Contents

Abstract

Lie systems form a class of systems of first-order ordinary differential equations whose general solutions can be described in terms of certain finite families of particular solutions and a set of constants, by means of a particular type of mapping: the so-called superposition rule. Apart from this fundamental property, Lie systems enjoy many other geometrical features and they appear in multiple branches of mathematics and physics. These facts, together with the authors' recent findings in the theory of Lie systems, led them to write this essay, which aims to describe the new achievements within a self-contained guide to the whole theory of Lie systems, their generalisations, and applications.

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1. The theory of Lie systems

1.1. Motivation and general scheme of the work. It is a little surprising that the theory of *Lie systems* [153, 154, 157, 224], which studies a very specific class of systems of first-order ordinary differential equations, can be employed to investigate a large variety of topics [8, 12, 53, 55, 59, 98, 144, 202, 212]. Indeed, although being a Lie system is an exception rather than a rule [128], these equations frequently turn up in multiple branches of mathematics and physics. For instance, linear systems of first-order differential equations, Riccati equations [86], and matrix Riccati equations [103, 116, 117, 131] are Lie systems that very frequently appear in the literature [62, 98, 112, 141, 207, 212, 234]. This obviously motivates the study of the theory of Lie systems as a means to investigate the properties of various remarkable differential equations and their applications.

The research of Lie systems involves the analysis of multiple interesting geometric and algebraic problems. For example, determination of Lie systems defined on a fixed manifold is related to the existence of finite-dimensional Lie algebras of vector fields over the manifold [157, 210]. Furthermore, the study of Lie systems leads to the investigation of foliations [35], generalised distributions [38], Lie group actions [141], finite-dimensional Lie algebras [40, 157, 210], etc. As a result, Lie systems provide methods to study the integrability of systems of first-order differential equations [40], control theory [32, 61, 79, 187], geometric phases [98], certain problems in quantum mechanics [46, 51], and other topics. Finally, it is remarkable that the theory of Lie systems has been investigated by means of different techniques and approaches, like Galois theory [17, 19] or differential geometry [38, 60, 186, 220].

When applying Lie systems to study more general systems of differential equations than merely first-order ones (see for instance [34, 35, 52, 77, 202]), the interest of their analysis becomes even more evident. For example, various systems of second-order differential equations, which very frequently appear in classical mechanics, can be studied by means of Lie systems. Dissipative Milne–Pinney equations [45], Milne–Pinney equations [52], Caldirola–Kanai oscillators [54], t-dependent frequency harmonic oscillators [55], or second-order Riccati equations [48, 225] are just some examples.

The relevance of the above studies, along with the determination of new applications of Lie systems, is twofold. On one hand, they allow us to obtain novel results about interesting differential equations. On the other hand, such examples may show us new features or generalisations of the notions appearing in the theory of Lie systems that have not been previously observed. Let us briefly provide a case in point. While studying second-order differential equations by means of Lie systems [52, 53, 202], a new type of ‘superposition-like’ expression describing the general solution of certain systems of second-order differential equations appeared. These papers led to the definition of a possible superposition rule for such systems whose main properties are still under analysis [48]. In addition, these works took different approaches to second-order differential equations: by means of SODE Lie systems [52] and through regular Lagrangians [54]. Relations between these approaches and the existence of new approaches are still an open question [48].

Apart from the investigation of the above open problems, perhaps the most active field of research into Lie systems is concerned with the development of new generalisations of Lie systems and superposition rules. Quasi-Lie systems [34, 35, 42], t-dependent superposition rules [34], PDE Lie systems [38, 172], SODE Lie systems [52], partial superposition rules [38, 153], quantum Lie systems [60], or stochastic Lie–Scheffers systems [144] are just a few such generalisations that have been carried out in order to analyse non-Lie systems with techniques similar to those developed for Lie systems. Indeed, the list of generalisations is much larger and even sometimes the term ‘superposition rule’ has been used with different, nonequivalent, meanings [197, 215].

In view of the above and many other reasons, the theory of Lie systems, along with its multiple generalisations, can be regarded as a multidisciplinary active field of research which involves the use of techniques from diverse branches of mathematics and physics as well as their applications to control theory [25, 26, 32, 59, 61, 79, 119, 187, 212], physics [39, 54, 58, 234], and other fields [31].

Our work starts by surveying briefly the historical development of the theory of Lie systems and several of their generalisations. In this way, we aim to provide a general overview of the subject, the main authors, trends, and the principal works dedicated to the major results. Special attention has been paid to provide a complete bibliography, which contains numerous references that cannot be easily found elsewhere. Furthermore, we provide a detailed account of the works of the main contributors to the theory of Lie systems: Lie [153–157], Vessiot [222–227], Winternitz [8, 9, 13, 112, 105, 173, 174, 233–236], Ibragimov [120–125], etc. Additionally, we present the main contents of some works which have been written in other languages than English, e.g. [153, 222, 223, 225].

After our brief overview of the history of Lie systems, the fundamental notions of this theory and other related topics are presented. More specifically, along with a recently developed differential geometric approach to the investigation of Lie systems [38], results about application of Lie systems to quantum mechanics, partial differential equations (PDEs), and systems of second- and higher-order differential equations are discussed. This, together with the historical introduction, furnishes a self-contained presentation of the topic which can be used both as an introduction to the subject and as a reference guide to Lie systems.

Later on, in Chapter 2, our survey focuses on methods of analysing second-order differential equations. Chapter 3 is concerned with various applications of Lie systems in quantum mechanics. Subsequently, we describe a theory of integrability of Lie systems in Chapter 4. This theory is employed to investigate some systems of differential equations appearing in classical mechanics in Chapter 5 and various Schrödinger equations in

Chapter 6. Finally, Chapters 7 and 8 describe the theory and applications of a new powerful technique, the *quasi-Lie schemes*, developed to apply the methods invented for Lie systems to a much larger set of systems of differential equations. In the same way as Lie systems, this method can straightforwardly be applied to second- and higher-order differential equations and quantum mechanics. Finally, diverse applications of this technique are presented in Chapter 8.

1.2. Historical introduction. It seems that Abel was the first to deal with the concept of superposition rule, while analysing linearisation of nonlinear operators [128]. Apart from this very early treatment, the fundamentals of the theory of Lie systems were laid down towards the end of the XIX century by the Norwegian mathematician Sophus Lie [153, 154, 155, 157] and the French one Ernest Vessiot [222–228]. Indeed, Lie systems are also frequently referred to as *Lie–Vessiot systems* to honour their contributions.

The first study that focused on analysing differential equations admitting a superposition rule was carried out by Königsberger [136] in 1883. He proved that the only first-order ordinary differential equations on the real line admitting a superposition rule that depends algebraically on the particular solutions are (up to a diffeomorphism) Riccati equations, linear and homogeneous linear differential equations. Later on, in 1885, Lie proposed a special class of systems of first-order ordinary differential equations [153, p. 128] whose general solutions can be obtained out of certain finite families of particular solutions and sets of constants [18, 220].

Despite the above mentioned achievements, these pioneering works did not draw much attention. Nevertheless, the situation changed from 1893. At that time, Vessiot and Guldberg proved, independently, a slightly more general form of Königsberger's main result. They demonstrated that (up to diffeomorphism) Riccati equations and linear differential equations are the only differential equations over the real line admitting a superposition rule [108, 122, 128, 222]. This result attracted Lie's attention [154], who claimed that their contribution is a simple consequence of his previous work [153]. More specifically, he stated that systems which admit a superposition rule are those he had defined in 1885 [155]. In view of these criticisms, Lie did not recognise the value of Vessiot and Guldberg's discovery [128]. Nevertheless, some credit to them must be given, as the theory of Lie does not easily lead to the case provided by Vessiot and Guldberg [128].

Lie's remarks gave rise to one of the most important results about the theory of Lie systems, today called the *Lie Theorem* [157, Theorem 44]. This theorem characterises systems of first-order ordinary differential equations admitting a superposition rule. In addition, it provides some information on the form of such a superposition rule. In [157], Lie and Scheffers presented the first detailed discussion of Lie systems. In recognition of that work, some authors also call Lie systems *Lie–Scheffers systems.*

In spite of this important success, the Lie Theorem, as stated by Lie, contains some small gaps in its proof as well as a slight lack of rigour about the definition of superposition rule. This was noticed and fixed at the beginning of the XXI century by Cariñena, Grabowski, Marmo, Blázquez, and Morales [18, 38].

After Lie's reply, Vessiot recognised the importance of Lie's work and proposed to call *Lie systems* those systems of first-order ordinary differential equations admitting a superposition rule [224]. Apart from this first 'trivial result', Vessiot furnished many new contributions to the theory of Lie systems [223, 224, 226, 228] and proposed various generalisations [225, 227, 228]. For instance, he showed that a superposition-like expression can be used to analyse particular types of second-order Riccati equations [225]. More specifically, he proved that for some of these equations, general solutions can be obtained from families of four particular solutions, their derivatives, and two real constants. As far as we know, this is the first result concerning superposition rules for nonlinear second-order differential equations.

After a deep initial study of superposition rules and Lie systems [108, 153–155, 222, 224–228], the topic was almost forgotten for nearly a century. Just a few works were devoted to superposition rules [76, 80–82, 149, 198]. In the seventies, nevertheless, the interest in the topic revived and many authors focused again on investigating Lie systems, their generalisations, and applications to mathematics, physics and control theory [127, 130, 175]. Among the reasons that motivated that rebirth of the theory of Lie systems, we can emphasise the importance of the works of Winternitz and Brockett. On one hand, Brockett analysed the rôle of certain Lie systems in control theory [25, 26], which initiated a research field that continues until the present [32, 59, 61, 79, 119, 185, 187, 201, 212]. On the other hand, Winternitz and his collaborators made a huge contribution to the theory of Lie systems and their applications to physics, mathematics and control theory [8, 9, 13–15, 21, 112, 114, 141, 234, 236].

Let us discuss in more detail some of Winternitz's results. Using diverse results derived by Lie [156, 157], Winternitz and his collaborators developed and applied a method of deriving superposition rules [202, 209, 235]. They also studied the problem of classification of Lie systems through transitive primitive Lie algebras [210], a concept that also appeared in some of Winternitz's works about the integrability of Lie systems [21, 22]. Winternitz also analysed discrete problems and numerical approximations of solutions by means of superposition rules [179, 188, 202, 219] and, finally, with collaborators, developed a new generalisation of superposition rule, the so-called *super-superposition rule*, in order to study the general solutions of various types of superequations [12, 13].

Besides these theoretical achievements, Winternitz et al. applied their methods to the analysis of multiple discrete and differential equations with applications to mathematics, physics and control theory. For instance, many superposition rules were derived for matrix Riccati equations [8, 112, 141, 174, 188, 212], which play an important rôle in control theory, as well as for diverse Lie systems, like projective Riccati equations [21], various superequations [12, 13], and others [9, 14, 15, 99, 114]. Finally, Winternitz's paper on Milne–Pinney equations [202] is also worth mentioning; it is one of the first papers devoted to analysing second-order differential equations through Lie systems.

Currently, many researchers investigate Lie systems and other closely related topics. Let us merely point out here some of them along with some of their works: Blázquez and Morales [17–19], Cariñena [34, 37, 38], Clemente [32], Grabowski [37–39], Ibragimov [120, 121, 122, 124], de Lucas [34, 35, 52], Lázaro-Camí and Ortega [144], Marmo [37, 38, 39], Odzijewicz and Grundland [172], Ramos [40, 59, 62], Rañada [43, 52, 53, 55] and Nasarre [57, 58]. As a result of their contributions, multiple interesting results about the fundamentals, applications, and generalisations of the theory of Lie systems were furnished.

Among the above works, we describe briefly the content of [34, 37, 38]. The book [37] presents an instructive geometric introduction to the basic topics of the theory of Lie systems. [38] provides multiple relevant contributions to the theory of Lie systems. First, it fixes a remarkable gap in the proof of the Lie Theorem. Additionally, it establishes that a superposition rule amounts to a certain type of flat connection, which substantially clarifies its properties. The furnished demonstration of the Lie Theorem shows that the Lie system notion can be naturally extended to the case of PDEs. Finally, this work led, more or less indirectly, to the characterisation of families of systems of first-order differential equations admitting a t-dependent superposition rule [35] and to the definition of mixed and partial superposition rules [38, 52]. Finally, we mention the usefulness of the *Lie scheme* concept provided in [34], which generalises Lie systems and leads to the discovery of new properties for various systems of differential equations, including non-Lie systems, appearing in physics and mathematics [34, 42, 45, 48, 56].

Let us now discuss some of the authors' contributions that led them to write this essay. On one hand, Cariñena and his collaborators investigated the integrability of Lie systems [40, 43, 47, 50, 54, 63], a generalisation of the Wei–Norman method for the study of Lie systems [57], application of Lie systems techniques to analyse systems of second-order differential equations [48, 49, 52, 53], and other topics like the analysis of certain Schrödinger equations [46, 51, 59]. In this way, they provided a continuation of diverse previous articles dedicated to some of these themes [77, 172, 202, 225] and they opened several new research lines [59].

Besides the above contributions, Cariñena and his collaborators also developed numerous applications of Lie systems to classical physics [39, 43–45, 52, 54, 55, 58, 62], quantum mechanics [46, 51, 59, 60], financial mathematics [31], and control theory [60, 61].

Apart from the aforementioned generalisations of Lie systems that are related to other works in the literature [6, 172, 202, 225], a new generalisation of Lie systems and superposition rules was carried out by Cariñena, Grabowski and de Lucas in the theory of quasi-Lie schemes [34]. One one hand, this approach provides us with a method to transform differential equations of a certain type into equations of the same type, e.g. Abel equations into Abel equations [56]. This can also be used to transform differential equations into Lie systems [34], which leads to the *quasi-Lie system* notion. Such systems inherit some properties of Lie systems and, for instance, they admit superposition rules showing an explicit dependence on the independent variable of the system [34, 48].

Quasi-Lie schemes admit multiple applications. They can be used not only to analyse the properties of Lie and quasi-Lie systems but also to investigate many other systems, e.g. nonlinear oscillators [34], Emden–Fowler equations [42], Mathews–Lakshmanan oscillators [34], dissipative and non-dissipative Milne–Pinney equations [45], and Abel equations [56]. As a consequence, various results about the integrability properties of such equations have been obtained and many others are being analysed at present. Furthermore, the appearance of t-dependent superposition rules led to the examination of the so-called *Lie families*, which cover, as particular cases, Lie systems and quasi-Lie schemes. Additionally, they can be used to analyse the exact solutions of very general families of differential equations [35].

As a result of all the above mentioned achievements, there exists today a vast collection of methods and procedures to analyse Lie systems from different points of view. All these tools can be used to provide interesting results in mathematics, physics, control theory, and other fields. At the same time, these applications motivate the development of new techniques, generalisations, and applications of this theory, which yields multiple and interesting topics for further research.

1.3. Fundamentals about Lie systems and superposition rules. Our main purpose in this section is to review the basic notions and fundamental results concerning the theory of Lie systems to be employed and analysed throughout our essay. Here, as well as in the major part of our essay, we mostly restrict ourselves to analysing differential equations on vector spaces and we assume that mathematical objects, e.g. flows of vector fields, are smooth, real, and globally defined. This will allow us to highlight the key points of our exposition and omit several irrelevant technical aspects that can be easily deduced from our presentation. Nonetheless, numerous differential equations on manifolds and diverse technical points will be presented when relevant.

Definition 1.1. Given the projections $\pi : (x, v) \in \mathrm{T}\mathbb{R}^n \mapsto x \in \mathbb{R}^n$ and $\pi_2 : (t, x) \in \mathbb{R} \times \mathbb{R}^n \mapsto x \in \mathbb{R}^n$, a *$t$-dependent vector field X* on $\mathbb{R}^n$ is a map $X : (t, x) \in \mathbb{R} \times \mathbb{R}^n \mapsto X(t, x) \in \mathrm{T}\mathbb{R}^n$ such that the diagram

$$\begin{tikzcd} & \mathrm{T}\mathbb{R}^n \arrow[d, "\pi"] \\ \mathbb{R}\times\mathbb{R}^n \arrow[ur, "X"] \arrow[r, "\pi_2"'] & \mathbb{R}^n \end{tikzcd}$$

is commutative, i.e. $\pi \circ X = \pi_2$.

In view of the above definition, $X(t, x) \in \pi^{-1}(x) = \mathrm{T}_x\mathbb{R}^n$ and hence $X_t : x \in \mathbb{R}^n \mapsto X_t(x) \equiv X(t, x) \in \mathrm{T}\mathbb{R}^n$ is a vector field over $\mathbb{R}^n$ for every $t \in \mathbb{R}$. Thus, it is immediate that each t-dependent vector field X is equivalent to a family $\{X_t\}_{t\in\mathbb{R}}$ of vector fields over $\mathbb{R}^n$.

The t-dependent vector field concept includes, as a particular instance, the standard vector field notion. Indeed, every vector field Y over $\mathbb{R}^n$ can be naturally regarded as a t-dependent vector field X of the form $X_t = Y$ for every $t \in \mathbb{R}$. Conversely, a ‘constant’ t-dependent vector field X over $\mathbb{R}^n$, i.e. $X_t = X_{t'}$ for every $t, t' \in \mathbb{R}$, can be considered as a vector field $Y = X_0$ over this space.

As vector fields, t-dependent vector fields also admit local integral curves (see [29]). For each t-dependent vector field X over $\mathbb{R}^n$, this gives rise to the *generalised flow* g^X, i.e. the map $g^X : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ such that $g^X(t, x) \equiv g^X_t(x) = \gamma_x(t)$ with $\gamma_x(t)$ being the unique integral curve of X such that $\gamma_x(0) = x$.

Definition 1.2. A t-dependent vector field X over $\mathbb{R}^n$ is said to be *projectable* under a projection $p : \mathbb{R}^n \to \mathbb{R}^{n'}$ if every X_t is projectable, as a usual vector field, under such a map.

The usage of t-dependent vector fields is fundamental in the theory of Lie systems. They provide us with a geometrical object which contains all information necessary to study systems of first-order differential equations. Let us start by showing how systems of first-order differential equations are described by means of t-dependent vector fields.

Definition 1.3. Given a t-dependent vector field

$$X(t,x) = \sum_{i=1}^{n} X^i(t,x) \frac{\partial}{\partial x^i}, \tag{1.1}$$

over $\mathbb{R}^n$, its *associated system* is the system of first-order differential equations determining its integral curves, that is,

$$\frac{dx^i}{dt} = X^i(t,x), \qquad i = 1, \ldots, n. \tag{1.2}$$

Note that there exists a one-to-one correspondence between t-dependent vector fields and systems of first-order differential equations of the form (1.2). That is, every t-dependent vector field has an associated system of first-order differential equations and each system of this type, in turn, determines the integral curves of a unique t-dependent vector field. Taking this into account, we can use X to refer to both a t-dependent vector field and the system of equations describing its integral curves. This simplifies our exposition and it does not lead to confusion as the difference of meaning is clear from context.

The following definition and lemma, whose proof is straightforward and will not be detailed, simplify the statements and proofs of various results in the theory of Lie systems.

Definition 1.4. Given a (possibly infinite) family $\mathcal{A}$ of vector fields on $\mathbb{R}^n$, we denote by $\mathrm{Lie}(\mathcal{A})$ the smallest Lie algebra V of vector fields on $\mathbb{R}^n$ containing $\mathcal{A}$.

Lemma 1.5. *Given a family $\mathcal{A}$ of vector fields, the linear space* $\mathrm{Lie}(\mathcal{A})$ *is spanned by the vector fields in*

$$\mathcal{A},\ [\mathcal{A},\mathcal{A}],\ [\mathcal{A},[\mathcal{A},\mathcal{A}]],\ [\mathcal{A},[\mathcal{A},[\mathcal{A},\mathcal{A}]]],\ldots$$

where $[\mathcal{A},\mathcal{B}]$ denotes the set of Lie brackets of elements of $\mathcal{A}$ and $\mathcal{B}$.

Throughout this work two different notions of linear independence are used frequently. For clarity, we provide the following definition.

Definition 1.6. Let us denote by $\mathfrak{X}(\mathbb{R}^n)$ the space of vector fields over $\mathbb{R}^n$. We say that the vector fields $X_1, \ldots, X_r$ on $\mathbb{R}^n$ are *linearly independent over* $\mathbb{R}$ if they are linearly independent as elements of $\mathfrak{X}(\mathbb{R}^n)$ when considered as an $\mathbb{R}$-vector space, i.e. whenever

$$\sum_{\alpha=1}^{r} \lambda_\alpha X_\alpha = 0$$

for certain constants $\lambda_1, \ldots, \lambda_r$, then $\lambda_1 = \cdots = \lambda_r = 0$. On the other hand, $X_1, \ldots, X_r$ are said to be *linearly independent at a generic point* if they are linearly independent as elements of $\mathfrak{X}(\mathbb{R}^n)$ when regarded as a $C^\infty(\mathbb{R}^n)$-module. That is, if

$$\sum_{\alpha=1}^{r} f_\alpha X_\alpha = 0$$

on $\mathbb{R}^n$ for certain functions $f_1, \ldots, f_r \in C^\infty(\mathbb{R}^n)$, then $f_1 = \cdots = f_r = 0$.

In this essay, we frequently deal with linear spaces of the form $\mathbb{R}^{n(m+1)}$. Such spaces are always considered as a product $\mathbb{R}^n \times \overset{m+1 \text{ times}}{\cdots} \times \mathbb{R}^n$. Each point of $\mathbb{R}^{n(m+1)}$ is denoted by $(x_{(0)}, \ldots, x_{(m)})$, where $x_{(j)}$ stands for a point in the jth copy of the manifold $\mathbb{R}^n$ within $\mathbb{R}^{n(m+1)}$.

Associated with $\mathbb{R}^{n(m+1)}$, there exists a group S_{m+1} of permutations whose elements, S_{ij}, with $i \leq j = 0, 1, \ldots, m$, act on $\mathbb{R}^{n(m+1)}$ by permuting the variables $x_{(i)}$ and $x_{(j)}$. Finally, let us define the projections

$$\mathrm{pr} : (x_{(0)}, \ldots, x_{(m)}) \in \mathbb{R}^{n(m+1)} \mapsto (x_{(1)}, \ldots, x_{(m)}) \in \mathbb{R}^{nm} \tag{1.3}$$

and

$$\mathrm{pr}_0 : (x_{(0)}, \ldots, x_{(m)}) \in \mathbb{R}^{n(m+1)} \mapsto x_{(0)} \in \mathbb{R}^n. \tag{1.4}$$

We now proceed to introduce the notion of *superposition rule*, which plays a central role in the study of Lie systems.

For each system of first-order ordinary homogeneous linear differential equations on $\mathbb{R}^n$ of the form

$$\frac{dy^i}{dt} = \sum_{j=1}^{n} A^i_j(t) y^j, \qquad i = 1, \ldots, n, \tag{1.5}$$

where $A^i_j(t)$, with $i, j = 1, \ldots, n$, is a family of t-dependent functions, its general solution $y(t)$ can be written as a linear combination of the form

$$y(t) = \sum_{j=1}^{n} k_j y_{(j)}(t), \tag{1.6}$$

with $y_{(1)}(t), \ldots, y_{(n)}(t)$ being a family of n generic (linearly independent) particular solutions, and $k_1, \ldots, k_n$ being a set of constants. The above expression is called a *linear superposition rule* for system (1.5).

Linear superposition rules allow us to reduce the search for the general solution of a linear system to the determination of a finite set of particular solutions. This property is not exclusive to homogeneous linear systems. Indeed, for each linear system

$$\frac{dy^i}{dt} = \sum_{j=1}^{n} A^i_j(t) y^j + B^i(t), \qquad i = 1, \ldots, n, \tag{1.7}$$

where $A^i_j(t), B^i(t)$, with $i, j = 1, \ldots, n$, are a family of t-dependent functions, its general solution $y(t)$ can be written as a linear combination of the form

$$y(t) = \sum_{j=1}^{n} k_j (y_{(j)}(t) - y_{(0)}(t)) + y_{(0)}(t), \tag{1.8}$$

with $y_{(0)}(t), \ldots, y_{(n)}(t)$ being a family of $n+1$ particular solutions such that $y_{(j)}(t) - y_{(0)}(t)$, with $j = 1, \ldots, n$, are linearly independent solutions of the homogeneous problem associated with (1.7), and $k_1, \ldots, k_n$ being a set of constants.

In a more general way, system (1.5) becomes a (generally) nonlinear system

$$\frac{dx^i}{dt} = X^i(t, x), \qquad i = 1, \ldots, n, \tag{1.9}$$

through a diffeomorphism $\varphi : \mathbb{R}^n \ni y \mapsto x = \varphi(y) \in \mathbb{R}^n$. In view of the linear superposition rule (1.6), the general solution $x(t)$ of the above system can be described in terms of a family of certain particular solutions $x_{(1)}(t), \ldots, x_{(m)}(t)$ as

$$x(t) = \varphi\Big(\sum_{j=1}^{n} k_j \varphi^{-1}(x_{(j)}(t))\Big).$$

This clearly shows that there exist many systems of first-order differential equations whose general solutions can be described, nonlinearly, in terms of certain families of particular solutions and sets of constants. Another relevant family of equations with this property are Riccati equations [4, 64, 102, 112, 170, 189, 212] of the form

$$\frac{dx}{dt} = b_1(t) + b_2(t)x + b_3(t)x^2, \tag{1.10}$$

with $x \in \bar{\mathbb{R}} \equiv \mathbb{R} \cup \{\infty\}$. More specifically, for each Riccati equation, its general solution $x(t)$ can be cast in the form

$$x(t) = \frac{x_1(t)(x_3(t) - x_2(t)) - kx_2(t)(x_3(t) - x_1(t))}{(x_3(t) - x_2(t)) - k(x_3(t) - x_1(t))}, \tag{1.11}$$

where $x_1(t)$, $x_2(t)$, $x_3(t)$ are three particular solutions of the equation and $k \in \bar{\mathbb{R}}$.

It is worth noting that, given a fixed family of three different particular solutions with initial conditions within $\mathbb{R}$, if we only choose k in $\mathbb{R}$, the above expression does not give the whole general solution of the Riccati equation, as $x_2(t)$ cannot be recovered.

The above examples show the existence of a certain type of expression, called a *global superposition rule*, which enables us to express the general solution of certain systems of first-order ordinary differential equations in terms of certain families of particular solutions and a set of constants. Let us state a rigorous definition of this notion for systems of differential equations in $\mathbb{R}^n$.

Definition 1.7. The system of first-order ordinary differential equations

$$\frac{dx^i}{dt} = X^i(t, x), \qquad i = 1, \ldots, n, \tag{1.12}$$

is said to admit a *global superposition rule* if there exists a t-independent map $\Phi : (\mathbb{R}^n)^m \times \mathbb{R}^n \to \mathbb{R}^n$ of the form

$$x = \Phi(x_{(1)}, \ldots, x_{(m)}; k_1, \ldots, k_n) \tag{1.13}$$

such that the general solution $x(t)$ of (1.12) can be written as

$$x(t) = \Phi(x_{(1)}(t), \ldots, x_{(m)}(t); k_1, \ldots, k_n), \tag{1.14}$$

with $x_{(1)}(t), \ldots, x_{(m)}(t)$ being any generic family of particular solutions of (1.12), and $k_1, \ldots, k_n$ being a set of n constants related to initial conditions.

To give a meaning to the above definition, it is necessary to specify the sense in which the term 'generic' is used. Precisely, expression (1.14) is said to be valid for any generic family of m particular solutions if there exists an open dense subset $U \subset (\mathbb{R}^n)^m$ such that (1.14) is satisfied for every set of particular solutions $x_1(t), \ldots, x_m(t)$ such that $(x_1(0), \ldots, x_m(0))$ lies in U.

Let us now show that the aforementioned examples admit a global superposition rule. Consider the function $\Phi : (\mathbb{R}^n)^n \times \mathbb{R}^n \to \mathbb{R}^n$ of the form

$$\Phi(x_{(1)}, \ldots, x_{(n)}; k_1, \ldots, k_n) = \sum_{j=1}^{n} k_j x_{(j)}. \tag{1.15}$$

This mapping is a superposition rule for system (1.5). Indeed, for each set of particular solutions $x_{(1)}(t), \ldots, x_{(m)}(t)$ of (1.5) such that $(x_{(1)}(0), \ldots, x_{(m)}(0))$ belongs to the open dense subset

$$U = \left\{ (x_{(1)}, \ldots, x_{(n)}) \in (\mathbb{R}^n)^n \;\middle|\; \det \begin{pmatrix} x^1_{(1)} & \cdots & x^1_{(n)} \\ \cdots & \cdots & \cdots \\ x^n_{(1)} & \cdots & x^n_{(n)} \end{pmatrix} \neq 0 \right\}$$

of $(\mathbb{R}^n)^n$, the general solution $x(t)$ of (1.5) can be written in the form (1.6). Likewise, a superposition rule can now be proved to exist for the systems (1.9) obtained from (1.5) by means of a diffeomorphism.

The function $\Phi : (\mathbb{R}^n)^{n+1} \times \mathbb{R}^n \to \mathbb{R}^n$ of the form

$$\Phi(x_{(0)}, \ldots, x_{(n)}; k_1, \ldots, k_n) = \sum_{j=1}^{n} k_j (x_{(j)} - x_{(0)}) + x_{(0)} \tag{1.16}$$

is a superposition rule for the system (1.7). In fact, for each set of particular solutions $x_{(0)}(t), \ldots, x_{(n)}(t)$ of (1.7) such that $(x_{(0)}(0), \ldots, x_{(n)}(0))$ belongs to the open dense subset

$$U = \left\{ (x_{(0)}, \ldots, x_{(n)}) \in (\mathbb{R}^n)^{n+1} \;\middle|\; \det \begin{pmatrix} x^1_{(1)} - x^1_{(0)} & \cdots & x^1_{(n)} - x^1_{(0)} \\ \cdots & \cdots & \cdots \\ x^n_{(1)} - x^n_{(0)} & \cdots & x^n_{(n)} - x^n_{(0)} \end{pmatrix} \neq 0 \right\}$$

of $(\mathbb{R}^n)^{n+1}$, the general solution $x(t)$ of (1.7) can be put in the form (1.8).

Finally, let us analyse the case of Riccati equations in $\bar{\mathbb{R}}$. This example differs a little from the previous ones, as it concerns a differential equation defined in the manifold $\bar{\mathbb{R}} \simeq S^1$. Nevertheless, the generalisation of Definition 1.7 to manifolds is obvious. It is only necessary to replace $\mathbb{R}^n$ by a manifold N. Then the map $\Phi : \bar{\mathbb{R}}^3 \times \bar{\mathbb{R}} \to \bar{\mathbb{R}}$ of the form

$$\Phi(x_{(1)}, x_{(2)}, x_{(3)}; k) = \frac{x_{(1)}(x_{(3)} - x_{(2)}) - k x_{(2)}(x_{(3)} - x_{(1)})}{(x_{(3)} - x_{(2)}) - k(x_{(3)} - x_{(1)})} \tag{1.17}$$

is a global superposition rule for Riccati equations in $\bar{\mathbb{R}}$. To verify this, it is sufficient to note that given one of these equations with three particular solutions, $x_{(1)}(t), x_{(2)}(t)$, $x_{(3)}(t)$, such that $(x_{(1)}(0), x_{(2)}(0), x_{(3)}(0)) \in U$, where

$$U = \{(x_{(1)}, x_{(2)}, x_{(3)}) \in \mathbb{R}^3 \,|\, x_{(1)} \neq x_{(2)},\ x_{(1)} \neq x_{(3)} \text{ and } x_{(2)} \neq x_{(3)}\},$$

its general solution can be cast in the form (1.11).

The aforementioned superposition rules illustrate that for each permutation of their arguments $x_{(1)},\ldots,x_{(m)}$, e.g. an interchange of the arguments $x_{(i)}$ and $x_{(j)}$, one has, in general,

$$\Phi(x_{(1)},\ldots,x_{(i)},\ldots,x_{(j)},\ldots,x_{(m)};k)\neq\Phi(x_{(1)},\ldots,x_{(j)},\ldots,x_{(i)},\ldots,x_{(m)};k).$$

Nevertheless, it can be proved (cf. [38]) that there exists a map $\varphi: k\in\mathbb{R}^n\mapsto\varphi(k)\in\mathbb{R}^n$ such that

$$\Phi(x_{(1)},\ldots,x_{(i)},\ldots,x_{(j)},\ldots,x_{(m)};k)=\Phi(x_{(1)},\ldots,x_{(j)},\ldots,x_{(i)},\ldots,x_{(m)};\varphi(k)).$$

It is interesting to note that, if we consider Riccati equations defined on the real line, a global superposition rule for such equations would be a map of the form $\Phi:\mathbb{R}^m\times\mathbb{R}\to\mathbb{R}$. Obviously, expression (1.17) does not give rise to a global rule of this form. Indeed, if we restrict (1.17) to $\mathbb{R}^3\times\mathbb{R}$, we will not be able to recover $x_{(2)}(t)$ from a set of different particular solutions, $x_{(1)}(t),x_{(2)}(t),x_{(3)}(t)$, for any $k\in\mathbb{R}$. Even more, the function (1.17) is not globally defined over $\mathbb{R}^3\times\mathbb{R}$. Nevertheless, such a function is what is known in the literature as a *superposition rule* for Riccati equations over the real line [108, 157, 222].

In the literature, superposition rules appear as a 'milder' version of the aforementioned global superposition rules. In other words, superposition rules have almost the same properties as global superposition rules but, for instance, they may fail to recover certain particular solutions. Although it is enough to bear in mind the above example of Riccati equations to understand the main difference between both notions, the precise definition of a local superposition rule is very technical (see [18]) and it does not provide, in practice, any much deeper information about Lie systems. That is why, as usual in the literature [37, 108, 122, 123, 153, 157, 222–234], we will assume hereafter that superposition rules recover general solutions and are globally defined. This simplifies our theoretical presentation considerably and it highlights the main features of superposition rules and Lie systems. Despite these assumptions, a fully rigorous treatment of the general case can be easily carried out and some technical remarks will be discussed when relevant.

A relevant question now arises: which systems of first-order ordinary differential equations admit a superposition rule? Several works have been devoted to investigating this question. Its analysis was accomplished by Königsberger [136], Vessiot [222], and Guldberg [108]. They proved that every system of first-order differential equations defined over the real line admitting a superposition rule is, up to a diffeomorphism, a Riccati equation or a first-order linear differential equation.

Apart from these preliminary results, it was Lie [153, 154, 157] who established the conditions ensuring that a system of first-order differential equations of the form (1.12) admits a superposition rule. His result, today named the *Lie Theorem*, reads in modern geometric terms as follows.

THEOREM 1.8 (Lie Theorem). *A system of first-order ordinary differential equations* (1.12) *admits a superposition rule* (1.13) *if and only if its corresponding t-dependent* *vector field* (1.1) *can be cast in the form*

$$X(t,x)=\sum_{\alpha=1}^{r} b_\alpha(t)X_\alpha(x), \tag{1.18}$$

with $X_1,\ldots,X_r$ *being a family of vector fields over* $\mathbb{R}^n$ *spanning an* r*-dimensional real Lie algebra* V *of vector fields.*

In the proof of his theorem [157, Theorem 44], Lie also claimed that the dimension of the decomposition (1.18) and the number m of particular solutions for the superposition rule are related. More specifically, he proved that the existence of a superposition rule depending on m particular solutions for a system (1.12) in $\mathbb{R}^n$ implies that there exists a decomposition (1.18) associated with a Lie algebra V satisfying $\dim V\le m\cdot n$, which is referred to as *Lie's condition*. Conversely, given a decomposition of the form (1.18), we can ensure the existence of a superposition rule for system (1.12) whose number of particular solutions obeys the same condition.

Although the Lie Theorem solves theoretically the problem of determining whether a system (1.12) admits a superposition rule, it does not answer many other questions concerning superposition rules. Let us briefly comment on some of these.

- From a practical point of view, it is not straightforward, using solely the Lie Theorem, to prove that a system of first-order differential equations does not admit a superposition rule. Later on in this section, we will sketch a procedure to do so.
- The Lie Theorem says nothing about the possible existence of multiple superposition rules for the same system. What is more, it does not explain explicitly how to determine any of such superposition rules (although its proof [157, Theorem 4] furnishes some key hints). These questions are addressed later in this chapter, where we review a recent geometrical approach to Lie systems developed in [38].
- A system $X(t,x)$ admitting a superposition rule may be written in the form (1.18) in one or, sometimes, several different ways. Each of these decompositions is related to a different finite-dimensional Lie algebra V of vector fields. Such Lie algebras are generally called the *Vessiot–Guldberg Lie algebras* associated with a system. The Lie Theorem does not explain the possible relations amongst all possible Vessiot–Guldberg Lie algebras of a system (1.12). In fact, only Lie's condition suggests that different Vessiot–Guldberg Lie algebras may be related to different superposition rules. We will discuss these questions, in a more extensive way, later in this section and the next.
- Finally, it is worth noting that the Lie Theorem cannot be used to characterise straightforwardly systems of first-order differential equations of the form $F^i(t,x,\dot{x}) = 0$ with $i=1,\ldots,n$. Indeed, this is an open question.

The discovery of the Lie Theorem [157] in 1893 established definitively the notion of Lie system, which, on the other hand, had already been suggested long time ago by Lie [153], and whose name was coined by Vessiot in [224] in recognition of Lie's success in characterising systems admitting a superposition rule. The definition goes as follows.

DEFINITION 1.9. A system of the form (1.12) is a *Lie system* if the corresponding t-dependent vector field (1.1) admits a decomposition of the form (1.18).

In view of the Lie Theorem, the above definition can be rephrased by saying that (1.12) is a Lie system if and only if it admits a superposition rule. Hence, it is obvious that the systems (1.5), (1.7) and (1.10), which admit the global superposition rules (1.15), (1.16) and (1.17), respectively, are Lie systems. Let us analyse these examples in more detail. This brings the opportunity to illustrate diverse characteristics of Lie systems and the Lie Theorem, to be discussed here and in the forthcoming sections.

Consider again the homogeneous linear system (1.5). It describes the integral curves of the t-dependent vector field

$$X(t,x)=\sum_{i,j=1}^{n} A^i{}_j(t)x^j\,\frac{\partial}{\partial x^i}, \tag{1.19}$$

which is a linear combination of vector fields of the form

$$X(t,x)=\sum_{i,j=1}^{n} A^i{}_j(t)X_{ij}(x), \tag{1.20}$$

with the n^2 vector fields

$$X_{ij}=x^j\,\frac{\partial}{\partial x^i},\qquad i,j=1,\ldots,n. \tag{1.21}$$

Furthermore,

$$[X_{ij},X_{lm}]=\delta^i_m X_{lj}-\delta^l_j X_{im},$$

where δ^i_m is the Kronecker delta, i.e. the vector fields (1.21) generate an n^2-dimensional Vessiot–Guldberg Lie algebra isomorphic to the Lie algebra $\mathfrak{gl}(n,\mathbb{R})$ (see [62]).

In view of decomposition (1.20), each system (1.5) is a Lie system. This is not a surprise, as each system (1.5) admits the superposition rule (1.15) and the Lie Theorem states that every system admitting a superposition rule must be a Lie system. Moreover, in view of Lie's condition, since homogeneous linear systems in $\mathbb{R}^n$ admit a superposition rule depending on n particular solutions, their associated t-dependent vector fields must take values in *some* Lie algebra of dimension at most n^2. Indeed, decomposition (1.20) shows that $X(t,x)$ takes values in a Lie algebra isomorphic to $\mathfrak{gl}(n,\mathbb{R})$, which clearly obeys Lie's condition corresponding to the superposition rule (1.15).

Note that we have italicised the last 'some' in the paragraph above. We did it because we wanted to stress that a Lie system can take values in different Lie algebras, some of which do not need to satisfy the same Lie's condition. This will become clearer in the next example.

Let us now turn to an inhomogeneous system of the form (1.7). It describes the integral curves of the t-dependent vector field

$$X(t,x)=\sum_{i=1}^{n}\Big(\sum_{j=1}^{n} A^i{}_j(t)\,x^j+B^i(t)\Big)\frac{\partial}{\partial x^i}, \tag{1.22}$$

which is a linear combination with t-dependent coefficients,

$$X_t=\sum_{i,j=1}^{n} A^i{}_j(t)X_{ij}+\sum_{i=1}^{n} B^i(t)X_i, \tag{1.23}$$

of the vector fields (1.21) and

$$X_i = \frac{\partial}{\partial x^i}, \qquad i = 1, \ldots, n. \tag{1.24}$$

The above vector fields satisfy the commutation relations

$$[X_i, X_j] = 0, \qquad i, j = 1, \ldots, n, \qquad [X_{ij}, X_l] = -\delta^{lj} X_i, \qquad i, j, l = 1, \ldots, n.$$

This shows that the vector fields (1.21) and (1.24) span a Lie algebra of vector fields isomorphic to the (n^2+n)-dimensional Lie algebra of the affine group [62]. Thus, in view of decomposition (1.23), systems (1.7) are Lie systems.

As systems (1.7) admit a superposition rule (1.16) depending on $n+1$ particular solutions, Lie's condition implies that their t-dependent vector fields must take values in some Lie algebra of dimension at most $n(n+1)$. In fact, the above results easily show that this is the case.

The previous example shows that a Lie system may admit various Vessiot–Guldberg Lie algebras. Recall that every homogeneous linear system (1.5) is related to a t-dependent vector field taking values in a Lie algebra isomorphic to $\mathfrak{gl}(n, \mathbb{R})$. Additionally, as a particular instance of system (1.7), its t-dependent vector field also takes values in the above defined n^2+n-dimensional Lie algebra of vector fields. In other words, linear systems admit at least two nonisomorphic Vessiot–Guldberg Lie algebras.

Now, we can illustrate how different superposition rules for the same system may be associated with multiple, nonisomorphic, Vessiot–Guldberg Lie algebras and lead to distinct Lie's conditions. We showed that linear systems admit a linear superposition rule, which leads, in view of Lie's condition, to the existence of an associated Vessiot–Guldberg Lie algebra of dimension at most n^2, which was determined. Nevertheless, the above-mentioned second Vessiot–Guldberg Lie algebra for linear systems does not satisfy this condition. On the contrary, this Lie algebra shows that there must exist a second superposition rule, namely (1.8), which, along with this Vessiot–Guldberg Lie algebra, satisfies a new Lie's condition.

To sum up, the Lie Theorem implies that a system admitting a superposition rule is related to the existence of, at least, one Vessiot–Guldberg Lie algebra satisfying the Lie's condition relative to this superposition rule. Nevertheless, the system can possess more Vessiot–Guldberg Lie algebras, some of which do not need to obey Lie's condition for the assumed superposition rule. In that case, the other Vessiot–Guldberg Lie algebras are related to other superposition rules for which a new Lie's condition is satisfied.

We now consider Riccati equations (1.10). They determine the integral curves of the t-dependent vector field on $\bar{\mathbb{R}}$ of the form

$$X(t, x) = (b_1(t) + b_2(t)x + b_3(t)x^2)\frac{\partial}{\partial x}. \tag{1.25}$$

As Riccati equations admit a global superposition rule, they must satisfy the assumptions detailed in the Lie Theorem. Indeed, note that X is a linear combination with t-dependent coefficients of the three vector fields

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial x}, \quad X_3 = x^2\frac{\partial}{\partial x}, \tag{1.26}$$

which generate a three-dimensional Lie algebra with defining relations

$$[X_1, X_2] = X_1, \quad [X_1, X_3] = 2X_2, \quad [X_2, X_3] = X_3. \tag{1.27}$$

Thus, as expected, Riccati equations obey the conditions given by Lie to admit a superposition rule. Moreover, Riccati equations are associated with a Vessiot–Guldberg Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Since this Lie algebra is three-dimensional and Riccati equations admit a superposition rule depending on three particular solutions, it is immediate that the equations (1.10) satisfy the corresponding Lie's condition.

The existence of different Vessiot–Guldberg Lie algebras for a system of first-order ordinary differential equations is an important question because their characteristics determine, among other features, the integrability by quadratures of Lie systems [31].

Let us now turn to determining when a system (1.12) is *not* a Lie system. In order to analyse this question, it is useful to rewrite the Lie Theorem in the following, abbreviated, form.

PROPOSITION 1.10 (Abbreviated Lie Theorem). *A system X on $\mathbb{R}^n$ is a Lie system if and only if* $\mathrm{Lie}(\{X_t\}_{t\in\mathbb{R}})$ *is finite-dimensional.*

In view of the above result, determining that (1.12) is not a Lie system reduces to showing that $\mathrm{Lie}(\{X_t\}_{t\in\mathbb{R}})$ is infinite-dimensional. The standard procedure to prove this consists in demonstrating that there exists an infinite chain $\{Z_j\}_{j\in\mathbb{N}}$ of linearly independent vector fields over $\mathbb{R}$ obtained through successive Lie brackets of elements in $\{X_t\}_{t\in\mathbb{R}}$. In order to illustrate how this is usually done, consider the particular example based on the study of the Abel equation of the first type

$$\frac{dx}{dt} = x^2 + b(t)x^3, \qquad b(t) \neq 0,$$

where $b(t)$ is additionally a nonconstant function. These equations describe the integral curves of the t-dependent vector field

$$X_t = (x^2 + b(t)x^3)\frac{\partial}{\partial x}.$$

Consider the chain of vector fields

$$Z_1 = x^2\frac{\partial}{\partial x}, \qquad Z_2 = x^3\frac{\partial}{\partial x}, \qquad Z_j = [X_1, X_{j-1}], \qquad j = 3, 4, 5, \ldots.$$

Since $Z_j = x^{j+1}\partial/\partial x$, it turns out that $\mathrm{Lie}(\{X_t\}_{t\in\mathbb{R}})$ admits the infinite chain of linearly independent vector fields $\{Z_j\}_{j\in\mathbb{R}}$ and so, in view of the abbreviated Lie Theorem, Abel equations of the above type are not Lie systems.

There are many other relevant Lie systems associated with important systems of differential equations appearing in the physical and mathematical literature. A nonexhaustive brief list of Lie systems includes:

1. Linear first-order systems and, more specifically, Euler systems [62, 98].
2. Riccati equations [47, 222, 234] and coupled Riccati equations of projective type [7].
3. Matrix Riccati equations [112, 141, 174, 188, 212, 234].
4. Bernoulli equations, several equations appearing in supermechanics [13], etc.

Apart from the above instances, there are other important systems of differential equations which can be studied through other Lie systems. Several of such Lie systems will be detailed in the next sections.

Determination of the general solution of any Lie system reduces to deriving a particular solution of a particular type of a Lie system defined on a Lie group. Let us analyse this in detail.

Consider a Lie system related to a t-dependent vector field (1.18) over $\mathbb{R}^n$ and associated, for simplicity, with a Vessiot–Guldberg Lie algebra V made up of complete vector fields. This gives rise to a Lie group action $\Phi : G \times \mathbb{R}^n \to \mathbb{R}^n$ whose fundamental vector fields are exactly those of V. Obviously, this implies that the Lie algebra $\mathfrak{g} \simeq \mathrm{T}_eG$ is isomorphic to V. Choose now a basis $\{\mathrm{a}_1, \ldots, \mathrm{a}_r\}$ of $\mathfrak{g}$ such that $\Phi : G \times \mathbb{R}^n \to \mathbb{R}^n$ and

$$\Phi(\exp(-s\mathrm{a}_\alpha), x) = g_s^{(\alpha)}(x), \qquad \alpha = 1, \ldots, r, \qquad s \in \mathbb{R}, \tag{1.28}$$

where $g^{(\alpha)} : (s, x) \in \mathbb{R} \times \mathbb{R}^n \mapsto g^{(\alpha)}(s, x) = g_s^{(\alpha)}(x) \in \mathbb{R}^n$ is the flow of the vector field X_α. In this way, each vector field X_α becomes the fundamental vector field corresponding to a_α and the map $\phi : \mathfrak{g} \to V$ such that $\phi(\mathrm{a}_\alpha) = X_\alpha$ for $\alpha = 1, \ldots, r$ is a Lie algebra isomorphism.

Let X_α^{R} be the right-invariant vector field on G with $(X_\alpha^{\mathrm{R}})_e = \mathrm{a}_\alpha$, i.e. $(X_\alpha^{\mathrm{R}})_g = R_{g*e}\mathrm{a}_\alpha$, where $R_g : g' \in G \mapsto g'g \in G$ is the right action of G on itself. Then the t-dependent right-invariant vector field

$$X^G(t, g) = -\sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha^{\mathrm{R}}(g) \tag{1.29}$$

defines a Lie system on G whose integral curves are the solutions of the system on G given by

$$\frac{dg}{dt} = -\sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha^{\mathrm{R}}(g). \tag{1.30}$$

Applying $R_{g^{-1}*g}$ to both sides of the equation, we see that its general solution $g(t)$ satisfies

$$R_{g^{-1}(t)*g(t)}\dot{g}(t) = -\sum_{\alpha=1}^{r} b_\alpha(t)\mathrm{a}_\alpha \in \mathrm{T}_eG. \tag{1.31}$$

Note that right-invariance implies that the knowledge of one particular solution of the above equation, e.g. $g_0(t)$ with $g_0(0) = g_0$, is enough to obtain the general solution of (1.31). Indeed, consider $g'(t) = R_{\bar{g}}g_0(t)$ for a given $\bar{g} \in G$. This curve satisfies

$$\frac{dg'}{dt}(t) = R_{\bar{g}*g_0(t)}\left(\frac{dg_0}{dt}(t)\right), \quad \text{i.e.} \quad \frac{dg'}{dt}(t) = R_{\bar{g}*g_0(t)}\Big(-\sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha^{\mathrm{R}}(g_0(t))\Big).$$

Taking into account that $R_{\bar{g}*g_0} X_\alpha^{\mathrm{R}}(g_0) = X_\alpha^{\mathrm{R}}(g_0\bar{g})$, one has

$$\frac{dg'}{dt}(t) = -\sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha^{\mathrm{R}}(R_{\bar{g}}g_0(t)) = -\sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha^{\mathrm{R}}(g'(t))$$

and $g'(t)$ is another particular solution of (1.29) with initial condition $g'(0) = R_{\bar{g}}g_0$.

Consequently, the general solution $g(t)$ of (1.31) can be written as

$$g(t) = R_{\bar{g}} g_0(t), \qquad \bar{g} \in G.$$

That is, system (1.29) admits a superposition rule and, according to the Lie Theorem, it must be a Lie system. This is not surprising, as the vector fields X_α^{R} span a Lie algebra of vector fields isomorphic to V and system (1.30) describes the integral curves of a t-dependent vector field taking values in a finite-dimensional Lie algebra of vector fields.

The relevance of the Lie system (1.31) relies on the fact that the integral curves of the t-dependent vector field $X(t,x)$ can be obtained from one particular solution of equation (1.31). More explicitly, the general solution $x(t)$ of the Lie system $X(t,x)$ reads $x(t) = \Phi(g_e(t), x_0)$, where x_0 is the initial condition of the particular solution and $g_e(t)$ is the particular solution of equation (1.31) with $g_e(0) = e$.

Note that, in view of Ado's Theorem [2], every finite-dimensional Lie algebra, e.g. the above Vessiot–Guldberg Lie algebra V, admits an isomorphic matrix Lie algebra. Related to this matrix Lie algebra, there exists a matrix Lie group $\bar{G}$. In this way, the system describing the t-dependent vector field (1.18) reduces to solving an equation of the form

$$\dot{A}(t)A^{-1}(t) = -\sum_{\alpha=1}^{r} b_\alpha(t) M_\alpha, \quad \text{so} \quad \dot{A} = -\sum_{\alpha=1}^{r} b_\alpha(t) M_\alpha A,$$

with $A(t)$ being a curve taking values in the matrix Lie group $\bar{G}$ and $M_1, \ldots, M_r$ being a basis with the same structure constants as $X_1, \ldots, X_r$. Obviously, the above equation becomes a homogeneous linear differential equation in the coefficients of the matrix A. Consequently, determining the general solution of a Lie system reduces to solving a linear problem.

Although the above process was described for Lie systems associated with Vessiot–Guldberg Lie algebras of complete vector fields, it can be proved that a similar process, with almost identical final results, can be applied to any Lie system $X(t,x)$. Indeed, this can be done by taking the compactification of $\mathbb{R}^n$ in order to make all vector fields complete (as in the case of the Riccati equation) or just by considering that the induced action is only a local one.

A generalisation of the method [57] used by Wei and Norman for linear systems [231, 232] is very useful for solving equations (1.31). Furthermore, there exist reduction techniques that can also be used [40]. Such techniques show, for instance, that Lie systems related to solvable Vessiot–Guldberg Lie algebras are integrable by quadratures ([40, Section 8]). Finally, as right-invariant vector fields X^{R} project onto the fundamental vector fields in each homogeneous space for G, the solution of equation (1.31) enables us to find the general solution for the corresponding Lie system in each homogeneous space. Conversely, the knowledge of particular solutions of the associated system in a homogeneous space gives us a method for reducing the problem to the corresponding isotropy group [40].

1.4. Geometric approach to superposition rules. Let us now review the modern geometrical approach to the theory of Lie systems introduced in [38]. Although we here basically point out the results given in that work, several slight improvements have been included in our presentation.

A fundamental notion in the geometrical description of Lie systems is the so-called *diagonal prolongation* of a t-dependent vector field. Its definition and most important properties are described below.

Definition 1.11. Given a t-dependent vector field over $\mathbb{R}^n$ of the form

$$X(t, x_{(0)}) = \sum_{i=1}^{n} X^i(t, x_{(0)}) \frac{\partial}{\partial x^i_{(0)}},$$

its *diagonal prolongation* to $\mathbb{R}^{n(m+1)}$ is the t-dependent vector field over this last space given by

$$\widehat{X}(t, x_{(0)}, \ldots, x_{(m)}) = \sum_{a=0}^{m} \sum_{i=1}^{n} X^i(t, x_{(a)}) \frac{\partial}{\partial x^i_{(a)}}.$$

Recall that every vector field X over $\mathbb{R}^n$ can be regarded as a t-dependent vector field in a natural way. Evidently, the above definition can also be applied to define diagonal prolongations for vector fields over $\mathbb{R}^n$. Obviously, such prolongations are vector fields over $\mathbb{R}^{n(m+1)}$ as well.

Note that diagonal prolongations can be redefined in an intrinsic, and equivalent, way as follows.

Definition 1.12. Given a t-dependent vector field X over $\mathbb{R}^n$, its *diagonal prolongation* to $\mathbb{R}^{n(m+1)}$ is the unique t-dependent vector field $\widehat{X}$ over $\mathbb{R}^{n(m+1)}$ such that:

- The t-dependent vector field $\widehat{X}$ is invariant under the action of the symmetry group S_{m+1} over $\mathbb{R}^{n(m+1)}$.
- The vector fields $\widehat{X}_t$ are projectable under the projection pr_0 given by (1.4) and $\mathrm{pr}_{0*}\widehat{X}_t = X_t$.

Lemma 1.13. *For any vector fields $X, Y \in \mathfrak{X}(\mathbb{R}^n)$, we have $[\widehat{X}, \widehat{Y}] = \widehat{[X, Y]}$. Therefore, given a Lie algebra of vector fields $V \subset \mathfrak{X}(\mathbb{R}^n)$, the prolongations of its elements to $\mathbb{R}^{n(m+1)}$ span an isomorphic Lie algebra of vector fields.*

Proof. The proof is straightforward and left to the reader. ■

Lemma 1.14. *Consider a family $X_1, \ldots, X_r$ of vector fields over $\mathbb{R}^n$ whose diagonal prolongations to $\mathbb{R}^{nm}$ are linearly independent at a generic point. Given the diagonal prolongations $\widehat{X}_1, \ldots, \widehat{X}_r$ to $\mathbb{R}^{n(m+1)}$, the vector field $\sum_{\alpha=1}^{r} b_\alpha \widehat{X}_\alpha$ with $b_\alpha \in C^\infty(\mathbb{R}^{n(m+1)})$ is also a diagonal prolongation if and only if the coefficients $b_1, \ldots, b_r$ are constant.*

Proof. Let us write in local coordinates

$$X_\alpha = \sum_{i=1}^{n} A^i_\alpha(x) \frac{\partial}{\partial x^i}, \qquad \alpha = 1, \ldots, r,$$

which implies that

$$\widehat{X}_\alpha = \sum_{i=1}^{n} \sum_{a=0}^{m} A^i_\alpha(x_{(a)}) \frac{\partial}{\partial x^i_{(a)}}, \qquad \alpha = 1, \ldots, r.$$

Then

$$\sum_{\alpha=1}^{r} b_\alpha(x_{(0)},\ldots,x_{(m)})\widehat{X}_\alpha=\sum_{\alpha=1}^{r}\sum_{i=1}^{n}\sum_{a=0}^{m} b_\alpha(x_{(0)},\ldots,x_{(m)})A^i_\alpha(x_{(a)})\frac{\partial}{\partial x^i_{(a)}},$$

which is a diagonal prolongation if and only if there exist functions $B^j : x\in\mathbb{R}^n\mapsto B^j(x)\in\mathbb{R}$, with $j=1,\ldots,n$, such that

$$\sum_{\alpha=1}^{r} b_\alpha(x_{(0)},\ldots,x_{(m)})A^i_\alpha(t,x_{(a)})=B^i(x_{(a)}),\qquad a=0,\ldots,m,\ i=1,\ldots,n.$$

In particular, the functions $b_\alpha(x_{(0)},\ldots,x_{(m)})$ with $\alpha=1,\ldots,r$ solve the subsystem of linear equations in the variables $u_1,\ldots,u_r$ given by

$$\sum_{\alpha=1}^{r} u_\alpha A^i_\alpha(x_{(a)})=B^i(x_{(a)}),\qquad a=1,\ldots,m,\ i=1,\ldots,n.$$

The coefficient matrix of the above system of $m\cdot n$ equations with r unknowns has rank $r\le m\cdot n$ since the $\mathrm{pr}_*(\widehat{X}_\alpha)$ are linearly independent. Hence, the solutions $u_1,\ldots,u_r$ are completely determined in terms of the functions $B^i(x_{(a)})$ with $a=1,\ldots,m$ and $i=1,\ldots,n$, and do not depend on $x_{(0)}$. But since the prolongations are invariant under the action of the symmetry group S_{m+1}, the functions $u_\alpha=b_\alpha(x_{(0)},\ldots,x_{(m)})$ with $\alpha=1,\ldots,r$ must satisfy this symmetry. Consequently, they cannot depend on the variables $x_{(1)},\ldots,x_{(m)}$, and therefore must be constant. ∎

Lemma 1.15. *For every family of vector fields $X_1,\ldots,X_r\in\mathfrak{X}(\mathbb{R}^n)$ linearly independent over $\mathbb{R}$, there exists an integer m such that their prolongations to $\mathbb{R}^{nm}$ are linearly independent at a generic point.*

Proof. Denote by $\widehat{X}^q_\alpha$ the diagonal prolongation of X_α to $\mathbb{R}^{nq}$ and define $\sigma(q)$ to be the maximum number of vector fields, among the $\widehat{X}^q_\alpha$, linearly independent at a generic point of $\mathbb{R}^{nq}$.

Assume towards a contradiction that each family $\widehat{X}^q_1,\ldots,\widehat{X}^q_r$ of diagonal prolongations are linearly dependent at a generic point of $\mathbb{R}^{qn}$, in other words, $1\le\sigma(q)<r$ for every q. Then the function $\sigma(q)$ must admit a maximum $p<r$ for a certain integer $\bar{m}$, i.e. $p=\sigma(\bar{m})$. We can assume, without loss of generality, that $\widehat{X}^{\bar{m}}_1,\ldots,\widehat{X}^{\bar{m}}_p$ are linearly independent at a generic point of $\mathbb{R}^{n\bar{m}}$. Moreover, $\widehat{X}^{\bar{m}+1}_1,\ldots,\widehat{X}^{\bar{m}+1}_p$ are also linearly independent at a generic point of $\mathbb{R}^{n(\bar{m}+1)}$ and, as $\sigma(\bar{m})$ is maximal, we must have $\sigma(\bar{m}+1)=\sigma(\bar{m})$. Consequently, there exist p uniquely defined functions $\bar{f}_1,\ldots,\bar{f}_p\in C^\infty(\mathbb{R}^{n(\bar{m}+1)})$ obeying the equation

$$\bar{f}_1\widehat{X}^{\bar{m}+1}_1+\cdots+\bar{f}_p\widehat{X}^{\bar{m}+1}_p=\widehat{X}^{\bar{m}+1}_{p+1}. \tag{1.32}$$

This forces the left-hand side to be a diagonal prolongation. Moreover, since $\widehat{X}^{\bar{m}}_1,\ldots,\widehat{X}^{\bar{m}}_p$, are linearly independent at a generic point, Lemma 1.14 applies and it turns out that $\bar{f}_1,\ldots,\bar{f}_p$ must be constant. Then, projecting the above expression by pr_0, it follows that $X_1,\ldots,X_{p+1}$ are linearly dependent over $\mathbb{R}$. This violates our initial assumption and thereby we conclude that our initial premise, i.e. $\sigma(q)<r$ for every q, must be false and there must exist an integer m such that the diagonal prolongations of $X_1\ldots,X_r$ to $\mathbb{R}^{nm}$ become linearly independent at a generic point, which proves our lemma. ∎

The above lemma already contains the key point to prove the following result.

