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Optimal estimates for the fractional Hardy operator

by

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Abstract. Let $A_\alpha f(x) = |B(0,|x|)|^{-\alpha/n} \int_{B(0,|x|)} f(t)\,dt$ be the n-dimensional fractional Hardy operator, where $0 < \alpha \le n$. It is well-known that A_α is bounded from L^p to L^{p_α} with $p_\alpha = np/(\alpha p - np + n)$ when $n(1-1/p) < \alpha \le n$. We improve this result within the framework of Banach function spaces, for instance, weighted Lebesgue spaces and Lorentz spaces. We in fact find a 'source' space $S_{\alpha,Y}$, which is strictly larger than X, and a 'target' space T_Y, which is strictly smaller than Y, under the assumption that A_α is bounded from X into Y and the Hardy–Littlewood maximal operator M is bounded from Y into Y, and prove that A_α is bounded from $S_{\alpha,Y}$ into T_Y. We prove optimality results for the action of A_α and the associate operator A'_α on such spaces, as an extension of the results of Mizuta et al. (2013) and Nekvinda and Pick (2011). We also study the duals of optimal spaces for A_α.

1. Introduction. Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space and Ω be an open subset of $\mathbb{R}^n$. For an integrable function u on a measurable set $E \subset \mathbb{R}^n$ of positive measure, we define the integral mean over E by

$$\fint_E u(x)\,dx = \frac{1}{|E|}\int_E u(x)\,dx,$$

where $|E|$ denotes the Lebesgue measure of E. We denote by $B(x,r)$ the open ball with center x and of radius $r > 0$, and by $|B(x,r)|$ its Lebesgue measure, i.e. $|B(x,r)| = \sigma_n r^n$, where σ_n is the volume of the unit ball in $\mathbb{R}^n$. For a locally integrable function f on Ω and $0 < \alpha \le n$, we consider the *fractional Hardy operator* A_α, defined by

$$A_\alpha f(x) = \frac{1}{|B(0,|x|)|^{\alpha/n}} \int_{B(0,|x|)} f(t)\,dt,$$

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the *Hardy averaging operator* A, defined by

$$Af(x) = \fint_{B(0,|x|)} f(t)\,dt,$$

and the *centered Hardy–Littlewood maximal operator* M, defined by

$$Mf(x) = \sup_{r>0} \fint_{B(x,r)} |f(y)|\,dy$$

by setting $f = 0$ outside Ω (for the fundamental properties of maximal functions, see Stein [14]). In the case $\alpha = n$, $A_\alpha f(x) = Af(x)$.

Let $1 < p < \infty$, $1/p + 1/p' = 1$ and

$$p_\alpha = \frac{np'}{\alpha p' - n} = \frac{np}{\alpha p - np + n}.$$

We know that A_α is bounded from L^p to L^{p_α} provided $n(1 - 1/p) < \alpha \le n$. Clearly, $p_\alpha \ge p > 1$.

In this paper we improve the result of the second author and Pick [12] in the case when $\alpha = n = 1$ and Ω is a bounded interval, and that of the authors [8] within the framework of generalized Banach function spaces. Let $\hookrightarrow$ denote continuous embedding and $\to$ denote boundedness of an operator. Under the assumptions that $A_\alpha : X \to Y$ and $M : Y \to Y$, we find a 'source' space $S_{\alpha,Y}$ and a 'target' space T_Y such that:

(i) the Hardy averaging operator A_α satisfies

$$A_\alpha : S_{\alpha,Y} \to T_Y;$$

(ii) this result improves the classical estimate

$$A_\alpha : X \to Y$$

in the sense that

$$X \hookrightarrow S_{\alpha,Y}, \qquad T_Y \hookrightarrow Y;$$

(iii) this result cannot be improved any further, at least not within the environment of generalized Banach function spaces in the sense that whenever Z is a generalized Banach function space strictly larger than $S_{\alpha,Y}$, then

$$A_\alpha : Z \not\to T_Y$$

and, likewise, when Z is a generalized Banach function space strictly smaller than T_Y, then

$$A_\alpha : S_{\alpha,Y} \not\to Z.$$

The paper is structured as follows. In Section 2, we introduce generalized Banach function spaces (briefly GBFS), and collect some of their properties. In Section 3, we introduce the spaces T_Y and $S_{\alpha,Y}$, and show that $A_\alpha : S_{\alpha,Y} \to T_Y$. In Section 4, we prove a key lemma to obtain optimality results for the action of A_α (see Lemma 4.2). In Section 5, we prove optimality results for the action of A_α on L^p spaces. In Section 6, we prove optimality results for the action of A_α on weighted Lebesgue spaces. In Section 7, we prove optimality results for the action of A_α on Lorentz spaces. In Section 8, we also prove optimality results for the action of the associate operator A'_α. In the last section, we study the duals of the optimal spaces for A_α. For related results, see [2] and [13].

2. Preliminaries. Throughout this paper, let C denote various constants independent of the variables in question, and $C(a, b, \ldots)$ a constant that depends on $a, b, \ldots$.

Let $\mathcal{M}(\mathbb{R}^n)$ denote the space of measurable functions on $\mathbb{R}^n$ with values in $[-\infty, \infty]$. Denote by χ_E the characteristic function of E. Let $|f|$ stand for the modulus of a function $f \in \mathcal{M}(\mathbb{R}^n)$.

Recall the frequently used definition of Banach function spaces which can be found for instance in [1].

Definition 2.1. We say that a normed linear space $(X, \|\cdot\|_X)$ is a Banach function space (BFS for short) if the following conditions are satisfied:

(2.1) $\|f\|_X$ is defined for all $f \in \mathcal{M}(\mathbb{R}^n)$ and
$f \in X$ if and only if $\|f\|_X < \infty$;

(2.2) $\|f\|_X = \|\,|f|\,\|_X$ for every $f \in \mathcal{M}(\mathbb{R}^n)$;

(2.3) if $0 \le f_n \nearrow f$ a.e. in $\mathbb{R}^n$, then $\|f_n\|_X \nearrow \|f\|_X$;

(2.4) if $E \subset \mathbb{R}^n$ is a measurable set of finite measure, then $\chi_E \in X$;

(2.5) for every measurable set $E \subset \mathbb{R}^n$ of finite measure, there exists a positive constant C_E such that $\int_E |f(x)|\,dx \le C_E \|f\|_X$.

Denote by $\mathfrak{B} = \mathfrak{B}(\mathbb{R}^n)$ the class of all BFSs defined on $\mathbb{R}^n$.

We will work with more general spaces where conditions (2.4) and (2.5) are omitted.

