\alpha\sigma\beta} = \sum_{\sigma \in S_n / S_n^{(\beta)}} q_{\alpha_1\beta_{\sigma 1}} \cdots q_{\alpha_n\beta_{\sigma n}}. \tag{8.12}$$

This shows, in particular, that $E_{K,\beta}$ $(\beta \in I_{m,n})$ and $E_{\mathbf{P},\beta}$ $(\beta \in I_{m,n}/S_n)$ are respectively contained in the Bose–Mesner algebras of $\mathcal{X}_K$ and $\mathcal{X}_{\mathbf{P}}$. Next, one checks easily that $\{E_{K,\beta}\}_{\beta \in I_{m,n}}$ and $\{E_{\mathbf{P},\beta}\}_{\beta \in I_{m,n}/S_n}$ respectively satisfy conditions (i)–(iii) in (3.2). So they are the irreducible idemponents for $\mathcal{X}_K$ and $\mathcal{X}_{\mathbf{P}}$, and (8.10) and (8.12) are the respective q-numbers.

Let λ be a fixed nontrivial additive character of $\mathbb{F}_q$, as before. Then

$$X = \mathbb{F}_q^{m \times n} \xrightarrow{\sim} X^* \quad (u \mapsto \lambda_u)$$

is an isomorphism, where

$$\lambda_u(v) = \lambda(\operatorname{tr} u\, {}^t v) = \lambda({}^t u^1 v^1 + \cdots + {}^t u^n v^n),$$

for $u = [u^1 \ \cdots \ u^n], v = [v^1 \ \cdots \ v^n] \in \mathbb{F}_q^{m \times n}$. Then, in addition, we have

$$E_{K,\beta} = \frac{1}{q^{mn}} \sum_{u \in \mathcal{O}_{\check{K}}(\beta)} \lambda_u {}^t\bar{\lambda}_u \quad (\beta \in I_{m,n}), \tag{8.13}$$

$$E_{\mathbf{P},\beta} = \frac{1}{q^{mn}} \sum_{u \in \mathcal{O}_{\check{\mathbf{P}}}(\beta)} \lambda_u {}^t\bar{\lambda}_u \quad (\beta \in I_{m,n}/S_n). \tag{8.14}$$

So Theorem 3.3(3) is satisfied for $E_{K,\beta}$ and $E_{\mathbf{P},\beta}$, and hence we may invoke Theorem 3.3 for the association schemes $\mathcal{X}_K$ and $\mathcal{X}_{\mathbf{P}}$. Indeed, restoring the indices $\mathbf{P}_1, \ldots, \mathbf{P}_n$ and using Theorem 8.1(2), we have

$$\begin{aligned} E_{\mathbf{P},\beta} &= \sum_{\sigma \in S_n/S_n^{(\beta)}} E_{\beta_{\sigma 1}} \otimes \cdots \otimes E_{\beta_{\sigma n}} \\ &= \sum_{\sigma \in S_n/S_n^{(\beta)}} \left(\frac{1}{q^m} \sum_{w_{\check{\mathbf{P}}_1}(u^1)=\beta_{\sigma 1}} \lambda_{u^1} {}^t\bar{\lambda}_{u^1} \right) \otimes \cdots \otimes \left(\frac{1}{q^m} \sum_{w_{\check{\mathbf{P}}_n}(u^n)=\beta_{\sigma n}} \lambda_{u^n} {}^t\bar{\lambda}_{u^n} \right) \\ &= \frac{1}{q^{mn}} \sum_{\sigma \in S_n/S_n^{(\beta)}} \sum_{u=[u^1 \ \cdots \ u^n] \in \mathcal{O}_{\check{K}}(\sigma\beta)} \lambda_u {}^t\lambda_u = \frac{1}{q^{mn}} \sum_{u \in \mathcal{O}_{\check{\mathbf{P}}}(\beta)} \lambda_u {}^t\bar{\lambda}_u. \end{aligned}$$

Let p_{ij} $(i, j = 0, \ldots, m)$ be the p-numbers of $\mathcal{X}_{\mathbf{P}_0}$. Then one easily shows that

$$p_{K,\alpha\beta} = p_{\alpha_1\beta_1} \cdots p_{\alpha_n\beta_n} \tag{8.15}$$

for $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n) \in I$, and

$$p_{\mathbf{P},\alpha\beta} = \sum_{\sigma \in S_n/S_n^{(\beta)}} p_{K,\alpha\beta} = \sum_{\sigma \in S_n/S_n^{(\beta)}} p_{\alpha_1\beta_{\sigma 1}} \cdots p_{\alpha_n\beta_{\sigma n}} \tag{8.16}$$

for $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n) \in I/S_n$.

We now summarize our results in this chapter as the following theorems. Here, as before, $A_0, A_1, \ldots, A_m$, $E_0, E_1, \ldots, E_m$, p_{ij} $(i, j = 0, 1, \ldots, m)$, q_{ij} $(i, j = 0, 1, \ldots, m)$ are respectively the adjacency matrices, the irreducible idempotents, p-numbers, and q-numbers of the association scheme $\mathcal{X}_{\mathbf{P}_0} = (\mathbb{F}_q^m, \mathcal{R}_{\mathbf{P}_0} = \{R_{\mathbf{P}_0,i}\}_{i=0}^m)$, which are given in Theorem 8.1.

Theorem 8.3. *Let $\mathcal{X}_K = (\mathbb{F}_q^{m\times n}, \mathcal{R}_K = \{R_{K,\beta}\}_{\beta \in I_{m,n}})$ be the association scheme given by*

$$(u, v) \in R_{K,\beta} \iff u - v \in \mathcal{O}_K(\beta)$$

(cf. (4.34)). *Then*

(1) $A_{K,\beta} = A_{\beta_1} \otimes \cdots \otimes A_{\beta_n}$ $(\beta = (\beta_1, \ldots, \beta_n) \in I_{m,n})$ *are the adjacency matrices.*
(2) $E_{K,\beta} = E_{\beta_1} \otimes \cdots \otimes E_{\beta_n} = (1/q^{mn}) \sum_{u \in \mathcal{O}_{\check{K}}(\beta)} \lambda_u {}^t\bar{\lambda}_u$ $(\beta = (\beta_1, \ldots, \beta_n) \in I_{m,n})$ *are the irreducible idempotents.*
(3) $q_{K,\alpha\beta} = q_{\alpha_1\beta_1} \cdots q_{\alpha_n\beta_n}$ $(\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in I_{m,n})$ *are the q-numbers.*
(4) $p_{K,\alpha\beta} = p_{\alpha_1\beta_1} \cdots p_{\alpha_n\beta_n}$ $(\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in I_{m,n})$ *are the p-numbers.*
(5) $(X = \mathbb{F}_q^{m\times n}, \mathcal{R}_{\check{K}} = \{R_{\check{K},\beta}\}_{\beta \in I_{m,n}}) \to (X^*, \mathcal{R}_K^*)$ $(u \mapsto \lambda_u)$ *is an isomorphism of association schemes. Here*

$$(u, v) \in \mathcal{R}_{\check{K},\beta} \iff u - v \in \mathcal{O}_{\check{K}}(\beta)$$

(cf. (4.35)).

Theorem 8.4. *Let $\mathcal{X}_{\mathbf{P}} = (\mathbb{F}_q^{m\times n}, \mathcal{R}_{\mathbf{P}} = \{R_{\mathbf{P},\beta}\}_{\beta \in I_{m,n}/S_n})$ be the association scheme given by*

$$(u, v) \in R_{\mathbf{P},\beta} \iff u - v \in \mathcal{O}_{\mathbf{P}}(\beta)$$

(cf. (4.36)). *Then*

(1) $A_{\mathbf{P},\beta} = \sum_{\sigma\in S_n/S_n^{(\beta)}} A_{K,\sigma\beta} = \sum_{\sigma\in S_n/S_n^{(\beta)}} A_{\beta_{\sigma 1}}\otimes\cdots\otimes A_{\beta_{\sigma n}}$ $(\beta = (\beta_1,\ldots,\beta_n) \in I_{m,n}/S_n)$ *are the adjacency matrices.*
(2) $E_{\mathbf{P},\beta} = \sum_{\sigma\in S_n/S_n^{(\beta)}} E_{K,\sigma\beta} = \sum_{\sigma\in S_n/S_n^{(\beta)}} E_{\beta_{\sigma 1}}\otimes\cdots\otimes E_{\beta_{\sigma n}} = (1/q^{mn})\sum_{u\in\mathcal{O}_{\check{\mathbf{P}}}(\beta)} \lambda_u{}^t\bar{\lambda}_u$ $(\beta = (\beta_1,\ldots,\beta_n) \in I_{m,n}/S_n)$ *are the irreducible idempotents.*
(3) $q_{\mathbf{P},\alpha\beta} = \sum_{\sigma\in S_n/S_n^{(\beta)}} q_{K,\alpha\sigma\beta} = \sum_{\sigma\in S_n/S_n^{(\beta)}} q_{\alpha_1\beta_{\sigma 1}}\cdots q_{\alpha_n\beta_{\sigma n}}$ $(\alpha = (\alpha_1,\ldots,\alpha_n), \beta = (\beta_1,\ldots,\beta_n) \in I_{m,n}/S_n)$ *are the q-numbers.*
(4) $p_{\mathbf{P},\alpha\beta} = \sum_{\sigma\in S_n/S_n^{(\beta)}} p_{K,\alpha\sigma\beta} = \sum_{\sigma\in S_n/S_n^{(\beta)}} p_{\alpha_1\beta_{\sigma 1}}\cdots p_{\alpha_n\beta_{\sigma n}}$ $(\alpha = (\alpha_1,\ldots,\alpha_n), \beta = (\beta_1,\ldots,\beta_n) \in I_{m,n}/S_n)$ *are the p-numbers.*
(5) $(X = \mathbb{F}_q^{m\times n}, \mathcal{R}_{\check{\mathbf{P}}} = \{R_{\check{\mathbf{P}},\beta}\}_{\beta\in I_{m,n}/S_n}) \to (X^*, \mathcal{R}^*_{\mathbf{P}})$ $(u \mapsto \lambda_u)$ *is an isomorphism of association schemes. Here*

$$(u,v) \in R_{\check{\mathbf{P}},\beta} \Leftrightarrow u - v \in \mathcal{O}_{\check{\mathbf{P}}}(\beta)$$

(cf. (4.36)).