LEMMA 1.16. *If $\sigma(q) < r$, then $\sigma(q) < \sigma(q+1)$.*

Proof. It is immediate that $\sigma(q) \leq \sigma(q+1)$. Now, if we assume $p = \sigma(q) < r$ and $\sigma(q) = \sigma(q+1)$, one can pick, among the $\widehat{X}^q_\alpha$, a family of p vector fields linearly independent at a generic point of $\mathbb{R}^{nq}$. We can assume, with no loss of generality, that they are $\widehat{X}^q_1, \ldots, \widehat{X}^q_p$. Consequently, as in the above lemma, we can write

$$\bar{f}_1 \widehat{X}^{q+1}_1 + \cdots + \bar{f}_p \widehat{X}^{q+1}_p = \widehat{X}^{q+1}_{p+1},$$

for certain uniquely defined functions $\bar{f}_1, \ldots, \bar{f}_r \in C^\infty(\mathbb{R}^{n(m+1)})$. As in the proof of the previous lemma, this implies that $X_1, \ldots, X_{p+1}$ are linearly dependent over $\mathbb{R}$. This is in contradiction with our initial assumption. ■

Taking into account the above two lemmas, it follows trivially that $\sigma(q)$ grows monotonically until it reaches the maximum r. This gives rise to the following proposition.

PROPOSITION 1.17. *For every family of vector fields $X_1, \ldots, X_r \in \mathfrak{X}(\mathbb{R}^n)$ linearly independent over $\mathbb{R}$, there exists an integer $m \leq r$ such that their prolongations to $\mathbb{R}^{nm}$ are linearly independent at a generic point.*

The above proposition constitutes an explicit proof for vector fields over $\mathbb{R}^n$ of the analogous result for vector fields over manifolds pointed out in [38]. Let us now turn to a geometric interpretation of superposition rules.

Consider a t-dependent vector field (1.1) associated with the system

$$\frac{dx^i}{dt} = X^i(t, x), \qquad i = 1, \ldots, n, \tag{1.33}$$

describing its integral curves. Recall that the above system admits a superposition rule if there exists a map $\Phi : \mathbb{R}^{n(m+1)} \to \mathbb{R}^n$ of the form $x = \Phi(x_{(1)}, \ldots, x_{(m)}; k_1, \ldots, k_n)$ such that its general solution $x(t)$ can be written as

$$x(t) = \Phi(x_{(1)}(t), \ldots, x_{(m)}(t); k_1, \ldots, k_n),$$

with $x_{(1)}(t), \ldots, x_{(m)}(t)$ being a generic family of particular solutions and $k_1, \ldots, k_n$ a set of constants associated with each particular solution.

The map $\Phi(x_{(1)}, \ldots, x_{(m)}; \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ can be inverted, at least locally around points of an open dense subset of $\mathbb{R}^{nm}$, to give rise to a map $\Psi : \mathbb{R}^{n(m+1)} \to \mathbb{R}^n$,

$$k = \Psi(x_{(0)}, \ldots, x_{(m)}),$$

where we write $x_{(0)}$ instead of x and $k = (k_1, \ldots, k_n)$ in order to simplify the notation. Note that the map Ψ is defined so that

$$k = \Psi(\Phi(x_{(1)}, \ldots, x_{(m)}; k), x_{(1)}, \ldots, x_{(m)}).$$

Hence, Ψ defines an n-codimensional foliation on the manifold $\mathbb{R}^{n(m+1)}$.

As the fundamental property of the map Ψ states that

$$k = \Psi(x_{(0)}(t), \ldots, x_{(m)}(t)) \tag{1.34}$$

for any $(m+1)$-tuple of generic particular solutions of system (1.33), the foliation determined by Ψ is invariant under permutations of its $(m+1)$ arguments, $x_{(0)}, \ldots, x_{(m)}$.

Moreover, differentiating expression (1.34) with respect to t, we get

$$\sum_{a=0}^{m}\sum_{j=1}^{n} X^j(t, x_{(a)}(t))\frac{\partial \Psi^k}{\partial x^j_{(a)}}(\bar{p}(t)) = \widehat{X}_t\Psi^k(\bar{p}(t)) = 0, \qquad k = 1, \dots, n,$$

where $(\Psi^1, \dots, \Psi^n) = \Psi$ and $\bar{p}(t) = (x_{(0)}(t), \dots, x_{(m)}(t))$. Thus, the functions $\Psi^1, \dots, \Psi^n$ are first integrals for the vector fields $\{\widehat{X}_t\}_{t\in\mathbb{R}}$ defining an n-codimensional foliation $\mathfrak{F}$ over $\mathbb{R}^{n(m+1)}$ such that the vector fields $\{\widehat{X}_t\}_{t\in\mathbb{R}}$ are tangent to its leaves.

The foliation $\mathfrak{F}$ has another important property. Given a leaf $\mathfrak{F}_k$ corresponding to the level set of Ψ determined by $k = (k_1, \dots, k_n) \in \mathbb{R}^n$ and a point $(x_{(1)}, \dots, x_{(m)}) \in \mathbb{R}^{mn}$, there exists a unique point $(x_{(0)}, x_{(1)}, \dots, x_{(m)}) \in \mathfrak{F}_k$, namely,

$$(\Phi(x_{(1)}, \dots, x_{(m)}; k), x_{(1)}, \dots, x_{(m)}) \in \mathfrak{F}_k.$$

Consequently, the projection onto the last $m \cdot n$ factors, i.e. the map pr given by (1.3), induces diffeomorphisms between $\mathbb{R}^{nm}$ and each of the leaves $\mathfrak{F}_k$. In other words, the foliation $\mathfrak{F}$ is horizontal with respect to the projection pr.

The foliation $\mathfrak{F}$ corresponds to a connection ∇ on the bundle $\mathrm{pr} : \mathbb{R}^{n(m+1)} \to \mathbb{R}^{nm}$ with zero curvature. Indeed, the restriction of the projection pr to a leaf gives a one-to-one map that gives rise to a linear map from vector fields on $\mathbb{R}^{nm}$ to 'horizontal' vector fields tangent to the leaf.

Note that the knowledge of this connection (foliation) gives us the superposition rule without referring to the map Ψ. If we fix a point $x_{(0)}(0)$ and m particular solutions, $x_{(1)}(t), \dots, x_{(m)}(t)$, then $x_{(0)}(t)$ is the unique point in $\mathbb{R}^n$ such that the point $(x_{(0)}(t), x_{(1)}(t), \dots, x_{(m)}(t))$ belongs to the same leaf as $(x_{(0)}(0), x_{(1)}(0), \dots, x_{(m)}(0))$. Thus, it is only $\mathfrak{F}$ that really matters when the superposition rule is concerned.

On the other hand, if we have a connection ∇ on the bundle

$$\mathrm{pr} : \mathbb{R}^{n(m+1)} \to \mathbb{R}^{nm},$$

with zero curvature, i.e. a horizontal distribution ∇ on $\mathbb{R}^{n(m+1)}$ that it is involutive and can be integrated to give a foliation on $\mathbb{R}^{n(m+1)}$ such that the vector fields $\widehat{X}_t$ belong to ∇, then the procedure described above determines a superposition rule for system (1.33). Indeed, let $k \in \mathbb{R}^n$ enumerate smoothly the leaves $\mathfrak{F}_k$ of the foliation $\mathfrak{F}$; then we can define $\Phi(x_{(1)}, \dots, x_{(m)}; k) \in \mathbb{R}^n$ to be the unique point $x_{(0)}$ of $\mathbb{R}^n$ such that

$$(x_{(0)}, x_{(1)}, \dots, x_{(m)}) \in \mathfrak{F}_k.$$

This gives rise to a superposition rule $\Phi : \mathbb{R}^{nm} \times \mathbb{R}^n \to \mathbb{R}^n$ for the system of first-order differential equations (1.33). To see this, let us observe the inverse relation

$$\Psi(x_{(0)}, \dots, x_{(m)}) = k,$$

which is equivalent to $(x_{(0)}, \dots, x_{(m)}) \in \mathfrak{F}_k$. If we fix k and take a generic family of particular solutions $x_{(1)}(t), \dots, x_{(m)}(t)$ of equation (1.33), then $x_{(0)}(t)$ defined by the condition $\Psi(x_{(0)}(t), \dots, x_{(m)}(t)) = k$ satisfies (1.33). In fact, let $x'_{(0)}(t)$ be the solution of (1.33) with initial value $x'_{(0)} = x_{(0)}$. Since the t-dependent vector fields $\widehat{X}(t, x)$ are tangent to $\mathfrak{F}$, the curve $(x_{(0)}(t), x_{(1)}(t), \dots, x_{(m)}(t))$ lies entirely within a leaf of $\mathfrak{F}$, so in $\mathfrak{F}_k$. But a point of a leaf is entirely determined by its projection under pr, so $x'_{(0)}(t) = x_{(0)}(t)$ and $x_{(0)}(t)$ is a solution.

PROPOSITION 1.18. *Giving a superposition rule depending on m generic particular solutions for a Lie system described by a t-dependent vector field X is equivalent to giving a zero curvature connection ∇ on the bundle* $\mathrm{pr} : \mathbb{R}^{(m+1)n} \to \mathbb{R}^{nm}$ *for which the vector fields $\{\widehat{X}_t\}_{t\in\mathbb{R}}$ are horizontal vector fields with respect to this connection.*

Although we decided not to investigate in full detail the difference between global superposition rules and superposition rules, we comment briefly on this theme here. Note that a rigorous analysis shows that a global or 'simple' superposition rule gives rise to a zero curvature connection. Nevertheless, on the contrary, a zero curvature connection *only* ensures the existence of a superposition rule that need not be global. This is due to the fact that the connection only guarantees the existence of a series of *local* first integrals that give rise to a superposition rule. In order to ensure the existence of a global superposition rule, some extra conditions on the connection must be imposed (see [18]).

1.5. Geometric Lie Theorem. Let us now prove the classical Lie theorem [157, Theorem 44] from a modern geometric perspective by using the previous results. The following theorem is a restatement of the geometric version of the Lie Theorem given in [38, Theorem 1]. Our aim is to include one of the main results of the theory of Lie systems and, at the same time, to furnish a slightly more detailed proof.

MAIN THEOREM 1.19 (Geometric Lie Theorem). *A system* (1.33) *admits a superposition rule depending on m generic particular solutions if and only if the t-dependent vector field X can be written as*

$$X_t = \sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha, \tag{1.35}$$

where the vector fields $X_1, \ldots, X_r$ form a basis for an r-dimensional real Lie algebra.

Proof. Suppose that system (1.33) admits a superposition rule (1.14) and let $\mathfrak{F}$ be its associated foliation over $\mathbb{R}^{n(m+1)}$. As the vector fields $\{\widehat{X}_t\}_{t\in\mathbb{R}}$ are tangent to the leaves of $\mathfrak{F}$, the vector fields in $\mathrm{Lie}(\{\widehat{X}_t\}_{t\in\mathbb{R}})$ span a generalised involutive distribution

$$\mathcal{D}_p = \{\widehat{Y}(t,p) \mid Y \in \mathrm{Lie}(\{\widehat{X}_t\}_{t\in\mathbb{R}})\} \subset \mathrm{T}_p\mathbb{R}^{n(m+1)},$$

whose elements are also tangent to the leaves of $\mathfrak{F}$. Since the Lie bracket of two prolongations is a prolongation, we can choose, among the elements of $\mathrm{Lie}(\{\widehat{X}_t\}_{t\in\mathbb{R}})$, a finite family $\widehat{X}_1, \ldots, \widehat{X}_r$ that gives rise to a local basis of diagonal prolongations for the distribution $\mathcal{D}$. As the map pr projects each leaf of the foliation $\mathfrak{F}$ into $\mathbb{R}^{nm}$ diffeomorphically, we find that the vector fields $\mathrm{pr}_*(\widehat{X}_\alpha)$ with $\alpha = 1, \ldots, r$ are linearly independent at a generic point of $\mathbb{R}^{nm}$. These vector fields satisfy the commutation relations

$$[\widehat{X}_\alpha, \widehat{X}_\beta] = \sum_{\gamma=1}^{r} f_{\alpha\beta\gamma} \widehat{X}_\gamma, \qquad \alpha, \beta = 1, \ldots, r,$$

for certain functions $f_{\alpha\beta\gamma} \in C^\infty(\mathbb{R}^{n(m+1)})$. In view of Lemma 1.14, these functions must be constant, say $f_{\alpha\beta\gamma} = c_{\alpha\beta\gamma}$, and, taking into account the properties of diagonal prolongations, one finds that $X_1, \ldots, X_r$ are linearly independent vector fields obeying the relations

$$[X_\alpha, X_\beta] = \sum_{\gamma=1}^{r} c_{\alpha\beta\gamma} X_\gamma, \qquad \alpha, \beta = 1, \ldots, r.$$

Since each $\widehat{X}_t$ is spanned by the vector fields $\widehat{X}_1, \ldots, \widehat{X}_r$, there are t-dependent functions $b_\alpha \in C^\infty(\mathbb{R} \times \mathbb{R}^{n(m+1)})$ with $\alpha = 1, \ldots, r$ such that

$$\widehat{X}_t = \sum_{\alpha=1}^{r} b_\alpha \widehat{X}_\alpha.$$

But each $\widehat{X}_t$ is a diagonal prolongation, so, using Lemma 1.14, one sees that the functions $b_1, \ldots, b_r$ depend only on time and thus

$$\widehat{X}_t = \sum_{\alpha=1}^{r} b_\alpha(t) \widehat{X}_\alpha. \tag{1.36}$$

Hence, it is immediate that (1.35) holds.

To prove the converse, assume that the t-dependent vector field X can be put in the form (1.35), where the vector fields $X_1, \ldots, X_r$ are linearly independent over $\mathbb{R}$ and span an r-dimensional Lie algebra.

As $X_1, \ldots, X_r$ are linearly independent over $\mathbb{R}$, there exists, in view of Proposition 1.17, a minimal number $m \leq r$ such that their diagonal prolongations to $\mathbb{R}^{nm}$ are linearly independent at a generic point (which yields $r \leq n \cdot m$). Moreover, the diagonal prolongations $\widehat{X}_1, \ldots, \widehat{X}_r$ to $\mathbb{R}^{n(m+1)}$ are linearly independent and form a basis for an involutive distribution $\mathcal{D}$. This distribution leads to an $(n(m+1) - r)$-codimensional foliation $\mathfrak{F}_0$ on $\mathbb{R}^{n(m+1)}$. As the codimension of $\mathfrak{F}_0$ is at least n, we can consider an n-codimensional foliation $\mathfrak{F}$ whose leaves include those of $\mathfrak{F}_0$. The leaves of this foliation project onto the last $m \cdot n$ factors diffeomorphically and they are at least n-codimensional. Hence, according to Proposition 1.18, the foliation $\mathfrak{F}$ defines a superposition rule depending on m particular solutions. ∎

Note that the converse part of the previous proof shows that all systems described by t-dependent vector fields of the form (1.36) share a common superposition rule. More specifically, all such t-dependent vector fields give rise to the same distribution $\mathcal{D}$ over the same space $\mathbb{R}^{n(m+1)}$, and this ensures the existence of a common superposition rule for all of them. This fact will be analysed more extensively in the second part of our work, where certain families of systems of differential equations that admit a t-dependent common superposition rule, referred to as *Lie families*, are investigated.

1.6. Determination of superposition rules. Note that the previous geometric demonstration of the Lie Theorem also contains information about the superposition rules associated with a Lie system. Let us analyse this more carefully.

Consider a Lie system in $\mathbb{R}^n$ associated with a t-dependent vector field X. In view of the Lie Theorem, X can be written in the form

$$X(t,x) = \sum_{i=1}^{n}\sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha^i(x)\frac{\partial}{\partial x^i},$$

where the vector fields $X_\alpha(x) = \sum_{i=1}^n X_\alpha^i(x)\partial/\partial x^i$ span an r-dimensional Lie algebra of vector fields. Now, the geometric proof of the Lie Theorem shows that the above decomposition gives rise to a superposition rule depending on m generic particular solutions with $r \leq m \cdot n$. More exactly, the number m coincides with the minimal integer that makes the diagonal prolongations of $X_1, \ldots, X_r$ to $\mathbb{R}^{mn}$ linearly independent at a generic point. In other words, the only functions $f_1, \ldots, f_r \in C^\infty(\mathbb{R}^{nm})$ such that

$$\sum_{\alpha=1}^{r} f_\alpha\, X_\alpha^i(x_{(a)}) = 0, \qquad a = 1,\ldots,m,\ i = 1,\ldots,n, \tag{1.37}$$

at a generic point $(x_{(1)}, \ldots, x_{(k)})$ are $f_1 = \cdots = f_r = 0$.

Let us illustrate the above comments by a simple example. Consider the Riccati equation

$$\dot{x} = b_1(t) + b_2(t)\,x + b_3(t)x^2,$$

which describes the integral curves of the t-dependent vector field

$$X_t = b_1(t)\frac{\partial}{\partial x} + b_2(t)x\frac{\partial}{\partial x} + b_3(t)x^2\frac{\partial}{\partial x}.$$

Recall that the vector fields $\{X_t\}_{t\in\mathbb{R}}$ take values in the three-dimensional Lie algebra V spanned by the vector fields

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = x\frac{\partial}{\partial x}, \qquad X_3 = x^2\frac{\partial}{\partial x}.$$

Consequently, we can determine the number of particular solutions for a superposition rule for Riccati equations by considering the minimal m such that corresponding system (1.37) admits only the trivial solution. For $m = 2$, this system reads

$$f_1 + f_2 x_{(1)} + f_3 x_{(1)}^2 = 0, \qquad f_1 + f_2 x_{(2)} + f_3 x_{(2)}^2 = 0,$$

and it has nontrivial solutions. Nevertheless, the system for the prolongations to $\mathbb{R}^3$, that is,

$$f_1 + f_2 x_{(1)} + f_3 x_{(1)}^2 = 0, \qquad f_1 + f_2 x_{(2)} + f_3 x_{(2)}^2 = 0, \qquad f_1 + f_2 x_{(3)} + f_3 x_{(3)}^2 = 0,$$

does not admit any nontrivial solution because the determinant of the coefficients, i.e.

$$\begin{vmatrix} 1 & x_{(1)} & x_{(1)}^2 \\ 1 & x_{(2)} & x_{(2)}^2 \\ 1 & x_{(3)} & x_{(3)}^2 \end{vmatrix} = (x_{(2)} - x_{(1)})(x_{(2)} - x_{(3)})(x_{(1)} - x_{(3)}),$$

is different from zero when the three points $x_{(1)}$, $x_{(2)}$, and $x_{(3)}$ are different. Thus, we see that $m = 3$ and the superposition rule for the Riccati equation depends on three particular solutions. Obviously, the relations $m \leq \dim V \leq m \cdot n$ are valid in this case.

Once the number m of particular solutions has been determined, the superposition rule can be worked out in terms of first integrals for the diagonal prolongations $\widehat{X}_1, \ldots, \widehat{X}_r$ over $\mathbb{R}^{n(m+1)}$. Finally, it is worth noting that when the vector fields $\widehat{X}_1, \ldots, \widehat{X}_r$ over $\mathbb{R}^{n(m+1)}$ admit more than n common first integrals, the system X admits more than one superposition rule (see [38]).

1.7. Mixed superposition rules and constants of motion. Roughly speaking, a *mixed superposition rule* is a t-independent map describing the general solution of a system of first-order differential equations in terms of a generic family of particular solutions of various systems (generically different) of first-order differential equations and a set of constants. Obviously, mixed superposition rules include, as particular instances, the standard superposition rules related to Lie systems.

DEFINITION 1.20. A *mixed superposition rule* for a system of first-order differential equations determined by a t-dependent vector field X over $\mathbb{R}^{n_0}$ is a t-independent map $\Phi : \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m} \times \mathbb{R}^{n_0} \to \mathbb{R}^{n_0}$ of the form

$$x = \Phi(x_{(1)}, \ldots, x_{(m)}; k_1, \ldots, k_{n_0}),$$

such that the general solution $x(t)$ of the system X can be written as

$$x(t) = \Phi(x_{(1)}(t), \ldots, x_{(m)}(t); k_1, \ldots, k_{n_0}),$$

with $x_{(1)}(t), \ldots, x_{(m)}(t)$ being a generic family of curves such that each $x_{(a)}(t)$ is a particular solution of the system determining the integral curves for a t-dependent vector field $X^{(a)}$ over $\mathbb{R}^{n_a}$ with $a = 1, \ldots, m$.

As an example of a mixed superposition rule, consider the linear system of differential equations

$$\frac{dx^i}{dt} = \sum_{j=1}^{n} A^i_j(t)x^j + B^i(t), \qquad i = 1, \ldots, n, \tag{1.38}$$

whose general solution $x(t)$ can be written as

$$x(t) = y_{(1)}(t) + \sum_{j=1}^{n} k_j z_{(j)}(t),$$

in terms of one particular solution $y_{(1)}(t)$ of (1.38), any family of n linearly independent particular solutions $z_{(1)}(t), \ldots, z_{(n)}(t)$ of the homogeneous linear system

$$\frac{dz^i}{dt} = \sum_{j=1}^{n} A^i_j(t)z^j, \qquad i = 1, \ldots, n,$$

and a set of n constants $k_1, \ldots, k_n$.

We aim to give a method to obtain a particular type of mixed superposition rule for a Lie system in terms of particular solutions of another Lie system. Additionally, we relate our results to the commentary given in [38, Remark 5], where it was briefly discussed that the solutions of a certain first-order differential equation on a manifold may be obtained in terms of solutions of other first-order systems by constructing a certain foliation.

Consider the system on $\mathbb{R}^{n_0}$ given by

$$\frac{dx^i}{dt} = \sum_{\alpha=1}^{r} b_\alpha(t)X^i_\alpha(x), \qquad i = 1, \ldots, n_0, \tag{1.39}$$

determining the integral curves of the t-dependent vector field

$$X(t, x) = \sum_{\alpha=1}^{r}\sum_{i=1}^{n_0} b_\alpha(t)X^i_\alpha(x)\frac{\partial}{\partial x^i}, \tag{1.40}$$

where the vector fields $X_\alpha(x) = \sum_{i=1}^{n_0} X_\alpha^i(x)\partial/\partial x^i$ generate an r-dimensional Lie algebra V, i.e. there exist r^3 constants $c_{\alpha\beta\gamma}$ such that

$$[X_\alpha, X_\beta] = \sum_{\gamma=1}^{r} c_{\alpha\beta\gamma} X_\gamma, \qquad \alpha, \beta = 1, \ldots, r.$$

We aim to derive a particular type of mixed superposition rule of the form $\Phi : (\mathbb{R}^{n_1})^m \times \mathbb{R}^{n_0} \to \mathbb{R}^{n_0}$ for the above Lie system in such a way that its general solution $x(t)$ can be expressed as

$$x(t) = \Phi(x_{(1)}(t), \ldots, x_{(m)}(t); k_1, \ldots, k_n),$$

where $x_{(1)}(t), \ldots, x_{(m)}(t)$ are a generic family of particular solutions of a Lie system determined by a t-dependent vector field $X^{(1)}$ on $\mathbb{R}^{n_1}$. Let us assume that $X^{(1)}$ takes the particular form

$$X_t^{(1)} = \sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha^{(1)}, \tag{1.41}$$

where the vector fields $X_\alpha^{(1)} \in \mathfrak{X}(\mathbb{R}^{n_1})$ obey the same commutation relations as the vector fields X_α, that is,

$$[X_\alpha^{(1)}, X_\beta^{(1)}] = \sum_{\gamma=1}^{r} c_{\alpha\beta\gamma} X_\gamma^{(1)}, \qquad \alpha, \beta = 1, \ldots r, \tag{1.42}$$

It is important to clarify when such an $X^{(1)}$ exists. Let us prove its existence. On one hand, Ado's Theorem states that for every finite-dimensional Lie algebra V, e.g. the one spanned by the vector fields X_α, there exists an isomorphic matrix Lie algebra V_M of $n_1 \times n_1$ square matrices. Now, since the homogeneous linear system

$$\dot{y} = A(t)y,$$

where $A(t)$ takes values in V_M, is a Lie system associated with a Lie algebra of vector fields isomorphic to V_M (see [31]), it follows immediately that we can always determine a family of linear vector fields on $\mathbb{R}^{n_1}$ obeying relations (1.42). In terms of this family, we can build a t-dependent vector field of the form (1.41). Apart from the t-dependent vector field $X_t^{(1)}$ constructed in the aforementioned way, there might exist others made from finite-dimensional Lie algebras of vector fields admitting a basis whose elements obey relations (1.42).

Proposition 1.17 ensures the existence of a minimal m such that the diagonal prolongations of the $X_\alpha^{(1)}$ to $\mathbb{R}^{n_1 m}$ are linearly independent at a generic point. Let us denote such prolongations by

$$\widetilde{X}_\alpha = \sum_{a=1}^{m} X_\alpha^{i(1)}(x_{(a)}) \frac{\partial}{\partial x_{(a)}^i}, \qquad \alpha = 1, \ldots, r,$$

and define vector fields on $\widetilde{N} = \mathbb{R}^{n_0} \times \mathbb{R}^{n_1 m}$ by

$$Y_\alpha = X_\alpha + \sum_{a=1}^{m} X_\alpha^{i(1)}(x_{(a)}) \frac{\partial}{\partial x_{(a)}^i}, \qquad \alpha = 1, \ldots, r,$$

where we have considered the vector fields X_α and $X_\alpha^{(1)}$ as vector fields on $\widetilde{N}$ in the natural way. From the above definition, one has

$$[Y_\alpha, Y_\beta] = \sum_{\gamma=1}^{r} c_{\alpha\beta\gamma} Y_\gamma, \qquad \alpha, \beta = 1, \dots, r.$$

Consequently, the system of differential equations that determines the integral curves of the t-dependent vector field

$$Y_t = \sum_{\alpha=1}^{r} b_\alpha(t) Y_\alpha$$

is a Lie system associated with a Vessiot–Guldberg Lie algebra isomorphic to V.

Define the involutive distribution $\widetilde{\mathcal{V}}$ on $\widetilde{N}$ by

$$\widetilde{\mathcal{V}}_{\tilde{x}} = \langle (Y_1)_{\tilde{x}}, \dots, (Y_r)_{\tilde{x}} \rangle, \qquad \tilde{x} \in \widetilde{N},$$

whose rank is r, around a generic point of $\widetilde{N}$. Additionally, as $r \leq m{\cdot}n_1$, we may choose, at least locally, n_0 common first integrals of the vector fields $Y_1, \dots, Y_r$, giving rise to an n_0-codimensional local foliation $\mathcal{F}$ over $\mathbb{R}^{n_0} \times \mathbb{R}^{n_1 m}$, whose leaves project diffeomorphically onto $\mathbb{R}^{n m_1}$ through the projection

$$p : (x, x_{(1)}, \dots, x_{(m)}) \in \widetilde{N} \mapsto (x_{(1)}, \dots, x_{(m)}) \in \mathbb{R}^{n_1 m}.$$

Additionally, the vector fields Y_α are tangent to the leaves of this foliation.

On one hand, it is immediate that the above results lead to defining a flat connection ∇ on the bundle $p : \widetilde{N} \to \mathbb{R}^{n_1 m}$. On the other hand, as it happened in the case of superposition rules (see Section 1.4), for every point $(x_{(1)}, \dots, x_{(m)}) \in \mathbb{R}^{n_1 m}$ and a leaf $\mathcal{F}_k$, with $k = (k_1, \dots, k_{n_0})$, of the foliation $\mathcal{F}$, there exists a unique point $x_{(0)}$ in $\mathbb{R}^{n_0}$ such that $(x_{(0)}, x_{(1)}, \dots, x_{(m)}) \in \mathcal{F}_k$. This gives rise to a map

$$x_{(0)} = \Phi(x_{(1)}, \dots, x_{(m)}; k_1, \dots, k_{n_0}).$$

Mutatis mutandis, the same arguments at the end of Section 1.4 apply here, and it can easily be proved that given a generic set of m particular solutions of system $X^{(1)}$, the general solution of X can be written as

$$x(t) = \Phi(x_{(1)}(t), \dots, x_{(m)}(t); k_1, \dots, k_{n_0}),$$

which shows that Φ is a particular type of mixed superposition rule. In this way, we have also shown that, as claimed in [38, Remark 5], a flat connection ∇ on a bundle of the form $N_0 \times N_1 \times \cdots \times N_m \to N_1 \times \cdots \times N_m$ can be used to obtain the solutions of a first-order system on N_0 by means of particular solutions of other first-order systems on $N_1, \dots, N_m$.

1.8. Differential geometry on Hilbert spaces. In order to provide some basic knowledge to develop the main applications of the theory of Lie systems to quantum mechanics, we report in this section some known concepts of differential geometry on infinite-dimensional manifolds. For further details one can consult [51, 60, 138].

As far as quantum mechanics is concerned, the separable complex Hilbert space of states $\mathcal{H}$ can be seen as an (infinite-dimensional) real manifold admitting a global chart [23]. Infinite-dimensional manifolds do not enjoy the same geometric properties as finite-dimensional ones, e.g. in the most general case, and given an open $U \subset \mathcal{H}$, there is not a one-to-one correspondence between derivations on $C^\infty(U,\mathbb{R})$ and sections of the tangent bundle TU. Therefore, some explanations must be given before dealing with such manifolds.

On one hand, given a point $\phi \in \mathcal{H}$, a *kinematic tangent vector* with foot point ϕ is a pair (ϕ,ψ) with $\psi \in \mathcal{H}$. We denote by $T_\phi\mathcal{H}$ the space of all kinematic tangent vectors with foot point ϕ. It consists of all derivatives $\dot{c}(0)$ of smooth curves $c : \mathbb{R} \to \mathcal{H}$ with $c(0) = \phi$. This justifies the word 'kinematic'.

From the concept of kinematic tangent vector we can provide the definition of smooth kinematic vector fields as follows: A *smooth kinematic vector field* is an element $X \in \mathfrak{X}(\mathcal{H}) \equiv \Gamma(\pi)$, with $T\mathcal{H}$ the *kinematic tangent bundle* and $\pi : \mathrm{T}\mathcal{H} \to \mathcal{H}$ the projection of this bundle. We define a *kinematic vector field* X as a map $X : \mathcal{H} \to \mathrm{T}\mathcal{H}$ such that $\pi \circ X = \mathrm{Id}_{\mathcal{H}}$. Given a $\psi \in \mathcal{H}$, we will write from now on $X(\psi) = (\psi, X_\psi)$, with X_ψ being the value of $X(\psi)$ in $T_\psi\mathcal{H}$.

As in the differential geometry on finite-dimensional manifolds, we say that a kinematic vector field X on $\mathcal{H}$ admits a *local flow* on an open subset $U \subset \mathcal{H}$ if there exists a map $Fl^X : \mathbb{R} \times U \to \mathcal{H}$ such that $Fl^X(0,\psi) = \psi$ for all $\psi \in U$ and

$$X_\psi = \frac{d}{ds}\bigg|_{s=0} Fl^X(s,\psi) = \frac{d}{ds}\bigg|_{s=0} Fl^X_s(\psi),$$

with $Fl^X_s(\psi) = Fl^X(s,x)$.

All these mathematical concepts are used to study quantum mechanics as a geometric theory. Note that the Abelian translation group on $\mathcal{H}$ provides an identification of the tangent space $T_\phi\mathcal{H}$ at any point $\phi \in \mathcal{H}$ with $\mathcal{H}$ itself. Furthermore, through such an identification of $\mathcal{H}$ with $T_\phi\mathcal{H}$ at any $\phi \in \mathcal{H}$, a continuous kinematic vector field is simply a continuous map $X\colon \mathcal{H} \to \mathcal{H}$.

Starting with a bounded $\mathbb{C}$-linear operator A on $\mathcal{H}$, we can define the kinematic vector field X^A by $X^A_\psi = A\psi \in \mathcal{H} \simeq T_\psi\mathcal{H}$. In other words, we have

$$X^A : \psi \in \mathcal{H} \mapsto (\psi, X\psi) \in \mathrm{T}\mathcal{H} \simeq \mathcal{H} \oplus \mathcal{H}.$$

Usually, operators in quantum mechanics are neither continuous nor defined on the whole space $\mathcal{H}$. The most relevant case happens when A is a skew-self-adjoint operator of the form $A = -iH$. The reason is that $\mathcal{H}$ can be endowed with a natural (strongly) symplectic structure, and then such skew-self-adjoint operators are singled out as the linear vector fields that are Hamiltonian. The integral curves of such a Hamiltonian vector field X^A are the solutions of the corresponding Schrödinger equation [23, 51]. Even when A is not bounded, if A is skew-self-adjoint it must be densely defined and, by Stone's Theorem, its integral curves are strongly continuous and defined in all $\mathcal{H}$.

Additionally, these kinematic vector fields related to skew-self-adjoint operators admit local flows, i.e. any skew-self-adjoint operator A has a local flow

$$Fl^A_s(\psi) = \exp(sA)(\psi) \quad \text{as} \quad \frac{d}{ds}Fl^A_s(\psi) = A\exp(sA)(\psi) = A(Fl^A_s(\psi)). \tag{1.43}$$

We remark that given two constants $\lambda,\mu \in \mathbb{R}$ and two skew-self-adjoint operators A and B, we get $X^{\lambda A+\mu B} = \lambda X^A + \mu X^B$. Moreover, skew-self-adjoint operators considered as vector fields are fundamental vector fields relative to the usual action of the unitary group $U(\mathcal{H})$ on the Hilbert space $\mathcal{H}$.

Let us define the Lie bracket of two kinematic vector fields X^A and X^B associated with two skew-self-adjoint operators A and B. To simplify notation, and as it shall be clear from the context, we hereafter denote both the commutator of operators, i.e. $[A,B] = AB - BA$, and the Lie bracket of vector fields $[X^A, X^B]$ in the same way. In view of the previous remarks, we can declare the Lie bracket of vector fields related to skew-self-adjoint operators to be

$$[X^A, X^B] = X^{[B,A]}.$$

It is worth noting that the above formula is equivalent to the standard one

$$[X,Y]_\psi = \frac{1}{2}\frac{d^2}{ds^2}\bigg|_{s=0}(Fl^Y_{-s}\circ Fl^X_{-s}\circ Fl^Y_s \circ F^X_s(\psi)) \tag{1.44}$$

in finite-dimensional differential geometry when the right-hand side is properly defined. Indeed, the above formula yields

$$\begin{aligned}[X^A, X^B]_\psi &= \frac{1}{2}\frac{d^2}{ds^2}\bigg|_{s=0} \exp(-sB)\exp(-sA)\exp(sB)\exp(sA)(\psi)\\
&= \frac{1}{2}\frac{d^2}{ds^2}\bigg|_{s=0}\left(\sum_{n_1=0}^{\infty}\frac{(-sB)^{n_1}}{n_1!}\right)\left(\sum_{n_2=0}^{\infty}\frac{(-sA)^{n_2}}{n_2!}\right)\\
&\quad\left(\sum_{n_3=0}^{\infty}\frac{(sB)^{n_3}}{n_3!}\right)\left(\sum_{n_4=0}^{\infty}\frac{(sA)^{n_4}}{n_4!}\right)(\psi)\\
&= \frac{1}{2}\frac{d^2}{ds^2}\bigg|_{s=0}(-s^2AB + s^2BA)(\psi) = \frac{1}{2}\frac{d^2}{ds^2}\bigg|_{s=0}(s^2[B,A])(\psi) = [B,A](\psi),\end{aligned}$$

when the above expressions are properly defined. Hence, we obtain again

$$[X^A, X^B] = -X^{[A,B]}, \tag{1.45}$$

just as we defined.

1.9. Quantum Lie systems. The theory of Lie systems can be applied to investigate a particular class of t-dependent Hamiltonians satisfying a specific set of conditions, the so-called *quantum Lie systems*. Let us now precisely define this notion and sketch some of its properties.

We define a *t-dependent Hamiltonian* $H(t)$ to be a t-parametric family of self-adjoint operators $H_t : \mathcal{H} \to \mathcal{H}$.

DEFINITION 1.21. We say that the t-dependent Hamiltonian $H(t)$ is a *quantum Lie system* if it can be written as

$$H(t) = \sum_{\alpha=1}^{r} b_\alpha(t) H_\alpha, \tag{1.46}$$

where the operators iH_α are a family of skew-self-adjoint operators on $\mathcal{H}$ giving rise to a basis of a real r-dimensional Lie algebra of operators V under the commutator of operators, i.e.

$$[iH_\alpha, iH_\beta] = \sum_{\gamma=1}^{r} c_{\alpha\beta\gamma}\, iH_\gamma, \qquad \alpha, \beta = 1, \ldots, r, \tag{1.47}$$

for certain r^3 real structure constants $c_{\alpha\beta\gamma}$. We call V a *quantum Vessiot–Guldberg Lie algebra* associated with $H(t)$.

Each quantum Lie system $H(t)$ leads to a Schrödinger equation

$$\frac{d\psi}{dt} = -iH(t)\psi = -\sum_{\alpha=1}^{r} b_\alpha(t) iH_\alpha \psi, \tag{1.48}$$

describing the integral curves for the kinematic t-dependent vector field on $\mathcal{H}$ given by

$$X_t = \sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha,$$

where X_α is the vector field associated with the operator $-iH_\alpha$. In view of the relation (1.45) and the commutation relations (1.47), we obtain

$$[X_\alpha, X_\beta] = -X^{[iH_\alpha, iH_\beta]} = \sum_{\gamma=1}^{r} c_{\alpha\beta\gamma} X_\gamma, \qquad \alpha, \beta = 1, \ldots, n. \tag{1.49}$$

Consequently, the vector fields X_α span an r-dimensional Lie algebra of vector fields. In addition, the structure constants for the basis $\{X_\alpha \mid \alpha = 1, \ldots, r\}$ coincide with those of the quantum Vessiot–Guldberg Lie algebra for the basis $\{iH_\alpha \mid \alpha = 1, \ldots, r\}$.

Given the Lie algebra V, consider an isomorphic Lie algebra $\mathfrak{g}$ corresponding to a connected Lie group G. Choose a basis $\{\mathrm{a}_\alpha \,|\, \alpha = 1, \ldots, r\}$ of the Lie algebra $T_eG \simeq \mathfrak{g}$ such that the Lie brackets of its elements, denoted by $[\cdot, \cdot]$, obey the relations

$$[\mathrm{a}_\alpha, \mathrm{a}_\beta] = \sum_{\gamma=1}^{r} c_{\alpha\beta\gamma} \mathrm{a}_\gamma, \qquad \alpha, \beta = 1, \ldots, r. \tag{1.50}$$

It can be proved that there exists a unitary action $\Phi : G \times \mathcal{H} \to \mathcal{H}$ such that each X_α is the fundamental vector field associated with the element a_α, according to the relation (1.50). Indeed, note that, for a fixed basis $\{\mathrm{a}_\alpha \mid \alpha = 1, \ldots, r\}$, each element g in a sufficiently small open U containing the neutral element of G can be written in a unique way as

$$g = \exp(-\mu_1 \mathrm{a}_1) \times \cdots \times \exp(-\mu_r \mathrm{a}_r).$$

Now, we define

$$\Phi(\exp(-\mu_\alpha \mathrm{a}_\alpha), \psi) = \exp(-i\mu_\alpha H_\alpha)\psi, \qquad \alpha = 1, \ldots, r.$$

As G is connected, every element can be written as a product of elements in U, which, in view of the above relations, gives rise to an action $\Phi : G \times \mathcal{H} \to \mathcal{H}$.

Similarly to the procedure carried out to show that solving a Lie system reduces to working out a particular solution for an equation in a Lie group (see Section 1.3), it can be proved that solving the Schrödinger equation for a quantum Lie system $H(t)$ reduces to determining the solution of the equation in G given by

$$R_{g^{-1}*g}\dot{g} = -\sum_{\alpha=1}^{r} b_\alpha(t) \mathrm{a}_\alpha \equiv \mathrm{a}(t), \qquad g(0) = e.$$

More specifically, the particular solution of the Schrödinger equation (1.48) with initial condition ψ_0 reads $\psi_t = \Phi(g(t), \psi_0)$, where $g(t)$ is the solution of the above equation.

1.10. Superposition rules for second and higher-order differential equations. Although the theory of Lie systems is mainly devoted to the study of first-order differential equations, it can also be applied to investigate various systems of second-order differential equations, e.g. the so-called SODE Lie systems. This allows us to derive t-dependent and t-independent constants of motion, exact solutions, superposition rules or mixed superposition rules for these equations, etc. Moreover, our methods can also be generalised to study systems of higher-order differential equations.

Vessiot pioneered the analysis of systems of second-order differential equations by means of the theory of Lie systems [225]. Additionally, this theme was also briefly examined by Winternitz, Chisholm and Common [77, 202]. Apart from these few works, the analysis of systems of second-order differential equations through the theory of Lie systems was not deeply analysed until the beginning of the XXI century, when the SODE Lie systems were defined and employed to investigate various systems of second-order differential equations [36, 44, 45, 48, 52, 53]. This allowed us to recover previous results from a new clarifying perspective as well as to obtain some new achievements.

The description of the general solution of systems of second-order differential equations in terms of certain families of particular solutions and sets of constants appears in the study of some systems in physics and mathematics [115, 194]. Nevertheless, these results are frequently obtained through *ad hoc* procedures that neither explain their theoretical meaning nor the possibility of their generalisation. This section is concerned with the application of the theory of Lie systems to SODE Lie systems to review, through a geometrical unifying approach, some results previously obtained in the literature. Not only does this provide a deeper theoretical understanding of those results, but it also offers several new ones.

Recall that the theory of Lie systems initially aimed to study systems of first-order differential equations with general solution admitting an expression in terms of certain families of particular solutions and a set of constants. Nevertheless, this property is not exclusive to systems of first-order differential equations. For instance, for each second-order differential equation of the form $\ddot{x} = a(t)x$, with $a(t)$ being a real function, the general solution $x(t)$ can be cast in the form

$$x(t) = k_1 x_{(1)}(t) + k_2 x_{(2)}(t), \tag{1.51}$$

with k_1, k_2 being constants and $x_{(1)}(t), x_{(2)}(t)$ particular solutions whose initial conditions $(x_{(1)}(0), \dot{x}_{(1)}(0))$ and $(x_{(2)}(0), \dot{x}_{(2)}(0))$ are linearly independent vectors of $\mathrm{T}\mathbb{R}$. Note also that such a superposition rule leads to the existence of many other nonlinear superposition rules for other systems of second-order differential equations. For instance, the change of variables $y = 1/x$ transforms the previous system into $y\ddot{y} - 2\dot{y}^2 = -a(t)y^2$ for which, in view of the above linear superposition rule and the above change of variable, the general solution can be written as

$$y(t) = (k_1 y_1^{-1}(t) + k_2 y_2^{-1}(t))^{-1} \tag{1.52}$$

in terms of a pair $y_{(1)}(t), y_{(2)}(t)$ of particular solutions and a pair of constants.

Consequently, in view of the previous examples and others that can be found, for instance, in [34, 43], it is natural to define superposition rules for second-order differential equations as follows.

DEFINITION 1.22. We say that a second-order differential equation

$$\ddot{x}^i = F^i(t, x, \dot{x}), \qquad i = 1, \ldots, n, \tag{1.53}$$

on $\mathbb{R}^n$ admits a global superposition rule if there exists a map $\Psi : \mathrm{T}\mathbb{R}^{mn} \times \mathbb{R}^{2n} \to \mathbb{R}^n$ such that its general solution $x(t)$ can be written as

$$x(t) = \Psi(x_{(1)}(t), \ldots, x_{(m)}(t), \dot{x}_{(1)}(t), \ldots, \dot{x}_{(m)}(t); k_1, \ldots, k_{2n}), \tag{1.54}$$

in terms of a generic family $x_{(1)}(t), \ldots, x_{(m)}(t)$ of particular solutions, their derivatives, and a set of $2n$ constants.

In order to understand the previous definition, it is necessary to establish the precise meaning of 'generic' in the above statement. Formally, we say that expression (1.54) is valid for a generic family of particular solutions when it holds for every family of particular solutions $x_1(t), \ldots, x_m(t)$ such that $(x_1(0), \dot{x}_1(0), \ldots, x_m(0), \dot{x}_m(0)) \in U$, with U being an open dense subset of $(\mathrm{T}\mathbb{R}^n)^m$.

There exists no characterisation for systems of SODEs of the form (1.53) admitting a superposition rule. In spite of this, there exists a special class of such systems, called *SODE Lie systems* [52], which have this property. Even though this fact has been broadly used in the literature, it has been proved very recently [48]. We next furnish the definition of a SODE Lie system along with a proof that every SODE Lie system admits a superposition rule. In addition, some remarks on the properties of this notion are given.

DEFINITION 1.23. We say that the system (1.53) of second-order differential equations is a *SODE Lie system* if the system of first-order differential equations

$$\begin{cases} \dot{x}^i = v^i, \\ \dot{v}^i = F^i(t, x, v), \end{cases} \qquad i = 1, \ldots, n, \tag{1.55}$$

obtained from (1.53) by defining the new variables $v^i = \dot{x}^i$ with $i = 1, \ldots, n$ is a Lie system.

PROPOSITION 1.24. *Every SODE Lie system* (1.53) *admits a superposition rule* $\Psi : (\mathrm{T}\mathbb{R}^n)^m \times \mathbb{R}^{2n} \to \mathbb{R}^n$ *of the form* $\Psi = \pi \circ \Phi$, *where* $\Phi : (\mathrm{T}\mathbb{R}^n)^m \times \mathbb{R}^{2n} \to \mathrm{T}\mathbb{R}^n$ *is a superposition rule for the system* (1.55) *and* $\pi : \mathrm{T}\mathbb{R}^n \to \mathbb{R}^n$ *is the projection of the tangent bundle* $\mathrm{T}\mathbb{R}^n$.

Proof. Each SODE Lie system (1.53) is associated with a first-order system of differential equations (1.55) admitting a superposition rule $\Phi : (\mathrm{T}\mathbb{R}^n)^m \times \mathbb{R}^{2n} \to \mathrm{T}\mathbb{R}^n$. This allows us to describe the general solution $(x(t), v(t))$ of (1.55) in terms of a generic set $(x_a(t), v_a(t))$, with $a = 1, \ldots, m$, of particular solutions and a set of $2n$ constants, i.e.

$$(x(t), v(t)) = \Phi(x_1(t), \ldots, x_m(t), v_1(t), \ldots, v_m(t); k_1, \ldots, k_{2n}). \tag{1.56}$$

Each solution $x_p(t)$ of (1.53) corresponds to a unique solution $(x_p(t), v_p(t))$ of (1.55) and vice versa. Furthermore, since $(x_p(t), v_p(t)) = (x_p(t), \dot{x}_p(t))$, the general solution $x(t)$ of (1.53) can be written as

$$x(t) = \pi \circ \Phi(x_1(t), \ldots, x_m(t), \dot{x}_1(t), \ldots, \dot{x}_m(t); k_1, \ldots, k_{2n}), \tag{1.57}$$

in terms of a generic family $x_a(t)$, with $a = 1, \ldots, n$, of particular solutions of (1.53). That is, the map $\Psi = \pi \circ \Phi$ is a superposition rule for (1.53). ∎

Since every autonomous system is related to a one-dimensional Vessiot–Guldberg Lie algebra [34], a corollary follows immediately.

COROLLARY 1.25. *Every autonomous system of second-order differential equations of the form* $\ddot{x}^i = F^i(x, \dot{x})$ *with* $i = 1, \ldots, n$ *admits a superposition rule.*

The above result is, in practice, almost useless. Actually, the superposition rule ensured by Proposition 1.24 relies on the derivation of a superposition rule for an autonomous first-order system of differential equations. Applying the method sketched in Section 1.6, it is found that determining this superposition rule implies working out all the integral curves of a vector field on $(\mathrm{T}\mathbb{R}^n)^2$. Although the solution of this problem is known to exist, its explicit description can be as difficult as solving the initial system (indeed, this is usually the case). Consequently, deriving explicitly a superposition rule for the above autonomous system frequently depends on the search of an alternative superposition rule for the associated first-order system.

Many superposition rules for second-order differential equations do not present an explicit dependence on the derivatives of the particular solutions. Consider, for instance, either the linear superposition rule (1.51) for the equation $\ddot{x} = a(t)x$, or the affine one,

$$x(t) = k_1(x_1(t) - x_2(t)) + k_2(x_2(t) - x_3(t)) + x_3(t),$$

for $\ddot{x} = a(t)x + b(t)$. Such superposition rules are called *velocity free superposition rules* or even *free superposition rules*. To find conditions ensuring the existence of such superposition rules is an interesting open problem. Let us provide a brief analysis of the existence of such superposition rules.

PROPOSITION 1.26. *Every system* (1.53) *of SODEs admitting a free superposition rule is a SODE Lie system.*

Proof. Suppose that (1.53) admits a superposition rule of the special form

$$x^i = \Phi_x^i(x_1, \ldots, x_m; k_1, \ldots, k_{2n}), \qquad i = 1, \ldots, n. \tag{1.58}$$

In that case, the general solution $x(t)$ of the system can be expressed as

$$x^i(t) = \Phi_x^i(x_1(t), \ldots, x_m(t); k_1, \ldots, k_{2n}), \qquad i = 1, \ldots, n. \tag{1.59}$$

Define $p(t) = (x_1(t), \ldots, x_m(t), \dot{x}_1(t), \ldots, \dot{x}_m(t))$ and $v^i = \dot{x}^i$ for $i = 1, \ldots, n$. Take the time derivative in the above expression. This yields

$$v^i(t) = \dot{x}^i(t) = \sum_{a=1}^{m} \sum_{j=1}^{n} \left(v_a^j(t) \frac{\partial \Phi_x^i}{\partial x_a^j}(p(t)) \right), \qquad i = 1, \ldots, n, \tag{1.60}$$

where we have used that $\partial \Phi_x^i / \partial v_a^j = 0$ for $i, j = 1, \ldots, n$, and $a = 1, \ldots, m$. Consequently, there exists a function

$$\Phi_v^i(x_1, \ldots, x_m, v_1, \ldots, v_m) = \sum_{a=1}^{m} \sum_{j=1}^{n} \left(v_a^j \frac{\partial \Phi_x^i}{\partial x_a^j} \right), \qquad i = 1, \ldots, n,$$

such that

$$\begin{cases} x^i(t) = \Phi^i_x(x_1(t), \ldots, x_m(t); k_1, \ldots, k_{2n}), \\ v^i(t) = \Phi^i_v(x_1(t), \ldots, x_m(t), v_1(t), \ldots, v_m(t); k_1, \ldots, k_{2n}), \end{cases} \qquad i = 1, \ldots, n.$$

Therefore, system (2.13) admits a superposition rule and (1.53) becomes a SODE Lie system. ■

Apart from SODE Lie systems, there exists another method to study certain second-order differential equations admitting a regular Lagrangian, like Caldirola–Kanai oscillators or Milne–Pinney equations [52, 97]. Although this method cannot be used to study all systems of second-order differential equations, it provides some additional information that cannot be derived by means of SODE Lie systems, e.g. on the t-dependent constants of motion of the system [97].

1.11. Superposition rules for PDEs. The geometrical formulation of the theory of Lie systems enables us to extend the notion of Lie system to partial differential equations. Here, we briefly analyse this generalisation and its properties [38, 185].

Consider the system of first-order PDEs of the form

$$\frac{\partial x^i}{\partial t^a} = X^i_a(t, x), \qquad x \in \mathbb{R}^n,\ t = (t^1, \ldots, t^s) \in \mathbb{R}^s, \tag{1.61}$$

whose solutions are maps $x(\cdot) : \mathbb{R}^s \to \mathbb{R}^n$. When $s = 1$, the above system of PDEs becomes the system of ordinary differential equations (1.33). The main difference between these systems is that for $s > 1$ there exists, in general, no solution with a given initial condition. For a better understanding of this problem, let us put (1.61) in a more general and geometric framework.

Let $P^s_{\mathbb{R}^n}$ be the trivial fibre bundle

$$P^s_{\mathbb{R}^n} = \mathbb{R}^s \times \mathbb{R}^n \to \mathbb{R}^s.$$

A connection $\bar{Y}$ on this bundle is a horizontal distribution over $\mathrm{T}P^s_{\mathbb{R}^n}$, i.e. an s-dimensional distribution transversal to the fibres. This distribution may be determined by the horizontal lifts of the vector fields $\partial/\partial t^a$ on $\mathbb{R}^s$, i.e.

$$\overline{X}_a(t, x) = \frac{\partial}{\partial t^a} + X_a(t, x),$$

where

$$X_a(t, x) = \sum_{i=1}^n X^i_a(t, x)\frac{\partial}{\partial x^i}.$$

The solutions of system (1.61) can be identified with integral submanifolds of the distribution $\overline{X}$,

$$(t, X_a(t, x)), \qquad t \in \mathbb{R}^s,\ x \in \mathbb{R}^n.$$

It is now clear that there is an (obviously unique) solution of (1.61) for every initial data if and only if the distribution $\overline{Y}$ is integrable, i.e. the connection has zero curvature. This means that

$$[\overline{X}_a, \overline{X}_b] = \sum_{c=1}^r f_{abc}\overline{X}_c$$

for some functions f_{abc} in $P^s_{\mathbb{R}^n}$. But the commutators $[\overline{X}_a, \overline{X}_b]$ are clearly vertical, while $\overline{X}_c$ are linearly independent horizontal vector fields, so $f_{abc} = 0$, which yields the integrability condition in the form of the system of equations $[\overline{X}_a, \overline{X}_b] = 0$, i.e. in local coordinates,

$$\frac{\partial X^i_b}{\partial t^a}(t,x) - \frac{\partial X^i_a}{\partial t^b}(t,x) + \sum_{j=1}^n \left(X^j_a(t,x)\frac{\partial X^i_b}{\partial x^j}(t,x) - X^j_b(t,x)\frac{\partial X^i_a}{\partial x^j}(t,x)\right) = 0. \qquad (1.62)$$

Let us assume now that we analyse a system of first-order PDEs of the form (1.61) that satisfies integrability conditions (1.62). Then, for a given initial value, there exists a unique solution of system (1.61). Furthermore, it is immediate that the geometrical interpretation for superposition rules for first-order systems described in Section 1.4 can be directly generalised to the case of PDEs. Consequently, Proposition 1.18 now takes the following form.

Proposition 1.27. *Giving a superposition rule for system* (1.61) *obeying the integrability condition* (1.62) *is equivalent to giving a connection on the bundle* $\mathrm{pr} : \mathbb{R}^{n(m+1)} \to \mathbb{R}^{nm}$ *with zero curvature such that the vector fields* $\{(X_a)_t \mid t \in \mathbb{R}^s,\ a = 1,\dots,s\}$ *are horizontal.*

Also the proof of the Lie Theorem remains unchanged. Therefore, we get the following analogue of the Lie Theorem for PDEs.

Theorem 1.28. *The system* (1.61) *of PDEs defined on* $\mathbb{R}^n$ *and satisfying the integrability condition* (1.62) *admits a superposition rule if and only if the vector fields* $\{(X_a)_t\}$ *on* $\mathbb{R}^n$ *depending on the parameter* $t \in \mathbb{R}^s$ *can be written in the form*

$$(X_a)_t = \sum_{\alpha=1}^r u^\alpha_a(t) X_\alpha, \qquad a = 1,\dots,s, \qquad (1.63)$$

where the vector fields X_α *span a finite-dimensional real Lie algebra.*

Note that the integrability condition for $Y_a(t,x)$ of the form (1.63) can be written as

$$\sum_{\alpha,\beta,\gamma=1}^r [(u^\gamma_b)'(t) - (u^\gamma_a)'(t) + u^\alpha_a(t) u^\beta_b(t) c^\gamma_{\alpha\beta}] X_\gamma = 0.$$

We now illustrate the above results by an example. Consider the following system of partial differential equations on $\mathbb{R}^2$ associated with the $SL(2,\mathbb{R})$-action on $\bar{\mathbb{R}}$:

$$\begin{aligned} u_x &= a(x,y)u^2 + b(x,y)u + c(x,y), \\ u_y &= d(x,y)u^2 + e(x,y)u + f(x,y). \end{aligned} \qquad (1.64)$$

This equation can be written in the form of a 'total differential equation'

$$(a(x,y)u^2 + b(x,y)u + c(x,y))\mathrm{d}x + (d(x,y)u^2 + e(x,y)u + f(x,y))\mathrm{d}y = \mathrm{d}u.$$

The integrability condition only states that the one-form

$$\omega = (a(x,y)u^2 + b(x,y)u + c(x,y))\mathrm{d}x + (d(x,y)u^2 + e(x,y)u + f(x,y))\mathrm{d}y$$

is closed for an arbitrary function $u = u(x,y)$. If this is the case, there is a unique solution with the initial condition $u(x_0,y_0) = u_0$ and there is a superposition rule giving a general solution as a function of three independent solutions exactly as in the case of Riccati equations:

$$u = \frac{(u_{(1)} - u_{(3)})u_{(2)}k + u_{(1)}(u_{(3)} - u_{(2)})}{(u_{(1)} - u_{(3)})k + (u_{(3)} - u_{(2)})}.$$

2. SODE Lie systems

We already pointed out that the theory of Lie systems is mainly dedicated to the analysis of systems of first-order differential equations. In spite of this, the theory can also be applied to studying a variety of systems of second-order differential equations. This can be done in several ways that rely, as a last resort, on using some kind of transformation to convert systems of second-order differential equations into first-order ones [52, 54, 77, 100, 202]. A class of systems that can be investigated by these techniques is the SODE Lie systems, which were theoretically analysed in Section 1.10. In this chapter, we focus on analysing several instances of SODE Lie systems in order to derive t-independent constants of motion, exact solutions, superposition rules, and other properties. This allows us not only to study the mathematical properties of such systems, but also to provide tools to analyse diverse physical or control systems modelled through such equations.

Among the above applications to SODEs, one must be emphasised: the use of *mixed superposition rules*. This recently described notion enables us to express the general solution of a SODE Lie system in terms of particular solutions of the same, or other, SODE Lie systems. In this way, this new concept can be employed to analyse the properties of the general solutions of certain SODEs appearing in the physical and mathematical literature [115, 194]. As a consequence, new results can be obtained and other known ones will be recovered, in a systematic way, which will enhance their understanding.

The following section is dedicated to the application of the theory of Lie systems to SODE Lie systems in order to review, through a geometrical unifying approach, some results previously obtained in the literature by means of *ad hoc* methods and to provide new ones. The whole chapter can be divided into two parts: The first one is devoted to the application of the geometric theory of Lie systems to derive superposition rules, constants of motion and exact solutions for various SODE Lie systems. More specifically, we study t-dependent harmonic oscillators, generalised Ermakov systems and Milne–Pinney equations, providing a new superposition rule for the latter. The second part is concerned with the study and application of mixed superposition rules.

2.1. The harmonic oscillator with t-dependent frequency. The one-dimensional t-dependent frequency harmonic oscillator is perhaps the simplest SODE which allows us to illustrate the application of SODE Lie systems. Let us make use of this fact to show how this notion applies and to analyse thoroughly the properties of such a system.

The equation of motion for a one-dimensional harmonic oscillator with t-dependent frequency $\omega(t)$ is $\ddot{x} = -\omega^2(t)x$. In view of Definition 1.23, this equation is a SODE Lie system if and only if the system of first-order differential equations

$$\begin{cases} \dot{x} = v, \\ \dot{v} = -\omega^2(t)x, \end{cases} \tag{2.1}$$

is a Lie system. This feature depends on the properties of the t-dependent vector field over $\mathrm{T}\mathbb{R}$ given by

$$X(t, x, v) = v\frac{\partial}{\partial x} - \omega^2(t)x\,\frac{\partial}{\partial v},$$

which describes the integral curves of system (2.1). It is immediate that

$$X_t = X_1 + \omega^2(t)X_3, \tag{2.2}$$

where

$$X_1 = v\frac{\partial}{\partial x}, \qquad X_3 = -x\frac{\partial}{\partial v}.$$

These vector fields obey the commutation relations

$$[X_1, X_3] = 2X_2, \qquad [X_2, X_3] = X_3, \qquad [X_1, X_2] = X_1, \tag{2.3}$$

with

$$X_2 = \frac{1}{2}\left(x\frac{\partial}{\partial x} - v\frac{\partial}{\partial v}\right).$$

From (2.3) and (2.2), it follows that X_t defines a Lie system associated with a Vessiot–Guldberg Lie algebra $V = \langle X_1, X_2, X_3\rangle$. Hence, one-dimensional harmonic oscillators with a t-dependent frequency are SODE Lie systems.