Definition 2.2. We say that a normed linear space $(X, \|\cdot\|_X)$ is a *generalized Banach function space* (briefly GBFS) if the following conditions are satisfied:

(2.6) $\|f\|_X$ is defined for all $f \in \mathcal{M}(\mathbb{R}^n)$, and
$f \in X$ if and only if $\|f\|_X < \infty$;

(2.7) $\|f\|_X = \|\,|f|\,\|_X$ for every $f \in \mathcal{M}(\mathbb{R}^n)$;

(2.8) if $0 \le f_n \nearrow f$ a.e. in $\mathbb{R}^n$, then $\|f_n\|_X \nearrow \|f\|_X$.

Denote by $\mathfrak{G} = \mathfrak{G}(\mathbb{R}^n)$ the class of all GBFSs defined on $\mathbb{R}^n$.

Recall that condition (2.8) immediately yields the following property:

$$\text{if} \quad 0 \le f \le g, \quad \text{then} \quad \|f\|_X \le \|g\|_X. \tag{2.9}$$

To see this it suffices to set $f_1 = f$, $f_n = g$ for $n \ge 2$ in (2.8). It is well-known that each BFS is complete, and so it is a Banach space (see [1, Theorem 1.6]). We know that each GBFS is complete (see [8]).

Let X, Y be Banach spaces (not necessarily generalized Banach function spaces). We write $X \hookrightarrow Y$ if $X \subset Y$ and there is $C > 0$ such that $\|f\|_Y \le C\|f\|_X$ for all $f \in X$. Well-known theorems on Banach function spaces (see [1, Theorem 1.8]) yield the implication

$$(\|f\|_X < \infty \Rightarrow \|f\|_Y < \infty) \ \Rightarrow \ X \hookrightarrow Y.$$

In what follows we need a generalization of this remark as in [8].

Definition 2.3. Let $(X, \|\cdot\|_X)$ be a GBFS. Say that a mapping $T : (X, \|\cdot\|_X) \to \mathcal{M}(\mathbb{R}^n)$ is a *sublinear nondecreasing operator* if the following conditions are satisfied for all $\alpha \in \mathbb{R}$ and $f, g \in X$:

(i) $T(\alpha f) = \alpha T(f)$ and $T(f + g) \le T(f) + T(g)$ almost everywhere;
(ii) $0 \le f \le g$ almost everywhere implies $0 \le Tf \le Tg$ almost everywhere.

Lemma 2.4 ([8, Lemma 2.7]). *Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be GBFSs and T a sublinear nondecreasing operator on $\mathcal{M}(\mathbb{R}^n)$. Then the following two conditions are equivalent:*

(i) $\|f\|_X < \infty \Rightarrow \|Tf\|_Y < \infty$;
(ii) *there is $C > 0$ such that $\|Tf\|_Y \le C\|f\|_X$ for all $f \in X$.*

3. Spaces T_Y, $S_{\alpha,Y}$ and boundedness of A_α from $S_{\alpha,Y}$. Given a measurable function f on $\mathbb{R}^n$ set

$$\widetilde{f}(x) = \operatorname*{ess\,sup}_{|t| \ge |x|} |f(t)|.$$

If x is a Lebesgue point of f, then $|f(x)| \le \widetilde{f}(x)$, so that

$$|f(x)| \le \widetilde{f}(x) \quad \text{a.e.} \tag{3.1}$$

Definition 3.1. Let Y be a GBFS and let f be a measurable function on $\mathbb{R}^n$. Set

$$\|f\|_{T_Y} = \|\widetilde{f}\|_Y$$

and define the corresponding space

$$T_Y = \{f : \widetilde{f} \in Y\}.$$

Note that T_Y is a GBFS [8, Lemma 3.2].

Lemma 3.2. *Let Y be a GBFS and $Y \ne 0$. Then $T_Y \hookrightarrow Y$, and we have $T_Y \subsetneq Y$ provided $\lim_{|E_n| \to 0} \|\chi_{E_n}\|_Y = 0$ for measurable sets $E_n \subset \mathbb{R}^n$.*

Proof. By [8, Theorem 3.3], the embedding $T_Y \hookrightarrow Y$ holds. Since $Y \neq 0$, there exist $x_0 \in \mathbb{R}^n$ and a nondecreasing sequence $0 \le a_1 \le a_2 \le \cdots$ such that $\|a_j \chi_{A_j}\|_Y \ge j$, where $A_j = B(x_0, 2^{-j}) \setminus B(x_0, 2^{-j-1})$. By our assumption, there is a sequence of numbers $2^{-j-1} < b_j < 2^{-j}$ such that $\|a_j \chi_{B_j}\|_Y \le 1/j^2$, where $B_j = B(x_0, 2^{-j}) \setminus B(x_0, b_j)$. Set

$$f(x) = \sum_{j=1}^{\infty} a_j \chi_{B_j}(x).$$

Then

$$\|f\|_Y \le \sum_{j=1}^{\infty} \|a_j \chi_{B_j}\|_Y \le \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty,$$

so that $f \in Y$.

Now, it is easy to see that

$$\widetilde{f}(x) = \sum_{j=1}^{\infty} a_j \chi_{A_j}(x).$$

Then

$$\|f\|_{T_Y} = \|\widetilde{f}\|_Y \ge \|a_j \chi_{A_j}(x)\|_Y \ge j$$

for each j, and so $f \notin T_Y$. ∎

LEMMA 3.3. *There is $C > 0$ with*

$$\widetilde{A_\alpha|f|}(x) \le CM(A_\alpha|f|)(x), \qquad x \in \mathbb{R}^n. \tag{3.2}$$

Proof. Fix $x \in \mathbb{R}^n$. If $|x| \le |y| \le 2|x|$, then

$$\begin{aligned} A_\alpha|f|(y) &= \frac{1}{|B(0,|y|)|^{\alpha/n}} \int_{B(0,|y|)} |f(w)|\,dw \\ &\ge \frac{C}{|x|^\alpha} \int_{B(0,|x|)} |f(w)|\,dw = CA_\alpha|f|(x). \end{aligned}$$

Now, for $|y| \ge |x|$ we have $B(0,2|y|) \subset B(x,3|y|)$, and therefore

$$\begin{aligned} M(A_\alpha|f|)(x) &\ge \fint_{B(x,3|y|)} A_\alpha|f|(w)\,dw \ge C|y|^{-n} \int_{B(0,2|y|)} A_\alpha|f|(w)\,dw \\ &\ge C|y|^{-n} \int_{\{w:\,|y|\le|w|\le 2|y|\}} A_\alpha|f|(w)\,dw \\ &\ge C|y|^{-n} \int_{\{w:\,|y|\le|w|\le 2|y|\}} A_\alpha|f|(y)\,dw \ge CA_\alpha|f|(y). \end{aligned}$$

Hence

$$\widetilde{A_\alpha|f|}(x) \le CM(A_\alpha|f|)(x)$$

for $x \in \mathbb{R}^n$, as desired. ∎

Lemma 3.4. *Let X, Y be GBFSs and suppose that*

$$A_\alpha : X \to Y, \qquad M : Y \to Y. \tag{3.3}$$

Then

$$A_\alpha : X \to T_Y.$$

Proof. By (3.2) and (3.3), we have

$$\|A_\alpha f\|_{T_Y} \le \|\widetilde{A_\alpha|f|}\|_Y \le C\|M(A_\alpha|f|)\|_Y \le C\|A_\alpha|f|\|_Y \le C\|f\|_X,$$

as desired. ■

Definition 3.5. Let Y be a GBFS and let f be a measurable function on $\mathbb{R}^n$. Set

$$\|f\|_{S_{\alpha,Y}} = \|A_\alpha|f|\,\|_{T_Y}$$

and consider the corresponding space

$$S_{\alpha,Y} = \{f : \widetilde{A_\alpha|f|} \in Y\}.$$

Note that $S_{\alpha,Y}$ is a GBFS. Indeed, we can prove this as in [8, proof of Lemma 3.6].

