1. Introduction

The present work is devoted to the study of the t-deformed free probability. The notion of the t-deformation of a measure and of a convolution, inspired by the conditionally free convolution of Bożejko, Leinert and Speicher [Bo1, BS2, BLS], was first introduced in the papers by Bożejko and Wysoczański [BW1, BW2]. We propose for the t-deformed free probability the name of Kesten probability, justified by the fact that the measure arising in the corresponding central limit theorem is, for a suitable choice of parameters, exactly the spectral measure of a random walk on the free group with a finite number of generators, as discovered by Kesten in [K] (see also [PS]).

In the second chapter we recall the necessary definitions and basic facts from probability, especially relating to the Cauchy transforms and their reciprocals, as they are needed for analytic descriptions of the convolutions in question. In the next chapter we gather the definitions of the t-deformation of measures and of the t-deformed free convolution $\boxed{t}$ and relate them to the free, conditionally free and boolean cases. We present the $R^{\boxed{t}}$ transform, which linearizes the convolution $\boxed{t}$. We then recall two fundamental limit theorems, the central limit theorem and the Poisson limit theorem. We recalculate the Cauchy transform of the Poisson limit measure from first principles, as the limit of convolution powers of $\mu_N = \left(1 - \frac{\lambda}{N}\right)\delta_0 + \frac{\lambda}{N}\delta_1$, and then get the explicit form of the measure from a result of Saitoh and Yoshida on the corresponding orthogonal polynomials. We then discuss the infinite divisibility with respect to the convolution $\boxed{t}$, and we establish a Lévy–Khinchin formula. We conclude the third chapter by proving that for $t > 0$ all probability measures μ have the Nica–Speicher property, that is, one can find their convolution power $\mu^{\boxed{t}s}$ for all $s \geq 1$. This behaviour is similar to the free case, as in the original paper of Nica and Speicher [NS], and different from the boolean case (when $t = 0$) for which the property is satisfied for all $s > 0$.

In the fourth chapter we construct generalized Brownian motions, parametrized by a pair (t, q), $0 < t < 1$, $-1 \leq q < 1$. Such a process is a family of operators $\omega(\tau)$, $\tau \in \mathbb{R}$, in an appropriate noncommutative probability space. To construct it we first consider a Fock-type Hilbert space on which we define the creation and annihilation algebra generated by the creation and annihilation operators $c^*(h)$ and $c(h)$, $h \in \mathcal{H} = \mathcal{K}_{\mathbb{C}}$, where $\mathcal{K}$ is a real Hilbert space and $\mathcal{K}_{\mathbb{C}}$ its complexification. We show that the vacuum state $\varrho(a) = \langle \Omega, a\Omega \rangle$ on this algebra is determined by a function on pair partitions of an ordered set. We note that from the form of this pairing prescription it follows that the Gaussian elements $\omega(k) = c^*(k) + c(k)$, $k \in \mathcal{K} \hookrightarrow \mathcal{K}_{\mathbb{C}}$, are Kesten-distributed for $q = -1$ and are q-gaussian as $t \to 0$. We then get the process by identifying $\mathcal{K}$ with $L^2(\mathbb{R})$ and defining $\omega(\tau) = \omega(\chi_{[0,\tau)})$, where $\chi_{[0,\tau)}$ is the characteristic function of the interval $[0, \tau)$. Later in that chapter we present a link between the Kesten-distributed generalized Brownian motions and the reduced free product of Voiculescu. The last part of the chapter is devoted to the calculation of the explicit form of the Mehler kernel for the Kesten measure and to a discussion of its positivity. The Mehler kernel, when positive, defines a classical Markov process, which can be seen as a classical version of the generalized stationary Brownian (Ornstein–Uhlenbeck) process. Moreover, we notice that for those values of the parameter t for which the kernel is not positive, it is impossible to construct the second quantization functor.

The fifth chapter is devoted to a generalization of the t-deformation of the free convolution $\boxplus$ to the t-deformation of the free product $\star$. We first notice that the construction of the conditionally free product can be adapted to give the definition of a t-product of algebras of polynomials in one variable, together with their corresponding states. We then show by a combinatorial approach that we can define products for states on algebras in many noncommutative variables arising in the above way, and that they are positive definite. We conclude this chapter by generalizing a recurrence formula for moments of t-deformed measures to a recurrence formula for moments of t-deformed states on algebras in many noncommutative variables.

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2. Noncommutative probability

The present work is concerned with a particular construction in the general framework of noncommutative probability theory.

In classical probability the fundamental object of study is the triple $(\Omega, \Sigma, \mathbf{P})$ where Ω is a sample space, Σ the σ-field of events and $\mathbf{P}$ a probability measure on (Ω, Σ). The random variables are real-valued measurable functions. The distribution of a random variable X is the measure μ_X defined on the Borel subsets of the real line given by $\mu_X(B) = \mathbf{P}(X^{-1}(B))$ for all Borel sets B. The expectation of a random variable X is thus the expectation of the distribution μ_X.

The random variables form a commutative algebra $\mathcal{A}$ on which one can define the expectation functional $\mathbb{E}$ associated to the probability measure. To determine the distribution of a random variable X one can look at its higher-order moments, that is, the values of the expectation functional $\mathbb{E}$ at $X^n \in \mathcal{A}$. The moment sequence, when all moments are finite, can determine the measure uniquely or not. This distinction was addressed for instance in [Ak]. Although various sufficient conditions are known, there is no explicit characterization of measures determined by their moment sequences. The sufficient condition most important to us is the compactness of the support of the measure.

In noncommutative probability the starting point is the above algebraic property of the random variables. The fundamental object of study is the algebra of random variables together with an expectation functional; they need not arise from any triple $(\Omega, \Sigma, \mathbf{P})$. What is more, one can consider noncommutative algebras. Thus, one defines a noncommutative probability space to be a pair $(\mathcal{A}, \varphi)$ where $\mathcal{A}$ is a unital complex $\star$-algebra and φ a linear positive functional such that $\varphi(1) = 1$. A noncommutative random variable is simply an element $X \in \mathcal{A}$. We shall almost always consider self-adjoint random variables $X = X^*$. By the distribution we then understand the moments $\varphi(X^n)$, $n = 0, 1, \ldots$. Since the sequence of moments is positive definite, there exists a probability measure μ on the real line such that $\varphi(X^n) = \int x^n \, d\mu(x)$. In most of what follows we shall assume that the algebra $\mathcal{A}$ is a $C^\star$-algebra, as a result the moment sequences generated by the expectation functional will correspond to measures with compact support. In this way the probability measure associated to any self-adjoint random variable from the algebra will be uniquely determined.

An essential concept in classical probability is independence of random variables and convolution of probability measures, which is the distribution of the sum of independent random variables. These notions can be carried over to the noncommutative framework, they are known there as tensor independence and classical convolution. However, several other kinds of independence together with corresponding convolutions were described:

- free independence together with the free convolution $\boxplus$, introduced by Voiculescu [V1, V2]; those concepts can be traced back to the paper [Av],
- boolean independence and boolean convolution $\uplus$, introduced by Speicher and Woroudi [SW]; they are closely related to the regular free product representation of free product groups of Bożejko (see [Bo2, BLS]),
- conditionally free independence together with the conditionally free convolution $\boxed{c}$ of pairs of measures, introduced by Bożejko, Leinert and Speicher in [BLS, BS3]; see also [Bo1]. The free and boolean cases are contained in this approach through an appropriate choice of the second measure of the pairs.

Before we recall the above notions and discuss the deformations of which the t-deformation of this paper is a prominent example, we need some preliminaries. A good and more comprehensive introduction to noncommutative probability can be found in [HP] or [VDN].

2.1. Basic notions. In the present work we shall be working with probability measures on the real line, the set of which we shall denote by $\mathbf{Prob}(\mathbb{R})$. Let us recall some of the basic definitions and facts that we shall need in the sequel. Since all the theorems and facts presented in this section are well known, we omit the proofs.

2.1.1. *Orthogonal polynomials.* Let $\mu \in \mathbf{Prob}(\mathbb{R})$ be a measure with finite moments of all orders, that is, for all $k \in \mathbb{N}$,

$$|m_\mu(k)| = \left| \int_{-\infty}^{\infty} x^k \, d\mu(x) \right| < \infty,$$

which we denote by $\mu \in \mathbf{Prob}^{(m)}(\mathbb{R})$. For such a measure we can define the corresponding orthonormal polynomials by the classical three-term recurrence formula [Ak]

$$\begin{aligned} p_0(x) &= 1, \qquad p_1(x) = x - a_0, \\ (x - a_n)p_n(x) &= b_n p_{n+1}(x) + b_{n-1}p_{n-1}(x), \end{aligned} \tag{2.1}$$

where we call the numbers $a_n, b_n \in \mathbb{R}$, $b_n \geq 0$, $n = 0, 1, \ldots$, the *Jacobi coefficients.* Symmetric measures with moments are characterized by the property $a_n = 0$. The orthonormal polynomials satisfy the relation

$$\int_{\operatorname{supp}(\mu)} p_j(x)p_k(x)\, d\mu(x) = \delta_{j,k}.$$

Most measures we will consider have compact support; such measures are uniquely determined by their moments. In that case the corresponding orthonormal polynomials form an orthonormal basis of the space $L^2(\mu)$.

2.1.2. *Cauchy transforms.* The most important tool to handle probability measures in noncommutative probability is the Cauchy transform.

DEFINITION 2.1. Let $\mu \in \mathbf{Prob}(\mathbb{R})$. Then the *Cauchy transform* of μ is defined by

$$G_\mu(z) = \int_{-\infty}^{\infty} \frac{d\mu(x)}{z - x} \quad \text{for } z \in \mathbb{C}^+.$$

PROPOSITION 2.2. *The Cauchy transform $G_\mu(z)$ is analytic in the upper half plane and takes values in the lower half plane, $G_\mu : \mathbb{C}^+ \to \mathbb{C}^-$.*

There is an important link between the Cauchy transform of a measure with finite moments of all orders and the corresponding recurrence coefficients of orthogonal polynomials. For such $\mu \in \mathbf{Prob}^{(m)}(\mathbb{R})$ the Cauchy transform can be written in the form of a formal continued fraction:

$$G_\mu(z) = \cfrac{1}{z - a_0 - \cfrac{\lambda_0}{z - a_1 - \cfrac{\lambda_1}{z - a_2 - \cfrac{\lambda_2}{\ddots}}}},$$

where $\lambda_n = (b_n)^2$ and the coefficients a_n are the same as in the recurrence formula (2.1).

If μ has compact support, which we denote $\mu \in \mathbf{Prob}^{(c)}(\mathbb{R})$, the continued fraction converges to the Cauchy transform (for proof see [C, Chapter III, Section 4]); moreover, we have the following theorem ([C, Chapter IV, Theorem 2.2]):

THEOREM 2.3. *A measure $\mu \in \mathbf{Prob}(\mathbb{R})$ has compact support if and only if the coefficients a_i and λ_i are bounded.*

The Cauchy transform $G_\mu(z)$ is also related to $M_\mu(z)$, the generating function of the moments $m_\mu(k)$:

$$\frac{1}{z} G_\mu\left(\frac{1}{z}\right) = M_\mu(z) = \sum_{k=0}^{\infty} m_\mu(k) z^k.$$

An important operation on measures that is well reflected in Cauchy transforms is the dilation.

Definition 2.4. We define the *dilation* of a measure $\mu \in \mathbf{Prob}(\mathbb{R})$ by a factor λ by setting $D_\lambda(\mu)(A) = \mu(\lambda^{-1}A)$ for all Borel subsets $A \subset \mathbb{R}$.

We then have

$$G_{D_\lambda(\mu)}(z) = \cfrac{1}{z - \lambda a_0 - \cfrac{\lambda^2\lambda_0}{z - \lambda a_1 - \cfrac{\lambda^2\lambda_1}{z - \lambda a_2 - \cfrac{\lambda^2\lambda_2}{\ddots}}}} = \frac{1}{\lambda} G_\mu\left(\frac{z}{\lambda}\right).$$

The moments of the measure μ can be calculated from the coefficients of the continued fraction with the use of Theorem 5.1 of [AB]:

Theorem 2.5. *For a probability measure μ with compact support we have*

$$m_\mu(n) = \sum_{\pi \in \mathrm{NC}_{1,2}(n)} \prod_{\substack{B_j \in \pi \\ |B_j|=2}} \lambda_{d(B_j)} \prod_{\substack{B_k \in \pi \\ |B_k|=1}} \alpha_{d(B_j)}$$

where $\mathrm{NC}_{1,2}(n)$ is the set of noncrossing partitions of $\{1, \dots, n\}$ such that for $\pi \in \mathrm{NC}_{1,2}(n)$ its blocks $B_j \in \pi$ have one or two elements, $|B_j|$ is the cardinality of the block B_j, and $d(B_j)$ is its depth.

Definition 2.6. A *partition* of the ordered set $A = \{1, \dots, n\}$ is a set of blocks $B_j \subset A$ such that $B_i \cap B_j = \emptyset$ if $i \neq j$ and $\bigcup B_j = A$. A *crossing* in a partition $V = \{B_1, \dots, B_m\}$ occurs if for some $1 \leq j \leq m$ and $k, l \in B_j, k < l$ there exists B_i and $r, s \in B_i, r < s$ such that $k < r < l < s$ or $r < k < s < l$. A partition is called *noncrossing* if it has no crossings. In a noncrossing partition one defines the *depth* of a block B_j as the number of blocks enveloping B_j, that is, $d(B_j) = \#\{B_i \mid \exists r, s \in B_i,\ r < B_j < s\}$.

2.1.3. *Reciprocals of Cauchy transforms*

Proposition 2.7. *The reciprocal of the Cauchy transform $F_\mu(z) = 1/G_\mu(z) : \mathbb{C}^+ \to \mathbb{C}^+$ is analytic in the upper half plane.*

Complex functions mapping analytically the upper half plane into itself are called *Pick functions*. An elementary introduction to this subject can be found in Chapter 2 of [D], a more detailed treatment is in [AG]. For our purposes, the most important property of Pick functions is the Nevanlinna integral representation theorem.

Theorem 2.8 (Nevanlinna). *A function $F(z)$ is a Pick function if and only if there exist $a, b \in \mathbb{R}$ with $b \geq 0$ and a finite positive measure ϱ such that*

$$F(z) = a + bz + \int_{-\infty}^{\infty} \frac{1 + xz}{x - z}\, d\varrho(x).$$

Moreover, a, b and ϱ are uniquely determined.

It is also possible to easily characterize reciprocals of Cauchy transforms in the class of Pick functions.

THEOREM 2.9 (Nevanlinna). *A function $F(z)$ is the reciprocal of the Cauchy transform of some probability measure $\mu \in \mathbf{Prob}(\mathbb{R})$ if and only if it is a Pick function and $b = 1$ in the Nevanlinna representation*:

$$F(z) = F_\mu(z) = a + z + \int_{-\infty}^{\infty} \frac{1 + xz}{x - z}\, d\varrho(x).$$

2.2. Free probability. We are now in a position to define the most prominent type of noncommutative probability, the free probability. To define independence in classical probability one passes through conditions on sub-σ-fields, it is thus natural that in the noncommutative theory one starts with subalgebras.

DEFINITION 2.10. A family of subalgebras $\mathcal{A}_i \subset \mathcal{A}$ is called *free* if

$$\varphi(a_1 \cdots a_n) = \varphi(a_1) \cdots \varphi(a_n) = 0 \tag{2.2}$$

whenever $\varphi(a_j) = 0$, $a_j \in \mathcal{A}_{i_j}$, $j = 1, \ldots, n$ and $i_1 \neq i_2 \neq \cdots \neq i_n$.

Two random variables are called *free* if they belong to two distinct free subalgebras. For measures μ and ν with compact support, their *free convolution* $\mu \boxplus \nu$ is defined as the distribution of $X + Y \in \mathcal{A}$ where $X, Y \in \mathcal{A}$ are free and have distributions μ and ν respectively. To this concept there corresponds the notion of the free product of noncommutative probability spaces. Given $(\mathcal{A}_1, \varphi_1)$ and $(\mathcal{A}_2, \varphi_1)$ we define $\mathcal{A} = \mathcal{A}_1 \star \mathcal{A}_2$ as the free product with amalgamation of units, that is, the $\star$-algebra generated by the unit and words of the form $a_1^{i_1} b_1^{j_1} \ldots a_n^{i_n} b_n^{j_n}$ where $a_k \in \mathcal{A}_1$, $b_k \in \mathcal{A}_2$, $k, i_k, j_k \in \mathbb{N}$, $i_k, j_k > 0$, $i_1, j_n \geq 0$. The state $\varphi = \varphi_1 \star \varphi_2$ is defined so as to satisfy the relation (2.2). Then we have $\varphi|_{\mathcal{A}_i} = \varphi_i$, the algebras $\mathcal{A}_i$ naturally embedded into $\mathcal{A}$ are free, and if $X \in \mathcal{A}_1$, $Y \in \mathcal{A}_2$ then $m_{\mu_X \boxplus \mu_Y}(n) = \varphi((X+Y)^n)$.

