1. Introduction

The main purpose of our study is to show some generalizations and applications of the Kantorovich–Rubinstein maximum principle. First we prove this principle for nonlinear functionals of Hammerstein type. This result is based on a series of lemmas concerning local changes of Lipschitzian functions. Then we show that the Kantorovich–Rubinstein maximum principle combined with the LaSalle invariance principle yields new sufficient conditions for the asymptotic stability of Markov semigroups. These criteria are applied to the semigroups generated by discrete time stochastically perturbed dynamical systems, Poisson driven stochastic differential equations and to the Tjon–Wu version of the Boltzmann equation.

The outline of the paper is as follows. In Chapter 2 we consider some properties of contractive functions which satisfy the inequality

$$|f(x) - f(y)| < \varrho(x, y) \quad \text{for } x, y \in X,\ x \neq y, \tag{1}$$

where (X, ϱ) is a metric space. It is shown that under some additional conditions concerning the space X a function f satisfying (1) may be locally changed (in a neighbourhood of a compact set) in such a way that the inequality (1) is preserved. The proofs are partially based on the McShane extension theorem (see [31, Theorem 1]).

In Chapter 3 we study nonlinear functionals Φ_μ of the form

$$\Phi_\mu(f) = \int_X k(x, f(x))\, \mu(dx) \quad \text{for } f \in L,$$

where L is the space of Lipschitzian functions with Lipschitz constant 1 and μ is a given finite signed measure. We show that if a function $f_0 \in L$ maximizes Φ_μ, then there exist two different points $x, y \in X$ such that

$$|f_0(x) - f_0(y)| = \varrho(x, y).$$

This is a nonlinear version of the classical Kantorovich–Rubinstein maximum principle. In the same chapter we prove maximum principles for functionals defined on the subset $\mathcal{F}$ of L of functions satisfying the additional condition $|f| \leq 1$. The maximum principles allow us to establish interesting properties of the Hutchinson and Fortet–Mourier metrics.

In Chapter 4 we use these properties to prove that some semigroups of Markov operators acting on the space of signed measures are contractive. In Chapter 5 we show a new version of the invariance principle for dynamical systems acting on a topological Hausdorff space. It generalizes the results of A. Lasota (see [20, Theorem 2.1]) and A. Lasota and J. Traple (see [26, Theorem 1.1]). We also give an application of the invariance principle in the theory of the Tjon–Wu equation

$$\frac{d\psi}{dt} + \psi = P\psi,$$

where the unknown function ψ takes values in the space of signed measures and P is a collision operator. Similar results for ψ with values in $L^1(\mathbb{R}_+)$ were proved by A. Lasota and J. Traple (see [26, Theorem 3.1]).

In Chapter 6 we show new sufficient conditions for the asymptotic stability for semigroups of Markov operators. They are formulated in terms of adjoint operators. This approach simplifies further applications. We use these criteria to study stochastically perturbed dynamical systems

$$x_{n+1} = S(x_n, \xi_n) \quad \text{for } n = 0, 1, \dots,$$

where ξ_n, $n = 1, 2, \dots$, are independent identically distributed random variables. Our results generalize theorems of A. Lasota and M. C. Mackey (see [23, Theorem 2]) and K. Łoskot and R. Rudnicki (see [29, Theorem 3]), the latter in the case of locally compact separable metric spaces.

Further, we consider stochastic differential equation of the form

$$d\xi(t) = a(\xi(t))dt + \int_{\Theta} \sigma(\xi(t), \theta)\,\mathcal{N}_p(dt, d\theta) \quad \text{for } t \geq 0,$$

where $\{\xi(t)\}_{t\geq 0}$ is a stochastic process with values in the d-dimensional real space $\mathbb{R}^d$ and $\mathcal{N}_p$ is a Poisson random measure with intensity λ. Our result intersects with those of Traple (see [37, Theorem 7.3]) and Szarek (see [34, Theorem 8.3.1]).

We close Chapter 6 by giving an application to the mathematical model of the cell cycle introduced by A. Lasota and M. C. Mackey [25].

The present paper is based on the results contained in [10–12]. However, many theorems are now stated in a more general form and some new results are included. In particular in Section 3.3 we prove a new nonlinear version of the maximum principle for the Fortet–Mourier norm (Theorem 3.3.1). Furthermore, the main result of Chapter 5, Theorem 5.1.2 concerning the invariance principle, has never been published before. Also some results on the asymptotic stability of the Poisson driven stochastic differential equation (Theorem 6.3.1) and the Tjon–Wu equation (Theorem 5.2.4) are new.

2. Local changes of Lipschitzian functions

The aim of this chapter is to show two lemmas concerning local changes of Lipschitzian functions. In Chapter 3 we will apply these results in the theory of nonlinear functionals of Hammerstein type (see [13, Theorem 4.4] and [12, Theorem 2]).

2.1. Local changes of contractive bounded functions. A function $f : X \to \mathbb{R}$ defined on a metric space (X, ϱ) will be called *contractive* if

$$|f(x) - f(y)| < \varrho(x, y) \quad \text{for } x, y \in X,\ x \neq y. \tag{2.1.1}$$

Recall that a separable metric space X is locally compact iff X is an increasing union of compact sets. One can then define an equivalent metric on X (compatible with the topology) such that every closed ball is compact. In this paper, a "locally compact separable metric space" will always mean a locally compact separable metric space such that every closed ball is compact.

Lemma 2.1.1. *Let (X, ϱ) be a locally compact separable metric space and let $f : X \to \mathbb{R}$ be a contractive function satisfying*

$$\inf f > -\infty. \tag{2.1.2}$$

Further let an open set $G \subset X$ and a compact set $K \subset G$ be given. Then there exists an $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there is a contractive function $\tilde{f} : X \to \mathbb{R}$ satisfying

$$\tilde{f}(x) = f(x) \quad \text{for } x \in X \setminus G, \qquad \tilde{f}(x) = f(x) + \varepsilon \quad \text{for } x \in K, \tag{2.1.3}$$

and

$$f(x) \le \tilde{f}(x) \le f(x) + \varepsilon \qquad \text{for } x \in G \setminus K. \tag{2.1.4}$$

Proof. We may assume that $K \ne \emptyset$ and $X \setminus G \ne \emptyset$; otherwise the theorem is trivial. Replacing f by $f - \inf f$ we may assume that $f(x) \ge 0$ for $x \in X$. Let

$$\delta = \inf\{\varrho(x, y) : x \in K,\ y \notin G\}. \tag{2.1.5}$$

Since K is compact, this number is positive. For every $a \in K$ we define $h_a : X \to \mathbb{R}$ by

$$h_a(x) = \inf\{f(u) + \varrho(u, x) : \varrho(u, a) \ge \delta\} \qquad \text{for } x \in X.$$

From the inequality $f(x) - f(u) < \varrho(x, u)$ it follows immediately that

$$h_a(x) \ge f(x) \qquad \text{for } x \in X. \tag{2.1.6}$$

Moreover

$$h_a(x) = f(x) \qquad \text{for } \varrho(x, a) \ge \delta. \tag{2.1.7}$$

It is also evident that

$$|h_a(x) - h_a(y)| \le \varrho(x, y) \qquad \text{for } x, y \in X. \tag{2.1.8}$$

We claim that

$$h_a(a) > f(a) \qquad \text{for } a \in K. \tag{2.1.9}$$

To prove this fix $a \in K$ and define

$$A = \{u \in X : \varrho(a, u) > r + 2\delta\}, \qquad B = \{u \in X : \delta \le \varrho(a, u) \le r + 2\delta\},$$

where $r = \max_{x \in K} f(x)$. Since $a \in K$ and $f(u) \ge 0$, we have

$$f(u) + \varrho(u, a) > f(a) + 2\delta \qquad \text{for } u \in A. \tag{2.1.10}$$

According to (2.1.1) the function $u \mapsto f(u) + \varrho(u, a) - f(a)$ is positive on B. Moreover, since B is compact, we have

$$f(u) + \varrho(u, a) \ge f(a) + \sigma \qquad \text{for } u \in B,$$

where σ is a positive constant. This and (2.1.10) imply (2.1.9).

Define

$$\bar{f}(x) = \sup\{h_a(x) : a \in K\}.$$

From (2.1.5)–(2.1.7) it follows that

$$f(x) \le \bar{f}(x) < \infty \quad \text{for } x \in X \quad \text{and } \bar{f}(x) = f(x) \quad \text{for } x \in X \setminus G.$$

Since $\bar{f}(x) < \infty$, condition (2.1.8) implies that

$$|\bar{f}(x) - \bar{f}(y)| \le \varrho(x,y) \qquad \text{for } x, y \in X.$$

Further from (2.1.9) it follows that $f(x) < \bar{f}(x)$ for $x \in K$.

Let $\varepsilon_0 = \frac{1}{2}\min_{x\in K}(\bar{f}(x) - f(x))$. Then for every $\varepsilon \in (0, \varepsilon_0)$ the desired function $\tilde{f}$ is given by the formula

$$\tilde{f}(x) = \tfrac{1}{2}(f(x) + \min\{\bar{f}(x), f(x) + 2\varepsilon\}). \ \blacksquare \tag{2.1.11}$$

The following example shows that in the statement of Lemma 2.1.1 assumption (2.1.2) is essential.

Example 2.1.1. Consider the set $X = \mathbb{N} \cup \{0\}$ of nonnegative integers with the metric

$$\varrho(n,m) = \begin{cases} n+m & \text{for } n \ne m, \\ 0 & \text{for } n = m. \end{cases}$$

Let $f : X \to \mathbb{R}$ be given by the formula

$$f(n) = \begin{cases} 0 & \text{for } n = 0, \\ n^{-1} - n & \text{for } n > 0. \end{cases}$$

It is easy to verify that f is contractive. In the space (X, ϱ) the one-point set $\{0\}$ is simultaneously open and compact. So we may take $K = G = \{0\}$ and except (2.1.2), all the assumptions of Lemma 2.1.1 are satisfied. Now fix $\varepsilon \in (0, \varepsilon_0)$ and consider the function

$$\tilde{f}(n) = \begin{cases} f(0) + \varepsilon & \text{for } n = 0, \\ f(n) & \text{for } n > 0, \end{cases}$$

as described in Lemma 2.1.1. For $n > 1/\varepsilon$ we have

$$|\tilde{f}(n) - \tilde{f}(0)| = |-n + n^{-1} - \varepsilon| > n = \varrho(n, 0),$$

which shows that $\tilde{f}$ is not contractive.

Replacing f by $-f$ we obtain from Lemma 2.1.1 the following result.

Remark 2.1.1. If $\sup f < \infty$ and $f : X \to \mathbb{R}$ is a contractive function then there exists an $\varepsilon_0 > 0$ such that for every $\varepsilon \in (-\varepsilon_0, 0)$ there is a contractive function $\tilde{f} : X \to \mathbb{R}$ satisfying conditions (2.1.3) and the inequality

$$f(x) + \varepsilon \le \tilde{f}(x) \le f(x) \qquad \text{for } x \in G \setminus K. \tag{2.1.12}$$

2.2. Local changes of contractive unbounded functions. Assumption (2.1.2) can be omitted if we assume that the space X has some additional properties. We say that a metric space (X, ϱ) is *metrically convex* if for any two different points $x, y \in X$ and $\lambda \in (0, \varrho(x,y))$ there exists a point $z \in X$ such that

$$\varrho(x,z) = \lambda \quad \text{and} \quad \varrho(x,y) = \varrho(x,z) + \varrho(z,y).$$

Lemma 2.2.1. *Let (X, ϱ) be a locally compact separable, metrically convex metric space and let $f : X \to \mathbb{R}$ be a contractive function. Moreover let an open set $G \subset X$ and a* *compact set $K \subset G$ be given. Then there exists an $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there is a contractive function $\tilde{f} : X \to \mathbb{R}$ satisfying conditions* (2.1.3) *and* (2.1.4).

Proof. Again we may assume that $K \neq \emptyset$ and $X \setminus G \neq \emptyset$. Let δ be given by (2.1.5). Then $G_0 = \{x \in X : \varrho(x, K) < \delta\}$ is a subset of G. We define an auxiliary function $\bar{f} : X \to \mathbb{R}$ by

$$\bar{f}(x) = \inf\{f(u) + \varrho(x, u) : u \in X \setminus G_0\}.$$

It is easy to verify that

$$\bar{f}(x) \geq f(x) \qquad \text{for } x \in X, \tag{2.2.1}$$

$$\bar{f}(x) = f(x) \qquad \text{for } x \in X \setminus G, \tag{2.2.2}$$

$$|\bar{f}(x) - \bar{f}(y)| \leq \varrho(x, y) \qquad \text{for } x, y \in X.$$

Let $r = \delta + \operatorname{diam} K$. We are going to show that

$$\bar{f}(x) = \bar{\bar{f}}(x) \qquad \text{for } x \in K, \tag{2.2.3}$$

where

$$\bar{\bar{f}}(x) = \inf\{f(u) + \varrho(x, u) : u \in X \setminus G_0 \text{ and } \varrho(x, u) \leq 2r\}. \tag{2.2.4}$$

It is obvious that $\bar{f}(x) \leq \bar{\bar{f}}(x)$ for every $x \in X$. To show the opposite inequality for $x \in K$ it is sufficient to prove the following claim. For every $x \in K$ and $u \in X \setminus G_0$ there exists $v \in X \setminus G_0$ such that $\varrho(x, v) \leq 2r$ and

$$f(v) + \varrho(x, v) \leq f(u) + \varrho(x, u). \tag{2.2.5}$$

If $\varrho(x, u) \leq 2r$ we may choose $v = u$ and condition (2.2.5) is satisfied. Now assume that $x \in K$ and $\varrho(x, u) > 2r$. Then due to the metric convexity of X there exists a point $v \in X$ such that $\varrho(x, v) = 2r$ and

$$\varrho(x, u) = \varrho(x, v) + \varrho(v, u). \tag{2.2.6}$$

Using the definition of r it is easy to verify that $v \in X \setminus G_0$. Moreover using (2.2.6) and the inequality $f(v) - f(u) \leq \varrho(v, u)$ we obtain

$$f(v) + \varrho(x, v) = f(v) + \varrho(x, u) - \varrho(v, u) \leq f(v) + \varrho(x, u) - [f(v) - f(u)],$$

which gives (2.2.5) and completes the proof of the claim. This in turn implies (2.2.3). Observe that for $x \in K$ and $u \in X \setminus G_0$ we have $x \neq u$ and consequently

$$f(u) + \varrho(x, u) > f(x).$$

Moreover for every $x \in K$ the set

$$\{u \in X \setminus G_0 : \varrho(x, u) \leq 2r\}$$

is compact. Consequently, $\bar{\bar{f}}(x) > f(x)$ for $x \in K$. From this and (2.2.3) it follows that $\bar{f}(x) > f(x)$ for $x \in K$. Since K is compact, there exists a constant $\varepsilon_0 > 0$ such that

$$\bar{f}(x) \geq f(x) + 2\varepsilon_0 \quad \text{for } x \in K.$$

For $\varepsilon \in (0, \varepsilon_0)$ the desired function $\tilde{f}$ is again given by formula (2.1.11). ■

The following example shows that in the statement of Lemma 2.2.1 the assumption that X is locally compact separable is essential.

EXAMPLE 2.2.1. Let $\mathbb{C}$ be the complex plane and let

$$A_n = \{z \in \mathbb{C} : |z| \le 4, \arg z = \pi/n\} \quad \text{for } n = 1, 2 \ldots.$$

In the space $X = \bigcup_{n\in\mathbb{N}} A_n$ we define the metric ϱ by the formula

$$\varrho(z, w) = \begin{cases} |z - w| & \text{if } z, w \in A_n \text{ for some } n \in \mathbb{N}, \\ |z| + |w| & \text{otherwise,} \end{cases}$$

so that (X, ϱ) is a metrically convex space. Now consider the function

$$f(z) = (n^{-1} - 1)|z| \quad \text{for } z \in A_n,\ n = 1, 2, \ldots.$$

Let $K = \{0\}$ and $G = \{z \in X : |z| < 2\}$. Evidently f is contractive. Now fix an arbitrary $\varepsilon > 0$ and assume that a function $\tilde{f} : X \to \mathbb{R}$ satisfies the conditions

$$\tilde{f}(0) = f(0) + \varepsilon = \varepsilon \quad \text{and} \quad \tilde{f}(z) = f(z) \quad \text{for } z \in X \setminus G.$$

We have

$$|\tilde{f}(z) - \tilde{f}(0)| = |(n^{-1} - 1)|z| - \varepsilon| \quad \text{for } z \in A_n,\ |z| \ge 2.$$

For $|z| = 2$ and $n > 2/\varepsilon$ the right hand side is larger than 2 and the function $\tilde{f}$ is not contractive.

REMARK 2.2.1. Under conditions of Lemma 2.2.1 there exists an $\varepsilon_0 > 0$ such that for every $\varepsilon \in (-\varepsilon_0, 0)$ there is a contractive function $\tilde{f} : X \to \mathbb{R}$ satisfying (2.1.3) and

$$f(x) + \varepsilon \le \tilde{f}(x) \le f(x) \quad \text{for } x \in G \setminus K.$$

3. Maximum principles

The purpose of this chapter is to present maximum principles for functionals of Hammerstein type defined on the space of Lipschitzian functions. Our proofs are based on the lemmas concerning local changes of Lipschitzian functions. Using this method we prove new versions of the maximum principles for the Hutchinson and Fortet–Mourier metrices.

3.1. Metrics and norms in the space of measures. Let (X, ϱ) be a Polish space, i.e., a separable, complete metric space. We denote by $\mathcal{B}_X$ the σ-algebra of Borel subsets of X and by $\mathcal{M}$ the family of all finite (nonnegative) Borel measures on X.

Let $\mathcal{M}_1$ denote the subset of those $\mu \in \mathcal{M}$ such that $\mu(X) = 1$. The elements of $\mathcal{M}_1$ will be called *distributions*. Further let

$$\mathcal{M}_{\mathrm{sig}} = \{\mu_1 - \mu_2 : \mu_1, \mu_2 \in \mathcal{M}\}$$

be the space of finite signed measures. For arbitrary $\mu \in \mathcal{M}_{\mathrm{sig}}$ we denote by μ_+ and μ_- the positive and negative parts of μ. Then we set

$$\mu_+ - \mu_- = \mu \quad \text{and} \quad \mu_+ + \mu_- = |\mu|, \tag{3.1.1}$$

where $|\mu|$ is called the *total variation* of the measure μ.

Let c be a fixed element of X. For every real number $\alpha \ge 1$ we define the sets $\mathcal{M}_{1,\alpha}$ and $\mathcal{M}_{\mathrm{sig},\alpha}$ by setting

$$\mathcal{M}_{1,\alpha} = \{\mu \in \mathcal{M}_1 : m_\alpha(\mu) < \infty\} \quad \text{and} \quad \mathcal{M}_{\mathrm{sig},\alpha} = \{\mu \in \mathcal{M}_{\mathrm{sig}} : m_\alpha(\mu) < \infty\}$$

where

$$m_\alpha(\mu) = \int_X (\varrho(x,c))^\alpha \, |\mu|(dx).$$

Evidently $\mathcal{M}_{\mathrm{sig},\alpha} \subset \mathcal{M}_{\mathrm{sig},\beta}$ for $\alpha \ge \beta$. Moreover, we denote by $\mathcal{M}^0_{\mathrm{sig},\alpha}$ the subset of those $\mu \in \mathcal{M}_{\mathrm{sig},\alpha}$ for which $\mu(X) = 0$. It is evident that these spaces do not depend on the choice of c.

As usual, $B(X)$ denotes the space of all bounded Borel measurable functions $f : X \to \mathbb{R}$, and $C(X)$ the subspace of all bounded continuous functions. Both spaces are endowed with the supremum norm

$$\|f\| = \sup_{x\in X} |f(x)|.$$

For every $f : X \to \mathbb{R}$ and $\mu \in \mathcal{M}_{\mathrm{sig}}$ we write

$$\langle f, \mu\rangle = \int_X f(x)\, \mu(dx), \tag{3.1.2}$$

whenever this integral exists.

In $\mathcal{M}_1$ we introduce the *Fortet–Mourier metric* (see [7, Proposition 8.2]) by the formula

$$\|\mu_1 - \mu_2\|_{\mathcal{F}} = \sup\{|\langle f, \mu_1 - \mu_2\rangle| : f \in \mathcal{F}\}, \tag{3.1.3}$$

where $\mathcal{F}$ is the set of functions $f : X \to \mathbb{R}$ satisfying

$$\|f\| \le 1 \quad \text{and} \quad |f(x) - f(y)| \le \varrho(x,y) \quad \text{for } x, y \in X.$$

Remark 3.1.1. It is known that $\mathcal{M}_1$ with the Fortet–Mourier metric is a complete metric space. Furthermore, if X has at least one accumulation point then $\mathcal{M}_{\mathrm{sig}}$ with this metric is not complete (see [9, Theorem 3.1.7]).

We say that a sequence $(\mu_n) \subset \mathcal{M}_1$ *converges weakly* to a measure $\mu \in \mathcal{M}_1$ if

$$\lim_{n\to\infty} \langle f, \mu_n\rangle = \langle f, \mu\rangle \qquad \text{for } f \in C(X). \tag{3.1.4}$$

Since X is a Polish space, condition (3.1.4) is equivalent to

$$\lim_{n\to\infty} \|\mu_n - \mu\|_{\mathcal{F}} = 0$$

(see [7, Theorem 8.3]).