Lemma 3.6. *Let X, Y be GBFSs and $A_\alpha : X \to T_Y$. Then $A_\alpha : S_{\alpha,Y} \to T_Y$ and $X \hookrightarrow S_{\alpha,Y}$.*

Proof. By the definitions of $S_{\alpha,Y}$ and T_Y, we have $A_\alpha : S_{\alpha,Y} \to T_Y$.

Let now $\|f\|_X < \infty$. Then

$$\|f\|_{S_{\alpha,Y}} = \|A_\alpha|f|\,\|_{T_Y} \le C\|f\|_X < \infty$$

by our assumption. ■

By Lemmas 3.4 and 3.6, we readily have the following result.

Lemma 3.7. *Let X, Y be GBFSs and $A_\alpha : X \to Y$, $M : Y \to Y$. Then $A_\alpha : S_{\alpha,Y} \to T_Y$ and $X \hookrightarrow S_{\alpha,Y}$.*

We recall the definition of a rearrangement invariant space. Given f on $\mathbb{R}^n$, the *symmetric decreasing rearrangement* of f is defined by

$$f^*(x) = \int_0^\infty \chi_{E_f(t)^*}(x)\,dt,$$

where $E^* = \{x : |B(0,|x|)| < |E|\}$ and $E_f(t) = \{y : |f(y)| > t\}$.

Note that:

(R1) $|E_f(t)| = |E_{f^*}(t)|$ for $t > 0$;
(R2) if $|f| \le |g|$, then $f^* \le g^*$;
(R3) $(cf)^* = |c|f^*$;
(R4) $(f+g)^*(x) \le (2f)^*(2^{-1/n}x) + (2g)^*(2^{-1/n}x)$,

when f, g are measurable on $\mathbb{R}^n$ and c is a real number.

(R1), (R2) and (R3) are easy. To show (R4), we first see that

$$\begin{aligned}|E_{f+g}(t)| &\le |E_f(t/2)| + |E_g(t/2)| \\ &= |E_{f^*}(t/2)| + |E_{g^*}(t/2)|,\end{aligned}$$

and hence

$$\begin{aligned}(f+g)^*(x) &= \int_0^\infty \chi_{\{t:\,|B(0,|x|)|\le|E_{(f+g)}(t)|\}}\,dt \\ &\le \int_0^\infty \chi_{\{t:\,|B(0,|x|)|\le|E_{(2f)^*}(t)|+|E_{(2g)^*}(t)|\}}\,dt \\ &= \int_0^\infty \chi_{\{t:\,|B(0,|x|)|\le 2|E_{(2f)^*}(t)|\}}\,dt + \int_0^\infty \chi_{\{t:\,|B(0,|x|)|\le 2|E_{(2g)^*}(t)|\}}\,dt \\ &\le (2f)^*(2^{-1/n}x) + (2g)^*(2^{-1/n}x),\end{aligned}$$

as required.

Definition 3.8. Let $X \in \mathfrak{G}$. Say that X is a *rearrangement invariant space* if $\|f\|_X = \|f^*\|_X$ for each f. Denote by $\mathfrak{R}$ the class of all rearrangement invariant spaces.

Theorem 3.9. *Let $X \in \mathfrak{G}$, and suppose*

(A) $$f(cx) \in X \quad \text{for all } f \in X \text{ and } c > 0.$$

Then there is a unique $Y \in \mathfrak{R}$ such that $T_X = T_Y$ and the norms in both spaces are equal. Moreover, if $Z \in \mathfrak{R}$ is such that $T_Z \hookrightarrow Y$, then $Z \hookrightarrow Y$.

Proof. Set $\|f\|_Y = \|f^*\|_X$ and consider the corresponding family

$$Y = \{f : f^* \in X\}.$$

By (A) we see that Y is a linear space. Since

$$\|f^*\|_Y = \|(f^*)^*\|_X = \|f^*\|_X = \|f\|_Y,$$

we have $Y \in \mathfrak{R}$. Since

$$\|f\|_{T_Y} = \|\widetilde{f}\|_Y = \|(\widetilde{f}\ )^*\|_X = \|\widetilde{f}\|_X = \|f\|_{T_X},$$

we have $T_Y = T_X$, which proves existence.

Assume that $Y_1, Y_2 \in \mathfrak{R}$, $T_{Y_1} = T_{Y_2}$ and $Y_1 \ne Y_2$. Suppose $Y_2 \setminus Y_1 \ne \emptyset$ without loss of generality, and take $f \in Y_2 \setminus Y_1$. Then $f^* \in Y_2 \setminus Y_1$ and so

$$\|f^*\|_{T_{Y_2}} = \|\widetilde{f^*}\|_{Y_2} = \|f^*\|_{Y_2} < \infty, \qquad \|f^*\|_{T_{Y_1}} = \|\widetilde{f^*}\|_{Y_1} = \|f^*\|_{Y_1} = \infty.$$

Consequently, T_{Y_1} and T_{Y_2} do not coincide.

Now, fix f. Then

$$\|f\|_Y = \|f^*\|_Y \le C\|f^*\|_{T_Z} = C\|\widetilde{f^*}\|_Z = C\|f^*\|_Z = C\|f\|_Z,$$

which proves $Z \hookrightarrow Y$. ∎

4. Optimal pairs

Definition 4.1. Let $\mathfrak{S} \subset \mathfrak{G}$. Assume $X, Y \in \mathfrak{S}$. Say that (X, Y) is an *optimal pair* for A_α with respect to $\mathfrak{S}$ if

$$A_\alpha : X \to Y, \tag{4.1}$$

$$\text{if } Z \in \mathfrak{S} \text{ with } A_\alpha : Z \to Y, \text{ then } Z \hookrightarrow X, \tag{4.2}$$

$$\text{if } Z \in \mathfrak{S} \text{ with } A_\alpha : X \to Z, \text{ then } Y \hookrightarrow Z. \tag{4.3}$$

Lemma 4.2. *Let $X, Y \in \mathfrak{G}$ and $A_\alpha : X \to T_Y$. Suppose*

$$A_\alpha[|x|^{\alpha-n} h(x)] \in T_Y \quad \text{for } h \in T_Y. \tag{4.4}$$

Then $(S_{\alpha,Y}, T_Y)$ is an optimal pair for A_α with respect to $\mathfrak{G}$.