Determining the general solution of every SODE Lie system reduces to working out the solution of an equation on a Lie group. Let us illustrate this in detail through the example of harmonic oscillators.

Since (2.1) is a Lie system, its general solution can be worked out by solving an equation on a certain Lie group (see Section 1.3). Recall that as the elements of V are complete, there exists a Lie group action $\Phi_L : G \times \mathrm{T}\mathbb{R} \to \mathrm{T}\mathbb{R}$ whose fundamental vector fields are exactly those corresponding to V. It is easy to check that this action can be chosen to be $\Phi_L : SL(2, \mathbb{R}) \times \mathrm{T}\mathbb{R} \to \mathrm{T}\mathbb{R}$, with

$$\Phi_L\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \begin{pmatrix} x \\ v \end{pmatrix}\right) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} \alpha x + \beta v \\ \gamma x + \delta v \end{pmatrix}.$$

Indeed, if we take the basis

$$\mathrm{a}_1 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad \mathrm{a}_2 = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathrm{a}_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{2.4}$$

of the Lie algebra of 2×2 traceless matrices (the usual representation of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$), its elements satisfy the same commutation relations as the vector fields X_1, X_2, X_3. Furthermore, it can be easily verified that X_1, X_2 and X_3 are the fundamental vector fields associated with the matrices $\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3$, according to our convention (1.28).

Once the action Φ_L is determined, it enables us to write the general solution $(x(t), v(t))$ of system (2.1) in the form

$$\begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \Phi_L\left(g(t), \begin{pmatrix} x_0 \\ v_0 \end{pmatrix}\right), \quad \text{with } \begin{pmatrix} x_0 \\ v_0 \end{pmatrix} \in \mathrm{T}\mathbb{R}, \tag{2.5}$$

where $g(t)$ is the solution of the Cauchy problem

$$R_{g^{-1}*}\dot{g} = -\sum_{\alpha=1}^{3} b_\alpha(t)\mathrm{a}_\alpha, \qquad g(0) = e,$$

on $SL(2,\mathbb{R})$. This immediately gives us the general solution $x(t)$ of the equation (2.1) from expression (2.5). Moreover, this process is easily generalised to every SODE Lie system.

Apart from the above Lie group approach, the SODE Lie system notion furnishes us with a second approach to investigate one-dimensional t-dependent frequency harmonic oscillators. This is based on determining a superposition rule for the Lie system (2.1).

Recall that a superposition rule for a Lie system can be worked out by means of a set of first integrals for certain diagonal prolongations of the vector fields of an associated Vessiot–Guldberg Lie algebra V. As discussed in Section 1.6, to obtain these first integrals requires determining the minimal integer m such that the prolongations to $\mathbb{R}^{nm}$ of the elements of a basis of the Lie algebra V become linearly independent at a generic point. This yields $\dim V \leq m \cdot n$. Additionally, if we consider the diagonal prolongations of such a basis to $\mathbb{R}^{n(m+1)}$, these elements are again linearly independent at a generic point and a family of $m \cdot n - r$ first integrals appears. These first integrals allow us to determine a superposition rule.

We next illustrate the above process by means of the study of harmonic oscillators. In addition, we analyse in parallel the problem of finding t-independent constants of motion for systems made of some copies of the initial system. This problem will be proved to be related to the above process and, in addition, will permit us to show interesting properties of harmonic oscillators.

Consider two copies of the same one-dimensional harmonic oscillator, i.e.

$$\begin{cases} \ddot{x}_1 = -\omega^2(t)x_1, \\ \ddot{x}_2 = -\omega^2(t)x_2. \end{cases} \tag{2.6}$$

This system of SODEs, which corresponds to a two-dimensional isotropic harmonic oscillator with a t-dependent frequency $\omega(t)$, is related to the following system of first-order differential equations:

$$\begin{cases} \dot{x}_1 = v_1, \\ \dot{x}_2 = v_2, \\ \dot{v}_1 = -\omega^2(t)x_1, \\ \dot{v}_2 = -\omega^2(t)x_2. \end{cases} \tag{2.7}$$

Its solutions are the integral curves of the t-dependent vector field

$$X_t^{2d} = v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial x_2} - \omega^2(t)x_1\frac{\partial}{\partial v_1} - \omega^2(t)x_2\frac{\partial}{\partial v_2},$$

which is a linear combination

$$X_t^{2d} = X_1^{2d} + \omega^2(t) X_3^{2d}, \tag{2.8}$$

with

$$X_1^{2d} = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2}, \qquad X_3^{2d} = -x_1 \frac{\partial}{\partial v_1} - x_2 \frac{\partial}{\partial v_2}$$

satisfying the commutation relations

$$[X_1^{2d}, X_3^{2d}] = 2\, X_2^{2d}, \qquad [X_2^{2d}, X_3^{2d}] = X_3^{2d}, \qquad [X_1^{2d}, X_2^{2d}] = X_1^{2d}, \tag{2.9}$$

where

$$X_2^{2d} = \frac{1}{2}\left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} - v_1 \frac{\partial}{\partial v_1} - v_2 \frac{\partial}{\partial v_2} \right).$$

The previous decomposition of the t-dependent vector field X^{2d} has been obtained by considering the new vector fields, $X_1^{2d}, X_2^{2d}, X_3^{2d}$, to be diagonal prolongations to $\mathrm{T}\mathbb{R}^2$ of the vector fields X_1, X_2, X_3. In this way, the commutation relations (2.9) are the same as (2.3) and, in view of decomposition (2.8), this t-dependent vector field defines a Lie system related to a Lie algebra of vector fields isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

The distribution associated with the Lie system X_t^{2d}, i.e.

$$\mathcal{V}_p^{2d} = \langle (X_1^{2d})_p, (X_2^{2d})_p, (X_3^{2d})_p \rangle, \qquad p \in \mathrm{T}\mathbb{R}^2,$$

has rank lower than or equal to the dimension of the Lie algebra V. More specifically, it has rank three in an open dense subset of $\mathrm{T}\mathbb{R}^2$. Hence, there exists a local nontrivial first integral common to all the vector fields of the above distribution. Furthermore, this first integral is a t-independent constant of motion of system (2.7). Let us analyse this statement more carefully. Given a constant of motion $F : (x_1, v_1, x_2, v_2) \in \mathrm{T}\mathbb{R}^2 \mapsto F(x_1, v_1, x_2, v_2) \in \mathbb{R}$ of system (2.7), it follows that

$$\frac{dF}{dt}(p(t)) = \sum_{j=1}^{2} \left(\frac{dx^i}{dt}(t) \frac{\partial F}{\partial x^i}(p(t)) + \frac{dv^i}{dt}(t) \frac{\partial F}{\partial v^i}(p(t)) \right) = X_t^{2d} I(p(t)) = 0,$$

where $p(t) = (x_1(t), v_1(t), x_2(t), v_2(t))$. If F is a first integral for the system (2.7), whatever $\omega(t)$ is, then F must be a first integral of the vector fields of X_1^{2d}, X_3^{2d} and, therefore, of X_2^{2d}.

Consequently, there exists, at least locally, a function F that is a constant of motion for every system (2.7) and such that dF is incident to the distribution generated by the $X_1^{2d}, X_2^{2d}, X_3^{2d}$, i.e. $dF(X_1^{2d}) = dF(X_2^{2d}) = dF(X_3^{2d}) = 0$ in a certain dense open subset U of $\mathrm{T}\mathbb{R}^2$.

As $X_3^{2d} F = 0$, there is a function $\bar{F}(\xi, x_1, x_2)$ such that $F(x_1, x_2, v_1, v_2) = \bar{F}(\xi, x_1, x_2)$ with $\xi = x_1 v_2 - x_2 v_1$. Next, in view of the condition $X_1^{2d} \bar{F} = 0$, we have

$$v_1 \frac{\partial \bar{F}}{\partial x_1} + v_2 \frac{\partial \bar{F}}{\partial x_2} = 0$$

and there exists a function $\widehat{F}(\xi)$ such that $\bar{F}(\xi, x_1, x_2) = \widehat{F}(\xi)$. As $2X_2^{2d} = [X_1^{2d}, X_3^{2d}]$, the conditions $X_1^{2d} \widehat{F} = X_3^{2d} \widehat{F} = 0$ imply $X_2^{2d} \widehat{F} = 0$ and hence $F(x_1, x_2, v_1, v_2) = x_1 v_2 - x_2 v_1$ is a first integral which physically corresponds to the angular momentum. Additionally, this first integral allows us to solve the second-order differential equation $\ddot{x} = -\omega^2(t) x$

by means of a particular solution. Actually, if $x_1(t)$ is a nonvanishing solution of this equation, any other particular solution $x_2(t)$ gives rise to a particular solution $(x_1(t), v_1(t), x_2(t), v_2(t))$ of system (2.7). As the first integral F is constant along this particular solution, it follows that $x_2(t)$ obeys the equation

$$x_1(t)\frac{dx_2}{dt} = k + \dot{x}_1(t)x_2,$$

whose solution reads

$$x_2(t) = k' x_1(t) + k\, x_1(t)\int^t \frac{d\zeta}{x_1^2(\zeta)}, \tag{2.10}$$

which gives us the general solution to the t-dependent frequency harmonic oscillator in terms of a particular solution.

In order to look for a superposition rule, we must consider a system made of some copies of (2.1) and obtain at least as many t-independent constants of motion as the dimension of the initial manifold. Also, it must be possible to obtain the dependent variables of one of the copies of (2.1) in terms of the dependent variables describing the remaining copies and such constants. Recall that the number m of particular solutions to obtain a superposition rule is such that the diagonal prolongations of the vector fields X_1, X_2 and X_3 to $\mathbb{R}^{nm}$ are linearly independent at a generic point.

In the case of two copies of the t-dependent harmonic oscillator, the condition on the prolongations of the vector fields X_1, X_2, X_3, that is, $f_1 X_1^{2d} + f_2 X_2^{2d} + f_3 X_3^{2d} = 0$, implies that $f_1 = f_2 = f_3 = 0$. Therefore, the one-dimensional oscillator admits a superposition rule involving two particular solutions and, in view of our previous results, we need to study three copies of the t-dependent harmonic oscillator (2.1) to obtain a superposition rule. Consider therefore the system of first-order ordinary differential equations

$$\begin{cases} \dot{x}_1 = v_1, \\ \dot{v}_1 = -\omega^2(t)x_1, \\ \dot{x}_2 = v_2, \\ \dot{v}_2 = -\omega^2(t)x_2, \\ \dot{x} = v, \\ \dot{v} = -\omega^2(t)x, \end{cases} \tag{2.11}$$

whose solutions are the integral curves for the t-dependent vector field

$$X_t^{3d} = v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial x_2} + v\frac{\partial}{\partial x} - \omega^2(t)x_1\frac{\partial}{\partial v_1} - \omega^2(t)x_2\frac{\partial}{\partial v_2} - \omega^2(t)x\frac{\partial}{\partial v},$$

which is a linear combination $X_t^{3d} = X_1^{3d} + \omega^2(t)X_3^{3d}$ with the vector fields

$$X_1^{3d} = v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial x_2} + v\frac{\partial}{\partial x}, \qquad X_3^{3d} = -x_1\frac{\partial}{\partial v_1} - x_2\frac{\partial}{\partial v_2} - x\frac{\partial}{\partial v}$$

obeying the commutation relations

$$[X_1^{3d}, X_3^{3d}] = 2X_2^{3d}, \qquad [X_2^{3d}, X_3^{3d}] = X_3^{3d}, \qquad [X_1^{3d}, X_2^{3d}] = X_1^{3d},$$

where

$$X_2^{3d} = \frac{1}{2}\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x\frac{\partial}{\partial x} - v_1\frac{\partial}{\partial v_1} - v_2\frac{\partial}{\partial v_2} - v\frac{\partial}{\partial v}\right).$$

We can determine the first integrals F for these three vector fields as solutions of the system of PDEs $X_1^{3d}F = X_3^{3d}F = 0$, because $2X_2^{3d} = [X_1^{3d}, X_3^{3d}]$ and the previous relations automatically imply $X_2^{3d}F = 0$. This last condition implies that there exists a function $\bar{F} : \mathbb{R}^5 \to \mathbb{R}^2$ such that $F(x_1, x_2, x, v_1, v_2, v) = \bar{F}(\xi_1, \xi_2, x_1, x_2, x)$ with $\xi_1(x_1, x_2, x, v_1, v_2, v) = xv_1 - x_1 v$ and $\xi_2(x_1, x_2, x, v_1, v_2, v) = xv_2 - x_2 v$. Hence, the condition $X_1^{3d}F = 0$ transforms into

$$v_1 \frac{\partial \bar{F}}{\partial x_1} + v_2 \frac{\partial \bar{F}}{\partial x_2} + v \frac{\partial \bar{F}}{\partial x} = 0,$$

i.e. the functions ξ_1 and ξ_2 are first integrals (of course, $\xi = x_1 v_2 - x_2 v_1$ is also a first integral). They produce a superposition rule, because from

$$\begin{cases} xv_2 - x_2 v = k_1, \\ x_1 v - v_1 x = k_2, \end{cases}$$

we get the expected superposition rule for two solutions

$$x = c_1 x_1 + c_2 x_2, \qquad v = c_1 v_1 + c_2 v_2, \qquad c_i = \frac{k_i}{k}, \qquad k = x_1 v_2 - x_2 v_1.$$

2.2. Generalised Ermakov system. Let us now study the so-called *generalised Ermakov system*

$$\begin{cases} \ddot{x} = \dfrac{1}{x^3} f(y/x) - \omega^2(t) x, \\ \ddot{y} = \dfrac{1}{y^3} g(y/x) - \omega^2(t) y, \end{cases} \tag{2.12}$$

which has been widely studied in [104, 191, 192, 193, 194, 205, 206]. Although this system is, in general, more complex than the standard Ermakov system, which will be discussed later, its analysis is easier from our point of view and it is therefore studied now. More exactly, our aim is to recover by means of our methods its known constant of motion, which is used next to study the Milne–Pinney equation and to obtain a superposition rule.

For simplicity, let us consider the generalised Ermakov system on $\mathbb{R}^2_+$. This system can be written as a system of first-order differential equations

$$\begin{cases} \dot{x} = v_x, \\ \dot{y} = v_y, \\ \dot{v}_x = -\omega^2(t) x + \dfrac{1}{x^3} f(y/x), \\ \dot{v}_y = -\omega^2(t) y + \dfrac{1}{y^3} g(y/x), \end{cases} \tag{2.13}$$

in $\mathrm{T}\mathbb{R}^2_+$ by introducing the new variables $v_x = \dot{x}$ and $v_y = \dot{y}$. Therefore, we can study its solutions as the integral curves for a t-dependent vector field X_t on $\mathrm{T}\mathbb{R}^2_+$ of the form

$$X_t = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + \left(-\omega^2(t) x + \frac{1}{x^3} f(y/x)\right) \frac{\partial}{\partial v_x} + \left(-\omega^2(t) y + \frac{1}{y^3} g(y/x)\right) \frac{\partial}{\partial v_y},$$

which can be written as a linear combination

$$X_t = N_1 + \omega^2(t) N_3,$$

where

$$N_1 = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + \frac{1}{x^3} f(y/x) \frac{\partial}{\partial v_x} + \frac{1}{y^3} g(y/x) \frac{\partial}{\partial v_y}, \qquad N_3 = -x \frac{\partial}{\partial v_x} - y \frac{\partial}{\partial v_y}.$$

Note that these vector fields generate a three-dimensional real Lie algebra with the third generator

$$N_2 = \frac{1}{2}\left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - v_x \frac{\partial}{\partial v_x} - v_y \frac{\partial}{\partial v_y} \right).$$

In fact, as

$$[N_1, N_3] = 2N_2, \qquad [N_1, N_2] = N_1, \qquad [N_2, N_3] = N_3,$$

they generate a Lie algebra of vector fields isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and thus the generalised Ermakov system is a SODE Lie system.

As Lie system (2.13) is associated with an integrable distribution of rank three at a generic point of a four-dimensional manifold, there exists, at least locally, a first integral $F : \mathrm{T}\mathbb{R}^2_+ \to \mathbb{R}$ for any $\omega^2(t)$. It satisfies $N_i F = 0$ for $i = 1, 2, 3$, but as $[N_1, N_3] = 2N_2$ it is sufficient to impose $N_1 F = N_3 F = 0$ to get $N_2 F = 0$. Then, if $N_3 F = 0$ we have

$$x \frac{\partial F}{\partial v_x} + y \frac{\partial F}{\partial v_y} = 0,$$

and the associated system of characteristics is

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dv_x}{x} = \frac{dv_y}{y}.$$

Hence, there exists a function $\bar{F} : \mathbb{R}^3 \to \mathbb{R}$ such that $F(x, y, v_x, v_y) = \bar{F}(x, y, \xi = x v_y - y v_x)$ and so the condition $N_1 F = 0$ reads

$$v_x \frac{\partial \bar{F}}{\partial x} + v_y \frac{\partial \bar{F}}{\partial y} + \left(-\frac{y}{x^3} f(y/x) + \frac{x}{y^3} g(y/x) \right) \frac{\partial \bar{F}}{\partial \xi} = 0.$$

We can therefore consider the associated system of characteristics

$$\frac{dx}{v_x} = \frac{dy}{v_y} = \frac{d\xi}{-\frac{y}{x^3} f(y/x) + \frac{x}{y^3} g(y/x)},$$

and using that

$$\frac{-y\, dx + x\, dy}{\xi} = \frac{dx}{v_x} = \frac{dy}{v_y},$$

we arrive at

$$\frac{-y\, dx + x\, dy}{\xi} = \frac{d\xi}{-\frac{y}{x^3} f(\frac{y}{x}) + \frac{x}{y^3} g(\frac{y}{x})},$$

i.e.

$$-\frac{y^2 d\left(\frac{x}{y}\right)}{\xi} = \frac{d\xi}{-\frac{y}{x^3} f(\frac{y}{x}) + \frac{x}{y^3} g(\frac{y}{x})}.$$

Integrating we obtain the first integral

$$\frac{1}{2}\xi^2 + \int^u \left[-\frac{1}{\zeta^3} f\left(\frac{1}{\zeta}\right) + \zeta\, g\left(\frac{1}{\zeta}\right)\right] d\zeta = C, \tag{2.14}$$

with $u = x/y$. This first integral allows us to determine, by means of quadratures, a solution of one subsystem in terms of a solution of another equation.

2.3. Milne–Pinney equation. The *Milne–Pinney equation* is the second-order ordinary nonlinear differential equation [163, 182]

$$\ddot{x} = -\omega^2(t)x + \frac{k}{x^3}, \tag{2.15}$$

where k is a nonzero constant. This equation describes the t-evolution of an isotonic oscillator [28, 181] (also called pseudo-oscillator), i.e. an oscillator with an inverse quadratic potential [204]. This oscillator shares with the harmonic one the property of having a period independent of the energy [68], i.e. they are isochronous systems and, in the quantum case, they have an equispaced spectrum [10]. The equation (2.15) appears in the study of certain Friedmann–Lemaître–Robertson–Walker spaces [85], certain scalar field cosmologies [115], and in many other works in physics and mathematics (see [147] and references therein).

The Milne–Pinney equation is defined on $\mathbb{R}^* \equiv \mathbb{R} - \{0\}$ and it is invariant under parity, i.e. if $x(t)$ is a solution, then so is $-x(t)$. That means that it is sufficient to restrict ourselves to analysing this equation in $\mathbb{R}_+$.

As usual, we can relate the Milne–Pinney equation to a system of first-order differential equations on $\mathrm{T}\mathbb{R}_+$

$$\begin{cases} \dot{x} = v, \\ \dot{v} = -\omega^2(t)x + \dfrac{k}{x^3}, \end{cases}$$

by introducing a new auxiliary variable $v \equiv \dot{x}$. Then the t-dependent vector field on $\mathrm{T}\mathbb{R}_+$ describing its integral curves reads

$$X_t = v\frac{\partial}{\partial x} + \left(-\omega^2(t)x + \frac{k}{x^3}\right)\frac{\partial}{\partial v}.$$

This is a Lie system because X_t can be written as $X_t = L_1 + \omega^2(t)L_3$, where the vector fields L_1 and L_3 are given by

$$L_1 = v\frac{\partial}{\partial x} + \frac{k}{x^3}\frac{\partial}{\partial v}, \qquad L_3 = -x\frac{\partial}{\partial v}$$

and satisfy

$$[L_1, L_3] = 2L_2, \qquad [L_1, L_2] = L_1, \qquad [L_2, L_3] = L_3$$

with

$$L_2 = \frac{1}{2}\left(x\frac{\partial}{\partial x} - v\frac{\partial}{\partial v}\right),$$

i.e. they span a 3-dimensional real Lie algebra of vector fields isomorphic to $\mathfrak{sl}(2,\mathbb{R})$.

Let us choose the basis (2.4) for $\mathfrak{sl}(2,\mathbb{R})$, which satisfies the same commutation relations as the vector fields L_1, L_2, L_3. Actually, it is possible to show that each L_α is the fundamental vector field corresponding to a_α with respect to the action $\Phi : (A,(x,v)) \in SL(2,\mathbb{R}) \times \mathrm{T}\mathbb{R}_+ \mapsto (\bar{x},\bar{v}) \in \mathrm{T}\mathbb{R}_+$ given by

$$\begin{cases} \bar{x} = \sqrt{\dfrac{k + \left[(\beta v + \alpha x)(\delta v + \gamma x) + k(\delta\beta/x^2)\right]^2}{(\delta v + \gamma x)^2 + k(\delta/x)^2}}, \\ \bar{v} = \kappa\sqrt{(\delta v + \gamma x)^2 + \dfrac{k\delta^2}{x^2}\left(1 - \dfrac{x^2}{\delta^2 \bar{x}^2}\right)}, \end{cases} \quad \text{with} \quad A \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where κ is ± 1 or 0, depending on the initial point (x,v) and the element of the group $SL(2,\mathbb{R})$ that acts on it. In order to obtain an explicit expression for κ in terms of A and (x,v), we can use the following decomposition for every element of the group $SL(2,\mathbb{R})$:

$$A = \exp(-\alpha_1 \mathrm{a}_1)\exp(\alpha_3 \mathrm{a}_3)\exp(-\alpha_2 \mathrm{a}_2) = \begin{pmatrix} 1 & \alpha_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \alpha_3 & 1 \end{pmatrix}\begin{pmatrix} e^{\alpha_2/2} & 0 \\ 0 & e^{-\alpha_2/2} \end{pmatrix},$$

from which we find that $\alpha_3 = \gamma\delta$ and $\alpha_1 = \beta/\delta$. As we know that

$$\Phi(\exp(-\alpha_2 \mathrm{a}_2), (x,v))$$

is the integral curve of the vector field L_2 starting from the point (x,v) parametrised by α_2, it is straightforward to check that

$$(x_1, v_1) \equiv \Phi(\exp(-\alpha_2 \mathrm{a}_2), (x,v)) = (\exp(\alpha_2/2)x, \exp(-\alpha_2/2)v),$$

and in a similar way

$$(x_2, v_2) \equiv \Phi(\exp(\alpha_3 \mathrm{a}_3), (x_1,v_1)) = (x_1, \alpha_3 x_1 + v_1).$$

Finally, we want to obtain $(\bar{x},\bar{v}) = \Phi(\exp(-\alpha_1 \mathrm{a}_1), (x_2,v_2))$, and taking into account that the integral curves of L_1 satisfy

$$\frac{x^3 dv}{k} = \frac{dx}{v} = d\alpha_1, \tag{2.16}$$

it turns out that when $k > 0$ we have $\bar{v}^2 + k/\bar{x}^2 = v_2^2 + k/x_2^2 \equiv \lambda$ with $\lambda > 0$. Using this fact and (2.16) we obtain

$$\frac{k^{1/2} dv}{(\lambda - v^2)^{3/2}} = d\alpha_1,$$

and integrating in v between v_2 and $\bar{v}$ yields

$$\frac{\bar{v}}{(\lambda - \bar{v}^2)^{1/2}} = \alpha_1 \frac{\lambda}{k^{1/2}} + \frac{v_2}{(\lambda - v_2^2)^{1/2}} = \frac{1}{k^{1/2}}(\alpha_1\lambda + v_2|x_2|).$$

As $\kappa = \mathrm{sign}[\bar{v}]$, we see that κ is given by

$$\kappa = \mathrm{sign}[\alpha_1\lambda + v_2|x_2|] = \mathrm{sign}\left[\frac{\beta}{\delta}(x\gamma + v\delta)^2 + \frac{k\delta\beta}{x^2} + \frac{|x|}{\delta}(v\delta + x\gamma)\right].$$

System (2.15) has no nontrivial first integrals independent of $\omega(t)$, i.e. there is no function $I : U \subset \mathrm{T}\mathbb{R}_+ \to \mathbb{R}$ such that $X_t I = 0$ for X determined by any function $\omega(t)$. This is equivalent to $dI(L_\alpha) = 0$ on an open U, with $\alpha = 1,2,3$. Thus, the first integrals we are looking for are such that dI_p is incident to the involutive distribution $\mathcal{V}_p \simeq \langle (L_1)_p, (L_2)_p, (L_3)_p \rangle$ generated by the fundamental vector fields L_α in U. At almost every point we obtain $\mathcal{V}_p = \mathrm{T}_p\mathrm{T}\mathbb{R}_+$. Then, as $dI_p = 0$ at a generic point $p \in U \subset \mathrm{T}\mathbb{R}_+$, the only possibility is $dI = 0$ and therefore I is a constant first integral.

2.4. A new superposition rule for the Milne–Pinney equation. Our aim now is to show that there exists a superposition rule for the Milne–Pinney equation (2.15) for the case $k > 0$ [53, 163, 182] in terms of a pair of its particular solutions [44]. The case $k < 0$ is analogous.

In fact, one sees from the first integral (2.14) that in the particular case of $f = g = k$, if a particular solution x_1 is known, there is a t-dependent constant of motion for the Milne–Pinney equation given by (see e.g. [53])

$$I_1 = (x_1\dot{x} - \dot{x}_1 x)^2 + k\left[\left(\frac{x}{x_1}\right)^2 + \left(\frac{x_1}{x}\right)^2\right]. \tag{2.17}$$

If another particular solution x_2 of the equation (2.15) is given, then we have another t-dependent constant of motion

$$I_2 = (x_2\dot{x} - \dot{x}_2 x)^2 + k\left[\left(\frac{x}{x_2}\right)^2 + \left(\frac{x_2}{x}\right)^2\right]. \tag{2.18}$$

Moreover, the two solutions x_1 and x_2 provide a function of t which is a constant of motion and generalises the Wronskian W of two solutions of (2.15),

$$I_3 = (x_1\dot{x}_2 - x_2\dot{x}_1)^2 + k\left[\left(\frac{x_2}{x_1}\right)^2 + \left(\frac{x_1}{x_2}\right)^2\right]. \tag{2.19}$$

Remark that for any real number α the inequality $(\alpha - 1/\alpha)^2 \geq 0$ implies

$$\alpha^2 + \frac{1}{\alpha^2} \geq 2,$$

with

$$\alpha^2 + \frac{1}{\alpha^2} = 2 \;\Leftrightarrow\; |\alpha| = 1.$$

Therefore, as we have assumed $k > 0$, we see that $I_i \geq 2k$ for $i = 1, 2, 3$. Moreover, as $x_1(t)$ and $x_2(t)$ are different solutions of the Milne–Pinney equation, it turns out that $I_3 > 2k$.

The knowledge of the two first integrals I_1 and I_2, together with the constant value of I_3 for a pair of solutions of (2.15), can be used to obtain a superposition rule for the Milne–Pinney equation. In fact, given two particular solutions x_1 and x_2, the first integral (2.18) allows us to write an explicit expression for $\dot{x}$ in terms of x, x_2 and I_2,

$$\dot{x} = \dot{x}_2\frac{x}{x_2} \pm \sqrt{-k\frac{x^2}{x_2^4} + I_2\frac{1}{x_2^2} - k\frac{1}{x^2}},$$

and using such an expression with the first integral (2.17), we see, after a careful computation, that x satisfies the fourth degree equation

$$\begin{aligned}(I_2^2 - 4k^2)x_1^4 - 2(I_1I_2 - 2I_3k)x_1^2x_2^2 + (I_1^2 - 4k^2)x_2^4 \\ - 2((I_2I_3 - 2I_1k)x_1^2 + (I_1I_3 - 2I_2k)x_2^2)x^2 + (I_3^2 - 4k^2)x^4 = 0,\end{aligned} \tag{2.20}$$

where we have used that I_3 is constant along pairs of solutions $x_1(t)$, $x_2(t)$ of the Milne–Pinney equation.

Hence, we can obtain from (2.20) the expression for the square of the solutions of the Milne–Pinney equation in terms of any pair of its particular positive solutions by means of the superposition rule

$$x^2 = k_1 x_1^2 + k_2 x_2^2 \pm 2\sqrt{\lambda_{12}[-k(x_1^4 + x_2^4) + I_3 x_1^2 x_2^2]}, \tag{2.21}$$

where the constants k_1 and k_2 are given by

$$k_1 = \frac{I_2 I_3 - 2I_1 k}{I_3^2 - 4k^2}, \qquad k_2 = \frac{I_1 I_3 - 2I_2 k}{I_3^2 - 4k^2},$$

and λ_{12} is the constant

$$\lambda_{12} = \lambda_{12}(k_1, k_2; I_3, k) = \frac{k_1 k_2 I_3 + k(-1 + k_1^2 + k_2^2)}{I_3^2 - 4k^2} = \varphi(I_1, I_2; I_3, k),$$

where the function φ is given by

$$\varphi(I_1, I_2; I_3, k) = \frac{I_1 I_2 I_3 - (I_1^2 + I_2^2 + I_3^2)k + 4k^3}{(I_3^2 - 4k^2)^2}.$$

It is important to remark that if $k_1 < 0$ then $k_2 > 0$. Indeed if $k_1 < 0$ then $I_2 I_3 < 2I_1 k$, and thus $I_2 < 2kI_1/I_3$. Therefore, $\lambda_2(I_3^2 - 4k^2) = I_1 I_3 - 2kI_2 > I_1 I_3 - 4k^2 I_1/I_3 = I_1(I_3^2 - 4k^2) > 0$, and thus, as $I_3 > 2k$, $k_2 > 0$. Similarly $k_2 < 0$ implies $k_1 > 0$.

The parity invariance of (2.15) is displayed by (2.21), which gives us the solutions

$$x^2 = k_1 x_1^2 + k_2 x_2^2 \pm 2\sqrt{\lambda_{12}[-k(x_1^4 + x_2^4) + I_3 x_1^2 x_2^2]}. \tag{2.22}$$

In order to ensure that the right-hand term of the above formula is positive, which gives rise to a real solution of the Milne–Pinney equation, the constants k_1 and k_2 in the preceding expression should satisfy some additional restrictions. In particular, they must obey

$$\lambda_{12}[-k(x_1^4(0) + x_2^4(0)) + I_3 x_1^2(0) x_2^2(0)] \geq 0$$

and

$$k_1 x_1^2(0) + k_2 x_2^2(0) \pm 2\sqrt{\lambda_{12}[-k(x_1^4(0) + x_2^4(0)) + I_3 x_1^2(0) x_2^2(0)]} > 0.$$

If these conditions are satisfied, then, differentiating expression (2.22) at $t = 0$ for $x_1 = x_1(t)$ and $x_2 = x_2(t)$ solutions of the Milne–Pinney equation (2.15), it can be checked that $\dot{x}(0)$ is also a real constant. As $x(t)$ is a solution with real initial conditions, $x(t)$ given by (2.22) is real in an interval of t and thus all the conditions obtained are valid in an interval of t.

If we take into account that we have considered $x_2 > 0$, we can simplify the study of such restrictions by writing (2.22) in terms of the variables x_2 and $z = (x_1/x_2)^2$ as

$$x^2 = x_2^2(k_1 z + k_2 \pm 2\sqrt{\lambda_{12}[-k(z^2 + 1) + I_3 z]}),$$

and the preceding conditions turn out to be $\lambda_{12}[-k(z^2 + 1) + I_3 z] \geq 0$ and $k_1 z + k_2 \pm 2\sqrt{\lambda_{12}[-k(z^2 + 1) + I_3 z]} > 0$.

Next, in order to get $\lambda_{12}[-k(z^2 + 1) + I_3 z] \geq 0$, we first notice that this expression is not definite because its discriminant is $\lambda_{12}^2(I_3^2 - 4k^2) \geq 0$, and this restricts the possible values of k_1 and k_2 for a given z. To see this we define the polynomial

$$P(z) = -k(z^2+1) + I_3 z,$$

with roots

$$z = z_\pm = \frac{I_3 \pm \sqrt{I_3^2 - 4k^2}}{2k},$$

which can be written in terms of the variable $\alpha_3 = I_3/2k$ as

$$z_\pm = \alpha_3 \pm \sqrt{\alpha_3^2 - 1}.$$

As $\alpha_3 > 1$, we have $\alpha_3 > \sqrt{\alpha_3^2 - 1} > 0$ and thus $z_\pm > 0$. The sign of the polynomial $P(z)$ is displayed in Fig. 1.

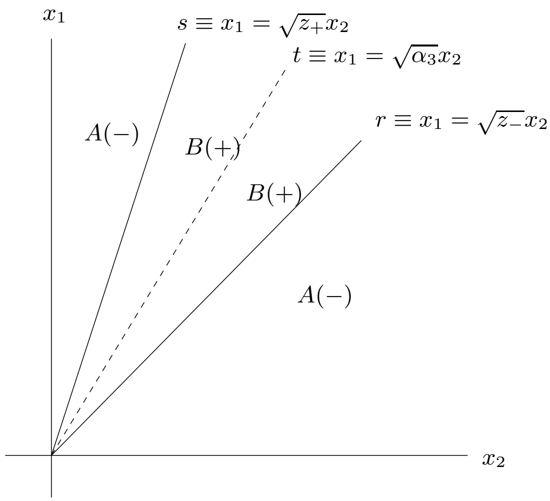

Fig. 1. Sign of the polynomial $P(x_1, x_2)$.

The region $\mathbb{R}_+ \times \mathbb{R}_+$ splits into three regions,

$$A = \{(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid x_1 > \sqrt{z_+}x_2\} \cup \{(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid x_1 < \sqrt{z_-}x_2\},$$
$$B = \{(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid \sqrt{z_-}x_2 < x_1 < \sqrt{z_+}x_2\}$$

separated by the union

$$C = \{(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid x_1 = \sqrt{z_+}x_2\} \cup \{(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid x_1 = \sqrt{z_-}x_2\}$$

of the straight lines $x_1 = \sqrt{z_+}x_2$ and $x_1 = \sqrt{z_-}x_2$. To make $\lambda_{12}P(z)$ nonnegative in region A, where the polynomial P takes negative values, we have to choose k_1 and k_2 so that $\lambda_{12}(k_1, k_2, I_3, k) \leq 0$. Similarly, as P is positive in region B we have to choose k_1 and k_2 such that $\lambda_{12}(k_1, k_2, I_3, k) \geq 0$. Finally, as P vanishes in region C, there is no restriction on the coefficients k_1 and k_2.

Once we have stated the conditions for $\lambda_{12}P(z)$ to be nonnegative we still have to impose the condition

$$k_1 z + k_2 \pm 2\sqrt{\lambda_{12}[-k(z^2+1) + I_3 z]} > 0. \tag{2.23}$$

In order to study these conditions, we study the sign of the polynomial

$$P_{I_3,k}(z,k_1,k_2)=(k_1z+k_2)^2-4\lambda_{12}[-k(z^2+1)+I_3z]=\frac{4P(z)I_3}{I_3^2-4k^2}+(ak_1+bk_2)^2,$$

where

$$a=\sqrt{-\frac{4P(z)k}{I_3^2-4k^2}+z^2},\qquad b=\sqrt{1-\frac{4P(z)k}{I_3^2-4k^2}}.$$

As we remarked before, the constants k_1,k_2 cannot be both negative. Let K denote the set

$$K=\mathbb{R}^2-\{(k_1,k_2)\in\mathbb{R}^2\mid k_1<0,k_2<0\}$$

and consider three cases:

1. If $(x_1,x_2)\in A$, then as $P(z)\leq 0$, we must have $\lambda_{12}\leq 0$ in order to satisfy $\lambda_{12}P(z)\geq 0$. In this case, set

$$K_1=\left\{(k_1,k_2)\in K\;\middle|\;\sqrt{-\frac{4P(z)I_3}{I_3^2-4k^2}}>|ak_1+bk_2|\right\},$$

$$K_2=\left\{(k_1,k_2)\in K\;\middle|\;\sqrt{-\frac{4P(z)I_3}{I_3^2-4k^2}}<|ak_1+bk_2|\right\}.$$

   We find the following particular cases:

   (a) If $(k_1,k_2)\in K_1$, then $P_{I_3,k}(z,k_1,k_2)>0$.
   (b) If $(k_1,k_2)\in K_2$, then $P_{I_3,k}(z,k_1,k_2)<0$.

   They can be summarised by means of Figure 2.

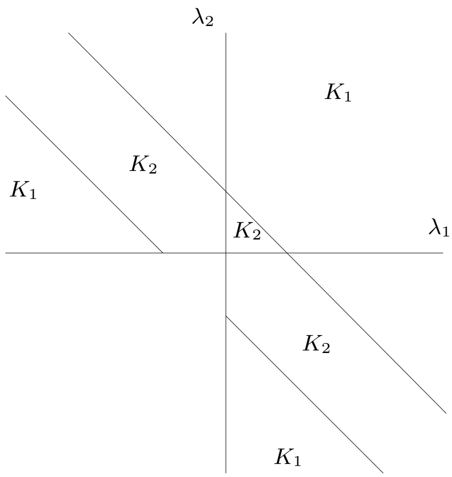

Fig. 2. Sign of the polynomial $P_{I_3,k}(z,k_1,k_2)$ in K.

2. If $(x_1,x_2)\in B$, as $P(z)$ is positive, then λ_{12} must also be positive, $\lambda_{12}\geq 0$. Thus for $(k_1,k_2)\in K_1\cup K_2$, $P_{I_3,k}(z,k_1,k_2)>0$.
3. If $(x_1,x_2)\in C$, then for $(k_1,k_2)\in K_1\cup K_2$, $P_{I_3,k}(z,k_1,k_2)>0$.

In those cases in which $P_{I_3,k}(z,k_1,k_2)>0$, we can assert that

$$|k_1z+k_2|>2\sqrt{\lambda_{12}[-k(z^2+1)+I_3z]}$$

but we still have to impose that $\lambda_1 z+\lambda_2>0$ for (2.23) to be positive. Nevertheless, this is very simple, because if the pair (k_1,k_2) does not satisfy $k_1z+k_2>0$, the pair of opposite elements $(-k_1,-k_2)$ does it, while the other conditions are invariant under the change $k_i\to -k_i$ with $i=1,2$.

In those cases in which $P_{I_3,k}(z,k_1,k_2)<0$ we can assert that

$$|k_1z+k_2|<2\sqrt{\lambda_{12}[-k(x_1^4+x_2^4)+I_3x_1^2x_2^2]}$$

and in this case the unique valid superposition rule is

$$x=|x_2|\big(k_1z+k_2+2\sqrt{\lambda_{12}[-k(z^2+1)+I_3z]}\big)^{1/2},$$

which is equivalent to

$$x=\big(k_1x_1^2+k_2x_2^2+2\sqrt{\lambda_{12}[-k(x_1^4+x_2^4)+I_3x_2^2x_1^2]}\big)^{1/2}.$$

Note that if we had considered no restriction on k_1,k_2, we would have obtained real and imaginary solutions of the Milne–Pinney equation.

Expression (2.22) provides us with a superposition rule for the positive solutions of the Pinney equation (2.15) in terms of two of its independent particular positive solutions. Therefore, once two particular solutions of the equation (2.15) are known, we can write its general solution. Note also that, because of the parity symmetry of (2.15), the superposition (2.22) can be used with both positive and negative solutions. In all these ways we obtain nonvanishing solutions of (2.15) when $k>0$. *Mutatis mutandis*, the above procedure can also be applied to analyse Milne–Pinney equations when $k<0$.

A similar superposition rule works for negative solutions of Milne–Pinney equation (2.15):

$$x=-\big(k_1x_1^2+k_2x_2^2\pm 2\sqrt{\lambda_{12}(-k(x_1^4+x_2^4)+I_3x_1^2x_2^2)}\big)^{1/2}, \tag{2.24}$$

where once again x_1 and x_2 are arbitrary solutions.

2.5. Painlevé–Ince equations and other SODE Lie systems. In this section we show a new relevant instance of SODE Lie systems including, as particular instances, some Painlevé–Ince equations [93]. In the process of analysing that this particular case of Painlevé–Ince is a SODE Lie system, we find a much larger family of SODE Lie systems which frequently occur in the mathematical and physical literature.

Consider the family of differential equations

$$\ddot{x}+3x\dot{x}+x^3=f(t), \tag{2.25}$$

with $f(t)$ being any t-dependent function. The interest in these equations is motivated by their frequent appearance in physics and mathematics [66, 71, 134]. The properties of these equations have been deeply analysed since their first analysis by Vessiot and Wallenberg [224, 229] as a particular case of second-order Riccati equations. For instance, these equations appear in [106] in the study of the Riccati chain. There, it is stated that such equations can be used to derive solutions for certain PDEs. In addition, equation

(2.25) also appears in the book by Davis [86], and the particular case with $f(t) = 0$ has recently been treated through geometric methods in [41, 66].

The results described in previous sections can be used to study differential equations (2.25). Let us first show that the above differential equations are SODE Lie systems and, in view of Proposition 1.24, they admit a superposition rule that is derived. According to Definition 1.23, equation (2.25) is a SODE Lie system if and only if the system

$$\begin{cases} \dot{x} = v, \\ \dot{v} = -3xv - x^3 + f(t), \end{cases} \tag{2.26}$$

determining the integral curves of the t-dependent vector field

$$X_{PI}(t, x, v) = X_1(x, v) + f(t)X_2(x, v), \tag{2.27}$$

with

$$X_1 = v\frac{\partial}{\partial x} - (3xv + x^3)\frac{\partial}{\partial v}, \qquad X_2 = \frac{\partial}{\partial v},$$

is a Lie system.

In view of the decomposition (2.27), all equations (2.25) are SODE Lie systems if the vector fields X_1 and X_2 are included in a finite-dimensional real Lie algebra V of vector fields. This happens if and only if $\mathrm{Lie}(\{X_1, X_2\})$ is a finite-dimensional linear space. We consider the family of vector fields on $\mathrm{T}\mathbb{R}$ given by

$$\begin{aligned}
&X_1 = v\frac{\partial}{\partial x} - (3xv + x^3)\frac{\partial}{\partial v}, && X_2 = \frac{\partial}{\partial v}, \\
&X_3 = -\frac{\partial}{\partial x} + 3x\frac{\partial}{\partial v}, && X_4 = x\frac{\partial}{\partial x} - 2x^2\frac{\partial}{\partial v}, \\
&X_5 = (v + 2x^2)\frac{\partial}{\partial x} - x(v + 3x^2)\frac{\partial}{\partial v}, && X_6 = 2x(v + x^2)\frac{\partial}{\partial x} + 2(v^2 - x^4)\frac{\partial}{\partial v}, \\
&X_7 = \frac{\partial}{\partial x} - x\frac{\partial}{\partial v}, && X_8 = 2x\frac{\partial}{\partial x} + 4v\frac{\partial}{\partial v}.
\end{aligned} \tag{2.28}$$

where $X_3 = [X_1, X_2]$, $-3X_4 = [X_1, X_3]$, $X_5 = [X_1, X_4]$, $X_6 = [X_1, X_5]$, $X_7 = [X_2, X_5]$, $X_8 = [X_2, X_6]$. The vector fields $X_1, \ldots, X_8$ are linearly independent over $\mathbb{R}$. Their commutation relations read

$$\begin{array}{llll}
[X_1, X_2] = X_3, & [X_1, X_3] = -3X_4, & [X_1, X_4] = X_5, & [X_1, X_5] = X_6, \\
[X_1, X_6] = 0, & [X_1, X_7] = \frac{1}{2}X_8, & [X_1, X_8] = -2X_1, & [X_2, X_3] = 0, \\
[X_2, X_4] = 0, & [X_2, X_5] = X_7, & [X_2, X_6] = X_8, & [X_2, X_7] = 0, \\
[X_2, X_8] = 4X_2, & [X_3, X_4] = -X_7, & [X_3, X_5] = -\frac{1}{2}X_8, & [X_3, X_6] = -2X_1, \\
[X_3, X_7] = -2X_2, & [X_3, X_8] = 2X_3, & [X_4, X_5] = -X_1, & [X_4, X_6] = 0, \\
[X_4, X_7] = X_3, & [X_4, X_8] = 0, & [X_5, X_6] = 0, & [X_5, X_7] = -3X_4, \\
[X_5, X_8] = -2X_5, & [X_6, X_7] = -2X_5, & [X_6, X_8] = -4X_6, & [X_7, X_8] = 2X_7.
\end{array} \tag{2.29}$$

In other words, the vector fields $X_1, \ldots, X_8$ span an eight-dimensional Lie algebra V of vector fields containing X_1 and X_2. Therefore, equation (2.25) is a SODE Lie system.

Moreover, the traceless real 3×3 matrices

$$M_1 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0. \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0. \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0. \end{pmatrix}, \qquad M_4 = -\frac{1}{3}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1. \end{pmatrix},$$

$$M_5 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0. \end{pmatrix}, \qquad M_6 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0. \end{pmatrix},$$

$$M_7 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0. \end{pmatrix}, \qquad M_8 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2. \end{pmatrix}$$

obey the same commutation relations as $X_1, \ldots, X_8$, i.e. the linear map $\rho : \mathfrak{sl}(3, \mathbb{R}) \to V$ such that $\rho(M_\alpha) = X_\alpha$ with $\alpha = 1, \ldots, 8$ is a Lie algebra isomorphism. Consequently, the systems of differential equations describing the integral curves for the t-dependent vector fields

$$X(t, x, v) = \sum_{\alpha=1}^{8} b_\alpha(t) X_\alpha(x, v), \tag{2.30}$$

are Lie systems related to a Vessiot–Guldberg Lie algebra isomorphic to $\mathfrak{sl}(3, \mathbb{R})$.

Many instances of the family of Lie systems (2.30) are associated with interesting SODE Lie systems with applications in physics or related to remarkable mathematical problems. In all these cases, the theory of Lie systems can be applied to investigate these second-order differential equations, recover some of their known properties, and, possibly, provide new results. Let us illustrate this by means of a few examples.

Another equation appearing in the physics literature [71, 72, 218] which can be analysed by means of our methods is

$$\ddot{x} + 3x\dot{x} + x^3 + \lambda_1 x = 0, \tag{2.31}$$

which is a special kind of the Liénard equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$, with $f(x) = 3x$ and $g(x) = x^3 + \lambda_1 x$. The above equation can also be related to a generalised form of an Emden equation occurring in the thermodynamical study of equilibrium configurations of spherical clouds of gas acting under the mutual attraction of their molecules [88].

As in the study of (2.25), by considering the new variable $v = \dot{x}$, equation (2.31) becomes the system

$$\begin{cases} \dot{x} = v, \\ \dot{v} = -3xv - x^3 - \lambda_1 x, \end{cases} \tag{2.32}$$

describing the integral curves of the vector field $X = X_1 - \lambda_1/2(X_7 + X_3)$ included in the family (6.14).

Finally, we can also treat the equation

$$\ddot{x} + 3x\dot{x} + x^3 + f(t)(\dot{x} + x^2) + g(t)x + h(t) = 0, \tag{2.33}$$

embracing, as particular cases, all the previous examples [134]. The system of first-order differential equations associated with this equation reads

$$\begin{cases} \dot{x} = v, \\ \dot{v} = -3xv - x^3 - f(t)(v + x^2) - g(t)x - h(t). \end{cases} \tag{2.34}$$

Hence, this system describes the integral curves of the t-dependent vector field

$$X_t = X_1 - h(t)X_2 - \frac{1}{4}f(t)(X_8 - 2X_4) - \frac{1}{2}g(t)(X_7 + X_3).$$

Therefore, equation (2.33) is a SODE Lie system and the theory of Lie systems can be used to analyse its properties.

Some particular cases of system (2.33) were pointed out in [72, 134]. Additionally, the case of $f(t) = 0$, $g(t) = \omega^2(t)$ and $h(t) = 0$ was studied in [71] and it is related to harmonic oscillators. The case of $g(t) = 0$ and $h(t) = 0$ appears in the catalogue of equations possessing the Painlevé property [126]. Additionally, our result generalises Vessiot's contribution [225] describing the existence of an expression determining the general solution of a system like (2.33) (but with constant coefficients) in terms of four of their particular solutions, their derivatives and two constants.

Finally, it is worth noting that the second-order differential equation (2.33) is a particular case of second-order Riccati equations [66, 106]. Such equations were analysed through Lie systems in [77]. The approach carried out in that paper is based on the use of certain *ad hoc* changes of variables which transform second-order Riccati equations into some Lie systems. The advantage of our approach here is that it allows us to study equations (2.33) without using such transformations. In addition, our presentation along with the theory of quasi-Lie schemes can be used to perform a quite complete study of second-order Riccati equations in a systematic way [48].

2.6. Mixed superposition rules and Ermakov systems. Let us now show how the theory developed in Section 1.7 for mixed superposition rules works. By adding some, probably different, Lie systems to an initial one, we get new Lie systems that admit constants of motion which do not depend on the t-dependent coefficients of these systems and relate different solutions of the constitutive Lie systems. Moreover, if we add enough copies, these constants of motion can be used to construct a mixed superposition rule.

We here investigate Ermakov systems. These systems are formed by a second-order homogeneous linear differential equation and a Milne–Pinney equation, i.e.

$$\begin{cases} \ddot{x} = -\omega^2(t)x + \dfrac{k}{x^3}, \\ \ddot{y} = -\omega^2(t)y, \end{cases} \qquad (x, y) \in \mathbb{R}^2_+.$$

These systems have been widely studied in physics and mathematics since their introduction until the present day. In physics they appear in the study of Bose–Einstein condensates and cosmological models [109, 115, 152] and in the solution of t-dependent harmonic or anharmonic oscillators [87, 96, 101, 150, 192, 204]. A lot of works have also been devoted to the usage of Hamiltonian or Lagrangian structures in the study of such systems (see e.g. [194]). Here we recover a constant of motion, the so-called *Lewis–Ermakov invariant* [150], which appears naturally.

In order to use the theory of Lie systems to analyse Ermakov systems, consider the system of ordinary first-order differential equations [87, 146]

$$\begin{cases} \dot{x} = v_x, \\ \dot{y} = v_y, \\ \dot{v}_x = -\omega^2(t)x + \dfrac{k}{x^3}, \\ \dot{v}_y = -\omega^2(t)y, \end{cases} \tag{2.35}$$

defined over $\mathrm{T}\mathbb{R}^2_+$ and built by adding the new variables $\dot{x} = v_x$ and $v_y = \dot{y}$ to the Ermakov systems and satisfying the conditions explained in Section 1.7. Its solutions are the integral curves for the t-dependent vector field

$$X_t = v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + \left(-\omega^2(t)x + \frac{k}{x^3}\right)\frac{\partial}{\partial v_x} - \omega^2(t)y\frac{\partial}{\partial v_y},$$

which is a linear combination with t-dependent coefficients, $X_t = X_1 + \omega^2(t)X_3$, of

$$X_1 = v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + \frac{k}{x^3}\frac{\partial}{\partial v_x}, \qquad X_3 = -x\frac{\partial}{\partial v_x} - y\frac{\partial}{\partial v_y}.$$

Taking into account the vector field

$$X_2 = \frac{1}{2}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - v_x\frac{\partial}{\partial v_x} - v_y\frac{\partial}{\partial v_y}\right),$$

the vector fields X_1, X_2 and X_3 span a three-dimensional Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. In this way, this system is a SODE Lie system related to a Lie algebra of vector fields isomorphic to $\mathfrak{sl}(2,\mathbb{R})$.

The vector fields L_1, L_2, L_3 associated with the Milne–Pinney equation (see Section 2.3) span a distribution of rank two on $\mathrm{T}\mathbb{R}_+$. Consequently, there is no local first integral I such that $(L_1 + \omega(t)^2(t)L_2)I = 0$ for any given $\omega(t)$. In other words, Milne–Pinney equations do not admit a common t-independent constant of motion.

By adding the other $\mathfrak{sl}(2,\mathbb{R})$ linear Lie system appearing in the Ermakov system, i.e. the harmonic oscillator with t-dependent angular frequency $\omega(t)$, the distribution spanned by X_1, X_2 and X_3 has rank three over a dense open subset of $\mathrm{T}\mathbb{R}^2_+$. Therefore, there is a local first integral. It can be obtained from $X_1F = X_3F = 0$. But $X_3F = 0$ implies that there exists a function $\bar{F} : \mathbb{R}^3 \to \mathbb{R}$ such that $F(x, y, v_x, v_y) = \bar{F}(x, y, \xi)$ with $\xi = yv_x - xv_y$, and then $X_1F = 0$ is written

$$v_x\frac{\partial\bar{F}}{\partial x} + v_y\frac{\partial\bar{F}}{\partial y} + k\frac{y}{x^3}\frac{\partial\bar{F}}{\partial\xi}$$

and we obtain the associated system of characteristics

$$k\frac{y\,dx - x\,dy}{\xi} = \frac{x^3\,d\xi}{y} \;\Rightarrow\; \frac{d(y/x)}{\xi} + \frac{x\,d\xi}{ky} = 0.$$

Hence, the following first integral is found [150]:

$$\psi(x,y,v_x,v_y) = k\left(\frac{y}{x}\right)^2 + \xi^2 = k\left(\frac{y}{x}\right)^2 + (yv_x - xv_y)^2,$$

which is the well-known Ermakov–Lewis invariant [87, 146, 192].

Once we have obtained a first integral, we can obtain new constants by adding new copies of any of the systems we have already used. For instance, consider the system of first-order differential equations

$$\begin{cases} \dot{x} = v_x, \\ \dot{y} = v_y, \\ \dot{z} = v_z, \\ \dot{v}_x = -\omega^2(t)x + \dfrac{k}{x^3}, \\ \dot{v}_y = -\omega^2(t)y, \\ \dot{v}_z = -\omega^2(t)z, \end{cases} \tag{2.36}$$

which corresponds to the vector field

$$X_t = v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z} + \frac{k}{x^3}\frac{\partial}{\partial v_x} - \omega^2(t)\left(x\frac{\partial}{\partial v_x} + y\frac{\partial}{\partial v_y} + z\frac{\partial}{\partial v_z}\right).$$

The t-dependent vector field X_t can be expressed as $X_t = N_1 + \omega^2(t)N_3$ where

$$N_1 = v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z} + \frac{k}{x^3}\frac{\partial}{\partial v_x}, \qquad N_3 = -x\frac{\partial}{\partial v_x} - y\frac{\partial}{\partial v_y} - z\frac{\partial}{\partial v_z}.$$

These vector fields generate a three-dimensional real Lie algebra together with the vector field

$$N_2 = \frac{1}{2}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} - v_x\frac{\partial}{\partial v_x} - v_y\frac{\partial}{\partial v_y} - v_z\frac{\partial}{\partial v_z}\right).$$

In fact, they span a Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ because

$$[N_1, N_3] = 2N_2, \qquad [N_1, N_2] = N_1, \qquad [N_2, N_3] = N_3.$$

The distribution spanned by these fundamental vector fields has rank three in an open dense subset of $\mathrm{T}\mathbb{R}^3_+$. Thus, there exist three local first integrals for all the vector fields of the latter distribution. In other words, system (2.36) admits three t-independent constants of motion which turn out to be the Ermakov invariant I_1 of the subsystem involving the variables x and y, the Ermakov invariant I_2 of the subsystem involving x and z, i.e.

$$I_1 = \frac{1}{2}\left((yv_x - xv_y)^2 + k\left(\frac{y}{x}\right)^2\right), \qquad I_2 = \frac{1}{2}\left((xv_z - zv_x)^2 + k\left(\frac{z}{x}\right)^2\right),$$

and the Wronskian $W = yv_z - zv_y$ of the subsystem involving y and z. They define a foliation with three-dimensional leaves. We can use this foliation to obtain a superposition rule. To do this we describe x in terms of y, z and the integrals I_1, I_2, W, i.e.

$$x = \frac{\sqrt{2}}{|W|}(I_2y^2 + I_1z^2 \pm \sqrt{4I_1I_2 - kW^2}yz)^{1/2}. \tag{2.37}$$

This can be interpreted, as pointed out by Pinney [182], as saying that there is a superposition rule allowing us to express the general solution of the Milne–Pinney equation in terms of two independent solutions of the corresponding harmonic oscillator with the same t-dependent angular frequency.

2.7. Relations between the new and the known superposition rule. We can now compare the known superposition rule for the Milne–Pinney equation

$$x(t) = \frac{\sqrt{2}}{|W|}(I_2 y_1^2(t) + I_1 y_2^2(t) \pm \sqrt{4I_1 I_2 - kW^2}\; y_1(t)y_2(t))^{1/2}, \tag{2.38}$$

where $y_1(t)$ and $y_2(t)$ are two independent solutions of

$$\ddot{y} = -\omega^2(t)y, \tag{2.39}$$

and (2.22) and check that actually the latter reduces to the former when x_1 and x_2 are obtained from solutions y_1 and y_2 of the associated harmonic oscillator equation.

Let y_1 and y_2 be two solutions of (2.39) and W their Wronskian. Consider the two particular positive solutions of the Milne–Pinney equation given by

$$x_1(t) = \frac{\sqrt{2}}{|\,W\,|}\sqrt{C_1 y_1^2(t) + C_2 y_2^2(t)}, \qquad x_2(t) = \frac{\sqrt{2}}{|\,W\,|}\sqrt{C_2 y_1^2(t) + C_1 y_2^2(t)}, \tag{2.40}$$

where $C_1 < C_2$ and we additionally impose

$$4C_1C_2 = kW^2. \tag{2.41}$$

The t-dependent constant of motion I_3 given by (2.19) for the two particular solutions of the Milne–Pinney equation can then be expressed as a function of the solutions y_1 and y_2 of the t-dependent harmonic oscillator and their Wronskian W. After a long computation I_3 turns out to be

$$I_3 = \frac{4(C_1^2 + C_2^2)}{W^2}, \tag{2.42}$$

and then using the explicit form (2.40) of the particular solutions and taking into account the constant (2.42) in (2.22) we obtain

$$\begin{aligned} k_1x_1^2 + k_2x_2^2 \pm 2\sqrt{\lambda_{12}(-k(x_1^4 + x_2^4) + I_3x_1^2x_2^2)} &= \frac{2}{W^2}(C_1k_1 + C_2k_2)y_1^2 \\ + (C_1k_2 + C_2k_1)y_2^2) \pm \frac{2}{W^2}&\sqrt{4(C_1k_1 + C_2k_2)(C_1k_2 + C_2k_1) - kW^2}y_1y_2. \end{aligned} \tag{2.43}$$

Consequently, from the superposition rule (2.22), we recover expression (2.37):

$$x = \frac{\sqrt{2}}{|W|}\sqrt{\mu_1 y_1^2 + \mu_2 y_2^2 \pm \sqrt{4\mu_1\mu_2 - kW^2}y_1y_2}, \tag{2.44}$$

where

$$\begin{cases} \mu_1 = C_1k_1 + C_2k_2, \\ \mu_2 = C_1k_2 + C_2k_1. \end{cases}$$

Once we have stated the superposition rule, we still have to analyse the possible values of λ_1 and λ_2 that we can use in this case. If we use the expression (2.42) we obtain after a short calculation the following values $z_\pm$:

$$z_+ = \frac{4C_2^2}{kW^2}, \quad z_- = \frac{4C_1^2}{kW^2}. \tag{2.45}$$

Now if we write y_1^2 and y_2^2 in terms of x_1^2, x_2^2 and W from the system (2.40) we obtain

$$\frac{1}{C_1^2 - C_2^2}\begin{pmatrix} C_1 & -C_2 \\ -C_2 & C_1 \end{pmatrix}\begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix} = \begin{pmatrix} y_1^2 \\ y_2^2 \end{pmatrix}. \tag{2.46}$$

Therefore, as $C_2 > C_1$ the condition of y_1^2 and y_2^2 being positive is

$$\begin{cases} C_1 x_1^2 \leq C_2 x_2^2, \\ C_2 x_1^2 \geq C_1 x_2^2, \end{cases} \tag{2.47}$$

and it is satisfied if $x_1^2/x_2^2 \leq C_2/C_1 = 4C_2^2/kW^2 = z_+$ and $x_1^2/x_2^2 \geq C_1/C_2 = 4C_1^2/kW^2 = z_-$, because of (2.41). Thus, $(x_1, x_2) \in B$ and therefore the only restrictions for k_1, k_2 are $\lambda_{12} \geq 0$ and $k_1 x_1^2 + k_2 x_2^2 \geq 0$. Obviously, by the change of variables (2.40) this last expression is equivalent to $\mu_1 y_1^2 + \mu_2 y_2^2 \geq 0$ and thus μ_1 and μ_2 cannot be simultaneously negative. Furthermore, $\lambda_{12}(I_3^2 - 4k^2) = 4\mu_1\mu_2 - kW^2$. As $\lambda_{12} \geq 0$ we have $4\mu_1\mu_2 \geq kW^2$, i.e. $\mu_1\mu_2$ is positive and thus, μ_1 and μ_2 are positive. In this way we recover the usual constants of the known superposition rule of the Milne–Pinney equation in terms of solutions of a harmonic oscillator.