In $\mathcal{M}_1$ we also introduce another metric called the *Hutchinson metric* (see [15, Definition 4.3.1]) by the formula

$$\|\mu_1 - \mu_2\|_{\mathcal{H}} = \sup\{|\langle f, \mu_1 - \mu_2\rangle| : f \in \mathcal{H}\} \qquad \text{for } \mu_1, \mu_2 \in \mathcal{M}_1, \tag{3.1.5}$$

where $\mathcal{H}$ is the set of functions $f : X \to \mathbb{R}$ which satisfy the condition

$$|f(x) - f(y)| \le \varrho(x,y) \qquad \text{for } x, y \in X.$$

Fix $c \in X$. It is easy to see that

$$\|\mu_1 - \mu_2\|_{\mathcal{H}} = \sup\{|\langle f, \mu_1 - \mu_2\rangle| : f \in \mathcal{H}_c\} \qquad \text{for } \mu_1, \mu_2 \in \mathcal{M}_1,$$

where $\mathcal{H}_c = \{f \in \mathcal{H} : f(c) = 0\}$.

It should be noted that the Hutchinson metric is strongly related by a duality principle to the Kantorovich–Rubinstein norm (see [32, Corollary 6.1.1]).

Denote by $B(x,r)$ the closed ball in X with centre $x \in X$ and radius r. Let $\mu \in \mathcal{M}_1$. We define the support of μ by setting

$$\operatorname{supp}\mu = \{x \in X : \mu(B(x,\varepsilon)) > 0 \text{ for every } \varepsilon > 0\}.$$

Remark 3.1.2. Every set $\mathcal{M}_{1,\alpha}$ for $\alpha \ge 1$ contains the subset of all measures $\mu \in \mathcal{M}_1$ with compact support. This subset is dense in $\mathcal{M}_1$ with respect to the Fortet–Mourier norm (see [3, Theorem 4, p. 237]).

3.2. Nonlinear version of the Kantorovich–Rubinstein maximum principle. The main result of this section is stimulated by the following classical Kantorovich–Rubinstein maximum principle. Let (X,ϱ) be a separable metric space and let L be the space of functions $f : X \to \mathbb{R}$ which satisfy the Lipschitz condition. The space L is considered with the seminorm

$$\|f\|_L = \sup\left\{\frac{|f(x)-f(y)|}{\varrho(x,y)} : x \ne y;\ x,y \in X\right\}. \tag{3.2.1}$$

If μ is a given finite signed measure, then the linear functional $\varphi_\mu : L \to \mathbb{R}$ defined by the formula

$$\varphi_\mu(f) = \int_X f(x)\,\mu(dx) \quad \text{for } f \in L \tag{3.2.2}$$

has the following properties (for details see [33, Corollary 6.2]):

Theorem 3.2.1 (Kantorovich–Rubinstein maximum principle). *For every $\mu_1,\mu_2 \in \mathcal{M}_{1,1}$, $\mu_1 \ne \mu_2$, there exists a function $f \in \mathcal{H}$ such that*

$$\varphi_{\mu_1-\mu_2}(f) = \|\mu_1-\mu_2\|_{\mathcal{H}}.$$

Moreover, every function f for which the distance is attained (with $\mu_1 \ne \mu_2$) satisfies

$$|f(x)-f(y)| = \varrho(x,y)$$

for some $x,y \in X, x \ne y$.

The aim of this part is to prove analogous results for a nonlinear functional $\Phi_\mu : L \to \mathbb{R}$ given by the formula

$$\Phi_\mu(f) = \int_X k(x,f(x))\,\mu(dx), \qquad f \in L, \tag{3.2.3}$$

where (see (3.1.1) for the definition of $|\mu|$)

(i) $\mu \in \mathcal{M}_{\mathrm{sig},1}$, $\mu(X) = 0$, $|\mu| > 0$,

(ii) the function $k : X \times \mathbb{R} \to \mathbb{R}$ is continuous, has continuous derivative k_y with respect to the second variable, and satisfies

$$|k(x,y)| \le \beta_0\varrho(x,c) + \beta_1|y| + \beta_2 \quad \text{for} \quad (x,y) \in X \times \mathbb{R}, \tag{3.2.4}$$

where β_0,β_1,β_2 are nonnegative constants and $c \in X$ is a given point.

REMARK 3.2.1. Conditions (i) and (ii) imply that for every $f \in \mathcal{H}_c$ the integral (3.2.3) exists and

$$\sup_{f\in\mathcal{H}_c} |\Phi_\mu(f)| < \infty.$$

This is an immediate consequence of the inequality

$$|k(x, f(x))| \le (\beta_0 + \beta_1)\varrho(x, c) + \beta_2 \quad \text{for } x \in X \text{ and } f \in \mathcal{H}_c \tag{3.2.5}$$

and the assumption that $\mu \in \mathcal{M}_{\mathrm{sig},1}$.

Functionals of this type are in general studied by methods of convex analysis in the case when X is a vector space (see [8]).

Now we are ready to state the main theorem of this section.

THEOREM 3.2.2. *Assume that the space* (X, ϱ) *is locally compact, separable, metrically convex and that* μ *and* k *satisfy conditions* (i), (ii). *Assume moreover that*

$$k_y(x, y) > 0 \quad \textit{for } (x, y) \in X \times \mathbb{R}. \tag{3.2.6}$$

Then

$$\Phi_\mu(f) < \sup_{g\in\mathcal{H}_c} \Phi_\mu(g) \tag{3.2.7}$$

for every contractive function $f \in \mathcal{H}_c$.

Proof. Suppose that there exists a contractive $f \in \mathcal{H}_c$ such that

$$\Phi_\mu(f) = \sup_{g\in\mathcal{H}_c} \Phi_\mu(g). \tag{3.2.8}$$

Let $\mu = \mu_+ - \mu_-$ be the Jordan decomposition of μ and let $X = X_+ \cup X_-$ be the corresponding Hahn decomposition. We start from the case when $c \notin X_+$. Since μ is a nontrivial measure and $\mu(X) = 0$, according to the Ulam theorem (see [3, Theorem 1.4]) there is a compact set $K \subset X_+$ such that

$$\mu_+(K) > 0 \quad \text{and} \quad \mu_-(K) = 0. \tag{3.2.9}$$

Define

$$\delta_0 = \inf\{k_y(x, f(x) + z) : x \in K,\ z \in [0, 1]\}, \tag{3.2.10}$$

$$\delta_1 = \sup\{k_y(x, f(x) + z) : (x, z) \in K \times \{0\}\}. \tag{3.2.11}$$

Using the compactness of K we can find a $\delta > 0$ such that

$$k_y(x, f(x) + z) \le \delta_1 + 1 \quad \text{for } x \in K_\delta,\ 0 \le z \le \delta, \tag{3.2.12}$$

where $K_\delta = \{x \in X : \varrho(x, K) < \delta\}$. Changing δ if necessary, we may assume that

$$\mu_-(K_\delta \setminus K) \le \frac{1}{2}\,\frac{\delta_0\mu_+(K)}{1 + \delta_1} \quad \text{and} \quad \varrho(c, K) > \delta. \tag{3.2.13}$$

Since $K \subset K_\delta$ and K_δ is an open set, by Lemma 2.2.1 there exists an $\varepsilon \le \min(\delta, 1)$ and a contractive function $\tilde f : X \to \mathbb{R}$ satisfying (2.1.3) and (2.1.4) with $G = K_\delta$. By the mean value theorem we have

$$\Phi_\mu(\tilde f) - \Phi_\mu(f) = \int_X k_y(x, f(x) + \theta(x)(\tilde f(x) - f(x)))(\tilde f(x) - f(x))\,\mu(dx),$$

where $\theta(x) \in (0,1)$. From (2.1.3), (2.1.4) and the equality $\mu_-(K) = 0$ it follows that

$$\Phi_\mu(\tilde{f}) - \Phi_\mu(f) \geq \varepsilon \int\limits_K k_y(x, f(x) + \theta(x)\varepsilon)\, \mu_+(dx)$$
$$- \varepsilon \int\limits_{K_\delta \setminus K} k_y(x, f(x) + \theta(x)(\tilde{f}(x) - f(x)))\, \mu_-(dx).$$

Now using (3.2.10), (3.2.11) and (3.2.12) we obtain

$$\Phi_\mu(\tilde{f}) - \Phi_\mu(f) \geq \varepsilon\delta_0\mu_+(K) - \varepsilon(\delta_1 + 1)\mu_-(K_\delta \setminus K),$$

which in virtue of (3.2.13) gives

$$\Phi_\mu(\tilde{f}) \geq \Phi_\mu(f) + \frac{\varepsilon\delta_0}{2}\mu_+(K).$$

Since $\tilde{f} \in \mathcal{H}_c$, this contradicts (3.2.8) and completes the proof in the case when $c \notin X_+$.

If $c \notin X_-$ the argument is similar. It is based on Remark 2.2.1. ∎

Remark 3.2.2. Theorem 3.2.2 remains true if the space $\mathcal{H}_c$ is replaced by $\mathcal{H}$. The proof is similar. However, in this case the value $\sup_{g\in\mathcal{H}} \Phi_\mu(g)$ may be infinite.

We close this section with the following nonlinear version of the Kantorovich–Rubinstein maximum principle.

Theorem 3.2.3. *Assume that the space* (X, ϱ) *is complete and separable and that* μ *and* k *satisfy conditions* (i), (ii) *and* (3.2.6). *Then there exists an* $f_0 \in \mathcal{H}_c$ *such that*

$$\Phi_\mu(f_0) = \sup_{f\in\mathcal{H}_c} \Phi_\mu(f). \tag{3.2.14}$$

Moreover, if (X, ϱ) *is a locally compact, separable and metrically convex space then every function* $f_0 \in \mathcal{H}_c$ *satisfying* (3.2.14) *is not contractive.*

Proof. From Remark 3.2.1 it follows immediately that there exists a sequence $(f_n) \subset \mathcal{H}_c$ satisfying

$$\lim_{n\to\infty} \Phi_\mu(f_n) = \sup_{f\in\mathcal{H}_c} \Phi_\mu(f) < \infty. \tag{3.2.15}$$

By the Ulam theorem we can choose an increasing sequence of compact sets $K_s \subset X$ such that

$$|\mu|(X \setminus K_s) < 1/s \quad \text{for every } s = 1, 2, \ldots.$$

We may also assume that $c \in K_s$ for every $s \in \mathbb{N}$. Using the Arzelà–Ascoli theorem and the diagonal Cantor process we can find a subsequence (f_{α_n}) which converges pointwise on the set

$$K = \bigcup_{s=1}^{\infty} K_s,$$

to a function $\hat{f} : K \to \mathbb{R}$. Evidently $\hat{f}$ satisfies the Lipschitz condition with constant 1 and $\hat{f}(c) = 0$.

According to the McShane theorem (see [31, Theorem 1]) there exists an extension f_0 of $\hat{f}$ defined on the space X and satisfying the Lipschitz condition with the same constant.

From inequality (3.2.5) it follows that the functions $k(\cdot, f_{\alpha_n}(\cdot))$ for $n \in \mathbb{N}$ are bounded by a $|\mu|$-integrable function. As the sequence (f_{α_n}) converges to f_0 on K and $|\mu|(X \setminus K) = 0$, by the Lebesgue dominated convergence theorem we have

$$\lim_{n\to\infty} \varPhi_\mu(f_{\alpha_n}) = \varPhi_\mu(f_0).$$

This and (3.2.15) imply (3.2.14). By Theorem 3.2.2 the function f_0 is not contractive. ■

3.3. Maximum principle for the Fortet–Mourier metric. In this section we will prove a maximum principle for functionals acting on the space of uniformly bounded and uniformly Lipschitzian functions. In Section 6.5 we will show applications of this result in the theory of the cell cycle (see [12, Proposition 2]). As before we consider a nonlinear functional of the form

$$\varPhi_\mu(f) = \int_X k(x, f(x))\, \mu(dx) \quad \text{for } f \in \mathcal{F}, \tag{3.3.1}$$

where (X, ϱ) is a metric space, $k : X \times [-1, 1] \to \mathbb{R}$ is a given function and $\mu \in \mathcal{M}_{\mathrm{sig}}$.

We will assume that the function $k : X \times \mathbb{R} \to \mathbb{R}$ and the signed measure μ satisfy the following conditions:

(i) $\mu = \mu_1 - \mu_2$, $\mu_1, \mu_2 \in \mathcal{M}_1$.

(ii) The function $k : X \times [-1, 1] \to \mathbb{R}$ is continuous and has a continuous derivative k_y with respect to the second variable. Moreover,

$$-\infty < \int_X k(x, -1)\, |\mu|(dx) \quad \text{and} \quad \int_X k(x, 1)\, |\mu|(dx) < \infty, \tag{3.3.2}$$

$$k_y(x, y) > 0 \quad \text{for } (x, y) \in X \times [-1, 1]. \tag{3.3.3}$$

THEOREM 3.3.1. *Assume that the space (X, ϱ) is complete and separable and that μ and k satisfy conditions* (i) *and* (ii). *Then there exists a function $f \in \mathcal{F}$ such that*

$$\varPhi_\mu(f) = \sup_{g \in \mathcal{F}} \varPhi_\mu(g). \tag{3.3.4}$$

Moreover, if (X, ϱ) is locally compact separable space, $|\mu| > 0$ and a function $f \in \mathcal{F}$ satisfies (3.3.4) *then it fulfills at least one of the following two conditions*:

1° *There exist two points $x, y \in X$, $x \neq y$, such that*

$$|f(x) - f(y)| = \varrho(x, y). \tag{3.3.5}$$

2° *The function f has the following properties*:

$$f(x) = 1 \qquad \textit{for } x \in \operatorname{supp} \mu_+, \tag{3.3.6}$$

$$f(x) = -1 \qquad \textit{for } x \in \operatorname{supp} \mu_-. \tag{3.3.7}$$

Proof. The proof of the existence of $f \in \mathcal{F}$ satisfying (3.3.4) is similar to that of Theorem 3.2.3.

To complete the proof note that we have two possibilities: either f is not contractive and then (3.3.5) holds for some x and y, or f is contractive. In the latter case assume that

$$f(x_0) < 1 \quad \text{for some } x_0 \in \operatorname{supp} \mu_+. \tag{3.3.8}$$

Then there is a closed ball $B(x_0, r_0)$ such that

$$f(x) < 1 \quad \text{for } x \in B(x_0, r_0).$$

Moreover

$$\mu_+(B(x_0, r_0)) > 0.$$

Let $X = X_+ \cup X_-$ be the Hahn decomposition corresponding to μ. As before, according to the Ulam theorem there is a compact set $K \subseteq B(x_0, r_0) \cap X_+$ such that

$$\mu_+(K) > 0, \quad \mu_-(K) = 0. \tag{3.3.9}$$

Define

$$K_\delta = \{x \in X : \varrho(x, K) < \delta\}.$$

Since K is compact, we can find a $\delta > 0$ such that

$$\mu_-(K_\delta \setminus K) \le \mu_+(K)\frac{\delta_0}{2\delta_1} \quad \text{and} \quad \sup_{x \in K_\delta} f(x) < 1, \tag{3.3.10}$$

where

$$\delta_0 = \inf\{k_y(x, y) : (x, y) \in K \times [-1, 1]\}, \tag{3.3.11}$$

$$\delta_1 = \sup\{k_y(x, y) : (x, y) \in K_\delta \times [-1, 1]\}, \quad \delta_1 < \infty. \tag{3.3.12}$$

Since $K \subset K_\delta$ and K_δ is open, by Lemma 2.1.1 there exists $\varepsilon > 0$ and a contractive function $\tilde{f} : X \to \mathbb{R}$ satisfying conditions (2.1.3), (2.1.4) with $G = K_\delta$ and the inequality

$$\varepsilon < 1 - \sup_{x \in K_\delta} f(x). \tag{3.3.13}$$

By the mean value theorem we have

$$\Phi_\mu(\tilde{f}) - \Phi_\mu(f) = \int_X k_y(x, f(x) + \theta(x)(\tilde{f}(x) - f(x)))(\tilde{f}(x) - f(x))\, \mu(dx),$$

where $\theta(x) \in (0, 1)$. From (2.1.3), (2.1.4) and the equality $\mu_-(K) = 0$ it follows that

$$\begin{aligned}\Phi_\mu(\tilde{f}) - \Phi_\mu(f) \ge{}& \varepsilon \int_K k_y(x, f(x) + \theta(x)\varepsilon)\mu_+(dx) \\ &- \varepsilon \int_{K_\delta \setminus K} k_y(x, f(x) + \theta(x)(\tilde{f}(x) - f(x)))\, \mu_-(dx).\end{aligned}$$

Now using (3.3.11) and (3.3.12) we obtain

$$\Phi_\mu(\tilde{f}) - \Phi_\mu(f) \ge \varepsilon\delta_0\mu_+(K) - \varepsilon\, \delta_1\mu_-(K_\delta \setminus K),$$

which in virtue of (3.3.10) gives

$$\Phi_\mu(\tilde{f}) - \Phi_\mu(f) \ge \frac{\varepsilon\delta_0}{2}\mu_+(K).$$

Further, from (3.3.13) it follows that $\tilde{f} \in \mathcal{F}$. Consequently, the last inequality contradicts (3.3.4) and finishes the proof in this case. If $f(x_0) > -1$ for some $x_0 \in \operatorname{supp} \mu_-$, the argument is similar, based on Remark 2.1.1. ∎

Given two nonempty sets $A, B \subset X$ we define

$$\operatorname{dist}(A, B) = \inf\{\varrho(x, y) : x \in A,\ y \in B\}.$$

Using Theorem 3.3.1 it is easy to prove the following corollary which will be applied in Subsections 4.3 and 6.4.

Corollary 3.3.1. *Let* $\mu = \mu_2 - \mu_1$, *where* $\mu_2, \mu_1 \in \mathcal{M}_1$, $\mu_1 \neq \mu_2$ *and*

$$\operatorname{dist}(\operatorname{supp}\mu_+, \operatorname{supp}\mu_-) < 2. \tag{3.3.14}$$

Then every $f_0 \in \mathcal{F}$ *satisfying* (3.3.4) *with* $\mu = \mu_2 - \mu_1$ *fulfills condition* 1°.

Proof. The proof is straightforward. Suppose, on the contrary, that there exists a contractive $f_0 \in \mathcal{F}$ such that

$$\Phi_\mu(f_0) = \sup_{g\in\mathcal{F}} \Phi_\mu(g). \tag{3.3.15}$$

Using (3.3.14) we can find $x_0 \in \operatorname{supp}\mu_+$ and $y_0 \in \operatorname{supp}\mu_-$ such that $\varrho(x_0, y_0) < 2$. On the other hand, by condition 2° of the maximum principle we have $f_0(x_0) - f_0(y_0) = 2$, which is impossible. ■

Observe that the special linear function $k(x, y) = y$ for $(x, y) \in X \times [-1, 1]$ satisfies condition (ii) of Theorem 3.3.1. In this case (3.3.1) reduces to $\Phi_\mu(f) = \langle f, \mu\rangle$. Using this fact we obtain

Remark 3.3.1. Assume that (X, ϱ) is a locally compact separable metric space and μ satisfies condition (i). Then there exists a function $f_0 \in \mathcal{F}$ such that

$$\langle f_0, \mu\rangle = \|\mu\|_{\mathcal{F}}. \tag{3.3.16}$$

Moreover, if $|\mu| > 0$ and a function $f_0 \in \mathcal{F}$ satisfies (3.3.16), then it fulfills at least one of the following two conditions:

1° There exist two points $x, y \in X$, $x \neq y$, such that

$$|f_0(x) - f_0(y)| = \varrho(x, y). \tag{3.3.17}$$

2° The function f_0 has the following properties:

$$f_0(x) = 1 \qquad \text{for } x \in \operatorname{supp}\mu_+, \tag{3.3.18}$$

$$f_0(x) = -1 \qquad \text{for } x \in \operatorname{supp}\mu_-. \tag{3.3.19}$$

4. Asymptotically contractive semigroups of Markov operators

In this chapter we study a class of asymptotically contractive locally Lipschitzian Markov semigroups acting on the space of signed measures. Our results are based on maximum principles. In Chapter 6 we will apply these criteria to the stability theory of Markov–Feller semigroups.

4.1. Markov operators. An operator $P : \mathcal{M} \to \mathcal{M}$ is called a *Markov operator* if it satisfies the following conditions:

(i) P is positively linear:

$$P(\lambda_1\mu_1 + \lambda_2\mu_2) = \lambda_1 P\mu_1 + \lambda_2\mu_2$$

for $\lambda_1, \lambda_2 \geq 0$ and $\mu_1, \mu_2 \in \mathcal{M}$,

(ii) P preserves the measure of the space:

$$P\mu(X) = \mu(X) \quad \text{for } \mu \in \mathcal{M}. \tag{4.1.1}$$

Remark 4.1.1. Every Markov operator P can be uniquely extended as a linear operator to the space of signed measures. Namely for every $\mu \in \mathcal{M}_{\mathrm{sig}}$ we define

$$P\mu = P\mu_1 - P\mu_2, \quad \text{where} \quad \mu = \mu_1 - \mu_2,\ \mu_1, \mu_2 \in \mathcal{M}.$$

It is easy to verify that this definition does not depend on the choice of μ_1, μ_2.

A Markov operator P is called *regular* if there exists an operator $U : B(X) \to B(X)$ on the space of bounded Borel measurable functions such that

$$\langle Uf, \mu\rangle = \langle f, P\mu\rangle \quad \text{for } f \in B(X),\ \mu \in \mathcal{M}. \tag{4.1.2}$$

The operator U is called *dual* to P. If in addition $Uf \in C(X)$ for $f \in C(X)$, then the regular operator P is called a *Markov–Feller operator*.