From Delsarte's results in (3.12) and (3.13), we obtain the following corollaries and thus complete the remaining half of the equivalent condition. To apply Delsarte's results, we have to use Theorems 8.3(5) and 8.4(5).

Corollary 8.5. *Let C be a linear code in $\mathbb{F}_q^{m\times n}$, and let $a_{K,\beta}(C) = |\mathcal{O}_K(\beta)\cap C|$, $a_{\check{K},\beta}(C^\perp) = |\mathcal{O}_{\check{K}}(\beta)\cap C^\perp|$, for $\beta\in I_{m,n}$. Then*

$$(a_{\check{K},\beta}(C^\perp))_{\beta\in I_{m,n}} = \frac{1}{|C|}(a_{K,\alpha}(C))_{\alpha\in I_{m,n}}(q_{K,\alpha\beta}),$$
$$(a_{K,\beta}(C))_{\beta\in I_{m,n}} = \frac{1}{|C^\perp|}(a_{\check{K},\alpha}(C^\perp))_{\alpha\in I_{m,n}}(p_{K,\alpha\beta}),$$

where $(q_{K,\alpha\beta})$, $(p_{K,\alpha\beta})$ are respectively given by (3) *and* (4) *of Theorem* 8.3.

Corollary 8.6. *Let C be a linear code in $\mathbb{F}_q^{m\times n}$, and let $a_{\mathbf{P},\beta}(C) = |\mathcal{O}_{\mathbf{P}}(\beta)\cap C|$, $a_{\check{\mathbf{P}},\beta}(C^\perp) = |\mathcal{O}_{\check{\mathbf{P}}}(\beta)\cap C^\perp|$, for $\beta\in I_{m,n}/S_n$. Then*

$$(a_{\check{\mathbf{P}},\beta}(C^\perp))_{\beta\in I_{m,n}/S_n} = \frac{1}{|C|}(a_{\mathbf{P},\alpha}(C))_{\alpha\in I_{m,n}/S_n}(q_{\mathbf{P},\alpha\beta}),$$
$$(a_{\mathbf{P},\beta}(C))_{\beta\in I_{m,n}/S_n} = \frac{1}{|C^\perp|}(a_{\check{\mathbf{P}},\alpha}(C^\perp))_{\alpha\in I_{m,n}/S_n}(p_{\mathbf{P},\alpha\beta}),$$

where $(q_{\mathbf{P},\alpha\beta})$ and $(p_{\mathbf{P},\alpha\beta})$ are respectively given by (3) *and* (4) *of Theorem* 8.4.

We are ready to state our grand theorem by combining Theorems 5.2, 6.1, 7.2, and Corollary 8.6.

Theorem 8.7. *Let $\mathbf{P} = \mathbf{P}_1 \dot{+} \cdots \dot{+} \mathbf{P}_n$ be the poset given by the disjoint sum of the posets $\mathbf{P}_j$ on the underlying set $[m]\times\{j\}$, for $j = 1,\ldots,n$. Assume further that $\mathbf{P}$ is not an antichain. Then the following are equivalent:*

(1) $\mathbf{P}$ *is a generalized Niederreiter–Rosenbloom–Tsfasman poset.*
(2) $(\mathbf{P}, \check{\mathbf{P}})$ *is a weak dual orbit pair.*
(3) *The group $G = \mathrm{Aut}(\mathbb{F}_q^{m\times n}, w_{\mathbf{P}})$ acts transitively on each set $\mathcal{O}_{\mathbf{P}}(\beta) \subseteq \mathbb{F}_q^{m\times n}$ $(\beta\in I_{m,n}/S_n)$.*
(4) *$\mathcal{X}_{\mathbf{P}} = (\mathbb{F}_q^{m\times n}, \mathcal{R}_{\mathbf{P}} = \{R_{\mathbf{P},\beta}\}_{\beta\in I_{m,n}/S_n})$ with $(x,y)\in R_{\mathbf{P},\beta} \Leftrightarrow x - y \in \mathcal{O}_{\mathbf{P}}(\beta)$ $(\beta\in I_{m,n}/S_n)$ is a symmetric association scheme.*

9. MacWilliams-type identity for fragments

Let H_0 be the subgroup of $G_0 = \mathrm{Aut}(\mathbb{F}_q^m, w_{\mathbf{P}_0})$, and $\check{H}_0$ the subgroup of $\check{G}_0 = \mathrm{Aut}(\mathbb{F}_q^m, w_{\check{\mathbf{P}}_0})$, given by

$$H_0 = \pi((\mathbb{F}_q^\times)^{m_1} \times \cdots \times (\mathbb{F}_q^\times)^{m_t}) \ltimes U, \tag{9.1}$$

$$\check{H}_0 = \pi((\mathbb{F}_q^\times)^{m_1} \times \cdots \times (\mathbb{F}_q^\times)^{m_t}) \ltimes \check{U} \tag{9.2}$$

(cf. (4.7), (4.9), (4.10)). In [8, Proposition 2.4], it is shown that the orbits under the action of H_0 on $\mathbb{F}_q^m$ are given by the fragments $F(\lambda^{(l)})$ with $\lambda^{(l)} \in J_0$ (cf. (4.28)), where

$$J_0 = \{\lambda^{(l)} \mid \lambda^{(l)} \in \{0,1\}_{\mathbf{P}}^{m_l},\ l = 1, \dots, t\} \tag{9.3}$$

(cf. (4.23)). Also, the orbits under the action of $\check{H}_0$ are given by the fragments $\check{F}(\mu^{(l)})$ with $\mu^{(l)} \in \check{J}_0$ (cf. (4.29)), where

$$\check{J}_0 = \{\mu^{(l)} \mid \mu^{(l)} \in \{0,1\}_{\check{\mathbf{P}}}^{m_l},\ l = 1, \dots, t\} \tag{9.4}$$

(cf. (4.24)).

Theorem 9.1. *Let $\mathcal{X}_{H_0} = (X = \mathbb{F}_q^m,\ \mathcal{R}_{H_0} = \{R_{H_0,\lambda^{(l)}}\}_{\lambda^{(l)} \in J_0})$ and $\mathcal{X}_{\check{H}_0} = (X = \mathbb{F}_q^m,\ \mathcal{R}_{\check{H}_0} = \{R_{\check{H}_0,\mu^{(l)}}\}_{\mu^{(l)} \in \check{J}_0})$ be the association schemes given by*

$$(x,y) \in R_{H_0,\lambda^{(l)}} \ \Leftrightarrow\ x - y \in F(\lambda^{(l)}),$$
$$(x,y) \in R_{\check{H}_0,\mu^{(l)}} \ \Leftrightarrow\ x - y \in \check{F}(\mu^{(l)})$$

(cf. (9.1)–(9.4)). *Then*

(1) $(X = \mathbb{F}_q^m,\ \mathcal{R}_{\check{H}_0} = \{R_{\check{H}_0,\mu^{(l)}}\}_{\mu^{(l)} \in \check{J}_0}) \to (X^*,\ \mathcal{R}^*_{H_0} = \{R^*_{H_0,\mu^{(l)}}\}_{\mu^{(l)} \in \check{J}_0})$ $(x \mapsto \lambda_x)$ *is an isomorphism of association schemes.*

(2) $E_{H_0,\mu^{(l)}} = |X|^{-1} \sum_{x \in \check{F}(\mu^{(l)})} \lambda_x {}^t\bar{\lambda}_x$, *for $\mu^{(l)} \in \check{J}_0$, are the irreducible idempotents for the scheme $\mathcal{X}_{H_0}$.*

(3) *The p-numbers are* (*here and below $\lambda^{(k)} \in J_0$, $\mu^{(l)} \in \check{J}_0$*)

$$p_{\mu^{(l)}\lambda^{(k)}} = \begin{cases} 0 & \text{if } k > l, \\ q^{m_1+\cdots+m_{k-1}}(-1)^{|\mathrm{Supp}(\lambda^{(k)}) \cap \mathrm{Supp}(\mu^{(k)})|} & \\ \quad \times (q-1)^{|\mathrm{Supp}(\lambda^{(k)}) - \mathrm{Supp}(\lambda^{(k)}) \cap \mathrm{Supp}(\mu^{(k)})|} & \text{if } k = l, \\ q^{m_1+\cdots+m_{k-1}}(q-1)^{|\mathrm{Supp}\,\lambda^{(k)}|} & \text{if } k < l. \end{cases} \tag{9.5}$$

(4) *The q-numbers are*

$$q_{\lambda^{(k)}\mu^{(l)}} = \begin{cases} 0 & \text{if } l < k, \\ q^{m_{l+1}+\cdots+m_t}(-1)^{|\mathrm{Supp}(\lambda^{(l)}) \cap \mathrm{Supp}(\mu^{(l)})|} & \\ \quad \times (q-1)^{|\mathrm{Supp}(\mu^{(l)}) - \mathrm{Supp}(\lambda^{(l)}) \cap \mathrm{Supp}(\mu^{(l)})|} & \text{if } l = k, \\ q^{m_{l+1}+\cdots+m_t}(q-1)^{|\mathrm{Supp}\,\mu^{(l)}|} & \text{if } l > k. \end{cases} \tag{9.6}$$

Proof. One shows first by computation that $f_{\mu^{(l)}}(a) = \sum_{x \in \check{F}(\mu^{(l)})} \lambda(a \cdot x)$ has constant value on each $F(\lambda^{(k)})$ $(\lambda^{(k)} \in J_0)$. Invoking Theorem 3.3 gives (1), (2) and (4). The p-numbers can be computed from (3.7), which, in view of the isomorphism in (1) above, says

$$p_{\mu^{(l)}\lambda^{(k)}} = \sum_{x\in F(\lambda^{(k)})} \lambda(a\cdot x) \quad \text{for } a\in \check{F}(\mu^{(l)}).$$

Theorem 8.1 could be proved along the same lines. ∎

The following corollary follows from Delsarte's results in (3.12) and (3.13).

