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Zero sums of products of Toeplitz and Hankel operators on the Hardy space

by

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Abstract. On the Hardy space of the unit disk, we consider operators which are finite sums of products of a Toeplitz operator and a Hankel operator. We then give characterizations for such operators to be zero. Our results extend several known results using completely different arguments.

1. Introduction. Let $\mathbb{T}$ be the boundary of the unit disk in the complex plane $\mathbb{C}$ and σ be the normalized Lebesgue measure on $\mathbb{T}$. We let $L^2 = L^2(\mathbb{T}, \sigma)$ denote the usual Lebesgue space of $\mathbb{T}$. The Hardy space H^2 is the closure of the polynomials in L^2. Let P denote the orthogonal projection from L^2 onto H^2. For a function $u \in L^\infty(\mathbb{T})$, the *Toeplitz operator* T_u and (*little*) *Hankel operator* H_u with symbol u are defined respectively by

$$T_u f = P(uf) \quad \text{and} \quad H_u f = PJ(uf)$$

for $f \in H^2$. Here J is the unitary operator on L^2 defined by $Jf(z) = \bar{z}f(\bar{z})$. Then clearly T_u and H_u are bounded linear operators on H^2. See [9, Chapter 10] for details.

For decades, algebraic properties of Toeplitz and Hankel operators have been studied. First Brown and Halmos [1] obtained a complete description of (semi)commuting Toeplitz operators. Also, they studied the problem of when a product of two Toeplitz operators is another Toeplitz operator. Later Stroethoff [7] gave a new proof of the criterion for $T_uT_v = T_\varphi T_\psi$ to hold and recovered the result of Brown and Halmos above. Also, the present author [4] later characterized when operators which are finite sums of products of two Toeplitz operators are zero and recovered the results of [1] and [7] mentioned above.

Also, products of Toeplitz and Hankel operators have been studied. Yoshino [8] determined when a product of two Hankel operators is another Hankel operator. Gu and Zheng [3] established when finite sums of products

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of two (big) Hankel operators are zero. Martínez-Avendaño [6] studied when a Toeplitz operator and a Hankel operator commute; note that he used a slightly different notion of Hankel operator. For given $u, v \in L^\infty(\mathbb{T})$ for which H_u is not zero, the result of Martínez-Avendaño shows that $H_u T_v = T_v H_u$ if and only if $v\hat{v}, v + \hat{v}$ are constants and $v + \alpha u \in H^\infty$ for some constant α. Here, H^∞ denotes the set of all bounded analytic functions on $\mathbb{T}$, and $\hat{u}(z) = u(\bar{z})$. Later Ding [2] studied when the product of a Toeplitz operator and a Hankel operator is another Hankel operator, and proved that given $u, v, \psi \in L^\infty(\mathbb{T})$, $H_u T_v = H_\psi$ if and only if either $u, \psi \in H^\infty$ or $v, uv - \psi \in H^\infty$.

Motivated by these results, we consider a more general class of operators which contains products of Toeplitz and Hankel operators. More explicitly, we consider operators of the form

$$\sum_{j=1}^{N} T_{u_j} H_{u_j} \quad \text{or} \quad \sum_{j=1}^{N} H_{u_j} T_{u_j} \tag{1}$$

where $u_j, v_j \in L^\infty(\mathbb{T})$. By using arguments completely different from those applied before, we characterize when operators of the type (1) are equal to zero. Together with these operators, we also consider operators of the form $H_u T_v + T_\varphi H_\psi$ and determine when such an operator is zero. Our results generalize several known results concerning products of Toeplitz operators or Hankel operators.

In Section 2, we collect some preliminary results which will be used in our characterizations. In Section 3, we characterize when operators of the type (1) are zero (Theorems 3.1 and 3.4). Also, a characterization when $H_u T_v + T_\varphi H_\psi = 0$ is given in Theorem 3.7. As immediate consequences, we recover several known results.

We mention that the corresponding characterizations on the Dirichlet space have been obtained in [5]. While the main scheme of our proofs is adapted from [5], we need to establish corresponding theories for Toeplitz and Hankel operators on the Hardy space.

2. Preliminaries. Given $f, g \in H^2$, the rank-one operator $f \otimes g$ is defined on H^2 by

$$(f \otimes g)h = \langle h, g \rangle f, \qquad h \in H^2,$$

where $\langle\ ,\ \rangle$ is the usual inner product on L^2 defined by

$$\langle \psi, \varphi \rangle = \int_{\mathbb{T}} \psi \overline{\varphi}\, d\sigma$$

for $\psi, \varphi \in L^2$. We note that the operator equation

$$S(f \otimes g)T = (Sf) \otimes (T^* g) \tag{2}$$

holds for any bounded operators S, T on H^2.