2.8. A new mixed superposition rule for the Pinney equation. In this section we derive a mixed superposition rule for the Milne–Pinney equation in terms of a Riccati equation. Consider again the t-dependent Riccati equation

$$\frac{dx}{dt} = b_1(t) + b_2(t)x + b_3(t)x^2 \tag{2.48}$$

which has been studied in [50, 63] from the perspective of the theory of Lie systems. We have already mentioned that it can be considered as the differential equation determining the integral curves for the t-dependent vector field (1.25). This vector field is a linear combination with t-dependent coefficients of the vector fields X_1, X_2, X_3 given by (1.26), which generate a three-dimensional real Lie algebra with defining relations (1.27). Consequently, this Lie algebra is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Note also that the commutation relations (1.27) are the same as (2.3).

Take now the following particular case of the Riccati equation:

$$\frac{dx}{dt} = 1 + \omega^2(t)x^2.$$

This is the equation of the integral curves of the t-dependent vector field $X_t = X_1 + \omega^2(t)X_3$. Thus, we can apply the procedure of Section 1.7 and consider the following differential equation in $\mathbb{R}^3 \times \mathrm{T}\mathbb{R}_+$:

$$\begin{cases} \dot{x}_1 = 1 + \omega^2(t)x_1^2, \\ \dot{x}_2 = 1 + \omega^2(t)x_2^2, \\ \dot{x}_3 = 1 + \omega^2(t)x_3^2, \\ \dot{x} = v, \\ \dot{v} = -\omega^2(t)x + \dfrac{k}{x^3}, \end{cases}$$

where $(x_1, x_2, x_3) \in \mathbb{R}^3$, $x \in \mathbb{R}_+$ and $(x, v) \in T_x\mathbb{R}_+$. According to our general recipe, consider the vector fields

$$\begin{aligned} M_1 &= \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + v\frac{\partial}{\partial x} + \frac{k}{x^3}\frac{\partial}{\partial v}, \\ M_2 &= x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3} + \frac{1}{2}\left(x\frac{\partial}{\partial x} - v\frac{\partial}{\partial v}\right), \\ M_3 &= x_1^2\frac{\partial}{\partial x_1} + x_2^2\frac{\partial}{\partial x_2} + x_3^3\frac{\partial}{\partial x_3} - x\frac{\partial}{\partial v}, \end{aligned}$$

which, by construction, satisfy the same commutation relations as before, i.e.

$$[M_1, M_3] = 2M_2, \qquad [M_1, M_2] = M_1, \qquad [M_2, M_3] = M_3,$$

and the full system of differential equations can be viewed as the system of differential equations for the determination of the integral curves of the t-dependent vector field $M(t) = M_1 + \omega^2(t)M_3$. The distribution associated with this Lie system has rank three at almost every point and so there exist locally two first integrals. As $2M_2 = [M_1, M_3]$, it is enough to find a common first integral for M_1 and M_3, i.e. a function $F : \mathbb{R}^5 \to \mathbb{R}$ such that $M_1F = M_3F = 0$.

We first look for first integrals independent of x_3, i.e. we suppose that F depends just on x_1, x_2, x and v. Using the method of characteristics, the condition $M_3F = 0$ implies that the characteristics system is

$$\frac{dx_1}{x_1^2} = \frac{dx_2}{x_2^2} = \frac{dv}{-x} = \frac{dx}{0},$$

That means that for a first integral for M_3 which depends on x_1, x_2, x and v, there is a function $\bar{F} : \mathbb{R}^3 \to \mathbb{R}$ such that $F(x_1, x_2, x, v) = \bar{F}(I_1, I_2, I_3)$, with I_1, I_2 and I_3 given by

$$I_1 = \frac{1}{x_1} - \frac{1}{x_2}, \qquad I_2 = \frac{1}{x_1} - \frac{v}{x}, \qquad I_3 = x.$$

Now, in terms of $\bar{F}$, the condition $M_1F = M_1\bar{F} = 0$ implies

$$v\left(-\frac{2I_1}{I_3}\frac{\partial\bar{F}}{\partial I_1} - \frac{2I_2}{I_3}\frac{\partial\bar{F}}{\partial I_2} + \frac{\partial\bar{F}}{\partial I_3}\right) + (I_1 - 2I_2)I_1\frac{\partial\bar{F}}{\partial I_1} - \left(I_2^2 + \frac{k}{I_3^4}\right)\frac{\partial\bar{F}}{\partial I_2} = 0. \tag{2.49}$$

Thus the linear term in v and the other one must vanish independently. The method of characteristics applied to the first term implies that there exists a map $\widehat{F} : \mathbb{R}^2 \to \mathbb{R}$ such that $\bar{F}(I_1, I_2, I_3) = \widehat{F}(K_1, K_2)$ where

$$K_1 = \frac{I_1}{I_2}, \qquad K_2 = I_2I_3^2.$$

Finally, taking into account the last result in $M_1\hat{F}=0$, we get

$$\left(-K_1^2-K_1+\frac{kK_1}{K_2^2}\right)\frac{\partial\widehat{F}}{\partial K_1}-\left(K_2+\frac{k}{K_2}\right)\frac{\partial\widehat{F}}{\partial K_2}=0,$$

and by the method of characteristics expression (2.49) yields

$$\frac{dK_1}{dK_2}=\frac{K_1^2+K_1-\frac{kK_1}{K_2^2}}{K_2+\frac{k}{K_2}},$$

which gives the first integral

$$C_1=K_2+\frac{k+K_2^2}{K_1K_2},$$

which in the initial variables reads

$$C_1=\left(x_2-\frac{v}{x}\right)x^2+\frac{k+(x_2-\frac{v}{x})^2x^4}{(x_1-x_2)x^2}.$$

If we repeat this procedure with the assumption that the integral does not depend on x_2 we obtain the first integral

$$C_2=\left(x_3-\frac{v}{x}\right)x^2+\frac{k+(x_3-\frac{v}{x})^2x^4}{(x_1-x_3)x^2}.$$

It is a long but easy calculation to check that both are first integrals of M_1, M_2 and M_3. We can now obtain the general solution x of the Milne–Pinney equation in terms of x_1, x_2, x_3, C_1, C_2, as

$$x=\sqrt{\frac{(C_1(x_1-x_2)-C_2(x_1-x_3))^2+k(x_2-x_3)^2}{(C_2-C_1)(x_2-x_3)(x_2-x_1)(x_1-x_3)}},$$

where C_1 and C_2 are constants such that, once $x_1(t)$, $x_2(t)$ and $x_3(t)$ have been fixed, they make $x(0)$ given by the latter expression real.

Thus we have obtained a new mixed superposition rule which enables us to express the general solution of the Pinney equation in terms of three solutions of Riccati equations and, of course, two constants related to initial conditions which determine each particular solution.

3. Applications of quantum Lie systems

In Sections 1.9 and 1.8, it is proved that we can make use of the geometric theory of Lie systems to treat a certain kind of Schrödinger equations, those related to the so-called quantum Lie systems. In this section we use this point of view to investigate quantum mechanics.

First, we develop the geometric theory of reduction for quantum Lie systems. Reduction techniques have already been put into practice to study Lie systems [40, 47, 50, 63]. In these works, a variety of reduction methods and other closely related topics are analysed. Most of these methods are based on the properties of a special type of Lie system in a Lie group associated with the Lie system under study. As quantum Lie systems can also be related to such Lie systems, we can apply most of the methods developed in the aforementioned works to analyse quantum Lie systems. This is the main purpose of the present section.

In detail, we start by analysing the reduction technique for quantum Lie systems and we complete some previous classic achievements. We next show that the interaction picture can be explained from this geometrical point of view in terms of this reduction technique. Furthermore, the method of unitary transformations is analysed from our perspective to exemplify that quantum Lie systems associated with solvable Lie algebras of linear operators, similarly to the classical case, can be exactly solved. On the other hand, systems related to nonsolvable Lie algebras can be solved in particular cases. Both cases can be analysed to reproduce some results on the method of unitary transformations in particular cases found in the literature.

3.1. The reduction method in quantum mechanics. We here review the reduction techniques explained, for example, in [40, 51, 63]. While in some previous works certain sufficient conditions to perform a reduction process were explained [40, 63], here we show that these conditions are also necessary [51]. Additionally, we use the geometric reduction technique to explain the interaction picture used in quantum mechanics and we review, from a geometric point of view, the method of unitary transformations.

In Section 1.3 it was shown that the study of Lie systems can be reduced to that of finding the solution of the equation

$$R_{g^{-1}*g}\dot{g} = -\sum_{\alpha=1}^{r} b_\alpha(t)\mathrm{a}_\alpha \equiv \mathrm{a}(t) \in \mathrm{T}_eG \tag{3.1}$$

with $g(0) = e$.

The reduction method developed in [40] shows that given a solution $\tilde{x}(t)$ of a Lie system on a homogeneous space G/H, the solution of the Lie system in the group G, and therefore the general solution in the given homogeneous space, can be reduced to that of a Lie system in the subgroup H. More specifically, if the curve $\tilde{g}(t)$ in G is such that $\tilde{x}(t) = \Phi(\tilde{g}(t), \tilde{x}(0))$, with Φ being the given action of G in the homogeneous space, then $g(t) = \tilde{g}(t)g'(t)$, where $g'(t)$ turns out to be a curve in H which is a solution of a Lie system in H. Actually, once the curve $\tilde{g}(t)$ in G has been fixed, the curve $g'(t)$, which takes values in H, satisfies the equation [40]

$$R_{g'^{-1}*g'}\dot{g}' = -\mathrm{Ad}(\tilde{g}^{-1})\Big(\sum_{\alpha=1}^{r} b_\alpha(t)\mathrm{a}_\alpha + R_{\tilde{g}^{-1}*\tilde{g}}\dot{\tilde{g}}\Big) \equiv \mathrm{a}'(t) \in \mathrm{T}_eH. \tag{3.2}$$

This transformation law can be understood in the language of the theory of connections. It has been shown in [40, 60] that Lie systems can be related to connections in a bundle and that the group of curves in G, which is the group of automorphisms of the principal bundle $G \times \mathbb{R}$ [60], acts on the left on the set of Lie systems on G, and defines an induced action on the set of Lie systems in each homogeneous space for G. More specifically, if $x(t)$ is a solution of a Lie system in a homogeneous space N defined by the curve $a(t)$ in $\mathfrak{g}$, then for each curve $\bar{g}(t)$ in G such that $\bar{g}(0) = e$ we see that $x'(t) = \Phi(\bar{g}(t), x(t))$ is a solution of the Lie system defined by the curve

$$\mathrm{a}'(t) = R_{\bar{g}^{-1}*\bar{g}}\dot{\bar{g}} + \mathrm{Ad}(\bar{g})\mathrm{a}(t), \tag{3.3}$$

which is the transformation law for a connection.

In conclusion, the aim of the reduction method is to find an automorphism $\bar{g}(t)$ such that the right-hand side in (3.3) belongs to $\mathrm{T}_eH \equiv \mathfrak{h}$ for a certain Lie subgroup H of G. The papers [40, 60] gave a sufficient condition for obtaining this result. In this section we study the above geometrical development in quantum mechanics and we determine a necessary condition for the right-hand side in (3.3) to belong to $\mathfrak{h}$.

Quantum Lie systems are those t-dependent self-adjoint Hamiltonians such that

$$H(t) = \sum_{\alpha=1}^{r} b_\alpha(t) H_\alpha, \tag{3.4}$$

with the iH_α spanning (under the commutator of operators) an r-dimensional real Lie algebra V of skew-self-adjoint operators. Therefore, by regarding these operators as fundamental vector fields of a unitary action of a connected Lie group G with Lie algebra $\mathfrak{g}$ isomorphic to V, we can relate the Schrödinger equation to a differential equation in G determined by curves in T_eG given by $\mathrm{a}(t) = -\sum_{\alpha=1}^{r} b_\alpha(t)\mathrm{a}_\alpha$ by considering $-iH_\alpha$ as fundamental vector fields of the basis of $\mathfrak{g}$ given by $\{\mathrm{a}_\alpha \mid \alpha = 1, \ldots, r\}$.

Now, the preceding methods enable us to transform the problem into a new one in the same group G, for each choice of the curve $\bar{g}(t)$ but with a new curve $\mathrm{a}'(t)$. The action of G on $\mathcal{H}$ is given by a unitary representation U, and therefore the t-dependent vector field determined by the original t-dependent Hamiltonian $H(t)$ becomes a new one with t-dependent Hamiltonian $H'(t)$. Its integral curves are the solutions of the equation

$$\frac{d\psi'}{dt} = -iH'(t)\psi',$$

where

$$-iH'(t) = -iU(\bar{g}(t))H(t)U^\dagger(\bar{g}(t)) + \dot{U}(\bar{g}(t))U^\dagger(\bar{g}(t)).$$

That is, from a geometric point of view, we have related a Lie system on the Lie group G to a certain curve $\mathrm{a}(t)$ in T_eG and the corresponding system in $\mathcal{H}$ determined by a unitary representation of G to another one with a different curve $\mathrm{a}'(t)$ in T_eG and its associated one in $\mathcal{H}$.

Let us choose a basis of T_eG given by $\{c_\alpha \mid \alpha = 1, \ldots, r\}$ with $r = \dim \mathfrak{g}$ such that $\{c_\alpha \mid \alpha = 1, \ldots, s\}$ is a basis of T_eH, where $s = \dim \mathfrak{h}$, and denote by $\{c^\alpha \mid \alpha = 1, \ldots, r\}$ the dual basis of $\{c_\alpha \mid \alpha = 1, \ldots, r\}$. In order to find $\bar{g}$ such that the right-hand term of (3.3) belongs to T_eH for all t, the condition on $\bar{g}$ is

$$c^\alpha(\mathrm{Ad}(\bar{g})\mathrm{a}(t) + R_{\bar{g}^{-1}*\bar{g}}\dot{\bar{g}}) = 0, \qquad \alpha = s+1, \ldots, r.$$

Now, if θ^α is the left invariant 1-form on G induced by c^α, the previous equation implies

$$\theta^\alpha_{\bar{g}^{-1}}\left(R_{\bar{g}^{-1}*e}\mathrm{a}(t) - \frac{d\bar{g}^{-1}}{dt}\right) = 0, \qquad \alpha = s+1, \ldots, r.$$

Let $\tilde{g} = \bar{g}^{-1}$. The above expression implies that $R_{\tilde{g}*e}\mathrm{a}(t) - \dot{\tilde{g}}$ is generated by left invariant vector fields on G from elements of $\mathfrak{h}$. Then, given $\pi^L : G \to G/H$, the kernel of π^L_* is spanned by the left invariant vector fields on G generated by elements of $\mathfrak{h}$. Then it follows that

$$\pi^L_{*\tilde{g}}(R_{\tilde{g}*e}\mathrm{a}(t) - \dot{\tilde{g}}) = 0. \tag{3.5}$$

Therefore, if we use that $\pi^L_* \circ X^R_\alpha = -X^L_\alpha \circ \pi^L$, where X^L_α denotes the fundamental vector field of the action of G in G/H and X^R_α denotes the right-invariant vector field in G whose value at e is a_α, we can prove that $\pi^L(\tilde{g})$ is a solution on G/H of the equation

$$\frac{d\pi^L(\tilde{g})}{dt} = \sum_{\alpha=1}^{r} b_\alpha(t) X^L_\alpha(\pi^L(\tilde{g})). \tag{3.6}$$

Thus, given a certain solution $g'(t)$ in $\mathfrak{h}$ related to the initial $g(t)$ by means of $\tilde{g}(t)$ according to $g(t) = \tilde{g}(t)g'(t)$, the projection to G/H of $\tilde{g}(t)$, i.e. $\pi^L(\tilde{g}(t))$, is a solution of (3.6). This shows that whenever $g'(t)$ is a curve in H, then $\tilde{g}(t)$ satisfies equation (3.6). Moreover, as shown in [40], if $\tilde{g}(t)$ satisfies (3.6), then $g'(t)$ is a curve in H satisfying (3.2). The previous result shows that the condition for (3.2) to hold is not only sufficient but also necessary. Thus, we provide a new result which completes the one found in [40].

Finally, it is worth noting that even though this last proof has been developed for quantum mechanics, it can also be applied to ordinary differential equations, because it appears as a consequence of the group structure of Lie systems which is the same for both quantum and ordinary Lie systems.

3.2. Interaction picture and Lie systems. As a first application of the reduction method for Lie systems, we analyse here how this theory can be applied to explain the interaction picture used in quantum mechanics. This picture has been proved to be very effective in the development of perturbation methods. It plays a rôle when the t-dependent Hamiltonian can be written as a linear combination with t-dependent coefficients of a simpler Hamiltonian H_1 and a perturbation $V(t)$. In the framework of Lie systems, we can analyse what happens when the t-dependent Hamiltonian is

$$H(t) = H_1 + V(t) = H_1 + \sum_{\alpha=2}^{r} b_\alpha(t) H_\alpha = \sum_{\alpha=1}^{r} b_\alpha(t) H_\alpha, \quad b_1(t) = 1,$$

where the set of skew-self-adjoint operators $\{-iH_\alpha \mid \alpha = 1, \ldots, r\}$ is closed under commutation and generates a finite-dimensional real Lie algebra. The situation is very similar to the case of control systems with a drift term (here H_1) that are linear in the control functions. The functions $b_\alpha(t)$ correspond to the control functions.

According to the theory of Lie systems, take a basis $\{\mathrm{a}_\alpha \mid \alpha = 1, \ldots, r\}$ of the Lie algebra with corresponding associated fundamental vector fields $-iH_\alpha$. The equation to be studied in T_eG is (3.1) and if we define $g'(t) = \bar{g}(t)g(t)$, where $\bar{g}(t)$ is a previously chosen curve, it obeys a similar equation to $g'(t)$ given by (3.3).

If, in particular, we choose $\bar{g}(t) = \exp(\mathrm{a}_1 t)$, we find the new equation in T_eG

$$\begin{aligned} R_{g'^{-1}*g'}\dot{g}' &= -\mathrm{Ad}(\exp(\mathrm{a}_1 t))\Big(\sum_{\alpha=2}^{r} b_\alpha(t)\mathrm{a}_\alpha\Big) \\ &= -\exp(\mathrm{ad}(\mathrm{a}_1)t)\Big(\sum_{\alpha=2}^{r} b_\alpha(t)\mathrm{a}_\alpha\Big). \end{aligned} \tag{3.7}$$

Correspondingly, the action of G on $\mathcal{H}$ by a unitary representation defines a transformation on $\mathcal{H}$ in which the state ψ_t transforms into $\psi'_t = \exp(iH_1 t)\psi_t$ and its dynamical evolution is given by the vector field corresponding to the right-hand side of (3.7). In particular, if $\{\mathrm{a}_2, \ldots, \mathrm{a}_r\}$ span an ideal of the Lie algebra $\mathfrak{g}$, the problem reduces to the corresponding normal subgroup in G.

3.3. The method of unitary transformations. A second application of the theory of Lie systems in quantum mechanics and, in particular, of the reduction method is to obtain information about how to proceed to solve a quantum Lie Hamiltonian. Let us discuss here a relevant general procedure to accomplish this task.

Every Schrödinger equation of Lie type is determined by a Lie algebra $\mathfrak{g}$, a unitary representation of its connected and simply connected Lie group G on $\mathcal{H}$, and a curve $\mathrm{a}(t)$ in T_eG. Depending on $\mathfrak{g}$, there are two cases. If $\mathfrak{g}$ is solvable, we can use the reduction method in quantum mechanics to obtain the general solution. If $\mathfrak{g}$ is not solvable, it is not known how to integrate the problem in terms of quadratures in the most general case. Nevertheless, it is possible to solve the problem completely for some specific curves as for instance it happens for the Caldirola–Kanai Hamiltonian [118]. A way of dealing with such systems is to try to transform the curve $\mathrm{a}(t)$ into another one $\mathrm{a}'(t)$, easier to handle, as has been done in the previous section for the interaction picture. In a more general case, although any two curves $\mathrm{a}(t)$ and $\mathrm{a}'(t)$ are always connected by an automorphism, the equation determining the transformation can be as difficult to solve as the initial problem. Because of this, it is of interest to find a curve that:

1. determines an easily solvable equation;
2. can be transformed through an explicitly known transformation into the curve associated with our initial problem.

This is the topic of the next three sections, where conditions for such Schrödinger equations are analysed. In any case, we can always express the solution of the initial problem in terms of a solution of the equation determining the transformation. In certain cases, for an appropriate choice of the curve $\bar{g}(t)$ the new curve $\mathrm{a}'(t)$ belongs to T_eH for all t, where H is a solvable Lie subgroup of G. In this case we can reduce the problem from $\mathfrak{g}$ to a certain solvable Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Of course, in order to do this, a solution of the equation of reduction is needed, but once this is known we can solve the problem completely. Other methods have also been used in the literature, like the Lewis–Riesenfeld (LR) method. However, this method seems to offer a complete solution only if $\mathfrak{g}$ is solvable. If $\mathfrak{g}$ is not solvable, the LR method offers a solution which depends on a solution of a system of differential equations, as in the method of reduction.

To sum up, given a Lie system associated with a Lie algebra $\mathfrak{g}$, whose Lie group G acts, by unitary operators, on $\mathcal{H}$, and determined by a curve $\mathrm{a}(t)$ in T_eG, the systematic procedure to be used is the following:

- If $\mathfrak{g}$ is solvable, we can solve the problem easily by quadratures as in [94, 107].
- If $\mathfrak{g}$ is not solvable, we can try to solve the problem for a given curve as for the Caldirola–Kanai Hamiltonian in [118], by choosing a curve $\bar{g}(t)$ transforming the curve $\mathrm{a}(t)$ into another one easier to solve, as in the interaction picture. If this does not work we can try to reduce the problem to an integrable case as for the t-dependent mass and frequency harmonic oscillator or quadratic one-dimensional Hamiltonian in [52, 96, 211, 238].

3.4. t-dependent operators for quantum Lie systems. In this section we apply our methods to obtain the t-dependent evolution operators of several problems found in the physics literature in an algorithmic way.

We first provide a simple example to illustrate the main points of our theory. Next, we analyse t-dependent quadratic Hamiltonians. These Hamiltonians describe a very large class of physical models. Sometimes, one of these physical models is described by a certain family of quadratic Hamiltonians associated with a Lie subalgebra of the Lie algebra of operators related to general quadratic Hamiltonians. If this Lie subalgebra is solvable, the differential equations related to it through the Wei–Norman methods are solvable too and the t-evolution operator can be explicitly obtained. In these cases, we can find the explicit solution of these problems in the literature using different methods for each case. We also describe some approaches to study these quantum Lie systems in the nonsolvable cases.

3.5. Initial examples. We start our investigation by studying the motion of a particle with a t-dependent mass under the action of a t-dependent linear potential term. The Hamiltonian describing this physical case is

$$H(t) = \frac{P^2}{2m(t)} + S(t)X.$$

The Lie algebra associated with this example is a central extension of the Heisenberg Lie algebra. A basis for the Lie algebra of vector fields related to this physical model is

$$Z_1 = i\frac{P^2}{2}, \qquad Z_2 = iP, \qquad Z_3 = iX, \qquad Z_4 = iI,$$

which generates a Lie algebra with the commutation relations

$$\begin{aligned}
&[Z_1, Z_2] = 0, \qquad &&[Z_1, Z_3] = 2Z_2, \qquad &&[Z_1, Z_4] = 0,\\
&[Z_2, Z_3] = Z_4, \qquad &&[Z_2, Z_4] = 0,\\
&[Z_3, Z_4] = 0.
\end{aligned}$$

This Lie algebra is solvable, and so the related equations obtained through the Wei–Norman method can be solved by quadratures for any pair of t-dependent coefficients $m(t)$ and $S(t)$. The solution of the associated Wei–Norman system allows us to obtain the t-evolution operator and the wave function solution of the t-dependent Schrödinger equation.

This t-dependent Hamiltonian has been studied in [221] for some particular cases using *ad hoc* methods and in general in [94]. Here, we investigate it through the Wei–Norman method. Its equation in the group G with $\mathrm{T}_eG \simeq V$ is

$$R_{g^{-1}*g}\dot{g} = -\frac{1}{m(t)}a_1 - S(t)a_3 \equiv a_{MS}(t),$$

where the $a_1, \ldots, a_4$ are a basis of $\mathfrak{g}$ with the same commutation relations as the operators $Z_1, \ldots, Z_4$. The factorisation

$$g(t) = \exp(v_2(t)a_2)\exp(-v_3(t)a_3)\exp(-v_4(t)a_4)\exp(-v_1(t)a_1)$$

allows us to solve the equation in G by the Wei–Norman method to get

$$\begin{aligned}
\dot{v}_1 &= \frac{1}{m(t)},\\
\dot{v}_2 &= \frac{v_3}{m(t)},\\
\dot{v}_3 &= S(t),\\
\dot{v}_4 &= -S(t)v_2 - \frac{v_3^2}{2m(t)},
\end{aligned}$$

with initial conditions $v_1(0) = v_2(0) = v_3(0) = v_4(0) = 0$. The solution of this system can be expressed using quadratures because the related group is solvable:

$$\begin{aligned}
v_1(t) &= \int_0^t \frac{du}{m(u)},\\
v_2(t) &= \int_0^t \frac{du}{m(u)}\left(\int_0^u S(v)\,dv\right),\\
v_3(t) &= \int_0^t S(u)\,du,\\
v_4(t) &= -\int_0^t S(u)\left(\int_0^u \frac{dv}{m(v)}\left(\int_0^v S(w)\,dw\right)\right)du - \int_0^t \frac{du}{2m(u)}\left(\int_0^u S(v)\,dv\right)^2,
\end{aligned} \tag{3.8}$$

and the t-evolution operator is

$$\begin{aligned}
U(g(t)) &= \exp(v_2(t)Z_2)\exp(-v_3(t)Z_3)\exp(-v_4(t)Z_4)\exp(-v_1(t)Z_1)\\
&= \exp(iv_2(t)P)\exp(-iv_3(t)X)\exp(-iv_4(t)I)\exp\left(-iv_1(t)\frac{P^2}{2}\right).
\end{aligned}$$

3.6. Quadratic Hamiltonians. After dealing with the above easy example, we can now proceed to the t-dependent quadratic Hamiltonian given by [237] (see [59])

$$H(t) = \alpha(t)\frac{P^2}{2} + \beta(t)\frac{XP+PX}{4} + \gamma(t)\frac{X^2}{2} + \delta(t)P + \epsilon(t)X + \phi(t)I, \tag{3.9}$$

where X and P are the position and momentum operators satisfying the commutation relation

$$[X, P] = iI.$$

It is important to solve this quantum quadratic Hamiltonian because it frequently appears in quantum mechanics.

In order to prove that (3.9) is a quantum Lie system, we must check that this t-dependent Hamiltonian can be written as a sum with t-dependent coefficients of some self-adjoint Hamiltonians generating a real finite-dimensional Lie algebra of operators.

As we can write

$$H(t) = \alpha(t)H_1 + \beta(t)H_2 + \gamma(t)H_3 - \delta(t)H_4 + \epsilon(t)H_5 + \phi(t)H_6$$

with the Hamiltonians

$$H_1 = \frac{P^2}{2},\quad H_2 = \frac{1}{4}(XP+PX),\quad H_3 = \frac{X^2}{2},\quad H_4 = -P,\quad H_5 = X,\quad H_6 = I,$$

satisfying the commutation relations

$$
\begin{array}{llll}
[iH_1, iH_2] = iH_1, & [iH_2, iH_3] = iH_3, & [iH_3, iH_4] = iH_5, & [iH_4, iH_5] = -iH_6, \\
[iH_1, iH_3] = 2iH_2, & [iH_2, iH_4] = -\frac{i}{2}H_4, & [iH_3, iH_5] = 0, & \\
[iH_1, iH_4] = 0, & [iH_2, iH_5] = \frac{i}{2}H_5, & & \\
[iH_1, iH_5] = -iH_4, & & &
\end{array}
$$

and $[iH_\alpha, iH_6] = 0$, $\alpha = 1, \ldots, 5$, we see that $H(t)$ is a quantum Lie system.

This means that the skew-self-adjoint operators iH_α generate a six-dimensional real Lie algebra V of operators. Now, we can relate them to the basis $\{\mathrm{a}_1, \ldots, \mathrm{a}_6\}$ for an abstract real Lie algebra isomorphic to the one spanned by the $-iH_\alpha$. This basis is chosen in such a way that

$$
\begin{array}{lllll}
[\mathrm{a}_1, \mathrm{a}_2] = \mathrm{a}_1, & [\mathrm{a}_2, \mathrm{a}_3] = \mathrm{a}_3, & [\mathrm{a}_3, \mathrm{a}_4] = \mathrm{a}_5, & [\mathrm{a}_4, \mathrm{a}_5] = -\mathrm{a}_6, & [\mathrm{a}_5, \mathrm{a}_6] = 0, \\
[\mathrm{a}_1, \mathrm{a}_3] = 2\mathrm{a}_2, & [\mathrm{a}_2, \mathrm{a}_4] = -\frac{1}{2}\mathrm{a}_4, & [\mathrm{a}_3, \mathrm{a}_5] = 0, & [\mathrm{a}_4, \mathrm{a}_6] = 0, & \\
[\mathrm{a}_1, \mathrm{a}_4] = 0, & [\mathrm{a}_2, \mathrm{a}_5] = \frac{1}{2}\mathrm{a}_5, & [\mathrm{a}_3, \mathrm{a}_6] = 0, & & \\
[\mathrm{a}_1, \mathrm{a}_5] = -\mathrm{a}_4, & [\mathrm{a}_2, \mathrm{a}_6] = 0, & & & \\
[\mathrm{a}_1, \mathrm{a}_6] = 0. & & & &
\end{array}
$$

This six-dimensional real Lie algebra is a semidirect sum of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ spanned by $\{\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3\}$ and the Heisenberg–Weyl Lie algebra generated by $\{\mathrm{a}_4, \mathrm{a}_5, \mathrm{a}_6\}$, which is an ideal.

In order to find the t-evolution provided by the t-dependent Hamiltonian (3.9) we should find the curve $g(t)$ in G, with $\mathrm{T}_eG \simeq V$, such that

$$
R_{g^{-1}*g}\dot{g} = -\sum_{\alpha=1}^{6} b_\alpha(t)\mathrm{a}_\alpha, \qquad g(0) = e,
$$

with

$$
b_1(t) = \alpha(t), \quad b_2(t) = \beta(t), \quad b_3(t) = \gamma(t), \quad b_4(t) = -\delta(t), \quad b_5(t) = \epsilon(t), \quad b_6(t) = \phi(t).
$$

This can be carried out by using the generalised Wei–Norman method, i.e. by writing the curve $g(t)$ in G in terms of a set of second class canonical coordinates. For instance,

$$
\begin{aligned}
g(t) = {} & \exp(-v_4(t)\mathrm{a}_4)\exp(-v_5(t)\mathrm{a}_5)\exp(-v_6(t)\mathrm{a}_6) \\
& \times \exp(-v_1(t)\mathrm{a}_1)\exp(-v_2(t)\mathrm{a}_2)\exp(-v_3(t)\mathrm{a}_3),
\end{aligned}
\tag{3.10}
$$

and a straightforward application of the above mentioned Wei–Norman method technique leads to the system

$$
\begin{cases}
\dot{v}_1 = b_1 + b_2v_1 + b_3v_1^2, & \dot{v}_4 = b_4 + \frac{1}{2}b_2v_4 + b_1v_5, \\
\dot{v}_2 = b_2 + 2b_3v_1, & \dot{v}_5 = b_5 - b_3v_4 - \frac{1}{2}b_2v_5, \\
\dot{v}_3 = e^{v_2}b_3, & \dot{v}_6 = b_6 - b_5v_4 + \frac{1}{2}b_3v_4^2 - \frac{1}{2}b_1v_5^2,
\end{cases}
\tag{3.11}
$$

with $v_1(0) = v_2(0) = v_3(0) = v_4(0) = v_5(0) = v_6(0) = 0$.

If we consider the vector fields

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial v_1} + v_5\frac{\partial}{\partial v_4} - \frac{1}{2}v_5^2\frac{\partial}{\partial v_6},\\
X_2 &= v_1\frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_2} + \frac{1}{2}v_4\frac{\partial}{\partial v_4} - \frac{1}{2}v_5\frac{\partial}{\partial v_5},\\
X_3 &= v_1^2\frac{\partial}{\partial v_1} + 2v_1\frac{\partial}{\partial v_2} + e^{v_2}\frac{\partial}{\partial v_3} - v_4\frac{\partial}{\partial v_5} + \frac{1}{2}v_4^2\frac{\partial}{\partial v_6},\\
X_4 &= \frac{\partial}{\partial v_4},\\
X_5 &= \frac{\partial}{\partial v_5} - v_4\frac{\partial}{\partial v_6},\\
X_6 &= \frac{\partial}{\partial v_6},
\end{aligned}
\tag{3.12}
$$

we can check that these vector fields satisfy the same commutation relations as the corresponding $\{\mathrm{a}_\alpha \mid \alpha = 1,\ldots,6\}$ and thus, system (3.11) is a Lie system related to a Vessiot–Guldberg Lie algebra isomorphic to the Lie algebra (of operators) associated with the t-dependent Hamiltonian (3.9) and to the Vessiot–Guldberg Lie algebra related to its corresponding equation on a Lie group.

Now, once the functions $v_\alpha(t)$, with $\alpha = 1,\ldots,6$, have been determined, the t-evolution of any state is given by

$$
\begin{aligned}
\psi_t = {}& \exp(-v_4(t)iH_4)\exp(-v_5(t)iH_5)\exp(-v_6(t)iH_6)\\
&\times \exp(-v_1(t)iH_1)\exp(-v_2(t)iH_2)\exp(-v_3(t)iH_3)\psi_0,
\end{aligned}
$$

and thus

$$
\begin{aligned}
\psi_t = {}& \exp(v_4(t)iP)\exp(-v_5(t)iX)\exp(-v_6(t)iI)\\
&\times \exp\left(-v_1(t)i\frac{P^2}{2}\right)\exp\left(-v_2(t)i\frac{PX+XP}{4}\right)\exp\left(-v_3(t)i\frac{X^2}{2}\right)\psi_0.
\end{aligned}
\tag{3.13}
$$

3.7. Particular cases. t-dependent quadratic Hamiltonians describe a very large class of physical models. Sometimes, one of these models is described by a family of quadratic Hamiltonians that can be regarded as a quantum Lie system related to a Lie subalgebra of the one given for general quadratic Hamiltonians. If they are associated with a Lie solvable subalgebra, then the system of differential equations related to it through the Wei–Norman method is solvable too and the t-evolution operator can be explicitly obtained. In this section we treat some instances of this case through a unified approach. In these instances, we can also find the explicit solutions of these problems in the literature, but obtained by different *ad hoc* methods.

Once we have obtained the solution for a generic quadratic Hamiltonian $H(t)$, we can study the solution for a system with constant mass and linear potential given by

$$
H(t) = \frac{P^2}{2m} + S(t)X, \tag{3.14}
$$

to obtain, in view of equations (3.11),

$$
\begin{aligned}
&v_1(t)=\frac{t}{m},\qquad v_2(t)=0,\qquad v_3(t)=0,\\
&v_4(t)=\frac{1}{m}\int_0^t\left(\int_0^u S(v)\,dv\right)du,\qquad v_5(t)=\int_0^t S(u)\,du,\\
&v_6(t)=-\frac{1}{m}\int_0^t\left(S(u)\int_0^u\left(\int_0^v S(w)\,dw\right)dv\right)du-\frac{1}{2m}\int_0^t\left(\int_0^u S(v)\,dv\right)^2du,
\end{aligned}
$$

which give the t-evolution operator if we substitute them into the t-evolution operator (3.13).

Now we can consider particular instances of this t-dependent Hamiltonian. For example, for the curves with constant mass m and $S(t)=q\epsilon_0+q\epsilon\cos(\omega t)$, studied in [107], we obtain

$$
\begin{gathered}
v_1(t)=\frac{t}{m},\qquad v_2(t)=0,\qquad v_3(t)=0,\\
v_4(t)=\frac{q}{2m\omega^2}(2\epsilon+\epsilon_0\omega^2t^2-2\epsilon\cos(\omega t)),\qquad v_5(t)=\frac{q}{\omega}(\epsilon_0\omega t+\epsilon\sin(\omega t)),
\end{gathered}
$$

and

$$
v_6(t)=\frac{-q^2}{12m\omega^3}\big(4\epsilon_0^2\omega^3t^3-3\epsilon(\epsilon-4\epsilon_0)\omega t+3\epsilon(4\epsilon+2\epsilon_0(\omega^2t^2-2)-3\epsilon\cos(\omega t))\sin(\omega t)\big).
$$

The procedure to obtain a solution with arbitrary nonconstant mass and $S(t)=q\epsilon_0+q\epsilon\cos(\omega t)$ was pointed out in [107] and solved in [94]. From our point of view, the most general solution comes directly from expression (3.8), because all cases in the literature are particular instances of our approach with general functions $m(t)$ and $S(t)$.

Now, we can obtain the wave function solution of this system. We know that the wave function solution ψ_t with initial condition ψ_0 is

$$
\begin{aligned}
\psi_t(x)&=U(g(t))\psi(x,0)\\
&=\exp(iv_6(t))\exp(-v_4(t)iP)\exp(-v_5(t)iX)\exp\left(-v_1(t)i\frac{P^2}{2}\right)\psi_0(x).
\end{aligned}
$$

However, if we express the initial wave function $\psi_0(x)$ in the momentum space as $\phi_0(p)$, the solution will take a similar form as before but with $U(g(t))$ in the momentum representation. In this case the solution with initial condition $\phi_0(p)$ is

$$
\begin{aligned}
\phi_t(p)&=U(g(t))\phi_0(p)\\
&=\exp(-iv_6(t))\exp(v_4(t)iP)\exp(-v_5(t)iX)\exp\left(-iv_1(t)\frac{P^2}{2}\right)\phi_0(p)\\
&=\exp(-iv_6(t))\exp(v_4(t)iP)\exp(-v_5(t)iX)\exp\left(-iv_1(t)\frac{p^2}{2}\right)\phi_0(p)\\
&=\exp(-iv_6(t))\exp(v_4(t)iP)\exp\left(-iv_1(t)\frac{(p+v_5(t))^2}{2}\right)\phi_0(p+v_5(t))\\
&=\exp\left(-iv_6(t)+iv_4(t)p-iv_1(t)\frac{(p+v_5(t))^2}{2}\right)\phi_0(p+v_5(t)).
\end{aligned}
$$

3.8. Nonsolvable Hamiltonians and particular instances. In the preceding section the differential equations associated with the t-dependent quantum Hamiltonians were Lie systems related to a solvable Lie algebra. Thus, it was proved that the differential equations obtained were integrable by quadratures through the Wei–Norman method. If this does not happen, it is not easy to obtain a general solution. Now, we describe some examples of 'nonsolvable' t-dependent quadratic Hamiltonians. In general we do not obtain a general solution in terms of t-dependent functions of quadratic Hamiltonians. Nevertheless, we show that for some instances, with coefficients satisfying certain integrability conditions [52, 54], the differential equations can be integrated.

As a first case, consider the Hamiltonian for a forced harmonic oscillator with t-dependent mass and frequency given by

$$H(t) = \frac{P^2}{2m(t)} + \frac{1}{2}m(t)\omega^2(t)X^2 + f(t)X.$$

This case, either with or without t-dependent frequency, has been studied in [78, 107, 238]. The equations describing the solutions of this Lie system by the method of Wei–Norman are

$$\begin{aligned}
\dot{v}_1 &= \frac{1}{m(t)} + m(t)\omega^2(t)v_1^2,\\
\dot{v}_2 &= 2m(t)\omega^2(t)v_1,\\
\dot{v}_3 &= e^{v_2}m(t)\omega^2(t),\\
\dot{v}_4 &= \frac{1}{m(t)}v_5,\\
\dot{v}_5 &= f(t) - m(t)\omega^2(t)v_4,\\
\dot{v}_6 &= \frac{1}{2}m(t)\omega^2(t)v_4^2 - f(t)v_4 - \frac{1}{2m(t)}v_5^2,
\end{aligned}$$

with initial conditions $v_1(0) = v_2(0) = v_3(0) = v_4(0) = v_5(0) = v_6(0) = 0$, where the factorisation (3.10) has been used. The solution of this system cannot be obtained by quadratures in the general case because the associated Lie algebra is not solvable. Nevertheless, we can consider a particular instance of this kind of Hamiltonian, the so-called Caldirola–Kanai Hamiltonian [118]. In this case, for $m(t) = e^{-rt}m_0$, $\omega(t) = \omega_0$ and $f(t) = 0$ the Hamiltonian reads

$$H(t) = \frac{P^2}{2m_0}e^{rt} + \frac{1}{2}m_0 e^{-rt}\omega_0^2 X^2.$$

The corresponding solution is completely known and is given by

$$\begin{aligned}
v_1(t) &= \frac{2e^{rt}}{m_0\big(r + \bar{\omega}_0\coth\big(\frac{t}{2}\bar{\omega}_0\big)\big)},\\
v_2(t) &= rt + 2\log\bar{\omega}_0 - 2\log\left(r\sinh\left(\frac{t}{2}\bar{\omega}_0\right) + \bar{\omega}_0\cosh\left(\frac{t}{2}\bar{\omega}_0\right)\right),\\
v_3(t) &= \frac{2m_0\omega_0^2}{r + \bar{\omega}_0\coth\big(\frac{t}{2}\bar{\omega}_0\big)}, \qquad v_4(t) = 0, \qquad v_5(t) = 0, \qquad v_6(t) = 0,
\end{aligned}$$

where $\bar{\omega}_0 = \sqrt{r^2 - 4\omega_0^2}$. This example shows that the problem can also be exactly solved for particular instances of curves in $\mathfrak{g}$ of Lie systems with nonsolvable Lie algebras. Another example is

$$H(t) = \frac{P^2}{2m} + \frac{m\omega_0^2}{2(t+k)^2} X^2,$$

for which the solution of the Wei–Norman system reads

$$\begin{aligned}
v_1(t) &= \frac{2(k+t)((k+t)^{\bar{\omega}_0} - k^{\bar{\omega}_0})}{m(k^{\bar{\omega}_0}(\bar{\omega}_0 - 1) + (k+t)^{\bar{\omega}_0}(\bar{\omega}_0 + 1))},\\
v_2(t) &= (1+\bar{\omega}_0)\log(k+t) - (1+\bar{\omega}_0)\log k + 2\log(2k^{\bar{\omega}_0}\bar{\omega}_0)\\
&\quad - 2\log(k^{\bar{\omega}_0}(\bar{\omega}_0 - 1) + (k+t)^{\bar{\omega}_0}(\bar{\omega}_0 + 1)),\\
v_3(t) &= \frac{2m\omega_0^2}{k} \frac{(k+t)^{\bar{\omega}_0} - k^{\bar{\omega}_0}}{k^{\bar{\omega}_0}(\bar{\omega}_0 - 1) + (k+t)^{\bar{\omega}_0}(\bar{\omega}_0 + 1)},\\
v_4(t) &= 0, \qquad v_5(t) = 0, \qquad v_6(t) = 0,
\end{aligned}$$

where now $\bar{\omega}_0 = \sqrt{1 - 4\omega_0^2}$.

Other examples of Hamiltonians which can be studied by our method can be found in [118]. We just mention two examples which can be completely solved:

$$\begin{aligned}
H_1(t) &= \frac{P^2}{2m_0} + \frac{1}{2}m_0(U + V\cos(\omega_0 t))X^2,\\
H_2(t) &= \frac{P^2}{2m_0}e^{rt} + \frac{1}{2}m_0 e^{-rt}\omega_0^2 X^2 + f(t)X.
\end{aligned}$$

The first one corresponds to a Paul trap which has been studied in [95] and admits a solution in terms of Mathieu's functions. The second one is a damped Caldirola–Kanai Hamiltonian analysed in [221].

3.9. Reduction in quantum mechanics. Quite often, when a quantum Lie system is related to a nonsolvable Lie algebra, it is interesting to solve it in terms of (unknown) solutions of differential equations. Next, we study some examples of how to use the method of reduction in this way. We find that the reduction method can be applied not only to analyse systems of differential equations but also to solve certain quantum problems in an algorithmic way.

Consider a harmonic oscillator with t-dependent frequency whose Hamiltonian is given by

$$H(t) = \frac{P^2}{2} + \frac{1}{2}\Omega^2(t)X^2.$$

As a particular case of the Hamiltonian described in Section 1.8, this example is related to an equation in the connected Lie group associated with the semidirect sum of $\mathfrak{sl}(2,\mathbb{R})$, spanned by the elements $\{\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3\}$, with the Heisenberg Lie algebra generated by the ideal $\{\mathrm{a}_4, \mathrm{a}_5, \mathrm{a}_6\}$:

$$R_{g^{-1}*g}\dot{g} = -\mathrm{a}_1 - \Omega^2(t)\mathrm{a}_3, \qquad g(0) = e. \tag{3.15}$$

Since the solution of this equation starts from the identity and $\{\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3\}$ generate an $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra, the t-dependent Hamiltonian $H(t)$ is related to the group $SL(2,\mathbb{R})$.

As a particular application of the reduction technique we will perform the reduction from $G = SL(2,\mathbb{R})$ to the Lie group related to the Lie subalgebra $\mathfrak{h} = \langle \mathrm{a}_1 \rangle$. We have shown in Section 3.1 that to obtain such a reduction, we have to solve an equation in G/H, namely

$$\frac{d\pi^L(\tilde{g})}{dt} = \sum_{\alpha=1}^{3} b_\alpha(t) X_\alpha^L(\pi^L(\tilde{g})) \tag{3.16}$$

where X_α^L are the fundamental vector fields of the action λ of G on G/H. Now, we are going to describe this equation in a set of local coordinates. First, we can write any element of an open neighbourhood U of $e \in G$ in a unique way as

$$g = \exp(-c_3\mathrm{a}_3)\exp(-c_2\mathrm{a}_2)\exp(-c_1\mathrm{a}_1), \tag{3.17}$$

where the matrices a_α, with $\alpha = 1, 2, 3$, are given by (2.4).

This decomposition allows us to establish a local diffeomorphism between an open neighbourhood $V \subset G/H$ and the set of matrices given by $\exp(-c_3\mathrm{a}_3)\exp(-c_2\mathrm{a}_2)$. Now, the decomposition (3.17) reads in matrix terms as

$$\begin{aligned}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ -c_3 & 1 \end{pmatrix}\begin{pmatrix} e^{c_2/2} & 0 \\ 0 & e^{-c_2/2} \end{pmatrix}\begin{pmatrix} 1 & c_1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} e^{c_2/2} & 0 \\ -c_3e^{c_2/2} & e^{-c_2/2} \end{pmatrix}\begin{pmatrix} 1 & c_1 \\ 0 & 1 \end{pmatrix}.\end{aligned}$$

If we express c_1, c_2, c_3 in terms of α, β, γ and δ, we obtain $c_3 = -\gamma/\alpha$, $c_2 = \log \alpha^2$, and $c_1 = \beta/\alpha$. Consequently, we get

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \gamma/\alpha & 1 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}\begin{pmatrix} 1 & \beta/\alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \gamma & \alpha^{-1} \end{pmatrix}\begin{pmatrix} 1 & \beta/\alpha \\ 0 & 1 \end{pmatrix}.$$

Thus, we can define the projection $\pi^L : U \subset G \to G/H$ by

$$\pi^L \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \gamma & \alpha^{-1} \end{pmatrix} H, \tag{3.18}$$

which allows us to represent elements of G/H, locally, as 2×2 lower triangular matrices with determinant one. Now, given $\lambda_g : g'H \in G/H \mapsto gg'H \in G/H$, as $\lambda_g \circ \pi^L = \pi^L \circ L_g$, the fundamental vector fields defined in G/H by a_1 and a_3 through the action $\lambda : (g, g'H) \in G \times G/H \mapsto \lambda_g(g'H) \in G/H$ are given by

$$\begin{aligned} X_1^L(\pi^L(g)) &= \left.\frac{d}{dt}\right|_{t=0} \pi^L\left(\exp(-t\mathrm{a}_1)\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right) = \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma/\alpha^2 \end{pmatrix}, \\ X_3^L(\pi^L(g)) &= \left.\frac{d}{dt}\right|_{t=0} \pi^L\left(\exp(-t\mathrm{a}_3)\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ -\alpha & 0 \end{pmatrix}, \end{aligned}$$

and the equation on $V \subset G/H$ is described by

$$\begin{pmatrix} \dot{\alpha} & 0 \\ \dot{\gamma} & -\dot{\alpha}\alpha^{-2} \end{pmatrix} = \begin{pmatrix} \gamma & 0 \\ -\Omega^2(t)\alpha & -\gamma\alpha^{-2} \end{pmatrix}.$$

Therefore, we need to obtain a solution of the system

$$\begin{cases} \ddot{\alpha} = -\Omega^2(t)\alpha, \\ \gamma = \dot{\alpha}. \end{cases} \tag{3.19}$$

Taking into account (3.18), if α_1 is a solution of (3.19), the curve $\tilde{g}(t)$ that satisfies $g(t) = \tilde{g}(t)h(t)$, where $h(t)$ is a solution of an equation defined on the Lie group with Lie algebra $\mathfrak{h} = \langle \mathrm{a}_1 \rangle$, reads

$$\begin{aligned} \tilde{g}(t) = \begin{pmatrix} \alpha_1 & 0 \\ \dot{\alpha}_1 & \alpha_1^{-1} \end{pmatrix} &= \begin{pmatrix} e^{c_2/2} & 0 \\ -c_3 e^{c_2/2} & e^{-c_2/2} \end{pmatrix} \\ &= \exp\left(\frac{\dot{\alpha}_1}{\alpha_1}\mathrm{a}_3\right)\exp(-2\log\alpha_1\mathrm{a}_2), \end{aligned}$$

and the curve which acts on the initial equation in $SL(2,\mathbb{R})$ to transform it into one in the above mentioned Lie subalgebra is given by $\bar{g}(t) = \tilde{g}^{-1}(t)$,

$$\bar{g}(t) = \exp(2\log\alpha_1\,\mathrm{a}_2)\exp\left(-\frac{\dot{\alpha}_1}{\alpha_1}\,\mathrm{a}_3\right).$$

This curve transforms the initial equation in the group given by (3.15) into the new one given by (3.3), i.e.

$$\mathrm{a}'(t) = -\frac{\mathrm{a}_1}{\alpha_1^2(t)},$$

which corresponds to the t-dependent Hamiltonian $H'(t) = P^2/(2\alpha_1^2(t))$. The induced transformation in the Hilbert space $\mathcal{H}$ that transforms $H(t)$ into $H'(t)$ is

$$\exp\left(i\frac{\log\alpha_1}{2}(PX+XP)\right)\exp\left(-i\frac{\dot{\alpha}_1}{2\alpha_1}X^2\right).$$

Both results can be found in [96].

There are other possibilities of choosing different Lie subalgebras of $\mathfrak{g}$ in order to perform the reduction, but the results are always given in terms of a solution of a differential equation.

4. Integrability conditions for Lie systems

The main aim of this chapter is to describe the main aspects of the integrability theory for Lie systems detailed in [47] and based on the geometrical understanding of Riccati equations.

The Riccati equation can be considered as the simplest nonlinear differential equation [40, 50]. It is, basically, the only first-order ordinary differential equation admitting a nonlinear superposition rule [157, 234]. In spite of its apparent simplicity, its general solution cannot be described by means of quadratures except in some very particular cases [63, 132, 169, 183, 214, 239].

The relevance of the Riccati equation becomes evident when we take into account its frequent appearance in many fields of mathematics and physics [57, 159, 176, 184, 203, 207, 216, 234]. This also implies the necessity of a theory of integrability providing all those integrable cases that might lead to solvable physical models.

4.1. Integrability of Riccati equations. In order to provide a first insight into integrability conditions for Riccati equations, we review here some very well-known results.

Recall that Riccati equations are first-order differential equations of the form

$$\frac{dx}{dt} = b_1(t) + b_2(t)x + b_3(t)x^2. \tag{4.1}$$

A first particular example of Riccati equation integrable by quadratures is the one with $b_3 = 0$. In fact, in that case, the Riccati equation reduces to an inhomogeneous linear equation, which can be explicitly integrated by means of two quadratures.

Additionally, the change of variable $w = -1/x$ transforms the above Riccati equation into

$$\frac{dw}{dt} = b_1(t)w^2 - b_2(t)w + b_3(t).$$

Consequently, if we suppose that $b_1 = 0$ in equation (4.1), that is, if we consider a Bernoulli equation, the above change of variable leads to an integrable linear equation.

Another known property is that given a particular solution $x_1(t)$ of (4.1), the change $x = x_1(t) + z$ transform the equation into a new one for which the coefficient of the term independent of z is zero, i.e.

$$\frac{dz}{dt} = (2b_3(t)x_1(t) + b_2(t))z + b_3(t)z^2,$$

and, as we pointed out previously, this reduces to an inhomogeneous linear equation with the change of variables $z = -1/u$. Consequently, the knowledge of a particular solution of a Riccati equation allows us to find its general solution by means of two quadratures. It is worth recalling that this property can be more generally understood by means of the theory of Lie systems. Indeed, this theory states that the knowledge of a particular solution of a Lie system enables us to reduce the initial equation into a 'simpler' one; see Section 1.2 or [40].

If we know two particular solutions, $x_1(t)$ and $x_2(t)$, of equation (4.1), its general solution can be determined with one quadrature. Indeed, the change of variable $z = (x - x_1(t))/(x - x_2(t))$ transforms the original equation into a homogeneous linear differential equation, and hence the general solution can be immediately found.

Finally, giving three particular solutions, $x_1(t), x_2(t), x_3(t)$, the general solution can be written, without making use of any quadrature, in terms of the superposition rule (1.11).

The simplest case of Riccati equation, i.e. the one with b_1, b_2 and b_3 being constant, has been fully studied and it is integrable by quadratures (see for example [64]). This can be viewed as a consequence of the existence of a constant (maybe complex) solution, permitting us to reduce the equation to an inhomogeneous linear one. Note also that, in a similar way, separable Riccati equations of the form

$$\frac{dx}{dt} = \varphi(t)(c_1 + c_2x + c_3x^2),$$

with $\varphi(t)$ being a nonvanishing function, are integrable, because they admit a constant solution again, which enables us to transform the equation into a linear inhomogeneous one. On the other hand, the integrability of the above equation can also be related to the existence of a t-reparametrisation, reducing the problem to an autonomous one.

4.2. Transformation laws of Riccati equations. We here describe an important property of Lie systems, in the particular case of Riccati equations, playing a relevant rôle in establishing integrability criteria: *The group $\mathcal{G}$ of curves in a Lie group G associated with a Lie system acts on the set of related Lie systems.*

More explicitly, consider a family X_1, X_2, X_3 of vector fields on $\overline{\mathbb{R}}$, e.g. the set given in (1.26), spanning the Vessiot–Guldberg Lie algebra of vector fields associated with Riccati equations and isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. In terms of this family, each Riccati equation (4.1) is related to a t-dependent vector field $X_t = b_1(t)X_1 + b_2(t)X_2 + b_3(t)X_3$, which can be considered as a curve $(b_1(t), b_2(t), b_3(t))$ in $\mathbb{R}^3$. Each element $\bar{A}$ of the group of smooth curves in $SL(2,\mathbb{R})$, i.e. $\bar{A} \in \mathcal{G} \equiv \mathrm{Map}(\mathbb{R}, SL(2,\mathbb{R}))$, transforms every curve $x(t)$ in $\overline{\mathbb{R}}$ into a new one $x'(t) = \Phi(\bar{A}(t), x(t))$ by means of the action $\Phi : (A, x) \in SL(2,\mathbb{R}) \times \overline{\mathbb{R}} \mapsto \Phi(A, x) \in \bar{\mathbb{R}}$ of the form

$$\Phi(A,x) = \begin{cases} \dfrac{\alpha x + \beta}{\gamma x + \delta} & x \neq -\dfrac{\delta}{\gamma},\ x \neq \infty, \\ \alpha/\gamma & x = \infty, \\ \infty & x = -\dfrac{\delta}{\gamma}, \end{cases} \qquad \text{where} \quad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \tag{4.2}$$

Moreover, the above t-dependent change of variables transforms the Riccati equation (4.1) into a new one with t-dependent coefficients b'_1, b'_2, b'_3 given by

$$\begin{cases} b'_3 = \delta^2 b_3 - \delta\gamma b_2 + \gamma^2 b_1 + \gamma\dot{\delta} - \delta\dot{\gamma}, \\ b'_2 = -2\beta\delta b_3 + (\alpha\delta + \beta\gamma) b_2 - 2\alpha\gamma b_1 + \delta\dot{\alpha} - \alpha\dot{\delta} + \beta\dot{\gamma} - \gamma\dot{\beta}, \\ b'_1 = \beta^2 b_3 - \alpha\beta b_2 + \alpha^2 b_1 + \alpha\dot{\beta} - \beta\dot{\alpha}. \end{cases} \tag{4.3}$$

Indeed, the above expressions define an affine action of the group $\mathcal{G}$ on the set of Riccati equations. In other words, given $A_1, A_2 \in \mathcal{G}$, transforming the coefficients of a general Riccati equation by means of two successive transformations of the above type, e.g. first by A_1 and then by A_2, gives exactly the same result as doing only one transformation with $A_2 \cdot A_1 \in \mathcal{G}$ (see [63, 151]).

The group $\mathcal{G}$ also acts on the set of equations of the form (1.31) on $SL(2,\mathbb{R})$. In order to show this, note first that $\mathcal{G}$ acts on the left on the set of curves in $SL(2,\mathbb{R})$ by left translations, i.e. given two curves $\bar{A}(t), A(t) \subset SL(2,\mathbb{R})$, the curve $\bar{A}(t)$ transforms the curve $A(t)$ into a new one $A'(t) = \bar{A}(t)A(t)$. Moreover, if $A(t)$ is a solution of (1.31), then $A'(t)$ satisfies a new equation like (1.31) but with a different right hand side $\mathrm{a}'(t)$. Differentiating the relation $A'(t) = \bar{A}(t)A(t)$ and taking into account the form of (1.31), we see that, in the basis (2.4), the relation between the curves $\mathrm{a}(t)$ and $\mathrm{a}'(t)$ in $\mathfrak{sl}(2,\mathbb{R})$ is

$$\mathrm{a}'(t) = \bar{A}(t)\mathrm{a}(t)\bar{A}^{-1}(t) + \dot{\bar{A}}(t)\bar{A}^{-1}(t) = -\sum_{\alpha=1}^{3} b'_\alpha(t)\mathrm{a}_\alpha, \tag{4.4}$$

which yields the expressions (4.3). Conversely, if $A'(t) = \bar{A}(t)A(t)$ is the solution for the equation corresponding to the curve $\mathrm{a}'(t)$ given by the transformation rule (4.4), then $A(t)$ is the solution of the equation (1.31) determined by the curve $\mathrm{a}(t)$.

Summarising, we have shown that it is possible to associate to each Riccati equation an equation on the Lie group $SL(2,\mathbb{R})$ and to define an infinite-dimensional group of transformations acting on the set of Riccati equations. Additionally, this process can be easily derived in a similar way for any Lie system (see [47]).