Setting $\mu = \delta_x$ in (4.1.2) we obtain

$$(Uf)(x) = \langle f, P\delta_x\rangle \quad \text{for } f \in B(X),\ x \in X, \tag{4.1.3}$$

where $\delta_x \in \mathcal{M}_1$ is the point (Dirac) measure supported at x.

From formula (4.1.3) it follows immediately that U is linear and

$$Uf \geq 0 \quad \text{for } f \geq 0,\ f \in B(X), \tag{4.1.4}$$

$$U\mathbf{1}_X = \mathbf{1}_X, \tag{4.1.5}$$

$$Uf_n \downarrow 0 \quad \text{for } f_n \downarrow 0,\ f_n \in B(X). \tag{4.1.6}$$

Here $f_n \downarrow 0$ means that the sequence (f_n) is decreasing and pointwise converges to 0.

Conditions (4.1.4)–(4.1.6) allow one to reverse the roles of P and U. Namely, if a linear operator U satisfying (4.1.4)–(4.1.6) is given we may define a Markov operator $P : \mathcal{M} \to \mathcal{M}$ by setting

$$P\mu(A) = \langle U\mathbf{1}_A, \mu\rangle \quad \text{for } \mu \in \mathcal{M},\ A \in \mathcal{B}_X. \tag{4.1.7}$$

A mapping $\pi : X \times \mathcal{B}_X \to [0,1]$ is called a *transition function* if $\pi(x,\cdot)$ is a probability measure for every $x \in X$ and $\pi(\cdot, A)$ is a measurable function for every $A \in \mathcal{B}_X$.

Having a transition function π we may define the corresponding Markov operator $P : \mathcal{M}_{\mathrm{sig}} \to \mathcal{M}_{\mathrm{sig}}$ by the formula

$$P\mu(A) = \int_X \pi(x, A)\, \mu(dx) \quad \text{for } \mu \in \mathcal{M}_{\mathrm{sig}},\ A \in \mathcal{B}_X \tag{4.1.8}$$

and its dual operator $U : B(X) \to B(X)$ by

$$Uf(x) = \int_X f(u)\, \pi(x, du). \tag{4.1.9}$$

Conversely, having a regular Markov operator P we may define $\pi : X \times \mathcal{B}_X \to [0,1]$ by setting

$$\pi(x, A) = P\delta_x(A). \tag{4.1.10}$$

Clearly π is a transition function such that conditions (4.1.8) and (4.1.9) are satisfied.

Thus, conditions (4.1.8), (4.1.10) yield a one-to-one correspondence between the regular Markov operators and transition functions.

Note that a Markov operator P is Markov–Feller if and only if its transition function has the following property:

$$x_n \to x \quad \text{implies} \quad \pi(x_n, \cdot) \to \pi(x, \cdot) \text{ (weakly)}.$$

If this condition is satisfied the transition function π is called *Feller*.

REMARK 4.1.2. Observe that a Markov–Feller operator is continuous with respect to weak convergence. Namely, the weak convergence of (μ_n) to a measure μ implies the weak convergence of $(P\mu_n)$ to $P\mu$. This is a straightforward consequence of (4.1.2).

The dual operator U has a unique extension to the set of all Borel measurable nonnegative (not necessarily bounded) functions on X, such that formula (4.1.2) holds. Namely for a Borel measurable function $f : X \to \mathbb{R}^+$ we write

$$Uf(x) = \lim_{n\to\infty} Uf_n(x),$$

where $(f_n) \subset B(X)$ is an increasing sequence of bounded Borel measurable functions converging pointwise to f. Since the sequence (Uf_n) is increasing the limit Uf exists. Further from the Lebesgue monotone convergence theorem it follows that Uf satisfies (4.1.2). This formula shows that the limit is defined in a unique way and does not depend on the choice of the sequence (f_n). Evidently this extension is positively linear and monotonic.

For given $c \in X$ define

$$\varrho_c(x) := \varrho(x, c) \qquad \text{for } x \in X.$$

An important role in the study of the asymptotic behaviour of a Markov–Feller operator P is played by the function $U\varrho_c$, where U denotes the dual operator to P. Since ϱ_c is continuous and nonnegative the function $U\varrho_c$ is well defined.

If in addition $U\varrho_c$ is finite, i.e.

$$U\varrho_c(x) < \infty \qquad \text{for } x \in X, \tag{4.1.11}$$

then the operator U can be extended to a linear space of functions satisfying an appropriate growth condition. To formulate this fact precisely we introduce the following notion:

A function $f : X \to \mathbb{R}$ will be called *linearly bounded* if there exist nonnegative constants A, B such that

$$|f(x)| \le A\,\varrho_c(x) + B \qquad \text{for } x \in X. \tag{4.1.12}$$

The family of linearly bounded functions will be denoted by $\mathcal{L}(X)$.

REMARK 4.1.3. If condition (4.1.11) is satisfied then for every $f \in \mathcal{L}(X)$ the functions Uf^+, Uf^- also belong to $\mathcal{L}(X)$. Therefore the function

$$Uf(x) := Uf^+(x) - Uf^-(x) \tag{4.1.13}$$

is well defined and belongs to $\mathcal{L}(X)$. Elementary calculations show that U defined by (4.1.13) has the following properties:

1. U maps $\mathcal{L}(X)$ into itself.
2. U is linear and nondecreasing.
3. $|Uf| \le U|f|$ for $f \in \mathcal{L}(X)$.

Using the above remark it is easy to prove the following proposition:

PROPOSITION 4.1. *Let $P : \mathcal{M}_{\text{sig}} \to \mathcal{M}_{\text{sig}}$ be a Markov–Feller operator and let U be its dual. Assume that $U\varrho_c$ is a linearly bounded function. Then*

$$P(\mathcal{M}_{\text{sig},1}) \subset \mathcal{M}_{\text{sig},1}. \tag{4.1.14}$$

Moreover, for every $f \in \mathcal{L}(X)$ and $\mu \in \mathcal{M}_{\text{sig},1}$ the integrals $\langle Uf, \mu\rangle$, $\langle f, P\mu\rangle$ are finite and

$$\langle Uf, \mu\rangle = \langle f, P\mu\rangle \quad \text{for } f \in \mathcal{L}(X),\ \mu \in \mathcal{M}_{\text{sig},1}. \tag{4.1.15}$$

Proof. Conditions (4.1.14) and (4.1.15) follow immediately from the fact that for $f \in \mathcal{L}(X)$ and $\mu \in \mathcal{M}_{\text{sig},1}$ the eight integrals $\langle f^+, P\mu^+\rangle, \dots \langle Uf^-, \mu^-\rangle$ exist and are finite. ∎

Let d be an arbitrary metric on $\mathcal{M}_{\text{sig}}$. A Markov operator $P : \mathcal{M}_{\text{sig}} \to \mathcal{M}_{\text{sig}}$ is called *Lipschitzian* with respect to d with constant $k > 0$ if

$$d(P\mu_1, P\mu_2) \le k\, d(\mu_1, \mu_2) \quad \text{for } \mu_1, \mu_2 \in \mathcal{M}_{\text{sig}}. \tag{4.1.16}$$

If $k \le 1$ then P is *nonexpansive*.

A Markov operator $P : \mathcal{M}_{\text{sig}} \to \mathcal{M}_{\text{sig}}$ is called *contractive in the class* $\widetilde{\mathcal{M}} \subset \mathcal{M}_{\text{sig}}$ with respect to d if

$$d(P\mu_1, P\mu_2) < d(\mu_1, \mu_2) \quad \text{for } \mu_1, \mu_2 \in \widetilde{\mathcal{M}}. \tag{4.1.17}$$

REMARK 4.1.4. Note that a regular operator $P : \mathcal{M}_{\text{sig}} \to \mathcal{M}_{\text{sig}}$ is nonexpensive with respect to the Fortet–Mourier metric if and only if $U(\mathcal{F}) \subset \mathcal{F}$. This is an immediate consequence of formula (4.1.2).

Let T be a *nontrivial semigroup* of nonnegative real numbers. More precisely we assume that $\{0\} \subsetneq T \subset \mathbb{R}_+$ and

$$t_1 + t_2 \in T,\ t_1 - t_2 \in T \quad \text{for } t_1, t_2 \in T,\ t_1 \ge t_2. \tag{4.1.18}$$

A family of Markov operators $(P^t)_{t\in T}$ is called a *semigroup* if

$$P^{t+s} = P^t\, P^s \quad \text{for } t, s \in T$$

and $P^0 = I$ where I is the identity operator.

If the Markov operators P^t are Markov–Feller for $t \in T$, we say that $(P^t)_{t\in T}$ is a *Markov–Feller semigroup.* We denote by $(U^t)_{t\in T}$ the semigroup of the dual operators to $(P^t)_{t\in T}$.

A Markov semigroup $(P^t)_{t\in T}$ is called *locally Lipschitzian with respect to d in the class* $\widetilde{\mathcal{M}} \subset \mathcal{M}_{\text{sig}}$ if there exists a locally bounded function $\mathbf{k} : T \to \mathbb{R}^+$ such that for every $t \in T$ and $\mu_1, \mu_2 \in \widetilde{\mathcal{M}}$,

$$d(P^t\mu_1, P^t\mu_2) \le \mathbf{k}(t) d(\mu_1, \mu_2). \tag{4.1.19}$$

If $\mathbf{k}(t) \le 1$ for $t \in T$, then $(P^t)_{t\in T}$ is a *nonexpansive semigroup.*

A nonexpansive semigroup $(P^t)_{t\in T}$ is called *asymptotically contractive with respect to d in the class $\widetilde{\mathcal{M}} \subset \mathcal{M}_{\mathrm{sig}}$* if for every $\mu_1, \mu_2 \in \widetilde{\mathcal{M}}$, $\mu_1 \neq \mu_2$, there is $t_0 \in T$ such that

$$d(P^{t_0}\mu_1, P^{t_0}\mu_2) < d(\mu_1, \mu_2).$$

4.2. Asymptotically contractive semigroups with respect to the Hutchinson metric. In this section we study Markov–Feller semigroups which are asymptotically contractive in the class $\mathcal{M}_{1,\alpha}$ with respect to the Hutchinson metric. To verify that some semigroups have the desired asymptotic properties we use the maximum principle formulated in Theorem 3.2.3.

THEOREM 4.2.1. *Let $(P^t)_{t\in T}$ be a Markov–Feller semigroup and $(U^t)_{t\in T}$ its dual semigroup. Assume that there is $t_0 \in T$ such that for every $f \in \mathcal{H}$,*

$$|U^t f(x) - U^t f(y)| \le \varrho(x,y) \quad \text{for } x, y \in X \text{ and } t \in T, \tag{4.2.1}$$

$$|U^{t_0} f(x) - U^{t_0} f(y)| < \varrho(x,y) \quad \text{for } x, y \in X,\ x \neq y. \tag{4.2.2}$$

Moreover, assume that there exists a constant $\alpha > 1$ such that

$$P^t(\mathcal{M}_{1,\alpha}) \subset \mathcal{M}_{1,\alpha} \quad \text{for } t \ge 0. \tag{4.2.3}$$

Then $(P^t)_{t\in T}$ is asymptotically contractive with respect to the Hutchinson metric in the class $\mathcal{M}_{1,\alpha}$.

Proof. From (4.2.1)–(4.2.3), it follows immediately that $(P^t)_{t\in T}$ is nonexpansive on $\mathcal{M}_{1,\alpha}$ with respect to the Hutchinson metric. Indeed, for $\mu_1, \mu_2 \in \mathcal{M}_{1,\alpha}$ and $t \in T$ we have

$$\begin{aligned} \|P^t\mu_1 - P^t\mu_2\|_{\mathcal{H}} &= \sup\{|\langle f, P^t\mu_1 - P^t\mu_2\rangle| : f \in \mathcal{H}\} \\ &= \sup\{|\langle U^t f, \mu_1 - \mu_2\rangle| : f \in \mathcal{H}\} \le \|\mu_1 - \mu_2\|_{\mathcal{H}}. \end{aligned} \tag{4.2.4}$$

Note that $\mathcal{M}_{1,\alpha} \subset \mathcal{M}_{1,1}$ for $\alpha > 1$ and fix $\mu_1, \mu_2 \in \mathcal{M}_{1,\alpha}$, $\mu_1 \neq \mu_2$. By Theorem 3.2.1 there exists $f \in \mathcal{H}$ such that

$$\langle f, P^{t_0}\mu_1 - P^{t_0}\mu_2\rangle = \|P^{t_0}\mu_1 - P^{t_0}\mu_2\|_{\mathcal{H}}. \tag{4.2.5}$$

This may be rewritten in the form

$$\langle U^{t_0} f, \mu_1 - \mu_2\rangle = \|P^{t_0}\mu_1 - P^{t_0}\mu_2\|_{\mathcal{H}}.$$

As $U^{t_0} f$ satisfies (4.2.1), by the second part of Theorem 3.2.1 we obtain

$$\langle U^{t_0} f, \mu_1 - \mu_2\rangle < \|\mu_1 - \mu_2\|_{\mathcal{H}}. \tag{4.2.6}$$

This inequality and (4.2.4) show that $(P^t)_{t\in T}$ is asymptotically contractive with respect to the Hutchinson metric in $\mathcal{M}_{1,\alpha}$. ■

REMARK 4.2.1. Sometimes (4.2.3) can be verified using a more explicit condition. Namely, let c be a fixed element of X and $\varrho_c^\alpha(x) := (\varrho(x,c))^\alpha$ for $x \in X$, $\alpha > 0$. If there exist constants $A, B \ge 0$ and $\alpha > 1$ such that

$$(U^t \varrho_c^\alpha)(x) \le A\varrho_c^\alpha(x) + B \quad \text{for } x \in X \text{ and } t \in T, \tag{4.2.7}$$

then the condition (4.2.3) is satisfied.

As a consequence of Theorem 4.2.1 we obtain the following

Corollary 4.2.1. *Let $P : \mathcal{M}_{\mathrm{sig}} \to \mathcal{M}_{\mathrm{sig}}$ be a Markov–Feller operator and let U be its dual operator. Assume that for every $f \in \mathcal{H}$,*

$$|Uf(x) - Uf(y)| < \varrho(x,y) \quad \text{for } x, y \in X,\ x \neq y. \tag{4.2.8}$$

Moreover, assume that there exists a constant $\alpha > 1$ such that $P(\mathcal{M}_{1,\alpha}) \subset \mathcal{M}_{1,\alpha}$. Then $(P^n)_{n\in\mathbb{N}}$ is asymptotically contractive with respect to the Hutchinson metric in the class $\mathcal{M}_{1,\alpha}$.

4.3. Asymptotically contractive semigroups with respect to the Fortet–Mourier metric. In this section we study Markov–Feller semigroups which are asymptotically contractive in the class $\mathcal{M}_1$ with respect to the Fortet–Mourier metric. The proofs are based on the maximum principle formulated in Remark 3.3.1.

Theorem 4.3.1. *Let $(P^t)_{t\in T}$ be a Markov–Feller semigroup and $(U^t)_{t\in T}$ its dual semigroup. Assume that the following conditions are satisfied:*

(i) *For every $t \in T$,*

$$|U^t f(x) - U^t f(y)| \le \varrho(x,y) \quad \text{for } x, y \in X. \tag{4.3.1}$$

There is $t_0 \in T$ such that for every $f \in \mathcal{F}$,

$$|U^{t_0} f(x) - U^{t_0} f(y)| < \varrho(x,y) \quad \text{for } x, y \in X,\ x \neq y. \tag{4.3.2}$$

(ii) *For every $\mu_1, \mu_2 \in \mathcal{M}_1$, $\mu_1 \neq \mu_2$, there exists $t_1 \in T$ such that either $P^{t_1}(\mu_1) = P^{t_1}(\mu_2)$ or*

$$\operatorname{dist}(\operatorname{supp}(P^{t_1}(\mu_1 - \mu_2))_+, \operatorname{supp}(P^{t_1}(\mu_1 - \mu_2))_-) < 2. \tag{4.3.3}$$

Then $(P^t)_{t\in T}$ is asymptotically contractive with respect to the Fortet–Mourier metric in the class $\mathcal{M}_1$.

Proof. From (4.3.1), it follows immediately that $U^t(\mathcal{F}) \subset \mathcal{F}$ for $t \in T$, and that $(P^t)_{t\in T}$ is nonexpansive. Indeed, for $\mu_1, \mu_2 \in \mathcal{M}_1$ and $t \in \mathbb{R}^+$ we have

$$\begin{aligned} \|P^t\mu_1 - P^t\mu_2\|_{\mathcal{F}} &= \sup\{|\langle U^t f, \mu_1 - \mu_2\rangle| : f \in \mathcal{F}\} \\ &\le \sup\{|\langle f, \mu_1 - \mu_2\rangle| : f \in \mathcal{F}\} = \|\mu_1 - \mu_2\|_{\mathcal{F}}. \end{aligned} \tag{4.3.4}$$

Fix $\mu_1, \mu_2 \in \mathcal{M}_1$, $\mu_1 \neq \mu_2$. By Remark 3.3.1 there exists $f_0 \in \mathcal{F}$ such that

$$\langle f_0, P^{t_0+t_1}\mu_1 - P^{t_0+t_1}\mu_2\rangle = \|P^{t_0+t_1}\mu_1 - P^{t_0+t_1}\mu_2\|_{\mathcal{F}}. \tag{4.3.5}$$

This may be rewritten in the form

$$\langle U^{t_0} f_0, P^{t_1}\mu_1 - P^{t_1}\mu_2\rangle = \|P^{t_0+t_1}\mu_1 - P^{t_0+t_1}\mu_2\|_{\mathcal{F}}. \tag{4.3.6}$$

If $P^{t_1}\mu_1 = P^{t_2}\mu_2$ then automatically

$$\|P^{t_0+t_1}\mu_1 - P^{t_0+t_1}\mu_2\|_{\mathcal{F}} < \|\mu_1 - \mu_2\|_{\mathcal{F}}. \tag{4.3.7}$$

If not, we can apply Remark 3.3.1 to the measure $P^{t_1}\mu_1 - P^{t_1}\mu_2$ and the contractive function $U^{t_0} f_0$. By Corollary 3.3.1 this gives

$$\langle U^{t_0} f_0, P^{t_1}\mu_1 - P^{t_1}\mu_2\rangle < \|P^{t_1}\mu_1 - P^{t_1}\mu_2\|_{\mathcal{F}}. \tag{4.3.8}$$

The last inequality and (4.3.4) again imply (4.3.7). ∎

THEOREM 4.3.2. *Let $(P^t)_{t\in T}$ be a Markov–Feller semigroup and $(U^t)_{t\in T}$ its dual semigroup. Assume that the following conditions are satisfied*:

(i) *For every* $t \in T$,

$$|U^t f(x) - U^t f(y)| \le \varrho(x,y) \quad \text{for } x,y \in X \text{ and } t \in T. \tag{4.3.9}$$

There is $t_0 \in \mathbb{R}^+$ *such that for every* $f \in \mathcal{F}$

$$|U^{t_0} f(x) - U^{t_0} f(y)| < \varrho(x,y) \quad \text{for } x,y \in X,\ x \neq y. \tag{4.3.10}$$

(ii) *There exist constants* $t_0, t_1, t_2 \in T$ *such that for every* $f \in \mathcal{F}$ *we have either*

$$U^{t_0+t_1} f(x) \in (-1,1] \quad \text{for } x \in X$$

or

$$U^{t_0+t_2} f(x) \in [-1,1) \quad \text{for } x \in X.$$

Then $(P^t)_{t\in T}$ *is asymptotically contractive in the class* $\mathcal{M}_1$ *with respect to the Fortet–Mourier metric.*

Proof. We may repeat the argument used in the proof of Theorem 4.3.1. However, in this case for $\mu_1, \mu_2 \in \mathcal{M}_1$, $\mu_1 \neq \mu_2$, equality (4.3.5) should be replaced by

$$\langle f_0, P^{t_0+\tilde t}\mu_1 - P^{t_0+\tilde t}\mu_2\rangle = \|P^{t_0+\tilde t}\mu_1 - P^{t_0+\tilde t}\mu_2\|_{\mathcal{F}}, \tag{4.3.11}$$

where $\tilde t = \min(t_1, t_2)$ and $f_0 \in \mathcal{F}$. This equality may be rewritten in the form

$$\langle U^{t_0+\tilde t} f_0, \mu_1 - \mu_2\rangle = \|P^{t_0+\tilde t}\mu_1 - P^{t_0+\tilde t}\mu_2\|_{\mathcal{F}}. \tag{4.3.12}$$

From (4.3.10), (4.3.9) it follows that

$$|U^{t_0+\tilde t} f_0(x) - U^{t_0+\tilde t} f_0(y)| < \varrho(x,y) \quad \text{for } x,y \in X,\ x \neq y. \tag{4.3.13}$$

According to Remark 3.3.1 conditions (ii), (4.3.13) and (4.3.12) imply

$$\|P^{t_0+\tilde t}\mu_1 - P^{t_0+\tilde t}\mu_2\| < \|\mu_1 - \mu_2\|.$$

This inequality and (4.3.9) show that the semigroup $(P^t)_{t\in T}$ is asymptotically contractive in $\mathcal{M}_1$ with respect to the Fortet–Mourier metric. ■

We may simplify the verification of condition (ii). Namely we have the following

PROPOSITION 4.3.1. *Let* $\pi : X \times \mathcal{B}_X \to [0,1]$ *be a transition function and let* U *be the corresponding dual operator. Assume that*

$$\operatorname{supp} \pi(x,\cdot) = X \quad \text{for } x \in X. \tag{4.3.14}$$

Then for every $f \in C(X)$, $\|f\| \le 1$ *either*

$$Uf(x) = 1 \quad \text{for } x \in X$$

or

$$Uf(x) = -1 \quad \text{for } x \in X$$

or

$$Uf(x) \in (-1,1) \quad \text{for } x \in X.$$

Proof. Fix $f \in C(X)$, $\|f\| \le 1$, and suppose that there exists an $x_1 \in X$ such that $Uf(x_1) = 1$. By the properties of the dual operator we have

$$U\,1_X(x_1) - U\,f(x_1) = \int_X [1_X(y) - f(y)]\,\pi(x_1, dy) = 0.$$

From this and the inequality $f \le 1_X$ it follows that

$$f(x) = 1 \quad \pi(x_1, \cdot)\text{-almost everywhere.} \tag{4.3.15}$$

This and condition (4.3.14) imply that $f(x) = 1$ for $x \in X$. Since U is the dual operator, we finally obtain $Uf(x) = 1$ for $x \in X$. If there exists an $x_2 \in X$ such that $Uf(x_2) = -1$ the argument is similar. ■

The following example shows that in the statement of Proposition 4.3.1 the assumption (4.3.14) is essential.