Proof. Let $Z \in \mathfrak{G}$ be such that $Z \setminus S_{\alpha,Y} \neq \emptyset$. Choose $f \in Z \setminus S_{\alpha,Y}$. Since $\|f\|_{S_{\alpha,Y}} = \|A_\alpha |f|\,\|_{T_Y} = \infty$, we have $A_\alpha : Z \not\to T_Y$.

Let $Z \in \mathfrak{G}$ be such that $T_Y \setminus Z \neq \emptyset$. Choose $h \in T_Y \setminus Z$. Then $\widetilde{h} \in Y \setminus Z$. Set $f(x) = |x|^{\alpha-n}\widetilde{h}(x)$. Since $\widetilde{h}$ is radially non-increasing, $A_\alpha f \geq c\widetilde{h}$ for some $c > 0$. Since $\widetilde{h} \notin Z$, $A_\alpha f \notin Z$. By the fact that $\widetilde{h} \in T_Y$ and our assumption (4.4), $A_\alpha f \in T_Y$. Hence $f \in S_{\alpha,Y}$, which implies $A_\alpha : S_{\alpha,Y} \not\to Z$. ∎

Remark 4.3. We note that (4.4) holds if and only if

$$\|A_\alpha[|x|^{\alpha-n} g]\|_Y \leq C\|g\|_Y$$

for every radial symmetric non-increasing function g. Inequalities such as (4.4) are investigated for many function spaces. See for example [4].

By Lemmas 3.4 and 4.2, we have the following lemma.

Lemma 4.4. *Let $X, Y \in \mathfrak{G}$ and $A_\alpha : X \to Y$, $M : Y \to Y$. Suppose* (4.4) *holds. Then $(S_{\alpha,Y}, T_Y)$ is an optimal pair for A_α with respect to $\mathfrak{G}$.*

5. L^p spaces and A_α.

In this section we discuss optimal pairs for A_α with respect to $\mathfrak{G}$ in Lemma 3.7. Recall that

$$1/p_\alpha = 1/p - (n-\alpha)/n.$$

Let us begin with the boundedness of A_α.

Lemma 5.1. *Let $p > 1$ and $n(1 - 1/p) < \alpha \leq n$. Then*

$$A_\alpha : L^p(\mathbb{R}^n) \to L^{p_\alpha}(\mathbb{R}^n).$$

Proof. Assume $\|f\|_{L^p(\mathbb{R}^n)} \leq 1$. If $0 < \delta < 2|x|$, then

$$\begin{aligned} A_\alpha|f|(x) &= C|x|^{n-\alpha} \fint_{B(0,|x|)} |f(y)|\,dy \leq C|x|^{n-\alpha}\Big(\fint_{B(0,|x|)} |f(y)|^p\,dy\Big)^{1/p} \\ &\leq C|x|^{n-\alpha-n/p} \leq C\delta^{n-\alpha-n/p}. \end{aligned}$$

If $\delta \geq 2|x|$, then

$$A_\alpha|f|(x) = C|x|^{n-\alpha} \fint_{B(0,|x|)} |f(y)|\,dy \leq C|x|^{n-\alpha} Mf(x) \leq C\delta^{n-\alpha} Mf(x),$$

so that

$$A_\alpha|f|(x) \leq C\delta^{n-\alpha} Mf(x) + C\delta^{n-\alpha-n/p}.$$

Now, letting $\delta = [Mf(x)]^{-p/n}$, we have

$$A_\alpha|f|(x) \leq C(Mf(x))^{1-(n-\alpha)p/n} = C(Mf(x))^{p/p_\alpha},$$

so that

$$\int_{\mathbb{R}^n} (A_\alpha|f|(x))^{p_\alpha}\,dx \leq C \int_{\mathbb{R}^n} (Mf(x))^p\,dx \leq C \int_{\mathbb{R}^n} |f(y)|^p\,dy = C,$$

as required. ■

LEMMA 5.2. *Suppose $q > 1$, $\alpha \leq n$ and $n < \alpha q$. Assume $h \in L^q(\mathbb{R}^n)$ and set $f(y) = |y|^{\alpha-n}|h(y)|$. Then*

$$\|\widetilde{A_\alpha f}\|_q \leq C\|h\|_q.$$

Proof. Set $f(y) = |y|^{\alpha-n}|h(y)|$ for $h \in L^q(\mathbb{R}^n)$. By (3.2) and Lemma 6.2 below, we have

$$\begin{aligned}
\|\widetilde{A_\alpha f}\|_q^q \leq C\|M(A_\alpha f)\|_q^q &\leq C \int_{\mathbb{R}^n} |A_\alpha f(x)|^q\,dx \\
&\leq C \int_{\mathbb{R}^n} \big(|x|^{n-\alpha} Mf(x) g\big)^q\,dx \leq C \int_{\mathbb{R}^n} \big(|y|^{n-\alpha} f(y)\big)^q\,dy \\
&= C \int_{\mathbb{R}^n} |h(y)|^q\,dy = C\|h\|_q^q,
\end{aligned}$$

as required. ■

THEOREM 5.3. *Let $p > 1$ and $n(1 - 1/p) < \alpha \leq n$. If $X = L^p(\mathbb{R}^n)$ and $Y = L^{p_\alpha}(\mathbb{R}^n)$, then $(S_{\alpha,Y}, T_Y)$ is an optimal pair for A_α.*

Proof. First we see from Lemmas 3.4 and 5.1 that $A_\alpha : X \to T_Y$. By Lemma 5.2 with $q = p_\alpha$, (4.4) holds. Hence it follows from Lemma 4.2 that $(S_{\alpha,Y}, T_Y)$ is an optimal pair for A_α. ■

6. Weighted Lebesgue spaces and A_α

DEFINITION 6.1. Let $q \geq 1$ and v be a weight. Recall that the *weighted Lebesgue space* $L^q(\mathbb{R}^n, v)$ is the set of all functions f with

$$\|f\|_{L^q(\mathbb{R}^n,v)} = \Big(\int_{\mathbb{R}^n} |f(x)|^q v(x)\,dx\Big)^{1/q} < \infty.$$

Recall the well-known result on the maximal operator (see Muckenhoupt [9]).

LEMMA 6.2. *Let $q > 1$ and $-n < \beta < n(q-1)$. Then*

$$M : L^q(\mathbb{R}^n, |x|^\beta) \to L^q(\mathbb{R}^n, |x|^\beta).$$

Proof. It suffices to verify that the weight $|x|^\beta$ belongs to the Muckenhoupt class $\mathcal{A}_q$. For this, see, for example, Heinonen, Kilpeläinen and Martio [6]. ∎

Now we prove the boundedness of A_α on weighted Lebesgue spaces.