COROLLARY 9.2. *Let C be a linear code in $\mathbb{F}_q^m$, and let $a_{H_0,\lambda^{(k)}}(C) = |F(\lambda^{(k)})\cap C|$ ($\lambda^{(k)}\in J_0$), $a_{\check{H}_0,\mu^{(l)}}(C^\perp) = |\check{F}(\mu^{(l)})\cap C^\perp|$ ($\mu^{(l)}\in \check{J}_0$)* (cf. (9.1)–(9.4)). *Then*

$$(a_{\check{H}_0,\mu^{(l)}}(C^\perp))_{\mu^{(l)}\in\check{J}_0} = \frac{1}{|C|}(a_{H_0,\lambda^{(k)}}(C))_{\lambda^{(k)}\in J_0}(q_{\lambda^{(k)}\mu^{(l)}}), \tag{9.7}$$

$$(a_{H_0,\lambda^{(k)}}(C))_{\lambda^{(k)}\in J_0} = \frac{1}{|C^\perp|}(a_{\check{H}_0,\mu^{(l)}}(C^\perp))_{\mu^{(l)}\in\check{J}_0}(p_{\mu^{(l)}\lambda^{(k)}}), \tag{9.8}$$

where $(p_{\mu^{(l)}\lambda^{(k)}})$, $(q_{\lambda^{(k)}\mu^{(l)}})$ are respectively given by (9.5) *and* (9.6).

REMARK 9.3. (1) As the orbits $\check{F}(\mu^{(l)})$ under the action of $\check{H}_0$ on $\mathbb{F}_q^m$ are indexed by $\mu^{(l)}\in\check{J}_0$, it is natural to denote the relations and idempotents in (1) and (2) of Theorem 9.1 by $R^*_{H_0,\mu^{(l)}}$ and $E_{H_0,\mu^{(l)}}$.

(2) The p-numbers and q-numbers in (9.5) and (9.6) are respectively the same as the ones in (4.16)–(4.19) and (4.21)–(4.24) of [16], while the MacWilliams-type identities in (9.7) and (9.8) are respectively the same as the ones in (5.3) and (5.2) of [16]. Those are derived there by using Poisson summation and a suitable linear change of variables.

Let $\{A_{\lambda^{(k)}}\}_{\lambda^{(k)}\in J_0}$, $\{E_{\mu^{(l)}}\}_{\mu^{(l)}\in\check{J}_0}$ be the adjacency matrices and the irreducible idempotents for the association scheme $\mathcal{X}_{H_0} = (X = \mathbb{F}_q^m, \mathcal{R}_{H_0} = \{R_{H_0,\lambda^{(l)}},\}_{\lambda^{(l)}\in J_0})$. Then, by proceeding just as in Chapter 8, we have the following results of which the details are left to the reader.

THEOREM 9.4. *Let $\mathcal{X}_H = (\mathbb{F}_q^{m\times n}, \mathcal{R}_H = \{R_{H,\lambda}\}_{\lambda\in J})$ be the association scheme given by*

$$(u,v)\in R_{H,\lambda} \;\Leftrightarrow\; u-v\in \mathcal{O}_H(\lambda)$$

(cf. (4.25), (4.28), (4.30)). *Then*

(1) $A_{H,\lambda} = A_{\lambda^{(k_1)}}\otimes\cdots\otimes A_{\lambda^{(k_n)}}$ ($\lambda = (\lambda^{(k_1)},\ldots,\lambda^{(k_n)})\in J$) *are the adjacency matrices.*
(2) $E_{H,\lambda} = E_{\mu^{(l_1)}}\otimes\cdots\otimes E_{\mu^{(l_n)}}$ ($\mu = (\mu^{(l_1)},\ldots,\mu^{(l_n)})\in\check{J}$) $= q^{-mn}\sum_{u\in\mathcal{O}_{\check{H}}(\mu)}\lambda_u{}^t\bar{\lambda}_u$ *are the irreducible idempotents* (cf. (4.26), (4.29), (4.30)).
(3) $q_{H,\lambda\mu} = q_{\lambda^{(k_1)}\mu^{(l_1)}}\cdots q_{\lambda^{(k_n)}\mu^{(l_n)}}$ ($\lambda = (\lambda^{(k_1)},\ldots,\lambda^{(k_n)})\in J$, $\mu = (\mu^{(l_1)},\ldots,\mu^{(l_n)})\in\check{J}$) *are the q-numbers, with $q_{\lambda^{(k)}\mu^{(l)}}$ given by* (9.6).
(4) $p_{H,\mu\lambda} = p_{\mu^{(l_1)}\lambda^{(k_1)}}\cdots p_{\mu^{(l_n)}\lambda^{(k_n)}}$ ($\lambda = (\lambda^{(k_1)},\ldots,\lambda^{(k_n)})\in J$, $\mu = (\mu^{(l_1)},\ldots,\mu^{(l_n)})\in\check{J}$) *are the p-numbers, with $p_{\mu^{(l)}\lambda^{(k)}}$ given by* (9.5).
(5) $(X = \mathbb{F}_q^{m\times n}, \mathcal{R}_{\check{H}} = \{R_{\check{H},\mu}\}_{\mu\in\check{J}}) \to (X^*, \mathcal{R}^*_H = \{R^*_{H,\mu}\}_{\mu\in\check{J}})$ $(u\mapsto\lambda_u)$ *is an isomorphism of association schemes. Here*

$$(u,v)\in R_{\check{H},\mu} \;\Leftrightarrow\; u-v\in\mathcal{O}_{\check{H}}(\mu).$$

THEOREM 9.5. *Let $\mathcal{X}_L = (\mathbb{F}_q^{m\times n}, \mathcal{R}_L = \{R_{L,\lambda}\}_{\lambda\in J/S_n})$ be the association scheme given by*

$$(u,v)\in R_{L,\lambda} \;\Leftrightarrow\; u-v\in\mathcal{O}_L(\lambda)$$

(cf. (4.27), (4.31)). *Then*

(1) $A_{L,\lambda} = \sum_{\sigma \in S_n/S_n^{(\lambda)}} A_{H,\sigma\lambda} = \sum_{\sigma \in S_n/S_n^{(\lambda)}} A_{\lambda^{(k_{\sigma 1})}} \otimes \cdots \otimes A_{\lambda^{(k_{\sigma n})}}$ ($\lambda = (\lambda^{(k_1)}, \ldots, \lambda^{(k_n)}) \in J/S_n$) *are the adjacency matrices.*

(2) $E_{L,\mu} = \sum_{\sigma \in S_n/S_n^{(\mu)}} E_{H,\sigma\mu} = \sum_{\sigma \in S_n/S_n^{(\mu)}} E_{\mu^{(l_{\sigma 1})}} \otimes \cdots \otimes E_{\mu^{(l_{\sigma n})}} = q^{-mn} \sum_{u \in \mathcal{O}_{\check{L}}(\mu)} \lambda_u {}^t\bar{\lambda}_u$ ($\mu = (\mu^{(l_1)}, \ldots, \mu^{(l_n)}) \in \check{J}/S_n$) *are the irreducible idempotents* (cf. (4.31)).

(3) $q_{L,\lambda\mu} = \sum_{\sigma \in S_n/S_n^{(\mu)}} q_{H,\lambda\sigma\mu} = \sum_{\sigma \in S_n/S_n^{(\mu)}} q_{\lambda^{(k_1)}\mu^{(l_{\sigma 1})}} \cdots q_{\lambda^{(k_n)}\mu^{(l_{\sigma n})}}$ ($\lambda = (\lambda^{(k_1)}, \ldots, \lambda^{(k_n)}) \in J/S_n$, $\mu = (\mu^{(l_1)}, \ldots, \mu^{(l_n)}) \in \check{J}/S_n$) *are the q-numbers, with* $q_{\lambda^{(k)}\mu^{(l)}}$ *given by* (9.6).

(4) $p_{L,\mu\lambda} = \sum_{\sigma \in S_n/S_n^{(\lambda)}} p_{H,\mu\sigma\lambda} = \sum_{\sigma \in S_n/S_n^{(\lambda)}} p_{\mu^{(l_1)}\lambda^{(k_{\sigma 1})}} \cdots p_{\mu^{(l_n)}\lambda^{(k_{\sigma n})}}$ ($\mu = (\mu^{(l_1)}, \ldots, \mu^{(l_n)}) \in \check{J}/S_n$, $\lambda = (\lambda^{(k_1)}, \ldots, \lambda^{(k_n)}) \in J/S_n$) *are the p-numbers, with* $p_{\mu^{(l)}\lambda^{(k)}}$ *given by* (9.5).