Given nonzero functions $f, g, h, k \in H^2$, we observe that $f \otimes g = h \otimes k$ if and only if there exists a nonzero constant α such that $f = \alpha h$ and $k = \bar{\alpha} g$. Generally, for zero sums of such rank one operators, we have the following lemma which is proved in [3, Proposition 4].

For a given positive integer N, we let $\mathbb{M}_N$ be the set of all $N \times N$ complex matrices and $\mathbb{S}_N$ be the set of all permutations on $\{1, \dots, N\}$. Also, I denotes the identity operator.

Lemma 2.1. *Let $x_j, y_j \in H^2$ for $j = 1, \dots, N$. Then*

$$\sum_{j=1}^{N} x_j \otimes y_j = 0$$

on H^2 if and only if there exist $A \in \mathbb{M}_N$ and $\sigma \in \mathbb{S}_N$ such that

$$[A - I] \begin{pmatrix} x_{\sigma(1)} \\ \vdots \\ x_{\sigma(N)} \end{pmatrix} = 0 \quad \text{and} \quad A^* \begin{pmatrix} y_{\sigma(1)} \\ \vdots \\ y_{\sigma(N)} \end{pmatrix} = 0.$$

In our proofs, we will often use the following elementary fact on zero Hankel operators:

$$H_u = 0 \iff u \in H^\infty \iff H_u 1 = 0 \iff H_u^* 1 = 0 \tag{3}$$

for $u \in L^\infty(\mathbb{T})$.

Given $u \in L^\infty(\mathbb{T})$, we let M_u denote the operator of multiplication by u. Also recall $Ju = \bar{z} u(\bar{z})$ and $\hat{u}(z) = u(\bar{z})$. Then J has the following useful property on L^2 which can be easily checked:

$$JP = (I - P)J. \tag{4}$$

In the following proposition, we have some useful connections between Toeplitz operators and Hankel operators. They are known but we include the proofs for completeness.

Proposition 2.2. *Let $u, v \in L^\infty(\mathbb{T})$. Then:*

(a) $H_{\hat{u}} H_v = T_{uv} - T_u T_v$.
(b) $T_{\hat{u}} H_v + H_u T_v = H_{uv}$.
(c) *If $u \in H^\infty$, then $T_{\hat{u}} H_v = H_{uv} = H_v T_u$.*

Proof. By (4), we first note $JPJ = (I - P)J^2 = I - P$. Since $H_{\hat{u}} = PM_u J$ and $H_v = PJM_v$, we have

$$\begin{aligned} H_{\hat{u}} H_v &= [PM_u J][PJM_v] = PM_u(I - P)M_v \\ &= PM_{uv} - PM_u PM_v = T_{uv} - T_u T_v, \end{aligned}$$

so (a) holds. To prove (b), we note $M_{\hat{u}} = JM_uJ$. It follows that

$$\begin{aligned} T_{\hat{u}}H_v &= PM_{\hat{u}}PJM_v = PJM_uJPJM_v = PJM_u(I-P)M_v \\ &= PJM_{uv} - PJM_uPM_v = H_{uv} - H_uT_v, \end{aligned}$$

which gives (b). Finally, if $u \in H^\infty$, then $T_v = M_v$ and $H_u = 0$ by (3) and so (c) is a consequence of (b). ∎

It is easy to see that $H_{\bar{z}}$ is a rank one operator. In fact,

$$H_{\bar{z}} = 1 \otimes 1 \quad \text{on } H^2. \tag{5}$$

The following proposition will be very useful in our characterizations. Note that $\hat{z} = \bar{z}$ for all $z \in \mathbb{T}$.

Proposition 2.3. *Let $u, v \in L^\infty(\mathbb{T})$. Then:*

(a) $H_uT_vT_z = T_{\bar{z}}H_uT_v + (H_u1) \otimes (H_v^*1)$.
(b) $T_uH_vT_z = T_{\bar{z}}T_uH_v - (H_{\hat{u}}1) \otimes (H_v^*1)$.