EXAMPLE 4.3.1. Let $X = \mathbb{R}$ with the Euclidean metric. Further, let $P : \mathcal{M}_{\mathrm{sig}} \to \mathcal{M}_{\mathrm{sig}}$ be the Markov–Feller operator defined by the formula

$$P\mu(A) = \mu(-A). \tag{4.3.16}$$

It is easy to verify that $\pi(x, A) = \delta_x(-A)$ and consequently

$$\operatorname{supp} \pi(1, \cdot) = \{-1\} \ne X \quad \text{and} \quad \operatorname{supp} \pi(-1, \cdot) = \{1\} \ne X$$

Further let $f_0 \in \mathcal{F}$ be given by the formula

$$f_0(x) = \begin{cases} 1 & \text{for } x \ge 1, \\ x & \text{for } x \in (-1, 1), \\ -1 & \text{for } x \le -1. \end{cases}$$

From the definition of the dual operator it follows immediately that

$$Uf_0(-1) = \langle f_0, P\,\delta_{-1} \rangle = \langle f_0, \delta_1 \rangle = 1 \quad \text{and} \quad Uf_0(1) = \langle f_0, P\,\delta_1 \rangle = \langle f_0, \delta_{-1} \rangle = -1.$$

5. Invariance principle

In 1976 J. P. LaSalle (see [19, Chapter 1, Theorem 10.7]) proved that every compact trajectory of a dynamical system $(S^t)_{t\in T}$ converges to the largest invariant subset of the set $\{x : \dot V(x) = 0\}$, where V is a Lyapunov–LaSalle function and $\dot V$ its derivative with respect to the system. This result is called the *invariance principle*. Various versions of the invariance principle were studied and used in the proofs of the asymptotic stability of dynamical systems (see for example [20, Theorem 2.1], [26, Theorem 1.1] and [38, Chapter IV Theorem 4.2]). We show a new version of this principle which generalizes the results of A. Lasota (see [20], Theorem 2.1) and A. Lasota and J. Traple (see [26, Theorem 1.1]).

5.1. Criteria for the asymptotic stability of trajectories. For the convenience of the reader we recall a few definitions from the theory of dynamical systems. (For details see [20].)

Let X be a Hausdorff topological space. Further, let T be a nontrivial semigroup of nonnegative real numbers as in Chapter 4, i.e., we assume that T satisfies condition (4.1.18).

A semigroup $(S^t)_{t\in T}$ of maps $X \to X$ is called a *semidynamical system* if $X \ni x \mapsto S^t x \in X$ is continuous for every $t \in T$.

If a semidynamical system $(S^t)_{t\in T}$ is given, then for every fixed $x \in X$ the function $T \ni t \mapsto S^t x \in X$ will be called the *trajectory* starting from x and denoted $(S^t x)$. A point $y \in X$ is called the *limiting point* of the trajectory $(S^t x)$ if there exists a sequence $(t_n) \subset T$ such that $t_n \to \infty$ and

$$\lim_{n\to\infty} S^{t_n} x = y.$$

The set of all limiting points of the trajectory $(S^t x)$ will be denoted $\Omega(x)$. Further, we write

$$\gamma(x) = \{S^t x : t \in T\} \quad \text{and} \quad \Gamma(x) = \gamma(x) \cup \Omega(x).$$

A set $C \subset X$ is called *invariant* with respect to $(S^t)_{t\in T}$ if $S^t(C) \subset C$ for $t \in T$.

REMARK 5.1.1. Let $(S^t)_{t\in T}$ be a semidynamical system and let $\{C_\lambda\}_{\lambda\in\Lambda}$ be a family of invariant sets. It is easy to see that their union and intersection are also invariant with respect to $(S^t)_{t\in T}$.

From Remark 5.1.1 it follows immediately that every set $C \subset X$ contains the maximal invariant subset M which is the union of all invariant subsets of C. The set M may of course be empty.

We say that a function $\varphi : T \to X$ *converges to a set* $A \subset X$ if for every open $G \supset A$ there exists $t_0 \in T$ such that

$$\varphi(t) \in G \quad \text{for } t \geq t_0,\ t \in T. \tag{5.1.1}$$

From this definition it follows that $A \neq \emptyset$. Observe that if $A \subset B \subset X$ and φ converges to A then it also converges to B.

If $A = \{a\}$ is a singleton then φ converges to $\{a\}$ if and only if $\lim_{t\to\infty} \varphi(t) = a$.

REMARK 5.1.2. If $(S^t)_{t\in T}$ is a semidynamical system then the sets $\gamma(x), \Omega(x)$ and $\Gamma(x)$ are invariant for every $x \in X$. It is easy to verify that $\Gamma(x)$ is the minimal invariant subset of X which contains x and $\Omega(x)$.

Let $(S^t)_{t\in T}$ be a semidynamical system and let $x \in X$. We say that a trajectory $(S^t x)$ is *sequentially compact* if for every sequence $(t_n) \subset T$ with $t_n \to \infty$, there exists a subsequence (t_{k_n}) such that $(S^{t_{k_n}} x)$ converges to a point $y \in X$.

REMARK 5.1.3. If the trajectory $(S^t x)$ is sequentially compact, then $\Omega(x)$ is a nonempty, sequentially compact set and $S^t x$ converges to $\Omega(x)$. Moreover in this case $\Omega(x)$ is *strictly invariant*, i.e.

$$S^t(\Omega(x)) = \Omega(x) \quad \text{for } t \in T.$$

A point $x_* \in X$ is called *stationary* (or *invariant*) with respect to a semidynamical system $(S^t)_{t\in T}$ if

$$S^t x_* = x_* \quad \text{for } t \in T. \tag{5.1.2}$$

A semidynamical system $(S^t)_{t\in T}$ is called *asymptotically stable* if there exists a stationary point $x_* \in X$ such that

$$\lim_{t\to\infty} S^t x = x_* \quad \text{for } x \in X. \tag{5.1.3}$$

REMARK 5.1.4. Since X is a Hausdorff space, an asymptotically stable dynamical system has exactly one stationary point.

Let a nonempty invariant set $A \subset X$ be given. A function $V : A \to \mathbb{R}$ is called a *Lyapunov–LaSalle function* for a semidynamical system $(S^t)_{t\in T}$ if V is continuous and

$$V(S^{t_1}(x)) \geq V(S^{t_2}(x)) \quad \text{for } x \in A \text{ and } t_1 \leq t_2,\ t_1, t_2 \in T \tag{5.1.4}$$

(see [19, Chapter I, Definition 6.1 and Definition 8.2]).

A function $d : X \times X \to \mathbb{R}^+$ is called a *distance* if d is continuous and

$$d(x, y) = 0 \ \Leftrightarrow\ x = y \quad \text{for } x, y \in X. \tag{5.1.5}$$

In the proof of our main result Theorem 5.1.2 we will use the following properties of Lyapunov–LaSalle functions:

THEOREM 5.1.1. *Let $(S^t)_{t\in T}$ be a semidynamical system and let $x_0 \in X$. Assume that $V : \Gamma(x_0) \to \mathbb{R}$ is a Lyapunov–LaSalle function. Then there exists $\beta \in \mathbb{R}_+$ such that*

$$\Omega(x_0) \subset V^{-1}(\beta). \tag{5.1.6}$$

Further, if the trajectory $(S^t x_0)$ is sequentially compact, then

$$\beta = \lim_{t\to\infty} V(S^t(x_0)) \tag{5.1.7}$$

satisfies (5.1.6) *and the trajectory $(S^t x_0)$ converges to the largest invariant subset $M(x_0)$ of $V^{-1}(\beta)$. In this case $\Omega(x_0) \subset M(x_0)$.*

The proof of Theorem 5.1.1 can be found in [20, pp. 113–114] and [38, pp. 168–170].

In order to formulate our theorem we consider a semidynamical system $(S^t)_{t\in T}$ which has at least one sequentially compact trajectory. Further, let d be an arbitrary distance on X. We denote by Z the set of all $z \in X$ such that the trajectory $(S^t z)$ is sequentially compact. Since $Z \neq \emptyset$ we have

$$\Omega = \bigcup_{z\in Z} \Omega(z) \neq \emptyset.$$

The main result of this chapter is the following.

THEOREM 5.1.2. *Let $x_* \in \Omega$ be fixed. Assume that for every $x \in \Omega$, $x \neq x_*$, there is $t(x) \in T$ such that*

$$d(S^{t(x)} x, S^{t(x)} x_*) < d(x, x_*). \tag{5.1.8}$$

Further assume that the semidynamical system $(S^t)_{t\in T}$ is nonexpansive with respect to d, i.e.,

$$d(S^t x, S^t y) \leq d(x, y) \quad \textit{for } x, y \in X \textit{ and } t \in T. \tag{5.1.9}$$

Then x_* is a stationary point of $(S^t)_{t\in T}$ and

$$\lim_{t\to\infty} d(S^t z, x_*) = 0 \quad \textit{for } z \in Z. \tag{5.1.10}$$

Proof. We break up the proof of Theorem 5.1.2 into three steps.

STEP I. Choose $x_0 \in Z$ such that $x_* \in \Omega(x_0)$. We claim that every point $y \in \Omega(x_0)$ is stationary with respect to $(S^t)_{t\in T}$. To prove this fix $r \in T$ and consider the function $V_r : \Gamma(x_0) \to \mathbb{R}^+$ given by the formula

$$V_r(x) = d(S^r x, x) \quad \text{for } x \in \Gamma(x_0).$$

Using (5.1.9) it is easy to verify that V_r is a Lyapunov–LaSalle function. In fact, for every $x \in \Gamma(x_0)$ and $t_1 \ge t_2$ $(t_1, t_2 \in T)$ we have

$$\begin{aligned} V_r(S^{t_1}x) &= d(S^{t_1+r}x, S^{t_1}x) = d(S^{t_1-t_2}(S^{t_2+r}x), S^{t_1-t_2}(S^{t_2}x)) \\ &\le d(S^{t_2+r}(x), S^{t_2}x) = V_r(S^{t_2}x). \end{aligned}$$

Since $x_0 \in Z$, the trajectory $(S^t x_0)$ is sequentially compact and converges to $\Omega(x_0)$ which is an invariant subset of the set

$$\{x : V_r(x) = \beta\} \quad \text{where} \quad \beta = \lim_{t\to\infty} V_r(S^t x_0).$$

Further, since $x_* \in \Omega(x_0)$ and V_r is continuous, we have $V_r(x_*) = \beta$. Now we are going to show that $\beta = 0$. Suppose not. Then $d(S^r x_*, x_*) > 0$ and $S^r x_* \ne x_*$. Using the invariance of $\Omega(x_0)$, the inclusion $\Omega(x_0) \subset V_r^{-1}(\beta)$ and the condition (5.1.8) we obtain

$$\beta = V_r(S^{t(S^r x_*)}x_*) = d(S^{t(S^r x_*)}(S^r x_*), S^{t(S^r x_*)}x_*) < d(S^r x_*, x_*) = V_r(x_*) = \beta,$$

which is impossible. Thus we get $\beta = 0$. Let $y \in \Omega(x_0)$ be given and let $t_n \to \infty$ be such that $y = \lim_{n\to\infty} S^{t_n}x_0$. We have $\lim_{n\to\infty} V(S^{t_n}x_0) = 0$ and consequently

$$d(S^r y, y) = \lim_{n\to\infty} d(S^{t_n+r}x_0, S^{t_n}x_0) = 0.$$

This shows that $S^r y = y$ for $y \in \Omega(x_0)$. Since $r \in T$ was arbitrary, the proof of the claim is complete.

STEP II. Since $x_* \in \Omega(x_0)$, we have $S^t(x_*) = x_*$ and so (5.1.9) yields

$$d(S^t x, x_*) \le d(x, x_*) \quad \text{for } x \in X. \tag{5.1.11}$$

Now we are going to prove that for every $z \in Z$ and $\varepsilon > 0$ for which the set $K_z(\varepsilon) = \Omega(z) \cap \{x \in X : d(x, x_*) \ge \varepsilon\} \ne \emptyset$ there exists a constant $t_0 > 0$ such that

$$d(S^{t_0}x, x_*) < d(x, x_*) \quad \text{for } x \in K_z(\varepsilon) \text{ and } t \ge t_0,\ t \in T. \tag{5.1.12}$$

Suppose not. Then for some $z \in Z$ and $\varepsilon > 0$ there exists a sequence $(x_n) \subset K_z(\varepsilon)$ and a sequence $(t_n) \subset T$ such that

$$d(S^t x_n, x_*) = d(x_n, x_*) \quad \text{for } n \ge 1 \quad \text{and} \quad \lim_{n\to\infty} t_n = \infty.$$

Passing to a subsequence if necessary, we may assume that (x_n) converges to a point $\bar{x} \in K_z(\varepsilon)$. Fix $t \in T$ and choose $\bar{n} \in \mathbb{N}$ such that $t_n > t$ for $n \ge \bar{n}$. It is evident that

$$d(S^t x_n, x_*) = d(x_n, x_*) \quad \text{for } n \ge \bar{n}.$$

From the continuity of d and S^t it follows immediately that

$$d(S^t \bar{x}, x_*) = d(\bar{x}, x_*) \quad \text{for } t \in T. \tag{5.1.13}$$

Moreover, from the definition of $K_z(\varepsilon)$ and the continuity of d it follows that $d(\bar{x}, x_*) \geq \varepsilon$ and $\bar{x} \neq x_*$. Thus according to (5.1.8) and (5.1.13) we have

$$d(\bar{x}, x_*) = d(S^{t(\bar{x})}\bar{x}, x_*) < d(\bar{x}, x_*),$$

which is impossible. The proof of the inequality (5.1.12) is complete.

STEP III. Now we are going to prove that all the sets $\Omega(z)$ are identical singletons:

$$\Omega(z) = \{x_*\} \quad \text{for } z \in Z. \tag{5.1.14}$$

Fix $z \in Z$ and suppose, on the contrary, that

$$\varepsilon = \sup\{d(v, x_*) : v \in \Omega(z)\} > 0. \tag{5.1.15}$$

Since the function $v \mapsto d(v, x_*) \in \mathbb{R}^+$ is continuous and $\Omega(z)$ is sequentially compact, there exists a point $\tilde{v} \in \Omega(z)$ such that

$$\varepsilon = d(\tilde{v}, x_*). \tag{5.1.16}$$

Evidently $\tilde{v} \in K_z(\varepsilon)$. Thus according to Step II there exists $t_0 \in T$ such that

$$d(S^{t_0}x, x_*) < d(x, x_*) \quad \text{for } x \in K_z(\varepsilon). \tag{5.1.17}$$

Fix $\tilde{t} > t_0 > 0$, $\tilde{t} \in T$. Since $\Omega(z)$ is strictly invariant, there exists $\tilde{u} \in \Omega(z)$ such that $S^{\tilde{t}}\tilde{u} = \tilde{v}$. Further, by (5.1.11) and (5.1.16) we have

$$\varepsilon = d(\tilde{v}, x_*) = d(S^{\tilde{t}}\tilde{u}, x_*) \leq d(\tilde{u}, x_*).$$

Since $\tilde{u} \in \Omega(z)$ the last inequality and (5.1.15) imply that $d(\tilde{u}, x_*) = \varepsilon$. Consequently, $\tilde{u} \in K_z(\varepsilon)$ and we can apply inequality (5.1.17) to the point $\tilde{u}$. This and (5.1.11) give

$$\varepsilon = d(\tilde{u}, x_*) > d(S^{t_0}\tilde{u}, x_*) \geq d(S^{\tilde{t}-t_0}(S^{t_0}\tilde{u}), x_*) = d(S^{\tilde{t}}\tilde{u}, x_*) = \varepsilon,$$

which is impossible. Therefore condition (5.1.14) is satisfied.

Since the trajectory $(S^t z)$ converges to $\Omega(z) = \{x_*\}$ for every $z \in Z$, this completes the proof. ∎

5.2. Asymptotic stability of a nonlinear Boltzmann-type equation. To illustrate the application of the results developed in Section 5.1 we will discuss an example drawn from the kinetic theory of gases. This example was stimulated by the problem of stability of solutions of the following version of the Boltzmann equation:

$$\frac{\partial u(t,x)}{\partial t} + u(t,x) = \int_x^\infty \frac{dy}{y} \int_0^y u(t, y-z)u(t,z)\,dz, \qquad t \geq 0,\ x \geq 0. \tag{5.2.1}$$

Due to the physical interpretation equation (5.2.1) is considered with the additional conditions

$$\int_0^\infty u(t,x)\,dx = \int_0^\infty xu(t,x)\,dx = 1. \tag{5.2.2}$$

Equation (5.2.1) was derived by J. A. Tjon and T. T. Wu from the Boltzmann equation (see [36]). Following Barnsley and Cornille [1] we call it the *Tjon–Wu equation.* It is easy to see that the function $u_*(t,x) := \exp(-x)$ is a (stationary) solution of (5.2.1).

M. F. Barnsley and G. Turchetti (see [2, p. 369]) proved that this solution is stable in the class of all initial functions $u_0 := u(0, \cdot)$ satisfying the condition

$$\int_0^\infty u_0(x) e^{x/2}\, dx < \infty. \tag{5.2.3}$$

This condition was replaced by T. Dłotko and A. Lasota (see [6], Theorem 3) by a less restrictive

$$\int_0^\infty x^n u_0(x)\, dx < \infty \qquad \text{for } n = 2, 3, \ldots. \tag{5.2.4}$$

In 1990 Z. Kiełek (see [18, Theorem 1.1]) succeeded in proving that the stationary solution u_* is asymptotically stable if (5.2.4) is satisfied for $n = 2$.

Equation (5.2.1) has a simple interpretation. For fixed $t \geq 0$ the function $u(t, \cdot)$ denotes the density distribution function of the energy of the particle in an ideal gas. In the time interval $(t, t+\Delta t)$ the particle changes its energy with the probability $\Delta t + o(\Delta t)$ and the change is equal to $[-u(t,x) + P(u(t,x))]\Delta t + o(\Delta t)$, where the operator P is given by the formula

$$(Pv)(x) = \int_x^\infty \frac{dy}{y} \int_0^y v(y-z) v(z)\, dz. \tag{5.2.5}$$

In order to understand the action of P consider three independent random variables ξ_1, ξ_2 and η such that ξ_1, ξ_2 have the same density distribution function v and η is uniformly distributed on the interval $[0, 1]$. Then Pv is the density distribution function of the random variable

$$\eta(\xi_1 + \xi_2). \tag{5.2.6}$$

Physically this means that the energies of the particles before a collision are independent and that a particle after collision takes the η part of the sum of the energies of the colliding particles.

The assumption that η has a density distribution function of the form $\mathbf{1}_{[0,1]}$ is quite restrictive. In general, if η has the density distribution h, then the random variable (5.2.6) has the density distribution function

$$(Pv)(x) = \int_0^\infty h\left(\frac{x}{y}\right) \frac{dy}{y} \int_0^y v(y-z)\, v(z)\, dz. \tag{5.2.7}$$

In 1999 A. Lasota and J. Traple (see [26, Theorem 1.1]) studied the asymptotic behaviour of solutions of the equation

$$u' + u = Pu, \tag{5.2.8}$$

where $u : \mathbb{R} \to L^1(\mathbb{R})$ is an unknown function and P is the operator given by (5.2.7). Equation (5.2.8) was studied in the spaces $L^p(\mathbb{R}_+)$ with $p = 1, 2$ and different weights.

In the proof the following conditions on h were used:

$$\int_0^\infty h(x)\,dx = 2\int_0^\infty xh(x)\,dx = 1, \qquad 2\int_0^\infty x^p h(x)\,dx < 1, \tag{5.2.9}$$

$$\sup_x \{xh(x) : x \geq 0\} < \infty, \tag{5.2.10}$$

$$h(x) > 0 \quad \text{for } 0 < x < x_0, \tag{5.2.11}$$

where $p > 1$ and $x_0 > 0$.