4.3. Lie structure of an equation describing transformations of Lie systems. Let us construct a Lie system describing the curves in $SL(2,\mathbb{R})$ which transform the Riccati equation associated with an equation on $SL(2,\mathbb{R})$ characterised by a curve $\mathrm{a}(t) \subset \mathfrak{sl}(2,\mathbb{R})$ into the Riccati equation associated with the curve $\mathrm{a}'(t) \subset \mathfrak{sl}(2,\mathbb{R})$. By means of this Lie system, we later explain the results derived in [47] in order to describe, from a unified point of view, the developments of [40, 50].

Multiply equation (4.4) on the right by $\bar{A}(t)$ to get

$$\dot{\bar{A}}(t) = \mathrm{a}'(t)\bar{A}(t) - \bar{A}(t)\mathrm{a}(t). \tag{4.5}$$

If we consider the above equation as a system of first-order differential equations for the coefficients of the curve $\bar{A}(t)$ in $SL(2,\mathbb{R})$, with

$$\bar{A}(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{pmatrix}, \qquad \alpha(t)\delta(t) - \beta(t)\gamma(t) = 1,$$

then system (4.5) reads

$$\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \\ \dot{\delta} \end{pmatrix} = \begin{pmatrix} \frac{b_2'-b_2}{2} & b_3 & b_1' & 0 \\ -b_1 & \frac{b_2'+b_2}{2} & 0 & b_1' \\ -b_3' & 0 & -\frac{b_2'+b_2}{2} & b_3 \\ 0 & -b_3' & -b_1 & -\frac{b_2'-b_2}{2} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}. \tag{4.6}$$

The solutions $y(t) = (\alpha(t), \beta(t), \gamma(t), \delta(t))$ of the above system relating two given Riccati equations are associated with curves in $SL(2,\mathbb{R})$, i.e. they are such that, at any time, $\alpha\delta - \beta\gamma = 1$. Nevertheless, we can drop such a restriction for the time being as it can be implemented by a restraint on the initial conditions for the solutions, and hence we can treat the variables $\alpha, \beta, \gamma, \delta$ in the system (4.6) as being independent. In this case, this linear system can be regarded as a Lie system linked to a Lie algebra of vector fields isomorphic to $\mathfrak{gl}(4,\mathbb{R})$. Nevertheless, it may also be understood as a Lie system related to a Lie algebra of vector fields isomorphic to a Lie subalgebra of $\mathfrak{gl}(4,\mathbb{R})$. Indeed, consider the vector fields

$$\begin{aligned}
N_1 &= -\alpha\frac{\partial}{\partial\beta} - \gamma\frac{\partial}{\partial\delta}, & N_1' &= \gamma\frac{\partial}{\partial\alpha} + \delta\frac{\partial}{\partial\beta}, \\
N_2 &= \frac{1}{2}\left(\beta\frac{\partial}{\partial\beta} + \delta\frac{\partial}{\partial\delta} - \alpha\frac{\partial}{\partial\alpha} - \gamma\frac{\partial}{\partial\gamma}\right), & N_2' &= \frac{1}{2}\left(\alpha\frac{\partial}{\partial\alpha} + \beta\frac{\partial}{\partial\beta} - \gamma\frac{\partial}{\partial\gamma} - \delta\frac{\partial}{\partial\delta}\right), \\
N_3 &= \beta\frac{\partial}{\partial\alpha} + \delta\frac{\partial}{\partial\gamma}, & N_3' &= -\alpha\frac{\partial}{\partial\gamma} - \beta\frac{\partial}{\partial\delta},
\end{aligned}$$

spanning a Vessiot–Guldberg Lie algebra of vector fields isomorphic to $\mathfrak{g} \equiv \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \subset \mathfrak{gl}(4,\mathbb{R})$. Consequently, the linear system of differential equation (4.6) is a Lie system on $\mathbb{R}^4$ associated with a Lie algebra of vector fields isomorphic to $\mathfrak{g}$ (see [47]).

If we denote $y \equiv (\alpha, \beta, \gamma, \delta) \in \mathbb{R}^4$, system (4.6) is a differential equation on $\mathbb{R}^4$ of the form

$$\frac{dy}{dt} = N(t, y), \tag{4.7}$$

with N being the t-dependent vector field

$$N_t = \sum_{\alpha=1}^{3} (b_\alpha(t) N_\alpha + b'_\alpha(t) N'_\alpha).$$

The vector fields $\{N_1, N_2, N_3, N'_1, N'_2, N'_3\}$ span a regular distribution $\mathcal{D}$ with rank three at almost every point of $\mathbb{R}^4$ and thus there exists, at least locally, a first integral for all the vector fields in the distribution $\mathcal{D}$. The method of characteristics allows us to determine that this first integral can be

$$I : y = (\alpha, \beta, \gamma, \delta) \in \mathbb{R}^4 \mapsto \det y \equiv \alpha\delta - \beta\gamma \in \mathbb{R}.$$

Moreover, this first integral is related to the determinant of a matrix $\bar{A} \in SL(2, \mathbb{R})$ with coefficients given by the components of $y = (\alpha, \beta, \gamma, \delta)$. Therefore, if we have a solution of the system (4.6) with initial condition such that $\det y(0) = \alpha(0)\delta(0) - \beta(0)\gamma(0) = 1$, then $\det y(t) = 1$ at any time t and the solution can be understood as a curve in $SL(2, \mathbb{R})$. Summarising, we have proved the following theorem.

Theorem 4.1. *The curves in* $SL(2, \mathbb{R})$ *transforming equation* (1.31) *into a new equation of the same form characterised by a curve* $\mathrm{a}'(t) = -\sum_{\alpha=1}^{3} b'_\alpha(t)\mathrm{a}_\alpha$ *are described through the solutions of the Lie system*

$$\frac{dy}{dt} = N(t, y) = \sum_{\alpha=1}^{3} b_\alpha(t) N_\alpha(y) + \sum_{\alpha=1}^{3} b'_\alpha(t) N'_\alpha(y). \tag{4.8}$$

such that $\det y(0) = 1$. *Furthermore, the above Lie system is related to a nonsolvable Vessiot–Guldberg Lie algebra isomorphic to* $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$.

A consequence of the above theorem is the following corollary, whose proof is left to the reader.

Corollary 4.2. *Given two Riccati equations associated with curves* $\mathrm{a}'(t)$ *and* $\mathrm{a}(t)$ *in* $\mathfrak{sl}(2, \mathbb{R})$, *there always exists a curve* $\bar{A}(t)$ *in* $SL(2, \mathbb{R})$ *transforming the Riccati equation related to* $\mathrm{a}(t)$ *into one associated with* $\mathrm{a}'(t)$. *Furthermore, if* $\bar{A}(0) = I$, *this curve is uniquely defined.*

Even if we know that given two equations on the Lie group $SL(2, \mathbb{R})$ there always exists a transformation relating them, in order to obtain such a curve we need to solve the differential equation (4.7) which, unfortunately, is a Lie system related to a nonsolvable Vessiot–Guldberg. Consequently, it is not easy to find its solutions in general, because, for instance, it is not integrable by quadratures.

Nevertheless, many known and new integrability conditions for Riccati equations can be determined by means of Theorem 4.1. Furthermore, the procedure to obtain the Lie system (4.7) can be generalised to deal with any Lie system related to a Lie group G with Lie algebra $\mathfrak{g}$ (cf. [47]).

4.4. Description of some known integrability conditions. Note that Lie systems on G of the form (1.31) determined by a constant curve, $\mathrm{a} = -\sum_{\alpha=1}^{3} c_\alpha \mathrm{a}_\alpha$, are integrable, and therefore the same happens for curves of the form $\mathrm{a}(t) = -D(\sum_{\alpha=1}^{3} c_\alpha \mathrm{a}_\alpha)$, where $D = D(t)$ is a nonvanishing function, as a t-reparametrisation reduces the problem to the previous one.

Our aim now is to determine the curves $\bar{A}(t)$ in $SL(2,\mathbb{R})$ transforming the equation on $SL(2,\mathbb{R})$ characterised by a curve $\mathrm{a}(t)$ into the equation on $SL(2,\mathbb{R})$ characterised by $\mathrm{a}'(t) = -D(c_1\mathrm{a}_1 + c_2\mathrm{a}_2 + c_3\mathrm{a}_3)$, with $D = D(t)$ a nonvanishing function and $c_1c_3 \neq 0$. As the final equation is associated with a solvable one-dimensional Vessiot–Guldberg Lie algebra, such a transformation allows us to find by quadratures the solution of the initial equation, and therefore the solution for its associated Riccati equation. In order to get the transformation between the Riccati equations linked to the above equations on $SL(2,\mathbb{R})$, we look for particular curves $\bar{A}(t)$ in $SL(2,\mathbb{R})$ satisfying certain conditions in order to get an integrable equation (4.6). Nevertheless, under the assumed restrictions, we may obtain a system of differential equations which does not admit any solution. In such a case, the conditions ensuring the existence of solutions will be integrability conditions. As an application we show that many known results on integrability of Riccati equations can be recovered and explained in this way.

We have already shown that Riccati equations (4.1), with $b_1b_3 \equiv 0$, are reducible to linear differential equations and therefore they are always integrable [57]. Hence, they are not interesting in the study of integrability conditions and we can focus on reducing Riccati equations with $b_1b_3 \neq 0$ into integrable ones by means of the action of a curve in $SL(2,\mathbb{R})$. To this end, consider the family of curves with $\beta = 0$ and $\gamma = 0$, i.e. curves in $SL(2,\mathbb{R})$ of the form

$$A(t) = \begin{pmatrix} \alpha(t) & 0 \\ 0 & \delta(t) \end{pmatrix} \subset SL(2,\mathbb{R}), \qquad \alpha(t)\delta(t) = 1.$$

The curve $\bar{A}(t)$ in $SL(2,\mathbb{R})$ determines a t-dependent change of variables in $\overline{\mathbb{R}}$ given by $x'(t) = \Phi(\bar{A}(t), x)$. In view of the action (4.2), and as $\alpha\delta = 1$, the previous change of variables reads

$$x' = \alpha^2(t)x = G(t)x, \qquad G(t) \equiv \frac{\alpha(t)}{\delta(t)} > 0. \tag{4.9}$$

In view of relations (4.3), the initial Riccati equation is transformed, by means of the curve $\bar{A}(t)$, into the new Riccati equation with t-dependent coefficients

$$b_1' = \alpha^2 b_1, \qquad b_2' = \alpha\delta b_2 + \dot{\alpha}\delta - \alpha\dot{\delta}, \qquad b_3' = \delta^2 b_3.$$

Moreover, the functions $\alpha(t)$ and $\delta(t)$ are solutions of (4.7), which in this case reduces to

$$\begin{pmatrix} \dot{\alpha} \\ 0 \\ 0 \\ \dot{\delta} \end{pmatrix} = \begin{pmatrix} \frac{b_2'-b_2}{2} & b_3 & b_1' & 0 \\ -b_1 & \frac{b_2'+b_2}{2} & 0 & b_1' \\ -b_3' & 0 & -\frac{b_2'+b_2}{2} & b_3 \\ 0 & -b_3' & -b_1 & -\frac{b_2'-b_2}{2} \end{pmatrix} \begin{pmatrix} \alpha \\ 0 \\ 0 \\ \delta \end{pmatrix}. \tag{4.10}$$

The existence of solutions for the above system that are related to elements of $SL(2,\mathbb{R})$

determines the integrability of the Riccati equation. Thus, let us analyse the existence of such solutions.

From the above system, we get

$$-b_1\alpha + b_1'\delta = 0, \qquad -b_3'\alpha + b_3\delta = 0.$$

As $\alpha(t) = 1$, these relations imply that $b_1'b_3' = b_1b_3$ and

$$\alpha^2 = \frac{b_1'}{b_1} = \frac{b_3}{b_3'} \equiv G > 0.$$

Hence, the transformation formulas (4.3) reduce to

$$b_3' = \alpha^{-2}b_3, \qquad b_2' = b_2 + 2\frac{\dot\alpha}{\alpha}, \qquad b_1' = \alpha^2 b_1. \tag{4.11}$$

Then, in order to get a t-dependent function D and two real constants c_1 and c_3, with $c_1c_3 \neq 0$, such that $b_3' = Dc_3$ and $b_1' = Dc_1$, the function D must be given by

$$D^2c_1c_3 = b_1b_3 \quad \text{so} \quad D = \pm\sqrt{\frac{b_1b_3}{c_1c_3}},$$

where we have used that $b_1'b_3' = b_1b_3$. On the other hand, as $b_1'/b_1 = \alpha^2 > 0$, we have to fix the sign κ of the function D in order to satisfy this relation, i.e. $\mathrm{sg}(c_1D) = \mathrm{sg}(b_1)$. Therefore,

$$\kappa = \mathrm{sg}(D) = \mathrm{sg}(b_1/c_1).$$

Also, as $b_1b_3 = b_1'b_3'$, we get $\mathrm{sg}(b_1b_3) = \mathrm{sg}(c_1c_3)$. Furthermore, in view of (4.11), α is determined, up to sign, by

$$\alpha = \sqrt{\frac{Dc_1}{b_1}} = \left(\frac{c_1}{c_3}\,\frac{b_3}{b_1}\right)^{1/4}. \tag{4.12}$$

and therefore the change of variables (4.9) reads

$$x' = \frac{D(t)c_1}{b_1(t)}x. \tag{4.13}$$

Finally, as a consequence of (4.11), in order for b_2' to be the product $b_2' = c_2D$, we see that

$$b_2 + 2\frac{\dot\alpha}{\alpha} = \kappa c_2\sqrt{\frac{b_1b_3}{c_1c_3}}. \tag{4.14}$$

Using (4.12) and the above equality, we see that the integrability condition is

$$\sqrt{\frac{c_1c_3}{b_1b_3}}\left[b_2 + \frac{1}{2}\left(\frac{\dot b_3}{b_3} - \frac{\dot b_1}{b_1}\right)\right] = \kappa c_2.$$

Conversely, if the above integrability condition is valid and $D^2c_1c_3 = b_1b_3$, the change of variables (4.13) transforms the Riccati equation (4.1) into $dx'/dt = D(t)(c_1 + c_2y' + c_3y'^2)$, with $c_1c_3 \neq 0$. To sum up, we have proved the following theorem.

Theorem 4.3. *Necessary and sufficient conditions for the existence of a transformation*

$$x' = G(t)x, \qquad G(t) > 0,$$

relating the Riccati equation

$$\frac{dx}{dt} = b_1(t) + b_2(t)x + b_3(t)x^2, \qquad b_1b_3 \neq 0,$$

to an integrable one given by

$$\frac{dx'}{dt} = D(t)(c_1 + c_2x' + c_3x'^2), \qquad c_1c_3 \neq 0, \qquad D(t) \neq 0, \tag{4.15}$$

where c_1, c_2, c_3 *are real numbers and* $D(t)$ *is a nonvanishing function, are*

$$D^2c_1c_3 = b_1b_3, \qquad \left(b_2 + \frac{1}{2}\left(\frac{\dot b_3}{b_3} - \frac{\dot b_1}{b_1}\right)\right)\sqrt{\frac{c_1c_3}{b_1b_3}} = \kappa c_2, \tag{4.16}$$

where $\kappa = \mathrm{sg}(D) = \mathrm{sg}(b_1/c_1)$. *The transformation is then uniquely defined by*

$$x' = \sqrt{\frac{b_3(t)c_1}{b_1(t)c_3}}x.$$

From previous results, the following corollary follows.

Corollary 4.4. *A Riccati equation* (4.15) *with* $b_1b_3 \neq 0$ *can be transformed into a Riccati equation of the form* (4.15) *by a* t*-dependent change of variables* $y' = G(t)y$*, with* $g(t) > 0$*, if and only if*

$$\frac{1}{\sqrt{|b_1b_3|}}\left(b_2 + \frac{1}{2}\left(\frac{\dot b_3}{b_3} - \frac{\dot b_1}{b_1}\right)\right) = K, \tag{4.17}$$

for some real constant K. *In that case, the Riccati equation* (4.1) *is integrable by quadratures.*

In view of Theorem 4.3, if we start with the integrable Riccati equation (4.15), we can obtain the set of all Riccati equations that can be reached from it by means of a transformation of the form (4.9).

Corollary 4.5. *Given an integrable Riccati equation*

$$\frac{dx}{dt} = D(t)(c_1 + c_2x + c_3x^2), \qquad c_1c_3 \neq 0,\ D(t) \neq 0,$$

with $D(t)$ *a nonvanishing function, the set of Riccati equations which can be obtained by a transformation* $x' = G(t)x$*, with* $G(t) > 0$*, are those of the form*

$$\frac{dx'}{dt} = b_1(t) + \left(\frac{\dot b_1(t)}{b_1(t)} - \frac{\dot D(t)}{D(t)} + c_2D(t)\right)x' + \frac{D^2(t)c_1c_3}{b_1(t)}x'^2,$$

and the function G *is then given by*

$$G = \frac{Dc_1}{\sqrt{b_1}}.$$

Therefore, starting with an integrable equation, we can generate a family of solvable Riccati equations whose coefficients are parametrised by a nonvanishing function b_1. Moreover, the integrability condition for a Riccati equation to belong to this family can be easily verified.

The previous results can now be used for a better comprehension of some integrability conditions found in the literature. Let us illustrate this claim by reviewing some well-known integrability conditions through our methods.

The case of Allen and Stein. The main results of [4] can be recovered through our more general approach. In that work, a Riccati equation (4.1), with $b_1b_3 > 0$ and b_0, b_2 being differentiable functions satisfying the condition

$$\frac{b_2 + \frac{1}{2}\left(\frac{\dot b_3}{b_3} - \frac{\dot b_1}{b_1}\right)}{\sqrt{b_1 b_3}} = C, \tag{4.18}$$

where C is a real constant, was transformed into the integrable one

$$\frac{dx'}{dt} = \sqrt{b_1(t)b_3(t)}(1 + Cx' + x'^2), \tag{4.19}$$

through a t-dependent linear transformation of the form

$$x' = \sqrt{\frac{b_3(t)}{b_1(t)}}x.$$

If a Riccati equation obeys the integrability condition (4.18), it also satisfies the assumptions of Corollary 4.4, and therefore, the integrability condition given in Theorem 4.3 with

$$c_1 = c_3 = 1, \qquad c_2 = C, \qquad D = \sqrt{b_1 b_3}.$$

Consequently, the t-dependent change of variables described by Theorem 4.3 reads

$$x' = \sqrt{\frac{b_3(t)}{b_1(t)}}x,$$

showing that the transformation in [4] is a particular case of our results. This is not surprising, as Theorem 4.3 shows that if such a t-dependent change of variables is used to transform a Riccati equation (4.1) into one of the form (4.15), this change of variables must be of the form (4.13) and the initial Riccati equation must satisfy the integrability conditions (4.16).

The case of Rao and Ukidave. Rao and Ukidave stated [190] that a Riccati equation (4.1), with $b_1 b_3 > 0$, can be transformed into an integrable one

$$\frac{dx'}{dt} = \sqrt{cb_1 b3}\left(1 + kx' + \frac{1}{c}x'^2\right),$$

through a t-dependent linear transformation

$$x' = \frac{1}{v(t)}x,$$

if there exist two real constants c and k such that the following integrability condition is satisfied:

$$b_3 = \frac{b_1}{cv^2}, \tag{4.20}$$

with $v(t)$ being a solution of the differential equation

$$\frac{dv}{dt} = kb_1(t) + b_2(t)v. \tag{4.21}$$

Note that, in view of (4.20), necessarily $c > 0$ and if (4.20) and (4.21) hold with constants c and k and a negative solution $v(t)$, the same conditions are valid for the constants c, $-k$ and a positive solution $-v(t)$. Consequently, we can restrict ourselves to studying the conditions (4.20) and (4.21) for positive solutions $v(t) > 0$. In such a case, the above method uses a t-dependent linear change of coordinates of the form (4.9) and the final Riccati equations are of the type described in (4.15). Therefore, the integrability conditions derived by Rao and Ukidave are a particular instance of the integrable cases described by Theorem 4.3.

Writing the value of $v(t)$ in terms of the constant c and the functions b_1 and b_3 obtained with the aid of (4.20) and (4.21), we get

$$\frac{1}{\sqrt{|b_1 b_3|}}\left(b_2+\frac{1}{2}\left(\frac{\dot b_3}{b_3}-\frac{\dot b_1}{b_1}\right)\right)=-k\mathrm{sg}(b_0)\sqrt{c}.$$

Hence, the Riccati equations satisfying (4.20) and (4.21) obey the integrability conditions of Corollary 4.5. Moreover, if we choose

$$D^2=cb_1b_3, \qquad c_1=1, \qquad c_2=-k, \qquad c_3=c^{-1},$$

then $D=\sqrt{cb_1b_3}$ and the only possible transformation (4.9) given by Theorem 4.3 reads

$$x'=\alpha^2(t)x=\sqrt{\frac{cb_3(t)}{b_1(t)}}x,$$

and thus

$$\frac{1}{v}=\sqrt{\frac{cb_3}{b_1}}.$$

In this way, we recover one of the results derived by Rao and Ukidave [190].

In short, many integrability conditions found in the literature can be described by our more general methods.

4.5. Integrability and reduction. Now we develop a similar procedure to the one above, but now we assume the solutions of system (4.6) to be included in a two-parameter subset of $SL(2,\mathbb{R})$. As a result, we recover some known integrability conditions and review, from a more general point of view, the integrability method described in [40].

As previously, let us try to relate the Riccati equation (4.1) to an integrable one associated, as a Lie system, with a curve $\mathrm{a}'(t)=-D(t)(c_1\mathrm{a}_1+c_2\mathrm{a}_2+c_3\mathrm{a}_3)$ with $c_3\neq 0$ and a nonvanishing function $D=D(t)$. We consider solutions of system (4.7) with $\gamma=0$, $\alpha>0$, and related to a curve in $SL(2,\mathbb{R})$, i.e. we analyse transformations

$$x'=\frac{\alpha(t)}{\delta(t)}x+\frac{\beta(t)}{\delta(t)}=\alpha^2(t)x+\frac{\beta(t)}{\delta(t)}.$$

In this case, using the expression of system (4.8) in coordinates (4.6), we get

$$\begin{pmatrix}\dot\alpha\\ \dot\beta\\ 0\\ \dot\delta\end{pmatrix}=\begin{pmatrix}\frac{b_2'-b_2}{2} & b_3 & b_1' & 0\\ -b_1 & \frac{b_2'+b_2}{2} & 0 & b_1'\\ -b_3' & 0 & -\frac{b_2'+b_2}{2} & b_3\\ 0 & -b_3' & -b_1 & -\frac{b_2'-b_2}{2}\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\\ 0\\ \delta\end{pmatrix}, \tag{4.22}$$

where $b_j'=Dc_j$ and $c_j\in\mathbb{R}$ for $j=1,2,3$. As we suppose $b_3'\neq 0$, the third equation of the above system yields

$$\frac{\alpha}{\delta}=\frac{b_3}{b_3'}=\frac{b_3}{Dc_3}.$$

Since $\alpha\delta = 1$ so that the solution of (4.8) is related to an element of $SL(2,\mathbb{R})$, and $b_3' = Dc_3$, the above expression implies

$$\alpha^2 = \frac{b_3}{Dc_3}. \tag{4.23}$$

Therefore, α is determined by the values of $b_3(t)$, D and c_3. Additionally, the first equation of (4.22) determines β in terms of α and the coefficients of the initial and final Riccati equations, i.e.

$$\beta = \frac{1}{b_3}\left(\dot{\alpha} - \frac{b_2' - b_2}{2}\alpha\right).$$

Taking into account (4.23) and as $\alpha\delta = 1$, we can define $M = \beta/\alpha$ and rewrite the above expression as

$$\frac{dD}{dt} = \left(b_2(t) + \frac{\dot{b}_3(t)}{b_3(t)}\right)D - c_2D^2 - 2b_3(t)MD.$$

Considering the differential equation on $\dot{\beta}$ in terms of M, we get the equation

$$\frac{dM}{dt} = -b_1(t) + \frac{c_1c_3}{b_3(t)}D^2 + b_2(t)M - b_3(t)M^2.$$

Finally, as $\delta\alpha = 1$ is a first integral of (4.8), if the system for the variables M and D and all the above mentioned conditions are satisfied, the value $\delta = \alpha^{-1}$ obeys the corresponding differential equations of the system (4.22). Summarising, we have the following theorem.

Theorem 4.6. *Given a Riccati equation* (4.1) *there exists a transformation*

$$x' = G(t)x + H(t), \qquad G(t) > 0,$$

relating it to an integrable equation

$$\frac{dx'}{dt} = D(t)(c_1 + c_2x' + c_3x'^2) \tag{4.24}$$

with $c_3 \neq 0$ and D a nonvanishing function if and only if there exist functions D and M satisfying the system

$$\begin{cases} \dfrac{dD}{dt} = \left(b_2(t) + \dfrac{\dot{b}_3(t)}{b_3(t)}\right)D - c_2D^2 - 2b_3(t)MD, \\ \dfrac{dM}{dt} = -b_1(t) + \dfrac{c_1c_3}{b_3(t)}D^2(t) + b_2(t)M - b_3(t)M^2. \end{cases}$$

The transformation is then given by

$$x' = \frac{b_3(t)}{D(t)c_3}(x + M(t)). \tag{4.25}$$

If we consider $c_1 = 0$ in equation (4.24), the system determining the curve in $SL(2,\mathbb{R})$ which performs the transformation of Theorem 4.6 reads

$$\begin{cases} \dfrac{dD}{dt} = \left(b_2(t) + \dfrac{\dot{b}_3(t)}{b_3(t)}\right)D - c_2D^2(t) - 2b_3(t)MD, \\ \dfrac{dM}{dt} = -b_1(t) + b_2(t)M - b_3(t)M^2. \end{cases} \tag{4.26}$$

Note that this system does not involve any integrability condition, since there always exists a solution for every initial condition. Nevertheless, finding such solutions can be as difficult as solving the initial Riccati equation. Therefore, we need to assume some simplification in order to find a particular solution. Let us put, for instance, $M = b_1/b_2$. In this case, the first differential equation of the above system does not depend on M and reduces to

$$\frac{dD}{dt} = \left(-b_2(t) + \frac{\dot b_3(t)}{b_3(t)}\right) D - c_2 D^2$$

whose solutions read

$$D(t) = \frac{\exp(\int_0^t A(t')\,dt')}{C + c_2 \int_0^t \exp(\int_0^{t''} A(t')\,dt')\,dt''}, \qquad A(t) = -b_2(t) + \frac{\dot b_3(t)}{b_3(t)}.$$

Meanwhile, as $M = b_2/b_3$ must satisfy the second equation in (4.26), we obtain

$$\frac{d}{dt}\left(\frac{b_2}{b_3}\right) = -b_1,$$

which gives rise to an integrability condition, considered in [189].

Let us recover, from our point of view, the result that establishes that the knowledge of a particular solution of the Riccati equation allows us to obtain its general solution. In fact, under the change of variables $M = -x$, the system (4.26) becomes

$$\begin{cases} \dfrac{dD}{dt} = \left(b_2(t) + \dfrac{\dot b_3(t)}{b_3(t)}\right) D - c_2 D^2 + 2 b_3(t) x D, \\ \dfrac{dx}{dt} = b_1(t) + b_2(t) x + b_3(t) x^2. \end{cases} \tag{4.27}$$

Each particular solution of (4.27) takes the form $(D_p(t), x_p(t))$, with $x_p(t)$ being a particular solution of the Riccati equation (4.1). Therefore, given such a particular solution $x_p(t)$, the function $D_p = D_p(t)$ satisfies the equation

$$\frac{dD_p}{dt} = \left(b_2(t) + \frac{\dot b_3(t)}{b_3(t)} + 2 b_3(t) x_p(t)\right) D_p - c_2 D_p^2, \tag{4.28}$$

which is a Bernoulli equation, and therefore is integrable by quadratures. Consequently, the knowledge of a particular solution $x_p(t)$ of the Riccati equation (4.1) allows us to determine a particular solution $(D_p(t), x_p(t))$ of (4.27) and, in view of the change of variables $x = -M$, a particular solution $(D_p(t), M_p(t)) = (D_p(t), -x_p(t))$ of (4.26). Finally, the functions $M_p(t)$ and $D(t)$ lead to the change of variables (4.25) described in Theorem 4.6 which transforms the initial Riccati equation (4.1) into another one related to a solvable Lie algebra of vector fields.

The above process describes a reduction process similar to the one derived in [40], but our method allows us to obtain a direct reduction to an integrable Riccati equation (4.24) through a particular solution.

There exist many ways to impose conditions on the coefficients of the second equation in (4.27) to obtain a particular solution easily. For instance, if there exists a real constant c such that for the t-dependent functions b_1, b_2 and b_3 we have $b_1 + b_2 c + b_3 c^2 = 0$, then c is a particular solution, for example:

1. $b_1 + b_2 + b_3 = 0$ implies that $c = 1$ is a particular solution.
2. $k_2^2 b_1 + k_2 k_3 b_2 + k_3^2 b_3 = 0$ means that $c = k_3/k_2$ is a particular solution.

This corresponds to some cases found in [40, 214].

As a first application of the above method, we can integrate the Riccati equation

$$\frac{dx}{dt} = -\frac{n}{t} + \left(1 + \frac{n}{t}\right)x - x^2, \tag{4.29}$$

related to Hovy's equation [200]. This Riccati equation admits the particular constant solution $x_p(t) = 1$. Using it in (4.28) and taking, for instance, $c_1 = 0$ and $c_2 = 0$, we obtain a particular solution for (4.28), $D_p(t) = t^n e^{-t}$. Hence, $(t^n e^{-t}, 1)$ is a particular solution of (4.27) related to equation (4.29) and $(t^n e^{-t}, -1)$ is a solution of (4.26). In this way, Theorem 4.6 states that the transformation (4.25), determined by $D_p(t) = t^n e^{-t}$ and $M_p(t) = -1$, of the form

$$x' = -t^{-n} e^t c_3^{-1}(x - 1), \tag{4.30}$$

relates the solutions of (4.29) to those of the integrable equation

$$\frac{dx'}{dt} = e^{-t} t^n c_3 x'^2.$$

If we fix $c_3 = 1$, the solution of the above equation reads

$$x'(t) = \frac{1}{K - \Gamma(1 + n, t)},$$

where K is an integration constant and $\Gamma(a, b)$ is the incomplete Euler's Gamma function

$$\Gamma(a, t) = \int_t^\infty t'^{a-1} e^{-t'}\, dt'.$$

In view of the change of variables (4.30), the solutions $x(t)$ of (4.29) and $x'(t)$ are related through the expression $x'(t) = -t^{-n} e^t c_3^{-1}(x(t) - 1)$. Therefore, if we substitute the general solution $x'(t)$ in this expression, we can derive the general solution for the Riccati equation (4.29), that is,

$$x(t) = 1 - \frac{e^{-t} t^n}{\Gamma(n + 1, t) + K}.$$

4.6. Linearisation of Riccati equations. To finish this chapter, we shall analyse the problem of linearisation of Riccati equations through fractional linear transformations (4.9). As a main result, we establish various integrability conditions ensuring that a Riccati equation can be transformed into a linear one by means of a diffeomorphism on $\overline{\mathbb{R}}$ associated with a fractional linear transformation of a certain class.

As a first insight, notice that Corollary 4.2 states that there exists a curve in $SL(2, \mathbb{R})$, and therefore a t-dependent fractional linear transformation on $\overline{\mathbb{R}}$, transforming each given Riccati equation into any other one (and, in particular, into a linear one). This clearly implies that Riccati equations are always linearisable by this class of transformations. However, as the Lie system (4.7) describing such transformations is related to a nonsolvable Lie algebra of vector fields, determining such a transformation can be as difficult as solving the Riccati equation to be linearised.

Let us try to transform a given Riccati equation into a linear differential equation by means of a fractional linear transformation (4.2) determined by a constant vector $(\alpha,\beta,\gamma,\delta)\in\mathbb{R}^4$ with $\alpha\delta-\beta\gamma=1$. In this case, determining the conditions ensuring the existence of solutions of system (4.7) performing such a transformation is an easy task. Moreover, as solving (4.7) also becomes straightforward, we can determine some linearisability conditions and, when these conditions hold, specify the corresponding change of variables.

Note that as $(\alpha,\beta,\gamma,\delta)$ is a constant, we have $\dot\alpha=\dot\beta=\dot\gamma=\dot\delta=0$ and, in view of (4.6), the diffeomorphism on $\overline{\mathbb{R}}$ performing the transformation is related to a vector in the kernel of the matrix

$$B=\begin{pmatrix}\frac{b_2'-b_2}{2} & b_3 & b_1' & 0\\ -b_1 & \frac{b_2'+b_2}{2} & 0 & b_1'\\ 0 & 0 & -\frac{b_2'+b_2}{2} & b_3\\ 0 & 0 & -b_1 & -\frac{b_2'-b_2}{2}\end{pmatrix}, \tag{4.31}$$

where we assume $b_1\neq 0, b_3\neq 0$. We omit the study of the case $b_1(t)b_3(t)=0$ in an open interval because, as shown in Section 4.1, this case is integrable.

A necessary and sufficient condition for $\ker B$ to be nontrivial is $\det B=0$. Therefore, a short calculation shows that $\dim\ker B>0$ if and only if $-b_2^2+b_3'^2(t)+4b_1b_3=0$. Thus, $b_3'=\pm\sqrt{b_2^2-4b_1b_3}$ and b_3' is fixed, up to sign, by the values of b_1, b_2 and b_3. Let us study the kernel of the matrix B in the positive and negative cases for b_2'.

Positive case. The kernel of matrix (4.31) is given by the vectors

$$\left(\delta\frac{b_1'}{b_1}+\beta\frac{b_2+\sqrt{b_2^2-4b_1b_3}}{2b_1},\beta,-\delta\frac{-b_2+\sqrt{b_2^2-4b_1b_3}}{2b_1},\delta\right),\qquad \delta,\beta\in\mathbb{R}.$$

Recall that we are only considering the constant elements of $\ker B$, therefore there should be two real constants K_1 and K_2 such that

$$\begin{aligned}K_1&=\delta\frac{b_1'}{b_1}+\beta\frac{b_2+\sqrt{b_2^2-4b_1b_3}}{2b_1},\\ K_2&=\frac{-b_2+\sqrt{b_2^2-4b_1b_3}}{2b_1}.\end{aligned}\tag{4.32}$$

Moreover, in order to relate these vectors to elements in $SL(2,\mathbb{R})$, we have to impose the condition $\det(K_1,\beta,-\delta K_2,\delta)=\delta(K_1+\beta K_2)=1$.

The second condition in (4.32) imposes a restriction on the coefficients of the initial Riccati equation to be linearisable by a constant fractional linear transformation (4.2). If this is satisfied, we can choose β,γ,K_1 and b_2' to satisfy the other conditions. Thus, the only linearisation condition is the second one in (4.32).

Negative case. In this case, $\ker B$ reads

$$\left(\frac{\delta b_1'}{b_1}+\beta\frac{b_2-\sqrt{b_2^2-4b_1b_3}}{2b_1},\beta,-\delta\frac{-b_2-\sqrt{b_2^2-4b_1b_3}}{2b_1},\delta\right),\qquad \delta,\beta\in\mathbb{R},$$

and now the new conditions reduce to the existence of two real constants K_1 and K_2

such that

$$K_1 = \frac{\delta b_1'}{b_1} + \beta \frac{b_2 - \sqrt{b_2^2 - 4b_1b_3}}{2b_1}, \qquad K_2 = \frac{-b_2 - \sqrt{b_2^2 - 4b_1b_3}}{2b_1},$$

with $\delta(K_1 + \beta K_2) = 1$. If the second expression of the above conditions is satisfied, we can proceed in a similar fashion as for the positive case to obtain the transformation that performs the linearisation of the initial Riccati equation.

Summarising:

THEOREM 4.7. *A necessary and sufficient condition for the existence of a fractional linear diffeomorphism of* $\bar{\mathbb{R}}$ *associated with a transformation on* $SL(2,\mathbb{R})$ *transforming the Riccati equation* (4.1) *into a linear differential equation is the existence of a real constant* K *such that*

$$K = \frac{-b_2 \pm \sqrt{b_2^2 - 4b_1b_3}}{2b_1}. \tag{4.33}$$

As a Riccati equation (4.1) satisfies the above condition if and only if K is a constant particular solution, we get the following corollary:

COROLLARY 4.8. *A Riccati equation can be linearised by means of a diffeomorphism on* $\overline{\mathbb{R}}$ *of the form* (4.2) *if and only if it admits a constant particular solution.*

Ibragimov showed that a Riccati equation (4.1) is linearisable by means of a change of variables $z = z(x)$ if and only if the equation admits a constant solution [123]. We have proved that in that case, the change of variables can be effected by a transformation of the type (4.2).

5. Lie integrability in classical physics

In spite of their apparent simplicity, the methods developed in the previous chapter reduce the analysis of certain integrability conditions for Riccati equations to studying integrability conditions for an equation on $SL(2,\mathbb{R})$. Moreover, these methods can also be applied to any other Lie system related to the same equation on $SL(2,\mathbb{R})$. For instance, we use the results on integrability of Riccati equations to study t-dependent (frequency and/or mass) harmonic oscillators (TDHOs), which are associated with the same kind of equations on $SL(2,\mathbb{R})$ as Riccati equations. As a particular application of our results, we supply t-dependent constants of motion for certain one-dimensional TDHOs and the solutions for a two-dimensional TDHO. Also, our approach provides a unifying framework which allows us to apply our developments to all Lie systems associated with equations in $SL(2,\mathbb{R})$ and to generalise our methods to study any Lie system.

5.1. TDHO as a SODE Lie system. Let us prove that every TDHO is a SODE Lie system (see [37, 43, 52]). Each TDHO is described by a t-dependent Hamiltonian of the form

$$H(t) = \frac{p^2}{2m(t)} + \frac{1}{2}F(t)\omega^2 x^2,$$

whose Hamilton equations read

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial p} = \dfrac{p}{m(t)}, \\ \dot{p} = -\dfrac{\partial H}{\partial x} = -F(t)\omega^2 x. \end{cases} \tag{5.1}$$

The solutions of the above system are integral curves for the t-dependent vector field

$$X_t = p\frac{\partial}{\partial x} - F(t)\omega^2 x\frac{\partial}{\partial p},$$

over $\mathrm{T}^*\mathbb{R}$. Let X_1^{HO}, X_2^{HO} and X_3^{HO} be the vector fields

$$X_1^{HO} = p\frac{\partial}{\partial x}, \quad X_2^{HO} = \frac{1}{2}\left(x\frac{\partial}{\partial x} - p\frac{\partial}{\partial p}\right), \quad X_3^{HO} = -x\frac{\partial}{\partial p}, \tag{5.2}$$

which satisfy the commutation relations

$$[X_1^{HO}, X_3^{HO}] = 2X_2^{HO}, \quad [X_1^{HO}, X_2^{HO}] = X_1^{HO}, \quad [X_2^{HO}, X_3^{HO}] = X_3^{HO},$$

and therefore span a Lie algebra of vector fields V^{HO} isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. The t-dependent vector field X^{HO} associated with system (5.1) can be written as

$$X^{HO}(t) = F(t)\omega^2 X_3^{HO} + \frac{1}{m(t)}X_1^{HO}, \tag{5.3}$$

i.e. it is a linear combination with t-dependent coefficients

$$X^{HO}(t) = \sum_{\alpha=1}^{3} b_\alpha(t) X_\alpha^{HO}, \tag{5.4}$$

with $b_1(t) = 1/m(t)$, $b_2(t) = 0$ and $b_3(t) = F(t)\omega^2$. Hence, TDHOs are SODE Lie systems.

Consider the basis $\{\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3\}$ for $\mathfrak{sl}(2,\mathbb{R})$ given in (2.4). Its elements satisfy the same commutation relations as the vector fields X_α^{HO}. Denote by $\Phi^{HO} : SL(2,\mathbb{R})\times\mathrm{T}^*\mathbb{R} \to \mathrm{T}^*\mathbb{R}$ the action that associates to each a_α the fundamental vector field X_α^{HO}, i.e. each one-parameter subgroup $\exp(-t\mathrm{a}_\alpha)$ acts on $\mathrm{T}^*\mathbb{R}$ with infinitesimal generator X_α^{HO}. It can be verified that this action reads

$$\Phi^{HO}\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \begin{pmatrix} x \\ p \end{pmatrix}\right) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} x \\ p \end{pmatrix}.$$

Obviously, the linear map $\rho^{HO} : \mathfrak{sl}(2,\mathbb{R}) \to V^{HO}$ that maps each a_α to X_α is a Lie algebra isomorphism.

The action Φ^{HO} allows us to relate (5.1) to an equation on $SL(2,\mathbb{R})$ given by

$$R_{A^{-1}*A}\dot{A} = -\sum_{\alpha=1}^{3} b_\alpha(t)\mathrm{a}_\alpha, \qquad A(0) = I. \tag{5.5}$$

Thus, if $A(t)$ is the solution of (5.5) and we denote $\xi = (x,p) \in \mathrm{T}^*\mathbb{R}$, then the solution starting from $\xi(0)$ is $\xi(t) = \Phi^{HO}(A(t), \xi(0))$ (see e.g. [40]). In summary, system (5.1) is a Lie system on $\mathrm{T}^*\mathbb{R}$ related to an equation on $SL(2,\mathbb{R})$ and the solution of (5.5) allows us to obtain the solutions of (5.1) in terms of the initial condition by means of the action Φ^{HO}.

5.2. Transformation laws of Lie equations on $SL(2,\mathbb{R})$. Each t-dependent harmonic oscillator (5.1) can be considered as a curve in $\mathbb{R}^3$ of the form $(b_1(t), b_2(t), b_3(t))$ through the decomposition (5.4). Then, we can transform each curve $\xi(t)$ in $\mathrm{T}^*\mathbb{R}$ by an element $\bar{A}(t)$ of $\mathcal{G}$ as follows:

$$\bar{A}(t) = \begin{pmatrix} \bar{\alpha}(t) & \bar{\beta}(t) \\ \bar{\gamma}(t) & \bar{\delta}(t) \end{pmatrix} \in \mathcal{G} \;\Rightarrow\; \Theta(\bar{A}, \xi)(t) = \begin{pmatrix} \bar{\alpha}(t)x(t) + \bar{\beta}(t)p(t) \\ \bar{\gamma}(t)x(t) + \bar{\delta}(t)p(t) \end{pmatrix}. \tag{5.6}$$

The above change of variables transforms the TDHO (5.1) into an analogous TDHO with new coefficients b'_1, b'_2, b'_3 given by

$$\begin{cases} b'_3 = \bar{\delta}^2 b_3 - \bar{\delta}\bar{\gamma} b_2 + \bar{\gamma}^2 b_1 + \bar{\gamma}\dot{\bar{\delta}} - \bar{\delta}\dot{\bar{\gamma}}, \\ b'_2 = -2\bar{\beta}\bar{\delta}\, b_3 + (\bar{\alpha}\bar{\delta} + \bar{\beta}\bar{\gamma}) b_2 - 2\bar{\alpha}\bar{\gamma} b_1 + \delta\dot{\bar{\alpha}} - \bar{\alpha}\dot{\bar{\delta}} + \bar{\beta}\dot{\bar{\gamma}} - \bar{\gamma}\dot{\bar{\beta}}, \\ b'_1 = \bar{\beta}^2 b_3 - \bar{\alpha}\bar{\beta} b_2 + \bar{\alpha}^2 b_1 + \bar{\alpha}\dot{\bar{\beta}} - \bar{\beta}\dot{\bar{\alpha}}. \end{cases}$$

The solutions of the transformed TDHO are of the form $\Theta(\bar{A}(t), \xi(t))$, with $\xi(t)$ being a solution of the initial TDHO. Additionally, the above expressions define an affine action (see e.g. [151] for the general definition) of the group $\mathcal{G}$ on the set of TDHOs [63]. This means that in order to transform the coefficients of a TDHO by means of two transformations of the above type, first A_1 and then A_2, it suffices to do the transformation induced by the product A_2A_1.

The result of this action of $\mathcal{G}$ can also be studied from the point of view of equations in $SL(2,\mathbb{R})$. First, $\mathcal{G}$ acts on the left on the set of curves in $SL(2,\mathbb{R})$ by left translations, i.e. a curve $\bar{A}(t)$ transforms the curve $A(t)$ into $A'(t) = \bar{A}(t)A(t)$. Therefore, if $A(t)$ is a solution of (5.5), characterised by a curve $\mathrm{a}(t) \in \mathfrak{sl}(2,\mathbb{R})$, then the new curve satisfies a new equation like (5.5) but with a different right-hand side, $\mathrm{a}'(t)$, and thus it corresponds to a new equation on $SL(2,\mathbb{R})$ associated with a new TDHO. Of course, $A'(0) = \bar{A}(0)A(0)$, and if we want $A'(0) = \mathrm{Id}$, we have to impose the additional condition $\bar{A}(0) = \mathrm{Id}$. In this way $\mathcal{G}$ acts on the set of curves in $T_I SL(2,\mathbb{R}) \simeq \mathfrak{sl}(2,\mathbb{R})$. It can be shown that the relation between the curves $\mathrm{a}(t)$ and $\mathrm{a}'(t)$ in $\mathfrak{sl}(2,\mathbb{R})$ is given by [40]

$$\mathrm{a}'(t) = -\sum_{\alpha=1}^{3} b'_\alpha(t)\mathrm{a}_\alpha = \bar{A}(t)\mathrm{a}(t)\bar{A}^{-1}(t) + \dot{\bar{A}}(t)\bar{A}^{-1}(t). \tag{5.7}$$

Summarising, it has been shown that it is possible to associate to any TDHO, in a one-to-one way, an equation in the Lie group $SL(2,\mathbb{R})$ and to define a group $\mathcal{G}$ of transformations on the set of such TDHOs induced by the natural linear action of $SL(2,\mathbb{R})$.

Recall that, in view of Theorem 4.1, system (5.7) can be regarded as a system of first-order ordinary differential equations in the coefficients of the curve in $SL(2,\mathbb{R})$ of the form

$$\bar{A}(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{pmatrix}.$$

Moreover, we can state the following results, which are a straightforward application to TDHOs of Theorem 4.1 and Corollary 4.2 formulated for the analysis of certain Lie systems on $SL(2,\mathbb{R})$ related to Riccati equations.

THEOREM 5.1. *The curves in $SL(2,\mathbb{R})$ transforming a TDHO related to an equation on this Lie group determined by a curve* $\mathrm{a}(t)$ *into a new TDHO associated with an equation on $SL(2,\mathbb{R})$ determined by the curve* $\mathrm{a}'(t)$*, with*

$$\mathrm{a}'(t) = -\sum_{\alpha=1}^{3} b'_\alpha(t)\mathrm{a}_\alpha, \qquad \mathrm{a}(t) = -\sum_{\alpha=1}^{3} b_\alpha(t)\mathrm{a}_\alpha,$$

are given by the integral curves of the t-dependent vector field

$$N(t) = \sum_{\alpha=1}^{3}(b_\alpha(t)N_\alpha + b'_\alpha(t)N'_\alpha), \tag{5.8}$$

such that $\det \bar{A}(0) = 1$*. This system is a Lie system associated with a nonsolvable Lie algebra of vector fields isomorphic to* $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$*. Moreover, such curves also transform the TDHO related to the curve* $\mathrm{a}(t)$ *into the new one linked to* $\mathrm{a}'(t)$*.*

COROLLARY 5.2. *Given two TDHOs associated with the curves* $\mathrm{a}(t)$ *and* $\mathrm{a}'(t)$ *in* $\mathfrak{sl}(2,\mathbb{R})$*, there always exists a curve in $SL(2,\mathbb{R})$ transforming one TDHO into the other.*

We must remark that even if we know that given two equations in the Lie group $SL(2,\mathbb{R})$ there always exists a transformation relating them, in order to find such a curve we need to solve the system of differential equations providing the integral curves of (5.8). This is the solution of a system of differential equations that is a Lie system related to a nonsolvable Lie algebra in general. Hence, it is not easy to find its solutions, i.e. it may not be integrable by quadratures.

The result of Theorem 5.1, i.e. that the system of differential equations describing the transformations of Lie systems on $SL(2,\mathbb{R})$ is a matrix Riccati equation associated, as a Lie system, with a Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$, suggests a method of finding sufficiency conditions for integrability of TDHOs to be explained next.

5.3. Description of some known integrability conditions. We now study some cases when it is possible to find curves $\bar{A}(t)$ in $SL(2,\mathbb{R})$ transforming a given TDHO related to an equation on $SL(2,\mathbb{R})$ characterised by a curve $\mathrm{a}(t)$ into a new TDHO associated with an equation on $SL(2,\mathbb{R})$ characterised by a curve of the type $\mathrm{a}'(t) = -D(t)(c_1\mathrm{a}_1 + c_2\mathrm{a}_2 + c_3\mathrm{a}_3)$. This is possible if the system determined by (5.8) can be easily solved. Such a transformation allows us to find the solution of the given equation by quadratures. We first restrict ourselves to cases in which the curve $\bar{A}(t)$ lies in a one-parameter subset of $SL(2,\mathbb{R})$. The results we give next are a direct translation of Theorem 4.1 to the framework of TDHO (see also [50]).

THEOREM 5.3. *Necessary and sufficient conditions for the existence of a transformation*

$$\xi' = \Phi^{HO}(\bar{A}_0(t),\xi), \qquad \xi = \begin{pmatrix} x \\ p \end{pmatrix},$$

with

$$\bar{A}_0(t) = \begin{pmatrix} \alpha(t) & 0 \\ 0 & \alpha^{-1}(t) \end{pmatrix}, \qquad \alpha(t) > 0, \tag{5.9}$$

relating the TDHO associated with the t-dependent vector field

$$X_t = b_1(t)X_1 + b_2(t)X_2 + b_3(t)X_3, \tag{5.10}$$

where $b_1(t)b_3(t)$ has a constant sign, i.e. $b_1(t)b_3(t) \neq 0$, to another integrable one given by

$$X'(t) = D(t)(c_1X_1 + c_2X_2 + c_3X_3), \tag{5.11}$$

with c_1, c_2, c_3 being real numbers such that $c_1c_3 \neq 0$, are

$$D^2(t)c_1c_3 = b_1(t)b_3(t), \qquad b_2(t) + \frac{1}{2}\left(\frac{\dot{b}_3(t)}{b_3(t)} - \frac{\dot{b}_1(t)}{b_1(t)}\right) = c_2\sqrt{\frac{b_1(t)b_3(t)}{c_1c_3}}.$$

In that case the transformation is uniquely defined by

$$\bar{A}_0(t) = \begin{pmatrix} \left(\frac{b_3(t)c_1}{b_1(t)c_3}\right)^{1/4} & 0 \\ 0 & \left(\frac{b_3(t)c_1}{b_1(t)c_3}\right)^{-1/4} \end{pmatrix}.$$

Note that one coefficient, either c_1 or c_3, can be reabsorbed by redefining D. As a straightforward application of the preceding theorem, which can be found in a similar way to those in [50], we obtain the following corollaries:

Corollary 5.4. *A TDHO* (5.1) *with $b_1(t)b_3(t) \neq 0$ is integrable by a t-dependent change of variables*

$$\xi' = \Phi^{HO}(\bar{A}_0(t), \xi),$$

with $\bar{A}_0$ given by (5.9)*, if and only if*

$$\sqrt{\frac{c_1c_3}{b_1(t)b_3(t)}}\left[b_2(t) + \frac{1}{2}\left(\frac{\dot{b}_3(t)}{b_3(t)} - \frac{\dot{b}_1(t)}{b_1(t)}\right)\right] = c_2 \tag{5.12}$$

for certain real constants c_1, c_2, and c_3. In this case

$$D(t) = \sqrt{\frac{b_1(t)b_3(t)}{c_1c_3}},$$

and the new system is

$$\frac{d\xi'}{dt} = D(t)\begin{pmatrix} c_2/2 & c_1 \\ -c_3 & -c_2/2 \end{pmatrix}\xi'. \tag{5.13}$$

Corollary 5.5. *Given an integrable TDHO characterised by a t-dependent vector field* (5.11)*, the set of TDHOs which can be obtained through a t-dependent transformation*

$$\xi' = \Phi^{HO}(\bar{A}_0(t), \xi),$$

with $\bar{A}_0$ given by (5.9)*, are those of the form*

$$X_t = b_1(t)X_1 + \left(\frac{\dot{b}_1(t)}{b_1(t)} - \frac{\dot{D}(t)}{D(t)} + c_2D(t)\right)X_2 + \frac{D^2(t)c_1c_3}{b_1(t)}X_3. \tag{5.14}$$

Thus, $\bar{A}_0(t)$ reads

$$\bar{A}_0(t) = \begin{pmatrix} \left(\frac{b_3(t)c_1}{b_1(t)c_3}\right)^{1/4} & 0 \\ 0 & \left(\frac{b_3(t)c_1}{b_1(t)c_3}\right)^{-1/4} \end{pmatrix}.$$

Therefore, starting from an integrable system we can find a family of t-dependent vector fields (5.14) describing solvable TDHO systems whose coefficients are parametrised by $b_1(t)$. Given a TDHO, it is easy to check whether it belongs to such a family and can be easily integrated.

The integrability conditions we have described here arise as requirements on the initial t-dependent functions b_α that allow us to solve the initial TDHO exactly by a t-dependent transformation of the form

$$\xi' = \Phi^{HO}(\exp(\Psi(t)v), \xi),$$

with some $v \in \mathfrak{sl}(2, \mathbb{R})$ and $\Psi(t)$, in such a way that the initial TDHO system (5.1) in the variable ξ is transformed into another one in the variable ξ' associated, as a Lie system, with a Vessiot–Guldberg Lie algebra isomorphic to an appropriate Lie subalgebra of $\mathfrak{sl}(2, \mathbb{R})$ in such a way that the equation in ξ' can be integrated by quadratures, and so the equation in ξ is solvable too.

5.4. Some applications of integrability conditions to TDHOs. As a first application, we show that the usual approach to the solution of the classical Caldirola–Kanai Hamiltonian [27, 133] can be explained through our method (the solution of the quantum case can be obtained in a similar way). Next, we will also apply our approach to get integrable TDHOs.

The Hamiltonian of a t-dependent harmonic oscillator is

$$H(t) = \frac{1}{2}\frac{p^2}{m(t)} + \frac{1}{2}m(t)\omega^2(t)x^2. \tag{5.15}$$

For instance, a harmonic oscillator with a damping term [27, 133] with equation of motion

$$\frac{d}{dt}(m_0\dot{x}) + m_0\mu\dot{x} + kx = 0, \qquad k = m_0\omega^2,$$

admits a Hamiltonian description, with a t-dependent Hamiltonian

$$H(t) = \frac{p^2}{2m_0}\exp(-\mu t) + \frac{1}{2}m_0\exp(\mu t)\omega^2x^2,$$

i.e. $m(t)$ in (5.15) corresponds to $m(t) = m_0\exp(\mu t)$. In this case equations (5.1) are

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial p} = \dfrac{1}{m_0}\exp(-\mu t)p, \\ \dot{p} = -\dfrac{\partial H}{\partial x} = -m_0\exp(\mu t)x, \end{cases} \tag{5.16}$$

and the t-dependent coefficients of the associated Lie system read

$$b_1(t) = \frac{1}{m_0}\exp(-\mu t), \qquad b_2(t) = 0, \qquad b_3(t) = m_0\omega^2\exp(\mu t).$$

Therefore, as $b_1(t)b_3(t) = \omega^2$, $b_2 = 0$ and

$$\frac{\dot{b}_3}{b_3} - \frac{\dot{b}_1}{b_1} = 2\mu,$$

we see that (5.12) holds if we set $c_1 = c_3 = 1, c_2 = \mu/\omega$ and the function D is a constant,

$D=\omega$. Hence, this example reduces to the system

$$\frac{d}{dt}\begin{pmatrix}x'\\p'\end{pmatrix}=\begin{pmatrix}\mu/2 & \omega\\ -\omega & -\mu/2\end{pmatrix}\begin{pmatrix}x'\\p'\end{pmatrix},$$

which can be easily integrated. If we put $\bar{\omega}^2=\mu^2/4-\omega^2$, we get

$$\begin{pmatrix}x'(t)\\p'(t)\end{pmatrix}=\begin{pmatrix}\cosh(\bar{\omega}t)+\dfrac{\mu}{2\bar{\omega}}\sinh(\bar{\omega}t) & \dfrac{\omega}{\bar{\omega}}\sinh(\bar{\omega}t)\\ -\dfrac{\omega}{\bar{\omega}}\sinh(\bar{\omega}t) & \cosh(\bar{\omega}t)-\dfrac{\mu}{2\bar{\omega}}\sinh(\bar{\omega}t)\end{pmatrix}\begin{pmatrix}x'(0)\\p'(0)\end{pmatrix}$$

and, in terms of the initial variables, we obtain

$$x(t)=\frac{e^{-\mu t/2}}{\sqrt{m_0\omega}}\left(\left(\cosh(\bar{\omega}t)+\frac{\mu}{2\bar{\omega}}\sinh(\bar{\omega}t)\right)\sqrt{m_0\omega}x_0+\frac{\omega}{\bar{\omega}}\sinh(\bar{\omega}t)\frac{p_0}{\sqrt{m_0\omega}}\right).$$

We can also study a TDHO described by the t-dependent Hamiltonian

$$H(t)=\frac{1}{2}p^2+\frac{1}{2}F(t)\omega^2x^2,\qquad F(t)>0,$$

where we assume, for simplicity, $m=1$. The t-dependent vector field X is

$$X_t=p\frac{\partial}{\partial x}-F(t)\omega^2x\frac{\partial}{\partial p},$$

which is a linear combination

$$X_t=F(t)\omega^2X_3^{HO}+X_1^{HO},$$

i.e. the t-dependent coefficients in (5.10) are

$$b_1(t)=1,\qquad b_2(t)=0,\qquad b_3(t)=F(t)\omega^2,$$

and the condition for F to satisfy (5.12) is

$$\frac{1}{2}\frac{\dot{F}}{F}=c_2\omega\sqrt{F}.$$

Therefore, F must be of the form

$$F(t)=\frac{1}{(L-c_2\omega t)^2}$$

and the Hamiltonian, which can be exactly integrated, is

$$H(t)=\frac{p^2}{2}+\frac{1}{2}\frac{\omega^2}{(L-c_2\omega t)^2}x^2.$$

The corresponding Hamilton equations are

$$\begin{cases}\dot{x}=p,\\ \dot{p}=-\dfrac{\omega^2}{(L-c_2\omega t)^2}x,\end{cases}$$

and the t-dependent change of variables to perform is

$$\begin{cases}x'=\sqrt{\dfrac{\omega}{L-c_2\omega t}}x,\\ p'=\sqrt{\dfrac{L-c_2\omega t}{\omega}}p.\end{cases}$$

Consequently,

$$\begin{cases} \dfrac{dx'}{dt} = \dfrac{\omega}{L - c_2\omega t}\left(\dfrac{c_2}{2}x' + p'\right), \\ \dfrac{dp'}{dt} = \dfrac{\omega}{L - c_2\omega t}\left(-x' - \dfrac{c_2}{2}p'\right), \end{cases} \tag{5.17}$$

and, under the t-reparametrisation

$$\tau(t) = \int_0^t \frac{\omega\, dt'}{L - c_2\omega t'} = \frac{1}{c_2}\ln\left(\frac{K'}{L - c_2\omega t}\right),$$

the system (5.17) becomes

$$\begin{cases} \dfrac{dx'}{d\tau} = \dfrac{c_2}{2}x' + p', \\ \dfrac{dp'}{d\tau} = -x' - \dfrac{c_2}{2}p', \end{cases}$$

which is equivalent to a transformed Caldirola–Kanai differential equation through the change $\tau \mapsto \omega t$ and $c_2 \mapsto \mu/\omega$. In any case, the solution is

$$x'(\tau) = \left(\cosh(\widetilde{\omega}\tau) + \frac{c_2}{2\widetilde{\omega}}\sinh(\widetilde{\omega}\tau)\right)x'(0) + \frac{1}{\widetilde{\omega}}\sinh(\widetilde{\omega}\tau)p'(0),$$

where $\widetilde{\omega} = \sqrt{c_2^2/4 - 1}$. Finally,

$$x(\tau(t)) = \sqrt{\frac{L - c_2\omega t}{\omega}}\left[\left(\cosh(\widetilde{\omega}\tau(t)) + \frac{c_2}{2\widetilde{\omega}}\sinh(\widetilde{\omega}\tau(t))\right)x'(0) + \frac{1}{\widetilde{\omega}}\sinh(\widetilde{\omega}\tau(t))p'(0)\right].$$

Let us analyse another integrability condition that, as the preceding one, arises as a compatibility condition for a restricted case of the system describing the integral curves of (5.8). Nevertheless, this time, the solution is restricted to a one-parameter set of matrices of $SL(2,\mathbb{R})$ that is not a group in general.