LEMMA 6.3. *Let $p, q > 1$ and $n(1-1/p) < \alpha \le n$. Then*

$$A_\alpha : L^q(\mathbb{R}^n, |x|^{n(q/p-1)}) \to L^q(\mathbb{R}^n, |x|^{n(q/p_\alpha-1)}).$$

Proof. Set $X = L^q(\mathbb{R}^n, |x|^{n(q/p-1)})$ and $Y = L^q(\mathbb{R}^n, |x|^{n(q/p_\alpha-1)})$. Since $p > 1$, $|x|^\beta \in \mathcal{A}_q$ with $\beta = n(q/p-1)$. By Lemma 6.2, we have

$$\begin{aligned}
\|A_\alpha f\|_Y^q &= \int\limits_{\mathbb{R}^n} |A_\alpha f(x)|^q |x|^{n(q/p_\alpha-1)}\,dx \\
&= \int\limits_{\mathbb{R}^n} \bigg(\frac{1}{|B(0,|x|)|^{\alpha/n}} \int\limits_{B(0,|x|)} |f(t)|\,dt\bigg)^q |x|^{n(q/p_\alpha-1)}\,dx \\
&= C\int\limits_{\mathbb{R}^n} \bigg(\frac{1}{|x|^n} \int\limits_{B(0,|x|)} |f(t)|\,dt\bigg)^q |x|^{n(q/p_\alpha-1)+q(n-\alpha)}\,dx \\
&= C\int\limits_{\mathbb{R}^n} \bigg(\frac{1}{|x|^n} \int\limits_{B(0,|x|)} |f(t)|\,dt\bigg)^q |x|^{\beta}\,dx \\
&\le C\int\limits_{\mathbb{R}^n} (Mf(x))^q |x|^\beta\,dx \le C\int\limits_{\mathbb{R}^n} |f(x)|^q |x|^\beta\,dx = C\|f\|_X^q,
\end{aligned}$$

as required. ∎

Setting $\alpha = n$ in the previous lemma we obtain the next remark.

REMARK 6.4. Let $p, q > 1$. Then

$$A : L^q(\mathbb{R}^n, |x|^{n(q/p-1)}) \to L^q(\mathbb{R}^n, |x|^{n(q/p-1)}).$$

As an immediate consequence of Lemmas 6.2, 6.3 and 3.4, we obtain the following lemma.

LEMMA 6.5. *Let $p, q > 1$ and $n(1-1/p) < \alpha \le n$. Then*

$$A_\alpha : L^q(\mathbb{R}^n, |x|^{n(q/p-1)}) \to T_{L^q(\mathbb{R}^n, |x|^{n(q/p_\alpha-1)})}. \tag{6.1}$$

Rewrite (6.1) as

$$\Big(\int\limits_{\mathbb{R}^n} (\widetilde{A_\alpha f}(x))^q |x|^{n(q/p_\alpha-1)}\,dx\Big)^{1/q} \le C\Big(\int\limits_{\mathbb{R}^n} |f(y)|^q |y|^{n(q/p-1)}\,dy\Big)^{1/q}. \tag{6.2}$$

In fact, inequality (6.2) can be derived as a special case of Theorem 4.1 from [3], but our proof is different and shorter.

LEMMA 6.6. *Let* $p, q > 1$, $n(1-1/p) < \alpha \le n$ *and* $Y = L^q(\mathbb{R}^n, |x|^{n(q/p_\alpha - 1)})$. *Assume* $h \in T_Y$ *and set* $f(x) = |x|^{\alpha-n} h(x)$. *Then*

$$\int_{\mathbb{R}^n} (\widetilde{A_\alpha f}(x))^q |x|^{n(q/p_\alpha-1)}\, dx \le C \int_{\mathbb{R}^n} \widetilde{h}(x)^q |x|^{n(q/p_\alpha-1)}\, dx.$$

Proof. Let $h \in T_Y$. By (6.2) with $f(x) = |x|^{\alpha-n} h(x)$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} (\widetilde{A_\alpha f}(x))^q |x|^{n(q/p_\alpha-1)}\, dx &\le C \int_{\mathbb{R}^n} |f(x)|^q |x|^{n(q/p-1)}\, dx \\
&= C \int_{\mathbb{R}^n} (|x|^{\alpha-n} |h(x)|)^q |x|^{n(q/p-1)}\, dx \\
&= C \int_{\mathbb{R}^n} |h(x)|^q |x|^{n(q/p_\alpha-1)}\, dx \\
&\le C \int_{\mathbb{R}^n} \widetilde{h}(x)^q |x|^{n(q/p_\alpha-1)}\, dx,
\end{aligned}$$

as required. ∎

We discuss optimal pairs for A_α with respect to $\mathfrak{G}$ in Lemma 3.7. By Lemmas 6.5, 6.6 and 4.2, we obtain the following theorem.

THEOREM 6.7. *Let* $p, q > 1$ *and* $n(1-1/p) < \alpha \le n$. *If* $X = L^q(\mathbb{R}^n, |x|^{n(q/p-1)})$ *and* $Y = L^q(\mathbb{R}^n, |x|^{n(q/p_\alpha-1)})$, *then* $(S_{\alpha,Y}, T_Y)$ *is an optimal pair for* A_α.

Proof. Note from Lemma 6.5 that $A_\alpha : X \to T_Y$. Let $h \in T_Y$ and $f(x) = |x|^{\alpha-n} h(x)$. By Lemma 6.6, (4.4) holds. Hence, Lemma 4.2 shows that $(S_{\alpha,Y}, T_Y)$ is an optimal pair for A_α. ∎

7. Lorentz spaces and A_α

DEFINITION 7.1. Let $p, q \ge 1$. Recall that the *Lorentz space* $L^{p,q}(\mathbb{R}^n)$ is the set of all functions f with

$$\|f\|_{L^{p,q}(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n} f^*(x)^q |x|^{n(q/p-1)}\, dx\Big)^{1/q} < \infty.$$

Note that

$$\|f\|_{L^{p,q}(\mathbb{R}^n)} \sim \Big(\int_0^\infty f_*(t)^q t^{q/p-1}\, dt\Big)^{1/q} < \infty,$$

where f_* denotes the usual one-dimensional nonincreasing rearrangement of f. (Here $f \sim g$ means that $C^{-1} g \le f \le Cg$ for a constant $C > 0$.)