Now, we will consider a generalized version of (5.2.8) in the space $\mathcal{M}_{\mathrm{sig}}(\mathbb{R}_+)$ of all signed measures on $\mathbb{R}_+$. Set

$$D := \{\mu \in \mathcal{M}_1 : m_1(\mu) = 1\}, \quad \text{where} \quad m_1(\mu) = \int_0^\infty x\,\mu(dx). \tag{5.2.12}$$

We study the asymptotic behaviour of solutions of the equation

$$\frac{d\psi}{dt} + \psi = P\psi \quad \text{for } t \geq 0 \tag{5.2.13}$$

with the initial condition

$$\psi(0) = \psi_0, \tag{5.2.14}$$

where $P : D \to D$ is a nonlinear operator on measures analogous to (5.2.7) and $\psi_0 \in D$. In order to define precisely P we will introduce several notations.

Recall that the *convolution of measures* $\mu, \nu \in \mathcal{M}_{\mathrm{sig}}$ is a unique measure $\mu * \nu$ satisfying

$$(\mu * \nu)(A) := \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} 1_A(x + y)\,\mu(dx)\,\nu(dy) \quad \text{for } A \in \mathcal{B}_X. \tag{5.2.15}$$

It is easy to verify that

$$\langle f, \mu * \nu\rangle = \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} f(x + y)\,\mu(dx)\,\nu(dy), \tag{5.2.16}$$

for every Borel measurable $f : \mathbb{R}_+ \to \mathbb{R}$ such that $(x, y) \mapsto f(x + y)$ is integrable with respect to the product of the measures $|\mu|$ and $|\nu|$. For every $n \in \mathbb{N}$ we define the *convolution operator of order* n, $P_{*n} : \mathcal{M}_{\mathrm{sig}} \to \mathcal{M}_{\mathrm{sig}}$, by the formula

$$P_{*1}\,\mu := \mu, \qquad P_{*(n+1)}\,\mu := \mu * P_{*n}\,\mu \quad \text{for } \mu \in \mathcal{M}_{\mathrm{sig}}. \tag{5.2.17}$$

Remark 5.2.1. Observe that P_{*n} is not the nth power of P_{*1} but $P_{*n}\mu$ is the nth convolution power of μ.

It is easy to verify that $P_{*n}(\mathcal{M}_1) \subset \mathcal{M}_1$ for every $n \in \mathbb{N}$. Moreover, $P_{*n}|_{\mathcal{M}_1}$ has a simple probabilistic interpretation. Namely, if $\xi_1, \ldots, \xi_n$ are independent random variables with the same distribution μ, then $P_{*n}\mu$ is the distribution of $\xi_1 + \ldots + \xi_n$.

Another class of operators we are going to study is related to multiplication of random variables (see [22, p. 302]). The formal definition is as follows. Given $\mu, \nu \in \mathcal{M}_{\mathrm{sig}}$, we define their *elementary product* $\mu \circ \nu$ by

$$(\mu \circ \nu)(A) := \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} 1_A(xy)\,\mu(dx)\,\nu(dy) \quad \text{for } A \in \mathcal{B}_{\mathbb{R}_+}. \tag{5.2.18}$$

It follows that

$$\langle f, \mu \circ \nu \rangle = \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} f(xy)\, \mu(dx)\, \nu(dy) \tag{5.2.19}$$

for every Borel measurable $f : \mathbb{R}_+ \to \mathbb{R}$ such that $(x, y) \mapsto f(xy)$ is integrable with respect to the product of $|\mu|$ and $|\nu|$. For fixed $\varphi \in \mathcal{M}_1$ we define a linear operator $P_\varphi : \mathcal{M}_{\mathrm{sig}} \to \mathcal{M}_{\mathrm{sig}}$ by

$$P_\varphi\, \mu := \varphi \circ \mu \quad \text{for } \mu \in \mathcal{M}_{\mathrm{sig}}. \tag{5.2.20}$$

Again, as in the case of convolution, $P_\varphi(\mathcal{M}_1) \subset \mathcal{M}_1$. For $\mu \in \mathcal{M}_1$ the measure $P_\varphi\, \mu$ has an immediate probabilistic interpretation. If φ and μ are the distributions of random variables ξ and η respectively, then $P_\varphi\, \mu$ is the distribution of the product $\xi\eta$.

Now we return to equation (5.2.13) and give a precise definition of P:

$$P := P_\varphi\, P_{*2}, \tag{5.2.21}$$

where $\varphi \in \mathcal{M}_1$ and $m_1(\varphi) = 1/2$. From (5.2.21) it follows that $P(\mathcal{M}_1) \subset \mathcal{M}_1$. Further using (5.2.17) and (5.2.20) it is easy to verify that for $\mu \in D$,

$$m_1(P_{*2}\mu) = 2 \quad \text{and} \quad m_1(P_\varphi \mu) = 1/2. \tag{5.2.22}$$

Remark 5.2.2. Evidently every fixed point of the operator P is a stationary solution of equation (5.2.13). ∎

We will show that if equation (5.2.13) has a stationary measure u_* such that $\operatorname{supp} u_* = \mathbb{R}_+$ (that is, $u_*(B(x, \varepsilon)) > 0$ for every $\varepsilon > 0$ and $x \geq 0$), then this measure is asymptotically stable.

A similar problem for (5.2.1) was studied by A. Lasota and J. Traple (see [26, Theorem 3.3]). The positivity of u_* plays an important role in the proof of the stability. Namely, it allows one to apply the maximum and invariance principle to show that the Hutchinson distance between u_* and an arbitrary solution u decreases in time. We start with two simple lemmas concerning the support of $P\mu$.

Lemma 5.2.1. *Assume that $\varphi \in \mathcal{M}_1$ satisfies*

$$\varphi \neq \delta_{1/2}, \tag{5.2.23}$$

$$m_1(\varphi) = 1/2. \tag{5.2.24}$$

Then there exists $\beta > 1$ such that

$$\text{if } v \in D \text{ and } \operatorname{supp} v \supset (a, b), \text{ then } \operatorname{supp} Pv \supset (\beta\, a, \beta\, b).$$

Proof. First we recall a well-known property of the support of convolution of measures. If $v \in D$ satisfies $\operatorname{supp} v \supset (a, b)$ then the support of $P_{*2}\, v = v * v$ contains the interval $(2a, 2b)$. In fact, fix $c \in (2a, 2b)$ and choose $x, y \in (a, b)$ such that $c = x + y$. Let $\varepsilon > 0$. An elementary calculation show that

$$P_{*2}\, v((c - \varepsilon, c + \varepsilon)) \geq v((x - \varepsilon/2, x + \varepsilon/2))v((y - \varepsilon/2, y + \varepsilon/2)) > 0, \tag{5.2.25}$$

and consequently $c \in \operatorname{supp} P_{*2}\, v$.

From (5.2.23) and (5.2.24), it follows immediately that there exists $\beta > 1$ such that

$$\varphi((\beta/2 - \varepsilon, \beta/2 + \varepsilon)) > 0 \quad \text{for } \varepsilon > 0. \tag{5.2.26}$$

Fix $z \in (\beta a, \beta b)$ and $\varepsilon > 0$. Setting $x = 2z/\beta$ we can choose positive numbers $\varepsilon_1 < x$ and $\varepsilon_2 < \beta/2$ such that

$$\varepsilon_1 \beta/2 + x\varepsilon_2 + \varepsilon_1 \varepsilon_2 < \varepsilon. \tag{5.2.27}$$

Now using (5.2.25) and (5.2.26) we obtain

$$Pv((z-\varepsilon, z+\varepsilon)) \geq \varphi((\beta/2-\varepsilon_2, \beta/2+\varepsilon_2))\, P_{*2}\, v((x-\varepsilon_1, x+\varepsilon_1)) > 0. \tag{5.2.28}$$

This finally gives $Pv((z-\varepsilon, z+\varepsilon)) > 0$, which shows that $z \in \operatorname{supp} Pv$ and completes the proof. ■

The following result may be proved in much the same way as Lemma 5.2.1.

LEMMA 5.2.2. *Assume that there is $\sigma_0 > 0$ such that $(0, \sigma_0) \subset \operatorname{supp} \varphi$. Then for every $v \in \mathcal{M}$ there exists $\sigma > 0$ such that*

$$\operatorname{supp} Pv \supset (0, \sigma) \quad \textit{whenever } v \neq \delta_0. \tag{5.2.29}$$

Proof. Fix $v \in \mathcal{M}$ and assume that $v \neq \delta_0$. Then there exists $x_1 > 0$ such that $x_1 \in \operatorname{supp} P_{*2}\, v$. Set $\sigma = x_1 \sigma_0$. Fix $z \in (0, \sigma)$ and $\varepsilon > 0$. Now we may repeat the construction used in the proof of Lemma 5.2.1. Let $x_2 = z/x_1$ and

$$\varepsilon_1 x_2 + \varepsilon_2 x_1 + \varepsilon_1 \varepsilon_2 < \varepsilon, \tag{5.2.30}$$

where $\varepsilon_1, \varepsilon_2 > 0$ with $\varepsilon_1 < x_1$ and $\varepsilon_2 < x_2$. Then

$$Pv((z-\varepsilon, z+\varepsilon)) \geq \varphi((x_2-\varepsilon_2, x_2+\varepsilon_2))\, P_{*2}\, v((x_1-\varepsilon_1, x_1+\varepsilon_1)) > 0. \tag{5.2.31}$$

Consequently, $z \in \operatorname{supp} Pv$, which finishes the proof. ■

We are in a position to formulate the following theorem.

THEOREM 5.2.1. *Let φ be a probability measure and let $m_1(\varphi) = 1/2$. Assume that:*

(i) *There is $\sigma_0 > 0$ such that*

$$(0, \sigma_0) \subset \operatorname{supp} \varphi. \tag{5.2.32}$$

(ii) *The operator P has a fixed point $v \in \mathcal{M}$ such that $v \neq \delta_0$.*

Then

$$\operatorname{supp} v = \mathbb{R}_+. \tag{5.2.33}$$

Proof. From Lemmas 5.2.2 and 5.2.1 it follows that $\operatorname{supp} v \supset (0, \beta^n \sigma)$ for $n \in \mathbb{N}$. Since $\beta > 1$, this completes the proof. ■

REMARK 5.2.3. If $\varphi \in \mathcal{M}_1$ and $m_1(\varphi) = 1/2$, then the operator P given by (5.2.21) is nonexpansive on D with respect to the Hutchinson norm, i.e.

$$\|Pv - Pw\|_{\mathcal{H}} \leq \|v - w\|_{\mathcal{H}} \quad \text{for } v, w \in D. \tag{5.2.34}$$

In fact, using the conditions $m_1(\varphi) = 1/2$, $m_1(v + w) = 2$ it is easy to show that the function $\tilde{f} : \mathbb{R}_+ \to \mathbb{R}$ given by

$$\tilde{f}(x) = \int\limits_{\mathbb{R}_+} \int\limits_{\mathbb{R}_+} f((x+y)z)\, \varphi(dz)\, (w(dy) + v(dy)) \quad \text{for } x \in \mathbb{R}_+$$

belongs to $\mathcal{H}$ for $f \in \mathcal{H}$. Furthermore,

$$\langle f, Pv - Pw\rangle = \langle \tilde{f}, v - w\rangle \quad \text{for } f \in \mathcal{H},\ v, w \in D.$$

Finally

$$\|Pv - Pw\|_{\mathcal{H}} = \sup\{|\langle f, Pv - Pw\rangle| : f \in \mathcal{H}\} \le \sup\{|\langle g, v - w\rangle| : g \in \mathcal{H}\} = \|v - w\|_{\mathcal{H}}. \ \blacksquare$$

Now we are ready to state the main theorem of this chapter.

THEOREM 5.2.2. *Let φ be a probability measure with $m_1(\varphi) = 1/2$ and let 0 be an accumulation point of* $\operatorname{supp}\varphi$. *Further let $v, w \in D$ be such that $v \ne w$ and*

$$\operatorname{supp}(v + w) = \mathbb{R}_+. \tag{5.2.35}$$

Then inequality (5.2.34) *is strict*, *i.e.*

$$\|Pv - Pw\|_{\mathcal{H}} < \|v - w\|_{\mathcal{H}}. \tag{5.2.36}$$

Proof. Suppose not. Then there exist two different measures $v, w \in D$ such that $\operatorname{supp}(v + w) = \mathbb{R}_+$ and

$$\|Pv - Pw\|_{\mathcal{H}} = \|v - w\|_{\mathcal{H}}. \tag{5.2.37}$$

By Theorem 3.2.1 applied to the measure $Pv - Pw$ there exists $f_0 \in \mathcal{H}$ such that

$$\|Pv - Pw\|_{\mathcal{H}} = \langle f_0, Pv - Pw\rangle. \tag{5.2.38}$$

Using the last equality and (5.2.37) we obtain

$$\begin{aligned}\|v - w\|_{\mathcal{H}} &= \langle f_0, Pv\rangle - \langle f_0, Pw\rangle \\ &= \int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} f_0((x+y)z)\,\varphi(dz)\,v(dx)\,v(dy) - \int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} f_0((x+y)z)\,\varphi(dz)\,w(dx)\,w(dy).\end{aligned}$$

This may be rewritten in the form

$$\|v - w\|_{\mathcal{H}} = \int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} g(x+y)\,(v(dy) + w(dy))\,(v(dx) - w(dx)). \tag{5.2.39}$$

where

$$g(r) = \int\limits_{\mathbb{R}_+} f_0(rz)\,\varphi(dz) \quad \text{for } r \in \mathbb{R}_+.$$

Introducing the function $f_1 : \mathbb{R}_+ \to \mathbb{R}$ by the formula

$$f_1(x) = \int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} f_0((x+y)z)\varphi(dz)(v(dy) + w(dy)) \quad \text{for } x \in \mathbb{R}_+, \tag{5.2.40}$$

it is easy to verify that

$$\|v - w\|_{\mathcal{H}} = \langle f_1, v - w\rangle. \tag{5.2.41}$$

The function f_1 is again an element of $\mathcal{H}$. By the maximum principle applied to the equality (5.2.41) there exist $x_1, x_2 \in \mathbb{R}_+$, $x_1 < x_2$ and constants θ, σ ($\theta^2 = 1$) such that

$$f_1(x) = \theta x + \sigma \quad \text{for } x \in (x_1, x_2).$$

It follows that

$$|f_1(x_1 + \varepsilon) - f_1(x_1)| = \varepsilon \quad \text{for } \varepsilon \in (0, x_2 - x_1).$$

Replacing f_0 by $-f_0$ if necessary we may assume that

$$f_1(x_1+\varepsilon)-f_1(x_1)=\varepsilon. \tag{5.2.42}$$

Now we are going to show that

$$f_0(x)=x+c \quad \text{for } x\in\mathbb{R}_+, \tag{5.2.43}$$

where $c\in\mathbb{R}$. Observe that $f_0\in\mathcal{H}$ and so to prove (5.2.43) it suffices to show that

$$f_0(u_2)-f_0(u_1)\ge u_2-u_1 \quad \text{for } \ 0\le u_1<u_2.$$

To prove this let $u_1,u_2\in\mathbb{R}_+$ with $u_1<u_2$ and suppose that

$$f_0(u_2)-f_0(u_1)<u_2-u_1. \tag{5.2.44}$$

Hence, we can find a point $\bar u\in(u_1,u_2)$ such that the upper right Dini derivative (see [35, p. 9]) of f_0 at $\bar u$ satisfies

$$D^+f_0(\bar u)<1. \tag{5.2.45}$$

According to the definition of the Dini derivative there is a $\delta_0>0$ such that

$$\frac{f_0(\bar u+\delta)-f_0(\bar u)}{\delta}<1 \quad \text{for } \delta\in(0,\delta_0). \tag{5.2.46}$$

Now consider the function

$$\begin{aligned} h(y,z,\varepsilon) \\ =\frac{f_0((x_1+\varepsilon+y)z)-f_0((x_1+y)z)}{\varepsilon\, z} \quad \text{for } (y,z,\varepsilon)\in\mathbb{R}_+\times(0,\infty)\times(0,\infty). \end{aligned} \tag{5.2.47}$$

By (5.2.42) and the definition of f_1 for all $\varepsilon\in(0,x_2-x_1)$ we have

$$1=\frac{f_1(x_1+\varepsilon)-f_1(x_1)}{\varepsilon}=\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} h(y,z,\varepsilon)z\,\varphi(dz)\,(v(dy)+w(dy)). \tag{5.2.48}$$

Let $A\times B\in\mathcal{B}_{\mathbb{R}_+\times\mathbb{R}_+}$. We define a measure q on $\mathcal{B}_{\mathbb{R}_+\times\mathbb{R}_+}$ by the formula

$$q(A\times B)=\iint_{A\times B} z\,\varphi(dz)\,(v(dy)+w(dy)).$$

Evidently q is a probability measure. Since 0 is an accumulation point of $\operatorname{supp}\varphi$, there is a $\bar z\in\operatorname{supp}\varphi$ such that $x_1\bar z<\bar u$. On the other hand, by (5.2.35) there exists $\bar y\in\operatorname{supp}(v+w)$ such that

$$\bar u-x_1\,\bar z=\bar y\,\bar z.$$

Finally, observe that for every $\bar\varepsilon\in(0,x_2-x_1)$ such that $\bar\varepsilon\,\bar z\le\delta_0$ we have

$$h(\bar y,\bar z,\bar\varepsilon)<1.$$

From this and continuity of h it follows that there are two closed balls $B(\bar y,r_{\bar y})$ and $B(\bar z,r_{\bar z})$ such that for $(y,z)\in B(\bar y,r_{\bar y})\times B(\bar z,r_{\bar z})$ we obtain

$$h(y,z,\bar\varepsilon)<1. \tag{5.2.49}$$

Moreover, it is easy to see that $q(B(\bar y,r_{\bar y})\times B(\bar z,r_{\bar z}))>0$. Consequently,

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} h(y,z,\bar\varepsilon)\,q(dy,dz)<1.$$

This contradicts (5.2.48). Therefore $f_0(x) = x + c$ for $x \in \mathbb{R}_+$, where c is a constant. Since Pv and Pw belong to D we have $\langle f_0, Pv - Pw\rangle = 0$. According to (5.2.37) this implies $v = w$, which is a contradiction. ■

We finish this chapter with a new sufficient condition for the asymptotic stability of solutions of a generalized version of the Tjon–Wu equation of the form (5.2.13). We show that this equation may by considered in a convex closed subset of a vector space of signed measures. This approach seems to be quite natural and it is related to the classical results concerning semigroups and differential equations on convex subsets of Banach spaces (see [4, 5]).

Before formulating the main result we recall some known results concerning existence and uniqueness of solutions of ordinary differential equations in Banach spaces.

Let $(E, \|\cdot\|)$ be a Banach space and let $\tilde{D}$ be a closed, convex, nonempty subset of E. In the space E we consider an *evolutionary differential equation*

$$\frac{du}{dt} = -u + \tilde{P}u \quad \text{for } t \in \mathbb{R}_+ \tag{5.2.50}$$

with the initial condition

$$u(0) = u_0, \quad u_0 \in \tilde{D}, \tag{5.2.51}$$

where $\tilde{P} : \tilde{D} \to \tilde{D}$ is a given operator.

A function $u : \mathbb{R}_+ \to E$ is called a solution of problem (5.2.50), (5.2.51) if it is strongly differentiable on $\mathbb{R}_+$, $u(t) \in \tilde{D}$ for all $t \in \mathbb{R}_+$ and u satisfies relations (5.2.50), (5.2.51).

We start from the following theorem which is usually stated in the case $E = \tilde{D}$.

THEOREM 5.2.3. *Assume that the operator $\tilde{P} : \tilde{D} \to \tilde{D}$ satisfies the Lipschitz condition*

$$\|\tilde{P}v - \tilde{P}w\| \le l\,\|v - w\| \quad \textit{for } u, w \in \tilde{D}, \tag{5.2.52}$$

where l is a nonnegative constant. Then for every $u_0 \in \tilde{D}$ there exists a unique solution u of problem (5.2.50), (5.2.51).

The standard proof of Theorem 5.2.3 is based on the fact that a function $u : \mathbb{R}_+ \to \tilde{D}$ is a solution of (5.2.50), (5.2.51) iff it is continuous and satisfies the integral equation

$$u(t) = e^{-t}\,u_0 + \int_0^t e^{-(t-s)}\tilde{P}u(s)\,ds \quad \text{for } t \in \mathbb{R}_+. \tag{5.2.53}$$

By completeness of $\tilde{D}$ the integral on the right hand side is well defined and equation (5.2.53) may be solved by the method of successive approximations.

Observe that thanks to the properties of $\tilde{D}$ for every $u_0 \in \tilde{D}$ and every continuous function $u : \mathbb{R}_+ \to \tilde{D}$ the right hand side of (5.2.53) is also a function with values in $\tilde{D}$.

The solutions of (5.2.53) generate a semigroup of operators $(\tilde{P}^t)_{t\ge 0}$ on $\tilde{D}$ given by

$$\tilde{P}^t\,u_0 = u(t) \quad \text{for } t \in \mathbb{R}_+,\ u_0 \in \tilde{D}. \tag{5.2.54}$$

Now we are going to apply Theorem 5.2.3 to problem (5.2.13), (5.2.14).

We start with the following observations:

1. From (5.2.12) it follows immediately that D is a convex subset of $\mathcal{M}_{\mathrm{sig},1}$.

2. It is known that D with the Hutchinson metric is a complete metric space (see [22, Theorem 2.1]).
3. If $\varphi \in \mathcal{M}_1$ and $m_1(\varphi) = 1/2$, then the operator P maps the set D into itself.

Note that the last condition corresponds to the condition (5.2.9) in the model of Lasota–Traple (see [26, Theorem 1.1]). In the classical Tjon–Wu equation φ has the density distribution function of the form $\mathbf{1}_{[0,1]}$.