(5) $(X = \mathbb{F}_q^{m\times n}, \mathcal{R}_{\check{L}} = \{R_{\check{L},\mu}\}_{\mu \in \check{J}/S_n}) \to (X^*, \mathcal{R}_L^* = \{R_{L,\mu}^*\}_{\mu \in \check{J}/S_n})$ $(u \mapsto \lambda_u)$ *is an isomorphism of association schemes. Here*

$$(u, v) \in R_{\check{L},\mu} \;\Leftrightarrow\; u - v \in \mathcal{O}_{\check{L}}(\mu).$$

COROLLARY 9.6. *Let C be a linear code in $\mathbb{F}_q^{m\times n}$, and let $a_{H,\lambda}(C) = |\mathcal{O}_H(\lambda) \cap C|$ $(\lambda \in J)$, $a_{\check{H},\mu}(C^\perp) = |\mathcal{O}_{\check{H}}(\mu) \cap C^\perp|$ $(\mu \in \check{J})$. Then*

$$(a_{\check{H},\mu}(C^\perp))_{\mu \in \check{J}} = \frac{1}{|C|}(a_{H,\lambda}(C))_{\lambda \in J}(q_{H,\lambda\mu}),$$

$$(a_{H,\lambda}(C))_{\lambda \in J} = \frac{1}{|C^\perp|}(a_{\check{H},\mu}(C^\perp))_{\lambda \in \check{J}}(p_{H,\mu\lambda}),$$

where $(q_{H,\lambda\mu})$ and $(p_{H,\mu\lambda})$ are respectively given by (3) *and* (4) *of Theorem* 9.4.

COROLLARY 9.7. *Let C be a linear code in $\mathbb{F}_q^{m\times n}$, and let $a_{L,\lambda}(C) = |\mathcal{O}_L(\lambda) \cap C|$ $(\lambda \in J/S_n)$, $a_{\check{L},\mu}(C^\perp) = |\mathcal{O}_{\check{L}}(\mu) \cap C^\perp|$ $(\mu \in \check{J}/S_n)$. Then*

$$(a_{\check{L},\mu}(C^\perp))_{\mu \in \check{J}/S_n} = \frac{1}{|C|}(a_{L,\lambda}(C))_{\lambda \in J/S_n}(q_{L,\lambda\mu}),$$

$$(a_{L,\lambda}(C))_{\lambda \in J/S_n} = \frac{1}{|C^\perp|}(a_{\check{L},\mu}(C^\perp))_{\mu \in \check{J}/S_n}(p_{L,\mu\lambda}),$$

where $(q_{L,\lambda\mu})$ and $(p_{L,\mu\lambda})$ are respectively given by (3) *and* (4) *of Theorem* 9.5.

Appendix: Recent developments in the theory of poset codes

1. Introduction

Let $\mathbb{F}_q$ be the finite field with q elements and $\mathbb{F}_q^n$ be the vector space of n-tuples over $\mathbb{F}_q$. Coding theory may be considered as the study of $\mathbb{F}_q^n$ when $\mathbb{F}_q^n$ is endowed with the Hamming metric. Since the late 1980's several attempts have been made to generalize the classical problems of coding theory by introducing a new non-Hamming metric on the vector space $\mathbb{F}_q^n$ (cf. [27, 28, 26]). These attempts led Brualdi et al. [4] to introduce the concept of poset codes. We start our journey with a brief introduction to the basic notions of poset codes. We refer to [4] for details.

Let $\mathbb{P}$ be a partially ordered set (abbreviated as a poset) on the underlying set $[n] = \{1, \ldots, n\}$ of coordinate positions of vectors in $\mathbb{F}_q^n$. As usual the partial order relation of $\mathbb{P}$ is denoted by $\leq$. A subset I of $\mathbb{P}$ is called an *order ideal* if $x \in I$ and $y \leq x$ imply $y \in I$. For a subset E of $\mathbb{P}$, $\langle E \rangle$ will denote the smallest order ideal of $\mathbb{P}$ containing E. The $\mathbb{P}$-*weight* of a vector $u = (u_1, \ldots, u_n)$ in $\mathbb{F}_q^n$ is defined as the cardinality

$$w_{\mathbb{P}}(u) = |\langle \mathrm{Supp}(u) \rangle|$$

of the smallest order ideal of $\mathbb{P}$ containing $\mathrm{Supp}(u)$, where $\mathrm{Supp}(u) = \{i \mid u_i \neq 0\}$. The $\mathbb{P}$-*distance* between elements u and v in $\mathbb{F}_q^n$ is defined as

$$d_{\mathbb{P}}(u, v) = w_{\mathbb{P}}(u - v).$$

If $\mathbb{P}$ is an antichain in which no two elements are comparable, the $\mathbb{P}$-weight and $\mathbb{P}$-distance become the Hamming weight and Hamming distance of the classical coding theory. It is known that the $\mathbb{P}$-distance $d_{\mathbb{P}}(\cdot, \cdot)$ gives a metric on $\mathbb{F}_q^n$. The metric $d_{\mathbb{P}}$ is called the $\mathbb{P}$-*metric* (or simply a *poset metric* when $\mathbb{P}$ is not specified). If $\mathbb{F}_q^n$ is endowed with the $\mathbb{P}$-metric, then a subset C of $\mathbb{F}_q^n$ is called a $\mathbb{P}$-*code* (or simply a *poset code*).

In the theory of poset codes, one is interested in coding-theoretic properties of poset codes. In principle, one can try to generalize any result in Hamming metric spaces to poset metric spaces. However some concepts can be generalized nicely while the others cannot. The purpose of this paper is to report some important recent results in the theory of poset codes mainly done by the second author and his colleagues within the activities of the Combinatorial and Computational Mathematics Research Center.

In Section 2, as a foundation of our investigation, we study metric properties of poset metric spaces. We first determine the structure of the automorphism group of a poset metric space. Next, we classify the posets which force the linearity of isometries.

In Section 3, we study MDS poset codes. As will be demonstrated, most of the properties of classical MDS codes can be generalized to an arbitrary MDS poset code. We first introduce the Singleton bound for poset codes and define MDS poset codes as linear codes which attain the Singleton bound. Secondly, we introduce the concept of I-perfect codes and relate MDS poset codes to I-perfect poset codes. Thirdly, we show that the weight distribution of an MDS poset code, as in the case of an MDS code in the Hamming metric space, is completely determined. Finally, we prove the duality theorem which states that a linear code C is an MDS $\mathbb{P}$-code if and only if $C^{\perp}$ is an MDS $\overline{\mathbb{P}}$-code, where $C^{\perp}$ is the dual code of C and $\overline{\mathbb{P}}$ is the dual poset of $\mathbb{P}$.

In Section 4, we study perfect poset codes. As will be seen, the theory of perfect codes becomes rich in poset codes. In the first subsection, the classification of perfect codes over a crown poset metric is studied. In the second subsection, we classify the posets which admit the extended binary Hamming code or the extended binary Golay code as a perfect code.

In Section 5, we study the MacWilliams identity for poset codes. All poset structures which admit the MacWilliams identity are classified. It is proved that being a hierarchical poset is a necessary and sufficient condition for a poset to admit the MacWilliams identity. An explicit relation is also derived between the $\mathbb{P}$-weight distribution of a hierarchical poset code and the $\overline{\mathbb{P}}$-weight distribution of the dual code.

2. The poset metric space $(\mathbb{F}_q^n, d_{\mathbb{P}})$

In this section, as a foundation of our investigation, we study metric properties of poset metric spaces. We first determine the structure of the automorphism group of a poset metric space. Next, we classify the posets which force the linearity of isometries.

A map of $\mathbb{F}_q^n$ into $\mathbb{F}_q^n$ is called a $\mathbb{P}$-*isometry* if it preserves $\mathbb{P}$-distance. It is easy to see that every $\mathbb{P}$-isometry is a bijection. A linear $\mathbb{P}$-isometry is called a $\mathbb{P}$-*automorphism*. We denote by $\mathrm{Iso}_{\mathbb{P}}(\mathbb{F}_q^n)$ (resp. $\mathrm{Aut}_{\mathbb{P}}(\mathbb{F}_q^n)$) the group of $\mathbb{P}$-isometries (resp. $\mathbb{P}$-automorphisms) of $(\mathbb{F}_q^n, d_{\mathbb{P}})$, i.e.,

$$\begin{aligned}
\mathrm{Iso}_{\mathbb{P}}(\mathbb{F}_q^n) &= \{f : \mathbb{F}_q^n \to \mathbb{F}_q^n \mid d_{\mathbb{P}}(f(u), f(v)) = d_{\mathbb{P}}(u, v) \text{ for all } u, v \in \mathbb{F}_q^n\},\\
\mathrm{Aut}_{\mathbb{P}}(\mathbb{F}_q^n) &= \{f : \mathbb{F}_q^n \to \mathbb{F}_q^n \mid f \text{ is linear and } w_{\mathbb{P}}(f(u)) = w_{\mathbb{P}}(u) \text{ for all } u \in \mathbb{F}_q^n\}.
\end{aligned}$$

Our goal is to study the structure of these groups.

2.1. The structure of automorphism groups. In [8] the authors developed a coding theory over the ρ-metric which is a poset metric where the poset is given by the disjoint union of chains of the same cardinality. They determine the structure of the automorphism group of a ρ-metric space implicitly. This result is explicitly verified in [23]. In [5], the authors computed the automorphism group of a poset metric space when the poset is a crown of cardinality $2m$ (see Figure 1).