Since the measure $\mu \boxplus \nu$ depends only on the measures μ and ν, it is essential to be able to describe it only in terms of μ and ν. This is done with the use of the R-transforms $R_\mu^\boxplus(z)$, $R_\nu^\boxplus(z)$, the analogues of the logarithm of the Fourier transform in classical probability. If we define

$$R_\mu^\boxplus(z) = G_\mu^{-1}(z) - 1/z, \tag{2.3}$$

where $G_\mu^{-1}(z)$ is the right inverse of $G_\mu(z)$ with respect to composition of functions, we have

$$R_{\mu\boxplus\nu}^\boxplus(z) = R_\mu^\boxplus(z) + R_\nu^\boxplus(z). \tag{2.4}$$

$G_\mu^{-1}(z)$ and $R_\mu^\boxplus(z)$ are well defined in some neighbourhood of zero. We can thus write the above equation in an alternative form

$$G_\mu(z) = \frac{1}{z - R_\mu^\boxplus(G_\mu(z))}. \tag{2.5}$$

Moreover, since $R_\mu^\boxplus(z)$ is analytic, it can be treated as a series $\sum_{k=0}^\infty R_\mu^\boxplus(k+1) z^k$. The coefficients $R_\mu^\boxplus(k)$ can be calculated by the results of Speicher [S1] from the combinatorial moment-cumulant formula

$$m_\mu(n) = \sum_{\substack{\pi\in\mathrm{NC}(n)\\ \pi=(\pi_1,\dots,\pi_k)}} \prod_{i=1}^{k} R_\mu^{\boxplus}(|\pi_i|), \tag{2.6}$$

where $\mathrm{NC}(n)$ is the set of noncrossing partitions of $\{1,\dots,n\}$, π_i, $i=1,\dots,k$, are blocks of the partition π, and $|\pi_i|$ is the cardinality of the block. Equivalently, one can start with reciprocals instead of Cauchy transforms and define

$$\varphi_\mu(z) = F_\mu^{-1}(z) - z$$

getting a similar linearity relation $\varphi_{\mu\boxplus\nu}(z) = \varphi_\mu(z) + \varphi_\nu(z)$. Moreover, this approach extends to measures with unbounded support and with infinite moments; one only has to find an appropriate domain for z. This has been done by Maassen [Ma] for the case of measures with finite variance and by Bercovici and Voiculescu without this assumption in [BV2]. Bercovici and Voiculescu prove that for any probability measure $\mu \in \mathbf{Prob}(\mathbb{R})$ and any $\alpha > 0$ there exists $\beta > 0$ such that the function $\varphi_\mu(z)$ is analytic in a domain of the form

$$\{z : |z| > \beta,\ \mathrm{Im}(z) > 0,\ \mathrm{Re}(z) < \alpha\,\mathrm{Im}(z)\}$$

and that such an analytic function determines a corresponding probability measure. Since the sum of two such functions is again analytic in such a truncated angle for β large enough, the corresponding measure is determined.

2.3. Boolean probability. The second well-known example of noncommutative independence is the boolean relation.

Definition 2.11. A family of subalgebras $\mathcal{A}_i \subset \mathcal{A}$ is called *boolean-independent* if

$$\varphi(a_1\cdots a_n) = \varphi(a_1)\cdots\varphi(a_n) \tag{2.7}$$

whenever

$$a_j \in \mathcal{A}_{i_j}, \quad j=1,\dots,n, \quad a_j \neq 1 \in \mathcal{A} \quad \text{and} \quad i_1 \neq i_2 \neq \cdots \neq i_n.$$

The boolean product has been introduced by Bożejko in [Bo1, Bo2] and is known under its name since the paper of Speicher and Woroudi [SW]. For our purposes the boolean product $(\mathcal{A}_1,\varphi_1) \star_b (\mathcal{A}_2,\varphi_2)$ can be thought of as a special case of the conditionally free product $(\mathcal{A}_1,\varphi_1,\psi_1) \star_c (\mathcal{A}_2,\varphi_2,\psi_2)$, where on the appropriate algebras we have $\psi_i(\alpha 1 \oplus \beta) = \alpha$. This setup is enough for studying the distributions of sums of random variables. In a full treatment of the boolean product and independence we would have to consider algebras $\widetilde{\mathcal{A}}_i = \mathcal{A}_i \oplus \mathbb{C}\,\widetilde{1}$ with artificially added units, together with states $\widetilde{\varphi}_i(\alpha\widetilde{1} \oplus \beta) = \alpha + \varphi(\beta)$ and $\psi_i(\alpha\widetilde{1} \oplus \beta) = \alpha$; this is, however, beyond the scope of the present paper.

As in the previous constructions, the boolean convolution is best described in terms of analytic functions. Let $\mu,\nu \in \mathbf{Prob}(\mathbb{R})$, and let

$$R_\mu^{\uplus}(z) = z - \frac{1}{G_\mu(z)}, \qquad R_\nu^{\uplus}(z) = z - \frac{1}{G_\nu(z)}. \tag{2.8}$$

We know from [SW] that $R_{\mu\uplus\nu}^{\uplus}(z) = R_\mu^{\uplus}(z) + R_\nu^{\uplus}(z)$. Speicher and Woroudi also show that this definition works for arbitrary probability measures, possibly with infinite moments, due to the Nevanlinna theorem. Another important property arising from the Nevanlinna theory is that every probability measure is infinitely divisible with respect to the boolean convolution. The easiest proof of this fact is by showing that for any probability measure $\mu \in \mathbf{Prob}(\mathbb{R})$ and $t \geq 0$ the function $tR_\mu^{\uplus}(z)$ is the boolean transform $R_{\mu_t}^{\uplus}(z)$ of some probability measure μ_t. Consequently, for any $N \in \mathbb{N}$ we have $\mu = \mu_{1/N} \uplus \cdots \uplus \mu_{1/N}$, where $R_{\mu_{1/N}}^{\uplus}(z) = (1/N)R_\mu^{\uplus}(z)$. Moreover, for any $\mu \in \mathbf{Prob}(\mathbb{R})$ we can define its t-th boolean convolution power $\mu^{\uplus t}$ for $t \geq 0$ by requiring $R_{\mu^{\uplus t}}^{\uplus}(z) = tR_\mu^{\uplus}(z)$.

2.4. Conditionally free probability. The conditionally free convolution has been introduced in the papers of Bożejko, Leinert and Speicher [BS2, BLS]. Similarly to the free case, we start by looking at conditionally free subalgebras. Let $\mathcal{A}$ be a $\star$-algebra with two states φ and ψ.

Definition 2.12. We say that the subalgebras $\mathcal{A}_1, \mathcal{A}_2 \subset (\mathcal{A}, \varphi, \psi)$ are *conditionally free* if they satisfy

$$\varphi(a_1 \cdots a_n) = \varphi(a_1) \cdots \varphi(a_n), \tag{2.9}$$

$$\psi(a_1 \cdots a_n) = \psi(a_1) \cdots \psi(a_n) = 0 \tag{2.10}$$

whenever

$$\psi(a_j) = 0, \quad a_j \in \mathcal{A}_{i_j}, \quad j = 1, \ldots, n \quad \text{and} \quad i_1 \neq i_2 \neq \cdots \neq i_n, \tag{2.11}$$

where (2.10) means that the subalgebras are free with respect to the second state ψ.

Consider random variables $X \in (\mathcal{A}_1, \varphi, \psi)$, $Y \in (\mathcal{A}_2, \varphi, \psi)$. To each of them there correspond two sequences of moments, with respect to the two states φ and ψ, hence two probability measures, $X \sim (\mu_X, \nu_X)$, $Y \sim (\mu_Y, \nu_Y)$. The pair of measures $X + Y \sim (\mu_{X+Y}, \nu_{X+Y})$ corresponding to the random variable $X+Y$ is called the *conditionally free convolution* of (μ_X, ν_X) and (μ_Y, ν_Y) and denoted $(\mu_{X+Y}, \nu_{X+Y}) = (\mu_X, \nu_X) \boxed{c} (\mu_Y, \nu_Y)$, where by (2.10) we have $\nu_{X+Y} = \nu_X \boxplus \nu_Y$. As in the free case, if we are given two noncommutative probability spaces $(\mathcal{A}_1, \varphi_1, \psi_1)$ and $(\mathcal{A}_2, \varphi_2, \psi_2)$, on $\mathcal{A} = \mathcal{A}_1 \star \mathcal{A}_2$, the free product with amalgamation of units, we can define states φ and ψ by requiring them to satisfy relations (2.9)–(2.11). We denote this by $(\mathcal{A}, \varphi, \psi) = (\mathcal{A}_1, \varphi_1, \psi_1) \star_c (\mathcal{A}_2, \varphi_2, \psi_2)$. The natural embeddings of $\mathcal{A}_1$ and $\mathcal{A}_2$ into $\mathcal{A}$ are conditionally free, $\varphi|_{\mathcal{A}_i} = \varphi$, $\psi|_{\mathcal{A}_i} = \psi$, and if $X \in \mathcal{A}_1$, $Y \in \mathcal{A}_2$ then $m_{\mu_{X+Y}}(n) = \varphi((X+Y)^n)$ and $m_{\nu_{X+Y}}(n) = \psi((X+Y)^n)$.

The conditionally free convolution can also be described in terms of R-transforms. Since the second measure of the pairs is convolved freely, it will be described by the free transform $R_\nu^{\boxplus}(z)$. For the first measure one uses a different function dependent on both measures; we denote it by $R_{\mu,\nu}^{\boxed{c}}(z)$. We also use the equation (2.5) defining $R_\nu^{\boxplus}(z)$, thus getting

$$G_\mu(z) = \frac{1}{z - R_{\mu,\nu}^{\boxed{c}}(G_\nu(z))}, \qquad G_\nu(z) = \frac{1}{z - R_\nu^{\boxplus}(G_\nu(z))} \tag{2.12}$$

and if $(\mu, \nu) = (\mu_1, \nu_1) \boxed{c} (\mu_2, \nu_2)$ then

$$R_{\mu,\nu}^{\boxed{c}}(z) = R_{\mu_1,\nu_1}^{\boxed{c}}(z) + R_{\mu_2,\nu_2}^{\boxed{c}}(z), \qquad R_\nu^{\boxplus}(z) = R_{\nu_1}^{\boxplus}(z) + R_{\nu_2}^{\boxplus}(z).$$

As in the free case, the R-transforms can be written as power series

$$R^{\boxed{c}}_{\mu,\nu}(z) = \sum_{k=0}^{\infty} R^{\boxed{c}}_{\mu,\nu}(k+1)z^k, \qquad R^{\boxplus}_{\nu}(z) = \sum_{k=0}^{\infty} R^{\boxplus}_{\nu}(k+1)z^k,$$

and have corresponding combinatorial moment-cumulant formulae

$$m_\nu(n) = \sum_{\substack{\pi\in \mathrm{NC}(n)\\ \pi=(\pi_1,\dots,\pi_k)}} \prod_{i=1}^{k} R^{\boxplus}_{\nu}(|\pi_i|), \tag{2.13}$$

$$m_\mu(n) = \sum_{\substack{\pi\in \mathrm{NC}(n)\\ \pi=(\pi_1,\dots,\pi_k)}} \prod_{\pi_i \text{ outer}} R^{\boxed{c}}_{\mu,\nu}(|\pi_i|) \prod_{\pi_j \text{ inner}} R^{\boxplus}_{\nu}(|\pi_j|), \tag{2.14}$$

where a block π_i is called *inner* when there exists another block π_j with $a, b \in \pi_j$ such that $a < p < b$ for all $p \in \pi_i$. All blocks which are not enveloped in such a way are called *outer*. Equivalently one can say that the outer blocks π_i have depth $d(\pi_i) = 0$ and the inner π_j have $d(\pi_j) > 0$.

The above definitions are well established in the case of measures with compact support. Only recently, after the main part of the present work was completed, Belinschi [Be] extended the theory of the conditionally free convolution to arbitrary probability measures. His approach, however, is not expressed in terms of R-transforms but uses the subordination functions of Biane [Bi]. We know that for any probability measures ν_1, ν_2 there exist functions $\omega_1(z), \omega_2(z)$ such that for all $z \in \mathbb{C} \setminus \mathbb{R}$,

$$G_{\nu_1}(\omega_1(z)) = G_{\nu_2}(\omega_2(z)) = G_{\nu_1 \boxplus \nu_2}(z).$$

Belinschi proved that if $(\mu, \nu) = (\mu_1, \nu_1) \boxed{c} (\mu_2, \nu_2)$ and with the notation

$$h_\xi = F_\xi(z) - z \qquad \text{for all measures } \xi,$$

we have

$$h_\mu(z) = h_{\mu_1}(\omega_1(z)) + h_{\mu_2}(\omega_2(z)).$$

It seems likely that the above considerations will be extended to include a formulation in terms of the R or ϕ transforms on appropriate domains, similarly to the paper [BV2] in which the authors develop the case of arbitrary probability measures in the free case. Since a number of our results are proved through properties of the transforms, we will still need bounded support in most cases.

2.5. Deformations. Let $T : \mathbf{Prob}(\mathbb{R}) \to \mathbf{Prob}(\mathbb{R})$. There are two ways of using such a deformation of measures to define deformations of convolutions that we are interested in.

The first uses the free convolution and is valid for any invertible map T:

Definition 2.13. The *T-deformed free convolution* $\boxed{T}$ is defined by

$$\mu \boxed{T} \nu = T^{-1}(T\mu \boxplus T\nu) \tag{2.15}$$

for any probability measures μ and ν.

For the second one we need to assume that T maps measures with compact support to measures with compact support, but no invertibility is required.

DEFINITION 2.14.

$$(\mu \boxplus_T \nu, T\mu \boxplus T\nu) = (\mu, T\mu) \boxed{c} (\nu, T\nu)$$

for compactly supported μ and ν.

REMARK 2.15. The free convolution $\boxplus$ in Definition 2.13 can be replaced by any associative convolution $\oplus$ (for instance by the classical convolution), producing another associative convolution $\oplus^{(T)}$. This is going to be the subject of a forthcoming paper.

We are interested mostly in transformations T preserving boundedness of support, satisfying the Bożejko property of the following definition.

DEFINITION 2.16. A transformation T of probability measures has the *Bożejko property* if whenever for probability measures μ, ν we write

$$(\xi, \eta) = (\mu, T\mu) \boxed{c} (\nu, T\nu),$$

then

$$\eta = T\xi. \tag{2.16}$$

There are two reasons that such deformations are of interest; we gather them in the following propositions.

PROPOSITION 2.17. *For transformations with the Bożejko property the above defined convolution $\boxplus_T$ is associative.*

Proof. By associativity of the conditionally free convolution we have

$$\begin{aligned}(\mu \boxplus_T \nu) \boxplus_T \xi &= (\mu \boxplus_T \nu, T(\mu \boxplus_T \nu)) \boxed{c} (\xi, T\xi) = (\mu \boxplus_T \nu, T\mu \boxplus T\nu) \boxed{c} (\xi, T\xi) \\ &= (\mu, T\mu) \boxed{c} (\nu, T\nu) \boxed{c} (\xi, T\xi) = \mu \boxplus_T (\nu \boxplus_T \xi). \ \blacksquare\end{aligned}$$

PROPOSITION 2.18. *If the transformation T with the Bożejko property is invertible, then the convolutions $\boxed{T}$ and $\boxplus_T$ coincide.*

Proof. We have

$$\mu \boxplus_T \nu = (\mu \boxplus_T \nu, T\mu \boxplus T\nu) = (T^{-1}(T\mu \boxplus T\nu), T\mu \boxplus T\nu) = \mu \boxed{T} \nu. \ \blacksquare$$

A problem of Bożejko was to find all transformations T with the Bożejko property (2.16).

The first known example was the t-deformation treated in depth in the present paper, introduced by Bożejko and Wysoczański in the papers [BW1, BW2] and further studied by the author in [W]. Its generalization, the (a, b)-deformation, was considered by Krystek and Yoshida in [KY2]. Further examples, the so-called pure convolutions, were given by Oravecz in [O1, O2]. Another attempt was the Δ deformation and its special cases, the r and s deformations (see [Bo3, KY1, Y1, Y2]); however, in [BKW] it was proved that this deformation has the Bożejko property only when it reduces to the identity or maps all measures to δ_0, and that in other cases it leads to nonassociative convolutions. In the papers [KW1, KW2] the authors introduce two more families of deformations, of which one is invertible and based on ideas similar to the t- and (a, b)-deformations, whereas the other is not invertible, and uses measures infinitely divisible with respect to the free convolution.

As mentioned at the beginning of this chapter, the classical convolution in the ordinary probability theory, and the various canonical convolutions of the noncommutative probability theory, that is, the classical, free, boolean and strongly and weakly monotonic ones (not discussed in this paper, introduced in the work of Muraki [Mu1]), arise as the distributions of sums of suitably distributed random variables that are independent in the corresponding sense. Speicher [S2] and Muraki [Mu2] proved that without further assumptions on the algebras of random variables the above five notions are the only possible.

However, it seems that to deal successfully with noncommutative random variables it would suffice to limit ourselves to algebras of noncommutative polynomials in many variables. The last chapter of the present paper is an attempt to substantiate this idea in the case of the t-deformed probability. Only recently, a major step in this direction for the general case has been made by Muraki [Mu3]. His idea is to derive a notion of independence from the notion of Fock space, and to work with orthogonal polynomials that are noncommutative. A reconciliation of the approach of Muraki and ours will be the subject of a future work.

3. Kesten probability

The object of the present paper is to study some problems arising in free probability theory around the concept of t-deformation of measures, convolutions, states and products. It was introduced by Bożejko and Wysoczański in [BW1, BW2].