We may summarize this discussion with the following

Corollary 5.2.1. *If $\varphi \in \mathcal{M}_1$ and $m_1(\varphi) = 1/2$ then for every $\psi_0 \in D$ there exists a unique solution u of problem* (5.2.13), (5.2.14). ∎

Denote by $(P^t)_{t\geq 0}$ the unique semigroup on D corresponding to (5.2.13), (5.2.14). We have the following result concerning the asymptotic stability of $(P^t)_{t\geq 0}$.

Theorem 5.2.4. *Let P be an operator given by* (5.2.21). *Moreover, let φ be a probability measure with $m_1(\varphi) = 1/2$ and let* 0 *be an accumulation point of* $\operatorname{supp}\varphi$. *If P has a fixed point $\psi_* \in D$ such that*

$$\operatorname{supp}\psi_* = \mathbb{R}_+, \tag{5.2.55}$$

then

$$\lim_{t\to\infty} \|\psi(t) - \psi_*\|_{\mathcal{H}} = 0 \tag{5.2.56}$$

for every compact solution ψ of (5.2.13), (5.2.14).

Proof. First we show that $(P^t)_{t\geq 0}$ is nonexpansive on D with respect to the Hutchinson metric. In fact, let $\eta_0, \vartheta_0 \in D$. For $t \in \mathbb{R}_+$ define $v(t) = P^t\eta_0 - P^t\vartheta_0$. Condition (5.2.53) implies that

$$v(t) = e^{-t}v(0) + \int_0^t e^{-(t-s)}(P(P^s\eta_0) - P(P^s\vartheta_0))\,ds \quad \text{for } t \in \mathbb{R}_+.$$

From this and (5.2.34), it follows immediately that

$$\|v(t)\|_{\mathcal{H}} \leq e^{-t}\,\|v(0)\|_{\mathcal{H}} + \int_0^t e^{-(t-s)}\,\|v(s)\|_{\mathcal{H}}\,ds \quad \text{for } t \in \mathbb{R}_+.$$

This may be rewritten in the form

$$f(t) \leq \|v(0)\|_{\mathcal{H}} + \int_0^t f(s)\,ds \quad \text{for } t \in \mathbb{R}_+,$$

where $f(t) = e^t\,\|v(t)\|_{\mathcal{H}}$. From the Gronwall inequality it follows that

$$f(t) \leq e^t\|v(0)\|_{\mathcal{H}}.$$

This is equivalent to the fact that $(P^t)_{t\geq 0}$ is nonexpansive on D with respect to the Hutchinson metric. Furthermore, from Theorem 5.2.2 we have

$$\|P^t\eta_0 - \psi_*\|_{\mathcal{H}} < e^{-t}\|\eta_0 - \psi_*\|_{\mathcal{H}} + \int_0^t e^{-(t-s)}\|P^s\eta_0 - \psi_*\|_{\mathcal{H}}\,ds \quad \text{for } \eta_0 \in D \text{ and } t > 0.$$

Consequently, from the nonexpansiveness of $(P^t)_{t\geq 0}$ we obtain

$$\|P^t\eta_0-\psi_*\|_{\mathcal{H}} < e^{-t}\|\eta_0-\psi_*\|_{\mathcal{H}}+(1-e^{-t})\|\eta_0-\psi_*\|_{\mathcal{H}} = \|\eta_0-\psi_*\|_{\mathcal{H}} \quad \text{for } \eta_0 \in D \text{ and } t>0.$$

An application of Theorem 5.1.2 completes the proof. ∎

Remark 5.2.4. By virtue of Theorem 5.2.1 assumption (5.2.55) can be replaced by the more effective condition (5.2.32). Observe that in the case of the classical Tjon–Wu equation (5.2.1) the measure φ is absolutely continuous with density $\mathbf{1}_{[0,1]}$. Moreover, $u_*(t,x) := \exp(-x)$ is the density function of the stationary solution of (5.2.1). This is a simple illustration of the situation described by Theorems 5.2.1 and 5.2.4.

For a general model including (5.2.13) existence of a stationary solution has been studied in [22].

Remark 5.2.5. It is interesting to note that if there exists a constant $r>1$ such that

$$2m_r(\varphi) < 1, \tag{5.2.57}$$

then for every $\psi_0 \in D$ the solution $\psi(t) = P^t\psi_0$ of (5.2.13), (5.2.14) is compact (see [22, Theorem 4.2], [26, Theorem 3.3] and [27, Theorem 6]).

6. Maximum principles in the stability theory of Markov semigroups

In this last chapter we present new sufficient conditions for the asymptotic stability of Markov–Feller operators on the space of signed measures $\mathcal{M}_{\mathrm{sig}}$. Our proofs are based on the invariance principle and the maximum principle. We will also show applications of these criteria in the proofs of the asymptotic stability of a stochastically perturbed dynamical system with discrete time and a semigroup generated by a Poisson driven stochastic differential equation (see [10, Proposition 4.1] and [11, Theorem 3]). Moreover, we will discuss the problem of the asymptotic stability of a Markov operator appearing in the theory of the cell cycle (see [12, Proposition 2], [17, Theorem 4] and [25, Theorem 3.2]). We use the notation of Chapter 4.

6.1. Applications of the Kantorovich–Rubinstein maximum principle. In this section we study the problem of the asymptotic stability of semigroups asymptotically contractive with respect to the Hutchinson metric in the class $\mathcal{M}_{1,\alpha}$. In particular we will discuss the problem of the asymptotic stability of locally Lipschitzian Markov semigroups. As before (X,ϱ) denotes a locally compact separable metric space.

We start with a simple method of proving the Prokhorov property. It is based on the notion of Lyapunov function and the Chebyshev inequality.

A continuous $V : X \to [0,\infty)$ is called a *Lyapunov function* if

$$\lim_{\varrho(x,x_0)\to\infty} V(x) = \infty \tag{6.1.1}$$

for some $x_0 \in X$. Of course this definition is meaningful only in the case when X is an unbounded space. It is evident that the validity of (6.1.1) does not depend on the choice of x_0.

A family Π of probability measures on X is said to be *tight* if for every positive ε there exists a compact set K such that

$$\mu(K) \geq 1-\varepsilon \quad \text{for all } \mu \in \Pi. \tag{6.1.2}$$

Using the Lyapunov function, it is easy to give a sufficient condition for the tightness of trajectories of a Markov semigroup. Again assume that $T \subset \mathbb{R}_+$ satisfies condition (4.1.18).

LEMMA 6.1.1. *Let $(P^t)_{t\in T}$ be a Markov–Feller semigroup and $(U^t)_{t\in T}$ its dual semigroup. Assume that there exists a Lyapunov function V such that*

$$U^t V(x) \leq A\,V(x) + B \quad \textit{for } x \in X \textit{ and } t \in T, \tag{6.1.3}$$

where A, B are nonnegative constants. Then for every $\mu \in \mathcal{M}_1$ the family of distributions $\{P^t\mu\}_{t\in T}$ is tight.

Proof. Fix $\varepsilon > 0$ and $\mu \in \mathcal{M}_1$. By the Ulam theorem we may choose a compact set $K \subset X$ such that $\mu(K) \geq 1-\varepsilon/2$. Set $V_K = \sup_{x\in K} V(x)$. We define a new measure $\bar{\mu}$ by the formula $\bar{\mu}(E) = \mu(E \cap K)$, where $E \in \mathcal{B}_X$. Let $Y = V^{-1}([0,q])$, where q is a positive number satisfying

$$q \geq \frac{2}{\varepsilon}(A\,V_K + B). \tag{6.1.4}$$

Using the Chebyshev inequality and the definition of $\bar{\mu}$ we have

$$P^t\mu(Y) \geq P^t\bar{\mu}(Y) \geq 1 - \frac{\varepsilon}{2} - \frac{1}{q}\int_X V(x)\,P^t\,\bar{\mu}(dx) = 1 - \frac{\varepsilon}{2} - \frac{1}{q}\int_X U^t\,V(x)\,\bar{\mu}(dx).$$

Now using inequality (6.1.3) we obtain

$$P^t\mu(Y) \geq 1 - \frac{\varepsilon}{2} - \frac{1}{q}\Big[A\int_X V(x)\,\bar{\mu}(dx) + B\bar{\mu}(K)\Big].$$

From this and (6.1.4) it follows that

$$P^t\mu(Y) \geq 1 - \frac{\varepsilon}{2} - \frac{1}{q}\,[A\,V_K + B] \geq 1-\varepsilon \quad \text{for } t \in T.$$

Since the set Y is bounded and closed, it is compact. ■

As before let c be a fixed element of X and let $\varrho_c^\alpha(x) := (\varrho(x,c))^\alpha$ for $x \in X$ and $\alpha > 0$.

THEOREM 6.1.1. *Let $(P^t)_{t\in T}$ be a Markov–Feller semigroup and $(U^t)_{t\in T}$ its dual semigroup. Assume that there is $t_0 \in T$ such that for every $f \in \mathcal{H}$ the following two conditions are satisfied:*

$$|U^t f(x) - U^t f(y)| \leq \varrho(x,y) \quad \textit{for } x,y \in X \textit{ and } t \in T, \tag{6.1.5}$$

$$|U^{t_0} f(x) - U^{t_0} f(y)| < \varrho(x,y) \quad \textit{for } x,y \in X,\ x \neq y. \tag{6.1.6}$$

Moreover, assume that there exist constants $A, B \geq 0$ and $\alpha > 1$ such that

$$(U^t\varrho_c^\alpha)(x) \leq A\varrho_c^\alpha(x) + B \quad \textit{for } x \in X \textit{ and } t \in T. \tag{6.1.7}$$

Then $(P^t)_{t\in T}$ is asymptotically stable with respect to the Hutchinson metric.

Proof. From Remark 4.2.1 it follows that $P^t(\mathcal{M}_{1,\alpha}) \subset \mathcal{M}_{1,\alpha}$ for $t \geq 0$, and, by Theorem 4.2.1, the semigroup $(P^t)_{t\in T}$ is asymptotically contractive with respect to the Hutchinson metric in the class $\mathcal{M}_{1,\alpha}$.

Now we are going to verify that for every $\mu \in \mathcal{M}_{1,\alpha}$ the trajectory $\{P^t\mu\}_{t\in T}$ is relatively compact in $\mathcal{M}_{1,\alpha}$. Fix $\mu \in \mathcal{M}_{1,\alpha}$. Let (t_n) denote a sequence of integers such that $t_n \to \infty$ and $t_n \in T$ for $n = 1, 2, \ldots$.

From Lemma 6.1.1 and condition (6.1.7) it follows that the family of distributions $\{P^{t_n}\mu\}_{n\in\mathbb{N}}$ is tight. So from the Prokhorov theorem (see [3, Chapter 1, §6]) it follows immediately that there exists a subsequence $(P^{t_{k_n}}\mu)$ which converges weakly to a measure $\mu_0 \in \mathcal{M}_1$. Now we are going to show that $\mu_0 \in \mathcal{M}_{1,\alpha}$ and $(P^{t_{k_n}}\mu)$ is convergent to μ_0 with respect to the Hutchinson metric. For given $r > 0$ define

$$g_r(x) = \begin{cases} \varrho_c^\alpha(x) & \text{for } x \in K(c,r), \\ r^\alpha & \text{for } x \notin K(c,r). \end{cases}$$

Condition (6.1.7) implies that

$$\langle g_r, P^{t_{k_n}}\mu\rangle = \langle U^{t_{k_n}} g_r, \mu\rangle \leq l, \quad \text{where} \quad l = A\langle \varrho_c^\alpha, \mu\rangle + B. \tag{6.1.8}$$

The function g_r is continuous and bounded. Consequently,

$$\lim_{n\to\infty} \langle g_r, P^{t_{k_n}}\mu\rangle = \langle g_r, \mu_0\rangle.$$

Since $r > 0$ was arbitrary, the last equality and (6.1.8) imply that $\mu_0 \in \mathcal{M}_{1,\alpha}$. So it suffices to verify that

$$\lim_{n\to\infty} \|P^{t_{k_n}}\mu - \mu_0\|_{\mathcal{H}} = 0.$$

Since $P^{t_{k_n}}\mu$ and μ_0 belong to $\mathcal{M}_{1,\alpha}$, an elementary calculation shows that

$$\int\limits_{X\setminus K(c,r)} \varrho_c(x)\, P^{t_{k_n}}\mu(dx) \leq \frac{l}{r^{\alpha-1}} \quad \text{and} \quad \int\limits_{X\setminus K(c,r)} \varrho_c(x)\, \mu_0(dx) \leq \frac{l}{r^{\alpha-1}}. \tag{6.1.9}$$

Fix $\varepsilon > 0$ and choose $r > 0$ such that $4l\,/r^{\alpha-1} \leq \varepsilon$. Define

$$\triangle = [-r, r] \quad \text{and} \quad \mathcal{F}_{\triangle,1} = \{f \in C(X) : |f(x)| \leq r \text{ and } |f(x) - f(y)| \leq \varrho(x,y)\}.$$

On the set $\mathcal{M}_1$ the metric

$$\|\mu_1 - \mu_2\|_{\mathcal{F}_{\triangle,1}} = \sup\{\langle f, \mu_1 - \mu_2\rangle;\ f \in \mathcal{F}_{\triangle,1}\},$$

is equivalent to the Fortet–Mourier metric. For $f \in \mathcal{H}$ define

$$f_r(x) = \max\{\min[f(x), r], -r\}.$$

Evidently $f_r \in \mathcal{F}_{\triangle,1}$. Furthermore for $f \in \mathcal{H}_c$ the function f_r has the following properties:

(a) $f_r(x) = f(x)$ for $x \in K(r,c)$,
(b) $|f(x) - f_r(x)| \leq 2\varrho_c(x)$ for $x \in X$.

From this and (6.1.9), it follows immediately that

$$\langle f, P^{t_{k_n}}\mu - \mu_0\rangle \leq \|P^{t_{k_n}}\mu - \mu_0\|_{\mathcal{F}_{\triangle,1}} + \frac{4l}{r^{\alpha-1}} \leq \|P^{t_{k_n}}\mu - \mu_0\|_{\mathcal{F}_{\triangle,1}} + \varepsilon$$

for $f \in \mathcal{H}_c$. This shows that $(P^{t_{k_n}}\mu)$ converges to μ_0 with respect to the Hutchinson norm. Thus the trajectory $\{P^t\mu\}_{t\in T}$ is compact on $\mathcal{M}_{1,\alpha}$. Therefore, according to

Theorem 5.1.2 the measure μ_0 is a stationary point of $(P^t)_{t\in T}$ and

$$\lim_{t\to\infty} \|P^t\mu - \mu_0\|_{\mathcal{H}} = 0 \quad \text{for } \mu \in \mathcal{M}_{1,\alpha}.$$

To complete the proof it is sufficient to observe that the set $\mathcal{M}_{1,\alpha}$ is dense in $\mathcal{M}_1$ and by (6.1.5) the Markov–Feller semigroup $(P^t)_{t\in T}$ is nonexpansive on $\mathcal{M}_1$ with respect to the Fortet–Mourier norm. ∎

It is not difficult to verify that in the case of locally Lipschitzian Markov semigroups (see (4.1.19)) Theorem 6.1.1 may be replaced by the following

THEOREM 6.1.2. *Let $(P^t)_{t\in T}$ be a locally Lipschitzian Markov semigroup on $\mathcal{M}_{\mathrm{sig}}$ and let $(U^t)_{t\in T}$ denote the semigroup dual to $(P^t)_{t\in T}$. Assume that there is $t_0 \in T$ such that for every $f \in \mathcal{H}$,*

$$|U^{t_0}f(x) - U^{t_0}f(y)| < \varrho(x,y) \quad \text{for } x,y \in X,\ x \neq y. \tag{6.1.10}$$

Moreover, assume that there exist constants $A, B \geq 0$ and $\alpha > 1$ such that

$$(U^{nt_0}\varrho_c^\alpha)(x) \leq A\varrho_c^\alpha(x) + B \quad \text{for } x \in X \text{ and } n = 0,1,2,\ldots. \tag{6.1.11}$$

Then $(P^t)_{t\in T}$ is asymptotically stable with respect to the Hutchinson metric. ∎

6.2. Discrete time stochastically perturbed dynamical systems. Let $(\Omega, \Sigma, \mathrm{prob})$ be a probability space, E the expectation on $(\Omega, \Sigma, \mathrm{prob})$ and $(Y, \mathcal{A})$ a measurable space. We consider a discrete time stochastically perturbed dynamical system on a locally compact separable space (X, ϱ) given by the recurrence formula

$$x_{n+1} = S(x_n, \xi_n) \quad \text{for } n = 0,1,\ldots, \tag{6.2.1}$$

where $\xi_n : \Omega \to Y$ is a sequence of random elements and $S : X \times Y \to X$ is a given deterministic transformation. In our study of the asymptotic behaviour of (6.2.1) we assume that the following conditions are satisfied:

(i) The function S is measurable on the product space $X \times Y$ and for every fixed $y \in Y$ the function $S(\cdot, y)$ is continuous.

(ii) The random elements $\xi_0, \xi_1, \ldots$ are independent and have the same distribution, i.e., the measure

$$\varphi(A) = \mathrm{prob}(\xi_n \in A) \quad \text{for } A \in \mathcal{A}$$

is the same for all n.

(iii) The initial value $x_0 : \Omega \to X$ is independent of the sequence (ξ_n).

It is easy to derive a recurrence formula for the measures

$$\mu_n(A) = \mathrm{prob}(x_n \in A), \quad A \in \mathcal{B}(X),$$

corresponding to the dynamical system (6.2.1). Namely $\mu_{n+1} = P\mu_n$, $n = 0,1,\ldots$, where the operator $P : \mathcal{M}_1 \to \mathcal{M}_1$ is given by the formula

$$P\mu(A) = \int_X \Big(\int_Y 1_A(S(x,y))\, \varphi(dy) \Big) \mu(dx). \tag{6.2.2}$$

The operator P is a Markov–Feller operator and its dual U has the form

$$Uf(x) = \int_Y f(S(x,y))\,\varphi(dy) \quad \text{for } f \in C(X). \tag{6.2.3}$$

Now define a sequence of functions S_n by setting

$$S_1(x,y_1) = S(x,y_1), \qquad S_n(x,y_1,\dots,y_n) = S(S_{n-1}(x,y_1,\dots,y_{n-1}),y_n).$$

Using this notation we have

$$U^n f(x) = \int_Y \cdots \int_Y f(S_n(x,y_1,\dots,y_n))\,\varphi(dy_1)\cdots\varphi(dy_n).$$

PROPOSITION 6.2.1. *Assume that the mapping $S : X \times Y \to X$ and the sequence of random elements (ξ_n) satisfy conditions* (i)–(iii). *Assume moreover that there is $n \in \mathbb{N}$ such that*

$$E(\varrho(S(x,\xi_n), S(\overline{x},\xi_n)) < \varrho(x,\overline{x}) \quad \textit{for } x,\overline{x} \in X,\ x \neq \overline{x}, \tag{6.2.4}$$

and there exist constants $\alpha > 1$ and $A, B \in \mathbb{R}^+$ such that

$$U^n \varrho_c^\alpha(x) \le A\varrho_c^\alpha(x) + B, \quad \textit{for } x \in X,\ n = 0,1,2,\dots \tag{6.2.5}$$

Then the operator P defined by (6.2.2) *is asymptotically stable with respect to the Hutchinson metric.*

Proof. It is sufficient to verify condition (6.1.10). According to (6.2.4), for $f \in \mathcal{H}$ and $x \neq \overline{x}$ we have

$$\begin{aligned}|Uf(x) - Uf(\overline{x})| &\le \int_Y |f(S(x,y) - f(S(\overline{x},y)|\,\varphi(dy) \\ &\le \int_Y \varrho(S(x,y), S(\overline{x},y))\,\varphi(dy) < \varrho(x,\overline{x}). \ \blacksquare\end{aligned}$$

Using Proposition 6.2.1 it is easy to obtain a few known results concerning the stability of Markov operators.

In fact from Proposition 6.2.1 we immediately obtain as a special case the stability theorem of Lasota–Mackey (see [23, Theorem 2]) where the conditions

$$E(|S(x,\xi_n) - S(z,\xi_n)|) < |x - z| \quad \text{for } x,z \in X \subset \mathbb{R}^d,\ x \neq z$$

and

$$E(|S(x,\xi_n)|^2) \le A|x|^2 + B \quad \text{for } x \in X \subset \mathbb{R}^d,$$

were assumed. The symbol $|\cdot|$ denotes an arbitrary, not necessarily Euclidean, norm in $\mathbb{R}^d$ and A and B are nonnegative constants with $A < 1$.

Furthermore, in the case when X is a locally compact separable metric space, Proposition 6.2.1 contains a result of Łoskot and Rudnicki (see [29, Theorem 3]). Namely, they proved the asymptotic stability of P if

$$\varrho(S(x,y), S(\overline{x},y)) \le \lambda(y)\varrho(x,\overline{x}) \quad \text{for } x,\overline{x} \in X$$

and

$$E\varrho_c(S(c,\xi_1)) < \infty,$$

where $\lambda : Y \to \mathbb{R}_+$ and $E\lambda(\xi_1) < 1$.