In [30] the authors determined the structure of the group $\mathrm{Aut}_{\mathbb{P}}(\mathbb{F}_q^n)$ for a general poset $\mathbb{P}$. A bijection $\sigma : \mathbb{P} \to \mathbb{P}$ is called an automorphism if σ and σ^{-1} preserve the order

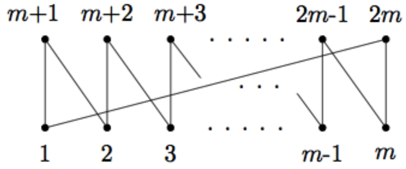

Fig. 1. The crown poset of cardinality $2m$

relation of $\mathbb{P}$. The group of automorphisms of $\mathbb{P}$ is denoted by $\mathrm{Aut}(\mathbb{P})$. Let $M_{n\times n}(\mathbb{F}_q)$ be the set of all $n\times n$ matrices over $\mathbb{F}_q$. The main result of [30] is

THEOREM 2.1. *The automorphism group of a poset metric space $(\mathbb{F}_q^n, d_{\mathbb{P}})$ is an internal semidirect product of the automorphism group of $\mathbb{P}$ by $G(\mathbb{P})$, i.e.,*

$$\mathrm{Aut}_{\mathbb{P}}(\mathbb{F}_q^n) \simeq G(\mathbb{P}) \rtimes \mathrm{Aut}(\mathbb{P}),$$

where

$$G(\mathbb{P}) = \left\{ (a_{ij}) \in M_{n\times n}(\mathbb{F}_q) \;\middle|\; \begin{array}{ll} \mathbb{F}_q & \text{if } i <_{\mathbb{P}} j \\ a_{ij} \in \mathbb{F}_q^* & \text{if } i = j \\ \{0\} & \text{otherwise} \end{array} \right\}.$$

2.2. Classification of posets forcing the linearity of isometries. Since the structure of the automorphism group of a poset metric space is determined, our next concern is the structure of the isometry group $\mathrm{Iso}_{\mathbb{P}}(\mathbb{F}_q^n)$. Let $\mathbf{0}$ denote the zero vector in $\mathbb{F}_q^n$. Since every $\mathbb{P}$-isometry g of $\mathbb{F}_q^n$ can be written uniquely as $g = f + g(\mathbf{0})$, where f is a $\mathbb{P}$-isometry which fixes the origin, it is enough to investigate the $\mathbb{P}$-isometries which fix the origin. Set

$$\mathrm{Iso}^0_{\mathbb{P}}(\mathbb{F}_q^n) = \{ f : \mathbb{F}_q^n \to \mathbb{F}_q^n \mid f \text{ is a } \mathbb{P}\text{-isometry and } f(\mathbf{0}) = \mathbf{0}\}.$$

It is clear that $\mathrm{Aut}_{\mathbb{P}}(\mathbb{F}_q^n)$ is a subgroup of $\mathrm{Iso}^0_{\mathbb{P}}(\mathbb{F}_q^n)$ and it is known ([14]) that $\mathrm{Aut}_{\mathbb{P}}(\mathbb{F}_q^n) = \mathrm{Iso}^0_{\mathbb{P}}(\mathbb{F}_q^n)$ if $\mathbb{P}$ is an antichain and $q = 2, 3$. This motivates the following definition.

DEFINITION 2.2. Let $\mathbb{P}$ be a poset on $[n]$. It is said that $\mathbb{P}$ *forces the linearity of isometries* over $\mathbb{F}_q$ if $\mathrm{Aut}_{\mathbb{P}}(\mathbb{F}_q^n) = \mathrm{Iso}^0_{\mathbb{P}}(\mathbb{F}_q^n)$.

In [14] the authors classified all posets which force the linearity of isometries over $\mathbb{F}_q$. To describe their main result, we need some terminology from the theory of posets.

DEFINITION 2.3. Let $\mathbb{P} = (X, \leq)$ and $\mathbb{Q} = (Y, \leq)$ be two posets.

(a) The *disjoint union* $\mathbb{P} \mathring{\cup} \mathbb{Q}$ of $\mathbb{P}$ and $\mathbb{Q}$ is the poset on $X \mathring{\cup} Y$ such that $x \leq y$ in $\mathbb{P} \mathring{\cup} \mathbb{Q}$ if one of the following conditions holds:

$$\text{(i) } x, y \in X \text{ and } x \leq y \text{ in } \mathbb{P}, \quad \text{(ii) } x, y \in Y \text{ and } x \leq y \text{ in } \mathbb{Q}.$$

(b) The *ordinal sum* $\mathbb{P} \oplus \mathbb{Q}$ of $\mathbb{P}$ and $\mathbb{Q}$ is the poset on $X \mathring{\cup} Y$ such that $x \leq y$ in $\mathbb{P} \oplus \mathbb{Q}$ if one of the following conditions holds:

$$\text{(i) } x, y \in X \text{ and } x \leq y \text{ in } \mathbb{P}, \quad \text{(ii) } x, y \in Y \text{ and } x \leq y \text{ in } \mathbb{Q}, \quad \text{(iii) } x \in X \text{ and } y \in Y.$$

We now state the main result of [14].

THEOREM 2.4. *Let $\mathbb{P}$ be a poset on $[n]$.*

(a) *A poset forces the linearity of isometries over $\mathbb{F}_2$ if and only if it is a disjoint union of chains of cardinality 2 and an antichain.*
(b) *A poset forces the linearity of isometries over $\mathbb{F}_3$ if and only if it is an antichain.*
(c) *There is no poset which forces the linearity of isometries over $\mathbb{F}_q$ if $q > 3$.*

We close this section with some additional research problems. The structure of the isometry group of a poset metric space is determined only for special cases. We refer to [9] for the classical case. We mention that in an unpublished paper [15], the structure of the isometry group is determined when $\mathbb{P}$ is the chain of cardinality n.

PROBLEM. Determine the structure of $\mathrm{Iso}_{\mathbb{P}}(\mathbb{F}_q^n)$.

So far, we have considered the structure of automorphism and isometry groups of the total space. We can generalize this concept to an arbitrary poset code, namely, if $C \subset (\mathbb{F}_q^n, d_{\mathbb{P}})$ is a poset code, we define its $\mathbb{P}$-automorphism (resp. $\mathbb{P}$-isometry) group by

$$\mathrm{Aut}_{\mathbb{P}}(C) = \{f : \mathbb{F}_q^n \to \mathbb{F}_q^n \mid f \text{ is linear, } f(C) = C,\ w_{\mathbb{P}}(x) = w_{\mathbb{P}}(f(x)) \text{ for all } x \in C\},$$
$$\mathrm{Iso}_{\mathbb{P}}(C) = \{f : \mathbb{F}_q^n \to \mathbb{F}_q^n \mid f(C) = C,\ w_{\mathbb{P}}(x) = w_{\mathbb{P}}(f(x)) \text{ for all } x \in C\}.$$

PROBLEM. Determine the structure of $\mathrm{Aut}_{\mathbb{P}}(C)$ and $\mathrm{Iso}_{\mathbb{P}}(C)$.

3. MDS poset codes

In [8] and [32], the authors introduced the notion of a maximum distance separable (or simply MDS) code in the ρ-metric space and studied the properties of such codes. Since the ρ-metric is a special type of a poset metric, it is natural to generalize the concept of MDS code to general poset metric spaces. This is done in this section. It turns out that most of the properties of MDS codes with respect to the ρ-metric discussed in [8] and [32] can be generalized to an arbitrary MDS poset code. We first introduce the Singleton bound for poset codes and define MDS poset codes as linear codes which attain the Singleton bound. Next, we introduce the concept of I-perfect codes and relate the MDS poset codes to I-perfect poset codes. We study the weight distribution of an MDS poset code. We show that the weight distribution of an MDS poset code, as in the case of an MDS code in the Hamming metric, is completely determined. We also prove the duality theorem which states that a linear code C is an MDS $\mathbb{P}$-code if and only if $C^{\perp}$ is an MDS $\overline{\mathbb{P}}$-code, where $C^{\perp}$ is the dual code of C and $\overline{\mathbb{P}}$ is the dual poset of $\mathbb{P}$. We first fix some notation which will be used throughout this section.

The minimum $\mathbb{P}$-distance between two distinct elements of C is denoted by $d_{\mathbb{P}}(C)$. In particular, if C is a subspace of of dimension k (resp. with $d_{\mathbb{P}}(C) = d_{\mathbb{P}}$), then C is called an $[n,k]_q$ (resp. $[n,k,d_{\mathbb{P}}]_q$) $\mathbb{P}$-*code.* Sometimes it is necessary to think of C as a code in the Hamming metric space. We use the term $[n,k]_q$ (resp. $[n,k,d_H]_q$) *code* for a linear code of length n and dimension k (resp. with the minimum Hamming distance d_H).

We start with a proposition which is called the *Singleton bound* for poset codes.

Proposition 3.1. *Let $\mathbb{P}$ be a poset on $[n]$ and C $(\subseteq \mathbb{F}_q^n)$ be a $\mathbb{P}$-code. Then*

$$|C| \leq q^{n-d_{\mathbb{P}}(C)+1}.$$

Definition 3.2. Let $\mathbb{P}$ be a poset on $[n]$. A linear code C over $\mathbb{F}_q$ is called a *maximum distance separable* $\mathbb{P}$*-code* (or simply *MDS* $\mathbb{P}$*-code*) if it attains the Singleton bound. Consequently, an $[n,k]_q$ code is an MDS $\mathbb{P}$-code if and only if $d_{\mathbb{P}}(C) = n-k+1$.

We now introduce the notion of I-ball (resp. I-sphere) and r-ball (resp. r-sphere) which will be useful in our investigations.

Definition 3.3. Let $\mathbb{P}$ be a poset on $[n]$, I an order ideal of $\mathbb{P}$, and r a nonnegative integer. For $u \in \mathbb{F}_q^n$, we define the *I-ball* (resp. *I-sphere*) centered at u to be the set

$$B_I(u) = \{v \in \mathbb{F}_q^n \mid \langle \mathrm{Supp}(u-v)\rangle \subseteq I\},$$
$$(\text{resp. } S_I(u) = \{v \in \mathbb{F}_q^n \mid \langle \mathrm{Supp}(u-v)\rangle = I\}).$$

Similarly, we define the r-ball (resp. r-sphere) centered at u to be the set

$$B_r(u) = \{v \in \mathbb{F}_q^n \mid |\langle \mathrm{Supp}(u-v)\rangle| \leq r\}$$
$$(\text{resp. } S_r(u) = \{v \in \mathbb{F}_q^n \mid |\langle \mathrm{Supp}(u-v)\rangle| = r\}).$$

In [1], [13], the authors defined an r-error-correcting perfect poset code as a code for which the r-balls centered at the codewords cover the whole space without overlapping. We modify this notion and introduce the notion of I-perfect poset codes, where I is an order ideal of $\mathbb{P}$.