We deal with a family of transformations

$$\bar{A}_0(t) = \begin{pmatrix} \frac{1}{V(t)} & 0 \\ -u_1 & V(t) \end{pmatrix}, \qquad V(t) > 0, \tag{5.18}$$

where u_1 is a constant, i.e. we want to relate the t-dependent vector field

$$X_t = X_1^{HO} + F(t)\omega^2 X_3^{HO},$$

characterised by the coefficients in (5.10)

$$b_1 = 1, \qquad b_2 = 0, \qquad b_3 = F(t)\omega^2,$$

to an integrable one characterised by b'_1, b'_2 and b'_3, or more explicitly, to the t-dependent vector field

$$X_t = D(t)(c_1 X_1 + c_3 X_3),$$

i.e. $b'_1 = Dc_1$, $b'_2 = 0$, and $b'_3 = Dc_3$. Moreover, if $c_1 \neq 0$, we can absorb its value redefining D and assuming $c_1 = 1$.

Under the action of (5.18), the original system transforms into

$$\begin{cases} b_3' = V^2 b_3 + u_1 V b_2 + u_1^2 b_1 - u_1 \dot{V}, \\ b_2' = b_2 + 2\dfrac{u_1}{V} b_1 - 2\dfrac{\dot{V}}{V}, \\ b_1' = \dfrac{1}{V^2} b_1. \end{cases}$$

As $b_2 = b_2' = 0$ and $b_1 = 1$, the second equation yields $\dot{V} = u_1$, i.e. $V(t) = u_1 t + u_0$ with $u_0 \in \mathbb{R}$. Moreover, using this condition in the first equation together with $b_1 = 1$, we get $b_3' = V^2 b_3$. Then, as the third equation gives $D = b_1' = 1/V^2$, we see that $b_3' = Dc_3 = V^2 F(t)\omega^2$. Therefore, F has to be proportional to $(u_1 t + u_0)^{-4}$,

$$F(t) = \frac{k}{(u_1 t + u_0)^4}, \qquad k = \frac{c_3}{\omega^2}.$$

Assume $k = 1$, and thus $c_3 = \omega^2$. Then the t-dependent transformation $\bar{A}_0(t)$ performing this reduction is

$$\begin{cases} x' = \dfrac{x}{V(t)}, \\ p' = -u_1 x + V(t) p. \end{cases}$$

Under this transformation, the initial system becomes

$$\begin{cases} \dfrac{dx'}{dt} = \dfrac{p'}{V^2(t)}, \\ \dfrac{dp'}{dt} = -\dfrac{\omega^2 x'}{V^2(t)}. \end{cases}$$

Using the t-reparametrisation

$$\tau(t) = \int_0^t \frac{dt'}{V^2(t')} = \frac{1}{u_1}\left(\frac{1}{u_0} - \frac{1}{V(t)}\right),$$

we get the autonomous linear system

$$\begin{cases} \dfrac{dx'}{d\tau} = p', \\ \dfrac{dp'}{d\tau} = -\omega^2 x', \end{cases}$$

whose solution is

$$\begin{pmatrix} x'(\tau) \\ p'(\tau) \end{pmatrix} = \begin{pmatrix} \cos(\omega\tau) & \frac{\sin(\omega\tau)}{\omega} \\ -\omega\sin(\omega\tau) & \cos(\omega\tau) \end{pmatrix} \begin{pmatrix} x'(0) \\ p'(0) \end{pmatrix}.$$

Thus, we obtain

$$x(t) = V(t)\left(\cos(\omega\tau(t))\frac{x_0}{u_0} + \frac{1}{\omega}\sin(\omega\tau(t))(-u_1 x_0 + u_0 p_0)\right).$$

5.5. Integrable TDHOs and t-dependent constants of motion. The autonomisations of the transformed integrable systems obtained above enable us to construct t-dependent constants of motion. Indeed, in previous cases, a TDHO was transformed into a Lie system related to an equation on $SL(2,\mathbb{R})$

$$R_{A^{-1}*A}\dot{A} = -D(t)(c_1M_0 + c_2\mathrm{a}_1 + c_3\mathrm{a}_1),$$

associated with a TDHO determined by the t-dependent vector field

$$X_t = D(t)(c_1X_1 + c_2X_2 + c_3X_3).$$

Each t-dependent first integral $I(t)$ of this differential equation satisfies

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + X_tI = 0.$$

Thus, I is a first integral of the vector field on $\mathbb{R}\times \mathrm{T}^*\mathbb{R}$

$$\overline{X}_t = c_1X_1(t) + c_2X_2(t) + c_3X_3(t) + \frac{1}{D(t)}\frac{\partial}{\partial t}.$$

As $\mathbb{R}\times\mathrm{T}^*\mathbb{R}$ is a three-dimensional manifold and the differential equation we are studying is determined by a distribution of dimension one, there exist (at least locally) two independent first integrals. Next, we analyse some integrable cases and the corresponding constants of motion.

Case $F(t) = (u_1t+u_0)^{-2}$. In this case, according to Theorem 5.3, the t-dependent vector field of the initial TDHO is transformed into

$$X_t = \frac{\omega}{u_1t+u_0}\left(X_1^{HO} - \frac{u_1}{\omega}X_2^{HO} + X_3^{HO}\right)$$

and thus, using the method of characteristics, we obtain the following constants of motion for this TDFHO:

$$I_1 = -\frac{u_1}{\omega}p'\,x' + x'^2 + p'^2, \qquad I_2 = \frac{(u_1+u_0t)^{\omega/u_1}}{((\frac{u_1}{\omega}x' - 2p') + 2\bar{\omega}x')^{1/\bar{\omega}}},$$

with $\bar{\omega} = \pm\sqrt{u_1^2/(4\omega^2)-1}$.

Case $F(t) = (u_1t+u_0)^{-4}$. In this case the t-dependent vector field of the initial TDHO is transformed into

$$X_t = \frac{1}{V^2(t)}(X_1^{HO} + \omega^2X_3^{HO}),$$

and thus, using the method of characteristics, we get the following t-dependent constants of motion for the initial TDHO:

$$\begin{aligned} I_1 &= \left(\frac{x\omega}{V(t)}\right)^2 + (V(t)p - u_1x)^2, \\ I_2 &= \arcsin\left(\frac{x\omega}{V(t)\sqrt{I_1}}\right) + \frac{\omega}{u_1V(t)}. \end{aligned} \tag{5.19}$$

As we have two t-dependent constants of motion over $\mathbb{R}\times\mathrm{T}^*\mathbb{R}$ and the solutions in this space are of the form $(t, x(t), p(t))$, we can obtain the solutions for our initial system.

5.6. Applications to two-dimensional TDHOs. In this section we apply our previous geometrical methods to analyse the two-dimensional t-dependent harmonic oscillator

$$H(t,x_1,x_2,p_1,p_2) = \frac{p_1^2}{2} + \frac{p_2^2}{2} + \frac{\omega_1^2x_1^2+\omega_2^2x_2^2}{2V^4(t)}$$

with ω_1 and ω_2 constant and $V(t) = u_1 t + u_0$. Nevertheless, our approach is also valid for the corresponding generalisation to n-dimensional TDHOs. This Hamiltonian is related to an uncoupled pair of TDHOs and therefore the developments of the last section apply again. In this way, we find that its Hamilton equations read

$$\begin{cases} \dot{x}_i = p_i, \\ \dot{p}_i = -\dfrac{\omega_i^2}{V^4(t)} x_i, \end{cases} \qquad i = 1, 2,$$

and can be transformed into

$$\begin{cases} \dfrac{dx_i'}{dt} = \dfrac{1}{V^2(t)} p_i', \\ \dfrac{dp_i'}{dt} = -\dfrac{\omega_i^2}{V^2(t)} x_i', \end{cases} \qquad i = 1, 2,$$

by means of the t-dependent change of variables

$$\begin{cases} x_i' = \dfrac{x_i}{V(t)}, \\ p_i' = -u_1 x_i + V(t) p_i, \end{cases} \qquad i = 1, 2.$$

The solutions of the last system are integral curves of a t-dependent vector field in the distribution generated by the vector field

$$X = -\omega_1^2 x_1' \frac{\partial}{\partial p_1'} + p_1' \frac{\partial}{\partial x_1'} - \omega_2^2 x_2' \frac{\partial}{\partial p_2'} + p_2' \frac{\partial}{\partial x_2'}.$$

If we consider the problem as a differential equation in $\mathrm{T}^*\mathbb{R}^2$, the constants of motion are first integrals for the vector field $X + \partial/\partial t$ over $\mathbb{R} \times \mathrm{T}^*\mathbb{R}^2$. Then, as we have a distribution of rank one over a five-dimensional manifold, there exist, at least locally, four functionally independent first integrals. Additionally, three of them can be chosen to be t-independent (in terms of the variables x_1', x_2', p_1', p_2'). The constants of motion for the initial TDHO corresponding to some of such first integrals read

$$I_i = \left(\frac{\omega_i x_i}{V(t)} \right)^2 + (V(t) p_i - u_1 x_i)^2, \qquad i = 1, 2,$$

and

$$I_{12} = \frac{1}{\omega_1} \arcsin\left(\frac{x_1 \omega_1}{V(t)\sqrt{I_1}} \right) - \frac{1}{\omega_2} \arcsin\left(\frac{x_2 \omega_2}{\sqrt{V(t) I_2}} \right).$$

This first integral is constant along the solutions. Nevertheless, in order for the function to be correctly defined, ω_1/ω_2 has to be rational. Finally, with the aid of (5.19), we can obtain two t-dependent constants of motion of the form

$$\bar{I}_i = \frac{\omega_i}{V(t) u_1} + \arcsin\left(\frac{x_i' \omega_i}{\sqrt{I_i}} \right), \qquad i = 1, 2.$$

As a consequence, we can explicitly obtain the t-evolution of the system. Indeed, either from $\bar{I}_1$ or $\bar{I}_2$, we reach the following solutions:

$$x_i(t) = \frac{V(t)\sqrt{I_i}}{\omega_i} \sin\left(\bar{I}_i - \frac{\omega_i}{V(t) u_1} \right), \qquad i = 1, 2.$$

Their properties become clearer when we write them as

$$x_i(t) = \frac{V(t)\sqrt{I_i}}{\omega_i} \sin\left(\bar{I}_i - \frac{\omega_i}{u_1(u_1 t + u_0)}\right), \qquad i = 1, 2,$$

and we realise that the quotient $x_1(t)/x_2(t)$ is a t-independent constant of motion if ω_1/ω_2 is rational.

These two equations can be viewed as a parametric representation of a curve on the configuration space $Q = \mathbb{R}^2$. In the general case x_1 and x_2 evolve in an independent way and the behaviour of the curve becomes blurred. In the rational case, the evolutions of x_1 and x_2 are correlated in such a way that the t-dependent coupling function I_{12} is preserved. The particular form of this curve will depend on the relation between u_1 and u_0. If $u_1 = 0$ it will be a Lissajous curve. If $u_1 \neq 0$ it can be considered as a curve obtained by the addition of growing amplitudes to the oscillations of the corresponding Lissajous curve. We can refer to them as 't-dependent Lissajous' figures. Nevertheless, it is not totally clear whether this term is appropriate, since these new curves are 'not closed'.

6. Integrability in quantum mechanics

Some papers have recently been devoted to applying the theory of Lie systems [38, 157, 234] to quantum mechanics [51, 60]. As a result, it has been proved that the theory of Lie systems can be used to treat some types of Schrödinger equations, the so-called quantum Lie systems, to obtain exact solutions, t-evolution operators, etc. One of the fundamental properties found is that quantum Lie systems can be investigated by means of equations in a Lie group. Through such an equation we can analyse the properties of the associated Schrödinger equation, e.g. the type of Lie group allows us to know if the Schrödinger equation can be integrated [51].

Lately, a lot of attention has also been dedicated to integrability of Lie systems and, in particular, of Riccati equations [40, 47, 50]. In these papers, as in previous sections, it has been shown that integrability conditions for Lie systems, in the case of Riccati equations, are related to some transformation properties of the associated equations in $SL(2, \mathbb{R})$. Nevertheless, as we have pointed out and as was shown in [47], the same procedure used to investigate Riccati equations can be applied to deal with any Lie system.

Therefore, in the case of a quantum Lie system, there exists an equation on a Lie group associated with it [51]. The transformation properties investigated in the theory of integrability of Lie systems can be used to study integrability conditions for quantum Lie systems. All results obtained in Chapter 4 can be generalised to the quantum case and some nontrivial integral models can be obtained. The aim of this chapter is to show how to apply the theory of integrability of Lie systems to quantum Lie systems. All our results are illustrated by the analysis of several types of spin Hamiltonians.

We stress the practical importance of this method: It enables us to obtain nontrivial exactly solvable t-dependent Schrödinger equations. This allows us to investigate physical models by means of nontrivial exact solutions. It also provides a procedure to avoid using numerical methods for studying Schrödinger equations in many cases.

6.1. Spin Hamiltonians. In this section we investigate a quantum mechanical system whose dynamics is given by the Schrödinger–Pauli equation [39]. We first prove that this Hamiltonian corresponds to a quantum Lie system and we next apply the theory of integrability of Lie systems to recover some exact known solutions and find some new ones.

The system under study is described by the t-dependent Hamiltonian

$$H(t) = B_x(t)S_x + B_y(t)S_y + B_z(t)S_z,$$

with S_x, S_y and S_z being the spin operators. Let us denote $S_1 = S_x$, $S_2 = S_y$ and $S_3 = S_z$. Then the t-dependent Hamiltonian $H(t)$ is a quantum Lie system, because the spin operators are such that

$$[iS_j, iS_k] = -\sum_{l=1}^{3} \epsilon_{jkl}\, iS_l, \qquad j,k = 1,2,3, \tag{6.1}$$

with ϵ_{jkl} being the components of the fully skew-symmetric Levi-Civita tensor and where we have assumed $\hbar = 1$. The Schrödinger equation corresponding to this t-dependent Hamiltonian is

$$\frac{d\psi}{dt} = -iB_x(t)S_x(\psi) - iB_y(t)S_y(\psi) - iB_z(t)S_z(\psi), \tag{6.2}$$

which can be seen as a differential equation determining the integral curves of the t-dependent vector field in a (maybe infinite-dimensional) Hilbert space $\mathcal{H}$ given by

$$X_t = B_x(t)X_1^{SH} + B_y(t)X_2^{SH} + B_z(t)X_3^{SH},$$

with

$$(X_1^{SH})_\psi = -iS_x(\psi), \qquad (X_2^{SH})_\psi = -iS_y(\psi), \qquad (X_3^{SH})_\psi = -iS_z(\psi).$$

The t-dependent vector field X can be written as a linear combination

$$X_t = \sum_{k=1}^{3} b_k(t) X_k^{SH}$$

of the vector fields X_k^{SH}, with $b_1(t) = B_x(t)$, $b_2(t) = B_y(t)$ and $b_3(t) = B_z(t)$, and therefore our Schrödinger equation is a Lie system related to a quantum Vessiot–Guldberg Lie algebra isomorphic to $\mathfrak{su}(2)$.

Take the basis for $\mathfrak{su}(2)$ given by the skew-self-adjoint 2×2 matrices

$$\mathrm{v}_1 \equiv \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

$$\mathrm{v}_2 \equiv \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\mathrm{v}_3 \equiv \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

These matrices satisfy the commutation relations

$$[\mathrm{v}_j, \mathrm{v}_k] = -\sum_{l=1}^{3} \epsilon_{jkl} \mathrm{v}_l, \qquad j,k = 1,2,3,$$

which are similar to (6.1). Hence, we can define an action $\Phi^{SH} : SU(2)\times\mathcal{H}\to\mathcal{H}$ such that

$$\Phi^{SH}(\exp(c_k \mathrm{v}_k), \psi) = \exp(c_k i H_k)(\psi), \qquad k = 1, 2, 3,$$

for any real constants c_1, c_2 and c_3. Moreover,

$$\left.\frac{d}{dt}\right|_{t=0} \Phi^{SH}(\exp(-it\mathrm{v}_k, \psi) = \left.\frac{d}{dt}\right|_{t=0} \exp(-itH_k)(\Phi) = -iH_k(\psi) = (X_k^{SH})_\psi,$$

showing that each X_k^{SH} is the fundamental vector field associated with v_k. Thus, the equation on $SU(2)$ related, by means of Φ^{SH}, to the Schrödinger equation (6.2) is

$$R_{g^{-1}*g}\dot{g} = -\sum_{k=1}^{3} b_k(t)\mathrm{v}_k \equiv \mathrm{a}(t) \in \mathfrak{su}(2), \qquad g(0) = e. \tag{6.3}$$

It was shown in [51], and previously in our work, that the group $\mathcal{G}$ of curves in the group of a Lie system, in this case $\mathcal{G} = \mathrm{Map}(\mathbb{R}, SU(2))$, acts on the set of Lie systems associated with an equation in the Lie group G in such a way that, in a similar way to what happened in [40], a curve $\bar{g} \in \mathcal{G}$ transforms the initial equation (6.3) into the new one characterised by the curve

$$\mathrm{a}'(t) \equiv -\mathrm{Ad}(\bar{g})\Big(\sum_{k=1}^{3} b_k(t)\mathrm{v}_k\Big) + R_{\bar{g}^{-1}*\bar{g}}\frac{d\bar{g}}{dt} = -\sum_{k=1}^{3} b'_k(t)\mathrm{v}_k. \tag{6.4}$$

Once again, this new equation is related to a new Schrödinger equation in $\mathcal{H}$ determined by a new Hamiltonian

$$H'(t) = \sum_{k=1}^{3} b'_k(t) S_k.$$

Additionally, the curve $\bar{g}(t)$ in $SU(2)$ induces a t-dependent unitary transformation $\bar{U}(t)$ on $\mathcal{H}$ transforming the initial t-dependent Hamiltonian $H(t)$ into $H'(t)$.

Summarising, the theory of Lie systems reduces the problem of determining the solution of Schrödinger equations related to spin Hamiltonians $H(t)$ to solving certain equations in the Lie group $SU(2)$. Then, the transformation properties of the equations in $SU(2)$ describe the transformation properties of $H(t)$ by means of certain t-dependent unitary transformations described by curves in $SU(2)$.

Note that the theory here developed for spin Hamiltonians can be directly employed to analyse any quantum Lie system. In this case, our procedure remains essentially the same. It is only necessary to replace $SU(2)$ by the Lie group G associated with the quantum Lie system under study.

6.2. Lie structure of an equation describing transformations of Lie systems. Our aim now is to prove that the curves in $SU(2)$ relating the equations defined by two curves $\mathrm{a}(t)$ and $\mathrm{a}'(t)$ in $T_I SU(2)$, respectively, can be found as solutions of a Lie system of differential equations.

Recall that the matrices in $SU(2)$ are of the form

$$\bar{g} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \qquad a, b \in \mathbb{C}, \tag{6.5}$$

with $|a|^2+|b|^2=1$ and that the elements of $\mathfrak{su}(2)$ are traceless skew-Hermitian matrices, namely, real linear combinations of the matrices $\{\mathrm{v}_k \mid k=1,2,3\}$. Then, the equation (6.4) becomes a matrix equation that can be written

$$\frac{d\bar{g}}{dt}\bar{g}^{-1} = -\sum_{k=1}^{3} b'_k(t)\mathrm{v}_k + \sum_{k=1}^{3} b_k(t)\bar{g}\mathrm{v}_k\bar{g}^{-1}. \tag{6.6}$$

Multiplying both sides of this equation by $\bar{g}$ on the right, we get

$$\frac{d\bar{g}}{dt} = -\sum_{k=1}^{3} b'_k(t)\mathrm{v}_k\bar{g} + \sum_{k=1}^{3} b_k(t)\bar{g}\mathrm{v}_k. \tag{6.7}$$

If we consider a reparametrisation of the t-dependent coefficients of $\bar{g}$,

$$\begin{aligned} a(t) &= x_1(t) + iy_1(t),\\ b(t) &= x_2(t) + iy_2(t), \end{aligned}$$

for real functions x_j and y_j, with $j=1,2$, a straightforward computation shows that (6.7) is a linear system of differential equations in the new variables x_1, x_2, y_1 and y_2:

$$\begin{pmatrix} \dot{x}_1\\ \dot{x}_2\\ \dot{y}_1\\ \dot{y}_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & b'_2-b_2 & -b_3+b'_3 & -b_1+b'_1\\ b_2-b'_2 & 0 & -b_1-b'_1 & b_3+b'_3\\ b_3-b'_3 & b'_1+b_1 & 0 & -b_2-b'_2\\ b_1-b'_1 & -b_3-b'_3 & b_2+b'_2 & 0 \end{pmatrix}\begin{pmatrix} x_1\\ x_2\\ y_1\\ y_2 \end{pmatrix}. \tag{6.8}$$

Only the solutions of the above system with $x_1^2+x_2^2+y_1^2+y_2^2=1$ describe curves in $SU(2)$ and, consequently, are related to solutions of (6.7). Nevertheless, we can forget this restriction for the time being, because it can be automatically implemented later in a more suitable way. Therefore, we can deal with the four variables in (6.8) as if they were independent. This linear system of differential equations is a Lie system associated with a Lie algebra of vector fields $\mathfrak{gl}(4,\mathbb{R})$, but the solutions with initial condition related to a matrix in the subgroup $SU(2)$ always remain in that subgroup. In fact, consider the set of vector fields

$$\begin{aligned}
N_1 &= \frac{1}{2}\left(-y_2\frac{\partial}{\partial x_1} - y_1\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial y_1} + x_1\frac{\partial}{\partial y_2}\right),\\
N_2 &= \frac{1}{2}\left(-x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} - y_2\frac{\partial}{\partial y_1} + y_1\frac{\partial}{\partial y_2}\right),\\
N_3 &= \frac{1}{2}\left(-y_1\frac{\partial}{\partial x_1} + y_2\frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial y_1} - x_2\frac{\partial}{\partial y_2}\right),\\
N'_1 &= \frac{1}{2}\left(y_2\frac{\partial}{\partial x_1} - y_1\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial y_1} - x_1\frac{\partial}{\partial y_2}\right),\\
N'_2 &= \frac{1}{2}\left(-x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} - y_2\frac{\partial}{\partial y_1} + y_1\frac{\partial}{\partial y_2}\right),\\
N'_3 &= \frac{1}{2}\left(y_1\frac{\partial}{\partial x_1} + y_2\frac{\partial}{\partial x_2} - x_1\frac{\partial}{\partial y_1} - x_2\frac{\partial}{\partial y_2}\right),
\end{aligned} \tag{6.9}$$

for which the nonzero commutation relations are

$$
\begin{aligned}
&[N_1, N_2] = -N_3, \quad [N_2, N_3] = -N_1, \quad [N_3, N_1] = -N_2,\\
&[N'_1, N'_2] = -N'_3, \quad [N'_2, N'_3] = -N'_1, \quad [N'_3, N'_1] = -N'_2.
\end{aligned}
$$

Note that $[N_j, N'_k] = 0$, for $j, k = 1, 2, 3$, and therefore (6.8) is a Lie system on $\mathbb{R}^4$ associated with a Lie algebra of vector fields isomorphic to $\mathfrak{g} \equiv \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, i.e. the Lie algebra decomposes into a direct sum of two Lie algebras isomorphic to $\mathfrak{su}(2, \mathbb{R})$, the first one generated by $\{N_1, N_2, N_3\}$ and the second one by $\{N'_1, N'_2, N'_3\}$.

If we denote $y \equiv (x_1, x_2, y_1, y_2) \in \mathbb{R}^4$, the system (6.8) can be written as a system of differential equations in $\mathbb{R}^4$:

$$
\frac{dy}{dt} = N(t, y), \tag{6.10}
$$

with N_t being the t-dependent vector field given by

$$
N(t, y) = \sum_{k=1}^{3} (b_k(t) N_k(y) + b'_k(t) N'_k(y)).
$$

The vector fields $\{N_1, N_2, N_3, N'_1, N'_2, N'_3\}$ span a distribution of rank three at almost every point of $\mathbb{R}^4$ and consequently there exists, at least locally, a first integral for all the vector fields (6.9). It can be verified that such a first integral is globally defined and reads $I(y) = x_1^2 + x_2^2 + y_1^2 + y_2^2$. Hence, given a solution $y(t)$ of (6.10) with an initial condition $I(y(0)) = x_1^2 + x_2^2 + y_1^2 + y_2^2 = 1$, we have $I(y(t)) = 1$ at any time t and this solution describes a curve in $SU(2)$. Therefore, we have found that the curves in $SU(2)$ relating two different equations on $SU(2)$ associated with two Schrödinger equations of the form (6.2) can be described by means of the solutions $y(t)$ of (6.10) with $I(y(0)) = 1$, and vice versa:

THEOREM 6.1. *The curves in $SU(2)$ relating two equations on the group $SU(2)$ characterised by the curves in $\mathfrak{su}(2)$ of the form*

$$
\mathrm{a}'(t) = -\sum_{k=1}^{3} b'_k(t) \mathrm{v}_k, \qquad \mathrm{a}(t) = -\sum_{k=1}^{3} b_k(t) \mathrm{v}_k
$$

are the solutions $y(t)$ of the system

$$
\frac{dy}{dt} = N(t, y)
$$

with

$$
N(t, y) = \sum_{k=1}^{3} (b_k(t) N_k(y) + b'_k(t) N'_k(y))
$$

and $I(y(0)) = 1$. This is a Lie system related to a Lie algebra of vector fields isomorphic to $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

COROLLARY 6.2. *Given two Schrödinger equations corresponding to two spin Hamiltonians, there always exists a curve in $SU(2)$ transforming one of them into the other.*

Although the above corollary ensures the existence of a t-dependent unitary transformation mapping a given spin Hamiltonian into any other one, obtaining such a transformation involves solving system (6.10) explicitly. This Lie system is related to a nonsolvable Lie algebra and so it is not easy to find its solutions in general. In view of this, it becomes interesting to determine integrability conditions which allow us to solve this system and obtain the corresponding transformation. This illustrates the interest of the integrability conditions derived in the next sections, which will be used to derive exact solutions for some physical problems involving spin Hamiltonians.

6.3. Integrability conditions for $SU(2)$ Schrödinger equations. Let $\bar{g}(t)$ be a curve in $SU(2)$ transforming the equation on $SU(2)$ defined by the curve $\mathrm{a}(t)$ into another characterised by $\mathrm{a}'(t)$ according to the rule (6.6). If $g'(t)$ is the solution of the equation in $SU(2)$ characterised by $\mathrm{a}'(t)$, then $g(t) = \bar{g}^{-1}(t)g'(t)$ is a solution for the equation in $SU(2)$ characterised by $\mathrm{a}(t)$.

If $\mathrm{a}'(t)$ lies in a solvable Lie subalgebra of $\mathfrak{su}(2)$, we can derive $g'(t)$ in many ways [40] and, once $g'(t)$ is obtained, the knowledge of the curve $\bar{g}(t)$ transforming $\mathrm{a}(t)$ into $\mathrm{a}'(t)$ provides the curve $g(t)$ solving the equation on $SU(2)$ determined by $\mathrm{a}(t)$.

Therefore, starting from a curve $\mathrm{a}'(t)$ in a solvable Lie subalgebra of $\mathfrak{su}(2)$ and using (6.10), with curves in a restricted family of curves in $SU(2)$, we can relate $\mathrm{a}'(t)$ to other possible curves $\mathrm{a}(t)$, finding, in this way a family of equations on $SU(2)$, and thus spin Schrödinger equations on $\mathcal{H}$, that can be exactly solved.

Let us assume some restrictions on the family of solution curves of the system (6.10), e.g. we choose $b = 0$. Consequently, there are instances of this system which do not admit a solution under these restrictions, i.e. it is not possible to connect the curves $\mathrm{a}(t)$ and $\mathrm{a}'(t)$ by a curve satisfying the assumed restrictions. This gives rise to some compatibility conditions for the existence of one of these special solutions, algebraic and/or differential ones, between the t-dependent coefficients of $\mathrm{a}'(t)$ and $\mathrm{a}(t)$ satisfied by explicitly solvable models found in the literature. Therefore, our approach is useful to provide exactly integrable models found in the literature and, as we will see next, to derive new ones.

The two main ingredients to be taken into account in the following sections are:

1. *The equations which are characterised by a curve* $\mathrm{a}'(t)$ *for which the solution can be obtained.* We here consider that $\mathrm{a}'(t)$ is associated with a one-dimensional Lie subalgebra of $\mathfrak{su}(2)$.
2. *The restriction on the set of curves considered as solutions of the equation (6.10).* We next look for solutions of (6.10) related to curves in a one-parameter subset of $SU(2)$.

Consider the following example: suppose that we want to connect a given $\mathrm{a}(t)$ with a final family of curves of the form $\mathrm{a}'(t) = -D(t)(c_1\mathrm{v}_1 + c_2\mathrm{v}_2 + c_3\mathrm{v}_3)$, with c_1, c_2, c_3 being real numbers. In this case, system (6.10), which describes the curves $\bar{g}(t) \subset SU(2)$ that transform the equation described by $\mathrm{a}(t)$ into the equation determined by $\mathrm{a}'(t)$, reads

$$\frac{dy}{dt} = \sum_{k=1}^{3} b_k(t)N_k(y) + D(t)\sum_{k=1}^{3} c_k N'_k(y) = N(t,y). \tag{6.11}$$

Note that the vector field

$$N' = \sum_{k=1}^{3} c_k N'_k$$

satisfies

$$[N_k, N'] = 0, \qquad k = 1, 2, 3.$$

Hence, the Lie system (6.11) is related to a Lie algebra of vector fields isomorphic to $\mathfrak{su}(2) \oplus \mathbb{R}$. As this Lie system is associated with a nonsolvable Vessiot–Guldberg Lie algebra, it is not integrable by quadratures and the solution cannot be easily found in the general case. Nevertheless, it is worth noting that (6.11) always has a solution.

In this way, we can consider some instances of (6.11) for which the resulting system of differential equations can be integrated by quadratures. We can assume that x is related to a one-parameter family of elements of $SU(2)$. Such a restriction implies that (6.11) not always has a solution, because sometimes it is not possible to connect $\mathrm{a}(t)$ and $\mathrm{a}'(t)$ by means of the chosen one-parameter family. This fact imposes differential and algebraic restrictions on the initial t-dependent functions b_k, with $k = 1, 2, 3$. These restrictions will describe known integrability conditions and other new ones. So, we can develop the ideas of [50, 55] in the framework of quantum mechanics. Moreover, from this point of view, we can find new integrability conditions that can be used to obtain exact solutions.

6.4. Application of integrability conditions in a $SU(2)$ Schrödinger equation. In this section we restrict ourselves to the case $\mathrm{a}'(t) = -D(t)\mathrm{v}_3$, i.e.

$$b_1'(t) = 0, \qquad b_2'(t) = 0, \qquad b_3'(t) = D(t). \tag{6.12}$$

Hence, the system of differential equations (6.8) describing the curves $\bar{g}$ relating a Schrödinger equation to $H'(t) = D(t)S_z$ is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & -b_2 & -b_3 + D & -b_1 \\ b_2 & 0 & -b_1 & b_3 + D \\ b_3 - D & b_1 & 0 & -b_2 \\ b_1 & -b_3 - D & b_2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix}. \tag{6.13}$$

We see that, according to the result of Theorem 6.1, the t-dependent vector field corresponding to such a system of differential equations can be written as a linear combination with t-dependent coefficients of the vector fields N_1, N_2, N_3 and N_3':

$$N(t, y) = \sum_{k=1}^{3} b_k(t) N_k(y) + D(t) N_3'(y).$$

Thus, system (6.13) is associated with a Lie algebra of vector fields isomorphic to $\mathfrak{u}(1) \oplus \mathfrak{su}(2)$. This Lie algebra is smaller than the initial one (6.8), but it is not solvable and the system is as difficult to solve as the initial Schrödinger equation. Therefore, in order to get exact solvable cases, we need to perform some kind of simplification once again, e.g. by imposing some extra assumptions on the variables. This may result in a system of differential equations whose solutions are incompatible with our additional conditions. Necessary and sufficient conditions on the t-dependent functions $b_1, b_2, b_3, b_1', b_2'$ and b_3' ensuring the existence of a solution compatible with the assumed restrictions on the variables give rise to integrability conditions for spin Hamiltonians.

For instance, suppose that we require the solutions to be in the one-parameter subset $A_\gamma \subset SU(2)$ given by

$$A_\gamma = \left\{ \begin{pmatrix} \cos\frac{\gamma}{2} & -e^{-bi}\sin\frac{\gamma}{2} \\ e^{bi}\sin\frac{\gamma}{2} & \cos\frac{\gamma}{2} \end{pmatrix} \,\middle|\, b \in [0, 2\pi) \right\} \tag{6.14}$$

where γ is a fixed real constant such that $\gamma \neq 2\pi n$, with $n \in \mathbb{Z}$, because in such a case $A_\gamma = \pm\mathrm{Id}$. In view of the definition of the sets A_γ and in terms of the parametrisation (6.5), we have

$$x_1 = \cos\frac{\gamma}{2}, \qquad y_1 = 0, \qquad x_2 = -\sin\frac{\gamma}{2}\cos b, \qquad y_2 = \sin\frac{\gamma}{2}\sin b. \tag{6.15}$$

The elements of A_γ are matrices in $SU(2)$ and we obtain the system of differential equations

$$\begin{pmatrix} 0 \\ \dot{x}_2 \\ 0 \\ \dot{y}_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -b_2 & -b_3+D & -b_1 \\ b_2 & 0 & -b_1 & b_3+D \\ b_3-D & b_1 & 0 & -b_2 \\ b_1 & -b_3-D & b_2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ 0 \\ y_2 \end{pmatrix}. \tag{6.16}$$

We get two integrability conditions for the system (6.16):

$$0 = -b_2 x_2 - b_1 y_2, \qquad 0 = (b_3 - D)x_1 + b_1 x_2 - b_2 y_2. \tag{6.17}$$

We can write the components $(B_x(t), B_y(t), B_z(t))$ of the magnetic field in polar coordinates,

$$\begin{aligned} B_x(t) &= B(t)\sin\theta(t)\cos\phi(t), \\ B_y(t) &= B(t)\sin\theta(t)\sin\phi(t), \\ B_z(t) &= B(t)\cos\theta(t), \end{aligned}$$

with $\theta \in [0, \pi)$ and $\phi \in [0, 2\pi)$.

The first algebraic integrability condition reads, in polar coordinates,

$$B(t)\sin\theta(t)\sin\frac{\gamma}{2}(\cos\phi(t)\sin b(t) - \sin\phi(t)\cos b(t)) = 0$$

and thus,

$$B(t)\sin\theta(t)\sin\frac{\gamma}{2}\sin(b(t) - \phi(t)) = 0,$$

so $b(t) = \phi(t)$. In such a case, the second algebraic integrability condition in (6.17) reduces to

$$(B_z - D)\cos\frac{\gamma}{2} - B\sin\frac{\gamma}{2}\sin\theta = 0$$

and then the t-dependent coefficient D is

$$D = \frac{B}{\cos\frac{\gamma}{2}}\cos\left(\frac{\gamma}{2} + \theta\right). \tag{6.18}$$

Finally, we have to take into account the differential integrability condition

$$\dot{x}_2 = \frac{1}{2}\left(b_2\cos\frac{\gamma}{2} + (b_3 + D)\sin\frac{\gamma}{2}\sin b\right),$$

which after some algebraic manipulation leads to

$$\dot{\phi} = \frac{B}{2}\left(\frac{\sin(\theta + \frac{\gamma}{2})}{\sin\frac{\gamma}{2}} + \frac{\cos(\frac{\gamma}{2} + \theta)}{\cos\frac{\gamma}{2}}\right),$$

and then

$$\dot{\phi}(t) = B(t)\frac{\sin(\theta(t)+\gamma)}{\sin\gamma}, \tag{6.19}$$

which is a far larger set of integrable Hamiltonians than the exactly solvable Hamiltonians of this type found in the literature. As a particular example, when θ and B are constant, we find

$$\dot{\phi} = B\frac{\sin(\theta+\gamma)}{\sin\gamma} \equiv \omega \tag{6.20}$$

and consequently,

$$\phi = \omega t + \phi_0.$$

Thus, the t-dependent spin Hamiltonian $H(t)$ determined by the magnetic vector field

$$\mathbf{B}(t) = B(\sin\theta\cos(\omega t), \sin\theta\sin(\omega t), \cos\theta)$$

is integrable.

Another interesting integrable case is that given by $\theta = \pi/2$, that is, the magnetic field moves in the XZ plane (see [20, 139, 140]). In such a case, in view of the integrability conditions (6.20), the angular frequency $\dot{\phi}$ is

$$\dot{\phi} = B\,\mathrm{cotan}\,\gamma = \omega.$$

The last one of the most known integrable cases of spin Hamiltonian is given by a magnetic field in a fixed direction, i.e. $\mathbf{B}(t) = B(t)(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Obviously, this case satisfies the integrability condition (6.20) for $\gamma = -\theta$.

Apart from the previous cases, the integrability condition (6.19) describes other integrable cases. For instance, consider the case with θ fixed and B nonconstant. In this case, the corresponding $H(t)$ is integrable if

$$\frac{\dot{\phi}}{B(t)} = \frac{\sin(\theta+\gamma)}{\sin\gamma},$$

that is, if we fix $\gamma = \pi/2$ we have

$$\omega = \dot{\phi} = B(t)\cos\theta, \quad \text{so} \quad \phi(t) = \cos\theta\int^t B(t')dt'.$$

Furthermore, we can consider $\theta(t) = t$ and B constant. In this case, the t-dependent Hamiltonian $H(t)$ is integrable if $\phi(t)$ satisfies the condition

$$\dot{\phi} = B\cos t, \quad \text{so} \quad \phi(t) = B\sin t.$$

Indeed, note that in this case the integrability condition (6.19) trivially follows for $\gamma = -1/2$.

To sum up, we have shown that there exists a large family of t-dependent integrable spin Hamiltonians that includes, as particular cases, many known integrable cases. Additionally, it is easy to check whether a t-dependent spin Hamiltonian satisfies the integrability condition (4.33) and can be integrated.

6.5. Applications to physics. Let us apply the above results to a t-dependent spin Hamiltonian

$$H(t) = \mathbf{B}(t)\cdot\mathbf{S},$$

which often appears in physics: the one characterised by a magnetic field

$$\mathbf{B}(t) = B(\sin\theta\cos(\omega t), \sin\theta\sin(\omega t), \cos\theta), \tag{6.21}$$

that is, a magnetic field with a constant modulus rotating along the z-axis with a constant angular velocity ω. Such Hamiltonians have been applied, for instance, to analyse spin precession in a transverse t-dependent magnetic field [208], investigate adiabatic approximation and the unitary of the t-evolution operator through such an approximation [160, 178], etc.

In the previous section we showed that this t-dependent Hamiltonian is integrable. Indeed, the integrability condition (6.20) can be written as

$$\tan\gamma = \frac{\sin\theta}{\dot\phi/B - \cos\theta}, \tag{6.22}$$

where we recall that γ has to be a real constant. In the case of our particular magnetic field (6.21) the angular frequency, $\omega = \dot\phi$, the angle θ and the modulus B are constants. Therefore γ is a properly defined constant, the integrability condition (6.20) holds and the value of γ is given by equation (6.22) in terms of the parameters B, θ and ω, which characterise the magnetic vector field (6.21).

We have already shown that if $B(t)$ satisfies (6.20), then $H(t)$ is integrable, because it can be transformed by means of a t-dependent change of variables determined by a curve $g(t)$ in the set A_γ into a directly integrable Schrödinger equation determined by a t-dependent Hamiltonian $H'(t) = D(t)S_z$. For simplicity, let us parametrise the elements of A_γ in a new way. Consider $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ and $\mathbf{n} \in \mathbb{R}^3$, where the matrices σ_i are the Pauli matrices, $\sigma_x, \sigma_y, \sigma_z$. We have

$$e^{i\sigma\cdot\mathbf{n}\phi} = \mathrm{Id}\cos\phi + i\sigma\cdot\mathbf{n}\sin\phi.$$

So, for $\mathbf{n} = (\alpha_1, \alpha_2, 0)/\sqrt{\alpha_1^2+\alpha_2^2}$ with real constants α_1, α_2 and taking into account that $\mathrm{v}_1 = i\sigma_x/2$, $\mathrm{v}_2 = i\sigma_y/2$ and $\mathrm{v}_3 = i\sigma_z/2$, we get

$$\exp(\alpha_1\mathrm{v}_1 + \alpha_2\mathrm{v}_2) = \exp\left(i\frac{\delta}{2}\sigma\cdot\mathbf{n}\right) = \begin{pmatrix} \cos\frac{\delta}{2} & -e^{-i\varphi}\sin\frac{\delta}{2} \\ e^{i\varphi}\sin\frac{\delta}{2} & \cos\frac{\delta}{2} \end{pmatrix} \tag{6.23}$$

with $\delta = \sqrt{\alpha_1^2+\alpha_2^2}$ and $-e^{-i\varphi} = (i\alpha_1+\alpha_2)/\sqrt{\alpha_1^2+\alpha_2^2}$. In terms of δ and φ the variables α_1 and α_2 can be written $\alpha_1 = \delta\sin\varphi$ and $\alpha_2 = -\delta\cos\varphi$. Hence, in view of (6.23), we can describe the elements of A_γ as

$$\begin{pmatrix} \cos\frac{\gamma}{2} & -e^{-bi}\sin\frac{\gamma}{2} \\ e^{bi}\sin\frac{\gamma}{2} & \cos\frac{\gamma}{2} \end{pmatrix} = \exp(\gamma\sin b\,\mathrm{v}_1 - \gamma\cos b\,\mathrm{v}_2), \tag{6.24}$$

where b and γ are real constants. For magnetic vector fields (6.21), the t-dependent change of variables transforming the initial $H(t)$ into an integrable $H'(t) = D(t)S_z$ is determined by a curve in A_γ with γ determined by equation (6.20) and $b(t) = \phi(t)$. Thus, such a curve in A_γ takes the form

$$t \mapsto \exp(\gamma\sin(\omega t)\mathrm{v}_1 - \gamma\cos(\omega t)\mathrm{v}_2). \tag{6.25}$$

We emphasise that the above t-dependent change of variables in $SU(2)$ transforms the equation in $SU(2)$ determined by the initial curve

$$\mathrm{a}(t) = -B_x(t)\mathrm{v}_1 - B_y(t)\mathrm{v}_2 - B_z(t)\mathrm{v}_3$$

into a new equation in $SU(2)$ determined by a curve $\mathrm{a}'(t) = -D(t)\mathrm{v}_3$. Such a t-dependent transformation in $SU(2)$ induces a t-dependent unitary change of variables in $\mathcal{H}$ transforming the initial Schrödinger equation determined by the t-dependent Hamiltonian $H(t)$, i.e.

$$\frac{\partial\psi}{\partial t} = -iH(t)(\psi),$$

into the new Schrödinger equation

$$\frac{\partial\psi'}{\partial t} = -iH'(t)(\psi') = -iD(t)S_z(\psi'). \tag{6.26}$$

The relation between ψ and ψ' is given by the corresponding t-dependent change of variables in $\mathcal{H}$ induced by curve (6.25), i.e.

$$\psi' = \exp(\gamma\sin(\omega t)\, iS_x - \gamma\cos(\omega t)\, iS_y)\psi. \tag{6.27}$$

In view of (6.18), we see that

$$D = B\left(\cos\theta - \tan\frac{\gamma}{2}\sin\theta\right),$$

and from (6.22) and the relations

$$\tan\gamma = \frac{2\tan\frac{\gamma}{2}}{1-\tan^2\frac{\gamma}{2}}, \quad \text{so} \quad \tan\frac{\gamma}{2} = \frac{-1\pm\sqrt{1+\tan^2\gamma}}{\tan\gamma},$$

we obtain

$$\tan\frac{\gamma}{2} = \frac{1}{\sin\theta}\left(-\frac{\omega}{B} + \cos\theta \pm \sqrt{\frac{\omega^2}{B^2} - 2\frac{\omega}{B}\cos\theta + 1}\right).$$

If we substitute the above expression in the expression for D, it turns out that

$$D = \omega \pm \sqrt{\omega^2 - 2\omega B\cos\theta + B^2}.$$

That is, D becomes a constant. Thus, the general solution ψ'_t for the Schrödinger equation (6.26) with initial condition ψ'_0 is

$$\psi'(t) = \exp(-itDS_z)\psi'_0,$$

and the solution for the initial Schrödinger equation with initial condition ψ_0 can be obtained by undoing the t-dependent change of variables (6.27) to get

$$\psi_t = \exp(-i\gamma\sin\omega t S_x + i\gamma\cos\omega t S_y)\exp(-iDtS_z)\psi_0.$$

7. The theory of quasi-Lie schemes and Lie families

7.1. Introduction. Several important systems of first-order ordinary differential equations can be studied through the theory of Lie systems. Moreover, this theory was recently applied to study SODE Lie systems, quantum Lie systems, some partial differential equations, etc. These last successes allow us to recover, from a unifying point of view, several results disseminated throughout the literature and to prove multiple new properties of systems of differential equations appearing in physics and mathematics. Apart from these successes, there are still some reasons to go further in the generalisation of the theory of Lie systems:

- *Lie systems are important but rather exceptional.* The theory of Lie systems investigates very interesting equations with many applications, e.g. t-dependent frequency harmonic oscillators, Milne–Pinney equations, Riccati equations, etc. Nevertheless, it fails to study many other (nonautonomous) interesting systems, like nonlinear oscillators, Abel equations, or Emden equations.
- *The theory of Lie systems does not allow us to investigate superposition rules involving an explicit t-dependence* which appears in various interesting systems, e.g. dissipative Milne–Pinney equation, Emden–Fowler equations [42], second-order Riccati equations [48, 126], whose properties are worth analysing.
- Lie systems have an associated group of t-dependent changes of variables enabling us to transform each particular Lie system into a new one of the same class, e.g. the group of curves in $SL(2,\mathbb{R})$ transforms a Riccati equation into a new Riccati equation. A similar property frequently applies to integrate differential equations, like Abel equations [74]. A natural question arises: Is there any kind of systems of differential equations more general than Lie systems admitting an analogous property?

The theory of quasi-Lie schemes [34] and the Generalised Lie Theorem [35], which gives rise to the *Lie family* notion, provide an answer to these problems. More specifically, quasi-Lie schemes, quasi-Lie systems and Lie families are interesting because:

- *The theory of quasi-Lie schemes and the Generalised Lie Theorem permit us to investigate a very large family of differential equations including Lie systems.* More specifically, this family includes, for instance, the following non-Lie systems: Emden–Fowler equations [34, 42], nonlinear oscillators [34], dissipative Milne–Pinney equations [34, 45], second-order Riccati equations [48], Abel equations [35], etc. Moreover, quasi-Lie schemes and Lie families can be applied to investigate not only systems of first-order ordinary differential equations, but also second-order differential equations [42, 45].
- *The theory of quasi-Lie schemes and the Generalised Lie Theorem treat, in a natural way, systems admitting a t-dependent superposition rule.* These theories show that many differential equations admit a t-dependent superposition rule, e.g. Abel equations [35], dissipative Milne–Pinney equations [34], Emden–Fowler equations [42], second-order Riccati equations [48], etc.
- *The quasi-Lie scheme concept permits us to transform a differential equation within a fixed family, e.g. a first-order Abel equation, into a new one with different t-dependent coefficients.* This feature generalises the transformation properties of Lie systems and enables us to derive integrability conditions for differential equations from a unified point of view.

Consequently, the theory of quasi-Lie schemes and the Generalised Lie Theorem represent powerful methods to study first- and higher-order differential equations.

7.2. Generalised flows and t-dependent vector fields. Recall that a nonautonomous system of first-order ordinary differential equations on $\mathbb{R}^n$ is represented in modern differential geometric terms by a t-dependent vector field $X = X(t, x)$ on such a space. On a noncompact manifold, the vector field $X_t(x) = X(t, x)$, for a fixed t, is generally not defined globally, but it is well defined on a neighbourhood of every point $x_0 \in \mathbb{R}^n$ for sufficiently small t. It is convenient to add the variable t to the manifold and to consider the *autonomisation* of our system, i.e. the vector field

$$\overline{X}(t, x) = \frac{\partial}{\partial t} + X(t, x),$$

defined on a neighbourhood U^X of $\{0\} \times \mathbb{R}^n$ in $\mathbb{R} \times \mathbb{R}^n$. The vector field X_t is then defined on the open set of $\mathbb{R}^n$,

$$U_t^X = \{x_0 \in \mathbb{R}^n \mid (t, x_0) \in U^X\},$$

for all $t \in \mathbb{R}$. If $U_t^X = \mathbb{R}^n$ for all $t \in \mathbb{R}$, we speak about a *global t-dependent vector field.* The system of differential equations associated with the t-dependent vector field $X(t, x)$ is written in local coordinates

$$\frac{dx^i}{dt} = X^i(t, x), \qquad i = 1, \dots, n,$$

where $X(t, x) = \sum_{i=1}^n X^i(t, x)\partial/\partial x^i$ is locally defined on the manifold for sufficiently small t.

A solution of this system is represented by a curve $s \mapsto \gamma(s)$ in $\mathbb{R}^n$ (integral curve) whose tangent vector $\dot{\gamma}$ at t, so at the point $\gamma(t)$ of the manifold, equals $X(t, \gamma(t))$. In other words,

$$\dot{\gamma}(t) = X(t, \gamma(t)). \tag{7.1}$$

It is well-known that, at least for smooth X we work with, for each x_0 there is a unique maximal solution $\gamma_X^{x_0}(t)$ of system (7.1) with initial value x_0, i.e. satisfying $\gamma_X^{x_0}(0) = x_0$. This solution is defined at least for t's from a neighbourhood of 0. In case $\gamma_X^{x_0}(t)$ is defined for all $t \in \mathbb{R}$, we speak about a *global t-solution.*

The collection of all maximal solutions of the system (7.1) gives rise to a (local) generalised flow g^X on $\mathbb{R}^n$. By a *generalised flow g* on $\mathbb{R}^n$ we understand a smooth t-dependent family g_t of local diffeomorphisms on $\mathbb{R}^n$, $g_t(x) = g(t, x)$, such that $g_0 = \mathrm{id}_{\mathbb{R}^n}$. More precisely, g is a smooth map from a neighbourhood U^g of $\{0\} \times \mathbb{R}^n$ in $\mathbb{R} \times \mathbb{R}^n$ into $\mathbb{R}^n$, such that g_t maps diffeomorphically the open submanifold $U_t^g = \{x_0 \in \mathbb{R}^n \mid (t, x_0) \in U^g\}$ onto its image, and $g_0 = \mathrm{id}_{\mathbb{R}^n}$. Again, for each $x_0 \in \mathbb{R}^n$ there is a neighbourhood U_{x_0} of x_0 in $\mathbb{R}^n$ and $\epsilon > 0$ such that g_t is defined on U_{x_0} for $t \in (-\epsilon, \epsilon)$ and maps U_{x_0} diffeomorphically onto $g_t(U_{x_0})$.

If $U_t^g = \mathbb{R}^n$ for all $t \in \mathbb{R}$, we speak about a *global generalised flow.* In this case $g : t \in \mathbb{R} \mapsto g_t \in \mathrm{Diff}(\mathbb{R}^n)$ may be viewed as a smooth curve in the diffeomorphism group $\mathrm{Diff}(\mathbb{R}^n)$ with $g_0 = \mathrm{id}_{\mathbb{R}^n}$.

Here it is also convenient to *autonomise* the generalised flow g extending it to a single local diffeomorphism

$$\overline{g}(t, x) = (t, g(t, x)) \tag{7.2}$$

defined on a neighbourhood U^g of $\{0\}\times\mathbb{R}^n$ in $\mathbb{R}\times\mathbb{R}^n$. The generalised flow g^X induced by the t-dependent vector field X is defined by

$$g^X(t,x_0)=\gamma_X^{x_0}(t). \tag{7.3}$$

Note that, for $g=g^X$, equation (7.3) can be rewritten in the form

$$X_t=X(t,x)=\dot g_t\circ g_t^{-1}. \tag{7.4}$$

In the above formula, we understand X_t and $\dot g_t$ as maps from $\mathbb{R}^n$ into $\mathrm{T}\mathbb{R}^n$, where $\dot g_t(x)$ is the vector tangent to the curve $s\mapsto g(s,x)$ at $g(t,x)$. Of course, the composition $\dot g_t\circ g_t^{-1}$, called sometimes the *right-logarithmic derivative* of $t\mapsto g_t$, is only defined for those points $x_0\in\mathbb{R}^n$ for which it makes sense. But this is always the case for sufficiently small t, at least locally.

Let us observe that equation (7.4) defines, in fact, a one-to-one correspondence between generalised flows and t-dependent vector fields modulo the observation that the domains of $\dot g_t\circ g_t^{-1}$ and X_t need not coincide. In any case, however, $\dot g_t\circ g_t^{-1}$ and X_t coincide in a neighbourhood of any point for sufficiently small t. One can simply say that the *germs* of X and $\dot g_t\circ g_t^{-1}$ coincide, where the germ in our context is understood as the class of corresponding objects that coincide on a neighbourhood of $\{0\}\times\mathbb{R}^n$ in $\mathbb{R}\times\mathbb{R}^n$.

Indeed, for a given g, the corresponding t-dependent vector field is defined by (7.4). Conversely, for a given X, the equation (7.4) determines the germ of the generalised flow $g(t,x)$ uniquely, as for each $x=x_0$ and for small t equation (7.4) implies that $t\mapsto g(t,x_0)$ is the solution of the system defined by X with initial value x_0. In this way we get the following.

THEOREM 7.1. *Equation* (7.4) *defines a one-to-one correspondence between the germs of generalised flows and the germs of t-dependent vector fields on $\mathbb{R}^n$.*

Any two generalised flows g and h can be composed: by definition $(g\circ h)_t=g_t\circ h_t$, where, as usual, we view $g_t\circ h_t$ as a local diffeomorphism defined for points for which the composition is defined. It is important to emphasise that in a neighbourhood of any point it really makes sense for sufficiently small t. As generalised flows correspond to t-dependent vector fields, this gives rise to an action of a generalised flow h on a t-dependent vector field X, giving rise to $h_\star X$, defined by the equation

$$g^{h_\star X}=h\circ g^X. \tag{7.5}$$

To obtain a more explicit form of this action, let us observe that

$$(h_\star X)_t=\frac{d(h\circ g^X)_t}{dt}\circ(h\circ g^X)_t^{-1}=(\dot h_t\circ g_t^X+Dh_t(\dot g_t^X))\circ(g^X)_t^{-1}\circ h_t^{-1},$$

and therefore

$$(h_\star X)_t=\dot h_t\circ h_t^{-1}+Dh_t(\dot g_t^X\circ(g^X)_t^{-1})\circ h_t^{-1},$$

i.e.

$$(h_\star X)_t=\dot h_t\circ h_t^{-1}+(h_t)_*(X_t), \tag{7.6}$$

where $(h_t)_*$ is the standard action of diffeomorphisms on vector fields. In a slightly different form, this can be written as an action of t-dependent vector fields on t-dependent vector fields:

$$(g_\star^Y X)_t = Y_t + (g_t^Y)_*(X_t). \tag{7.7}$$

For global t-dependent vector fields on compact manifolds, this defines a group structure in global t-dependent vector fields. This is an infinite-dimensional analogue of a group structure on paths in a finite-dimensional Lie algebra, which has been used as a source for a nice construction of the corresponding Lie group in [90]. Since every generalised flow has an inverse, $(g^{-1})_t = (g_t)^{-1}$, the generalised flows, or rather the corresponding germs, form a group and the formula (7.7) allows us to compute the t-dependent vector field (right-logarithmic derivative) X_t^{-1} associated with the inverse. It is the t-dependent vector field

$$X_t^{-1} = -(g_t^X)_*^{-1}(X_t). \tag{7.8}$$

For t-independent vector fields, $X_t = X_0$ for all t, we have $(g_t^X)_* X = X$ and also we get the well-known formula

$$X^{-1} = -X.$$

Note that, by definition, the integral curves of $h_\star X$ are of the form $h_t(\gamma(t))$, where $\gamma(t)$ are integral curves of X. We can summarise our observation as follows.

THEOREM 7.2. *The equation* (7.6) *defines a natural action of generalised flows on t-dependent vector fields. This action is a group action in the sense that*

$$(g \circ h)_\star X = g_\star(h_\star X).$$

The integral curves of $h_\star X$ are of the form $h_t(\gamma(t))$, for $\gamma(t)$ being an arbitrary integral curve for X.

The above action of generalised flows on t-dependent vector fields can also be defined in an elegant way by means of the corresponding autonomisations. Namely it is easy to check the following.

THEOREM 7.3. *For any generalised flow h and any t-dependent vector field X on a manifold $\mathbb{R}^n$, the standard action $\overline{h}_*\overline{X}$ of the diffeomorphism $\overline{h}$ (the autonomisation of h) on the vector field $\overline{X}$ (the autonomisation of X) is the autonomisation of the t-dependent vector field $h_\star X$:*

$$\overline{h}_*\overline{X} = \overline{h_\star X}.$$

7.3. Quasi-Lie systems and schemes. By a *quasi-Lie system* we understand a pair (X, g) consisting of a t-dependent vector field X on a manifold $\mathbb{R}^n$ (the *system*) and a generalised flow g on $\mathbb{R}^n$ (the *control*) such that $g_\star X$ is a Lie system.

Since for the Lie system $g_\star X$ we are able to obtain the general solution from a number of known particular solutions, the knowledge of the control makes it possible to apply a similar procedure for our initial system. Indeed, let $\Phi = \Phi(x_1, \ldots, x_m; k_1, \ldots, k_n)$ be a superposition function for the Lie system $g_\star X$, so that, knowing m solutions $\bar{x}_{(1)}, \ldots, \bar{x}_{(m)}$, of $g_\star X$, we can derive the general solution of the form

$$\bar{x}_{(0)} = \Phi(\bar{x}_{(1)}, \ldots, \bar{x}_{(m)}; k_1, \ldots, k_n).$$

If we now know m independent solutions, $x_{(1)},\dots,x_{(m)}$, of X, then, according to Theorem 7.3, $\bar{x}_a(t) = g_t(x_a(t))$ are solutions of $g_\star X$, producing a general solution of $g_\star X$ in the form $\Phi(\bar{x}_{(1)},\dots,\bar{x}_{(m)};k_1,\dots,k_n)$. It is now clear that

$$x_{(0)}(t) = g_t^{-1}\circ\Phi(g_t(x_{(1)}(t)),\dots,g_t(x_{(m)}(t));k_1,\dots,k_n) \tag{7.9}$$

is a general solution of X. In this way we have obtained a *t-dependent superposition rule* for the system X. We can summarise the above considerations as follows.

THEOREM 7.4. *Any quasi-Lie system (X, g) admits a t-dependent superposition rule of the form* (7.9)*, where Φ is a superposition function for the Lie system $g_\star X$.*

Of course, the above t-dependent superposition rule is practically useless for finding the general solution of a system X unless the generalised flow g is explicitly known. An alternative abstract definition of a quasi-Lie system as a t-dependent vector field X for which there exists a generalised flow g such that $g_\star X$ is a Lie system does not make much sense, as every X would be a quasi-Lie system in this context. For instance, given a t-dependent vector field X, the pair $(X,(g^X)^{-1})$ is a quasi-Lie system because $(g^X)_t^{-1}\circ g_t^X = \mathrm{id}_{\mathbb{R}^n}$, thus $(g^X)_\star^{-1}X = 0$, which is a Lie system trivially. On the other hand, finding $(g^X)^{-1}$ is nothing but solving our system X completely, so we just reduce to our original problem. In practice, it is therefore crucial that the control g comes from a system which can be effectively integrated. There are, however, many cases when our procedure works and provides a geometrical interpretation of many *ad hoc* methods of integration. Consider, for instance, the following scheme that can lead to 'nice' quasi-Lie systems.