3.1. ***t*-deformation of measures.** We will use the language introduced in the previous chapter to define the most fundamental idea in our study, the t-deformation of a measure. Let $t \geq 0$. For a measure with compact support $\mu \in \mathbf{Prob}^{(c)}(\mathbb{R})$ the Cauchy transform has the convergent continued fraction representation with bounded coefficients:

$$G_\mu(z) = \cfrac{1}{z - a_0 - \cfrac{\lambda_0}{z - a_1 - \cfrac{\lambda_1}{z - a_2 - \cfrac{\lambda_2}{\ddots}}}}.$$

We define the *t-deformed measure* denoted $U_t\mu$ or μ_t as the measure for which the continued fraction representation of the Cauchy transform is

$$G_{U_t\mu}(z) = \cfrac{1}{z - ta_0 - \cfrac{t\lambda_0}{z - a_1 - \cfrac{\lambda_1}{z - a_2 - \cfrac{\lambda_2}{\ddots}}}}.$$

Since the recurrence coefficients remain bounded, $U_t\mu$ is again a compactly supported measure and $G_{U_t\mu}(z)$ is well defined for $z \in \mathbb{C}^+$. The above definition seems intuitive and instructive, but it is preferable to use the algebraic expression relating the reciprocals of the above Cauchy transforms, since by the crucial observation of [BW1], U_t thus defined extends to all probability measures on the real line. Hence we make the following

Definition 3.1. The *t-deformation* of a measure $\mu \in \mathbf{Prob}(\mathbb{R})$ is the measure $U_t\mu$ corresponding to the reciprocal of the Cauchy transform given by

$$F_{U_t\mu}(z) = tF_\mu(z) + (1-t)z. \tag{3.1}$$

Let us recall the argument that allows this extension:

Proposition 3.2. *For all $\mu \in \mathbf{Prob}(\mathbb{R})$ and $t \ge 0$ the function $F_{U_t\mu}(z)$ defined in* (3.1) *is the reciprocal of the Cauchy transform of a unique measure $\mu_t = U_t\mu \in \mathbf{Prob}(\mathbb{R})$.*

Proof. By the Nevanlinna theorem, for μ there exist $a \in \mathbb{R}$ and a positive finite measure ϱ such that

$$\begin{aligned} F_{U_t\mu}(z) = tF_\mu(z) + (1-t)z &= t\Big(a + z + \int_{-\infty}^{\infty} \frac{1+xz}{x-z}\,d\varrho(x)\Big) + (1-t)z \\ &= ta + z + \int_{-\infty}^{\infty} \frac{1+xz}{x-z}\,d(t\varrho)(x), \end{aligned}$$

where ta and $t\varrho$ satisfy again the conditions of the Nevanlinna theorem. ■

Let us also recall from [BW1] the basic properties that follow from this definition.

Proposition 3.3. *For any $\mu \in \mathbf{Prob}(\mathbb{R})$ and $t, s \ge 0$ the following properties are satisfied:*

- *$(U_t)_{t\ge 0}$ is a multiplicative semigroup: $U_s(U_t(\mu)) = U_{st}(\mu)$;*
- *dilations of measures commute with U_t: $D_\lambda(U_t(\mu)) = U_t(D_\lambda(\mu))$;*
- *U_t and $U_{1/t}$ for $t > 0$ are inverses of each other;*
- *$U_t(\mu) \xrightarrow{t\to 1} \mu$ in the $\star$-weak topology;*
- *U_t is continuous in the $\star$-weak topology: if $\mu_n \to \mu$ then $U_t(\mu_n) \to U_t(\mu)$.*

Let us now see the action of U_t on a couple of elementary examples.

Example 3.4. Since for a single point measure δ_a we have $F_{\delta_a}(z) = z - a$, we get $U_t(\delta_a) = \delta_{ta}$.

Example 3.5. For a two-point measure $\omega = p\delta_a + q\delta_b$, $p + q = 1$ we have

$$\begin{aligned} G_{U_t(\omega)}(z) &= \frac{1}{\frac{t}{G_\omega(z)} + (1-t)z} = \frac{1}{(1-t)z + \frac{t}{\frac{p}{-a+z} + \frac{q}{-b+z}}} \\ &= \frac{p(z-b) + q(z-a)}{zp(1-t)(z-b) + zq(1-t)(z-a) + t(z-a)(z-b)} = \frac{W_1(z)}{W_2(z)}, \end{aligned}$$

where the degrees of the polynomials $W_1(z)$ and $W_2(z)$ are 1 and 2, respectively. This means that $U_t(\omega)$ is again a two-point measure $P\delta_A + Q\delta_B$ and its Cauchy transform can be multiplied in the numerator and in the denominator by c, the reciprocal of the coefficient of z^2 in $W_2(z)$, so that

$$G_{U_t(\omega)}(z) = \frac{cW_1(z)}{cW_2(z)} = \frac{P}{z-A} + \frac{Q}{z-B},$$

where $P+Q=1$ and A, B are the zeros of $W_2(z)$. This leads to the following solution:

$$\begin{aligned} A &= \frac{1}{2}\Big(b(1+q(-1+t)) + a(q+t-qt) \\ &\qquad + \sqrt{-4abt + (b(1+q(-1+t)) + a(q+t-qt))^2}\Big), \\ B &= \frac{1}{2}\Big(b(1+q(-1+t)) + a(q+t-qt) \\ &\qquad - \sqrt{-4abt + (b(1+q(-1+t)) + a(q+t-qt))^2}\Big), \\ P &= \frac{1}{2} + \frac{b(-1+q+qt) - a(q-t+qt)}{2\sqrt{-4abt + (b(1+q(-1+t)) + a(q+t-qt))^2}}, \\ Q &= \frac{1}{2} - \frac{b(-1+q+qt) - a(q-t+qt)}{2\sqrt{-4abt + (b(1+q(-1+t)) + a(q+t-qt))^2}}. \end{aligned}$$

Note that the solution given on page 740 of [BW2] is erroneous.

As observed by Bożejko and Wysoczański in [BW2], the relation between the Cauchy transform $G_\mu(z)$ and $M_\mu(z)$, the generating function of moments, allows the derivation of a recurrence formula for the moments of the deformed measure $U_t\mu$:

$$m_{U_t\mu}(n) = t m_\mu(n) + \sum_{k=1}^{n-1} m_\mu(k) m_{U_t\mu}(n-k). \tag{3.2}$$

In particular, if the first moment $m_\mu(1)$ of the measure μ vanishes, then also $m_{U_t\mu}(1) = 0$ and $m_{U_t\mu}(2) = t m_\mu(2)$.

3.2. t-deformed free convolution. We are now in a position to define the t-deformed free (or simply t-free) convolution.

Definition 3.6. Given two probability measures $\mu, \nu \in \mathbf{Prob}(\mathbb{R})$ and $t > 0$ we define their *t-free convolution* as

$$\mu \boxed{t} \nu = U_{1/t}((U_t\mu) \boxplus (U_t\nu)).$$

Remark 3.7. The convolution $\boxed{t}$ is clearly associative, since

$$(\mu \boxed{t} \nu) \boxed{t} \varrho = U_{1/t}\big[U_t\big(U_{1/t}(U_t\mu \boxplus U_t\nu)\big) \boxplus U_t\varrho\big] = U_{1/t}(U_t\mu \boxplus U_t\nu \boxplus U_t\varrho).$$

We would also like to be able to describe our convolution with the help of some transform $R^{\boxed{t}}_\mu(z)$ that would have the linearization property with respect to the convolution $\boxed{t}$. Since by the above definition $U_t(\mu \boxed{t} \nu) = (U_t\mu) \boxplus (U_t\nu)$, a natural choice would be $R^{\boxed{t}}_\mu(z) = R^{\boxplus}_{U_t\mu}(z)$; however, for reasons that will become clear after the next section, on the connection between the t-free convolution and the conditionally free convolution of Bożejko, Leinert and Speicher [BLS], we prefer to multiply it by a factor $1/t$, thus getting the following

Definition 3.8. The $R_\mu^{\boxed{t}}(z)$ *transform* of a measure $\mu \in \mathbf{Prob}(\mathbb{R})$ is given by

$$R_\mu^{\boxed{t}}(z) = \frac{1}{t}\, R_{U_t\mu}^{\boxplus}(z).$$

This way we have the desired linearization property $R_{\mu\boxed{t}\nu}^{\boxed{t}}(z) = R_\mu^{\boxed{t}}(z) + R_\nu^{\boxed{t}}(z)$. We can also define the $\varphi_\mu^{\boxed{t}}(z)$ transform by $\varphi_\mu^{\boxed{t}}(z) = \frac{1}{t}\varphi_{U_t\mu}(z)$.

Example 3.9. Let us calculate the $R^{\boxed{t}}(z)$ transform of the two-point measure considered above.

$$R_\omega^{\boxed{t}}(z) = \frac{1}{t}\, R_{U_t\omega}^{\boxplus}(z).$$

By definition of $R^{\boxplus}$ we get

$$R_{U_t\omega}^{\boxplus}(z) = G_{U_t\omega}^{-1}(z) - \frac{1}{z},$$

where $G_{U_t\omega}^{-1}(z)$ is the inverse of $G_{U_t\omega}(z)$ with respect to composition of functions. To calculate it we need to solve a quadratic equation. A straightforward calculation thus gives

$$_{\pm}G_{U_t\omega}^{-1}(z) = \frac{1 + z - zq + tzq \pm \sqrt{-4z(1-q) + (-1 - z + zq - tzq)^2}}{2z}.$$

It can be easily seen that $G_{U_t\omega}(z) \xrightarrow{z\to 0} \infty$, so the branch of the square root in the definition of $G_{U_t\omega}^{-1}(z)$ is chosen so that $G_{U_t\omega}^{-1}(z) \xrightarrow{z\to\infty} 0$, that is,

$$G_{U_t\omega}^{-1}(z) = \frac{1 + z - zq + tzq - \sqrt{-4z(1-q) + (-1 - z + zq - tzq)^2}}{2z}.$$

Consequently,

$$R_\omega^{\boxed{t}}(z) = \frac{-1 + z - zq + tzq - \sqrt{-4z(1-q) + (-1 - z + zq - tzq)^2}}{2zt}. \tag{3.3}$$

3.3. Connection with the conditionally free convolution. We can now recall the following observation from [BW1] and give it an analytic proof:

Theorem 3.10. *Let $\varrho, \eta \in \mathbf{Prob}^{(c)}(\mathbb{R})$ and $(\mu,\nu) = (\varrho, U_t\varrho) \boxed{c} (\eta, U_t\eta)$. Then*

$$(\mu,\nu) = (\varrho, U_t\varrho) \boxed{c} (\eta, U_t\eta) = (\varrho \boxed{t} \eta, U_t\varrho \boxplus U_t\eta).$$

Proof. The equation $\nu = U_t\varrho \boxplus U_t\eta$ follows trivially from the definition of the convolution $\boxed{c}$. To prove $\mu = \varrho \boxed{t} \eta$ we shall look at the respective R-transforms. From (2.12) and the definition of U_t we have

$$R_{\varrho,U_t\varrho}^{\boxed{c}}(G_{U_t\varrho}(z)) = z - \frac{1}{G_\varrho(z)} = z - \left(\left(1 - \frac{1}{t}\right)z + \frac{\frac{1}{t}}{G_{U_t\varrho}(z)}\right) = \frac{1}{t}\left(z - \frac{1}{G_{U_t\varrho}(z)}\right),$$

thus

$$R_{\varrho,U_t\varrho}^{\boxed{c}}(z) = \frac{1}{t}\left(G_{U_t\varrho}^{-1}(z) - \frac{1}{z}\right) = \frac{1}{t}\, R_{U_t\varrho}^{\boxplus}(z). \tag{3.4}$$

From (2.12) we can find the Cauchy transform of the measure μ:

$$\begin{aligned}
G_\mu(z) &= \frac{1}{z - R^{\boxed{c}}_{\varrho,U_t\varrho}(G_{U_t\varrho\boxplus U_t\eta}(z)) - R^{\boxed{c}}_{\eta,U_t\eta}(G_{U_t\varrho\boxplus U_t\eta}(z))} \\
&= \frac{1}{z - \frac{1}{t}R^{\boxplus}_{U_t\varrho}(G_{U_t\varrho\boxplus U_t\eta}(z)) - \frac{1}{t}R^{\boxplus}_{U_t\eta}(G_{U_t\varrho\boxplus U_t\eta}(z))} \\
&= \frac{1}{z - \frac{1}{t}R^{\boxplus}_{U_t\varrho\boxplus U_t\eta}(G_{U_t\varrho\boxplus U_t\eta}(z))} = \frac{1}{\frac{1}{t}(z - R^{\boxplus}_{U_t\varrho\boxplus U_t\eta}(G_{U_t\varrho\boxplus U_t\eta}(z))) + \left(1-\frac{1}{t}\right)z} \\
&= \frac{1}{\frac{1}{t}\cdot\frac{1}{G_{U_t\varrho\boxplus U_t\eta}(z)} + \left(1-\frac{1}{t}\right)z} = G_{U_{1/t}(U_t\varrho\boxplus U_t\eta)}(z) = G_{\varrho\boxed{t}\eta}(z). \quad \blacksquare
\end{aligned}$$

Remark 3.11. As we see in equation (3.4), the conditionally free transform $R^{\boxed{c}}_{\mu,U_t\mu}(z)$ is proportional to the free transform $R^{\boxplus}_{U_t\mu}(z)$ of the deformed measure $U_t\mu$, similarly to the t-free transform $R^{\boxed{t}}_{\mu}(z)$. This justifies the choice of the constant $1/t$ in Definition 3.8, so that $R^{\boxed{t}}_{\mu}(z) = R^{\boxed{c}}_{\mu,U_t\mu}(z)$.

Remark 3.12. From the preceding remark and equations (2.13) and (2.14) we get the following moment-cumulant formulae:

$$m_\mu(n) = \sum_{\substack{\pi\in\mathrm{NC}(n)\\ \pi=(\pi_1,\ldots,\pi_k)}} t^{-\#\mathrm{outer}(\pi)}\prod_{i=1}^{k} R^{\boxplus}_{U_t\mu}(|\pi_i|) = \sum_{\substack{\pi\in\mathrm{NC}(n)\\ \pi=(\pi_1,\ldots,\pi_k)}} t^{\#\mathrm{inner}(\pi)}\prod_{i=1}^{k} R^{\boxed{t}}_{\mu}(|\pi_i|)$$

3.4. Connection with the boolean convolution. The boolean convolution of measures with compact support can also be seen to be a special case of the conditionally free convolution, namely $(\mu\uplus\nu,\delta_0) = (\mu,\delta_0)\,\boxed{c}\,(\nu,\delta_0)$. As a consequence of this and of the continuity properties of the conditionally free convolution we know that if

$$\varrho_t \xrightarrow{t\to 0} \delta_0, \quad \eta_t \xrightarrow{t\to 0} \delta_0 \quad \text{and} \quad (\zeta_t,\theta_t) = (\mu,\varrho_t)\,\boxed{c}\,(\nu,\eta_t)$$

then

$$\zeta_t \xrightarrow{t\to 0} \mu\uplus\nu \quad \text{and} \quad \theta_t \xrightarrow{t\to 0} \delta_0,$$

all convergences considered in the weak-$\star$ topology.

The connection between the t-free convolution and the boolean convolution is twofold:

Remark 3.13. First of all, the deformation U_t is defined for all $t\geq 0$, whereas the convolution $\boxed{t}$ only for $t>0$, since it involves $U_{1/t}$. However, if we let $t\to 0$ then for any $\mu,\nu\in\mathbf{Prob}^{(c)}(\mathbb{R})$ we have $U_t\mu, U_t\nu \xrightarrow{t\to 0} \delta_0$ and by the above remarks

$$(\mu,U_t\mu)\,\boxed{c}\,(\nu,U_t\nu) = (\mu\,\boxed{t}\,\nu, U_t\mu\boxplus U_t\nu) \xrightarrow{t\to 0} (\mu\uplus\nu,\delta_0).$$

We use this convergence property to extend the t-free convolution to the case $t=0$, and to say that $\boxed{t}$ interpolates between the free convolution $\boxplus$ for $t=1$ and the boolean convolution $\uplus$ for $t=0$.

REMARK 3.14. Secondly, for any $t \geq 0$ and $\mu \in \mathbf{Prob}(\mathbb{R})$, we have, from Definition 3.1 and equation (2.8),

$$R^{\uplus}_{\mu^{\uplus t}}(z) = tR^{\uplus}_{\mu}(z) = tz - \frac{t}{G_\mu(z)} = z - \frac{1}{G_{U_t\mu}(z)} = R^{\uplus}_{U_t\mu}(z),$$

which means that the U_t transformation of a probability measure is nothing else than its t-th boolean convolution power.

3.5. t-free central limit theorem. Let us now recall the fundamental observation of [BW1], the central limit theorem.

THEOREM 3.15. *Let $\mu \in \mathbf{Prob}(\mathbb{R})$ be such that $m_\mu(1) = 0$, $m_\mu(2) = 1$ and let $t > 0$. Then*

$$D_{1/\sqrt{n}}\mu \boxed{t} \cdots \boxed{t} D_{1/\sqrt{n}}\mu \stackrel{n\to\infty}{\longrightarrow} \kappa_t$$

in the weak-$\star$ topology, where the limiting measure κ_t is related to the standard Wigner measure ω, appearing in the free central limit theorem, by $\kappa_t = U_{1/t}D_{\sqrt{t}}\omega$.