Definition 3.4. Let $\mathbb{P}$ be a poset on $[n]$ and I an order ideal of $\mathbb{P}$. A linear $\mathbb{P}$-code C over $\mathbb{F}_q$ of length n is called *I-perfect* if the I-balls centered at the codewords of C are pairwise disjoint and their union is $\mathbb{F}_q^n$, i.e. $\mathbb{F}_q^n = \mathring{\bigcup}_{x\in C} B_I(x)$, where the union is disjoint.

Now, we are in a position to relate MDS poset codes to I-perfect poset codes. We denote by $\mathcal{I}^s(\mathbb{P})$ the set of order ideals of $\mathbb{P}$ of cardinality s.

Theorem 3.5. *Let $\mathbb{P}$ be a poset on $[n]$ and C an $[n,k]_q$ $\mathbb{P}$-code. Then C is an MDS $\mathbb{P}$-code if and only if C is an I-perfect $\mathbb{P}$-code for all $I \in \mathcal{I}^{n-k}(\mathbb{P})$.*

Let $\mathbb{P}$ be a poset on $[n]$ and C be a $\mathbb{P}$-code of length n over $\mathbb{F}_q$. The number of codewords of $\mathbb{P}$-weight r is denoted by $A_{r,\mathbb{P}}(C)$, i.e.,

$$A_{r,\mathbb{P}}(C) = |\{x \in C \mid w_{\mathbb{P}}(x) = r\}| = |S_{r,\mathbb{P}}(\mathbf{0}) \cap C|.$$

We call the $(n+1)$-tuple $\{A_{0,\mathbb{P}}, A_{1,\mathbb{P}}, \ldots, A_{n,\mathbb{P}}\}$ the $\mathbb{P}$-*weight distribution* of C. The next theorem states, as in the case of the Hamming metric, that the $\mathbb{P}$-weight distribution of an MDS $\mathbb{P}$-code is completely determined.

Theorem 3.6. *Let $\mathbb{P}$ be a poset on $[n]$ and C an $[n,k,d_{\mathbb{P}}]_q$ $\mathbb{P}$-code. Suppose that C is an MDS $\mathbb{P}$-code. Then*

$$A_{r,\mathbb{P}}(C) = \begin{cases} 1 & \text{if } r = 0, \\ 0 & \text{if } 1 \leq r \leq d_{\mathbb{P}}-1, \\ (q-1)\displaystyle\sum_{I\in\mathcal{I}^r(\mathbb{P})}\sum_{j=0}^{r-d_{\mathbb{P}}}(-1)^j\binom{|M(I)|-1}{j}q^{r-d_{\mathbb{P}}-j} & \text{if } r \geq d_{\mathbb{P}}. \end{cases}$$

To describe the main result of [13], we need some terminology from the theory of posets.

Definition 3.7. For a given poset $\mathbb{P}$, the *dual poset* $\overline{\mathbb{P}}$ is defined as follows: $\mathbb{P}$ and $\overline{\mathbb{P}}$ have the same underlying set and $x \le y$ in $\overline{\mathbb{P}} \Leftrightarrow y \le x$ in $\mathbb{P}$.

The main result of [13] is

Theorem 3.8. *Let $\mathbb{P}$ be a poset on $[n]$ and $\overline{\mathbb{P}}$ be its dual poset. Let C be an $[n,k]_q$ $\mathbb{P}$-code. Then the following statements are equivalent:*

(a) *C is an MDS $\mathbb{P}$-code.*
(b) *C is an I-perfect $\mathbb{P}$-code for all $I \in \mathcal{I}^{n-k}(\mathbb{P})$.*
(c) *$C^{\perp}$ is an MDS $\overline{\mathbb{P}}$-code.*

Moreover, the $\mathbb{P}$-weight distribution of an MDS code is completely determined.

In [7] the authors introduced the concept of near-MDS codes and showed that many good properties of classical MDS codes can be extended to the class of near-MDS codes.

Problem. Develop the theory of near-MDS poset codes.

4. Perfect poset codes

A code is called *perfect* if the spheres of the same radius centered at the codewords cover the whole space without overlapping. The problem of classifying all perfect codes is one of the basic research problems in coding theory, and it still contains many interesting unsolved problems. One drawback in the theory of perfect codes is that perfect codes are very rare. As will be seen, the theory of perfect codes becomes rich in poset codes.

There are two types of problems in the theory of perfect poset codes:

(a) Let $\{\mathbb{P}_n\}$ be a family of posets parameterized by $n \in \mathbb{N}$. Classify all perfect $\mathbb{P}_n$-codes.
(b) (dual of (a)) Let C be a subset of $\mathbb{F}_q^n$ of 'good' shape. Classify all posets which admit C as a perfect code.

We start with the definition of a perfect poset code.

Let x be a vector in $\mathbb{F}_q^n$ and r a nonnegative integer. The $\mathbb{P}$-*sphere* with center x and radius r is defined as the set

$$S_{\mathbb{P}}(x;r) = \{y \in \mathbb{F}_q^n \mid d_{\mathbb{P}}(x,y) \le r\}.$$

Definition 4.1. A poset code C is called an *r-error-correcting perfect $\mathbb{P}$-code* if the $\mathbb{P}$-spheres of radius r centered at the codewords of C are pairwise disjoint, and their union is $\mathbb{F}_q^n$.

Let C be a perfect $\mathbb{P}$-code and $\mathbb{P}'$ be a poset equivalent to $\mathbb{P}$ (resp. let C be a perfect $\mathbb{P}$-code and C' be a code equivalent to C). Then, in general, it is not true that C is also a perfect $\mathbb{P}'$-code (resp. C' is also a perfect $\mathbb{P}$-code). We say that C is a *strongly perfect $\mathbb{P}$-code* if every code equivalent to C is a perfect $\mathbb{P}$-code (or equivalently, C is $\mathbb{P}'$-perfect for every poset $\mathbb{P}'$ which is equivalent to $\mathbb{P}$).

The following proposition gives a necessary and sufficient condition for a given linear code to be an r-error-correcting perfect $\mathbb{P}$-code. We refer to [13] for a proof.

Proposition 4.2. *Let C be an $[n,k]$ binary linear code. Then C is an r-error-correcting perfect $\mathbb{P}$-code if and only if the following two conditions are satisfied:*

(i) (*the sphere packing condition*) $|S_{\mathbb{P}}(0;r)| = 2^{n-k}$,

(ii) (*the partition condition*) *for any nonzero codeword c and any partition $\{x,y\}$ of c, either $w_{\mathbb{P}}(x) \geq r+1$ or $w_{\mathbb{P}}(y) \geq r+1$.*

In the next two subsections, we treat each type of problems separately.

4.1. Classification of perfect codes over the crown poset metric. In this subsection, we wish to classify all perfect $\mathbb{P}_n$-codes where $\{\mathbb{P}_n\}$ is a family of posets parameterized by $n \in \mathbb{N}$. If $\mathbb{P}_n$ is the antichain of cardinality n, the problem is reduced to the classical problem of the classification of perfect codes (cf. [24]). From now on, we consider the case where $\mathbb{P}_n$ is a crown poset described in Section 2. In the remainder of this subsection, $\mathbb{P}$ denotes a crown poset.

For the single-error-correcting case, we have the following theorem. We refer to [1] for a proof.

Theorem 4.3. *Every 1-error-correcting perfect linear $\mathbb{P}$-code over $\mathbb{F}_q$ has parameters $[2m, 2m-l]$, $m = (q^l-1)/(q-1)$ and $l \geq 2$. Furthermore, for $l \geq 2$, there exists a 1-error-correcting perfect linear P-code over $\mathbb{F}_q$ with parameters $[2m, 2m-l, 3]$.*

For the double-error-correcting case, we can derive the following two theorems by solving Ramanujan–Nagell type Diophantine equations which are derived from the sphere packing condition.

Theorem 4.4. *Every 2-error-correcting perfect linear $\mathbb{P}$-code over $\mathbb{F}_2$ has parameters $[4,2]$ or $[10,6]$. Moreover, there is a 2-error-correcting perfect linear $\mathbb{P}$-code over $\mathbb{F}_2$ with the given parameters.*

Theorem 4.5. *Every 2-error-correcting perfect linear $\mathbb{P}$-code over $\mathbb{F}_3$ has parameters $[4,2]$ or $[22,17]$. Moreover there is a 2-error-correcting perfect linear P-code over $\mathbb{F}_3$ with the given parameters.*

By an analysis of the parity check matrix and an application of the Johnson bound, we obtain the following result for the triple-error-correcting case. We refer to [20] for a proof.

Theorem 4.6. *There are no triple-error-correcting perfect binary linear $\mathbb{P}$-codes.*

By an extension of techniques developed in [20], one can prove that there are no 4-error-correcting binary perfect $\mathbb{P}$-codes (cf. [19]). This suggests the following problem.

Problem. Classify all perfect linear codes with a crown poset structure.

4.2. Classification of posets for which a given code is perfect. In [4], the authors gave an example of a poset for which the extended binary Hamming code of length 8 is a double-error-correcting perfect code. This example motivates the following problem which is the theme of this subsection:

For a given poset code C, classify all posets for which C is a perfect code.

We present some result along these lines for the case when C is the extended binary Hamming code or the extended binary Golay code. To state the result, it is necessary to fix some notation.

Let $\mathbb{P}$ be a poset with the ground set $[n] = \{1, \ldots, n\}$. As usual we use a Hasse diagram to represent P graphically. To describe $\mathbb{P}$ in words, we introduce the following subsets of $\mathbb{P}$. For an integer i, $1 \leq i \leq n$, and elements $a_1, \ldots, a_l$ of $[n]$, we define

$$\begin{aligned}
\Gamma^{(i)}(\mathbb{P}) &= \{x \in \mathbb{P} \mid |\langle x \rangle| = i\},\\
\Gamma_{a_1,\ldots,a_l}(\mathbb{P}) &= \{x \in \mathbb{P} \mid x > a_j,\ j = 1, \ldots, l\},\\
\Gamma^{(i)}_{a_1,\ldots,a_l}(\mathbb{P}) &= \Gamma^{(i)}(\mathbb{P}) \cap \Gamma_{a_1,\ldots,a_l}(\mathbb{P}).
\end{aligned}$$

The extended binary Hamming codes. In this subsection, we classify up to equivalence the poset structures on $[n]$, $n = 2^m$, for which the extended binary Hamming code $\widetilde{H}_m$ $(m \geq 2)$ is a double-error-correcting perfect P-code.