In view of Hardy's inequality (see [7]), if $q > 1$ and $\alpha < n/q'$, then for nonnegative measurable functions f on $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \Big(|y|^{\alpha-n} \int_{B(0,|y|)} f(x)|x|^{-\alpha}\, dx\Big)^q dy \le C \int_{\mathbb{R}^n} f(x)^q\, dx, \tag{7.1}$$

and if $q > 1$ and $\alpha > n/q'$, then

$$\int_{\mathbb{R}^n} \Big(|y|^{\alpha-n} \int_{\mathbb{R}^n \setminus B(0,|y|)} f(x)|x|^{-\alpha}\,dx \Big)^q dy \le C \int_{\mathbb{R}^n} f(x)^q\,dx. \tag{7.2}$$

Note from (7.1) that if $q > 1$ and $\alpha < n/q'$, then

$$\|A_{n-\alpha}(|x|^{-\alpha} f)\|_q \le C\|f\|_q. \tag{7.3}$$

Lemma 7.2. *Let $p > 0$, $q > 1$ and $n(1-1/p) < \alpha \le n$. Then*

$$\int_{\mathbb{R}^n} (\widetilde{A_\alpha} f(x))^q |x|^{n(q/p_\alpha - 1)}\,dx \le C \int_{\mathbb{R}^n} (Af(x))^q |x|^{n(q/p-1)}\,dx$$

for nonnegative measurable functions f on $\mathbb{R}^n$.

Proof. We have

$$\begin{aligned}
\widetilde{A_\alpha} f(x) &= \operatorname*{ess\,sup}_{|y|\ge|x|} \frac{1}{|B(0,|y|)|^{\alpha/n}} \int_{B(0,|y|)} f(t)\,dt \\
&\le C \sum_{j=0}^{\infty} (2^j|x|)^{-\alpha} \int_{B(0,2^{j+1}|x|)} f(t)\,dt \\
&\le C \int_{\{y:\,|y|\ge|x|\}} \Big(|y|^{-\alpha} \int_{B(0,|y|)} f(t)\,dt \Big) |y|^{-n}\,dy \\
&= C \int_{\{y:\,|y|\ge|x|\}} |y|^{-\alpha} Af(y)\,dy.
\end{aligned}$$

Note here that $\alpha + n(1/p - 1/q) > n/q'$ by our assumption $\alpha > n/p'$, and $\alpha + n(1/p - 1/q) - n = n(1/p_\alpha - 1/q)$. Hence, in view of (7.2), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} (\widetilde{A_\alpha} f(x))^q |x|^{n(q/p_\alpha-1)}\,dx \\
&\quad\le C \int_{\mathbb{R}^n} \Big(\int_{\{y:\,|y|\ge|x|\}} |y|^{-\alpha} Af(y)\,dy \Big)^q |x|^{n(q/p_\alpha-1)}\,dx \\
&\quad= C \int_{\mathbb{R}^n} \Big(\int_{\{y:\,|y|\ge|x|\}} |y|^{-\{\alpha+n(1/p-1/q)\}} Af(y) |y|^{n(1/p-1/q)}\,dy \Big)^q |x|^{n(q/p_\alpha-1)}\,dx \\
&\quad\le C \int_{\mathbb{R}^n} (Af(x))^q |x|^{n(q/p-1)}\,dx,
\end{aligned}$$

as required. ∎

In view of Lemma 7.2, we can prove the boundedness of A_α for Lorentz spaces.

Lemma 7.3. *Let $p, q > 1$. Let $n(1-1/p) < \alpha \le n$. Then $A_\alpha : L^{p,q}(\mathbb{R}^n) \to T_{L^{p_\alpha,q}(\mathbb{R}^n)}$.*

Proof. Let $f \geq 0$ be measurable. Since $Af(x) \leq A(f^*)(x)$, by Lemma 7.2 and Remark 6.4 we have

$$\begin{aligned}
\|A_\alpha f\|^q_{T_{L^{p_\alpha,q}(\mathbb{R}^n)}} &= \|\widetilde{A_\alpha f}\|^q_{L^{p_\alpha,q}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} ((\widetilde{A_\alpha f})^*(x))^q |x|^{n(q/p_\alpha - 1)}\,dx \\
&= \int_{\mathbb{R}^n} (\widetilde{A_\alpha f}(x))^q |x|^{n(q/p_\alpha - 1)}\,dx \leq C \int_{\mathbb{R}^n} (Af(x))^q |x|^{n(q/p-1)}\,dx \\
&\leq C \int_{\mathbb{R}^n} (A(f^*)(x))^q |x|^{n(q/p-1)}\,dx \\
&\leq C \int_{\mathbb{R}^n} f^*(x)^q |x|^{n(q/p-1)}\,dx = C\|f\|^q_{L^{p,q}(\mathbb{R}^n)},
\end{aligned}$$

as desired. ∎

We discuss optimal pairs for A_α.

THEOREM 7.4. *Let $p, q > 1$. Let $n(1-1/p) < \alpha \leq n$. If $X = L^{p,q}(\mathbb{R}^n)$ and $Y = L^{p_\alpha,q}(\mathbb{R}^n)$, then $(S_{\alpha,Y}, T_Y)$ is an optimal pair for A_α.*

Proof. Note from Lemma 7.3 that $A_\alpha : X \to T_Y$. Let $h \in T_Y$. Then $\widetilde{h} \in Y$. Set $f(x) = |x|^{\alpha-n}\widetilde{h}(x)$. By Lemma 7.3, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} ((\widetilde{A_\alpha f})^*(x))^q |x|^{n(q/p_\alpha - 1)}\,dx &\leq C \int_{\mathbb{R}^n} f^*(x)^q |x|^{n(q/p-1)}\,dx \\
&= C \int_{\mathbb{R}^n} ((|x|^{\alpha-n}\widetilde{h}(x))^*)^q |x|^{n(q/p-1)}\,dx \\
&= C \int_{\mathbb{R}^n} (|x|^{\alpha-n}\widetilde{h}(x))^q |x|^{n(q/p-1)}\,dx \\
&= C \int_{\mathbb{R}^n} \widetilde{h}(x)^q |x|^{n(q/p_\alpha - 1)}\,dx \\
&= C \int_{\mathbb{R}^n} ((\widetilde{h})^*(x))^q |x|^{n(q/p_\alpha - 1)}\,dx.
\end{aligned}$$

Since $\widetilde{h} \in T_Y$, (4.4) holds. Hence, Lemma 4.2 implies that $(S_{\alpha,Y}, T_Y)$ is an optimal pair for A_α. ∎

8. Associate operator A'_α. Note that the associate operator A'_α to A_α is given by

$$A'_\alpha f(y) = \sigma_n^{-\alpha/n} \int_{\{x:\,|y|\leq|x|\}} |x|^{-\alpha} f(x)\,dx$$

for a locally integrable function f on $\mathbb{R}^n$, where σ_n is the volume of the unit ball in $\mathbb{R}^n$.

In fact,

$$\int_{\mathbb{R}^n} A_\alpha g(x) f(x)\,dx = \int_{\mathbb{R}^n} g(y)\Big(\sigma_n^{-\alpha/n} \int_{\{x:\,|y|\le|x|\}} |x|^{-\alpha} f(x)\,dx\Big)\,dy \tag{8.1}$$

for nonnegative measurable functions f and g on $\mathbb{R}^n$.