Proof. From the definition of the convolution $\boxed{t}$ we have

$$D_{1/\sqrt{n}}\mu \boxed{t} \cdots \boxed{t} D_{1/\sqrt{n}}\mu = U_{1/t}(D_{1/\sqrt{n}}U_t\mu \boxplus \cdots \boxplus D_{1/\sqrt{n}}U_t\mu).$$

Moreover, we know from (3.2) that $m_{U_t\mu}(1) = 0$ and $m_{U_t\mu}(2) = t$. We may thus use the free central limit theorem to get

$$D_{1/\sqrt{n}}U_t\mu \boxplus \cdots \boxplus D_{1/\sqrt{n}}U_t\mu \stackrel{n\to\infty}{\longrightarrow} D_{\sqrt{t}}\omega. \ \blacksquare$$

REMARK 3.16. The measure κ_t for $t = 1-1/(2N)$ where $N \in \mathbb{N}$ appeared first in a paper by Harry Kesten [K], where it is shown that this is the spectral measure of a random walk on the free group with N generators.

DEFINITION 3.17. We shall call κ_t the *Kesten measure* with parameter t.

The Kesten distribution κ_t has been calculated for instance in [BW2]. It has a part absolutely continuous with respect to the Lebesgue measure, denoted $\widetilde{\kappa}_t$, and for $t < 1/2$ a discrete part $\widehat{\kappa}_t$ with two atoms:

$$\widetilde{\kappa}_t = \frac{1}{2\pi} \cdot \frac{\sqrt{4t - x^2}}{1-(1-t)x^2}\,\chi_{[-2\sqrt{t},2\sqrt{t}]}(x)\,dx,$$

$$\widehat{\kappa}_t = \frac{1-2t}{2-2t}\,(\delta_{-1/\sqrt{1-t}} + \delta_{1/\sqrt{1-t}}) \quad \text{for } t < 1/2,$$

and its Cauchy transform in the continued fraction representation has the form

$$G_{\kappa_t}(z) = \cfrac{1}{z - \cfrac{1}{z - \cfrac{t}{z - \cfrac{t}{\ddots}}}}.$$

3.6. t-free Poisson limit theorem. The second important type of limit theorem is the Poisson limit theorem, specifying the weak-$\star$ limit of $\mu_N = \big(1 - \frac{\lambda}{N}\big)\delta_0 + \frac{\lambda}{N}\delta_1$ as $N \to \infty$.

Let us first write in more detail the argument that allows the definition in [BW2] of the t-free Poisson measure $\pi_\lambda^{(t)}$ as one having constant cumulants (this relates also to the conditionally free Poisson distribution of [BLS]).

THEOREM 3.18. *Let $\lambda > 0$ and $\mu_N = \left(1 - \frac{\lambda}{N}\right)\delta_0 + \frac{\lambda}{N}\delta_1$. Then*

$$\lim_{N\to\infty} R^{\boxed{t}}_{\mu_N^{\boxed{t}N}}(z) = R^{\boxed{t}}_{\pi_\lambda^{(t)}}(z) = \frac{\lambda}{1-z}.$$

Proof. For simplicity of notation set $\epsilon = \lambda/N$. By the linearization property of the $R^{\boxed{t}}(z)$ transform we have

$$R^{\boxed{t}}_{\mu_N^{\boxed{t}N}}(z) = N R^{\boxed{t}}_{\mu_N}(z).$$

By equation (3.3) of Example 3.9 we have

$$\begin{aligned} R^{\boxed{t}}_{\mu_N}(z) &= \frac{-1 + z - z\epsilon + tz\epsilon - \sqrt{-4z(1-\epsilon) + (-1 - z + z\epsilon - tz\epsilon)^2}}{2zt} \\ &= \frac{z - 1 - \sqrt{(-1-z)^2 - 4z}}{2zt} + \frac{-z + tz - \frac{4z + 2(-1-z)(z-tz)}{2\sqrt{(-1-z)^2-4z}}}{2zt}\epsilon + \mathrm{O}(\epsilon^2) \\ &= \frac{\epsilon}{1-z} + \mathrm{O}(\epsilon^2), \end{aligned}$$

where the two terms in the second line are the first two terms of the Taylor expansion of $R^{\boxed{t}}_{\mu_N}(z)$, considered as a function of ϵ, around 0, and where $\mathrm{O}(\epsilon^2)$ stands for the remainder of the expansion. We thus have

$$R^{\boxed{t}}_{\pi_\lambda^{(t)}}(z) = \lim_{N\to\infty} N R^{\boxed{t}}_{\mu_N}(z) = \frac{\lambda}{1-z}. \blacksquare$$

Let us note that $\frac{t\lambda}{1-z}$ is the free $R^{\boxplus}$-transform of the free Poisson measure $\pi_{t\lambda}$ (see for instance [HP]). Thus, we have on one hand

$$tR^{\boxed{t}}_{\pi_\lambda^{(t)}}(z) = \frac{t\lambda}{1-z} = R^{\boxplus}_{\pi_{t\lambda}}(z),$$

on the other hand, from the definition of the $R^{\boxed{t}}$ transform we get

$$tR^{\boxed{t}}_{\pi_\lambda^{(t)}}(z) = R^{\boxplus}_{U_t\pi_\lambda^{(t)}}(z),$$

and hence $U_t\pi_\lambda^{(t)} = \pi_{t\lambda}$, which means that the t-free Poisson measure is a deformation of the free Poisson measure: $\pi_\lambda^{(t)} = U_{1/t}\pi_{t\lambda}$. Calculation of the explicit form of the measure $\pi_\lambda^{(t)}$ can now be done by calculating $G^{-1}_{U_{1/t}\pi_{t\lambda}}(z)$, inverting it and using the Stieltjes inversion formula. We can get it another way by using the continued fraction representation of Cauchy transforms and by a result by Saitoh and Yoshida [SY]. We know that

$$G_{\pi_{t\lambda}}(z) = \cfrac{1}{z - t\lambda - \cfrac{t\lambda}{z - (t\lambda + 1) - \cfrac{t\lambda}{z - (t\lambda+1) - \cfrac{t\lambda}{\ddots}}}}$$

and by the properties of U_t

$$G_{\pi_\lambda^{(t)}}(z) = G_{U_{1/t}\pi_{t\lambda}}(z) = \cfrac{1}{z-\lambda-\cfrac{\lambda}{z-(t\lambda+1)-\cfrac{t\lambda}{z-(t\lambda+1)-\cfrac{t\lambda}{\ddots}}}}.$$

Note that although the transform $G_{\pi_\lambda^{(t)}}(z)$ in [BW2] was calculated correctly, the corresponding continued fraction forms (11.39) and (11.40) were erroneous. From the above continued fraction representation we get the coefficients in the recursion formula for monic polynomials orthogonal with respect to the measure $\pi_\lambda^{(t)}$:

$$\begin{aligned} p_0(x) &= 1, \quad p_1(x) = x-\lambda, \\ p_2(x) &= (x-(t\lambda+1))p_1(x) - \lambda p_0(x), \\ p_{n+1}(x) &= (x-(t\lambda+1))p_n(x) - t\lambda p_{n-1}(x). \end{aligned}$$

Saitoh and Yoshida considered measures orthogonalizing systems of orthogonal monic polynomials defined by

$$\begin{aligned} q_0(x) &= c, \quad q_1(x) = x-\alpha, \\ q_{n+1}(x) &= (x-a)q_n(x) - bq_{n-1}(x). \end{aligned}$$

It can be easily seen that for $\alpha = \lambda$, $a = t\lambda+1$, $b = t\lambda$ and $c = 1/t$ these relations produce for $n \geq 1$ the polynomials $p_n(x)$ corresponding to the t-free Poisson measure $\pi_\lambda^{(t)}$. For $n = 0$ we get $p_0(x) = tq_0(x)$, but this does not spoil the orthogonality relations. Saitoh and Yoshida calculate the unique probability measure ν orthogonalizing the above system of polynomials:

$$\nu = \widetilde{\nu} + \widehat{\nu},$$

where

$$\begin{aligned} f(x) &= (1-c)(x-a)^2 + (c-2)(\alpha-a)(x-a) + (\alpha-a)^2 + bc^2, \\ d\widetilde{\nu}(x) &= \frac{c\sqrt{4b-(x-a)^2}}{2\pi f(x)} \cdot \chi_{[a-2\sqrt{b},a+2\sqrt{b}]}(x)\,dx, \end{aligned}$$

and

$$d\widehat{\nu}(x) = \begin{cases} 0 & \text{if } f(x) \text{ has no real roots,} \\ \max\left(0, 1-\frac{b}{(\alpha-a)^2}\right)\delta_y & \text{if } f(x) \text{ has one real root } y = \alpha + \frac{b}{\alpha-a}, \\ w_1\delta_{y_1} + w_2\delta_{y_2} & \text{if } f(x) \text{ has two real roots } y_1 \text{ and } y_2, \end{cases}$$

where

$$w_i = \frac{1}{\sqrt{(\alpha-a)^2 - 4b(1-c)}} \max\left(0, \frac{bc}{|y_i-\alpha|} - \frac{|y_i-\alpha|}{c}\right).$$

We therefore get the absolutely continuous part of the t-free Poisson measure, as well as its discrete part:

$$\widetilde{\pi}_\lambda^{(t)} = \frac{\sqrt{4t\lambda - (1 - x + t\lambda)^2}}{2\pi x(1 + (-1 + t)x + \lambda - t\lambda)} \chi_{[t\lambda+1-2\sqrt{t\lambda}, t\lambda+1+2\sqrt{t\lambda}]}(x)\, dx,$$

$$\widehat{\pi}_\lambda^{(t)} = \begin{cases} \max(0, 1 - \lambda)\delta_0 & \text{for } t = 1, \\ w_1\delta_{y_1} + w_2\delta_{y_2} & \text{for } t \neq 1. \end{cases}$$

3.7. t-infinite divisibility and t-free Lévy–Khinchin formula. We already briefly mentioned the term "infinite divisibility" in the context of the properties of the boolean convolution. Let us recall the definition:

Definition 3.19. We say that a probability measure μ is *infinitely divisible* with respect to a convolution $\star$ if for every $N \in \mathbb{N}$ there exists a measure μ_N such that $\mu = \mu_N^{\star N}$.

This can be rewritten equivalently in terms of $R^\star$-transforms: a probability measure μ is infinitely divisible with respect to a convolution $\star$ if for every $N \in \mathbb{N}$ there exists a measure μ_N such that $R_\mu^\star(z) = NR_{\mu_N}^\star(z)$. In the case of the t-free convolution $\boxed{t}$ we prefer to use the $\varphi_\mu^{\boxed{t}}(z)$ transform, since it allows us to treat measures with unbounded support. We have an analogue of the free Levy–Khinchin formula:

Theorem 3.20. *A measure μ is $\boxed{t}$-infinitely divisible if and only if there exist $\alpha \in \mathbb{R}$ and a finite positive measure ϱ such that for all $z \in \mathbb{C}^+$,*

$$\varphi_\mu^{\boxed{t}}(z) = \alpha + \int_{-\infty}^{\infty} \frac{1 + zx}{z - x}\, d\varrho(x).$$

Proof. In the case of the t-free convolution a measure μ is $\boxed{t}$-infinitely divisible if for every $N \in \mathbb{N}$ there exists a measure μ_N such that on some truncated angle domain

$$\varphi_\mu^{\boxed{t}}(z) = N\varphi_{\mu_N}^{\boxed{t}}(z).$$

By a double application of the definition of the $\varphi^{\boxed{t}}$ transform to the left and right hand side of the above equation we get

$$\frac{1}{t}\, \varphi_{U_t\mu}^{\boxplus}(z) = \varphi_\mu^{\boxed{t}}(z) = N\varphi_{\mu_N}^{\boxed{t}}(z) = N\,\frac{1}{t}\, \varphi_{U_t\mu_N}^{\boxplus}(z),$$

hence, the measure μ is $\boxed{t}$-infinitely divisible if and only if

$$\varphi_{U_t\mu}^{\boxplus}(z) = N\varphi_{U_t\mu_N}^{\boxplus}(z),$$

which is equivalent to $\boxplus$-infinite divisibility of $U_t\mu$ (all equalities hold on some truncated angle domains). By the free Levy–Khinchin formula (see [Ma], [BV1] and [BV2]) the measure $U_t\mu$ is freely infinitely divisible if and only if there exist $\widetilde{\alpha} \in \mathbb{R}$ and a positive measure $\widetilde{\varrho}$ such that for all $z \in \mathbb{C}^+$,

$$\varphi_{U_t\mu}^{\boxplus}(z) = \widetilde{\alpha} + \int_{-\infty}^{\infty} \frac{1 + zx}{z - x}\, d\widetilde{\varrho}(x).$$

Hence

$$\varphi_\mu^{\boxed{t}}(z) = \frac{1}{t}\, \varphi_{U_t\mu}^{\boxplus}(z) = \alpha + \int_{-\infty}^{\infty} \frac{1 + zx}{z - x}\, d\varrho(x),$$

where $\alpha = \widetilde{\alpha}/t$ and $\varrho = \widetilde{\varrho}/t$. ∎

3.8. $\boxplus_t$-convolution powers of probability measures. Nica and Speicher [NS] proved the following

THEOREM 3.21. *For any probability measure μ, possibly not infinitely divisible, and any number $s \geq 1$ there exists a probability measure μ_s such that*

$$\mu_s = \mu^{\boxplus s},$$

which is understood as

$$\varphi^{\boxplus}_{\mu_s}(z) = s\varphi^{\boxplus}_{\mu}(z).$$

In the case of the boolean convolution, since all probability measures are infinitely divisible, the same property holds for $s \geq 0$. We are interested in finding a similar result for the case of the t-free convolution $\boxed{t}$. Clearly, we could not expect such a property only for $s \geq s_0$, for some $0 < s_0 < 1$, since then, by iterating the convolution power, we would have the property for all $s > 0$. In fact, we have the following

THEOREM 3.22. *For an arbitrary probability measure $\mu \in \mathbf{Prob}(\mathbb{R})$ there exists a measure $\mu_s \in \mathbf{Prob}(\mathbb{R})$ such that $\mu = \mu_s^{\boxed{t}s}$ for $s \geq 1$.*

Proof. By definition

$$\varphi^{\boxed{t}}_{\mu}(z) = \frac{1}{t}\,\varphi^{\boxplus}_{U_t\mu}(z) = \frac{1}{t}\,\varphi^{\boxplus}_{\nu}(z),$$

where $U_t\mu = \nu \in \mathbf{Prob}(\mathbb{R})$, hence for $s \geq 1$ there exists a measure ν_s such that $\nu = \nu_s^{\boxplus s}$, which gives

$$\frac{1}{t}\,\varphi^{\boxplus}_{\nu}(z) = \frac{s}{t}\,\varphi^{\boxplus}_{\nu_s}(z) = s\varphi^{\boxed{t}}_{U_{1/t}\nu_s}(z).$$

We have therefore found a measure $U_{1/t}\nu_s$ which has the required property $(U_{1/t}\nu_s)^{\boxed{t}s} = \mu$ for $s \geq 1$. It is not possible to improve on s, since that would imply that the free convolution version would also hold for $s \geq 0$, which is known to be false. ∎

4. Generalized Brownian motion

4.1. Introduction. In [BS3] Bożejko and Speicher consider the generalized Brownian motions. Such a process is a family of operators $\omega(t)$, $t \in \mathbb{R}$, in an appropriate noncommutative probability space. The construction of such processes usually follows several steps. First we consider a separable complex infinite-dimensional Hilbert space $\mathcal{H}$ and a unital $\star$-algebra $\mathcal{C}(\mathcal{H})$ with generators $c^*(h), c(h)$ for all $h \in \mathcal{H}$ satisfying the relations

$$c^*(af + bg) = ac^*(f) + bc^*(g), \qquad (c(f))^\star = c^*(f),$$

for all $f, g \in \mathcal{H}$ and $a, b \in \mathbb{C}$. We call $c^*(h)$ *creation operators* and $c(h)$ *annihilation operators*, and the algebra $\mathcal{C}(\mathcal{H})$ the *creation and annihilation algebra*. We also consider a unital $\star$-algebra $\mathcal{A}(\mathcal{K})$ generated by $\omega(h)$, $h \in \mathcal{K}$, with the relations

$$\omega(af + bg) = a\omega(f) + b\omega(g), \qquad \omega(f) = (\omega(f))^\star,$$

for all $f, g \in \mathcal{K}$ and $a, b \in \mathbb{R}$, where $\mathcal{K}$ is a real Hilbert space. A natural way to relate the two objects above is to require the complex Hilbert space $\mathcal{H}$ to be the complexification of the real $\mathcal{K}$, written $\mathcal{H} = \mathcal{K}_{\mathbb{C}}$. Let $k \in \mathcal{K} \hookrightarrow \mathcal{H}$. Then the $\star$-subalgebra of $\mathcal{C}(\mathcal{H})$ generated by $c^*(k) + c(k)$ is isomorphic to $\mathcal{A}(\mathcal{K})$. The algebra $\mathcal{A}(\mathcal{K})$ could be called the algebra of increments, since if we identify $\mathcal{K}$ with $L^2(\mathbb{R}, dx)$ then the indicator functions $\chi_{[0,t)}(x)$ are in $L^2(\mathbb{R})$ and the process $\omega(t)$, $t \geq 0$, can be defined as $\omega(\chi_{[0,t)})$. In order to turn the algebras considered into noncommutative probability spaces we need yet to specify states on both of them.