Before considering the double-error-correcting case, we give a simple observation on the single-error-correcting case. Note that $S_{\mathbb{P}}(c; 1) \subseteq S_H(c; 1)$ for any poset $\mathbb{P}$, where $S_H(c; 1)$ denotes the Hamming sphere of radius 1 centered at c. Since the Hamming spheres of radius 1 centered at the codewords of $\widetilde{H}_m$ do not cover the whole space $\mathbb{F}_2^n$ in the Hamming metric, there are no poset structures on $[n]$ for which the extended binary Hamming code is a single-error-correcting perfect $\mathbb{P}$-code.

We now state the main result on the double-error-correcting case.

Theorem 4.7. *Let $m \geq 3$ be an integer, and $\widetilde{H}_m$ denote the extended binary Hamming code with parameters $[n = 2^m, 2^m - 1 - m, 4]_H$.*

(a) *$\widetilde{H}_m$ is a double-error-correcting strongly perfect $\mathbb{P}$-code if and only if P is equivalent to a poset depicted in* (i) *or* (ii) *of Figure* 2.
(b) *$\widetilde{H}_m$ is a double-error-correcting perfect but not strongly perfect $\mathbb{P}$-code if and only if $\mathbb{P}$ is equivalent to a poset depicted in* (iii) *of Figure* 2.

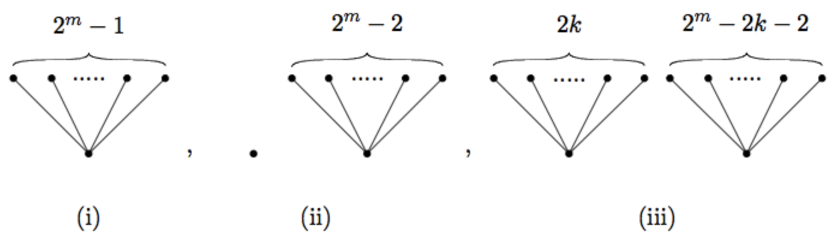

Fig. 2. The poset structures for which the extended binary Hamming code is a double-error-correcting perfect code

We next consider the triple-error-correcting case. It is well-known (cf. [24]) that the codewords of $\widetilde{H}_m$ of weight 4 form a Steiner system $S(3, 2^m, 4)$. Let $\mathbb{P}$ be a poset on $[n]$, $n = 2^m$, for which $\widetilde{H}_m$ is a 3-error-correcting perfect $\mathbb{P}$-code. For a positive integer $r \geq 3$, we define a set $M_r(\mathbb{P})$ (or simply M_r) as follows: $M_r = \{x \mid x \text{ is an } r\text{-subset of } [n] \text{ such that } w_P(x) \le r\}$. Using the fact that the codewords of $\widetilde{H}_m$ of weight 4 form a Steiner system, we define a map $\Phi : M_3 \to \bigcup_{i=4}^{\infty} \Gamma^{(i)}$ as follows: For $x = \{i,j,k\} \in M_3$, there is a unique l such that $\{i,j,k,l\}$ is a codeword of $\widetilde{H}_m$. By the partition condition in Proposition 4.2, $l \in \bigcup_{i=4}^{\infty} \Gamma^{(i)}$. We define $\Phi(x) = l$.

The following theorem is useful in our classification.

THEOREM 4.8. *The extended binary Hamming code $\widetilde{H}_m$ is a 3-error-correcting perfect $\mathbb{P}$-code if and only if the following conditions are satisfied:*

(i) $|\Gamma^{(1)}| \le 3$,
(ii) *the map $\Phi : M_3 \to \bigcup_{i=4}^{\infty} \Gamma^{(i)}$ gives a one-to-one correspondence,*
(iii) *for any codeword c of $\widetilde{H}_m$ of weight 4, and any partition $\{x,y\}$ of c such that $w_H(x) = w_H(y) = 2$, we have either $w_{\mathbb{P}}(x) \ge 4$ or $w_{\mathbb{P}}(y) \ge 4$.*

As an illustration of our theorem, we classify the poset structures for which the extended binary Hamming code $\widetilde{H}_m$ is a 3-error-correcting perfect P-code when $m = 3, 4$. It follows from Theorem 4.8 that $\mathbb{P}$-perfectness of $\widetilde{H}_m$ mainly depends on the substructure $\bigcup_{i=1}^{3} \Gamma^{(i)}(\mathbb{P})$ of $\mathbb{P}$. Therefore, we introduce the structure vector (a,b,c) of $\mathbb{P}$, where $a = |\Gamma^{(1)}|$, $b = |\Gamma^{(2)}|$, $c = |\Gamma^{(3)}|$. By an analysis of structure vectors, we can list all possible poset structures for which the extended binary Hamming code $\widetilde{H}_m$ is a 3-error-correcting perfect $\mathbb{P}$-code when $m = 3$. Recall that $\widetilde{H}_3$ is an $[8,4,4]_H$ code with the parity check matrix H_3, where H_3 is given by

$$\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}. \tag{1}$$

We summarize our calculation as in the following theorem.

THEOREM 4.9. *Let $\widetilde{H}_3$ denote the binary extended $[8,4,4]_H$ Hamming code with the parity check matrix given in* (1). *Then $\widetilde{H}_3$ is a 3-error-correcting perfect $\mathbb{P}$-code if and only if the Hasse diagram of $\bigcup_{i=1}^{3} \Gamma^{(i)}(\mathbb{P})$ is equivalent to one of the seven possibilities depicted in Figure 3 below.*

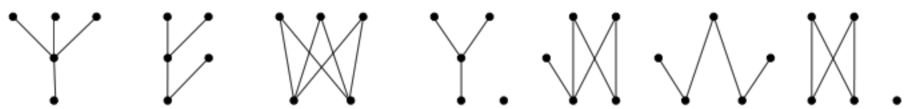

Fig. 3. The $\Gamma^{(1)} \cup \Gamma^{(2)} \cup \Gamma^{(3)}$ structure of posets for which $\widetilde{H}_3$ is a 3-error-correcting perfect code

Finally, we list all possible poset structures for which the extended binary Hamming code $\widetilde{H}_m$ is a 3-error-correcting perfect $\mathbb{P}$-code when $m = 4$. Recall that $\widetilde{H}_4$ is an

$[16,11,4]_H$ code with the parity check matrix H_4, where H_4 is given by

$$\begin{array}{c}
\begin{array}{cccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \end{array} \\
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1
\end{pmatrix}
\end{array}. \tag{2}$$

We summarize our calculation in the following theorem.

THEOREM 4.10. *Let $\widetilde{H}_4$ denote the binary extended $[16,11,4]_H$ Hamming code with the parity check matrix* (2). *Then $\widetilde{H}_4$ is a* 3*-error-correcting perfect $\mathbb{P}$-code if and only if the Hasse diagram of $\bigcup_{i=1}^{3} \Gamma^{(i)}(\mathbb{P})$ is equivalent to one of the* 17 *possibilities depicted in Figure* 4 *below.*

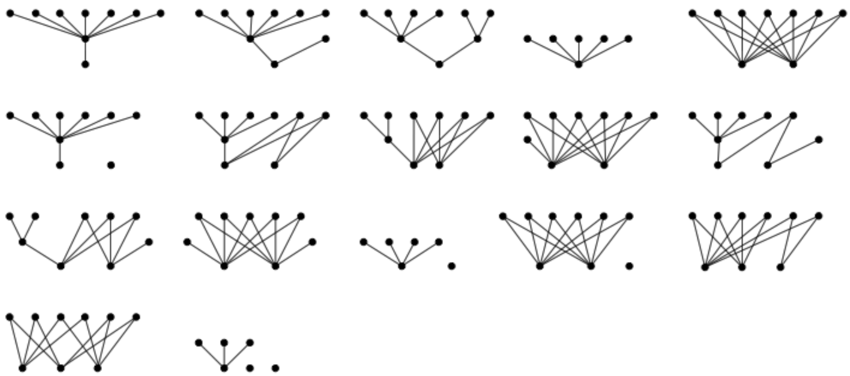

Fig. 4. The $\Gamma^{(1)} \cup \Gamma^{(2)} \cup \Gamma^{(3)}$ structure of posets for which $\widetilde{H}_4$ is a 3-error-correcting perfect code

The extended binary Golay code. In this subsection, we classify up to equivalence the poset structures on $[24] = \{1, 2, \ldots, 24\}$ for which the extended binary Golay code is a 4 or 5-error-correcting perfect code.

As in the case of extended Hamming code, one can easily verify that there are no poset structures on [24] for which the extended binary Golay code is an r-error-correcting perfect code when $1 \leq r \leq 3$.

THEOREM 4.11. *Let $\mathcal{G}_{24}$ denote the extended binary Golay code of length* 24*. Then $\mathcal{G}_{24}$ is a* 4*-error-correcting perfect $\mathbb{P}$-code if and only if $\mathbb{P}$ is equivalent to $\mathbb{P}_s$ for some s, $1 \leq s \leq 4$. Moreover, the perfectness is strong.*

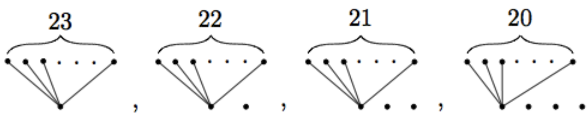

Fig. 5. The poset structures for which the extended Golay code $\mathcal{G}_{24}$ is a 4-error-correcting perfect code

The following is the main theorem of [11].