Lemma 8.1. *Let $p > 1$ and $n(1-1/p) < \alpha \le n$. Then*

$$\int_{\mathbb{R}^n} A'_\alpha f(x)^{p_\alpha}\,dx \sim \int_{\mathbb{R}^n} A_\alpha f(x)^{p_\alpha}\,dx$$

for nonnegative measurable functions f on $\mathbb{R}^n$.

Proof. Integrating by parts, we find

$$A'(A_\alpha f)(x) = C \int_{\{z:\,|x|\le|z|\}} A_\alpha f(z)|z|^{-n} dz \ge C A'_\alpha f(x).$$

By the boundedness of A' (see, e.g., [8, Lemma 8.1]), we have

$$\int_{\mathbb{R}^n} A'_\alpha f(x)^{p_\alpha}\,dx \le C \int_{\mathbb{R}^n} (A'(A_\alpha f)(x))^{p_\alpha}\,dx \le C \int_{\mathbb{R}^n} A_\alpha f(x)^{p_\alpha}\,dx.$$

We show the converse inequality. By Fubini's theorem, we find

$$A_\alpha f(x) \le C|x|^{-\alpha} \int_{\{y:\,|y|\le|x|\}} A'_\alpha f(y)|y|^{\alpha-n}\,dy \le C A_\alpha(|x|^{\alpha-n} A'_\alpha f)(x).$$

By (7.3) with α and q replaced by $n-\alpha$ and p_α respectively, we have

$$\|A_\alpha(|x|^{\alpha-n} A'_\alpha f)\|_{p_\alpha} \le C\|A'_\alpha f\|_{p_\alpha}.$$

Hence

$$\int_{\mathbb{R}^n} A_\alpha f(x)^{p_\alpha}\,dx \le C \int_{\mathbb{R}^n} A'_\alpha f(y)^{p_\alpha}\,dy,$$

as required. ∎

Note that $S_{\alpha,p_\alpha}(\mathbb{R}^n) \equiv S_{\alpha,L^{p_\alpha}(\mathbb{R}^n)} = \{f \in \mathcal{M}(\mathbb{R}^n) : A_\alpha|f| \in L^{p_\alpha}(\mathbb{R}^n)\}$, in view of (3.2). Set $U_{\alpha,p_\alpha}(\mathbb{R}^n) = \{f \in \mathcal{M}(\mathbb{R}^n) : A'_\alpha|f| \in L^{p_\alpha}(\mathbb{R}^n)\}$.

By Lemma 8.1, we have the following lemma.

Lemma 8.2. *If $p > 1$ and $n(1-1/p) < \alpha \le n$, then*

$$S_{\alpha,p_\alpha}(\mathbb{R}^n) = U_{\alpha,p_\alpha}(\mathbb{R}^n).$$

By Lemmas 8.1 and 5.1, we have the following lemma.

Lemma 8.3. *Let $p > 1$ and $n(1-1/p) < \alpha \le n$. Then*

$$A'_\alpha : L^p(\mathbb{R}^n) \to L^{p_\alpha}(\mathbb{R}^n).$$

Theorem 8.4. *Let $p > 1$ and $n(1-1/p) < \alpha \le n$. If $X = L^p(\mathbb{R}^n)$ and $Y = L^{p_\alpha}(\mathbb{R}^n)$, then $(S_{\alpha,Y}, T_Y)$ is an optimal pair for A'_α.*

Proof. First we see from Lemmas 3.4 and 8.3 that $A'_\alpha : X \to T_Y$. Let $h \in T_Y$. Set $f(y) = |y|^{\alpha-n}|h(y)|$. By Lemmas 8.1 and 5.2 with $q = p_\alpha$, (4.4) holds. Hence Lemma 4.2 shows that $(S_{\alpha,Y}, T_Y)$ is an optimal pair for A'_α. ∎

9. Duals. Recall the well-known fact (following from Muckenhoupt's condition):

$$\int_{\mathbb{R}^n} (Mf(x))^p|x|^\beta\,dx \le C \int_{\mathbb{R}^n} |f(x)|^p|x|^\beta\,dx, \tag{9.1}$$

if and only if $-n < \beta < n(p-1)$.

Recall also the Hardy inequality (it can be easily obtained from the 1-dimensional version): If f is a nonnegative radial function and $\alpha > -n$ then

$$\int_{\mathbb{R}^n} \bigg(\int_{|y|\ge|x|} \frac{f(y)}{|y|^n}\,dy\bigg)^p |x|^\alpha\,dx \le C \int_{\mathbb{R}^n} f(x)^p|x|^\alpha\,dx. \tag{9.2}$$

In fact, since $n+\alpha/p > n/p'$ by our assumption $\alpha > -n$ and $\{(n+\alpha/p) - n\}p = \alpha$, we obtain by (7.2),

$$\begin{aligned}\int_{\mathbb{R}^n} \bigg(\int_{|y|\ge|x|} \frac{f(y)}{|y|^n}\,dy\bigg)^p |x|^\alpha\,dx &= \int_{\mathbb{R}^n} \bigg(\int_{\{y:\,|y|\ge|x|\}} |y|^{-(n+\alpha/p)} f(y)|y|^{\alpha/p}\,dy\bigg)^p |x|^\alpha\,dx \\ &\le C \int_{\mathbb{R}^n} f(x)^p|x|^\alpha\,dx.\end{aligned}$$

For simplicity, write

$$X^{p,q}(\mathbb{R}^n) = L^q(\mathbb{R}^n, |x|^{n(q/p-1)})$$

and

$$\|f\|_{X^{p,q}(\mathbb{R}^n)} = \|f\|_{L^q(\mathbb{R}^n,|x|^{n(q/p-1)})}.$$

Note that the associate operator A' to A is given by

$$A'f(y) = \sigma_n^{-1} \int_{\{x:\,|y|\le|x|\}} |x|^{-n} f(x)\,dx$$

for a locally integrable function f on $\mathbb{R}^n$. In the case $\alpha = n$, $A'_\alpha f(y) = A'f(y)$.

Theorem 9.1. *Let $n(1-1/p) < \alpha \le n$ and assume $q'/p' < q$. Then*

$$(T_{X^{p',q'}(\mathbb{R}^n)})' = S_{\alpha,X^{p_\alpha,q}(\mathbb{R}^n)}.$$

Remark 9.2. The referee kindly suggested that Theorem 9.1 can be obtained by the methods in [11] for the one-dimensional case (see also [10]).

We here give a proof of Theorem 9.1 by a careful application of our results above.