Definition 4.1. A *Fock state* on $\mathcal{C}(\mathcal{H})$ is a positive normalized linear functional $\varrho_{\mathbf{t}} : \mathcal{C}(\mathcal{H}) \to \mathbb{C}$ given by

$$\varrho_{\mathbf{t}}[c^{\sharp_1}(f_1) \cdots c^{\sharp_n}(f_n)] = \sum_{V \in \mathcal{P}_2(n)} \mathbf{t}(V) \prod_{(k,l) \in V} \langle f_k, f_l \rangle \cdot Q(\sharp_k, \sharp_l), \tag{4.1}$$

where the symbols $\sharp_i \in (1, \star)$ indicate creation or annihilation, and Q is a two by two matrix with $Q(1, \star) = 1$ and 0 in all other entries:

$$Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and where $\mathbf{t}$ is a function on the set of pair partitions, $t : \mathcal{P}_2(n) \to \mathbb{C}$.

Definition 4.2. A *Gaussian state* $\widetilde{\varrho}_{\mathbf{t}}$ on $\mathcal{A}(\mathcal{K})$ is a positive normalized linear functional given by

$$\widetilde{\varrho}_{\mathbf{t}}[\omega(f_1) \cdots \omega(f_n)] = \sum_{V \in \mathcal{P}_2(n)} \mathbf{t}(V) \prod_{(k,l) \in V} \langle f_k, f_l \rangle. \tag{4.2}$$

Remark 4.3. The Gaussian state $\widetilde{\varrho}_{\mathbf{t}}$ on $\mathcal{A}(\mathcal{K})$ is the restriction of the Fock state $\varrho_{\mathbf{t}}$ on $\mathcal{C}(\mathcal{K}_{\mathbb{C}})$ to the subalgebra $\mathcal{A}(\mathcal{K})$.

Remark 4.4. Not every "pairing prescription" $\mathbf{t}(V)$ gives rise to a positive functional in the above definitions. However, Guţă and Maassen proved in Theorem 2.6 of [GM] that a function $\mathbf{t}(V)$ produces a positive Gaussian state $\widetilde{\varrho}_{\mathbf{t}}$ if and only if it also produces a positive Fock state $\varrho_{\mathbf{t}}$. We call such functions positive definite.

Remark 4.5. The GNS representation associated to the pair $(\mathcal{C}(\mathcal{H}), \varrho_{\mathbf{t}})$ is a $\star$-algebra of creation and annihilation operators acting on a Hilbert space $\mathcal{F}_{\mathbf{t}}(\mathcal{H})$ having a Fock-type structure

$$\mathcal{F}_{\mathbf{t}}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n.$$

Conversely, we can start by defining an appropriate Fock-type space and then take as $C(\mathcal{H})$ the $\star$-algebra generated by creation and annihilation operators on this Fock space; we follow this approach later in this chapter.

When $\widetilde{\varrho}_{\mathbf{t}}$ is indeed a state, the sequences $\widetilde{\varrho}_{\mathbf{t}}[\omega(f)^k]$, $k = 0, 1, \ldots$, are moment sequences of probability measures. One can consider measures only for $f \in \mathcal{K}$ such that $\langle f, f \rangle = 1$, since moments arising from other elements correspond to their dilations.

Example 4.6. Various examples of positive definite "pairing prescriptions" $\mathbf{t}(V)$ have been given, resulting in different probabilities:

1. when $\mathbf{t}(V) = \mathbf{t}_q^{(I)}(V) = q^{\#I(V)}$, $-1 \le q \le 1$, where $\#I(V)$ is the number of crossings in the partition V, the moments $\widetilde{\varrho}_{\mathbf{t}}[\omega(f)^k]$, $k = 0, 1, \ldots$, correspond to the q-Gaussian measure [BKS],
2. the case $\mathbf{t}(V) = \mathbf{t}_{1-t}^{(\mathrm{cc})}(V) = (1-t)^{\#V - \mathrm{cc}(V)}$, $0 \le t \le 1$, where $\mathrm{cc}(V)$ is the number of connected components of the partition V, and $\#V$ the number of blocks of the partition V, has been considered in [BS3]; the corresponding measure is known for $1 - t = 1/N$, $N \in \mathbb{N}$, and equals $D_{\sqrt{1/N}}g \boxplus \cdots \boxplus D_{\sqrt{1/N}}g$ where g is the standard classical Gaussian measure,
3. when $\mathbf{t}(V) = \mathbf{t}_{t,-1}(V) = \mathbf{t}_{1-t}^{(\mathrm{cc})}(V) \cdot \mathbf{t}_{-1}^{(I)}(V) = (1-t)^{\#V - \#\mathrm{cc}(V)} \cdot (-1)^{\#I(V)}$ the moments $\widetilde{\varrho}_{\mathbf{t}}[\omega(f)^k]$, $k = 0, 1, \ldots$, correspond to the Kesten measure κ_t.

The fact that the function $\mathbf{t}_{t,-1}(V)$ in point 3 above is positive definite is a consequence of Corollary 1 of [BS3], which states that the pointwise product of two positive definite functions $\mathbf{t}_1(V) \cdot \mathbf{t}_2(V)$ is again positive definite. Our aim in the remaining part of this section is to complete the results of [BS3] by presenting a detailed construction of the corresponding Fock-type space. Moreover, by the above mentioned corollary, we may consider the more general pointwise product $\mathbf{t}_q^{(I)}(V) \cdot \mathbf{t}_{1-t}^{(\mathrm{cc})}(V)$ instead of the special case $\mathbf{t}_{-1}^{(I)}(V) \cdot \mathbf{t}_{1-t}^{(\mathrm{cc})}(V)$.

Definition 4.7. For a partition $V \in \mathcal{P}_2(n)$ let

$$\mathbf{t}_{t,q}(V) = \mathbf{t}_{1-t}^{(\mathrm{cc})}(V) \cdot \mathbf{t}_q^{(I)}(V) = (1-t)^{\#V - \#\mathrm{cc}(V)} \cdot q^{\#I(V)},$$

where $\#\mathrm{cc}(V)$ is the number of connected components, $\#I(V)$ is the number of crossings and $\#V$ is the number of blocks of the partition V, and $0 < t < 1$, $-1 \le q < 1$.

Proposition 4.8. *The function $\mathbf{t}_{t,q}(V)$ is multiplicative, that is, if the partition V decomposes into connected components $(V_0, \ldots, V_k)$ then $\mathbf{t}_{t,q}(V) = \mathbf{t}_{t,q}(V_0) \cdots \mathbf{t}_{t,q}(V_k)$.*

Proof. Since $\mathbf{t}_{t,q}(V) = \mathbf{t}_{1-t}^{(\mathrm{cc})}(V) \cdot \mathbf{t}_q^{(I)}(V)$ and both factors are multiplicative, so is their product. ■

4.2. Fock space. We shall construct our Brownian motion by first constructing a Fock space on which we shall define appropriate operators.

Definition 4.9. Let $\mathcal{H}$ be an infinite-dimensional separable complex Hilbert space. Let us denote by $\mathcal{F}_0$ the algebraic Fock space consisting of a distinguished vector Ω and of vectors of the form $(f_1 \otimes \cdots \otimes f_n, A)$, where $n \in \mathbb{N}$, $f_i \in \mathcal{H}$, and $A \subset \{1, \ldots, n-1\}$ together with a pre-scalar product given by bilinear extension of

$$\langle \Omega, \Omega \rangle_{t,q} = 1,$$

$$\langle \Omega, (f_1 \otimes \cdots \otimes f_n, A) \rangle_{t,q} = 0,$$

$$\begin{aligned} &\langle (f_1 \otimes \cdots \otimes f_n, A), (g_1 \otimes \cdots \otimes g_m, B) \rangle_{t,q} \\ &\qquad = \delta_{mn} \sum_{\pi \in S_n} \langle f_1, g_{\pi(1)} \rangle \cdots \langle f_n, g_{\pi(n)} \rangle (t-1)^{(n-1) - \#[A \cap B \cap b(\pi)]} \cdot q^{\#I(\pi)} \end{aligned}$$

where for $\pi \in S_r$

$$b(\pi) = \{r - k \mid 1 \le k \le r-1,\ \pi(B_k) = B_k\} \subset \{1, \ldots, r-1\}.$$

The sets attached to simple tensors can be visualized as interval partitions grouping the elements of the tensor into interval blocks. The numbers in the sets indicate the tensor symbol, counted from the right, on which one interval ends and another begins. This notation may seem a little cumbersome, but allows for a very concise notation of interval partitions of sets of different cardinality.

THEOREM 4.10. *The bilinear form $\langle\,,\rangle_{t,q}$ is positive for $0 < t < 1$ and $-1 \le q < 1$.*

Proof. Fix $n \in \mathbb{N}$ and set $\widehat{f} = f_1 \otimes \cdots \otimes f_n \in \mathcal{H}^{\otimes n}$. We have to show that for all possible choices of $M \in \mathbb{N}$, $\widehat{f}_1, \ldots, \widehat{f}_M$ and $A_1, \ldots, A_M \subset \{1, \ldots, n-1\}$ we have

$$L := \Big\langle \sum_{i=1}^{M} (\widehat{f}_i, A_i), \sum_{j=1}^{M} (\widehat{f}_j, A_j) \Big\rangle_{t,q} \ge 0.$$

We have

$$\begin{aligned} L &= \sum_{i,j=1}^{M} \sum_{\pi \in S_n} \langle \widehat{f}_i, \pi(\widehat{f}_j) \rangle (1-t)^{(n-1)-\#[A_i \cap A_j \cap b(\pi)]} q^{\#I(\pi)} \\ &= (1-t)^{n-1} \frac{1}{n!} \sum_{i,j=1}^{M} \sum_{\pi,\sigma \in S_n} \langle \sigma(\widehat{f}_i), \pi(\widehat{f}_j) \rangle (1-t)^{-\#[A_i \cap A_j \cap b(\sigma^{-1}\pi)]} \, q^{\#I(\sigma^{-1}\pi)} \end{aligned}$$

where $\pi(\widehat{f})$ denotes $\pi(f_1 \otimes \cdots \otimes f_n) = f_{\pi(1)} \otimes \cdots \otimes f_{\pi(n)}$. We know that the kernels F, G, H on $\{1, \ldots, M\} \times S_n$ given by

$$\begin{aligned} F((i,\sigma),(j,\pi)) &= (1-t)^{-\#[A_i \cap A_j \cap b(\sigma^{-1}\pi)]}, \\ G((i,\sigma),(j,\pi)) &= \langle \sigma(\widehat{f}_i), \pi(\widehat{f}_j) \rangle, \\ H((i,\sigma),(j,\pi)) &= q^{\#I(\sigma^{-1}\pi)} \end{aligned}$$

are positive definite by [BS3], and so is their pointwise product. ∎

DEFINITION 4.11. To finish the construction of the space $\mathcal{F}$ we first divide the algebraic Fock space $\mathcal{F}_0$ by the kernel of the pre-scalar product $\langle\,,\rangle_{t,q}$ and take the completion of the result with respect to the scalar product $\langle\,,\rangle_{t,q}$.

4.3. Creation and annihilation algebra

DEFINITION 4.12. For each $f \in \mathcal{H}$ let us define a creation operator $c^*(f)$ and an annihilation operator $c(f)$ by linear extension of

$$\begin{aligned} c^*(f)\Omega &= (f, \emptyset), \\ c^*(f)(f_1 \otimes \cdots \otimes f_n, A) &= (f \otimes f_1 \otimes \cdots \otimes f_n, A \cup \{n\}), \end{aligned}$$

and

$$\begin{aligned} c(f)\Omega &= 0, \\ c(f)(f_1, \emptyset) &= \langle f, f_1 \rangle \Omega, \\ c(f)(f_1 \otimes \cdots \otimes f_n, A) &= \sum_{i=1}^{n} \langle f, f_i \rangle (f_1 \otimes \cdots \otimes \check{f}_i \otimes \cdots \otimes f_n, A|_i) \cdot (1-t)^{z(i,A)} \cdot q^{i-1}, \end{aligned}$$

where

$$z(i,A)=\begin{cases}0 & \text{if } i=1 \text{ and } n-1\in A,\\ 1 & \text{otherwise,}\end{cases}$$

$$A|_i=\begin{cases}A\setminus\{n-1\} & \text{if } i=1 \text{ and } n-1\in A,\\ A\cap\{1,\ldots,n-i\} & \text{otherwise,}\end{cases}$$

and with the usual convention that

$$f_1\otimes\cdots\otimes \check{f}_i\otimes\cdots\otimes f_n=f_1\otimes\cdots\otimes f_{i-1}\otimes f_{i+1}\otimes\cdots\otimes f_n.$$

THEOREM 4.13. *For all $\eta,\xi\in\mathcal{F}$ and all $f\in\mathcal{H}$ we have*

$$\langle c^*(f)\eta,\xi\rangle_{t,q}=\langle\eta,c(f)\xi\rangle_{t,q}.$$

Proof. It is enough to show for all $n\in\mathbb{N}$, all $f_1,\ldots,f_n,g_1,\ldots,g_n\in\mathcal{H}$, all $A\subset\{1,\ldots\ldots,n-2\}$ and all $B\subset\{1,\ldots,n-1\}$

$$\langle c^*(f_1)(f_2\otimes\cdots\otimes f_n,A),(g_1\otimes\cdots\otimes g_n,B)\rangle_{t,q} = \langle(f_2\otimes\cdots\otimes f_n,A),c(f_1)(g_1\otimes\cdots\otimes g_n,B)\rangle_{t,q}.$$

Let us calculate both sides:

$$\begin{aligned}\mathrm{LHS}&=\langle(f_1\otimes\cdots\otimes f_n,A\cup\{n-1\}),(g_1\otimes\cdots\otimes g_n,B)\rangle_{t,q}\\ &=\sum_{\pi\in S_n}\langle f_1,g_{\pi(1)}\rangle\cdots\langle f_n,g_{\pi(n)}\rangle(1-t)^{(n-1)-\#[(A\cup\{n-1\})\cap B\cap b(\pi)]}\cdot q^{\#I(\pi)},\end{aligned}$$

$$\begin{aligned}\mathrm{RHS}&=\\ &=\sum_{i=1}^{n}\langle f_1,g_i\rangle\langle(f_2\otimes\cdots\otimes f_n,A),(g_1\otimes\cdots\otimes\check{g}_i\otimes\cdots\otimes g_n,B|_i)\rangle_{t,q}(1-t)^{z(i,B)}\cdot q^{i-1}\\ &=\sum_{i=1}^{n}\sum_{\sigma\in S_{n-1}^{(i)}}\langle f_1,g_i\rangle\langle f_2,g_{\sigma(2)}\rangle\cdots\langle f_n,g_{\sigma(n)}\rangle(1-t)^{(n-2)-\#[A\cap B|_i\cap b(\sigma)]+z(i,B)}\cdot q^{i-1+\#I(\sigma)},\end{aligned}$$

where $S_{n-1}^{(i)}$ is the set of all bijections from $\{2,\ldots,n\}$ to $\{1,\ldots,\check{i},\ldots,n\}$ and $b(\sigma)$ and $I(\sigma)$ are defined by considering σ in the canonical way as an element of S_{n-1}. For given i and σ define $\pi\in S_n$ by $\pi(1)=i$, $\pi(j)=\sigma(j)$. The assertion follows if

$$(n-1)-\#[(A\cup\{n-1\})\cap B\cap b(\pi)]=(n-2)-\#[A\cap B|_i\cap b(\sigma)]+z(i,B)\tag{4.3}$$

and

$$\#I(\pi)=i-1+\#I(\sigma).\tag{4.4}$$

Condition (4.3) has been proven in [BS3]. To show (4.4) note that for $\sigma\in S_{n-1}^{(i)}$ and its canonical counterpart $\sigma'\in S_{n-1}$:

$$\begin{array}{ccc}\{2,\ldots,n\} & \xrightarrow{\ \sigma\ } & \{1,\ldots,\check{i},\ldots,n\}\\ {\scriptstyle 1\text{-}1}\uparrow\downarrow & & \uparrow\downarrow{\scriptstyle 1\text{-}1}\\ \{1,\ldots,n-1\} & \xrightarrow{\ \sigma'\ } & \{1,\ldots,n-1\}\end{array}$$

the sets of inversions $I(\sigma) = \{(k,l) \mid k < l \text{ and } \sigma(k) > \sigma(l)\}$ and $I(\sigma') = \{(k',l') \mid k' < l' \text{ and } \sigma'(k') > \sigma'(l')\}$ are also in 1-1 correspondence, hence of the same cardinality. Moreover, for $\pi \in S_n$ defined above we have $I(\pi) = I(\sigma') \cup \{(1,l) \mid 1 < l \text{ and } i = \sigma'(1) > \sigma'(l)\}$, thus $\#I(\pi) = \#I(\sigma) + i - 1$. ∎