THEOREM 4.12. *There are no poset structures for which the extended binary Golay code is a 5-error-correcting perfect poset code.*

It is known that there is a poset for which the extended binary Golay code is an r-error-correcting code when $r = 10, 11, 12$.

PROBLEM. Classify posets on $[24] = \{1, 2, \ldots, 24\}$ for which the extended binary Golay code is a perfect code.

5. MacWilliams identity for poset codes

In this subsection, we study the MacWilliams identity for poset weight enumerators. All poset structures which admit the MacWilliams identity are classified. It will be proved that being a hierarchical poset is a necessary and sufficient condition for a poset to admit the MacWilliams identity. An explicit relation is also derived between the $\mathbb{P}$-weight distribution of a hierarchical poset code and the $\overline{\mathbb{P}}$-weight distribution of the dual code.

Let $\mathcal{C} \subseteq \mathbb{F}_q^n$ be a linear code in the Hamming metric space. One useful way to study C is to study its weight enumerator

$$W_C(x) = \sum_{u \in C} x^{w_H(u)} = \sum_{i=0}^{n} A_i x^i,$$

where A_i denotes the number of codewords of weight i. We introduce the $\mathbb{P}$-weight enumerator of C by

$$W_{C,\mathbb{P}}(x) = \sum_{u \in C} x^{w_{\mathbb{P}}(u)} = \sum_{i=0}^{n} A_{i,\mathbb{P}} x^i,$$

where $A_{i,\mathbb{P}} = |\{u \in C \mid w_{\mathbb{P}}(u) = i\}|$.

The MacWilliams identity for linear codes over $\mathbb{F}_q$ is one of the most important identities in coding theory; it expresses the Hamming weight enumerator of the dual code $\mathcal{C}^{\perp}$ of a linear code $\mathcal{C}$ over $\mathbb{F}_q$ in terms of the Hamming weight enumerator of $\mathcal{C}$. Since the Hamming metric is a special case of poset metrics, it is natural to attempt to obtain MacWilliams-type identities for certain $\mathbb{P}$-weight enumerators. Essentially, what enables us to obtain the MacWilliams identity for the Hamming metric is that the Hamming weight enumerator of the dual code $\mathcal{C}^{\perp}$ is uniquely determined by that of $\mathcal{C}$. The following example suggests that we need some modification to generalize the MacWilliams identity to poset weight enumerators.

EXAMPLE 5.1. Let $\mathbb{P} = \{1, 2, 3\}$ be a poset with order relation $1 < 2 < 3$ and $\overline{\mathbb{P}} = \{1, 2, 3\}$ be a poset with order relation $1 > 2 > 3$. Consider the following binary linear $\mathbb{P}$-codes:

$$\mathcal{C}_1 = \{(0,0,0), (0,0,1)\}, \qquad \mathcal{C}_2 = \{(0,0,0), (1,1,1)\}.$$

It is easy to check that the $\mathbb{P}$-weight enumerators of $\mathcal{C}_1$ and $\mathcal{C}_2$ are given by

$$W_{\mathcal{C}_1,\mathbb{P}}(x) = 1 + x^3 = W_{\mathcal{C}_2,\mathbb{P}}(x).$$

The dual codes of $\mathcal{C}_1$ and $\mathcal{C}_2$ are

$$\mathcal{C}_1^\perp = \{(0,0,0),(1,0,0),(0,1,0),(1,1,0)\},$$
$$\mathcal{C}_2^\perp = \{(0,0,0),(1,1,0),(1,0,1),(0,1,1)\}.$$

The $\mathbb{P}$-weight enumerators of $\mathcal{C}_1^\perp$ and $\mathcal{C}_2^\perp$ are

$$W_{\mathcal{C}_1^\perp,\mathbb{P}}(x) = 1 + x + 2x^2, \qquad W_{\mathcal{C}_2^\perp,\mathbb{P}}(x) = 1 + x^2 + 2x^3,$$

while the $\overline{\mathbb{P}}$-weight enumerators of $\mathcal{C}_1^\perp$ and $\mathcal{C}_2^\perp$ are

$$W_{\mathcal{C}_1^\perp,\overline{\mathbb{P}}}(x) = 1 + x^2 + 2x^3 = W_{\mathcal{C}_2^\perp,\overline{\mathbb{P}}}(x).$$

As it is seen above, although the $\mathbb{P}$-weight enumerators of the codes $\mathcal{C}_1$ and $\mathcal{C}_2$ are the same, the $\mathbb{P}$-weights of the dual codes may be different. Fortunately, however, the $\overline{\mathbb{P}}$-weight enumerators of the dual codes are the same. This motivates the following definition.

Definition 5.2. Let $\mathbb{P}$ be a poset on $[n]$. We say that $\mathbb{P}$ *admits the MacWilliams identity* if the $\overline{\mathbb{P}}$-weight enumerator of the dual code $\mathcal{C}^\perp$ of a linear code $\mathcal{C}$ over $\mathbb{F}_q$ is uniquely determined by the $\mathbb{P}$-weight enumerator of $\mathcal{C}$, where $\overline{\mathbb{P}}$ denotes the dual poset of $\mathbb{P}$.

Our first goal is to derive a necessary condition for a poset to admit the MacWilliams identity. A hierarchical poset is introduced as an ordinal sum of antichains. Let $n_1, \ldots, n_t$ be positive integers with $n_1 + \cdots + n_t = n$. We define the poset $\mathbb{H}(n; n_1, \ldots, n_t)$ on the set $\{(i,j) \mid 1 \le i \le t,\ 1 \le j \le n_i\}$ whose order relation is given by

$$(i,j) < (l,m) \;\Leftrightarrow\; i < l.$$

The poset $\mathbb{H}(n; n_1, \ldots, n_t)$ is called a *hierarchical poset* with t levels and n elements. For each $1 \le i \le t$, the subset $\{(i,j) \mid 1 \le j \le n_i\}$ of $\mathbb{H}(n; n_1, \ldots, n_t)$ is called the ith *level set* of $\mathbb{H}(n; n_1, \ldots, n_t)$, and it is denoted by $\Gamma^i(\mathbb{H})$. Note that $\Gamma^i(\mathbb{H})$ is an antichain with cardinality n_i.

Using combinatorial and inductive arguments, we obtain the following theorem.

Theorem 5.3. *If $\mathbb{P}$ admits the MacWilliams identity, then $\mathbb{P}$ is a hierarchical poset.*

Next we will derive the MacWilliams identity for a hierarchical poset code. Since hierarchical poset codes have multi-levels, it makes the problem difficult. To overcome this, we first introduce the 'leveled' $\mathbb{P}$-weight enumerator $W_{\mathcal{C},\mathbb{P}}(x : y_0, y_1, \ldots, y_t)$ and obtain an equation which relates $W_{\mathcal{C}^\perp,\overline{\mathbb{P}}}(x : z_{t+1}, z_t, \ldots, z_1)$ to variations of the leveled $\mathbb{P}$-weight enumerator of $\mathcal{C}$. By specializing this equation, we can obtain the MacWilliams identity for a hierarchical poset code, and prove that our necessary condition in Theorem 5.3 is also sufficient.

The following theorem is the main theorem of [21].

Theorem 5.4. *A poset $\mathbb{P}$ admits the MacWilliams identity if and only if $\mathbb{P}$ is a hierarchical poset.*

In [22] the authors proved that P is a hierarchical poset if and only if it admits the association scheme. They use this fact to derive the second proof of Theorem 5.4.

Problem. Derive the proof for the MacWilliams identity for hierarchical posets codes using the theory of matroids.

Finally we give an explicit relation between the $\mathbb{P}$-weight distribution of a hierarchical poset code and the $\overline{\mathbb{P}}$-weight distribution of the dual code.

Let $\mathbb{P} = \mathbb{H}(n; n_1, \ldots, n_t)$ be a hierarchical poset of n elements with t levels and $\overline{\mathbb{P}}$ be its dual poset. Let $\mathcal{C}$ be a linear $\mathbb{P}$-code of length n over $\mathbb{F}_q$, and let $\{A_{i,\mathbb{P}}\}_{i=0,\ldots,n}$ (resp. $\{A'_{i,\overline{\mathbb{P}}}\}_{i=0,\ldots,n}$) be the weight distributions of the $\mathbb{P}$(resp. $\overline{\mathbb{P}}$)-code $\mathcal{C}$ (resp. $\mathcal{C}^{\perp}$), that is, $A_{i,\mathbb{P}} = |\{u \in \mathcal{C} \mid w_{\mathbb{P}}(u) = i\}|$ while $A'_{i,\overline{\mathbb{P}}} = |\{v \in \mathcal{C}^{\perp} \mid w_{\overline{\mathbb{P}}}(v) = i\}|$.

Theorem 5.5. *Let $\mathbb{P} = \mathbb{H}(n; n_1, \ldots, n_t)$ be a hierarchical poset of n elements with t levels and $\mathcal{C}$ be a linear $\mathbb{P}$-code of length n over $\mathbb{F}_q$. Then, for each $0 \leq i \leq t-1$, $1 \leq k \leq n_{i+1}$,*

$$A'_{n_t+\cdots+n_{i+2}+k,\overline{\mathbb{P}}} = \frac{q^{\widehat{n_{i+1}}}}{|\mathcal{C}|}\left(\sum_{j=1}^{n_{i+1}} P_k(j : n_{i+1}) A_{n_1+\cdots+n_i+j,\mathbb{P}} + \sum_{j=0}^{n_1+\cdots+n_i} \binom{n_{i+1}}{k} \gamma^k A_{j,\mathbb{P}}\right),$$

where $P_k(x : n)$ denotes the Krawtchouk polynomial.

Remark 5.6. In [16], the author derived the MacWilliams identity for hierarchical poset codes by a different method.

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