Proof of Theorem 9.1. An easy calculation gives, for each $0 \neq y \in \mathbb{R}^n$,

$$\int_{B(0,|2y|)\setminus B(0,|y|)} \frac{1}{|x|^n}\,dx = \omega_{n-1}\log 2,$$

where ω_{n-1} stands for the $(n-1)$-Hausdorff measure of the unit sphere. Thus, by Fubini's theorem we have

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{1}{|x|^n} \int_{B(0,|2x|)\setminus B(0,|x|)} h(y)\,dy\,dx &= \int_{\mathbb{R}^n} h(y) \int_{B(0,|y|)\setminus B(0,|y|/2)} \frac{1}{|x|^n}\,dx\,dy \\ &= \omega_{n-1}\log 2 \int_{\mathbb{R}^n} h(y)\,dy.\end{aligned}$$

Setting $h(y) = f(y)g(y)$ for $f, g \geq 0$ on $\mathbb{R}^n$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)g(x)\,dx &= \frac{1}{\omega_{n-1}\log 2}\int_{\mathbb{R}^n} \frac{1}{|x|^n} \int_{B(0,|2x|)\setminus B(0,|x|)} f(y)g(y)\,dy\,dx \\ &\leq \frac{1}{\omega_{n-1}\log 2}\int_{\mathbb{R}^n} \tilde{f}(x)\left(\frac{1}{|x|^n}\int_{B(0,2|x|)\setminus B(0,|x|)} g(y)\,dy\right)dx.\end{aligned}$$

Hence, by Hölder's inequality we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)g(x)\,dx &\leq C\int_{\mathbb{R}^n} |x|^{\alpha-n}\tilde{f}(x)A_\alpha g(2x)\,dx \\ &= C\int_{\mathbb{R}^n} (|x|^{n(1/p'-1/q')}\tilde{f}(x))(|x|^{n(1/p_\alpha-1/q)}A_\alpha g(2x))\,dx \\ &\leq C\left(\int_{\mathbb{R}^n} |x|^{n(q'/p'-1)}(\tilde{f}(x))^{q'}\,dx\right)^{1/q'}\left(\int_{\mathbb{R}^n} |x|^{n(q/p_\alpha-1)}(A_\alpha g(2x))^q\,dx\right)^{1/q} \\ &\leq C\|\tilde{f}\|_{X^{p',q'}(\mathbb{R}^n)}\left(\int_{\mathbb{R}^n} |x|^{n(q/p_\alpha-1)}(A_\alpha g(x))^q\,dx\right)^{1/q} \\ &\leq C\|\tilde{f}\|_{X^{p',q'}(\mathbb{R}^n)}\left(\int_{\mathbb{R}^n} |x|^{n(q/p_\alpha-1)}(\widetilde{A_\alpha g}(x))^q\,dx\right)^{1/q} \\ &= C\|\tilde{f}\|_{X^{p',q'}(\mathbb{R}^n)}\|g\|_{S_{\alpha,X^{p_\alpha,q}(\mathbb{R}^n)}},\end{aligned}$$

so that

$$\begin{aligned}\|g\|_{(T_{X^{p',q'}(\mathbb{R}^n)})'} &= \sup_{\|f\|_{T_{X^{p',q'}(\mathbb{R}^n)}}\leq 1}\int_{\mathbb{R}^n} f(x)g(x)\,dx \\ &\leq C\|g\|_{S_{\alpha,X^{p_\alpha,q}(\mathbb{R}^n)}}.\end{aligned}$$

Conversely, letting $\|g\|_{S_{\alpha,X^{p_\alpha,q}(\mathbb{R}^n)}} = 1$, we set

$$|x|^{n(1/p'-1/q')} f(x) = (|x|^{n(1/p_\alpha - 1/q)} A_\alpha g(x))^{q-1}. \tag{9.3}$$

Then $f(x) = (CA_{nq'/p'} g(x))^{q-1}$, so that by Lemma 3.3,

$$\tilde{f}(x) \le C(M(A_{nq'/p'} g)(x))^{q-1}.$$

Hence, by the assumption $q'/p' < q$ and (9.1) we have

$$\int_{\mathbb{R}^n} (M(A_{nq'/p'} g)(x))^q |x|^{n(q'/p'-1)}\, dx \le C \int_{\mathbb{R}^n} (A_{nq'/p'} g(x))^q |x|^{n(q'/p'-1)}\, dx,$$

and so

$$\begin{aligned}
(\|f\|_{T_{X^{p',q'}(\mathbb{R}^n)}})^{q'} &= \int_{\mathbb{R}^n} \tilde{f}(x)^{q'} |x|^{n(q'/p'-1)}\, dx \\
&\le C \int_{\mathbb{R}^n} (M(A_{nq'/p'} g)(x))^q |x|^{n(q'/p'-1)}\, dx \\
&\le C \int_{\mathbb{R}^n} (A_{nq'/p'} g(x))^q |x|^{n(q'/p'-1)}\, dx \\
&= C \int_{\mathbb{R}^n} (A_\alpha g(x))^q |x|^{n(q/p_\alpha - 1)}\, dx \\
&\le C \int_{\mathbb{R}^n} (\widetilde{A_\alpha g}(x))^q |x|^{n(q/p_\alpha - 1)}\, dx \\
&= C(\|g\|_{S_{\alpha,X^{p_\alpha,q}(\mathbb{R}^n)}})^q = C.
\end{aligned}$$

Consequently, by (9.2) we have

$$\|A'\tilde{f}\|_{T_{X^{p',q'}(\mathbb{R}^n)}} \le C\|\tilde{f}\|_{T_{X^{p',q'}(\mathbb{R}^n)}} \le C.$$

Again by Lemma 3.3 and (9.1) we can write

$$\begin{aligned}
(\|g\|_{S_{\alpha,X^{p_\alpha,q}(\mathbb{R}^n)}})^q &= \int_{\mathbb{R}^n} \big(|y|^{n(1/p_\alpha - 1/q)} \widetilde{A_\alpha g}(y)\big)^q\, dy \\
&\le \int_{\mathbb{R}^n} (M(A_\alpha g)(y))^q |y|^{n(q/p_\alpha - 1)}\, dy \le C \int_{\mathbb{R}^n} (A_\alpha g(y))^q |y|^{n(q/p_\alpha - 1)}\, dy.
\end{aligned}$$

Thus, by (9.3)

$$\begin{aligned}
\|g\|_{(T_{X^{p',q'}(\mathbb{R}^n)})'} &\ge C \int_{\mathbb{R}^n} (A'\tilde{f}(x)) g(x)\, dx \\
&\ge C \int_{\mathbb{R}^n} |y|^{\alpha-n} \tilde{f}(y) A_\alpha g(y)\, dy \\
&\ge C \int_{\mathbb{R}^n} (|y|^{n(1/p_\alpha - 1/q)} A_\alpha g(y))^q\, dy \\
&\ge C(\|g\|_{S_{\alpha,X^{p_\alpha,q}(\mathbb{R}^n)}})^q = C.
\end{aligned}$$

This implies that

$$\|g\|_{(T_{X^{p'},q'(\mathbb{R}^n)})'} \geq C\|g\|_{S_{\alpha,X^{p_\alpha,q}(\mathbb{R}^n)}}$$

for all $g \in S_{\alpha,X^{p_\alpha,q}(\mathbb{R}^n)}$. ∎

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