Take a finite-dimensional real vector space V of vector fields on $\mathbb{R}^n$ and consider the family $V(\mathbb{R})$ of all t-dependent vector fields X on $\mathbb{R}^n$ such that X_t belongs to V on its domain, i.e. $X_t \in V_{|U_t^X}$ or, for short, $X \in V(\mathbb{R})$. We will say that these t-dependent vector fields take values in V. The t-dependent vector fields of $V(\mathbb{R})$ depend on a finite family of control functions. For example, take a basis $\{X_1,\dots,X_r\}$ of V and consider a general t-dependent system with values in V determined by $b = b(t) = (b_1(t),\dots,b_r(t))$ as

$$(X^b)_t = \sum_{\alpha=1}^{r} b_\alpha(t)X_\alpha.$$

On the other hand, the nonautonomous systems of differential equations associated with $X \in V|_{U_t^X}$ are not Lie systems in general, if V is not a Lie algebra itself. If we additionally have a finitely parametrised family of local diffeomorphisms, say $\underline{g} = \underline{g}(a_1,\dots,a_k)$, then any curve $a = a(t) = (a_1(t),\dots,a_k(t))$ in the control parameters, defined for small t, gives rise to a generalised flow $g_t^a = \underline{g}(a(t))$. Let us additionally assume that there is a Lie algebra V_0 of vector fields contained in V. We can look for control functions $a(t)$ such that for certain $b(t)$, $g_\star^a X^b$ has values in V_0 for each t. We then write

$$g_\star^a X^b \in V_0(\mathbb{R}). \tag{7.10}$$

Consequently, each pair (X^b, g^a) becomes a quasi-Lie system and we can get a t-dependent superposition rule for the corresponding system X^b.

Let us observe that in the case when all the generalised flows g^a preserve V, i.e. for each t-dependent vector field $X^b \in V(\mathbb{R})$ also $g^a_\star X^b \in V(\mathbb{R})$, the inclusion (7.10) becomes a differential equation for the control functions $a(t)$ in terms of the functions $b(t)$. This situation is not as rare as it may seem at first sight. Suppose, for instance, that we find a Lie algebra $W \subset V$ such that $[W, V] \subset V$ and that the t-dependent vector fields with values in W can be effectively integrated to generalised flows. In this case, any t-dependent vector field Y^a with values in W gives rise to a generalised flow g^a which, in view of the transformation rule (7.7), preserves the set of t-dependent vector fields with values in V. For each $b = b(t)$ the inclusion (7.10) becomes therefore a differential equation for the control function $a = a(t)$ which can often be effectively solved.

DEFINITION 7.5. Let W, V be finite-dimensional real vector spaces of vector fields on $\mathbb{R}^n$. We say that they form a *quasi-Lie scheme* $S(W, V)$ if the following conditions are satisfied:

1. W is a vector subspace of V.
2. W is a Lie algebra of vector fields, i.e. $[W, W] \subset W$.
3. W normalises V, i.e. $[W, V] \subset V$.

If V is a Lie algebra of vector fields, we simply call the quasi-Lie scheme $S(V, V)$ a *Lie scheme* $S(V)$.

NOTE 7.6. Although the normaliser of V in V is the largest Lie algebra of vector fields that we can use as W, for practical purposes it is sometimes useful to consider smaller Lie subalgebras.

DEFINITION 7.7. We define the *group of the scheme* $S(W, V)$ to be the group $\mathcal{G}(W)$ of generalised flows corresponding to the t-dependent vector fields with values in W.

MAIN THEOREM 7.8 (Main property of a scheme). *Given a quasi-Lie scheme $S(W, V)$, we have $g_\star X \in V(\mathbb{R})$ for every t-dependent vector field $X \in V(\mathbb{R})$ and each generalised flow $g \in \mathcal{G}(W)$.*

This is obvious and follows directly from the fact that if g^Y is the generalised flow of a t-dependent vector field $Y \in W(\mathbb{R})$ and X takes values in V, then, according to the formula (7.7), $g^Y_\star X$ takes values in V as well, as $[W, V] \subset V$ and V is finite-dimensional.

In some applications, it turns out to be interesting to use a more general class of transformations than those described by $\mathcal{G}(W)$. Nevertheless, such transformations keep the main property of the generalised flows $\mathcal{G}(W)$, namely, for a given scheme $S(W, V)$ they transform elements of $V(\mathbb{R})$ into elements of this space.

Recall that given a Lie algebra of vector fields $W \subset \mathfrak{X}(\mathbb{R}^n)$, there always exists, at least locally in $\mathbb{R}^n$, a group action $\Phi : G \times U \to U$, with G a Lie group with Lie algebra $\mathfrak{g}$, whose fundamental vector fields are those of W (cf. [144] and Section 1.2). For simplicity, we shall suppose, as usual, that this action is globally defined on $\mathbb{R}^n$, and we will write $\Phi : G \times \mathbb{R}^n \to \mathbb{R}^n$ and define the restriction map $\Phi_g : x \in \mathbb{R}^n \mapsto \Phi_g(x) = \Phi(g, x) \in \mathbb{R}^n$ for every $g \in G$.

LEMMA 7.9. *Given a scheme $S(W, V)$, an element $g \in \exp(\mathfrak{g})$, and a vector field $X \in V(\mathbb{R})$, we have $\Phi_{g*} X \in V(\mathbb{R})$.*

Proof. As $g \in \exp(\mathfrak{g})$, there exists an element $\mathrm{a} \in \mathfrak{g}$ such that $g = \exp(\mathrm{a})$. Consider the curve $h : s \in [0,1] \mapsto \exp(s\mathrm{a}) \in G$. By means of the action $\Phi : G \times \mathbb{R}^n \to \mathbb{R}^n$, whose fundamental vector fields are the Lie algebra W of vector fields, the curve $h(s)$ induces the generalised flow $h^Y_s : x \in \mathbb{R}^n \mapsto \Phi(\exp(s\,\mathrm{a}), x) \in \mathbb{R}^n$ of the vector field

$$Y(x) = \left.\frac{\partial}{\partial s}\right|_{s=0} h^Y_s(x) = \left.\frac{\partial}{\partial s}\right|_{s=0} \Phi(\exp(s\mathrm{a}), x)$$

and, obviously, $Y \in W$. Taking into account the relation [1, p. 91]

$$\frac{\partial}{\partial s} h^Y_{-s*} X = h^Y_{-s*}[Y, X],$$

we define, for each s, the vector field $Z^{(0)}_{-s} = h^Y_{-s*}X$ to get

$$(h^Y_{-s*}X)_x = X_x + \int_0^s \frac{\partial}{\partial s'} Z^{(0)}_{-s'}(x)\, ds' = X_x + \int_0^s (h^Y_{-s'*}[Y, X])_x\, ds'.$$

If we set $Z^{(1)}_{-s} = h^Y_{-s*}([Y, X])$ and apply the above expression to $[Y, X]$, we get

$$(h^Y_{-s*}[Y, X])_x = [Y, X]_x + \int_0^s \frac{\partial}{\partial s'} Z^{(1)}_{-s'}(x)\, ds' = [Y, X]_x + \int_0^s (h^Y_{-s'*}[Y, [Y, X]])_x\, ds'.$$

Defining $Z^{(k)}_{-s}$ in an analogous way and applying all these results to the initial formula for $h^Y_{-s*}X$ we obtain

$$(h^Y_{-s*}X)_x = X_x + [Y, X]_x s + \frac{1}{2!}[Y, [Y, X]]_x s^2 + \frac{1}{3!}[Y, [Y, [Y, X]]]_x s^3 + \cdots .$$

From the properties of the scheme, we see that each term belongs to $V(\mathbb{R})$, i.e.

$$[Y, [Y, \ldots, [Y, X] \ldots]] \in V(\mathbb{R}),$$

and therefore

$$\Phi_{g*}X = h^Y_{1*}X \in V(\mathbb{R}). \ \blacksquare$$

Note that every curve $g(t)$ in G determines a diffeomorphism on $\mathbb{R} \times \mathbb{R}^n$ of the form $\overline{\Phi}_{g(t)} : (t, x) \in \mathbb{R} \times \mathbb{R}^n \mapsto (t, \Phi_{g(t)}x) \in \mathbb{R} \times \mathbb{R}^n$. Therefore, given a t-dependent vector field $X \in \mathfrak{X}_t(\mathbb{R}^n)$ and a curve $g(t)$, this curve transforms X into a new vector field X' such that $X' = \overline{\Phi}_{g(t)}\overline{X}$. For simplicity, we denote $X' = g_\star X$ and $g_t : x \in \mathbb{R}^n \mapsto \Phi_{g(t)}x \in \mathbb{R}^n$. Obviously, as in (7.6), we have $(g_\star X)_t = \dot{g}_t \circ g_t^{-1} + g_{t*}(X)$ and the set of curves in G is an infinite-dimensional group acting on $\mathfrak{X}_t(\mathbb{R}^n)$.

Proposition 7.10. *Given a scheme $S(W, V)$, a curve $g(t)$ in G, and a t-dependent vector field $X \in V(\mathbb{R})$, we have $g_\star X \in V(\mathbb{R})$.*

Proof. As formula (7.6) remains valid for the action of curves $g(t)$ included in $\exp(\mathfrak{g})$, proving that $g_\star X$ belongs to $V(\mathbb{R})$ can be reduced to checking that the corresponding terms $\dot{g}_t \circ g_t^{-1}$ and $g_{t*}X$ are in $V(\mathbb{R})$. On one hand, $\dot{g}_t \circ g_t^{-1} \in W(\mathbb{R}) \subset V(\mathbb{R})$ and, by Lemma 7.9, $g_{t*}X \in V(\mathbb{R})$ for each t. Consequently, $g_\star X \in V(\mathbb{R})$. Since every curve $g(t) \subset G$ decomposes as a product $g = g_1 \cdot \ldots \cdot g_p$ of curves $g_j \subset \exp(\mathfrak{g})$ with $j = 1, \ldots, p$, it follows that $g_\star X \in V(\mathbb{R})$ for every curve $g(t) \subset G$. $\blacksquare$

Definition 7.11. Given a scheme $S(W,V)$, we define the *symmetry group* of the scheme, $\mathrm{Sym}(W)$, to be the set of t-dependent transformations $\Phi_{g(t)}$ induced by the curves $g(t)$ in G and an action Φ associated with the Lie algebra W of vector fields.

In order to simplify the notation, we denote the t-dependent transformation $\Phi_{g(t)}$ just by g.

Definition 7.12. Given a quasi-Lie scheme $S(W,V)$ and a t-dependent vector field $X \in V(\mathbb{R})$, we say that X is a *quasi-Lie system with respect to* $S(W,V)$ if there exists a t-dependent transformation $g \in \mathrm{Sym}(W)$ and a Lie algebra of vector fields $V_0 \subset V$ such that

$$g_\star X \in V_0(\mathbb{R}).$$

We emphasise that if X is a quasi-Lie system with respect to the scheme $S(W,V)$, it automatically admits a t-dependent superposition rule (7.9).

7.4. t-dependent superposition rules. Minor modifications in the geometric approach to Lie systems detailed in Section 1.5 allow us to derive a new theory, based on the *Lie family* concept, in order to treat a much larger family of systems of differential equations including Lie and quasi-Lie systems. Roughly speaking, Lie families are sets of systems of differential equations admitting a common superposition rule with t-dependence. This theory clearly generalises the superposition rule notion and provides a characterisation, described by the *Generalised Lie Theorem*, of families of systems admitting such a property. Next, we provide a brief description of this theory and summarise its main results. For further details, see [35].

Consider the family of nonautonomous systems of first-order ordinary differential equations on $\mathbb{R}^n$, parametrised by the elements d of a set Λ, of the form

$$\frac{dx^i}{dt} = Y_d^i(t,x), \qquad i = 1,\ldots,n,\ d \in \Lambda. \tag{7.11}$$

describing the integral curves of the family of t-dependent vector fields $\{Y_d\}_{d\in\Lambda}$ given by

$$Y_d(t,x) = \sum_{i=1}^{n} Y_d^i(t,x)\frac{\partial}{\partial x^i}.$$

Let us state the fundamental concept to be studied along this section:

Definition 7.13. We say that the family of nonautonomous systems (7.11) admits a *common t-dependent superposition rule* if there exists a map $\Phi : \mathbb{R}\times\mathbb{R}^{n(m+1)} \to \mathbb{R}^n$, i.e.

$$x = \Phi(t, x_{(1)}, \ldots, x_{(m)}; k_1, \ldots, k_n), \tag{7.12}$$

such that the general solution, $x(t)$, of any system Y_d of the family (7.11) can be written, at least for sufficiently small t, as

$$x(t) = \Phi(t, x_{(1)}(t), \ldots, x_{(m)}(t); k_1, \ldots, k_n),$$

with $\{x_{(a)}(t) \mid a = 1,\ldots,m\}$ being any generic family of particular solutions of Y_d and the set $\{k_1,\ldots,k_n\}$ being n arbitrary constants associated with each particular solution. A family of systems (7.11) admitting a common t-dependent superposition is called a *Lie family*.

DEFINITION 7.14. Given a t-dependent vector field $Y = \sum_{i=1}^n Y^i(t,x)\partial/\partial x^i$ on $\mathbb{R}^n$, we define its *prolongation* to $\mathbb{R}\times\mathbb{R}^{n(m+1)}$ as the vector field on $\mathbb{R}\times\mathbb{R}^{n(m+1)}$ given by

$$Y^\wedge(t,x_{(0)},\ldots,x_{(m)}) = \sum_{a=0}^m\sum_{i=1}^n Y^i(t,x_{(a)})\frac{\partial}{\partial x^i_{(a)}},$$

and its *autonomisation*, $\widetilde{Y}$, as the vector field on $\mathbb{R}\times\mathbb{R}^{n(m+1)}$ of the form

$$\widetilde{Y}(t,x_{(0)},\ldots,x_{(m)}) = \frac{\partial}{\partial t} + \sum_{a=0}^m\sum_{i=1}^n Y^i(t,x_{(a)})\frac{\partial}{\partial x^i_{(a)}}.$$

The Implicit Function Theorem states that, given a common t-dependent superposition rule $\Phi:\mathbb{R}\times\mathbb{R}^{n(m+1)}\to\mathbb{R}^n$ of a Lie family $\{Y_d\}_{d\in\Lambda}$, the map $\Phi(t,x_{(1)},\ldots,x_{(m)};\,):\mathbb{R}^n\to\mathbb{R}^n$, given by $x_{(0)}=\Phi(t,x_{(1)},\ldots,x_{(m)};k)$, can be inverted to give rise to a map $\Psi:\mathbb{R}\times\mathbb{R}^{n(m+1)}\to\mathbb{R}^n$ given by

$$k=\Psi(t,x_{(0)},\ldots,x_{(m)}),$$

with $k=(k_1,\ldots,k_n)$ being the only point in $\mathbb{R}^n$ such that

$$x_{(0)}=\Phi(t,x_{(1)},\ldots,x_{(m)};k).$$

As the fundamental property of the map Ψ says that $\Psi(t,x_{(0)}(t),\ldots,x_{(m)}(t))$ is constant for any $(m+1)$-tuple of particular solutions of any system of the family (7.11), the foliation determined by Ψ is invariant under the permutation of its $m+1$ arguments $\{x_{(a)}\,|\,a=0,\ldots,m\}$ and differentiating the preceding expression we get

$$\frac{\partial\Psi^j}{\partial t}+\sum_{a=0}^m\sum_{i=1}^n Y^i_d(t,x_{(a)}(t))\frac{\partial\Psi^j}{\partial x^i_{(a)}}=0,\qquad j=1,\ldots,n,\, d\in\Lambda, \tag{7.13}$$

with $\Psi=(\Psi^1,\ldots,\Psi^n)$.

The relation (7.13) shows that the functions of the set $\{\Psi^i\,|\,i=1,\ldots,n\}$ are first integrals for the vector fields $\widetilde{Y}_d$, that is, $\widetilde{Y}_d\Psi^i=0$ for $i=1,\ldots,n$. Therefore, they generically define an n-codimensional foliation $\mathfrak{F}$ on $\mathbb{R}\times\mathbb{R}^{n(m+1)}$ such that the vector fields $\widetilde{Y}_d$ are tangent to the leaves $\mathfrak{F}_k$ of this foliation for $k\in\mathbb{R}^n$.

The foliation $\mathfrak{F}$ has another important property. Given the level set $\mathfrak{F}_k$ of the map Ψ corresponding to $k=(k_1,\ldots,k_n)\in\mathbb{R}^n$ and a generic point $(t,x_{(1)},\ldots,x_{(m)})$ of $\mathbb{R}\times\mathbb{R}^{mn}$, there is only one point $x_{(0)}\in\mathbb{R}^n$ such that $(t,x_{(0)},x_{(1)},\ldots,x_{(m)})\in\mathfrak{F}_k$. Then, the projection onto the last $m\cdot n$ coordinates and the time,

$$\pi:(t,x_{(0)},\ldots,x_{(m)})\in\mathbb{R}\times\mathbb{R}^{n(m+1)}\mapsto(t,x_{(1)},\ldots,x_{(m)})\in\mathbb{R}\times\mathbb{R}^{nm},$$

induces local diffeomorphisms on the leaves $\mathfrak{F}_k$ of $\mathfrak{F}$ into $\mathbb{R}\times\mathbb{R}^{nm}$.

This property can also be seen as the fact that the foliation $\mathfrak{F}$ corresponds to a zero curvature connection ∇ on the bundle $\pi:\mathbb{R}\times\mathbb{R}^{n(m+1)}\to\mathbb{R}\times\mathbb{R}^{nm}$. Indeed, the restriction of the projection π to a leaf gives a one-to-one map. In this way, we get a linear map from vector fields on $\mathbb{R}\times\mathbb{R}^{nm}$ to 'horizontal' vector fields tangent to a leaf.

Note that the knowledge of this connection (foliation) gives us the common t-dependent superposition rule without referring to the map Ψ. If we fix the point $x_{(0)}(0)$ and m particular solutions $x_{(1)}(t),\ldots,x_{(m)}(t)$ for a system of the family, then $x_{(0)}(t)$ is the unique curve in $\mathbb{R}^n$ such that

$$(t, x_{(0)}(t), x_{(1)}(t), \ldots, x_{(m)}(t)) \in \mathbb{R} \times \mathbb{R}^{nm}$$

belongs to the same leaf as the point $(0, x_{(0)}(0), x_{(1)}(0), \ldots, x_{(m)}(0))$. Thus, it is only the foliation $\mathfrak{F}$ that really matters when the common t-dependent superposition rule is concerned.

On the other hand, if we have a zero curvature connection ∇ on the bundle

$$\pi : \mathbb{R} \times \mathbb{R}^{n(m+1)} \to \mathbb{R} \times \mathbb{R}^{nm},$$

i.e. if we have an involutive horizontal distribution ∇ on $\mathbb{R} \times \mathbb{R}^{n(m+1)}$ that can be integrated to give a foliation $\mathfrak{F}$ on $\mathbb{R} \times \mathbb{R}^{n(m+1)}$ and such that the vector fields $\widetilde{Y}_d$ are tangent to the leaves of the foliation, then the procedure described above determines a common t-dependent superposition rule for the family of nonautonomous systems of first-order differential equations (7.11).

Indeed, let $k \in \mathbb{R}^n$ enumerate smoothly the leaves $\mathfrak{F}_k$ of $\mathfrak{F}$, i.e. there exists a smooth map $\iota : \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^{n(m+1)}$ such that $\iota(\mathbb{R}^n)$ intersects every $\mathfrak{F}_k$ in a unique point. Then, if $x_{(0)} \in \mathbb{R}^n$ is the unique point such that

$$(t, x_{(0)}, x_{(1)}, \ldots, x_{(m)}) \in \mathfrak{F}_k,$$

this fact gives rise to a t-dependent superposition rule

$$x_{(0)} = \Phi(t, x_{(1)}, \ldots, x_{(m)}; k)$$

for the family of nonautonomous systems of first-order ordinary differential equations (7.11). To see this, let us observe that the Implicit Function Theorem shows that there exists a function $\Psi : \mathbb{R} \times \mathbb{R}^{n(m+1)} \to \mathbb{R}$ such that

$$\Psi(t, x_{(0)}, \ldots, x_{(m)}) = k,$$

which is equivalent to saying that $(t, x_{(0)}, \ldots, x_{(m)}) \in \mathfrak{F}_k$. If we fix a $k \in \mathbb{R}^n$ and take solutions $x_{(1)}(t), \ldots, x_{(m)}(t)$ of a particular instance of (7.11), then $x_{(0)}(t)$ defined by the condition $\Psi(t, x_{(0)}(t), \ldots, x_{(m)}(t)) = k$ also satisfies that instance. Indeed, let $x'_{(0)}(t)$ be the solution with initial value $x'_{(0)}(0) = x_{(0)}$. Since the vector fields $\widetilde{Y}_d$ are tangent to $\mathfrak{F}$, the curve

$$t \mapsto (t, x_{(0)}(t), x_{(1)}(t), \ldots, x_{(m)}(t))$$

lies entirely in a leaf of $\mathfrak{F}$, so in $\mathfrak{F}_k$. But the point of one leaf is entirely determined by its projection π, so $x'_{(0)}(t) = x_{(0)}(t)$ and $x_{(0)}(t)$ is a solution.

PROPOSITION 7.15. *Giving a t-dependent superposition rule* (7.12) *for a family of systems of differential equations* (7.11) *is equivalent to giving a zero curvature connection on the bundle $\pi : \mathbb{R} \times \mathbb{R}^{(m+1)n} \to \mathbb{R} \times \mathbb{R}^{nm}$ for which the $\widetilde{Y}_d$ are 'horizontal' vector fields.*

In general it is difficult to determine whether a family of differential equations admits a common t-dependent superposition rule by means of the above proposition. It is therefore of interest to find a characterisation of Lie families by a more convenient criterion, e.g. through an easily verifiable condition based on the properties of the t-dependent vector fields $\{Y_a\}_{a\in\Lambda}$. Finding such a criterion is the main result of the theory of Lie families. It is formulated as the Generalised Lie Theorem and based on the lemmas given below. The first two are straightforward, and a complete detailed proof for the third can be found in [35].

LEMMA 7.16. *Given two t-dependent vector fields X and Y on $\mathbb{R}^n$, the commutator $[\widetilde{X}, \widetilde{Y}]$ on $\mathbb{R} \times \mathbb{R}^{n(m+1)}$ is the prolongation of a t-dependent vector field Z on $\mathbb{R}^n$, $[\widetilde{X}, \widetilde{Y}] = Z^\wedge$.*

LEMMA 7.17. *Given a family of t-dependent vector fields $X_1, \ldots, X_r$ on $\mathbb{R}^n$, their autonomisations satisfy the relations*

$$[\bar{X}_j, \bar{X}_k](t,x) = \sum_{l=1}^{r} f_{jkl}(t)\bar{X}_l(t,x), \qquad j,k = 1,\ldots,r,$$

for some t-dependent functions $f_{jkl} : \mathbb{R} \to \mathbb{R}$, if and only if their t-prolongations to $\mathbb{R} \times \mathbb{R}^{n(m+1)}$, $\widetilde{X}_1, \ldots, \widetilde{X}_r$, obey analogous relations

$$[\widetilde{X}_j, \widetilde{X}_k](t,x) = \sum_{l=1}^{r} f_{jkl}(t)\widetilde{X}_l(t,x), \qquad j,k = 1,\ldots,r.$$

Moreover, $\sum_{l=1}^{r} f_{jkl}(t) = 0$ for all $j,k = 1,\ldots,r$.

LEMMA 7.18. *Consider a family of t-dependent vector fields $Y_1, \ldots, Y_r$ with t-prolongations $\widetilde{Y}_1, \ldots, \widetilde{Y}_r$ to $\mathbb{R} \times \mathbb{R}^{n(m+1)}$ such that their projections $\pi_*(\widetilde{Y}_j)$ are linearly independent at a generic point in $\mathbb{R} \times \mathbb{R}^{nm}$. Then $\sum_{j=1}^{r} b_j \widetilde{Y}_j$ with $b_j \in C^\infty(\mathbb{R} \times \mathbb{R}^{nm})$ is of the form $Y^\wedge$ (resp. $\widetilde{Y}$) for a t-dependent vector field Y on $\mathbb{R}^n$ if and only if the functions b_j only depend on the variable t, that is, $b_j = b_j(t)$, and $\sum_{j=1}^{r} b_j = 0$ (resp., $\sum_{j=1}^{r} b_j = 1$).*

MAIN THEOREM 7.19 (Generalised Lie Theorem). *The family of systems* (7.11) *admits a* common t-dependent superposition rule *if and only if the vector fields $\{\overline{Y}_d\}_{d \in \Lambda}$ can be written in the form*

$$\overline{Y}_d(t,x) = \sum_{\alpha=1}^{r} b_{d\alpha}(t)\overline{X}_\alpha(t,x), \qquad d \in \Lambda,$$

where $b_{d\alpha}$ are functions of the single variable t such that $\sum_{\alpha=1}^{r} b_{d\alpha} = 1$ and $X_1, \ldots, X_r$ are t-dependent vector fields satisfying

$$[\overline{X}_\alpha, \overline{X}_\beta](t,x) = \sum_{\gamma=1}^{r} f_{\alpha\beta\gamma}(t)\overline{X}_\gamma(t,x), \qquad \alpha,\beta = 1,\ldots,r, \tag{7.14}$$

for certain functions $f_{\alpha\beta\gamma} : \mathbb{R} \to \mathbb{R}$.

The name of the above theorem comes from the following proposition, which shows that each Lie system can be embedded into a Lie family. In order to formulate this result, let us denote by $S_g(W,V;V_0)$ the set of quasi-Lie systems of the scheme $S(W,V)$ such that there exists a g satisfying that $g_\star X \in V_0(\mathbb{R})$ with V_0 a Lie algebra of vector fields included in V. Again, a complete proof of this proposition can be found in [35].

PROPOSITION 7.20. *The family $S_g(W,V;V_0)$ of quasi-Lie systems is a Lie family admitting the common t-dependent superposition rule of the form*

$$\bar{\Phi}_g(t, x_{(1)}, \ldots, x_{(m)}, k) = g_t^{-1} \circ \Phi(g_t(x_{(1)}, \ldots, g_t)x_{(m)}, k)$$

for any t-independent superposition rule Φ associated with the Lie algebra of vector fields V_0 by the Lie Theorem.

8. Applications of quasi-Lie schemes and Lie families

The theory of quasi-Lie schemes and quasi-Lie systems [34] and the theory of Lie families [35] can be used to investigate a very large set of differential equations, including nonlinear oscillators [34], dissipative Milne–Pinney equations [34, 35, 45], second-order Riccati equations [48], Abel equations [35], Emden equations [34, 42], etc. As we showed in the previous section, these theories enable us to obtain t-dependent superposition rules, constants of motion, exact solutions, integrability conditions, etc. The main aim in this chapter is to show that the possibilities of application of these methods are very wide and we can obtain a large set of results from a unified point of view.

More exactly, in previous sections it was proved that Milne–Pinney could be studied by means of the theory of Lie systems (see also [43]). Nevertheless, there exist dissipative Milne–Pinney equations that cannot be studied directly through this theory. In this section, we provide a quasi-Lie scheme to treat these dissipative Milne–Pinney equations. We use this quasi-Lie scheme to relate these equations to usual Milne–Pinney equations. By means of this relation, we obtain a t-dependent superposition rule for dissipative Milne–Pinney equations.

Apart from dissipative Milne–Pinney equations, we also investigate nonautonomous nonlinear oscillators. We show that some of them can be transformed into autonomous nonlinear oscillators. This result was already derived by Perelomov [180], but here we recover it from a more general point of view. More specifically, we show that the nonautonomous nonlinear oscillators analysed by Perelomov can be seen as differential equations obeying an integrability condition derived by means of a quasi-Lie scheme.

As a last application of quasi-Lie schemes, we extensively analyse Emden equations. We provide a quasi-Lie scheme to obtain t-dependent constants of motion by means of particular solutions that obey an integrability condition. The method developed also enables us to obtain Emden equations with a fixed t-dependent constant of motion. Kummer–Liouville transformations are also obtained by means of our scheme and many other properties are recovered.

Finally, in the last two sections of this chapter, we apply common t-dependent superposition rules to study some first- and second-order differential equations. In this way, we can analyse equations which cannot be studied by means of the usual theory of Lie systems. Additionally, some new results on Abel and Milne–Pinney equations are provided.

8.1. Dissipative Milne–Pinney equations. In this section, we study the so-called dissipative Milne–Pinney equations. We show that the first-order ordinary differential equations associated with these second-order equations in the usual way, i.e. by considering velocities as new variables, are not Lie systems. However, the theory of quasi-Lie schemes can be used to deal with such first-order systems. Here we provide a scheme which enables us to transform a certain kind of dissipative Milne–Pinney equations, considered as first-order systems, into some first-order Milne–Pinney equations already studied by means of the theory of Lie systems [53]. As a result we get a t-dependent superposition rule for some of these dissipative Milne–Pinney equations.

Consider the family of dissipative Milne–Pinney equations of the form

$$\ddot{x} = a(t)\dot{x} + b(t)x + c(t)\frac{1}{x^3}. \tag{8.1}$$

We are mainly interested in the case $c(t) \neq 0$, so we assume that $c(t)$ has a constant sign for the set of values of t that we analyse.

Usually, we associate to such a second-order differential equation a system of first-order differential equations with a new variable v,

$$\begin{cases} \dot{x} = v, \\ \dot{v} = a(t)v + b(t)x + c(t)\dfrac{1}{x^3}. \end{cases} \tag{8.2}$$

Let us search for a quasi-Lie scheme to handle the above system. Remember that we need to find linear spaces $W_{\rm DisM}$ and $V_{\rm DisM}$ of vector fields such that

1. $W_{\rm DisM} \subset V_{\rm DisM}$.
2. $[W_{\rm DisM}, W_{\rm DisM}] \subset W_{\rm DisM}$.
3. $[W_{\rm DisM}, V_{\rm DisM}] \subset V_{\rm DisM}$.

Also, in order to treat system (8.2) through this scheme, we have to ensure that the t-dependent vector field

$$X_t = v\frac{\partial}{\partial x} + \left(a(t)v + b(t)x + \frac{c(t)}{x^3}\right)\frac{\partial}{\partial v},$$

whose integral curves are solutions for (8.2), is such that $X_t \in V_{\rm DisM}$ for every t in an open interval of $\mathbb{R}$.

Consider the vector space $V_{\rm DisM}$ spanned by the vector fields

$$X_1 = v\frac{\partial}{\partial v}, \qquad X_2 = x\frac{\partial}{\partial v}, \qquad X_3 = \frac{1}{x^3}\frac{\partial}{\partial v}, \qquad X_4 = v\frac{\partial}{\partial x}, \qquad X_5 = x\frac{\partial}{\partial x}$$

and the two-dimensional vector subspace $W_{\rm DisM} \subset V_{\rm DisM}$ generated by

$$Y_1 = X_1 = v\frac{\partial}{\partial v}, \qquad Y_2 = X_2 = x\frac{\partial}{\partial v}.$$

It can be seen that $W_{\rm DisM}$ is a Lie algebra,

$$[Y_1, Y_2] = -Y_2,$$

and, additionally, as

$$\begin{aligned} &[Y_1, X_3] = -X_3, \quad &&[Y_1, X_4] = X_4, \quad &&[Y_1, X_5] = 0, \\ &[Y_2, X_3] = 0, \quad &&[Y_2, X_4] = X_5 - X_1, \quad &&[Y_2, X_5] = -X_2, \end{aligned}$$

the linear space $V_{\rm DisM}$ is invariant under the action of the Lie algebra $W_{\rm DisM}$ on $V_{\rm DisM}$, i.e. $[W_{\rm DisM}, V_{\rm DisM}] \subset V_{\rm DisM}$. Thus, the vector spaces

$$V_{\rm DisM} = \langle X_1, \ldots, X_5\rangle \quad \text{and} \quad W_{\rm DisM} = \langle Y_1, Y_2\rangle$$

of vector fields form a quasi-Lie scheme $S(W_{\rm DisM}, V_{\rm DisM})$. Let us observe that

$$X_t = a(t)X_1 + b(t)X_2 + c(t)X_3 + X_4$$

and thus $X \in V_{\rm DisM}(\mathbb{R})$.

We stress that the vector space $V_{\rm DisM}$ is not a Lie algebra, because the commutator $[X_3, X_4]$ does not belong to $V_{\rm DisM}$. Moreover, $V'' = \langle X_1, \ldots, X_4\rangle$ is not a Lie algebra for a similar reason: $[X_3, X_4] \notin V''$. Additionally, there exists no finite-dimensional real Lie algebra V' containing V''. Thus, (8.2) is not a Lie system, but we can use the quasi-Lie scheme $S(W_{\rm DisM}, V_{\rm DisM})$ to investigate it.

The key tool provided by the scheme $S(W_{\rm DisM}, V_{\rm DisM})$ is the infinite-dimensional group $\mathcal{G}(W_{\rm DisM})$ of generalised flows for the t-dependent vector fields with values in W, i.e. $\alpha_1(t)Y_1 + \alpha_2(t)Y_2$, which leads to the group of t-dependent changes of variables

$$\mathcal{G}(W_{\rm DisM}) = \left\{ g(\alpha(t), \beta(t)) = \begin{cases} x = x' \\ v = \alpha(t)v' + \beta(t)x' \end{cases} \middle| \, \alpha(t) > 0,\ \beta(0) = 0,\ \alpha(0) = 1 \right\}.$$

According to the general theory of quasi-Lie schemes, these t-dependent changes of variables enable us to transform system (8.2) into a new one taking values in $V_{\rm DisM}$,

$$X'_t = a'(t)X_1 + b'(t)X_2 + c'(t)X_3 + d'(t)X_4 + e'(t)X_5. \tag{8.3}$$

The new coefficients are

$$\begin{cases} a'(t) = a(t) - \beta(t) - \dfrac{\dot\alpha(t)}{\alpha(t)}, \\ b'(t) = \dfrac{b(t)}{\alpha(t)} + a(t)\dfrac{\beta(t)}{\alpha(t)} - \dfrac{\beta^2(t)}{\alpha(t)} - \dfrac{\dot\beta(t)}{\alpha(t)}, \\ c'(t) = \dfrac{c(t)}{\alpha(t)}, \\ d'(t) = \alpha(t), \\ e'(t) = \beta(t). \end{cases}$$

The integral curves for the t-dependent vector field (8.3) are solutions of the system

$$\begin{cases} \dfrac{dx'}{dt} = \beta(t)x' + \alpha(t)v', \\ \dfrac{dv'}{dt} = \left(\dfrac{b(t)}{\alpha(t)} + a(t)\dfrac{\beta(t)}{\alpha(t)} - \dfrac{\beta^2(t)}{\alpha(t)} - \dfrac{\dot\beta(t)}{\alpha(t)}\right)x' \\ \qquad\qquad + \left(a(t) - \beta(t) - \dfrac{\dot\alpha(t)}{\alpha(t)}\right)v' + \dfrac{c(t)}{\alpha(t)}\dfrac{1}{x'^3}. \end{cases} \tag{8.4}$$

As mentioned in Section 7.3, we use schemes to transform the corresponding systems of first-order differential equations into Lie ones. So, in this case, we must find a Lie algebra $V_0 \subset V_{\rm DisM}$ and a generalised flow $g \in \mathcal{G}(W_{\rm DisM})$ such that $g_\star X \in V_0(\mathbb{R})$. This leads to a system of ordinary differential equations for the functions $\alpha(t)$, $\beta(t)$ and some integrability conditions on the initial functions $a(t), b(t)$ and $c(t)$ for such a t-dependent change of variables to exist.

In order to find a proper Lie algebra $V_0 \subset V$, note that Milne–Pinney equations studied in [53] are Lie systems in the family of differential equations defined by systems (8.2) and therefore it is natural to look for conditions needed to transform a given system of (8.2), described by the t-dependent vector field X_t, into a system of first-order Milne–

Pinney equations of the form

$$\begin{cases} \dot{x} = f(t)v, \\ \dot{v} = -\omega(t)x + f(t)\dfrac{k}{x^3}, \end{cases} \tag{8.5}$$

where k is a constant, i.e. a system describing the integral curves for a t-dependent vector field with values in the Lie algebra [53]

$$V_0 = \left\langle X_4 + kX_3, X_2, \frac{1}{2}(X_5 - X_1) \right\rangle.$$

As a result, we get $\beta = 0$, $\alpha = f$ and, furthermore, the functions α, a and c must satisfy

$$k\alpha^2 = c, \qquad \dot{\alpha} - a\alpha = 0, \tag{8.6}$$

so c and k have the same sign. The second condition is a differential equation for α and the first one determines c in terms of α. Therefore, both conditions lead to a relation between c and a providing the integrability condition

$$c(t) = k \exp\left(2 \int a(t)\, dt\right) \tag{8.7}$$

and showing, in view of (8.4)–(8.6), that

$$\alpha(t) = \exp\left(\int a(t)\, dt\right) \quad \text{and} \quad \omega(t) = -b(t) \exp\left(- \int a(t)\, dt\right),$$

where we choose the constants of integration to get $\alpha(0) = 1$ as required.

Summarising the preceding results, under the integrability condition (8.7), the first-order Milne–Pinney equation

$$\begin{cases} \dot{x} = v, \\ \dot{v} = a(t)v + b(t)x + c(t)\dfrac{1}{x^3}, \end{cases}$$

can be transformed into the system

$$\begin{cases} \dfrac{dx'}{dt} = \exp\left(\int a(t)\, dt\right) v', \\ \dfrac{dv'}{dt} = b(t) \exp\left(- \int a(t)\, dt\right) x' + \exp\left(\int a(t)\, dt\right) \dfrac{k}{x'^3}, \end{cases}$$

by means of the t-dependent change of variables

$$g\left(\exp\left(\int a(t)\, dt\right), 0\right) = \begin{cases} x' = x, \\ v' = \exp\left(\int a(t)\, dt\right) v. \end{cases}$$

We stress that this change of variables is a particular instance of the so-called Liouville transformation [164].

The final Milne–Pinney equation can be rewritten through the t-reparametrisation

$$\tau(t) = \int \exp\left(\int a(t)\, dt\right) dt,$$

as

$$\begin{cases} \dfrac{dx'}{d\tau} = v', \\ \dfrac{dv'}{d\tau} = \exp\left(-2\displaystyle\int a(t)\,dt\right) b(t(\tau))x' + \dfrac{k}{x'^3}. \end{cases}$$

These systems were analysed in [50], where it was shown through the theory of Lie systems that they admit the constant of motion

$$I = (\bar{x}v' - \bar{v}x')^2 + k\left(\frac{\bar{x}}{x'}\right)^2,$$

where $(\bar{x}, \bar{v})$ is a solution of the system

$$\begin{cases} \dfrac{d\bar{x}}{d\tau} = \bar{v}, \\ \dfrac{d\bar{v}}{d\tau} = \exp\left(-2\displaystyle\int a(t)\,dt\right) b(t)\bar{x}, \end{cases}$$

which can be written as a second-order differential equation

$$\frac{d^2\bar{x}}{d\tau^2} = \exp\left(-2\int a(t)\,dt\right) b(t)\bar{x}.$$

If we invert the t-reparametrisation, we obtain the equation

$$\ddot{\bar{x}} - a(t)\dot{\bar{x}} - b(t)\bar{x} = 0, \tag{8.8}$$

which is the linear differential equation associated with the initial Milne–Pinney equation.

As shown in [53], we can obtain, by means of the theory of Lie systems, the following superposition rule:

$$x' = \frac{\sqrt{2}}{|\bar{x}_1\bar{v}_2 - \bar{v}_1\bar{x}_2|}(I_2\bar{x}_1^2 + I_1\bar{x}_2^2 \pm \sqrt{4I_1I_2 - k(\bar{x}_1\bar{v}_2 - \bar{v}_1\bar{v}_2)^2}\,\bar{x}_1\bar{x}_2)^{1/2},$$

and as the t-dependent transformation performed does not change the variable x, we get the t-dependent superposition rule

$$x = \frac{\sqrt{2}\alpha(t)}{|\bar{x}_1\dot{\bar{x}}_2 - \dot{\bar{x}}_1\bar{x}_2|}\left(I_2\bar{x}_1^2 + I_1\bar{x}_2^2 \pm \sqrt{4I_1I_2 - \frac{k}{\alpha^2(t)}(\bar{x}_1\dot{\bar{x}}_2 - \dot{\bar{x}}_1\bar{x}_2)^2\,\bar{x}_1\bar{x}_2}\right)^{1/2},$$

in terms of a set of solutions of the second-order linear system (8.8).

Summing up, application of our scheme to the family of dissipative Milne–Pinney equations

$$\ddot{x} = a(t)\dot{x} + b(t)x + \exp\left(2\int a(t)\,dt\right)\frac{k}{x^3}$$

shows that this family admits a t-dependent superposition rule

$$x = \frac{\sqrt{2}\alpha(t)}{|y_1\dot{y}_2 - y_2\dot{y}_1|}\left(I_2y_1^2 + I_1y_2^2 \pm \sqrt{4I_1I_2 - \frac{k}{\alpha^2(t)}(y_1\dot{y}_2 - y_2\dot{y}_1)^2\,y_1y_2}\right)^{1/2},$$

in terms of two independent solutions y_1, y_2 of the differential equation

$$\ddot{y} - a(t)\dot{y} - b(t)y = 0.$$

So, we have fully detailed a particular application of the theory of quasi-Lie schemes to dissipative Milne–Pinney equations. As a result, we provide a t-dependent superposition rule for a family of such systems. Another paper with such an approach to dissipative Milne–Pinney equations and explaining some of their properties is [45].

8.2. Nonlinear oscillators. As a second application of our theory, we use quasi-Lie schemes to deal with a certain kind of nonlinear oscillators. The main objective of this section is to explain several properties of a family of t-dependent nonlinear oscillators studied by Perelomov in [180]. We also furnish a new, as far as we know, constant of motion for these systems.

Consider the following subset of the family of nonlinear oscillators investigated in [180]:

$$\ddot{x} = b(t)x + c(t)x^n, \qquad n \neq 0, 1.$$

The cases $n = 0, 1$ are omitted because they can be handled with the usual theory of Lie systems. As in the section above, we link the above second-order ordinary differential equation to the first-order system

$$\begin{cases} \dot{x} = v, \\ \dot{v} = b(t)x + c(t)x^n. \end{cases} \tag{8.9}$$

Let us provide a quasi-Lie scheme to deal with systems (8.9). Consider the vector space V_{NO} spanned by the linear combinations of the vector fields

$$X_1 = x\frac{\partial}{\partial v}, \qquad X_2 = x^n\frac{\partial}{\partial v}, \qquad X_3 = v\frac{\partial}{\partial x}, \qquad X_4 = v\frac{\partial}{\partial v}, \qquad X_5 = x\frac{\partial}{\partial x}$$

on $\mathrm{T}\mathbb{R}$ and take the vector subspace $W_{NO} \subset V_{NO}$ generated by

$$Y_1 = X_4 = v\frac{\partial}{\partial v}, \qquad Y_2 = X_1 = x\frac{\partial}{\partial v}, \qquad Y_3 = X_5 = x\frac{\partial}{\partial x}.$$

Therefore, W_{NO} is a solvable Lie algebra of vector fields,

$$[Y_1, Y_2] = -Y_2, \qquad [Y_1, Y_3] = 0, \qquad [Y_2, Y_3] = -Y_2,$$

and taking into account that

$$\begin{aligned} &[Y_1, X_2] = -X_2, && [Y_1, X_3] = X_3, && [Y_2, X_2] = 0, \\ &[Y_2, X_3] = X_5 - X_4, && [Y_3, X_2] = nX_2, && [Y_3, X_3] = -X_3, \end{aligned}$$

we see that V_{NO} is invariant under the action of W_{NO}, i.e. $[W_{NO}, V_{NO}] \subset V_{NO}$. In this way we get the quasi-Lie scheme $S(W_{NO}, V_{NO})$.

Now, we have to check whether the solutions of system (8.9) are integral curves for a t-dependent vector field $X \in V_{NO}(\mathbb{R})$. For this, note that the system (8.9) describes the integral curves for the t-dependent vector field

$$X_t = v\frac{\partial}{\partial x} + (b(t)x + c(t)x^n)\frac{\partial}{\partial v},$$

which can be written as

$$X_t = b(t)X_1 + c(t)X_2 + X_3. \tag{8.10}$$

Note also that $[X_2, X_3] \notin V_{NO}$ and $V'' = \langle X_1, X_2, X_3\rangle$ is not only a Lie algebra of vector fields, but also there is no finite-dimensional Lie algebra V' including V''. Thus, X cannot be considered as a Lie system and we conclude that the first-order nonlinear oscillator

$$\begin{cases} \dot{x} = v, \\ \dot{v} = b(t)x + c(t)x^n, \end{cases}$$

describing integral curves of the t-dependent vector field (8.10) (which is not a Lie system) can be described by means of the quasi-Lie scheme $S(W_{NO}, V_{NO})$.

Now, the group $\mathcal{G}(W_{NO})$ of generalised flows associated with $S(W_{NO}, V_{NO})$ is formed by the t-dependent transformations

$$g(\alpha(t), \beta(t), \gamma(t)) = \begin{cases} x = \gamma(t)x', \\ v = \beta(t)v' + \alpha(t)x', \end{cases} \qquad \beta(t), \gamma(t) > 0,\ \beta(0) = \gamma(0) = 1,\ \alpha(0) = 0.$$

Let us restrict ourselves to the case $\alpha(t) = \dot{\gamma}(t)$ and $\beta(t) = 1/\gamma(t)$ and apply these transformations to the system (8.9). The theory of quasi-Lie systems tells us that

$$g(\alpha(t), \beta(t), \gamma(t))_\star X \in V_{NO}(\mathbb{R}).$$

Indeed, these t-dependent transformations lead to the systems

$$\begin{cases} \dfrac{dx'}{dt} = \dfrac{1}{\gamma^2(t)} v', \\ \dfrac{dv'}{dt} = (\gamma^2(t)b(t) - \ddot{\gamma}(t)\gamma(t))x' + c(t)\gamma^{n+1}(t)x'^n, \end{cases} \tag{8.11}$$

which are related to the second-order differential equations

$$\gamma^2(t)\ddot{x}' = -2\gamma(t)\dot{\gamma}(t)\dot{x}' + (\gamma^2(t)b(t) - \ddot{\gamma}(t)\gamma(t))x' + c(t)\gamma^{n+1}(t)x'^n.$$

But the theory of quasi-Lie schemes is based on the search of a generalised flow $g \in \mathcal{G}(W_{NO})$ such that $g_\star X$ becomes a Lie system, i.e. there exists a Lie algebra of vector fields $V_0 \subset V_{NO}$ such that $g_\star X \in V_0(\mathbb{R})$. For instance, we can try to transform a particular instance of the systems (8.11) into a first-order differential equation associated with a nonlinear oscillator with a zero t-dependent angular frequency, for example, into the first-order system

$$\begin{cases} \dfrac{dx'}{dt} = f(t)v', \\ \dfrac{dv'}{dt} = f(t)c_0 x'^n, \end{cases} \tag{8.12}$$

related to the nonlinear oscillator

$$\frac{d^2x'}{d\tau^2} = c_0 x'^n,$$

with $d\tau/dt = f(t)$.

The conditions ensuring such a transformation are

$$\gamma(t)b(t) - \ddot{\gamma}(t) = 0, \qquad c(t) = c_0\gamma^{-(n+3)}(t), \tag{8.13}$$

with $f(t) = \gamma_1^{-2}(t)$, where γ_1 is a nonvanishing particular solution for $\gamma(t)b(t) - \ddot{\gamma}(t) = 0$. We must emphasise that only particular solutions with $\gamma_1(0) = 1$ and $\dot{\gamma}_1(0) = 0$ are related to generalised flows in $\mathcal{G}(W_{\rm NO})$. Nevertheless, any other particular solution can also be used to transform a nonlinear oscillator into a Lie system as we stated. The Lie system (8.12) is the system associated with the t-dependent vector field

$$X_t = \frac{1}{\gamma_1^2(t)}\left(v'\frac{\partial}{\partial x'} + c_0 x'^n \frac{\partial}{\partial v'}\right).$$

By standard methods in the theory of Lie systems [52], we join two copies of the above system in order to get the first integrals

$$I_i = \frac{1}{2}v_i'^2 - \frac{c_0}{n+1}x_i'^{n+1}, \qquad i = 1, 2,$$

and

$$\begin{aligned} I_3 = {} & \frac{x_1'}{\sqrt{I_1}}\mathrm{Hyp}\left(\frac{1}{n+1}, \frac{1}{2}, 1 + \frac{1}{n+1}, -\frac{c_0 x_1'^{n+1}}{I_1(n+1)}\right) \\ & - \frac{x_2'}{\sqrt{I_2}}\mathrm{Hyp}\left(\frac{1}{n+1}, \frac{1}{2}, 1 + \frac{1}{n+1}, -\frac{c_0 x_2'^{n+1}}{I_2(n+1)}\right), \end{aligned}$$

where $\mathrm{Hyp}(a, b, c, d)$ denotes the corresponding hypergeometric functions. In terms of the initial variables these first integrals for $g_\star X$ read

$$I_i = \frac{1}{2}(\gamma_1(t)\dot{x}_i - \dot{\gamma}_1(t)x_i)^2 - \frac{c_0}{\gamma_1^{n+1}(t)(n+1)}x_i^{n+1}, \qquad i = 1, 2, \tag{8.14}$$

and

$$\begin{aligned} I_3 = {} & \frac{1}{\gamma_1(t)}\left(\frac{x_1}{\sqrt{I_1}}\mathrm{Hyp}\left(\frac{1}{n+1}, \frac{1}{2}, 1 + \frac{1}{n+1}, -\frac{c_0 x_1^{n+1}}{\gamma_1^{n+1}(t)I_1(n+1)}\right)\right. \\ & \left. - \frac{x_2}{\sqrt{I_2}}\mathrm{Hyp}\left(\frac{1}{n+1}, \frac{1}{2}, 1 + \frac{1}{n+1}, -\frac{c_0 x_2^{n+1}}{\gamma_1^{n+1}(t)I_2(n+1)}\right)\right). \end{aligned} \tag{8.15}$$

As a particular application of conditions (8.13), we can consider the following example of [180], where the t-dependent Hamiltonian

$$H(t) = \frac{1}{2}p^2 + \frac{\omega^2(t)}{2}x^2 + c^2\gamma_1^{-(s+2)}(t)x^s$$

with γ_1 such that $\ddot{\gamma}_1(t) + \omega^2(t)\gamma_1(t) = 0$ is studied. The corresponding Hamilton equations are

$$\begin{cases} \dot{x} = p, \\ \dot{p} = -sc^2\gamma_1^{-(s+2)}(t)x^{s-1} - \omega^2(t)x, \end{cases} \tag{8.16}$$

which are associated with the second-order differential equation for the variable x given by

$$\ddot{x} = -sc^2\gamma_1^{-(s+2)}(t)x^{s-1} - \omega^2(t)x. \tag{8.17}$$

Note that here the variable p plays the same rôle as v in our theoretical development and the last differential equation is a particular case of our Emden equations with

$$b(t) = -\omega^2(t), \qquad c(t) = -sc^2\gamma_1^{-(s+2)}(t), \qquad n = s - 1. \tag{8.18}$$

Let us prove that the above coefficients satisfy the conditions (8.13):

1. By assumption, $\omega^2(t)\gamma_1(t) + \ddot{\gamma}_1(t) = 0$. As $\omega^2(t) = -b(t)$, then $\gamma_1(t)b(t) - \ddot{\gamma}_1(t) = 0$.
2. If we fix $c_0 = -sc^2$, in view of conditions (8.18), we obtain $c(t) = c_0\gamma_1^{-(n+3)}(t)$.

Therefore, the t-dependent frequency nonlinear oscillator (8.17) can be transformed into a new one with zero frequency, i.e.

$$\frac{d^2x'}{d\tau^2} = -sc^2x'^{s-1},$$

with

$$\tau = \int \frac{dt}{\gamma_1^2(t)},$$

recovering the result of Perelomov [180]. The choice of the t-dependent frequencies is such that it is possible to transform the initial t-dependent nonlinear oscillator into the final autonomous nonlinear oscillator. Thus, we recover here such frequencies as a result of an integrability condition. Moreover, in view of the expressions (8.14), (8.15) and (8.18), we get a new t-dependent constant of motion for these nonlinear oscillators.

8.3. Dissipative Mathews–Lakshmanan oscillators. In this section we provide a simple application of the theory of quasi-Lie schemes to the t-dependent dissipative Mathews–Lakshmanan oscillator

$$(1+\lambda x^2)\ddot{x} - F(t)(1+\lambda x^2)\dot{x} - (\lambda x)\dot{x}^2 + \omega(t)x = 0, \qquad \lambda > 0. \tag{8.19}$$

More specifically, we supply some integrability conditions to relate the above dissipative oscillator to the Mathews–Lakshmanan oscillator [65, 67, 142, 161]

$$(1+\lambda x^2)\ddot{x} - (\lambda x)\dot{x}^2 + kx = 0, \qquad \lambda > 0, \tag{8.20}$$

and by means of such a relation we get a new t-dependent constant of motion.

Consider the system of first-order differential equation related to equation (8.19) in the usual way, i.e.

$$\begin{cases} \dot{x} = v, \\ \dot{v} = F(t)v + \dfrac{\lambda x v^2}{1+\lambda x^2} - \omega(t)\dfrac{x}{1+\lambda x^2}, \end{cases} \tag{8.21}$$

and determining the integral curves for the t-dependent vector field

$$X_t = \left(F(t)v + \frac{\lambda x v^2}{1+\lambda x^2} - \omega(t)\frac{x}{1+\lambda x^2}\right)\frac{\partial}{\partial v} + v\frac{\partial}{\partial x}.$$

Let us provide a scheme to handle the system (8.21). Consider the vector space V spanned by the vector fields

$$X_1 = v\frac{\partial}{\partial x} + \frac{\lambda x v^2}{1+\lambda x^2}\frac{\partial}{\partial v}, \qquad X_2 = \frac{x}{1+\lambda x^2}\frac{\partial}{\partial v}, \qquad X_3 = v\frac{\partial}{\partial v}, \tag{8.22}$$

and the linear space $W = \langle X_3 \rangle$. The commutation relations

$$[X_3, X_1] = X_1, \qquad [X_3, X_2] = -X_2$$

imply that the linear spaces W, V make up a quasi-Lie scheme $S(W, V)$. As the t-dependent vector field X_t reads in terms of the basis (8.22)

$$X_t = F(t)X_3 - \omega(t)X_2 + X_1,$$

we see that $X_t \in V(\mathbb{R})$.

Integration of X_3 shows that

$$\mathcal{G}(W) = \left\{ g(\alpha(t)) = \left\{ \begin{aligned} x &= x' \\ v &= \alpha(t)v' \end{aligned} \right. \middle|\, \alpha(t) > 0,\, \alpha(0) = 1 \right\},$$

and the t-dependent changes of variables related to the controls of $\mathcal{G}(W)$ transform the system (8.21) into

$$\begin{cases} \dot{x}' = \alpha(t)v', \\ \dot{v}' = \left(F(t) - \dfrac{\dot{\alpha}(t)}{\alpha(t)} \right) v' - \dfrac{\omega(t)}{\alpha(t)} \dfrac{x'}{1+\lambda x'^2} + \alpha(t) \dfrac{\lambda x' v'^2}{1+\lambda x'^2}. \end{cases}$$

Suppose that we fix $\dot{\alpha} - F(t)\alpha = 0$. Then the above becomes

$$\begin{cases} \dot{x}' = \alpha(t)v', \\ \dot{v}' = -\dfrac{\omega(t)}{\alpha(t)} \dfrac{x'}{1+\lambda x'^2} + \alpha(t) \dfrac{\lambda x' v'^2}{1+\lambda x'^2}. \end{cases}$$

Let us try to search conditions ensuring that the above system determines the integral curves for a t-dependent vector field of the form $X(t,x) = f(t)\bar{X}(x)$ with $\bar{X} \in V$, e.g.

$$\begin{cases} \dot{x}' = f(t)v', \\ \dot{v}' = f(t) \left(\dfrac{x'}{1+\lambda x'^2} + \dfrac{\lambda x' v'^2}{1+\lambda x'^2} \right). \end{cases}$$

In such a case, $\alpha(t) = f(t)$, $\omega(t) = -\alpha^2(t)$ and therefore $\omega(t) = -\exp(2\int F(t)\,dt)$. The t-reparametrisation $d\tau = f(t)\,dt$ transforms the previous system into the autonomous one

$$\begin{cases} \dfrac{dx'}{d\tau} = v', \\ \dfrac{dv'}{d\tau} = \dfrac{x'}{1+\lambda x'^2} + \dfrac{\lambda x' v'^2}{1+\lambda x'^2}, \end{cases}$$

determining the integral curves for the vector field $X = X_1 + X_2$ and related to a Mathews–Lakshmanan oscillator (8.20) with $k = 1$. The method of characteristics shows, after brief calculation, that this system has a first integral

$$I(x', v') = \frac{1+\lambda x'^2}{1+\lambda v'^2},$$

which reads in terms of the initial variables and the variable t as a new t-dependent constant of motion

$$I(t,x,v) = \frac{\alpha^2(t) + \lambda\alpha^2(t)x^2}{\alpha^2(t) + \lambda v^2}$$

for the t-dependent dissipative Mathews–Lakshmanan oscillator (8.19).

8.4. The Emden equation. In this and the following sections we analyse, from the perspective of the theory of quasi-Lie schemes, the so-called Emden equations of the form

$$\ddot{x} = a(t)\dot{x} + b(t)x^n, \qquad n \neq 1. \tag{8.23}$$

These equations can be associated with the system of first-order differential equations

$$\begin{cases} \dot{x} = v, \\ \dot{v} = a(t)v + b(t)x^n. \end{cases} \tag{8.24}$$

This system was already studied in [34, 42] by means of quasi-Lie schemes. We summarise some of the results of those papers, which concern the determination of t-dependent constants of motion by means of particular solutions, reducible particular cases of Emden equations, etc.

Consider the real vector space $V_{\rm Emd}$ spanned by the vector fields

$$X_1 = x\frac{\partial}{\partial v}, \quad X_2 = x^n\frac{\partial}{\partial v}, \quad X_3 = v\frac{\partial}{\partial x}, \quad X_4 = v\frac{\partial}{\partial v}, \quad X_5 = x\frac{\partial}{\partial x}.$$

The t-dependent vector field determining the dynamics of system (8.24) can be written as a linear combination

$$X_t = a(t)X_4 + X_3 + b(t)X_2.$$

Moreover, the linear space $W_{\rm Emd} \subset V_{\rm Emd}$ spanned by the complete vector fields

$$Y_1 = X_4 = v\frac{\partial}{\partial v}, \quad Y_2 = X_1 = x\frac{\partial}{\partial v}, \quad Y_3 = X_5 = x\frac{\partial}{\partial x}$$

is a three-dimensional real Lie algebra of vector fields with respect to the ordinary Lie bracket:

$$[Y_1, Y_2]_{LB} = -Y_2, \quad [Y_1, Y_3]_{LB} = 0, \quad [Y_2, Y_3]_{LB} = -Y_2.$$

Also $[W_{\rm Emd}, V_{\rm Emd}]_{LB} \subset V_{\rm Emd}$ because

$$\begin{aligned} &[Y_1, X_2]_{LB} = -X_2, && [Y_1, X_3]_{LB} = X_3, && [Y_2, X_2]_{LB} = 0, \\ &[Y_2, X_3]_{LB} = X_5 - X_4, && [Y_3, X_2]_{LB} = nX_2, && [Y_3, X_3]_{LB} = -X_3. \end{aligned}$$

So we get a quasi-Lie scheme $S(W_{\rm Emd}, V_{\rm Emd})$ which can be used to treat the Emden equations (8.24). This suggests that if we perform the t-dependent change of variables associated with this quasi-Lie scheme, namely,

$$\begin{cases} x = \gamma(t)x', \\ v = \beta(t)v' + \alpha(t)x', \end{cases} \qquad \gamma(t)\beta(t) > 0, \ \forall t, \tag{8.25}$$

the original system transforms into

$$\begin{cases} \dfrac{dx'}{dt} = \left(\dfrac{\alpha(t)}{\gamma(t)} - \dfrac{\dot{\gamma}(t)}{\gamma(t)}\right)x' + \dfrac{\beta(t)}{\gamma(t)}v', \\ \dfrac{dv'}{dt} = \left(a(t) - \dfrac{\alpha(t)}{\gamma(t)} - \dfrac{\dot{\beta}(t)}{\beta(t)}\right)v' + \dfrac{\alpha(t)}{\beta(t)}\left(a(t) - \dfrac{\alpha(t)}{\gamma(t)} - \dfrac{\dot{\alpha}(t)}{\alpha(t)} + \dfrac{\dot{\gamma}(t)}{\gamma(t)}\right)x' \\ \qquad + \dfrac{b(t)\gamma^n(t)}{\beta(t)}x'^n. \end{cases} \tag{8.26}$$

The key point of our method is to choose functions α, β and γ in such a way that (8.26) becomes a Lie system. A possible way to do so is to choose α, β and γ so that the above system becomes determined by a t-dependent vector field $X_t = f(t)\bar{X}$, where $\bar{X}$ is a true vector field and $f(t)$ is a nonvanishing function (on the interval of t under study).