In the special case when $Y = \{1, \ldots, N\}$, the stochastic dynamical system (6.2.1) reduces to an *iterated function system*

$$(S_1, \ldots, S_N; p_1, \ldots, p_N) \quad \text{where} \quad S_k(x) = S(x, k) \quad \text{and} \quad p_k = \mathrm{prob}(\xi_n = k).$$

Now the operators (6.2.2) and (6.2.3) have the form

$$P\mu(A) = \sum_{k=1}^{N} p_k \mu(S_k^{-1}(A)) \quad \text{and} \quad Uf(x) = \sum_{k=1}^{N} p_k f(S_k(x)). \tag{6.2.6}$$

We will assume the following conditions:

$$\sum_{k=1}^{N} p_k \varrho(S_k(x), S_k(\overline{x})) < \varrho(x, \overline{x}) \quad \text{for } x, \overline{x} \in X,\ x \neq \overline{x}, \tag{6.2.7}$$

$$\varrho(S_k(x), c) \leq L_k \varrho(x, c) \quad \text{for } x \in X,\ k = 1, \ldots, N, \tag{6.2.8}$$

where c is a given point in X and the L_k are nonnegative constants.

In this case Proposition 6.2.1 implies the following result

COROLLARY 6.2.1. *If the IFS $(S_1, \ldots, S_N; p_1, \ldots, p_N)$ satisfies conditions* (6.2.7), (6.2.8) *and there exists a constant $\alpha > 1$ such that*

$$\sum_{k=1}^{N} p_k L_k^{\alpha} < 1, \tag{6.2.9}$$

then this system is asymptotically stable.

In the case when there exist $i, j \in \{1, \ldots, N\}$ such that

$$\varrho(S_i(x), S_i(y)) \neq \varrho(S_j(x), S_j(y)) \quad \text{for } x, y \in X,\ x \neq y, \tag{6.2.10}$$

the strict inequality (6.2.7) may be replaced by

$$\sum_{k=1}^{N} p_k \varrho(S_k(x), S_k(y)) \leq \varrho(x, y). \tag{6.2.11}$$

In fact, for every $d \in (0, 1)$ the function $\varrho^d : X \times X \to \mathbb{R}_+$ given by

$$\varrho^d(x, y) = [\varrho(x, y)]^d$$

is again a metric on X and conditions (6.2.10), (6.2.11) imply

$$\sum_{k=1}^{N} p_k \varrho^d(S_k(x), S_k(y)) < \varrho^d(x, y) \quad \text{for } x, y \in X,\ x \neq y.$$

These observations generalize the sufficient conditions of the asymptotic stability of Markov operators generated by iterated function systems given in [21, Theorem 3.2].

6.3. Semigroups generated by Poisson driven differential equations. In this section we will apply Theorem 6.1.2 to the semigroup $(P^t)_{t \geq 0}$ of Markov operators generated by a Poisson driven stochastic differential equation. This equation has the form

$$d\xi(t) = a(\xi(t))dt + \int_{\Theta} \sigma(\xi(t), \theta)\, \mathcal{N}_p(dt, d\theta) \quad \text{for } t \geq 0 \tag{6.3.1}$$

and will be considered with the initial condition

$$\xi(0) = \xi_0, \tag{6.3.2}$$

where $\{\xi(t)\}_{t\geq 0}$ is a stochastic process with values in $\mathbb{R}^d$. In the special case $\xi(0) = x$ a.s. this solution will be denoted by ξ_x.

In order to formulate precise conditions concerning equation (6.3.1) and the formal definitions of the semigroup $(P^t)_{t\geq 0}$ we denote by $\|\cdot\|$, $(\cdot|\cdot)$ the Euclidean norm and scalar product in $\mathbb{R}^d$. As before, $B(\mathbb{R}^d)$ denotes the space of all bounded Borel measurable functions defined on $\mathbb{R}^d$, and $C(\mathbb{R}^d)$ the subspace of all bounded continuous functions. Both spaces are endowed with the supremum norm. Further $C_0^1(\mathbb{R}^d)$ denotes the space of all functions with compact support and continuous first derivatives.

In our study of solutions of (6.3.1), (6.3.2) we make the following assumptions:

(i) The coefficient $a : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitzian with Lipschitz constant l_a, i.e.,

$$\|a(x) - a(y)\| \leq l_a\|x - y\| \quad \text{for } x, y \in \mathbb{R}^d.$$

(ii) $(\Theta, \mathcal{G}, \tilde{n})$ is a finite measure space with $\tilde{n}(\Theta) = 1$.

(iii) The perturbation coefficient $\sigma : \mathbb{R}^d \times \Theta \to \mathbb{R}^d$ is $\mathcal{B}_{\mathbb{R}^d} \times \mathcal{G}/\mathcal{B}_{\mathbb{R}^d}$-measurable. Further $\sigma(z, \cdot) \in L^2(\tilde{n})$ for each $z \in \mathbb{R}^d$ and there exists $l_\sigma > 0$ such that

$$\|\sigma(x, \cdot) - \sigma(y, \cdot)\|_{L^2(\tilde{n})} \leq l_\sigma\|x - y\| \quad \text{for } x, y \in \mathbb{R}^d. \tag{6.3.3}$$

(iv) The mapping $q : \mathbb{R}^d \times \Theta \to \mathbb{R}^d$ given by

$$q(z, \theta) = z + \sigma(z, \theta) \qquad \text{for } z \in \mathbb{R}^d,\ \theta \in \Theta \tag{6.3.4}$$

is such that $q(z, \cdot) \in L^1(\tilde{n})$ for $z \in \mathbb{R}^d$. Moreover there exists a positive constant l_q such that

$$|q(x, \cdot) - q(y, \cdot)|_{L^1(\tilde{n})} \leq l_q\|x - y\| \quad \text{for } x, y \in \mathbb{R}^d. \tag{6.3.5}$$

(v) There is a probability space $(\Omega, \mathcal{F}, \mathrm{prob})$, a sequence $(t_i)_{i\in\mathbb{N}_0}$ of nonnegative random variables and a sequence $(\theta_i)_{i\in\mathbb{N}}$ of random elements with values in Θ. The variables $\Delta t_i = t_i - t_{i-1}$ $(t_0 = 0)$ are nonnegative, independent and identically distributed with probability density function $\lambda e^{-\lambda t}$ for $t \geq 0$. The elements θ_i are independent identically distributed with distribution $\tilde{n}$. The sequences (t_i) and (θ_i) are also independent. Under this condition the mapping

$$\Omega \ni \omega \mapsto p(\omega) = (t_i(\omega), \theta_i(\omega))_{i\in\mathbb{N}}$$

defines a stationary Poisson point process (see [16, Chapter I, §9]).

(vi) For every $\mu \in \mathcal{M}_1$ there is an $\mathbb{R}^d$-valued random vector ξ_μ defined on Ω, independent of p and having the distribution μ.

Condition (v) implies that for every measurable set $Z \subset (0, \infty) \times \Theta$ the variable

$$\mathcal{N}_p(Z) = \#\{i : (t_i, \theta_i) \in Z\}$$

is Poisson distributed. It is called the *Poisson random counting measure.*

Denote by E the expectation on the probability space $(\Omega, \mathcal{F}, \mathrm{prob})$. It can be proved that

$$E(\mathcal{N}_p((0, t] \times K)) = \lambda t\tilde{n}(K)$$

for $t \in (0, \infty), K \in \mathcal{G}$.

By a solution of (6.3.1), (6.3.2) we mean a stochastic process $(\xi(t))_{t\geq 0}$ with values in $\mathbb{R}^d$ such that with probability one the following two conditions are satisfied:

(a) The sample paths are right-continuous functions such that for $t>0$ the limit

$$\xi(t-) = \lim_{s\to t-0} \xi(s)$$

exists and

(b)

$$\xi(t) = \xi_0 + \int_0^t a(\xi(s))ds + \int_0^t\int_\Theta \sigma(\xi(s-),\theta)\mathcal{N}_p(ds,d\theta) \quad \text{for } t\geq 0,$$

where

$$\int_0^t\int_\Theta \sigma(\xi(s-),\theta)\mathcal{N}_p(ds,d\theta) = \sum_{t_n\leq t} \sigma(\xi(t_n-),\theta_n) \quad \text{for } t\geq 0 \quad \text{and} \quad p=(t_i,\theta_i)_{i\in\mathbb{N}},$$

(see [16, Chapter II, §3]). It is easy to write explicitly the formula for the solution of (6.3.1), (6.3.2). Denote by π^t the dynamical system defined by

(6.3.6) $$\pi^t(x) = y(t) \quad \text{for } t\in\mathbb{R}^+,$$

where y is the solution of the ordinary differential equation

(6.3.7) $$y'(t) = a(y(t)) \quad \text{for } t\in\mathbb{R}^+,$$

with the initial condition

(6.3.8) $$y(0) = x.$$

Then for every fixed value of $p=(t_i,\theta_i)_{i\in\mathbb{N}}$ the solution of (6.3.1), (6.3.2) is given by

$$\xi(t) = \pi^{t-t_i}(\xi(t_i)) \quad \text{for } t\in[t_i,t_{i+1}),\ i\in\mathbb{N}_0,$$

where

$$\xi(0)=\xi_0, \qquad \xi(t_i) = \xi(t_i-)+\sigma(\xi(t_i-),\theta_i) \quad \text{for } i\in\mathbb{N}.$$

For $x\in\mathbb{R}^d$ denote by $(\xi_x(t))_{t\geq 0}$ the solution of the initial value problem (6.3.1), (6.3.2) with $\xi_0 = x$. For every $t\geq 0$ and $f\in C(\mathbb{R}^d)$ define

(6.3.9) $$U^t f(x) = E(f(\xi_x(t))) \quad \text{for } t\geq 0.$$

REMARK 6.3.1. The classical theory of equation (6.3.1) ensures that under conditions (i)–(vi), $(\xi_x(t))_{t\geq 0}$ is a homogeneous-in-time Markov process and $(U^t)_{t\geq 0}$ is a continuous semigroup of bounded linear operators acting on the space $C(\mathbb{R}^d)$.

Analogously for given $\mu\in\mathcal{M}_1$ we can find a solution $\xi_\mu(t)$, $t\geq 0$, of (6.3.1), (6.3.2) such that $\xi_\mu(0)$ has the distribution μ. For every $t\geq 0$ we define $P^t\mu$ as the distribution of $\xi_\mu(t)$, i.e.,

(6.3.10) $$P^t\mu(A) = \operatorname{prob}(\xi_\mu(t)\in A) \quad \text{for } t\geq 0,\ A\in\mathcal{B}_{\mathbb{R}^d}.$$

The operators P^t and U^t satisfy the duality condition

(6.3.11) $$\langle f, P^t\mu\rangle = \langle U^t f,\mu\rangle \quad \text{for } t\geq 0,\ f\in C,\ \mu\in\mathcal{M}_1.$$

Using (6.3.11) the semigroup $(P^t)_{t\geq 0}$ can be easily extended to the vector space $\mathcal{M}_{\mathrm{sig}}$. It is locally Lipschitzian and weakly continuous. Moreover, using the Phillips perturbation theorem it is easy to find a formula for $(U^t)_{t\geq 0}$.

In fact, let G_0 be a linear operator given by the formula

$$G_0 f(x) = \int_{\Theta} f(q(x,\theta))\,\tilde{n}(d\theta) \quad \text{for } f \in C(\mathbb{R}^d),\ x \in \mathbb{R}^d, \tag{6.3.12}$$

and let $(T^t)_{t\geq 0}$ be the semigroup corresponding to the unperturbed system (6.3.7), i.e.

$$T^t f(x) = f(\pi^t(x)) \quad \text{for } f \in C(\mathbb{R}^d),\ x \in \mathbb{R}^d. \tag{6.3.13}$$

Then (see [37, p. 170])

$$U^t f = e^{-\lambda t} \sum_{n=0}^{\infty} U_n^t f \quad \text{for } f \in C(\mathbb{R}^d), \tag{6.3.14}$$

where

$$\begin{aligned} U_{n+1}^t f &= \lambda \int_0^t T^{t-s}\, G_0\, U_n^s f\, ds, \quad n = 0,1,\ldots, \\ U_0^t f &= T^t f \quad \text{for } t \geq 0. \end{aligned} \tag{6.3.15}$$

Many different criteria for the asymptotic stability of the flow of measures generated by equation (6.3.1) are known. Here we mention only a few of them which are related to our methods. J. Malczak (see [30, Proposition 7.1]) studied the asymptotic stability of the flow of the densities of the measures $\{P^t\mu\}$. His results were based on the lower bound technique. Using a double contraction principle A. Lasota (see [21, Proposition 5.1]) proved the asymptotic stability of the semigroup $(P^t)_{t\geq 0}$ acting on the space of signed measures. His result were generalized by J. Traple (see [37, Theorem 7.3]) who considered the case when the intensity λ of the Poisson process depends on the position of the solution. Another generalization was given by T. Szarek (see [34, Theorem 7.8.3]) who studied equation (6.3.1) in a Banach space. In all these results an important role was played by the following two conditions:

$$\|\pi^t x - \pi^t y\| \leq e^{\gamma t}\|x - y\| \quad \text{for } x, y \in \mathbb{R}^d,\ t \geq 0, \tag{6.3.16}$$

$$l_q < \exp\{-\gamma/\lambda\}. \tag{6.3.17}$$

Using (6.3.16) and (6.3.17) it is possible to prove the asymptotic stability of $(P^t)_{t\geq 0}$ by the invariance principle. However, this principle can also be useful in some cases when inequality (6.3.17) is not satisfied. We illustrate this fact by the following

THEOREM 6.3.1. *Assume that assumptions* (i)–(vi) *are satisfied with a given* $\lambda > 0$ *and* $l_q = 1$. *Further, assume that*

$$\|\pi^t x - \pi^t y\| < \|x - y\| \quad \textit{for } x,\ y \in \mathbb{R}^d,\ x \neq y \textit{ and } t > 0. \tag{6.3.18}$$

Assume moreover that there exist constants $\alpha_0, \beta_0 \in \mathbb{R}$ *such that*

$$(a(x)|2x) + \lambda \int_{\Theta} (\sigma(x,\theta)|x)\,\tilde{n}(d\theta) \leq \alpha_0\|x\|^2 + \beta_0 \quad \textit{for } x \in \mathbb{R}^d, \tag{6.3.19}$$

and

$$2\alpha_0 < -\lambda l_\sigma^2. \tag{6.3.20}$$

Then the semigroup $(P^t)_{t\geq 0}$ *defined by* (6.3.10) *is asymptotically stable with respect to the Hutchinson metric.*

Proof. We are going show that the semigroup $(U^t)_{t\geq 0}$ satisfies the assumptions of Theorem 6.1.2. First we prove by induction that for every $f \in \mathcal{H}$,

$$|(U_n^t f)(x) - (U_n^t f)(y)| < \frac{(\lambda t)^n}{n!}\|x-y\| \quad \text{for } x, y \in \mathbb{R}^d,\ n \in \mathbb{N}\cup\{0\},\ t > 0. \tag{6.3.21}$$

For $n = 0$ from (6.3.13) and (6.3.15) we obtain

$$\begin{aligned}|(U_0^t f)(x) - (U_0^t f)(y)| &\leq |f(\pi^t x) - f(\pi^t y)| \\ &\leq \|\pi^t x - \pi^t y\| < \|x-y\| \quad \text{for } x, y \in \mathbb{R}^d,\ t > 0.\end{aligned}$$

Now let (6.3.21) be satisfied for some integer $n \geq 0$. From (6.3.12), (6.3.13) and (6.3.15) it follows immediately that

$$\begin{aligned}|(G_0 U_n^s f)(x) - (G_0 U_n^s f)(y)| &\leq \int_\Theta |(U_n^s f)(q(x,\theta)) - (U_n^s f)(q(y,\theta))|\,\tilde{n}(d\theta) \\ &< \frac{(\lambda s)^n}{n!}\int_\Theta \|q(x,\theta) - q(y,\theta)\|\,\tilde{n}(d\theta) \\ &\leq \frac{(\lambda s)^n}{n!}\,\|x-y\| \quad \text{for } x, y \in \mathbb{R}^d \text{ and } s \in (0,t].\end{aligned}$$

For $s \in (0,t]$ and $f \in \mathcal{H}$ we also have

$$T^{t-s}G_0 U_n^s f(x) - T^{t-s}G_0 U_n^s f(y)| < \frac{(\lambda s)^n}{n!}\,\|x-y\| \quad \text{for } x, y \in \mathbb{R}^d.$$

This and (6.3.15) complete the induction argument.

From (6.3.14) and (6.3.21) we obtain

$$|U^t f(x) - U^t f(y)| < \|x-y\|, \quad x, y \in \mathbb{R}^d,\ f \in \mathcal{H}. \tag{6.3.22}$$

Therefore condition (6.1.10) of Theorem 6.1.2 is satisfied.

To prove (6.1.11) consider the function $V(x) = \|x\|^2$. Following the proof of Theorem 3 in [14] (see p. 236) it is easy to deduce that for every $t > 0$ there exists a constant k_t such that

$$E\|\xi_x(s)\|^2 \leq e^{k_t s}V(x) + 1 \quad \text{for } x \in \mathbb{R}^d \text{ and } s \leq t.$$

The last inequality may be rewritten in the form

$$U^s V(x) \leq e^{k_t s}V(x) + 1 \quad \text{for } x \in \mathbb{R}^d \text{ and } s \leq t. \tag{6.3.23}$$

Hence, the mapping $t \mapsto U^t V(x)$ is locally bounded for all $x \in \mathbb{R}^d$.

Now for the semigroup $(U^t)_{t\geq 0}$ we can write the formula

$$U^t f(x) = f(x) + \int_0^t U^s A_U f(x)\,ds \quad \text{for } x \in \mathbb{R}^d,\ f \in C^1(\mathbb{R}^d) \tag{6.3.24}$$

using its infinitesimal operator

$$A_U f(x) = (a(x)|f_x(x)) - \lambda f(x) + \lambda \int_\Theta f(x + \sigma(x,\theta))\, \tilde{n}(d\theta). \tag{6.3.25}$$

Consequently,

$$U^t V(x) = V(x) + \int_0^t U^s\, \psi(x)\, ds \quad \text{for } x \in \mathbb{R}^d, \tag{6.3.26}$$

where

$$\psi(x) = (a(x)|2x) + \lambda \int_\Theta (\|x + \sigma(x,\theta)\|^2 - \|x\|^2)\, \tilde{n}(d\theta). \tag{6.3.27}$$

By (6.3.20) there exists a constant $c > 0$ such that

$$\alpha = 2\alpha_0 + \lambda l_\sigma^2 + \lambda c l_\sigma \|\sigma(0,\cdot)\|_{L^2(\tilde{n})} < 0. \tag{6.3.28}$$

Now, we will verify that

$$\psi(x) \le \alpha\, V(x) + \beta \quad \text{for } x \in \mathbb{R}^d, \tag{6.3.29}$$

where

$$\beta = \lambda(1 + 1/c)(1 + \|\sigma(0,\cdot)\|_{L^2(\tilde{n})}) + \lambda c l_\sigma + \beta_0.$$

In fact, by the definition of ψ for every $x \in \mathbb{R}^d$ we have

$$\psi(x) = 2\Big((a(x)|x) + \lambda \int_\Theta (\sigma(x,\theta)|x)\, \tilde{n}(d\theta)\Big) + \lambda \int_\Theta \|\sigma(x,\theta)\|^2\, \tilde{n}(d\theta). \tag{6.3.30}$$

Further, from inequality (6.3.3) it follows immediately that

$$\int_\Theta \|\sigma(x,\theta)\|^2\, \tilde{n}(d\theta) \le l_\sigma \|x\| + 2 \int_\Theta \|\sigma(0,\theta)\|\, \|\sigma(x,\theta) - \sigma(0,\theta)\|\, \tilde{n}(d\theta)$$
$$+ \int_\Theta \|\sigma(0,\theta)\|^2\, \tilde{n}(d\theta) \quad \text{for } x \in \mathbb{R}^d.$$

Since $b \le \frac{c}{2} \cdot b^2 + \frac{1}{2c}$ for every $b \in \mathbb{R}$, the last inequality implies that

$$\int_\Theta \|\sigma(x,\theta)\|^2\, \tilde{n}(d\theta)$$
$$\le (l_\sigma^2 + c l_\sigma \|\sigma(0,\cdot)\|_{L^2(\tilde{n})})\|x\|^2 + ((1 + 1/c)(1 + \|\sigma(0,\cdot)\|_{L^2(\tilde{n})}) + c l_\sigma)$$

This inequality and conditions (6.3.19), (6.3.30) imply (6.3.29).

Now using (6.3.26) and (6.3.29) we obtain the inequality

$$\frac{d}{dt} U^t\, V(x) \le \alpha U^t\, V(x) + \beta. \tag{6.3.31}$$

From (6.3.31) we conclude that

$$U^t\, V(x) \le V(x) e^{\alpha t} + \frac{\beta}{\alpha}\,(e^{\alpha t} - 1) \quad \text{for } x \in \mathbb{R},\ t \ge 0. \tag{6.3.32}$$

Since $\alpha < 0$, this implies (6.1.11) with $c = 0$ and $\varrho_0^2(x) = V(x)$. Thus by Theorem 6.1.2 the semigroup $(P^t)_{t \ge 0}$ is asymptotically stable. ∎

6.4. Applications of the maximum principle for the Fortet–Mourier metric. Again let (X, ϱ) be a locally compact separable space. The relationship between the maximum principle for the Fortet–Mourier metric and the stability theory of the Markov–Feller semigroups is given in the following

THEOREM 6.4.1. *Let $(P^t)_{t\in T}$ be a Markov–Feller semigroup and $(U^t)_{t\in T}$ its dual semigroup. Assume that there is $t_0 \in T$ such that for every $f \in \mathcal{F}$:*

(i)

$$|U^t f(x) - U^t f(y)| \le \varrho(x, y) \quad \text{for } x, y \in X \text{ and } t \in T, \tag{6.4.1}$$

$$|U^{t_0} f(x) - U^{t_0} f(y)| < \varrho(x, y) \quad \text{for } x, y \in X,\ x \neq y. \tag{6.4.2}$$

(ii) *For every $\mu_1, \mu_2 \in \mathcal{M}_1$, $\mu_1 \neq \mu_2$, there exists $t_1 \in T$ that*

$$\operatorname{dist}(\operatorname{supp}(P^{t_1}(\mu_1 - \mu_2))_+, \operatorname{supp}(P^{t_1}(\mu_1 - \mu_2))_-) < 2. \tag{6.4.3}$$

(iii) *There exists a Lyapunov function V such that*

$$U^t V(x) \le A V(x) + B \quad \text{for } x \in X \text{ and } t \in T, \tag{6.4.4}$$

where A, B are nonnegative constants.