DEFINITION 4.14. Let $\mathcal{C}(\mathcal{H})$ be the unital $*$-algebra generated by all $c^*(f), c(f)$ for $f \in \mathcal{H}$ and define on $\mathcal{C}(\mathcal{H})$ the state

$$\varrho_{t,q}(a) = \langle \Omega, a\Omega \rangle_{t,q}.$$

THEOREM 4.15. *For all $n \in \mathbb{N}$ and all $f_1, \ldots, f_n \in \mathcal{H}$ we have*

$$\varrho_{t,q}[c^{\sharp_1}(f_1) \cdots c^{\sharp_n}(f_n)] = \begin{cases} 0 & \text{if } n \text{ odd}, \\ \displaystyle\sum_{V \in \mathcal{P}_2(n)} \mathbf{t}_{t,q}(V) \prod_{(k,l) \in V} \langle f_k, f_l \rangle \cdot Q(\sharp_k, \sharp_l) & \text{if } n = 2r, \end{cases} \tag{4.5}$$

where

$$Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Proof. A nonzero vacuum expectation is only possible if the number of creators equals that of annihilators, hence the odd moments vanish. By an observation of Bożejko and Speicher [BS3, p. 144], to prove the theorem it is enough to consider the case where the f_i form an orthonormal basis of $\mathcal{H}$ and where each f_i appears exactly twice in $\{f_1, \ldots, f_{2r}\}$, which means that in the sum only one term corresponding to a partition denoted V_0 survives. If for some $1 \leq i < m < 2r$ we have $\langle f_j, f_k \rangle = 0$ for all $j = i, \ldots, m$ and $k = 1, \ldots, i-1, m+1, \ldots, 2r$ then by orthogonality and the definition of the creation and annihilation operators we get

$$\begin{aligned} &[c^{\sharp_1}(f_1) \cdots c^{\sharp_{i-1}}(f_{i-1})] c^{\sharp_i}(f_i) \cdots c^{\sharp_m}(f_m) [c^{\sharp_{m+1}}(f_{m+1}) \cdots c^{\sharp_{2r}}(f_{2r})]\Omega \\ &\quad = [c^{\sharp_1}(f_1) \cdots c^{\sharp_{i-1}}(f_{i-1})] \langle \Omega, c^{\sharp_i}(f_i) \cdots c^{\sharp_m}(f_m)\Omega \rangle_{t,q} [c^{\sharp_{m+1}}(f_{m+1}) \cdots c^{\sharp_{2r}}(f_{2r})]\Omega \\ &\quad = \varrho_{t,q}(c^{\sharp_i}(f_i) \cdots c^{\sharp_m}(f_m)) [c^{\sharp_1}(f_1) \cdots c^{\sharp_{i-1}}(f_{i-1}) c^{\sharp_{m+1}}(f_{m+1}) \cdots c^{\sharp_{2r}}(f_{2r})]\Omega, \end{aligned}$$

which means that the state $\mathbf{t}_{t,q}$ is multiplicative. Thus, it is enough to consider the case when V_0 is a single connected component; the general case will follow by multiplicative extension. In such a case we have $\mathbf{t}_{t,q}(V_0) = (1-t)^{r-1} q^{\#I(V_0)} =$ RHS of (4.5). To see that this is equal to LHS of (4.5) $= \varrho_{t,q}[c^{\sharp}(f_1) \cdots c^{\sharp}(f_n)]$ we need to show that the exponents of $1-t$ and of q in $\mathbf{t}_{t,q}(V_0) =$ RHS will correspond to those coming from a direct computation of LHS. The two exponents behave exactly as in models where only one of them is present. The exponent of $1-t$ is exactly $r-1$, since each annihilation operator apart from $c(f_1)$ gives a factor $1-t$. The exponent of q is indeed equal to the number of crossings of V_0: this follows from the corresponding result on the q-Fock space considered in [BS1, Proposition 2]. ∎

4.4. Connection with the reduced free product. The next theorem gives a link between the generalized Brownian motion considered in the present work and the reduced free product of Voiculescu. However, we need to assume $q = -1$ and use the notation $\varrho_t = \varrho_{t,-1}$.

THEOREM 4.16. *Let $\{c_i = c(f_i), c_i^* = c^*(f_i) \mid i \in \mathbb{N}\}$ denote a distinguished set of generators of the unital $*$-algebra $\mathcal{C}^0 = \langle\{c_i = c(f_i)\}\rangle$, for some orthonormal basis $\{f_i\}$ of the underlying Hilbert space $\mathcal{K}$, let t_1, t_2 be real numbers with $0 \le t_1, t_2 \le 1$ and $s_1 = 1 - t_1$, $s_2 = 1 - t_2$. Define s by*

$$\frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2} \quad \text{and} \quad t = 1 - s = \frac{1 - t_1 t_2}{2 - t_1 - t_2}.$$

Embed $\mathcal{C}^0$ in $\mathcal{C}^0 \star \mathcal{C}^0$ (free product with identification of units) via

$$c_i \mapsto \sqrt{\frac{s}{s_1}}\, j_1(c_i) + \sqrt{\frac{s}{s_2}}\, j_2(c_i), \qquad c_i^* \mapsto \sqrt{\frac{s}{s_1}}\, j_1(c_i^*) + \sqrt{\frac{s}{s_2}}\, j_2(c_i^*),$$

and let ϱ be the restriction of $\varrho_{t_1} \star \varrho_{t_2}$ to $\mathcal{C}^0$. Then $\varrho = \varrho_t$.

Proof. We need to show that for all $n \in \mathbb{N}$ and all $i(1), \ldots, i(n) \in \mathbb{N}$,

$$\varrho_{t_1} \star \varrho_{t_2}\Big[\Big(\sqrt{\frac{s}{s_1}}\, j_1(c^\#_{i(1)}) + \sqrt{\frac{s}{s_2}}\, j_2(c^\#_{i(1)})\Big), \ldots, \Big(\sqrt{\frac{s}{s_1}}\, j_1(c^\#_{i(n)}) + \sqrt{\frac{s}{s_2}}\, j_2(c^\#_{i(n)})\Big)\Big] = \varrho_t[c^\#_{i(1)} \cdots c^\#_{i(n)}]. \tag{4.6}$$

To do this we shall show the equality of free cumulants of the left and right sides of the above equation. For a given state φ on a unital $*$-algebra $\mathcal{B}$ its multilinear free cumulants r_φ are defined via the relation ($a_i \in \mathcal{B}$)

$$\varphi(a_1 \cdots a_n) = \sum_{V = \{V_1, \ldots, V_p\} \in \mathcal{P}_2(n)} r_\varphi[a_{V_1}] \cdots r_\varphi[a_{V_p}], \tag{4.7}$$

where $r_\varphi[a_{V_i}] = r_\varphi[a_{v_1}, \ldots, a_{v_s}]$ for $V_i = (v_1, \ldots, v_s)$. Let us first consider the cumulants of the right-hand side of (4.6), that is, of $r_{\varrho_t}[c^\sharp_{i(1)} \cdots c^\sharp_{i(n)}]$. For odd n this quantity vanishes, since by the defining equation (4.7) it is a sum of products of "shorter" cumulants, at least one term in each of those products being of odd length, and cumulants of length one are the original state, hence are zero on single creators and annihilators. Moreover, since the moments of ϱ_t are expressed in Theorem 4.15 by a formula involving summation over all 2-partitions, it can be seen by induction that the free cumulant can be obtained from the following formula involving only connected pair partitions:

$$\begin{aligned} r_{\varrho_t}[c^\sharp_{i(1)}, \ldots, c^\sharp_{i(2r)}] &= \sum_{\substack{\mathcal{V}_0 = \{V_1, \ldots, V_r\} \in \mathcal{P}_2(2r) \\ \mathcal{V}_0 \text{ connected}}} \mathbf{t}_t(\mathcal{V}_0) \prod_{(k,l) \in \mathcal{V}_0} \langle f_k, f_l\rangle \cdot Q(\sharp_k, \sharp_l) \\ &= \sum_{\substack{\mathcal{V}_0 = \{V_1, \ldots, V_r\} \in \mathcal{P}_2(2r) \\ \mathcal{V}_0 \text{ connected}}} s^{r-1} \cdot (-1)^{\# I(\mathcal{V}_0)} \prod_{(k,l) \in \mathcal{V}_0} \langle f_k, f_l\rangle \cdot Q(\sharp_k, \sharp_l). \end{aligned}$$

To evaluate the free cumulant of the left-hand side of the equation (4.6) we use the fact that free cumulants linearize the free product:

$$r_{\varrho_{t_1} \star \varrho_{t_2}}\Big[\Big(\sqrt{\frac{s}{s_1}}\, j_1(c^\sharp_{i(1)}) + \sqrt{\frac{s}{s_2}}\, j_2(c^\sharp_{i(1)})\Big), \ldots, \Big(\sqrt{\frac{s}{s_1}}\, j_1(c^\sharp_{i(2r)}) + \sqrt{\frac{s}{s_2}}\, j_2(c^\sharp_{i(2r)})\Big)\Big]$$

$$= r_{\varrho_{t_1}}\left[\sqrt{\frac{s}{s_1}}\,c^{\sharp}_{i(1)},\dots,\sqrt{\frac{s}{s_1}}\,c^{\sharp}_{i(2r)}\right] + r_{\varrho_{t_2}}\left[\sqrt{\frac{s}{s_2}}\,c^{\sharp}_{i(1)},\dots,\sqrt{\frac{s}{s_2}}\,c^{\sharp}_{i(2r)}\right]$$
$$= \left(\frac{s}{s_1}\right)^r r_{\varrho_{t_1}}[c^{\sharp}_{i(1)},\dots,c^{\sharp}_{i(2r)}] + \left(\frac{s}{s_2}\right)^r r_{\varrho_{t_2}}[c^{\sharp}_{i(1)},\dots,c^{\sharp}_{i(2r)}]$$
$$= \sum_{\substack{\mathcal{V}_0=\{V_1,\dots,V_r\}\in\mathcal{P}_2(2r)\\ \mathcal{V}_0 \text{ connected}}} s^{r-1}\cdot(-1)^{\#I(\mathcal{V}_0)} \prod_{(k,l)\in\mathcal{V}_0} \langle f_k, f_l\rangle\cdot Q(\sharp_k,\sharp_l). \ \blacksquare$$

4.5. Second quantization. An important subject in the study of the generalized Brownian motions is the existence and properties of the second quantization functor. In the previous section we considered the $\star$-algebra $\mathcal{A}(\mathcal{K})$, generated by the Gaussian elements $\omega(k)$ where $k\in\mathcal{K}$ and $\mathcal{K}$ is a real Hilbert space, together with a Gaussian state $\widetilde{\varrho}_{\mathbf{t}}$ given by a pairing prescription $\mathbf{t}$. Alternatively, we can consider the von Neumann algebra $\Gamma(\mathcal{K})$ generated by the embeddings of $\omega(k)=c^*(k)+c(k)$ into the creation and annihilation algebra $\mathcal{C}(\mathcal{K}_{\mathbb{C}})$, together with the vacuum expectation Fock state $\varrho_{\mathbf{t}}(\cdot)=\langle\Omega,\cdot\,\Omega\rangle_t$ given by the same pairing prescription $\mathbf{t}$. The paper [R] studies the properties of the algebra $\Gamma(\mathcal{K})$. When the Fock state is tracial, we have the following definition.

Definition 4.17. Let $\mathcal{K}^{(1)},\mathcal{K}^{(2)}$ be two real Hilbert spaces and $T:\mathcal{K}^{(1)}\to\mathcal{K}^{(2)}$ any contraction. A unital trace preserving completely positive map $\Gamma(T):(\Gamma(\mathcal{K}^{(1)}),\langle\Omega,\cdot\,\Omega\rangle_t)\to(\Gamma(\mathcal{K}^{(2)}),\langle\Omega,\cdot\,\Omega\rangle_t)$ is called a *second quantization functor*.

A detailed study of this notion can be found in [GM]. The most important result of that paper is the existence of Γ under the condition of multiplicativity of $\mathbf{t}$ and faithfulness of $\varrho_{\mathbf{t}}$ for $\Gamma(l^2_{\mathbb{R}}(\mathbb{Z}))$. It is also proved that in the case of the Kesten type pairing prescription

$$\mathbf{t}(V)=\mathbf{t}_{t,-1}(V)=\mathbf{t}^{(\mathrm{cc})}_{1-t}(V)\cdot\mathbf{t}^{(I)}_{-1}(V)=(1-t)^{\#V-\#\mathrm{cc}(V)}\cdot(-1)^{\#I(V)},\qquad 0<t<1,$$

the corresponding vacuum Fock state $\langle\Omega,\cdot\,\Omega\rangle_t$ is tracial, and that the assumptions of the existence theorem are satisfied.

In our calculations we consider a particular choice of the Hilbert spaces and of the contraction in the above construction. Namely, we take the same one-dimensional Hilbert space $\mathcal{K}=\mathcal{K}^{(1)}=\mathcal{K}^{(2)}$, and the simplest possible contraction $T=e^{-\tau}I=sI$ where $\tau>0$, $0<s<1$. The algebra $\Gamma(\mathcal{K})$ is $L^\infty(\mathrm{supp}(\kappa_t),\kappa_t)$. Since the support of the Kesten measure κ_t is compact, we have $L^\infty(\mathrm{supp}(\kappa_t),\kappa_t)\subset L^2(\mathrm{supp}(\kappa_t),\kappa_t)$. Any $\gamma\in L^2(\kappa_t)$ can be written as $\gamma(x)=\sum_{k=0}^\infty\alpha_k p_k(x)$, where $p_k(x)$, $k=0,1,\dots,$ is the sequence of polynomials orthonormal with respect to the Kesten measure κ_t. The action of the operator $\Gamma(T)$ is

$$\Gamma(T)\gamma(x)=\sum_{k=0}^\infty s^k\alpha_k p_k(x). \tag{4.8}$$

The operator $\Gamma(T)$ can be expressed by a kernel, $\Gamma(T)\gamma(x)=\int k_s(x,y)\gamma(y)\,d\kappa_t(y)$, and the kernel can be defined with the use of orthonormal polynomials:

$$k_s(x,y)=\sum_{k=0}^\infty s^k p_k(x)p_k(y)$$

(see [J]). Moreover, if the function γ is bounded, $\gamma \in L^\infty(\kappa_t) \hookrightarrow L^2(\kappa_t)$, then from Theorem 2 of [J] it follows that $\Gamma(T)\gamma(x)$ is also bounded, provided that the kernel is nonnegative. Thus, the kernel, when nonnegative, defines an operator $\Gamma(T) : \Gamma(\mathcal{K}) \to \Gamma(\mathcal{K})$. The requirement of complete positivity of $\Gamma(T)$ reduces in this case to positivity, and in terms of the kernel to the question whether $k_s(x,y)$ is positive for x, y in the support of κ_t. This choice of the Hilbert space and of the operator T was meant as a test case for the general problem of existence of the second quantization functor and was carried out before the paper [GM] appeared. The results of Guţă and Maassen are, however, limited to the case $0 \le t \le 1$. Our calculations, in addition to providing a closed form formula for the kernel, show that it remains positive for $0 < t < (1+\sqrt{2})/2$ and is no longer so for greater t. This means that no second quantization functor can exist in this case; it is also an answer to a question of Janson [J] on the existence of kernels without the positivity property.

4.6. The Mehler kernel for the Kesten measure. We recall that the Cauchy transform of the Kesten measure κ_t is the following:

$$G_{\kappa_t}(z) = \cfrac{1}{z - \cfrac{1}{z - \cfrac{t}{z - \cfrac{t}{\ddots}}}}.$$

We can therefore use the coefficients of the continued fraction to define the recurrence coefficients of the orthonormal polynomials.

Definition 4.18. Let us denote by $p_k(x)$ the system of polynomials othonormal with respect to μ_t given by the following recurrence relations:

$$p_0(x) = 1, \qquad p_1(x) = x,$$
$$xp_n(x) = \lambda_n p_{n+1}(x) + \lambda_{n-1} p_{n-1}(x),$$

where

$$1 = \lambda_0, \qquad \sqrt{t} = \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \cdots .$$

Hence

$$p_2(x) = \frac{x^2 - 1}{\sqrt{t}}.$$

Definition 4.19. We denote by $k_s(x,y)$ the t-deformed Mehler kernel

$$k_s(x,y) = \sum_{k=0}^{\infty} s^k p_k(x) p_k(y),$$

and by $k_s^a(x,y)$, $a = x, y$, the shifted sums

$$k_s^x(x,y) = \sum_{k=0}^{\infty} s^k p_{k+1}(x) p_k(y), \qquad k_s^y(x,y) = \sum_{k=0}^{\infty} s^k p_k(x) p_{k+1}(y).$$

We would like to see that the above series are convergent for any $x, y \in \mathrm{supp}(\kappa_t)$. The coefficients λ_k of the recurrence relation for polynomials orthogonal with respect to the Kesten measure are constant for $k \geq 1$, we may thus use the following boundedness result proven in [N, Chapter 3, Theorem 12] for polynomials with convergent coefficients.

THEOREM 4.20. *The sequence* $\{|p_k(x)|\}$ *is uniformly bounded on any closed interval* $\Delta \subset (-2\sqrt{t}, 2\sqrt{t})$.

Moreover, we can explicitly calculate the values of $p_k(x)$ when x is one of the atoms or one of the endpoints of the above interval, which is the support of the absolutely continuous part of the Kesten measure.