As shown in the next section, this cannot always be done and some conditions must be imposed on α, β and γ. These restrictions lead to integrability conditions.

Suppose, for the time being, that this is the case. Therefore, system (8.26) is

$$\left\{\begin{aligned} \frac{dx'}{dt} &= f(t)(c_{11}x' + c_{12}v'), \\ \frac{dv'}{dt} &= f(t)(c_{22}x'^n + c_x x' + c_{21}v') \end{aligned}\right. \tag{8.27}$$

and it is determined by the t-dependent vector field

$$X_t = f(t)\bar{X}$$

with

$$\bar{X} = (c_{11}x' + c_{12}v')\frac{\partial}{\partial x'} + (c_{22}x'^n + c_x x' + c_{21}v')\frac{\partial}{\partial v'}.$$

Under the t-reparametrisation

$$\tau = \int^t f(t')\, dt',$$

system (8.27) is autonomous. It is determined by the vector field $\bar{X}$ on $T\mathbb{R}$ and therefore there exists a first integral. It can be obtained by the method of characteristics, which provides the characteristic curves where the first integrals for such a vector field $\bar{X}$ are constant. These characteristic curves are determined by

$$\frac{dx'}{c_{11}x' + c_{12}v'} = \frac{dv'}{c_{21}v' + c_x x' + c_{22}x'^n},$$

which can be written as

$$(c_{21}v' + c_x x' + c_{22}x'^n)\, dx' - (c_{11}x' + c_{12}v')\, dv' = 0. \tag{8.28}$$

This expression can be directly integrated if

$$\frac{\partial}{\partial v'}(c_{21}v' + c_x x' + c_{22}x'^n) = -\frac{\partial}{\partial x'}(c_{11}x' + c_{12}v'), \quad \text{so} \quad c_{21} = -c_{11}. \tag{8.29}$$

Under this condition we obtain a constant of motion for (8.28), namely

$$I = -c_{12}\frac{v'^2}{2} + c_x\frac{x'^2}{2} + c_{21}v'x' + c_{22}\frac{x'^{n+1}}{n+1}. \tag{8.30}$$

Finally, if we write the latter expression in terms of the initial variables x, v and t, we get a constant of motion for the initial differential equation.

If we do not wish to impose condition (8.29), we can alternatively integrate equation (8.28) by means of an integrating factor, i.e. we look for a function $\mu(x', v')$ such that

$$\frac{\partial}{\partial v'}(\mu(c_{21}v' + c_x x' + c_{22}x'^n)) = \frac{\partial}{\partial x'}(-\mu(c_{11}x' + c_{12}v')).$$

Thus the integrating factor satisfies the partial differential equation

$$\frac{\partial \mu}{\partial v'}(c_{21}v' + c_x x' + c_{22}x'^n) + \frac{\partial \mu}{\partial x'}(c_{11}x' + c_{12}v') = -\mu(c_{11} + c_{21}).$$

If $c_{11} + c_{21} = 0$, the integral factor can be chosen to be $\mu = 1$ and we get the first integral (8.30). On the other hand, if $c_{11} + c_{21} \neq 0$, we can still look for a solution to the partial differential equation for μ and obtain a new first integral.

8.5. t-dependent constants of motion and particular solutions for Emden equations. The main purpose of this section is to show that the knowledge of a particular solution of the Emden equation allows us to transform it into a Lie system and to derive a t-dependent constant of motion.

If we restrict ourselves to the case $\alpha(t) = 0$, system (8.26) reduces to

$$\left\{\begin{aligned} \frac{dx'}{dt} &= -\frac{\dot\gamma(t)}{\gamma(t)}x' + \frac{\beta(t)}{\gamma(t)}v', \\ \frac{dv'}{dt} &= \left(a(t) - \frac{\dot\beta(t)}{\beta(t)}\right)v' + \frac{b(t)\gamma^n(t)}{\beta(t)}x'^n. \end{aligned}\right. \tag{8.31}$$

In order to transform the original Emden–Fowler differential equation into a Lie system by means of our quasi-Lie scheme, we try to write the transformed differential equation in the form

$$\left\{\begin{aligned} \frac{dx'}{dt} &= f(t)(c_{11}x' + c_{12}v'), \\ \frac{dv'}{dt} &= f(t)(c_{22}x'^n + c_{21}v'), \end{aligned}\right. \tag{8.32}$$

where the c_{ij} are constants. This system can be reduced to an autonomous one, since under the t-dependent change of variables

$$\tau = \int^t f(t')\,dt',$$

it becomes

$$\left\{\begin{aligned} \frac{dx'}{d\tau} &= c_{11}x' + c_{12}v', \\ \frac{dv'}{d\tau} &= c_{22}x'^n + c_{21}v'. \end{aligned}\right. \tag{8.33}$$

In order for system (8.31) to be similar to (8.32), we look for α, β and γ satisfying

$$\left\{\begin{aligned} f(t)c_{11} &= -\frac{\dot\gamma(t)}{\gamma(t)}, &\qquad f(t)c_{12} &= \frac{\beta(t)}{\gamma(t)}, \\ f(t)c_{22} &= b(t)\frac{\gamma^n(t)}{\beta(t)}, &\qquad f(t)c_{21} &= a(t) - \frac{\dot\beta(t)}{\beta(t)}. \end{aligned}\right. \tag{8.34}$$

The conditions in the first line lead to

$$\beta(t) = -\frac{c_{12}}{c_{11}}\dot\gamma(t), \tag{8.35}$$

and using this equation in the last relation we obtain

$$f(t) = \frac{a(t)}{c_{21}} - \frac{1}{c_{21}}\frac{\ddot\gamma(t)}{\dot\gamma(t)}. \tag{8.36}$$

On the other hand from the three first relations in (8.34) we get

$$f(t) = -\frac{b(t)c_{11}}{c_{22}c_{12}}\frac{\gamma^n(t)}{\dot\gamma(t)}. \tag{8.37}$$

The equality of the right-hand sides of (8.36) and (8.37) leads to

$$\ddot\gamma = a(t)\dot\gamma + \frac{c_{11}c_{21}}{c_{22}c_{12}}b(t)\gamma^n.$$

Suppose that we make the choice, with $c_{21} = -c_{11}$ as indicated in (8.29),

$$c_{22} = -1, \qquad c_{11} = 1, \qquad c_{21} = -1, \qquad c_{12} = 1 \tag{8.38}$$

and thus $(c_{11}c_{22})/(c_{21}c_{12}) = 1$. Therefore we find that γ must be a solution of the initial equation (8.23). In other words, if we suppose that a particular solution $x_p(t)$ of the Emden equation is known, we can choose $\gamma(t) = x_p(t)$. Then, according to (8.35) and our choice (8.38), the corresponding function β turns out to be

$$\beta(t) = -\dot{x}_p(t).$$

Finally, in view of conditions (8.34), we get

$$\frac{-\dot{\gamma}(t)}{c_{11}\gamma(t)} = b(t)\frac{\gamma^n(t)}{c_{22}\beta(t)}$$

and taking into account (8.38) and $\gamma(t) = x_p(t)$, we obtain the condition satisfied by the particular solution:

$$x_p^{n+1}(t) = \dot{x}_p^2(t). \tag{8.39}$$

The system of differential equations (8.32) for such a choice (8.38) of the constants $\{c_{ij} \mid i, j = 1, 2\}$ is the equation for the integral curves of the t-dependent vector field

$$X_t = f(t)\left((x' + v')\frac{\partial}{\partial x'} - (v' + x'^n)\frac{\partial}{\partial v'}\right).$$

The method of characteristics can be used to find the following first integral for this vector field, in view of (8.30):

$$I(x', v') = \begin{cases} \dfrac{1}{n+1}x'^{n+1} + \dfrac{1}{2}v'^2 + x'v', & n \notin \{-1, 1\}, \\ \log x' + \dfrac{1}{2}v'^2 + x'v', & n = -1. \end{cases}$$

If we express this first integral in terms of the initial variables and t, we obtain a new t-dependent constant of motion for the initial Emden equation

$$I(t, x, v) = \begin{cases} \dfrac{x^{n+1}}{(n+1)x_p^{n+1}(t)} + \dfrac{v^2}{2\dot{x}_p^2(t)} - \dfrac{xv}{x_p(t)\dot{x}_p(t)}, & n \notin \{-1, 1\}, \\ \log\left(\dfrac{x}{x_p(t)}\right) + \dfrac{v^2}{2\dot{x}_p^2(t)} - \dfrac{xv}{x_p(t)\dot{x}_p(t)}, & n = -1. \end{cases} \tag{8.40}$$

So, the knowledge of a particular solution for the Emden equation enables us first to obtain a constant of motion and then to reduce the initial Emden equation to a Lie system. Thus, all Emden equations are quasi-Lie systems with respect to the above mentioned scheme.

8.6. Applications of particular solutions to study Emden equations. This section is devoted to illustrating the usefulness of the previous theory about Emden equations. More specifically, we detail several Emden equations for which one is able to find a particular solution satisfying an integrability condition, and we make use of such a solution to derive t-dependent constants of motion. In this way we recover several results appearing in the literature about Emden–Fowler equations from a unified point of view [42].

We start with a particular case of the Lane–Emden equation

$$\ddot{x} = -\frac{2}{t}\dot{x} - x^5. \tag{8.41}$$

The more general Lane–Emden equation is generally written as

$$\ddot{x} = -\frac{2}{t}\dot{x} + f(x)$$

and the example here considered corresponds to $f(x) = -x^n$, $n \neq 1$, which is one of the most interesting cases, together with that of $f(x) = -e^{-\beta x}$. Equation (8.41) appears in the study of the thermal behaviour of a spherical cloud of gas [135] and also in astrophysical applications. A particular solution for (8.41) satisfying (8.39) is $x_p(t) = (2t)^{-1/2}$. If we substitute this expression for $x_p(t)$ and the corresponding one for $\dot{x}_p(t)$ into the t-dependent constant of motion (8.40), we find that

$$I'(t,x,v) = \frac{4t^3x^6}{3} + 4t^3v^2 + 4t^2xv$$

is a t-dependent constant of motion proportional to (8.40) and also proportional to the t-dependent constants of motion found in [11, 34, 158].

We study from this new perspective other Emden equations investigated in [145]. Consider

$$\ddot{x} = -\frac{5}{t+K}\dot{x} - x^2.$$

A particular solution for this Emden equation satisfying (8.39) is

$$x_p(t) = \frac{4}{(t+K)^2}.$$

In this case a t-dependent constant of motion is

$$I'(t,x,v) = \frac{1}{3}x^3(t+K)^6 + \frac{1}{2}v^2(t+K)^6 + 2\,x\,v(t+K)^5,$$

which is proportional to the one found by Leach in [145].

Another Emden equation found in [145],

$$\ddot{x} = -\frac{3}{2(t+K)}\dot{x} - x^9,$$

admits the particular solution

$$x_p(t) = \frac{1}{\sqrt{2}(t+K)^{1/4}},$$

which satisfies (8.39). The corresponding t-dependent constant of motion is given by

$$I'(t,x,v) = (K+t)^{3/2}(10(K+t)v^2 + 5vx + 2(K+t)x^{10})$$

which is proportional to that given in [145].

Let us turn now to the Emden equation

$$\ddot{x} = -\frac{5}{3(t+K)}\dot{x} - x^7,$$

which admits the particular solution

$$x_p(t) = \frac{1}{3^{1/3}(t+K)^{1/3}},$$

which obeys (8.39) and leads to the t-dependent constant of motion

$$I'(t,x,v) = (K+t)^{5/3}(12(K+t)v^2 + 8vx + 3x^8(K+t)).$$

Finally we apply our development to obtain a t-dependent constant of motion for the Emden equation

$$\ddot{x} = -\frac{1}{K_1 + K_3 t}\dot{x} - x^n \tag{8.42}$$

with

$$K_3 = \frac{n-1}{n+3}.$$

We can find a particular solution of the form

$$x_p(t) = \frac{K_2}{(K_1 + K_3 t)^\nu}, \qquad \nu \neq 0.$$

In order for $x_p(t)$ to be a particular solution we must have the relation

$$\frac{(\nu+1)\nu K_2 K_3^2}{(K_1+K_3 t)^{\nu+2}} = \frac{\nu K_2 K_3}{(K_1+K_3 t)^{\nu+2}} - \frac{K_2^n}{(K_1+K_3 t)^{n\nu}}$$

and thus

$$\nu + 2 = n\nu \quad \text{and} \quad \nu(\nu+1)K_3^2 K_2 = \nu K_2 K_3 - K_2^n.$$

From these equations we get

$$\nu = \frac{2}{n-1}, \qquad K_2^{n-1} = \frac{2^2}{(n+3)^2}.$$

Under these conditions it can be easily verified that $\dot{x}_p^2(t) = x_p^{n+1}(t)$. Thus, a t-dependent constant of motion is

$$I'(t,x,v) = (K_1+K_3 t)^{2(n+1)/(n-1)}\left(\frac{x^{n+1}}{n+1} + \frac{v^2}{2}\right) + (K_1+K_3 t)^{(n+3)/(n-1)}\frac{2vx}{n+3}, \tag{8.43}$$

which can also be found in [145].

Another advantage of our method is that it allows us to obtain Emden equations admitting a preassigned t-dependent constant of motion.

Suppose that we want to construct an Emden equation admitting a given particular solution $x_p(t)$ satisfying $\dot{x}_p^2(t) = x_p^{n+1}(t)$ for certain $n \in \mathbb{Z} - \{1,-1\}$. We can integrate this equation to get all possible particular solutions which can be used by means of our method, i.e.

$$x_p(t) = \left(K + \frac{1-n}{2}t\right)^{-2/(n-1)}.$$

We consider functions $a(t)$ and $b(t)$ such that

$$\ddot{x}_p = a(t)\dot{x}_p + b(t)x_p^n.$$

For simplicity, we can assume that $b(t) = -1$. Then we get

$$a(t) = \frac{\ddot{x}_p + x_p^n}{\dot{x}_p}.$$

If we substitute the chosen particular solution in the above expression, we obtain

$$a(t) = \frac{3+n}{2(K + \frac{1-n}{2}t)},$$

which leads to an Emden equation equivalent to (8.42) and the t-dependent constant of motion for this equation is again (8.43). In this way we recover the cases studied in this section.

8.7. The Kummer–Liouville transformation for a general Emden–Fowler equation. As far as we know, the most general form of the Emden–Fowler equation considered nowadays is

$$\ddot{x} + p(t)\dot{x} + q(t)x = r(t)x^n. \tag{8.44}$$

This generalisation arises naturally as a consequence of our scheme. Indeed, the above second-order differential equation is associated with the system of first-order differential equations

$$\begin{cases} \dot{x} = v, \\ \dot{v} = -p(t)v - q(t)x + r(t)x^n, \end{cases} \tag{8.45}$$

which determines the integral curves for the t-dependent vector field

$$X_t = -p(t)X_4 - q(t)X_1 + r(t)X_2 + X_3.$$

This vector field is a generalisation of one studied in a previous section. Under the set of transformations (8.25), the initial system (8.45) becomes

$$\begin{cases} \dfrac{dx'}{dt} = \left(\dfrac{\alpha(t)}{\gamma(t)} - \dfrac{\dot{\gamma}(t)}{\gamma(t)}\right)x' + \dfrac{\beta(t)}{\gamma(t)}v', \\ \dfrac{dv'}{dt} = \left(-p(t) - \dfrac{\alpha(t)}{\gamma(t)} - \dfrac{\dot{\beta}(t)}{\beta(t)}\right)v' + \dfrac{\alpha(t)}{\beta(t)}\left(-p(t) - \dfrac{\alpha(t)}{\gamma(t)} - \dfrac{\dot{\alpha}(t)}{\alpha(t)} + \dfrac{\dot{\gamma}(t)}{\gamma(t)} - q(t)\dfrac{\gamma(t)}{\alpha(t)}\right)x' \\ \qquad + \dfrac{r(t)\gamma^n(t)}{\beta(t)}x'^n. \end{cases}$$

If we choose $\alpha = \dot{\gamma}$, the system reduces to

$$\begin{cases} \dfrac{dx'}{dt} = \dfrac{\beta(t)}{\gamma(t)}v', \\ \dfrac{dv'}{dt} = \left(-p(t) - \dfrac{\dot{\gamma}(t)}{\gamma(t)} - \dfrac{\dot{\beta}(t)}{\beta(t)}\right)v' + \dfrac{\dot{\gamma}(t)}{\beta(t)}\left(-p(t) - \dfrac{\ddot{\gamma}(t)}{\dot{\gamma}(t)} - q(t)\dfrac{\gamma(t)}{\dot{\gamma}(t)}\right)x' + \dfrac{r(t)\gamma^n(t)}{\beta(t)}x'^n. \end{cases}$$

When the function $\gamma(t)$ is chosen in such a way that $\ddot{\gamma} = -q(t)\gamma - p(t)\dot{\gamma}$, i.e. γ is a solution of the associated linear equation, we obtain

$$\begin{cases} \dfrac{dx'}{dt} = \dfrac{\beta(t)}{\gamma(t)}v', \\ \dfrac{dv'}{dt} = \left(-p(t) - \dfrac{\dot{\gamma}(t)}{\gamma(t)} - \dfrac{\dot{\beta}(t)}{\beta(t)}\right)v' + \dfrac{r(t)\gamma^n(t)}{\beta(t)}x'^n. \end{cases} \tag{8.46}$$

Finally, if the function $\beta(t)$ is such that

$$-p(t) - \frac{\dot{\gamma}(t)}{\gamma(t)} - \frac{\dot{\beta}(t)}{\beta(t)} = 0,$$

we obtain

$$\begin{cases} \dfrac{dx'}{dt} = \dfrac{\beta(t)}{\gamma(t)}v', \\ \dfrac{dv'}{dt} = \dfrac{r(t)\gamma^n(t)}{\beta(t)}x'^n, \end{cases} \tag{8.47}$$

which is related to the second-order differential equation

$$\frac{d^2x'}{d\tau^2} = r(t)\frac{\gamma^{n+1}(t)}{\beta^2(t)}x'^n,$$

with

$$\tau(t) = \int^t \frac{\beta(t')}{\gamma(t')}dt'.$$

The new form of the differential equation is called the canonical form of the generalised Emden–Fowler equation.

This fact is obtained by means of an appropriate Kummer–Liouville transformation in the literature, but we obtain it here as a straightforward application of the transformation properties of quasi-Lie schemes, thereby providing a theoretical explanation of such a Kummer–Liouville transformation.

8.8. Constants of motion for sets of Emden–Fowler equations. In this section we show that under certain assumptions on the t-dependent coefficients $a(t)$ and $b(t)$ the original Emden equation can be reduced to a Lie system and then we can obtain a first integral which provides us with a t-dependent constant of motion for the original system.

In fact consider the system of first-order differential equations

$$\begin{cases} \dfrac{dx'}{dt} = \left(\dfrac{\alpha(t)}{\gamma(t)} - \dfrac{\dot\gamma(t)}{\gamma(t)}\right)x' + \dfrac{\beta(t)}{\gamma(t)}v', \\ \dfrac{dv'}{dt} = \left(a(t) - \dfrac{\alpha(t)}{\gamma(t)} - \dfrac{\dot\beta(t)}{\beta(t)}\right)v' + \dfrac{\alpha(t)}{\beta(t)}\left(a(t) - \dfrac{\alpha(t)}{\gamma(t)} - \dfrac{\dot\alpha(t)}{\alpha(t)} + \dfrac{\dot\gamma(t)}{\gamma(t)}\right)x' + \dfrac{b(t)\gamma^n(t)}{\beta(t)}x'^n. \end{cases}$$

This system embraces all the systems of differential equations that can be obtained by means t-dependent transformations we get through the scheme $S(W_{\rm Emd}, V_{\rm Emd})$. We recall that the t-dependent change of variable which we use to relate the Emden equation (8.24) to the last system of differential equations is

$$\begin{cases} x = \gamma(t)x', \\ v = \beta(t)v' + \alpha(t)x'. \end{cases}$$

As in previous papers on this topic, we try to relate this system of differential equations to a Lie system determined by a t-dependent vector field of the form $X'(t,x) = f(t)\bar X(x)$ and we suppose that $f(t)$ does not vanish in the interval under study. So the system of differential equations determining the integral curves for this t-dependent vector field is a Lie system and we can use the theory of Lie systems to analyse its properties.

As a first example, we just use the set of transformations with $\gamma(t)=1$ and $\alpha(t)=0$. In this case system (8.25) is

$$\left\{\begin{aligned}\frac{dx'}{dt}&=\beta(t)v',\\ \frac{dv'}{dt}&=\left(a(t)-\frac{\dot\beta(t)}{\beta(t)}\right)v'+\frac{b(t)}{\beta(t)}x'^n.\end{aligned}\right.$$

We fix $\beta(t)$ such that

$$a(t)-\frac{\dot\beta(t)}{\beta(t)}=0,$$

i.e. $\beta(t)$ is (proportional to)

$$\beta(t)=\exp\left(\int^t a(t')\,dt'\right).$$

Therefore we get

$$\left\{\begin{aligned}\frac{dx'}{dt}&=\exp\left(\int^t a(t')\,dt'\right)v',\\ \frac{dv'}{dt}&=b(t)\exp\left(-\int^t a(t')\,dt'\right)x'^n.\end{aligned}\right.$$

For this system of differential equations to describe the integral curves for a t-dependent vector field $X'(t,x)=f(t)\bar X(x)$ for a given function $a(t)$, a necessary and sufficient condition is

$$b(t)\exp\left(-2\int^t a(t')\,dt'\right)=K,$$

with K being a real constant. Under this assumption the last system becomes

$$\left\{\begin{aligned}\frac{dx'}{dt}&=\exp\left(\int^t a(t')\,dt'\right)v',\\ \frac{dv'}{dt}&=\exp\left(\int^t a(t')\,dt'\right)Kx'^n.\end{aligned}\right.$$

We introduce the t-reparametrisation

$$\tau(t)=\int^t\exp\left(\int^{t'} a(t'')\,dt''\right)dt'$$

and the system becomes

$$\left\{\begin{aligned}\frac{dx'}{d\tau}&=v',\\ \frac{dv'}{d\tau}&=Kx'^n,\end{aligned}\right.$$

which admits a constant of motion

$$I=\frac{1}{2}v'^2-K\frac{x'^{n+1}}{n+1}.$$

In terms of the initial variables, the corresponding t-dependent constant of motion is

$$I=\exp\left(-2\int^t a(t')\,dt'\right)\left(\frac{1}{2}\dot y^2-b(t)\frac{x^{n+1}}{n+1}\right),$$

which is similar to that found in [16].

Suppose that we restrict the transformations (8.25) to the case $\alpha(t)=0$. In this case system (8.26) becomes

$$\begin{cases} \dfrac{dx'}{dt} = -\dfrac{\dot\gamma(t)}{\gamma(t)}x' + \dfrac{\beta(t)}{\gamma(t)}v', \\ \dfrac{dv'}{dt} = \left(a(t) - \dfrac{\dot\beta(t)}{\beta(t)}\right)v' + \dfrac{b(t)\gamma^n(t)}{\beta(t)}x'^n. \end{cases}$$

For this system to determine the integral curves of a t-dependent vector field of the form $X'(t,x) = f(t)\bar X(x)$ we need that

$$\begin{cases} c_{11}f(t) = -\dfrac{\dot\gamma(t)}{\gamma(t)}, & c_{12}f(t) = \dfrac{\beta(t)}{\gamma(t)}, \\ c_{21}f(t) = a(t) - \dfrac{\dot\beta(t)}{\beta(t)}, & c_{22}f(t) = \dfrac{b(t)\gamma^n(t)}{\beta(t)}. \end{cases} \tag{8.48}$$

From these relations, or more exactly from those of the first row, we get

$$f(t) = -\frac{1}{c_{11}}\frac{\dot\gamma(t)}{\gamma(t)} = \frac{1}{c_{12}}\frac{\beta(t)}{\gamma(t)}$$

and therefore

$$\dot\gamma(t) = -\frac{c_{11}}{c_{12}}\beta(t).$$

We choose $c_{11}=-1$ and $c_{12}=1$ so that

$$\beta(t) = \dot\gamma(t). \tag{8.49}$$

In view of this and using the third and second relations from (8.48) we get

$$\frac{c_{21}}{c_{12}}\frac{\beta(t)}{\gamma(t)} = a(t) - \frac{\dot\beta(t)}{\beta(t)}$$

and thus, as a consequence of (8.49), the last differential equation becomes

$$\frac{c_{21}}{c_{12}}\frac{\dot\gamma(t)}{\gamma(t)} = a(t) - \frac{\ddot\gamma(t)}{\dot\gamma(t)}$$

and, as $c_{12}=1$ and fixing $c_{21}=1$, we obtain

$$\frac{d}{dt}\log(\dot\gamma\gamma) = a(t),$$

which can be rewritten as

$$\frac{1}{2}\frac{d}{dt}\gamma^2(t) = \exp\left(\int^t a(t')\,dt'\right).$$

Hence we have

$$\gamma(t) = \sqrt{2\int^t \exp\left(\int^{t'} a(t'')\,dt''\right)dt'}$$

and in view of (8.49),

$$\beta(t) = \frac{1}{\sqrt{2\int^t \exp(\int^{t'} a(t'')\,dt'')\,dt'}}\exp\left(\int^t a(t')\,dt'\right).$$

So far we have only used three of the four relations we found. The fourth and second relations lead to an integrability condition: there exists a constant $c_{22} = K$ such that

$$K\frac{\beta(t)}{\gamma(t)} = \frac{b(t)\gamma^n(t)}{\beta(t)}.$$

Therefore, using the above expressions for $\gamma(t)$ and $\beta(t)$, we get

$$b(t)\exp\Big(-2\int^t a(t)\,dt'\Big)\Big(2\int^t \exp\Big(\int^{t'} a(t'')\,dt''\Big)\Big)^{(n+3)/2} = K. \tag{8.50}$$

So under this assumption we have connected the initial Emden equation with the Lie system

$$\begin{cases} \dfrac{dx'}{dt} = f(t)(-x'+v'), \\ \dfrac{dv'}{dt} = f(t)(v'+Kx'^n), \end{cases}$$

and then the method of characteristics shows that it admits the first integral

$$I' = -\frac{1}{2}v'^2 + \frac{K}{n+1}x'^{n+1} + v'x'.$$

In terms of the initial variables the corresponding constant of motion is

$$\begin{aligned} I = {} & \Big(\frac{1}{2}\dot{x}^2 - \frac{b(t)}{n+1}x^{n+1}\Big)\exp\Big(-2\int^t a(t')\,dt'\Big)\int^t \exp\Big(\int^{t'} a(t'')\,dt''\Big)\,dt' \\ & -\frac{1}{2}x\dot{x}\exp\Big(-\int^t a(t')\,dt'\Big) \end{aligned} \tag{8.51}$$

and in this way we recover the result found in [16]. If we now consider the particular case $n = -3$ we see that the integrability condition (8.50) implies that there is a constant K such that

$$b(t)\exp\Big(-2\int^t a(t)\,dt'\Big) = K,$$

and the corresponding t-dependent constant of motion is

$$\begin{aligned} I = {} & \Big(\frac{1}{2}\dot{x}^2 + \frac{b(t)}{2}x^{-2}\Big)\exp\Big(-2\int^t a(t')\,dt'\Big)\int^t \exp\Big(\int^{t'} a(t'')\,dt''\Big)\,dt' \\ & -\frac{1}{2}x\dot{x}\exp\Big(-\int^t a(t')\,dt'\Big), \end{aligned}$$

which is equivalent to the one found in [16].

8.9. A t-dependent superposition rule for Abel equations. Let us now illustrate the results of our theory of Lie families by deriving a common t-dependent superposition rule for a Lie family of Abel equations, whose elements do not admit a standard superposition rule except for a few particular instances. In this way, we show that our theory provides new tools for investigating solutions of nonautonomous systems of differential equations that cannot be investigated by means of the theory of Lie systems.

We analyse the so-called Abel equations of the first type [24, 74],

$$\frac{dx}{dt} = a_0(t) + a_1(t)x + a_2(t)x^2 + a_3(t)x^3, \tag{8.52}$$

with $a_3(t) \neq 0$. Abel equations appear in the analysis of several cosmological models [73, 111, 148] and other fields in physics [70, 84, 91, 92, 177, 240]. Additionally, the study of integrability conditions for Abel equations is of current interest in mathematics and the properties of their solutions have been thoughly investigated [5, 69, 74, 75, 215].

Note that, apart from its inherent mathematical interest, the knowledge of particular solutions of Abel equations allows us to study the properties of those physical systems that such equations describe. Thus, expressions enabling us to easily obtain new solutions of Abel equations from several particular ones, like common t-dependent superposition rules, are of interest.

Unfortunately, all the expressions describing the general solution of Abel equations presently known can only be applied to study autonomous instances and, moreover, they depend on families of particular conditions satisfying certain extra conditions (see [75, 215]). Taking this into account, common t-dependent superposition rules represent an improvement, as they enable us to treat nonautonomous Abel equations and they do not require the usage of particular solutions obeying additional conditions.

Recall that, according to Theorem 7.19, the existence of a common t-dependent superposition rule for a family of t-dependent vector fields $\{Y_d\}_{d\in\Lambda}$ requires the existence of a system of generators, i.e. a set of t-dependent vector fields $X_1,\ldots,X_r$ satisfying relations (7.14). Conversely, given such a set, the family of t-dependent vector fields Y whose autonomisations can be written in the form

$$\bar{Y}_c(t,x)=\sum_{j=1}^{r} b_{cj}(t)\bar{X}_j(t,x),\qquad \sum_{j=1}^{r} b_{cj}(t)=1,$$

admits a common t-dependent superposition rule and becomes a Lie family.

Consequently, a Lie family of Abel equations can be determined, for instance, by finding two t-dependent vector fields of the form

$$\begin{aligned} X_1(t,x)&=(b_0(t)+b_1(t)x+b_2(t)x^2+b_3(t)x^3)\frac{\partial}{\partial x},\\ X_2(t,x)&=(b'_0(t)+b'_1(t)x+b'_2(t)x^2+b'_3(t)x^3)\frac{\partial}{\partial x},\qquad b'_3(t)\neq 0, \end{aligned}\tag{8.53}$$

such that

$$[\bar{X}_1,\bar{X}_2]=2(\bar{X}_2-\bar{X}_1).\tag{8.54}$$

Let us analyse the existence of such X_1 and X_2. In coordinates, the Lie bracket $[\bar{X}_1,\bar{X}_2]$ reads

$$\begin{aligned} [(b'_3b_2-b'_2b_3)x^4&+(2(b'_3b_1-b_3b'_1)-\dot{b}_3+\dot{b}'_3)x^3+(-3(b'_0b_3-b_0b'_3)+(b'_2b_1-b_2b'_1)\\ &-\dot{b}_2+\dot{b}'_2)x^2+(-2b'_0b_2+2b_0b'_2-\dot{b}_1+\dot{b}'_1)x-b'_0b_1+b_0b'_1-\dot{b}_0+\dot{b}'_0]\frac{\partial}{\partial x}. \end{aligned}$$

Hence, to have condition (8.54), we must have $b'_3b_2-b'_2b_3=0$, e.g. we may fix $b_2=b_3=0$. Additionally, for simplicity, we assume $b'_3=1$. In this case, the previous expression takes the form

$$[2b_1x^3+(3b_0+b'_2b_1+\dot{b}'_2)x^2+(2b_0b'_2-\dot{b}_1+\dot{b}'_1)x-b'_0b_1+b_0b'_1-\dot{b}_0+\dot{b}'_0]\frac{\partial}{\partial x},$$

and, taking into account the values chosen for b_2, b_3 and b_3', assumption (8.54) yields $b_1 = 1$ and

$$\begin{cases} b_2' = 3b_0 + \dot{b}_2', \\ 2(b_1' - 1) = 2b_0 b_2' + \dot{b}_1', \\ 2(b_0' - b_0) = -b_0' + b_0 b_1' - \dot{b}_0 + \dot{b}_0'. \end{cases}$$

As this system has more variables than equations, we can try to fix some values of the variables in order to simplify it and obtain a particular solution. For $b_0(t) = t$, the above system reads

$$\begin{cases} \dot{b}_2' = b_2' - 3t, \\ \dot{b}_1' = 2(b_1' - 1) - 2tb_2', \\ \dot{b}_0' = 2(b_0' - t) + b_0' - tb_1' + 1. \end{cases}$$

This system is integrable by quadratures and one can check that it admits the particular solution

$$b_2'(t) = 3(1+t), \qquad b_1'(t) = 3(1+t)^2 + 1, \qquad b_0'(t) = (1+t)^3 + t.$$

Summing up, we have proved that the t-dependent vector fields

$$\begin{cases} X_1(t,x) = (t+x)\dfrac{\partial}{\partial x}, \\ X_2(t,x) = ((1+t)^3 + t + (3(1+t)^2+1)x + 3(1+t)x^2 + x^3)\dfrac{\partial}{\partial x}, \end{cases} \tag{8.55}$$

satisfy (8.54), and therefore the family of t-dependent vector fields

$$Y_{b(t)}(t,x) = (1 - b(t))X_1(x) + b(t)X_2(x)$$

is a Lie family. The corresponding family of Abel equations is

$$\frac{dx}{dt} = (t+x) + b(t)(1+t+x)^3. \tag{8.56}$$

According to the results of Section 1.5, to determine a common t-dependent superposition rule for the above Lie family, we have to determine a first integral for the vector fields of the distribution $\mathcal{D}$ spanned by the t-prolongations $\widetilde{X}_1$ and $\widetilde{X}_2$ on $\mathbb{R}\times\mathbb{R}^{n(m+1)}$ for a certain m so that the t-prolongations of X_1 and X_2 to $\mathbb{R}\times\mathbb{R}^{nm}$ are linearly independent at a generic point. Taking into account (8.55), the prolongations of X_1 and X_2 to $\mathbb{R}\times\mathbb{R}^2$ are linearly independent at a generic point and, in view of (8.54), the t-prolongations $\widetilde{X}_1$ and $\widetilde{X}_2$ to $\mathbb{R}\times\mathbb{R}^3$ span an involutive generalised distribution $\mathcal{D}$ with two-dimensional leaves in a dense subset of $\mathbb{R}\times\mathbb{R}^3$. Finally, a first integral for the vector fields in the distribution $\mathcal{D}$ will provide us a common t-dependent superposition rule for the Lie family (8.56).

Since, in view of (8.54), the vector fields $\widetilde{X}_1$ and $\widetilde{X}_2$ span the distribution $\mathcal{D}$, a function $G : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ is a first integral of the vector fields of the distribution $\mathcal{D}$ if and only if G is a first integral of $\widetilde{X}_1$ and $\widetilde{X}_1 - \widetilde{X}_2$, i.e. $\widetilde{X}_1 G = (\widetilde{X}_2 - \widetilde{X}_1)G = 0$.

The condition $\widetilde{X}_1 G = 0$ reads

$$\frac{\partial G}{\partial t} + (t+x_0)\frac{\partial G}{\partial x_0} + (t+x_1)\frac{\partial G}{\partial x_1} = 0,$$

and, using the method of characteristics [129], we find that the characteristics are solutions of the system

$$dt = \frac{dx_0}{t+x_0} = \frac{dx_1}{t+x_1}, \quad \text{so} \quad \frac{dx_i}{dt} = t + x_i, \qquad i = 0, 1,$$

which read $x_i(t) = \xi_i e^t - t - 1$, with $i = 0, 1$ and $\xi_0, \xi_1 \in \mathbb{R}$. Furthermore, these solutions are determined by the implicit equations $\xi_0 = e^{-t}(x_0 + t + 1)$ and $\xi_1 = e^{-t}(x_1 + t + 1)$. Therefore, there exists a function $G_2 : \mathbb{R}^2 \to \mathbb{R}$ such that $G(t, x_0, x_1) = G_2(\xi_0, \xi_1)$. In other words, each first integral G of $\widetilde{X}_1$ depends only on ξ_0 and ξ_1.

Now, we look for simultaneous first integrals of the vector fields $\widetilde{X}_2 - \widetilde{X}_1$ and $\widetilde{X}_1$, that is, for solutions of the equation $(\widetilde{X}_2 - \widetilde{X}_1)G = 0$ with G depending on ξ_0 and ξ_1. Using the expression of $\widetilde{X}_2 - \widetilde{X}_1$ in the coordinates $\{t, \xi_0, \xi_1\}$, we get

$$(\widetilde{X}_2 - \widetilde{X}_1)G = \xi_0^3 \frac{\partial G_2}{\partial \xi_0} + \xi_1^3 \frac{\partial G_2}{\partial \xi_1} = 0,$$

and, applying again the method of characteristics, we find that there exists a function $G_3 : \mathbb{R} \to \mathbb{R}$ such that $G(t, x_0, x_1) = G_2(\xi_0, \xi_1) = G_3(\Delta)$, where $\Delta = e^{2t}((x_0 + t + 1)^{-2} - (x_1 + t + 1)^{-2})$. Finally, using this first integral, we see that the common t-dependent superposition rule for the Lie family (8.56) reads

$$k = e^{2t}((x_0 + t + 1)^{-2} - (x_1 + t + 1)^{-2}),$$

with k being a real constant. Therefore, given any particular solution $x_1(t)$ of a particular instance of the family of first-order Abel equations (8.58), the general solution $x(t)$ of this instance is

$$x(t) = ((x_1(t) + t + 1)^{-2} + ke^{-2t})^{-1/2} - t - 1.$$

Note that our procedure can be directly generalised to derive common t-dependent superposition rules for generalised Abel equations [166] of the form

$$\frac{dx}{dt} = a_0(t) + a_1(t)x + a_2(t)x^2 + \cdots + a_n(t)x^n, \qquad n \geq 3.$$

Actually, their study can be approached by analysing the existence of vector fields

$$Y_1(t, x) = (b_0(t) + b_1(t)x + \cdots + b_n(t)x^n)\frac{\partial}{\partial x},$$
$$Y_2(t, x) = (b_0'(t) + b_1'(t)x + \cdots + b_n'(t)x^n)\frac{\partial}{\partial x}, \qquad b_n'(t) \neq 0,$$

obeying $[\bar{Y}_1, \bar{Y}_2] = 2(\bar{Y}_2 - \bar{Y}_1)$, and by following a procedure similar to the one above.

8.10. Lie families and second-order differential equations. Common t-dependent superposition rules describe solutions of nonautonomous systems of first-order differential equations. Nevertheless, we shall now illustrate how this new kind of superposition rule can also be applied to analyse families of second-order differential equations. More specifically, we shall derive a common t-dependent superposition rule in order to express the general solution of any instance of a family of Milne–Pinney equations [30, 75, 196, 195] in terms of each generic pair of particular solutions, two constants, and the variable t, i.e. time. In this way, we provide a generalization to the setting of dissipative Milne–Pinney equations of the expression derived in [44].

Consider the family of dissipative Milne–Pinney equations [89, 196, 195, 217] of the form

$$\ddot{x} = -\dot{F}\dot{x} + \omega^2 x + e^{-2F}x^{-3}, \tag{8.57}$$

with a fixed t-dependent function $F = F(t)$, and parametrised by an arbitrary t-dependent function $\omega = \omega(t)$. The physical motivation for the study of dissipative Milne–Pinney equations comes from its appearance in dissipative quantum mechanics [3, 113, 171, 213], where, for instance, their solutions are used to obtain Gaussian solutions of nonconservative t-dependent quantum oscillators [171]. Moreover, the mathematical properties of the solutions of dissipative Milne–Pinney equations have been studied from different points of view [34, 44, 45, 83, 110, 196, 195, 230]. The works [45, 196] outline the state-of-the-art of the investigation of dissipative and nondissipative Milne–Pinney equations. One of the main achievements in this topic (see [196, Corollary 5]) is an expression describing the general solution of a particular class of these equations in terms of a pair of generic particular solutions of a second-order linear differential equation and two constants. Recently, the theory of quasi-Lie schemes and the theory of Lie systems have enabled us to recover this last result and others from a geometric point of view [34, 52].

Note that introducing a new variable $v \equiv \dot{x}$, we transform the family (8.57) of second-order differential equations into a family of first-order ones,

$$\begin{cases} \dot{x} = v, \\ \dot{v} = -\dot{F}v + \omega^2 x + e^{-2F}x^{-3}, \end{cases} \tag{8.58}$$

whose dynamics is described by the family of t-dependent vector fields on $\mathrm{T}\mathbb{R}$ parametrised by ω of the form

$$Y_\omega = (-\dot{F}v + e^{-2F}x^{-3} + \omega^2 x)\frac{\partial}{\partial v} + v\frac{\partial}{\partial x}, \qquad \omega \in \Lambda = C^\infty(t).$$

Let us show that it is a Lie family whose common superposition rule can be used to analyse the solutions of (8.57).

In view of Theorem 7.19, if the family of systems related to the above family of t-dependent vector fields is a Lie family, that is, it admits a common t-dependent superposition rule in terms of m particular solutions, then the family of vector fields on $\mathbb{R} \times \mathbb{R}^{n(m+1)}$ given by $\mathrm{Lie}(\{Y_\omega\}_{\omega\in\Lambda})$ spans an involutive generalised distribution with leaves of rank $r \leq n \cdot m + 1$.

Note that the distribution spanned by all $\widetilde{Y}_\omega$ is generated by the vector fields $\widetilde{Y}_1$ and $\widetilde{Y}_2$ with

$$Y_1 = (-\dot{F}v + e^{-2F}x^{-3} + x)\frac{\partial}{\partial v} + v\frac{\partial}{\partial x}, \qquad Y_2 = (-\dot{F}v + e^{-2F}x^{-3})\frac{\partial}{\partial v} + v\frac{\partial}{\partial x},$$

since $\widetilde{Y}_\omega = (1-\omega^2)\widetilde{Y}_2 + \omega^2\widetilde{Y}_1$. The prolongation $[\widetilde{Y}_1, \widetilde{Y}_2]$ is not spanned by $\widetilde{Y}_1$ and $\widetilde{Y}_2$, and so we have to include the prolongation $Y_3^\wedge = [\widetilde{Y}_1, \widetilde{Y}_2]$, where

$$Y_3 = x\frac{\partial}{\partial x} - (v + x\dot{F})\frac{\partial}{\partial v}.$$

In the case $m = 0$, the distribution spanned by the vector fields $\widetilde{Y}_1, \widetilde{Y}_2, Y_3^\wedge$ does not admit a nontrivial first integral. In the case $m > 0$, the vector fields $\widetilde{Y}_1, \widetilde{Y}_2, Y_3^\wedge$ do not span the linear space $\mathrm{Lie}(\{\widetilde{Y}_\omega\}_{\omega\in\Lambda})$ and we need to add a new prolongation $Y_4^\wedge = [\widetilde{Y}_1, [\widetilde{Y}_1, \widetilde{Y}_2]]$ to the previous set, with

$$Y_4 = (2v + x\dot{F})\frac{\partial}{\partial x} + (2e^{-2F}x^{-3} - 2x - \dot{F}(v + x\dot{F}) - x\ddot{F})\frac{\partial}{\partial v}.$$

The vector fields $\widetilde{Y}_1, \widetilde{Y}_2, Y_3^\wedge, Y_4^\wedge$ satisfy the commutation relations

$$\begin{aligned}
[\widetilde{Y}_1, \widetilde{Y}_2] &= Y_3^\wedge,\\
[\widetilde{Y}_1, Y_3^\wedge] &= Y_4^\wedge,\\
[\widetilde{Y}_1, Y_4^\wedge] &= (4 + \dot{F}^2 + 2\ddot{F})Y_3^\wedge - (\dot{F}\ddot{F} + \dddot{F})(\widetilde{Y}_1 - \widetilde{Y}_2),\\
[\widetilde{Y}_2, Y_3^\wedge] &= 2(\widetilde{Y}_1 - \widetilde{Y}_2) + Y_4^\wedge,\\
[\widetilde{Y}_2, Y_4^\wedge] &= (2 + \dot{F}^2 + 2\ddot{F})Y_3^\wedge - (\dot{F}\ddot{F} + \dddot{F})(\widetilde{Y}_1 - \widetilde{Y}_2),\\
[Y_3^\wedge, Y_4^\wedge] &= -2Y_4^\wedge - 2(\widetilde{Y}_1 - \widetilde{Y}_2)(4 + \dot{F}^2 + 2\ddot{F}).
\end{aligned}$$

Consequently, the vector fields $\widetilde{Y}_1, \widetilde{Y}_2, Y_3^\wedge, Y_4^\wedge$ span the linear space $\mathrm{Lie}(\{\widetilde{Y}_\omega\}_{\omega\in\Lambda})$. Adding $\widetilde{Y}_1$ to each prolongation of the previous set, that is, considering the vector fields $\widetilde{X}_1 = \widetilde{Y}_1$, $\widetilde{X}_2 = \widetilde{Y}_2$, $\widetilde{X}_3 = \widetilde{Y}_1 + Y_3^\wedge$, and $\widetilde{X}_4 = \widetilde{Y}_1 + Y_4^\wedge$, we get a family of t-prolongations $\widetilde{X}_1, \widetilde{X}_2, \widetilde{X}_3, \widetilde{X}_4$ which spans the vector fields of the family $\mathrm{Lie}(\{\widetilde{Y}_\omega\}_{\omega\in\Lambda})$. The commutation relations among them are

$$\begin{aligned}
[\widetilde{X}_1, \widetilde{X}_2] &= \widetilde{X}_3 - \widetilde{X}_1,\\
[\widetilde{X}_1, \widetilde{X}_3] &= \widetilde{X}_4 - \widetilde{X}_1,\\
[\widetilde{X}_1, \widetilde{X}_4] &= -(\dot{F}\ddot{F} + \dddot{F} + 4 + \dot{F}^2 + 2\ddot{F})\widetilde{X}_1 + (\dot{F}\ddot{F} + \dddot{F})\widetilde{X}_2 + (4 + \dot{F}^2 + 2\ddot{F})\widetilde{X}_3,\\
[\widetilde{X}_2, \widetilde{X}_3] &= 2\widetilde{X}_1 - 2\widetilde{X}_2 - \widetilde{X}_3 + \widetilde{X}_4,\\
[\widetilde{X}_2, \widetilde{X}_4] &= -(1 + \dot{F}^2 + 2\ddot{F} + \dot{F}\ddot{F} + \dddot{F})\widetilde{X}_1 + (\dot{F}\ddot{F} + \dddot{F})\widetilde{X}_2 + (1 + \dot{F}^2 + 2\ddot{F})\widetilde{X}_3,\\
[\widetilde{X}_3, \widetilde{X}_4] &= -3\widetilde{X}_4 + (4 + \dot{F}^2 + 2\ddot{F})\widetilde{X}_3 + (8 + \dddot{F} + \dot{F}\ddot{F} + 2\dot{F}^2 + 4\ddot{F})\widetilde{X}_2\\
&\quad + (-9 - 3\dot{F}^2 - 6\ddot{F} - \dot{F}\ddot{F} - \dddot{F})\widetilde{X}_1.
\end{aligned}$$

As a consequence of Lemma 7.17, the vector fields $\bar{X}_1$, $\bar{X}_2$ $\bar{X}_3$ and $\bar{X}_4$ satisfy the same commutation relations as $\widetilde{X}_1$, $\widetilde{X}_2$, $\widetilde{X}_3$, $\widetilde{X}_4$. Hence, in view of Theorem 7.19, the family (8.58) is a Lie family and the knowledge of nontrivial first integrals of the vector fields of the distribution $\mathcal{D}$ spanned by $\widetilde{X}_1$, $\widetilde{X}_2$, $\widetilde{X}_3$, $\widetilde{X}_4$ provides us with a common t-dependent superposition rule.

Let us now determine this superposition rule. As the vector fields $\widetilde{X}_1$, $\widetilde{X}_1 - \widetilde{X}_2$ and their successive Lie brackets span the whole distribution $\mathcal{D}$, a function $G : \mathbb{R} \times \mathrm{T}\mathbb{R}^3 \to \mathbb{R}$ is a first integral for the vector fields of $\mathcal{D}$ if and only if it is a first integral for $\widetilde{X}_1$ and $\widetilde{X}_2 - \widetilde{X}_1$. Therefore, we can reduce the problem to finding common first integrals G for $\widetilde{X}_1$ and $\widetilde{X}_1 - \widetilde{X}_2$.

Let us analyse the implications of G being a first integral of the vector field

$$\widetilde{X}_1 - \widetilde{X}_2 = \sum_{i=0}^{2} x_i \frac{\partial}{\partial v_i}.$$

The characteristics of the above vector field are solutions of the system

$$\frac{dv_0}{x_0} = \frac{dv_1}{x_1} = \frac{dv_2}{x_2}, \qquad dx_0 = 0, \qquad dx_1 = 0, \qquad dx_2 = 0, \qquad dt = 0,$$

that is, the solutions are curves in $\mathbb{R}\times\mathrm{T}\mathbb{R}^3$ of the form $s\mapsto(t, x_0, x_1, x_2, v_0(s), v_1(s), v_2(s))$, with $\xi_{02} = x_0v_2(s) - x_2v_0(s)$ and $\xi_{12} = x_1v_2(s) - x_2v_1(s)$ for two real constants ξ_{02} and ξ_{12}. Thus, there exists a function $G_2 : \mathbb{R}^6 \to \mathbb{R}$ such that $G(p) = G_2(t, x_0, x_1, x_2, \xi_{02}, \xi_{12})$, with $p \in \mathbb{R} \times \mathrm{T}\mathbb{R}^3$, $\xi_{02} = x_0v_2 - x_2v_0$, and $\xi_{12} = x_1v_2 - v_1x_2$. In other words, G is a function of $t, x_0, x_1, x_2, \xi_{02}, \xi_{12}$.

The function G also satisfies the condition $\widetilde{X}_1G = 0$, which, in terms of the coordinate system $\{t, x_0, x_1, x_2, \xi_{02}\xi_{12}, v_2\}$, reads

$$\begin{aligned}\widetilde{X}_1G = {}& \frac{\partial G}{\partial t} + \frac{x_0v_2 - \xi_{02}}{x_2}\frac{\partial G}{\partial x_0} + \frac{x_1v_2 - \xi_{12}}{x_2}\frac{\partial G}{\partial x_1} + v_2\frac{\partial G}{\partial x_2} \\ & - \left[\dot{F}\xi_{12} + e^{-2F}\left(\frac{x_2}{x_1^3} - \frac{x_1}{x_2^3}\right)\right]\frac{\partial G}{\partial \xi_{12}} - \left[\dot{F}\xi_{02} + e^{-2F}\left(\frac{x_2}{x_0^3} - \frac{x_0}{x_2^3}\right)\right]\frac{\partial G}{\partial \xi_{02}} = 0.\end{aligned}$$

That is, if we define the vector fields

$$\begin{aligned}\Xi_1 = {}& \frac{\partial}{\partial t} - \frac{\xi_{12}}{x_2}\frac{\partial}{\partial x_1} - \frac{\xi_{02}}{x_2}\frac{\partial}{\partial x_0} + \left[-\dot{F}\xi_{12} - e^{-2F}\left(\frac{x_2}{x_1^3} - \frac{x_1}{x_2^3}\right)\right]\frac{\partial}{\partial \xi_{12}} \\ & + \left[-\dot{F}\xi_{02} - e^{-2F}\left(\frac{x_2}{x_0^3} - \frac{x_0}{x_2^3}\right)\right]\frac{\partial}{\partial \xi_{02}}, \\ \Xi_2 = {}& \frac{x_0}{x_2}\frac{\partial}{\partial x_0} + \frac{x_1}{x_2}\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2},\end{aligned}$$

the condition $\widetilde{X}_1G = 0$ implies that $\Xi_1G_2 + v_2\Xi_2G_2 = 0$, and as G_2 does not depend on v_2, the function G must simultaneously be a first integral for Ξ_1 and Ξ_2, i.e. $\Xi_1G = 0$ and $\Xi_2G = 0$.

Applying the method of characteristics to the vector field Ξ_2, we see that F can just depend on the variables $t, \xi_{02}, \xi_{12}, \Delta_{02} = x_0/x_2$ and $\Delta_{12} = x_1/x_2$. In other words, there exists a function $G_3 : \mathbb{R}^5 \to \mathbb{R}$ such that $G(t, x_0, x_1, x_2, v_0, v_1, v_2) = G_2(t, x_0, x_1, x_2, \xi_{02}, \xi_{12})$ $= G_3(t, \xi_{02}, \xi_{12}, \Delta_{02}, \Delta_{12})$.

It remains to check the implications of the equation $\Xi_1G = 0$. With the aid of the coordinate system $\{t, \xi_{02}, \xi_{12}, \Delta_{02}, \Delta_{12}, v_2, x_2\}$, this equation can be recast in the form $\Xi_1G = \frac{1}{x_2^2}\Upsilon_1G_3 + \Upsilon_2G_3 = 0$, where

$$\begin{aligned}\Upsilon_1 &= \sum_{i=0}^{1}\left(-\xi_{i2}\frac{\partial}{\partial \Delta_{i2}} - e^{-2F}(\Delta_{i2}^{-3} - \Delta_{i2})\frac{\partial}{\partial \xi_{i2}}\right), \\ \Upsilon_2 &= -\dot{F}\xi_{12}\frac{\partial}{\partial \xi_{12}} - \dot{F}\xi_{02}\frac{\partial}{\partial \xi_{02}} + \frac{\partial}{\partial t}.\end{aligned}$$

As G_3 only depends on $t, \Delta_{02}, \Delta_{12}, \xi_{12}, \xi_{02}$, we have $\Upsilon_1G = 0$ and $\Upsilon_2G = 0$. Repeating *mutatis mutandis* the previous procedure in order to determine the implications of being a first integral of Υ_1 and Υ_2, we finally deduce that the first integrals of the distribution

$\mathcal{D}$ are functions of I_1, I_2 and I, with

$$I_i = e^{2F}(x_0 v_i - x_i v_0)^2 + \left[\left(\frac{x_0}{x_i}\right)^2 + \left(\frac{x_i}{x_0}\right)^2\right], \qquad i = 1, 2,$$
$$I = e^{2F}(x_1 v_2 - x_2 v_1)^2 + \left[\left(\frac{x_1}{x_2}\right)^2 + \left(\frac{x_2}{x_1}\right)^2\right].$$

Defining $\bar{v}_2 = e^F v_2, \bar{v}_1 = e^F v_1$ and $\bar{v}_0 = e^F v_0$, the above first integrals read

$$I_i = (x_0 \bar{v}_i - x_i \bar{v}_0)^2 + \left[\left(\frac{x_0}{x_i}\right)^2 + \left(\frac{x_i}{x_0}\right)^2\right], \qquad i = 1, 2,$$
$$I = (x_1 \bar{v}_2 - x_2 \bar{v}_1)^2 + \left[\left(\frac{x_1}{x_2}\right)^2 + \left(\frac{x_2}{x_1}\right)^2\right].$$

Note that these first integrals have the same form as the ones considered in [52] for $k = 1$. Therefore, we can apply the procedure there to obtain

$$x_0 = \sqrt{k_1 x_1^2 + k_2 x_2^2 + 2\sqrt{\lambda_{12}[-(x_1^4 + x_2^4) + I\, x_1^2 x_2^2]}}, \tag{8.59}$$

with λ_{12} being a function of the form

$$\lambda_{12}(k_1, k_2, I) = \frac{k_1 k_2 I + (-1 + k_1^2 + k_2^2)}{I^2 - 4},$$

and where the constants k_1 and k_2 satisfy special conditions to ensure that x_0 is real [44].

Expression (8.59) permits us to determine the general solution $x(t)$ of any instance of family (8.57) in the form

$$x(t) = \sqrt{k_1 x_1^2(t) + k_2 x_2^2(t) + 2\sqrt{\lambda_{12}[-(x_1^4(t) + x_2^4(t)) + I x_1^2(t) x_2^2(t)]}}, \tag{8.60}$$

with

$$I = e^{2F(t)}(x_1(t)\dot{x}_2(t) - x_2(t)\dot{x}_1(t))^2 + \left[\left(\frac{x_1(t)}{x_2(t)}\right)^2 + \left(\frac{x_2(t)}{x_1(t)}\right)^2\right],$$

in terms of two of its particular solutions $x_1(t)$, $x_2(t)$, their derivatives, the constants k_1 and k_2, and the variable t (included in the constant of motion I).

Note that the rôle of the constant I in (8.60) differs from the rôles played by k_1 and k_2. Indeed, the value of I is fixed by the particular solutions $x_1(t)$, $x_2(t)$ and their derivatives, while, for every pair of generic solutions $x_1(t)$ and $x_2(t)$, the values of k_1 and k_2 range within certain intervals ensuring that $x(t)$ is real.

It is clear that the method illustrated here can also be applied to analyse solutions of any other family of second-order differential equations related to a Lie family by introducing the new variable $v = \dot{x}$. Additionally, it is worth noting that in the case $F(t) = 0$ the family of dissipative Milne–Pinney equations (8.57) reduces to a family of Milne–Pinney equations often appearing in the literature (see [147] and references therein), and the expression (8.60) takes the form of the expression obtained in [44] for these equations.

9. Conclusions and outlook

Apart from providing a quite self-contained introduction to the theory of Lie systems, this essay describes most of the results concerning this theory and its generalisations developed by the authors and other collaborators in recent years. In this way, our work presents the state-of-the-art of the subject and establishes the foundations for our present research activity. Let us here discuss some of the topics which we aim to analyse in the near future and their relations to the contents of this essay.

The theory of superposition rules for second- and higher-order differential equations has just been initiated [48, 49, 52, 77, 202, 225] and many questions have to be clarified. As an example, we point out that there exist several approaches to study systems of second-order differential equations by means of the theory of Lie systems. For instance, one can use SODE Lie systems [52], which allows one to study a particular type of systems of second-order differential equations. In addition, if an equation admits a regular Lagrangian, the corresponding Hamiltonian formulation can lead to a system of first-order differential equations which can also be a Lie system [54]. Analysing the relations between the results obtained through both approaches is still an open problem.

As a consequence, it has become of interest to study a class of Lie systems describing the Hamilton equations of certain t-dependent Hamiltonians on symplectic manifolds. This structure provides us with new tools, which can be employed to study the integrability and super-integrability of these Lie systems. We hope to analyse such relations in depth in the future.

After analysing Lie systems defined on symplectic manifolds, a natural question arises: What are the properties of Lie systems describing the solutions of a system on a Poisson manifold $(N, \{\cdot,\cdot\})$ of the form

$$\frac{dx}{dt} = \{x, h_t\}, \qquad x \in N,$$

where, for every $t \in \mathbb{R}$, the function $h_t : N \to \mathbb{R}$ belongs to a finite-dimensional Lie algebra of functions (with respect to the Poisson bracket). This challenging question leads to the analysis of the properties of such Lie systems by means of the Poisson structure of the manifold.

In [12, 13] Winternitz et al. proposed a new type of superposition rule, referred to as *super-superposition rules*, that describe the general solution of a particular family of systems of first-order differential equations on supermanifolds. These articles gave rise to many interesting questions. Although it seems that the geometric theory developed in [38] could easily be generalised to describe the properties of super-superposition rules, multiple nontrivial technical problems arise. We hope to solve such problems in the future and to develop a geometric theory of Lie systems on graded manifolds.

In [38, Remark 5], it was proposed to study Bäcklund transformations through a slight modification of the methods carried out to analyse superposition rules geometrically, i.e., by means of a certain type of flat connection. This topic deserves a further analysis.

Since their first appearance in [34], quasi-Lie schemes have been employed to investigate multiple systems of differential equations: nonlinear oscillators [34], Mathews–

Lakshmanan oscillators [34], Emden equations [42], Abel equations [56], dissipative Milne–Pinney equations [45], etc. There are still many other applications to be performed, e.g. we expect to apply this theory to study Abel equations in depth. In addition, it would be interesting to continue the analysis of the theory of quasi-Lie schemes and, for instance, to develop new generalisations of this theory. Indeed, we are already investigating a generalisation for the analysis of certain quantum systems, e.g. the quantum Calogero–Moser system. In addition, it would be interesting to study generalisations of this theory to analyse stochastic Lie–Scheffers systems [144] or control Lie systems [79].

As we pointed out at the beginning of this essay, being a Lie system is an exception rather than a rule. In addition, just a few, but relevant, Lie systems are known to have applications in physics, mathematics and other branches of science. Consequently, one of our main purposes remains to find new instances of Lie systems with remarkable applications.

To finish, we hope to have succeeded in showing that the theory of Lie systems, after more than a century of existence, is still an active and interesting field of research.

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