Then $(P^t)_{t\in T}$ is asymptotically stable with respect to the Fortet–Mourier metric.

Proof. From (6.4.1), it follows immediately that $U^t(\mathcal{F}) \subset \mathcal{F}$ for $t \in T$ and, by Theorem 4.3.1, the semigroup $(P^t)_{t\in T}$ is asymptotically contractive with respect to the Fortet–Mourier metric in the class $\mathcal{M}_1$.

To complete the proof it is sufficient to verify that for every $\mu \in \mathcal{M}_1$ the trajectory $\{P^t\mu\}_{t\ge 0}$ is compact with respect to the Fortet–Mourier metric. Let (t_n) be a sequence of numbers such that $t_n \to \infty$ and $t_n \in T$ for $n \in \mathbb{N}$. From Lemma 6.1.1 and condition (6.4.4) it follows that the family of distributions $\{P^{t_n}\mu\}_{n\in\mathbb{N}}$ is tight. From the Prokhorov theorem it follows immediately that there exists a subsequence $(P^{t_{k_n}}\mu)_{n\in\mathbb{N}}$ which converges weakly to a measure $\mu_0 \in \mathcal{M}_1$.

We have verified that $(P^t)_{t\ge 0}$ is asymptotically contractive with respect to the Fortet–Mourier metric in the class $\mathcal{M}_1$ and that the orbits are compact. According to the invariance principle the semigroup $(P^t)_{t\ge 0}$ is asymptotically stable. ■

For locally Lipschitzian Markov semigroups the following version of Theorem 6.4.1 can be proved similarly:

THEOREM 6.4.2. *Let $(P^t)_{t\in T}$ be a locally Lipschitzian Markov semigroup on $\mathcal{M}_{\mathrm{sig}}$ and let $(U^t)_{t\in T}$ denote the semigroup dual to $(P^t)_{t\in T}$. Assume that:*

(i) *There exists $t_0 \in T$ such that for every $f \in \mathcal{F}$,*

$$|U^{t_0} f(x) - U^{t_0} f(y)| < \varrho(x, y) \quad \text{for } x, y \in X,\ x \neq y. \tag{6.4.5}$$

(ii) *For every $\mu_1, \mu_2 \in \mathcal{M}_1$, $\mu_1 \neq \mu_2$, there exists $n_0 \in \mathbb{N}$ such that*

$$\operatorname{dist}(\operatorname{supp}(P^{n_0 t_0}(\mu_1 - \mu_2))_+, \operatorname{supp}(P^{n_0 t_0}(\mu_1 - \mu_2))_-) < 2. \tag{6.4.6}$$

(iii) *There exists a Lyapunov function V such that*

$$U^{nt_0} V(x) \le AV(x) + B \quad \text{for } x \in X,\ n \ge 0, \tag{6.4.7}$$

where A, B are nonnegative constants.

Then $(P^t)_{t\in T}$ is asymptotically stable with respect to the Fortet–Mourier metric.

We complete this series of sufficient conditions for the asymptotic stability of Markov semigroups with the following

THEOREM 6.4.3. *Let $(P^t)_{t\in T}$ be a Markov–Feller semigroup and $(U^t)_{t\in T}$ its dual semigroup. Assume that there is $t_0 \in T$ such that for every $f \in \mathcal{F}$:*

(i)

$$|U^t f(x) - U^t f(y)| \le \varrho(x,y) \quad \text{for } x, y \in X \text{ and } t \in T, \tag{6.4.8}$$

$$|U^{t_0} f(x) - U^{t_0} f(y)| < \varrho(x,y) \quad \text{for } x, y \in X,\ x \ne y. \tag{6.4.9}$$

(ii) *There exist constants $t_0, t_1, t_2 \in T$ such that for every $f \in \mathcal{F}$ either*

$$U^{t_0+t_1} f(x) \in (-1, 1] \quad \text{for } x \in X$$

or

$$U^{t_0+t_2} f(x) \in [-1, 1) \quad \text{for } x \in X.$$

(iii) *There exists a Lyapunov function V such that*

$$U^t V(x) \le A V(x) + B \quad \text{for } x \in X,\ t \in T, \tag{6.4.10}$$

where A, B are nonnegative constants.

Then $(P^t)_{t\in T}$ is asymptotically stable with respect to the Fortet–Mourier metric.

Proof. Again the proof is similar to that of Theorem 6.4.1. In this case we can use Theorem 4.3.2 to verify that the semigroup $(P^t)_{t\in T}$ is asymptotically contractive with respect to the Fortet–Mourier metric in the class $\mathcal{M}_1$. ∎

As a consequence of Proposition 4.3.1 and Theorem 6.4.3 we obtain the following

COROLLARY 6.4.1. *Let $P : \mathcal{M}_{\mathrm{sig}} \to \mathcal{M}_{\mathrm{sig}}$ be a Markov–Feller operator and let U be its dual. Assume that:*

(i) *For every $f \in \mathcal{F}$,*

$$|Uf(x) - Uf(y)| < \varrho(x,y) \quad \text{for } x, y \in X,\ x \ne y. \tag{6.4.11}$$

(ii) *The transition $\pi : X \times \mathcal{B}_X \to [0,1]$ corresponding to P, given by* (4.1.10), *satisfies*

$$\operatorname{supp} \pi(x, \cdot) = X \quad \text{for } x \in X. \tag{6.4.12}$$

(iii) *There exists a Lyapunov function V such that*

$$U^n V(x) \le A V(x) + B \quad \text{for } x \in X,\ n \ge 0, \tag{6.4.13}$$

where A, B are nonnegative constants.

Then $(P^n)_{n\in\mathbb{N}}$ is asymptotically stable with respect to the Fortet–Mourier metric. ∎

6.5. Applications in a mathematical model of the cell cycle. In order to illustrate the utility of Theorem 6.4.3 we show a sufficient condition for the asymptotic stability of a special Markov operator introduced by A. Lasota and M. C. Mackey in the theory describing the division and stability of cellular populations (see [25, Theorem 3.2]). Again, let (X, ϱ) be a locally compact separable metric space. Further, let (I, κ) be another metric space, which will be considered as the space of indices. We consider a continuous transformation

$$S : X \times I \to X$$

and a function

$$F : X \times \mathcal{B}_I \to [0,1].$$

where $\mathcal{B}_I$ denotes the σ-algebra of Borel subsets of I. We assume that:

(1) For every $x \in X$ the mapping $F(x,\cdot) : \mathcal{B}_I \to [0,1]$ is a probability measure.
(2) For every $A \in \mathcal{B}_I$ the function $F(\cdot, A) : X \to X$ is measurable.

Now we present an imprecise description of the process considered in this example.

Choose an arbitrary point $x_0 \in X$ and randomly select a point $i_0 \in I$ according to the distribution $F(x_0,\cdot)$. When the point t_0 is drawn we define $x_1 = S(x_0, i_0)$. Having x_1 we select $i_1 \in I$ according to the distribution $F(x_1,\cdot)$ and we define $x_2 = S(x_1, i_1)$ and so on. Denoting by $\mu_n, n = 0, 1, \ldots,$ the distribution of x_n, i.e. $\mu_n(A) = \mathrm{prob}(x_n \in A)$, we define P as the transition operator such that $\mu_{n+1} = P\mu_n$.

The above procedure can be easily formalized. To do this fix $x \in X$ and set $\mu_0 = \delta_x$. According to the description of our process and from the definition of the dual operator U we have

$$Uf(x) = \langle Uf, \delta_x\rangle = \langle f, P\,\delta_x\rangle = \langle f, \mu_1\rangle \quad \text{for } f \in B(X).$$

This means that $Uf(x)$ is the expectation of $f(x_1)$ if $x_0 = x$ is fixed. On the other hand, according to our description, the expectation of $f(x_1)$ is equal to

$$\int_I f(S(x,i))\, F(x,di).$$

Since x was arbitrary this implies

$$Uf(x) = \int_I f(S(x,i))\, F(x,di) \quad \text{for } x \in X. \tag{6.5.1}$$

We admit formula (6.5.1) as the precise formal definition of the operator U. It is easy to verify that the operator given by (6.5.1) satisfies conditions (4.1.5) and (4.1.7). Thus we can define P to be the Markov operator corresponding to U. It is the unique operator satisfying

$$\langle f, P\mu\rangle = \langle Uf, \mu\rangle. \tag{6.5.2}$$

The transition function $\pi : X \times \mathcal{B}_X \to [0,1]$ corresponding to P is defined by

$$\pi(x, A) = P\delta_x(A) = \int_I \mathbf{1}_A(S(x,i))\, F(x,di) \quad \text{for } (x,A) \in X \times \mathcal{B}_X. \tag{6.5.3}$$

To formulate sufficient conditions of the asymptotic stability of P we introduce the following notations.

Consider the class Φ of functions $\varphi : [0,\infty) \to [0,\infty)$ satisfying the following three conditions:

(a) φ is continuous and $\varphi(0) = 0$;
(b) φ is nondecreasing and concave;
(c) $\varphi(x) > 0$ for $x > 0$ and $\lim_{x\to\infty} \varphi(x) = \infty$.

We denote by Φ_0 the family of functions satisfying conditions (a), (b).

An important role in the study of the asymptotic behaviour of Markov operator P is played by the inequality

$$\omega(t) + \varphi(r(t)) \le \varphi(t) \quad \text{for } t \ge 0, \tag{6.5.4}$$

where $r, \omega \in \Phi_0$ are given functions. In [28, pp. 58–60] Lasota and Yorke discussed the cases when the functional inequality (6.5.4) has a solution belonging to Φ.

We are not going to recall all these results. However, it is worthwhile to note that if the function ω satisfies the Dini condition:

$$\int_0^\varepsilon \frac{\omega(t)}{t}\,dt < \infty \qquad \text{for some } \varepsilon > 0$$

and $r(t) = \lambda t$ $(0 \le \lambda < 1)$ then (6.5.4) has a solution $\varphi \in \Phi$.

Finally, denote by $\|\cdot\|_T$ the total variation norm in the space $\mathcal{M}_{\mathrm{sig}}(I)$. Following [24, Subsection 12.2], if $\{A_1, \ldots, A_n\}$ is a *measurable partition* of X, that is,

$$X = \bigcup_{i=1}^n A_i, \qquad A_i \cap A_j = \emptyset \quad \text{for } i \ne j, \qquad A_i \in \mathcal{B}_X,$$

then for $\mu \in \mathcal{M}_{\mathrm{sig}}$ we set

$$\|\mu\|_T = \sup\Big\{\sum_{i=1}^n |\mu(A_i)|\Big\}, \tag{6.5.5}$$

where the supremum is taken over all possible measurable partitions of X (with arbitrary n). In the special case where $\mu \in \mathcal{M}_1$ we have $\|\mu\|_T = 1$. The value $\|\mu\|_T$ is called the *total variation norm* of the measure μ, and the convergence with respect to this norm is called the *strong convergence of measures.*

THEOREM 6.5.1. *Let $\omega, r \in \Phi_0$ and let $0 \le r(x) < x$. Assume that the functional inequality* (6.5.4) *has a solution in the class Φ. Moreover, assume that:*

$$\int_I \varrho(S(x,i), S(y,i))F(x,di) \le r(\varrho(x,y)) \qquad \textit{for } x, y \in X, \tag{6.5.6}$$

$$\|F(x,\cdot) - F(y,\cdot)\|_T \le \omega(\varrho(x,y)) \qquad \textit{for } x, y \in X, \tag{6.5.7}$$

$$\sup_{x\in X} \int_I \varrho(x_0, S(x_0,i))\; F(x,di) < \infty \tag{6.5.8}$$

for some $x_0 \in X$ and

$$\operatorname{supp} \pi(x,\cdot) = X \qquad \textit{for } x \in X, \tag{6.5.9}$$

where π is the transition function given by (6.5.3). *Then the operator P given by* (6.5.1) *and* (6.5.2) *is asymptotically stable with respect to the Fortet–Mourier metric.*

Proof. Consider a solution $\tilde{\varphi} \in \Phi$ of (6.5.4) corresponding to the pair (ω, r). Since $r(t) < t$ the function $\varphi(t) = \tilde{\varphi}(t) + t$ satisfies

$$\omega(t) + \varphi(r(t)) < \varphi(t) \quad \text{for } t \geq 0. \tag{6.5.10}$$

Now using properties (a)–(c) it is easy to verify that the function ϱ_φ given by the formula

$$\varrho_\varphi(x,y) = \varphi(\varrho(x,y)) \quad \text{for } x, y \in X \tag{6.5.11}$$

is again a metric on X. Denote by $\|\cdot\|_\varphi$ the Fortet–Mourier norm generated by ϱ_φ, i.e.

$$\|\mu\|_{\mathcal{F}_\varphi} = \sup\{|\langle f, \mu\rangle| : f \in \mathcal{F}_\varphi\} \quad \text{for } \mu \in \mathcal{M}_{\text{sig}},$$

where $\mathcal{F}_\varphi \subset C(X)$ is the set of all f such that $|f| \leq 1$ and

$$|f(x) - f(y)| \leq \varrho_\varphi(x,y) \quad \text{for } x, y \in X.$$

Now fix $f \in \mathcal{F}_\varphi$. We are going to show that Uf is a contractive function with respect to the metric ϱ_φ. Using (6.5.1), (6.5.7) and the continuity of S it is easy to verify that $Uf \in C(X)$ and that $|Uf| \leq 1$. Moreover for $x, y \in X$, $x \neq y$ we have

$$\begin{aligned} |Uf(x) - Uf(y)| &= \Big|\int_I f(S(x,i))\, F(x,di) - \int_I f(S(y,i))\, F(y,di)\Big| \\ &\leq \|F(x,\cdot) - F(y,\cdot)\|_T + \int_I |f(S(x,i)) - f(S(y,i))|\, F(x,di). \end{aligned}$$

From this and (i) it follows that

$$\begin{aligned} |Uf(x) - Uf(y)| &\leq \omega(\varrho(x,y)) + \int_I \varphi(\varrho(S(x,i), S(y,i)))\, F(x,di) \\ &\leq \omega(\varrho(x,y)) + \varphi\Big(\int_I \varrho(S(x,i), S(y,i))\, F(x,di)\Big) \\ &\leq \omega(\varrho(x,y)) + \varphi(r(\varrho(x,y))). \end{aligned}$$

According to (6.5.10), the last inequality implies

$$|Uf(x) - Uf(y)| < \varrho_\varphi(x,y). \tag{6.5.12}$$

Now, we will verify that

$$U^n V(x) \leq r(1)V(x) + B \quad \text{for } x \in X \text{ and } n \in \mathbb{N}, \tag{6.5.13}$$

where $V(x) = \varrho(x, x_0)$ and

$$B = (1 - r(1))^{-1}\Big(r(1) + \sup_{x \in X} \int_I \varrho(x_0, S(x_0, i))\, F(x,di)\Big).$$

In fact from (6.5.6) it follows that

$$\int_I \varrho(S(x,i), x_0)\, F(x,di) \leq r(\varrho(x,x_0)) + \int_I \varrho(x_0, S(x_0,i))\, F(x,di). \tag{6.5.14}$$

Moreover, since r is nondecreasing, concave and $r(0) = 0$, we have

$$r(x) \leq r(1)\, x + r(1).$$

The last inequality and conditions (6.5.1) and (6.5.14) imply (6.5.13).

By virtue of Corollary 6.4.1 the operator P is asymptotically stable with respect to the Fortet–Mourier metric $\|\cdot\|_{\mathcal{F}_\varphi}$ generated by the metric ϱ_φ.

Finally, since the classes of convergent sequences in both spaces $(\mathcal{M}_{\mathrm{sig}}, \|\cdot\|_{\mathcal{F}_\varphi})$ and $(\mathcal{M}_{\mathrm{sig}}, \|\cdot\|_{\mathcal{F}})$ are the same, the operator P is asymptotically stable with respect to the Fortet–Mourier metric $\|\cdot\|_{\mathcal{F}}$. This completes the proof. ∎

Acknowledgements. I am grateful to the anonymous referees for their contribution to improving this manuscript.

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Notation and symbols

Let (X, ϱ) be a metric space. Given $c \in X$ and $\alpha > 0$ we denote by ϱ_c and ϱ_c^α, respectively, $\varrho_c(x) := \varrho(x,c)$ and $\varrho_c^\alpha(x) := (\varrho(x,c))^\alpha$ for $x \in X$. The notation $f_n \downarrow 0$ means that the sequence (f_n) of real-valued functions is decreasing and pointwise converges to 0.

The following is a list of the most commonly used symbols and their meaning:

a.e.	almost everywhere
$\mathcal{B}_X$	σ-algebra of Borel subsets of the space X
$B(X)$	space of bounded Borel measurable functions $f : X \to \mathbb{R}$
$B(x,r)$	closed ball in X with centre $x \in X$ and radius r
$C(X)$	space of bounded continuous functions $f : X \to \mathbb{R}$
$C_0^1(\mathbb{R}^d)$	space of functions $f : \mathbb{R}^d \to \mathbb{R}$ with compact supports and continuous first derivatives
$\mathrm{dist}(A,B)$	distance between sets A and B, 18
D^+	upper right Dini derivative
δ_x	point (Dirac) measure supported at x
$E\xi$	expectation of the random variable ξ
$\langle f, \mu\rangle$	Lebesgue integral of $f : X \to \mathbb{R}$ with respect to the measure μ
$\mathcal{F}$	set of test functions $f : X \to \mathbb{R}$ for the Fortet–Mourier metric, 13
$\mathcal{H}$	set of test functions $f : X \to \mathbb{R}$ for the Hutchinson metric, 13
$\mathcal{H}_c$	subset of $f \in \mathcal{H}$ for which $f(c) = 0$, 13
$\mathbf{1}_A$	characteristic function of the set A
L	space of Lipschitzian functions $f : X \to \mathbb{R}$
$\mathcal{L}(X)$	set of linearly bounded functions, 21
$\mu * \nu$	convolution of the measures $\mu, \nu \in \mathcal{M}_{\mathrm{sig}}$, 32
$\mu \circ \nu$	elementary product of the measures $\mu, \nu \in \mathcal{M}_{\mathrm{sig}}$, 32
$\Vert\mu\Vert_{\mathcal{F}}$	Fortet–Mourier norm of the measure μ, 13
$\Vert\mu\Vert_{\mathcal{F}_\varphi}$	Fortet–Mourier norm of the measure μ generated by the metric ϱ_φ, 54
$\Vert\mu\Vert_{\mathcal{H}}$	Hutchinson norm of the measure μ, 13
$\Vert\mu\Vert_T$	total variation norm of the measure μ, 53
μ_+, μ_-	positive part and negative part of the measure μ
$\vert\mu\vert$	total variation of the measure μ
$\mathcal{M}$	family of finite (nonnegative) Borel measures
$\mathcal{M}_1$	space of probability measures, 12
$\mathcal{M}_{\mathrm{sig}}$	space of finite signed measures, 12
$m_\alpha(\mu)$	αth moment of the measure $\mu \in \mathcal{M}_1$, 13

$m_\alpha(\lvert\mu\rvert)$	αth moment of the measure $\mu \in \mathcal{M}_{\mathrm{sig}}$, 13
$\mathcal{M}_{1,\alpha}$	subset of measures $\mu \in \mathcal{M}_1$ such that $m_\alpha(\mu) < \infty$, 12
$\mathcal{M}_{\mathrm{sig},\alpha}$	subset of measures $\mu \in \mathcal{M}_{\mathrm{sig}}$ such that $m_\alpha(\lvert\mu\rvert) < \infty$, 12
$\mathbb{N}$	positive integers
$\mathcal{N}_p$	Poisson random counting measure, 45
$(\Omega, \Sigma, \mathrm{prob})$	probability space
$\Omega(x)$	set of limiting points of the trajectory $(S^t x)$, 27
P	Markov operator, 19
P_{*n}	convolution operator of order n, 32
$(P^t)_{t\in T}$	semigroup of Markov operators, 22
π	transition function, 20
ϱ_φ	Fortet–Mourier metric corresponding to the pair (ϱ, φ), 54
$\mathbb{R}$	real numbers
$\mathbb{R}_+$	nonnegative real numbers
$\mathbb{R}^d$	d-dimensional real space
$\lVert \cdot \rVert$	Euclidean norm in $\mathbb{R}^d$
$(\cdot\vert\cdot)$	scalar product in $\mathbb{R}^d$
$(S^t)_{t\in T}$	semidynamical system, 27
$(S^t x)$	trajectory starting from x, 27
$\operatorname{supp}\mu$	support of measure μ, 14
$(S_1, \ldots, S_N; p_1, \ldots, p_N)$	iterated function system, 44
T	nontrivial semigroup of nonnegative real numbers
U	dual operator to P, 20
$(U^t)_{t\in T}$	dual semigroup corresponding to $(P^t)_{t\in T}$, 22

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