LEMMA 4.21. *For any* $t > 0$, *at the endpoints of the interval we have*

$$p_0(2\sqrt{t}) = 1, \qquad p_1(2\sqrt{t}) = 2\sqrt{t}, \qquad p_k(2\sqrt{t}) = \frac{2kt - (k-1)}{\sqrt{t}},$$

$$p_0(-2\sqrt{t}) = 1, \qquad p_1(-2\sqrt{t}) = -2\sqrt{t}, \qquad p_k(-2\sqrt{t}) = \frac{2kt - (k-1)}{\sqrt{t}} \cdot (-1)^k,$$

and for $0 < t < 1/2$ *at the atoms we have*

$$p_0\left(\frac{1}{\sqrt{1-t}}\right) = 1, \quad p_1\left(\frac{1}{\sqrt{1-t}}\right) = \frac{1}{\sqrt{1-t}}, \quad p_k\left(\frac{1}{\sqrt{1-t}}\right) = \frac{1}{\sqrt{1-t}}\left(\frac{\sqrt{t}}{\sqrt{1-t}}\right)^k,$$

$$p_0\left(\frac{-1}{\sqrt{1-t}}\right) = 1, \quad p_1\left(\frac{-1}{\sqrt{1-t}}\right) = \frac{-1}{\sqrt{1-t}}, \quad p_k\left(\frac{-1}{\sqrt{1-t}}\right) = \frac{1}{\sqrt{1-t}}\left(\frac{-\sqrt{t}}{\sqrt{1-t}}\right)^k.$$

Proof. It is a simple verification that the sequences $p_k(x)$ satisfy the recurrence formulae of Definition 4.18 for the appropriate values of x. ■

THEOREM 4.22. *The series* $k_s(x, y)$, $k_s^x(x, y)$ *and* $k_s^y(x, y)$ *are convergent for all* $-1 < s < 1$, $x, y \in \operatorname{supp}(\kappa_t)$, $t > 0$.

Proof. First observe that the sequences $p_k(x)$ when x is one of the atoms are bounded whenever the atoms show up in the measure, i.e. for $0 < t < 1/2$. Hence, for $-1 < s < 1$ the series $k_s(x, y)$, $k_s^x(x, y)$ and $k_s^y(x, y)$ are convergent for all $x, y \in (-2\sqrt{t}, 2\sqrt{t}) \cup \{-1/\sqrt{1-t}, 1/\sqrt{1-t}\}$ (the discrete part appearing only when $0 < t < 1/2$).

The remaining task consists in checking the case $x \in \{\pm 2\sqrt{t}\}$, $y \in (-2\sqrt{t}, 2\sqrt{t}) \cup \{-1/\sqrt{1-t}, 1/\sqrt{1-t}\}$ and also the case $x, y \in \{\pm 2\sqrt{t}\}$. But in the first case, the desired radius of convergence can be easily calculated from the Cauchy criterion, and in the second case from the d'Alembert criterion. ■

THEOREM 4.23. *The Mehler kernel* $k_s(x, y)$ *for the measure* κ_t *is given by*

$$k_s(x, y) = \frac{(s^2 - 1)(s^2(t-1) - t - s(t-1)xy)}{(s^2-1)^2 t + s(-xy - s^2 xy + s(x^2 + y^2))}$$

for any $-1 < s < 1$, $x, y \in \operatorname{supp}(\kappa_t)$.

Proof. We will use the recurrence relation defining the polynomials $p_k(x)$ to find relationships between the series $k_s(x,y)$, $k_s^x(x,y)$ and $k_s^y(x,y)$:

$$
\begin{aligned}
k_s(x,y) &= \sum_{k=0}^{\infty} s^k p_k(x)p_k(y) = 1 + sxy + s^2\,\frac{(x^2-1)(y^2-1)}{t} + \sum_{k=3}^{\infty} s^k p_k(x)p_k(y)\\
&= 1 + sxy + s^2\,\frac{(x^2-1)(y^2-1)}{t} + \sum_{k=3}^{\infty} s^k\left(\frac{xp_{k-1}(x)}{\sqrt{t}} - p_{k-2}(x)\right)p_k(y)\\
&= 1 + sxy + s^2\,\frac{(x^2-1)(y^2-1)}{t} + \frac{x}{\sqrt{t}}\sum_{k=3}^{\infty} s^k p_{k-1}(x)p_k(y) - \sum_{k=3}^{\infty} s^k p_{k-2}(x)p_k(y)\\
&= 1 + sxy + s^2\,\frac{(x^2-1)(y^2-1)}{t} + \frac{sx}{\sqrt{t}}\sum_{k=2}^{\infty} s^k p_k(x)p_{k+1}(y) - s^2\sum_{k=1}^{\infty} s^k p_k(x)p_{k+2}(y)\\
&= 1 + sxy + s^2\,\frac{(x^2-1)(y^2-1)}{t} - \frac{sx}{\sqrt{t}}\left(y + sx\,\frac{y^2-1}{\sqrt{t}}\right) + \frac{sx}{\sqrt{t}}\sum_{k=0}^{\infty} s^k p_k(x)p_{k+1}(y)\\
&\quad - s^2\sum_{k=1}^{\infty} s^k p_k(x)\left(\frac{yp_{k+1}(y)}{\sqrt{t}} - p_k(y)\right)\\
&= 1 + sxy - \frac{s^2(y^2-1)}{t} - \frac{sxy}{\sqrt{t}} + \frac{sx}{\sqrt{t}}\,k_s^y(x,y) - \frac{s^2y}{\sqrt{t}}\sum_{k=1}^{\infty} s^k p_k(x)p_{k+1}(y)\\
&\quad + s^2\sum_{k=1}^{\infty} s^k p_k(x)p_k(y)\\
&= 1 + sxy - s^2 - \frac{s^2(y^2-1)}{t} - \frac{sxy + s^2y^2}{\sqrt{t}} + \frac{sx - s^2y}{\sqrt{t}}\,k_s^y(x,y) + s^2k_s(x,y).
\end{aligned}
$$

Hence

$$
k_s(x,y) = \frac{1 + sxy - s^2 - \frac{s^2(y^2-1)}{t} - \frac{sxy+s^2y^2}{\sqrt{t}} + \frac{sx-s^2y}{\sqrt{t}}k_s^y(x,y)}{1-s^2} \tag{4.9}
$$

and symmetrically

$$
k_s(x,y) = \frac{1 + sxy - s^2 - \frac{s^2(x^2-1)}{t} - \frac{sxy+s^2x^2}{\sqrt{t}} + \frac{sy-s^2x}{\sqrt{t}}k_s^x(x,y)}{1-s^2}. \tag{4.10}
$$

Moreover

$$
\begin{aligned}
k_s^y(x,y) &= \sum_{k=0}^{\infty} s^k p_k(x)p_{k+1}(y)\\
&= y + sx\,\frac{y^2-1}{\sqrt{t}} + \sum_{k=2}^{\infty} s^k p_k(x)p_{k+1}(y)\\
&= y + sx\,\frac{y^2-1}{\sqrt{t}} + \sum_{k=2}^{\infty} s^k p_k(x)\left(\frac{yp_k(y)}{\sqrt{t}} - p_{k-1}(y)\right)
\end{aligned} \tag{4.11}
$$

$$= y + sx\,\frac{y^2-1}{\sqrt{t}} + \frac{y}{\sqrt{t}}\sum_{k=2}^{\infty} s^k p_k(x)p_k(y) - \sum_{k=2}^{\infty} s^k p_k(x)p_{k-1}(y)$$

$$= y + sx\,\frac{y^2-1}{\sqrt{t}} - \frac{y}{\sqrt{t}}(1+sxy) + \frac{y}{\sqrt{t}}\sum_{k=0}^{\infty} s^k p_k(x)p_k(y) - s\sum_{k=1}^{\infty} s^k p_{k+1}(x)p_k(y)$$

$$= y + \frac{sxy^2 - sx - y - sxy}{\sqrt{t}} + sx + \frac{y}{\sqrt{t}}\,k_s(x,y) - s\sum_{k=0}^{\infty} s^k p_{k+1}(x)p_k(y)$$

$$= y + sx + \frac{sxy^2 - sx - y - sxy}{\sqrt{t}} + \frac{y}{\sqrt{t}}\,k_s(x,y) - sk_s^x(x,y).$$

Solving (4.9), (4.10) and (4.11) for $k_s(x,y)$ we get the desired formula. ■

4.7. Positivity of the t-deformed Mehler kernel

THEOREM 4.24. *The Mehler kernel $k_s(x,y)$ is nonnegative for all choices of* $0 \le s < 1$, $x, y \in \operatorname{supp}(\kappa)$, $0 < t < (1+\sqrt{2})/2$.

Proof. We use the fact that the kernel is symmetric, $k_s(x,y) = k_s(y,x)$. We shall split the proof into four cases: 1. both x and y are in the atoms; 2. one of them is in the positive atom and the other in the continuous part of the support; 3. one in the negative atom and the other in the continuous part; 4. both x and y in the continuous part.

1. When both $x, y \in \{-1/\sqrt{1-t}, 1/\sqrt{1-t}\}$ and the measure admits atoms, that is, for $0 \le t < 1/2$, we get

$$k_s\left(\frac{-1}{\sqrt{1-t}}, \frac{-1}{\sqrt{1-t}}\right) = k_s\left(\frac{1}{\sqrt{1-t}}, \frac{1}{\sqrt{1-t}}\right) = 1 - \frac{s}{t-1+st} \ge 1$$

since $t - 1 + st \le 0$, and

$$k_s\left(\frac{-1}{\sqrt{1-t}}, \frac{1}{\sqrt{1-t}}\right) = 1 - \frac{s}{1-t+st} \ge 0.$$

2. When $x = 1/\sqrt{1-t}$ and $y \in [-2\sqrt{t}, 2\sqrt{t}]$, $0 \le t < 1/2$ we get

$$k_s\left(\frac{1}{\sqrt{1-t}}, y\right) = \frac{(1-s^2)(s^2(1-t) + t - s\sqrt{1-t}\,y)}{t\big((s^2-1)^2 - \frac{s(1+s^2)y}{\sqrt{1-t}\,t} + \frac{s^2y^2}{t} - \frac{s^2}{(t-1)t}\big)}.$$

To check positivity we shall consider separately the numerator and denominator. Since $1 - s^2 > 0$, only the second factor of the numerator is relevant. This is a linear function in y and assumes negative values for $y > \frac{s^2 - s^2t + t}{s\sqrt{1-t}}$.

We may ignore the positive factor t in the denominator and the remaining part is a quadratic function of y and assumes negative values for

$$\frac{s^2 - s^2t + t}{s\sqrt{1-t}} < y < \frac{1 - t + s^2t}{s\sqrt{1-t}}.$$

Since under the assumptions on s, y and t we have

$$y \le 2\sqrt{t} < \frac{s^2 - s^2t + t}{s\sqrt{1-t}} < \frac{1 - t + s^2t}{s\sqrt{1-t}},$$

which means that for the relevant values of y both the numerator and the denominator are positive, the positivity of the whole kernel is established in this case.

3. When $x=-1/\sqrt{1-t}$ and $y\in[-2\sqrt{t},2\sqrt{t}]$, $0\le t<1/2$ we get

$$k_s\left(\frac{-1}{\sqrt{1-t}},y\right)=\frac{(1-s^2)(s^2(1-t)+t+s\sqrt{1-t}\,y)}{t\big((s^2-1)^2+\frac{s(1+s^2)y}{\sqrt{1-t}\,t}+\frac{s^2y^2}{t}-\frac{s^2}{(t-1)t}\big)}.$$

By an argument similar to the previous point, the numerator is positive for all $y>\frac{s^2t-s^2-t}{\sqrt{1-t}}$, whereas the denominator assumes negative values for

$$y\in\left(\frac{t-1-s^2t}{s\sqrt{1-t}},\frac{s^2t-s^2-t}{s\sqrt{1-t}}\right).$$

But since

$$\frac{s^2t-s^2-t}{s\sqrt{1-t}}<-2\sqrt{t}\le y,$$

both the numerator and denominator remain positive for all $y\in[-2\sqrt{t},2\sqrt{t}]$, hence the positivity is established.

4. When both $x,y\in[-2\sqrt{t},2\sqrt{t}]$ and $t>0$ we get

$$k_s(x,y)=\frac{(1-s^2)(-s^2(t-1)+t+s(t-1)xy)}{t\big((s^2-1)^2+s\big(-\frac{xy}{t}-\frac{s^2xy}{t}+\frac{s(x^2+y^2)}{t}\big)\big)}.$$

Observe that the denominator satisfies

$$t\left((s^2-1)^2+s\left(-\frac{xy}{t}-\frac{s^2xy}{t}+\frac{s(x^2+y^2)}{t}\right)\right)$$
$$=t\left(\left((1-s^2)+\frac{s^2x^2-sxy}{2t}\right)^2+\left(1-\frac{x^2}{4t}\right)\left(\frac{s^2x-sy}{\sqrt{t}}\right)^2\right),$$

so it is nonnegative. Denote by $w(s,t,x,y)$ the second factor of the numerator:

$$w(s,t,x,y)=-(s^2(-1+t))+t+s(-1+t)xy=s^2(1-t)+s(t-1)xy+t.$$

It remains to check when the polynomial $w(s,t,x,y)$ is nonnegative. First assume that $t\in(0,1)$. We shall prove nonnegativity of w in this case:

$$s^2(1-t)+s(t-1)xy+t>s^2(1-t)+s(t-1)4t+t,$$

because since $s(t-1)<0$ the left hand side expression is the smallest for $xy=4t$, furthermore, evaluation of the right hand side at the lowest point of the parabola gives

$$s^2(1-t)+s(t-1)4t+t>t+\frac{4(t-1)^2t^2}{t-1}=(1-2t)^2t>0.$$

Hence $k_s(x,y)$ is nonnegative in this case.

Now assume $t\in[1,\infty)$. The minimum of the polynomial $w(s,t,x,y)$ is attained at one of the points when $xy=-4t$, and since the coefficient of s^2 is negative, when $s=0$ or $s\to1$. Then

$$w(0,t,-2\sqrt{t},2\sqrt{t})=t>0$$

and

$$w(s,t,-2\sqrt{t},2\sqrt{t})\xrightarrow{s\to1}-4t^2+4t+1>0\quad\text{ for }t<\frac{1+\sqrt{2}}{2}.$$

Hence, for $t>(1+\sqrt{2})/2$ the kernel $k_s(x,y)$ admits negative values for some s,x,y within the appropriate domain. ■

5. Multidimensional boolean cumulants

5.1. t-free product of states. From the previous considerations we know that the t-free convolution of measures can be interpreted as a special case of the conditionally free convolution of Bożejko, Leinert and Speicher [BLS]. However, the construction presented in that paper consists of first constructing a conditionally free product of pairs of states $(\Phi_1, \Phi_2) = (\mu_1, \mu_2) \star_c (\nu_1, \nu_2)$ on $\mathcal{A} = \mathcal{A}_1 \star \mathcal{A}_2 = \mathbb{C}\langle X_1, X_2\rangle$ for two given pairs of states (μ_1, μ_2) on $\mathcal{A}_1 = \mathbb{C}\langle X_1\rangle$ and (ν_1, ν_2) on $\mathcal{A}_2 = \mathbb{C}\langle X_2\rangle$, and only then defining $(\phi_1, \phi_2) = (\mu_1, \mu_2) \boxplus_c (\nu_1, \nu_2)$ on $\mathbb{C}\langle X\rangle$ by linear extension of $(\phi_1, \phi_2)(X^n) = (\Phi_1, \Phi_2)((X_1 + X_2)^n)$. Using those observations, we would now like to define the t-deformed free product of states.

To this end we shall define a family of transformations $U_t^{(n)}$ acting on states on algebras of polynomials in n noncommuting variables, such that for $n = 1$ we get the transformation of measures U_t discussed previously and that for

$$(\Phi, \Psi) = \mathop{\star_c}_{i=1}^{n} (\mu_i, U_t(\mu_i)), \tag{5.1}$$

where μ_i are states on $\mathbb{C}\langle X_i\rangle$ and the second state Ψ is defined as the free product

$$\Psi = \mathop{\star}_{i=1}^{n} U_t(\mu_i), \tag{5.2}$$

we have

$$(\Phi, \Psi) = (\Phi, U_t^{(n)}(\Phi)). \tag{5.3}$$

We have the following assiociativity lemma, proved by Młotkowski in [Mł, Proposition 2]:

Lemma 5.1. *Assume that $I = \bigcup_{j\in J} I_j$ is a partition of I. Then*

$$\mathop{\star_c}_{j\in J} \big(\mathop{\star_c}_{i\in I_j} (\mu_i, \nu_i)\big) = \mathop{\star_c}_{i\in I} (\mu_i, \nu_i). \tag{5.4}$$

Therefore, we also get

$$(\Psi, U_t^{(n)}(\Psi)) \star_c (\Phi, U_t^{(m)}(\Phi)) = (\Theta, U_t^{(n+m)}(\Theta)), \tag{5.5}$$

for the states Ψ on $\mathbb{C}\langle X_1, \dots, X_n\rangle$, Φ on $\mathbb{C}\langle X_{n+1}, \dots, X_{n+m}\rangle$ and Θ on $\mathbb{C}\langle X_1, \dots, X_{n+m}\rangle$. This allows for the following

Definition 5.2. We shall call the state Θ arising in equation (5.5) the *t-deformed free product* of the states Ψ and Φ.

5.1.1. *Boolean cumulants and interval partitions.* Since we are dealing with algebras of noncommutative polynomials in many variables, every $f \in \mathbb{C}\langle X_1, \dots, X_n\rangle$ can be written as a finite sum $f = \sum \alpha_{i_1,\dots,i_k} X_{i_1} \cdots X_{i_k}$. Linear functionals can thus be defined on simple words $X_{i_1} \cdots X_{i_k}$ and then extended to the whole algebra.

We now get to the details of the definition of the transformation $U_t^{(n)}$ and to the discussion of whether it produces states, i.e. positive functionals. In the previous chapters we saw that the one-dimensional deformation of measures U_t is the t-th boolean convolution power, which followed from the properties of the corresponding Cauchy transforms. We shall see that $U^{(n)}$ can also be seen as a kind of boolean convolution power. We need the following

Definition 5.3. Let η be a state on $\mathbb{C}\langle X_1, \ldots, X_n\rangle$. Then we define the *boolean R-transform* R^B_η on simple words $X_{i_1}\cdots X_{i_k}$ by the relation

$$\eta(X_{i_1}\cdots X_{i_k}) = \sum_{\substack{V\in B(n)\\ V=(V_1,\ldots,V_k)}} R^B_\eta(X_{V_1})\cdots R^B_\eta(X_{V_k})$$

where $B(n)$ is the set of interval partitions of the set $\{1,\ldots,n\}$, that is, containing only outer blocks, and $X_{V_k} = (X_{v_{k-1}+1},\ldots,X_{v_k})$ if $V_k = (v_{k-1}+1,\ldots,v_k)$. We shall denote the interval $(v_{k-1}+1,\ldots,v_k)$ by $[v_{k-1}+1, v_k]$.

Definition 5.4. We shall call a state $\eta^{\uplus t}$ on $\mathbb{C}\langle X_1,\ldots,X_n\rangle$ the t-th *boolean product power* of some other state η on $\mathbb{C}\langle X_1,\ldots,X_n\rangle$, $t\in\mathbb{R}$, when

$$\eta^{\uplus t}(X_{i_1}\cdots X_{i_k}) = \sum_{\substack{V\in B(n)\\ V=(V_1,\ldots,V_k)}} tR^B_\eta(X_{V_1})\cdots tR^B_\eta(X_{V_k}). \tag{5.6}$$

Following Lehner [L] we shall make use of the following definitions:

Definition 5.5. A partition π is *irreducible* if the elements 1 and n are in the same connected component.

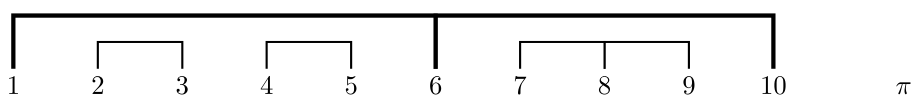
Definition 5.6. The *interval closure* of a given partition π is the smallest interval partition $\overline{\pi}$ dominating π, that is, for every block $\pi_i\in\pi$ there exists an interval block $B_i\in\overline{\pi}$ such that $\pi_i\subset B_i$.

In the following we will make use of the moment-cumulant relations for the conditionally free product, which involve summation over only noncrossing partitions. Thus we will need the above definitions only in the noncrossing context, along with some properties gathered in the following proposition:

Proposition 5.7. *Let π be a noncrossing partition of the set $[1,n]$. Then*:

1. *π is irreducible if the elements 1 and n are in the same block.*
2. *Any π can be decomposed into irreducible factors P_i, $i=1,\ldots,k$.*
3. *Any such irreducible factor P_i consists of blocks $\pi_{j_k}\in\pi$ such that $\bigcup\pi_{j_k}=[r_i,s_i]$ and exactly one of those blocks is outer and contains r_i and s_i, both the ends of the spanned interval*; *we denote this block by* $\operatorname{outer}(P_i)$.
4. *The interval closure of the partition π is the interval partition $\overline{\pi}$ with blocks $([r_1=1,s_1],[r_2=s_1+1,s_2],\ldots,[r_l,s_l=n])$ corresponding to the ends of intervals spanned by the irreducible components.*

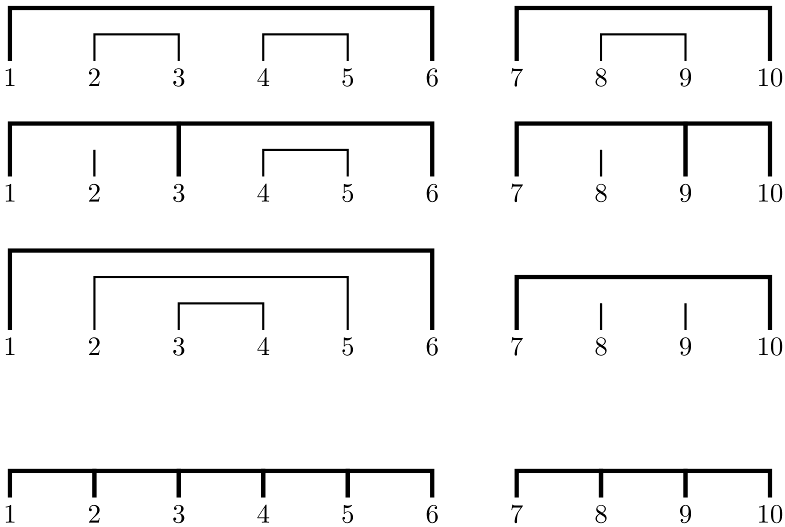
Proof. Instead of a formal algebraic proof we shall present the notions and ideas on diagrams. Let $n=10$ and consider a noncrossing partition π of the set $[1,10]$ such that one of the blocks π_i of π contains both 1 and 10; it could be for instance



where $\pi_i = (1, 6, 10)$. The smallest interval partition dominating π must have an interval block that dominates the block π_i containing the endpoints, hence $\overline{\pi}$ must be composed of one block containing all the points:

$\overline{\pi}$

Conversely, since we are now dealing with noncrossing partitions only, if the smallest interval partition dominating π is a one-block interval, there must be a block π_i in π containing both 1 and n, otherwise we have a situation like one of the following:



π_1

π_2

π_3

$\vdots$

$\overline{\pi}_i$

We have marked the outer blocks of all the partitions in bold line. Let *irreducible factors* be the subpartitions consisting of one outer block together with all inner blocks supported on points from between the ends of the outer block. As an illustration take π_2 from the above examples. Clearly $\pi_2 = ((1, 3, 6), (2), (4, 5), (7, 9, 10), (8))$ and it decomposes into $P_1 = ((1, 3, 6), (2), (4, 5))$ and $P_2 = ((7, 9, 10), (8))$. Every such irreducible factor is mapped into an interval in the interval closure.

It is also clear that summations involving all noncrossing partitions of an ordered set can be written as composite summations, first over interval partitions ω and then over all noncrossing partitions π such that $\overline{\pi} = \omega$.

5.1.2. *Main theorem*

THEOREM 5.8. *For states μ_i on $\mathbb{C}\langle X_i \rangle$, $i = 1, \ldots, n$, and $(\Phi, \Psi) = \star c_{i=1}^{n}(\mu_i, U_t(\mu_i))$ we have $\Phi^{\uplus t} = \Psi$, and write $U_t^{(n)}(\Phi) = \Psi$.*

Proof. The conditionally free product of states is linearized by the multilinear cumulants, which for a pair (μ, ν) of states on $\mathbb{C}\langle X_i \rangle$ are defined through

$$\nu(a_1\cdots a_k)=\sum_{\pi\in\mathrm{NC}(k)}\prod_{\substack{V_i\in\pi\\ V_i\text{ outer}}} r_\nu[V_i]\prod_{\substack{V_j\in\pi\\ V_j\text{ inner}}} r_\nu[V_j], \tag{5.7}$$

$$\mu(a_1\cdots a_k)=\sum_{\pi\in\mathrm{NC}(k)}\prod_{\substack{V_i\in\pi\\ V_i\text{ outer}}} R_{\mu,\nu}[V_i]\prod_{\substack{V_j\in\pi\\ V_j\text{ inner}}} r_\nu[V_j], \tag{5.8}$$

where for a partition block $V=\{v_1,\dots,v_j\}$ we denote by $r_\nu[V]=r_\nu[a_{v_1},\dots,a_{v_j}]$ the free cumulant of the state ν and by $R_{\mu,\nu}[V]=R_{\mu,\nu}[a_{v_1},\dots,a_{v_j}]$ the conditional cumulant with respect to the states μ and ν. In this definition we assumed $a_j\in\mathbb{C}\langle X_i\rangle$, $j=1,\dots,k$, and we extend it to $\mathbb{C}\langle X_1,\dots,X_n\rangle$ by putting $r_\nu[a_1,\dots,a_k]=R_{\mu,\nu}[a_1,\dots,a_k]=0$ whenever any $a_j\notin\mathbb{C}\langle X_i\rangle$.

We note that for the specific choice $a_1,\dots,a_k=X_i$ we have $\nu(a_1\cdots a_k)=\nu(X_i^k)$ and $\mu(a_1\cdots a_k)=\mu(X_i^k)$ and the above equations (5.7), (5.8) define the same recurrence relation as (2.13), (2.14) for linear cumulants, which means that $r_\nu[a_1,\dots,a_k]=R_\nu^{\boxplus}(k)$ and $R_{\mu,\nu}[a_1,\dots,a_k]=R_{\mu,\nu}^{\boxed{c}}(k)$. As a consequence, for $(\mu,\nu)=(\mu_i,U_t(\mu_i))$ we get

$$R_{\mu_i,U_t(\mu_i)}[a_1,\dots,a_k]=\frac{1}{t}\,r_{U_t(\mu_i)}[a_1,\dots,a_k] \tag{5.9}$$

for our specific choice $a_1,\dots,a_k=X_i$.

The transforms corresponding to the conditionally free product (Φ,Ψ) are the sums of the transforms corresponding to the pairs $(\mu_i,U_t(\mu_i))$, extended to $\mathbb{C}\langle X_1,\dots,X_n\rangle$:

$$R_V=R_{\Phi,\Psi}[a_{v_1},\dots,a_{v_j}]=\sum_{i=1}^{n}R_{\mu_i,U_t(\mu_i)}[a_{v_1},\dots,a_{v_j}], \tag{5.10}$$

$$r_V=r_\Psi[a_{v_1},\dots,a_{v_j}]=\sum_{i=1}^{n}r_{\mu_i}[a_{v_1},\dots,a_{v_j}], \tag{5.11}$$

and the moments with respect to the states (Φ,Ψ) are recovered through the moment-cumulant formulae for the conditionally free product established in [BLS]:

$$\Psi(a_1\cdots a_k)=\sum_{\pi\in\mathrm{NC}(k)}\prod_{\substack{V_i\in\pi\\ V_i\text{ outer}}} r_{V_i}\prod_{\substack{V_j\in\pi\\ V_j\text{ inner}}} r_{V_j}, \tag{5.12}$$

and

$$\Phi(a_1\cdots a_k)=\sum_{\pi\in\mathrm{NC}(k)}\prod_{\substack{V_i\in\pi\\ V_i\text{ outer}}} R_{V_i}\prod_{\substack{V_j\in\pi\\ V_j\text{ inner}}} r_{V_j}. \tag{5.13}$$

To prove the assertion we only need to consider $a_1\cdots a_n$ of the form $X_{i_1}\cdots X_{i_k}$. For this choice, all cumulants appearing in the right-hand sides of the above equations will satisfy $R_V=(1/t)r_V$, since they are sums of cumulants for which (5.9) holds. Let us now group the noncrossing partitions π in the summation in (5.13), according to the structure of their irreducible components, which is reflected by the interval partitions $\overline{\pi}$ arising as interval closures:

$$\begin{aligned}
\Phi(X_{i_1}\ldots X_{i_k}) &= \sum_{\omega\in \mathrm{B}(k)} \sum_{\substack{\pi\in \mathrm{NC}(k)\\ \overline{\pi}=\omega}} \prod_{\substack{V_i\in\pi\\ V_i \text{ outer}}} R_{V_i} \prod_{\substack{V_j\in\pi\\ V_j \text{ inner}}} r_{V_j}\\
&= \sum_{\substack{\omega\in \mathrm{B}(k)\\ \omega=\omega_1,\ldots,\omega_m}} \sum_{\substack{\pi=C_1\cup\cdots\cup C_m\\ C_i\in \mathrm{NC}(\mathrm{supp}(\omega_i))\\ \overline{C}_i=\omega_i}} \prod_{i=1}^{m} R_{\mathrm{outer}(C_i)} \prod_{\substack{V_j\in C_i\\ V_j \text{ inner}}} r_{V_j}\\
&= \sum_{\substack{\omega\in \mathrm{B}(k)\\ \omega=\omega_1,\ldots,\omega_m}} \sum_{\substack{C_1\in \mathrm{NC}(\mathrm{supp}(\omega_1))\\ \overline{C}_1=\omega_1}} \cdots \sum_{\substack{C_m\in \mathrm{NC}(\mathrm{supp}(\omega_m))\\ \overline{C}_m=\omega_m}} \prod_{i=1}^{m} R_{\mathrm{outer}(C_i)} \prod_{\substack{V_j\in C_i\\ V_j \text{ inner}}} r_{V_j}.
\end{aligned}$$

Set for convenience

$$K_\Phi(\omega_i) = \sum_{\substack{C_i\in \mathrm{NC}(\mathrm{supp}(\omega_i))\\ \overline{C}_i=\omega_i}} R_{\mathrm{outer}(C_i)} \prod_{\substack{V_j\in C_i\\ V_j \text{ inner}}} r_{V_j}.$$

Then

$$\Phi(X_{i_1}\cdots X_{i_k}) = \sum_{\substack{\omega\in \mathrm{B}(k)\\ \omega=\omega_1,\ldots,\omega_m}} \prod_{i=1}^{m} K_\Phi(\omega_i),$$

which is exactly the expression (5.6) defining boolean cumulants, thus $K_\Phi(\omega_i) = R^B_\Phi(\omega_i)$. Transforming equation (5.12) in the same way as (5.13) above we get

$$\Psi(X_{i_1}\cdots X_{i_k}) = \sum_{\substack{\omega\in \mathrm{B}(k)\\ \omega=\omega_1,\ldots,\omega_m}} \prod_{i=1}^{m} K_\Psi(\omega_i),$$

where

$$K_\Psi(\omega_i) = \sum_{\substack{C_i\in \mathrm{NC}(\mathrm{supp}(\omega_i))\\ \overline{C}_i=\omega_i}} r_{\mathrm{outer}(C_i)} \prod_{\substack{V_j\in C_i\\ V_j \text{ inner}}} r_{V_j},$$

hence

$$K_\Psi(\omega_i) = tK_\Phi(\omega_i).$$

Thus $tR^B_\Phi(\omega_i) = R^B_\Psi$, which completes the proof. ∎

5.2. Recurrence formula for moments. Earlier in this work we mentioned a result by Bożejko and Wysoczański from [BW2] where the authors give a recurrence formula for the moments of $U_t(\mu)$:

$$m_{U_t(\mu)}(n) = tm_\mu(n) + \sum_{k=1}^{n-1} m_{U_t(\mu)}(k)m_\mu(n-k). \tag{5.14}$$

We shall extend it to the states Φ and Ψ. By factoring the "leftmost" (respectively "rightmost") interval term out of the product and grouping similar terms in the definition of the boolean cumulants we get the following

PROPOSITION 5.9. *For any state η on $\mathbb{C}\langle X_1,\ldots,X_n\rangle$ we have*

$$\begin{aligned}\eta(X_{i_1}\ldots X_{i_k}) &= \sum_{\omega\in \mathrm{B}(k)}\prod_i R^B_\eta(\omega_i)\\ &= R^B_\eta([1,k]) + \sum_{j=1}^{k-1} R^B_\eta([1,j])\eta(X_{i_{j+1}}\cdots X_{i_k})\\ &= R^B_\eta([1,k]) + \sum_{j=1}^{k-1} \eta(X_{i_1}\cdots X_{i_j})R^B_\eta([j+1,k]).\end{aligned}$$

If $k=1$ only $R^B_\eta([1])$ survives.

THEOREM 5.10.

$$\Psi(X_{i_1}\cdots X_{i_k}) = t\Phi(X_{i_1}\cdots X_{i_k}) + (t-1)\sum_{j=1}^{k-1}\Psi(X_{i_1}\cdots X_{i_j})\Phi(X_{i_{j+1}}\cdots X_{i_k}).$$

Proof. We apply Proposition 5.9 to the RHS of the above a number of times:

$$\begin{aligned}\mathrm{RHS} = {} & tR^B_\Phi([1,k])\\ & + \sum_{j=1}^{k-1} tR^B_\Phi([1,j])\Phi(X_{i_{j+1}}\cdots X_{i_k})\\ & + \sum_{\iota=1}^{k-1}\Psi(X_{i_1}\cdots X_{i_\iota})tR^B_\Phi([\iota+1,k])\\ & + \sum_{\iota=1}^{k-2}\Psi(X_{i_1}\cdots X_{i_\iota})\sum_{j=\iota+1}^{k-1} tR^B_\Phi([\iota+1,j])\Phi(X_{i_{j+1}}\cdots X_{i_k})\\ & - \sum_{j=1}^{k-1}\Psi(X_{i_1}\cdots X_{i_j})\Phi(X_{i_{j+1}}\cdots X_{i_k})\\ = {} & R^B_\Psi([1,k]) + \sum_{\iota=1}^{k-1}\Psi(X_{i_1}\cdots X_{i_\iota})R^B_\Psi([\iota+1,k])\end{aligned}$$

$$+ \sum_{j=1}^{k-1} R^B_\Psi([1,j])\Phi(X_{i_{j+1}}\cdots X_{i_k}) \tag{5.15}$$

$$+ \sum_{j=2}^{k-1}\Big(\sum_{\iota=1}^{j-1}\Psi(X_{i_1}\cdots X_{i_\iota})R^B_\Psi([\iota+1,j])\Big)\Phi(X_{i_{j+1}}\cdots X_{i_k}) \tag{5.16}$$

$$- \sum_{j=1}^{k-1}\Psi(X_{i_1}\cdots X_{i_j})\Phi(X_{i_{j+1}}\cdots X_{i_k}) \tag{5.17}$$

$$= \Psi(X_{i_1}\cdots X_{i_k}) = \mathrm{LHS}$$

because by Proposition 5.9 the terms (5.15) and (5.16) cancel with (5.17). ∎

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