Proof. By Proposition 2.2, (5) and (2), we see that

$$\begin{aligned} H_uT_vT_z &= H_uT_{zv} = H_u[T_zT_v + H_{\bar{z}}H_v] = T_{\bar{z}}H_uT_v + H_u(1\otimes 1)H_v \\ &= T_{\bar{z}}H_uT_v + (H_u1) \otimes (H_v^*1), \end{aligned}$$

so (a) holds. To prove (b), we first note from Proposition 2.2(a) and (3) that $T_{\bar{z}u} - T_{\bar{z}}T_u = H_zH_u = 0$ and thus $T_{\bar{z}}T_u = T_{\bar{z}u}$. Now by a similar argument to the proof of (a), we see that

$$\begin{aligned} T_uH_vT_z &= T_uT_{\bar{z}}H_v = [T_{\bar{z}u} - H_{\hat{u}}H_{\bar{z}}]H_v = T_{\bar{z}}T_uH_v - H_{\hat{u}}(1\otimes 1)H_v \\ &= T_{\bar{z}}T_uH_v - (H_{\hat{u}}1) \otimes (H_v^*1), \end{aligned}$$

so (b) holds. ∎

3. Main results. In this section, we characterize the zero property of operators which are finite sums of products of Toeplitz and Hankel operators. We first consider operators which are sums of operators of the form H_uT_v.

Given a set X and an integer $N \geq 1$, we let X^N be the set of all N-tuples $(x_1, \ldots, x_N)$ where $x_j \in X$ for each j.

Theorem 3.1. *Let $u_j, v_j \in L^\infty(\mathbb{T})$ for $j = 1, \ldots, N$. Then*

$$\sum_{j=1}^{N} H_{u_j}T_{v_j} = 0 \tag{6}$$

on H^2 if and only if there exist $A \in \mathbb{M}_N$ and $\sigma \in \mathbb{S}_N$ such that:

(a) $[A - I]U_\sigma^T \in (H^\infty)^N$.
(b) $\bar{A}^*V_\sigma^T \in (H^\infty)^N$.
(c) $V_\sigma A U_\sigma^T \in H^\infty$.

Here $U_\sigma = (u_{\sigma(1)}, \ldots, u_{\sigma(N)})$ and $V_\sigma = (v_{\sigma(1)}, \ldots, v_{\sigma(N)})$ for simplicity.

Proof. First assume (6) holds. By Proposition 2.3(a), we see that

$$H_{u_j}T_{v_j}T_z = T_{\bar{z}}H_{u_j}T_{v_j} + (H_{u_j}1)\otimes(H_{v_j}^*1)$$

for each j and by (6),

$$\sum_{j=1}^{N} H_{u_j}1\otimes H_{v_j}^*1 = 0.$$

By Lemma 2.1 there exist $A=[a_{ij}]\in\mathbb{M}_N$ and $\sigma\in\mathbb{S}_N$ such that

$$[A-I](H_{u_{\sigma(1)}}1,\ldots,H_{u_{\sigma(N)}}1)^T = 0, \tag{7}$$

$$A^*(H_{v_{\sigma(1)}}^*1,\ldots,H_{v_{\sigma(N)}}^*1)^T = 0. \tag{8}$$

It follows from (7) that

$$H_{\sum_{j=1}^N a_{ij}u_{\sigma(j)}}1 = \sum_{j=1}^{N} a_{ij}H_{u_{\sigma(j)}}1 = H_{u_{\sigma(i)}}1,$$

so

$$H_{\sum_{j=1}^N a_{ij}u_{\sigma(j)}-u_{\sigma(i)}}1 = 0$$

for each $i=1,\ldots,N$. By (3), we have

$$\sum_{j=1}^{N} a_{ij}u_{\sigma(j)} - u_{\sigma(i)} \in H^\infty$$

for each i. This shows that $[A-I]U_\sigma^T\in(H^\infty)^N$ where $U_\sigma=(u_{\sigma(1)},\ldots,u_{\sigma(N)})$ and (a) holds. Also, since $H_{\alpha g}^* = \bar{\alpha}H_g^*$ for all $g\in L^\infty(\mathbb{T})$ and $\alpha\in\mathbb{C}$, we have, using (8),

$$H^*_{\sum_{i=1}^N a_{ij}v_{\sigma(i)}}1 = \sum_{i=1}^{N}\overline{a_{ij}}H_{v_{\sigma(i)}}^*1 = 0$$

and hence

$$\sum_{i=1}^{N} a_{ij}v_{\sigma(i)} \in H^\infty$$

for each j by (3), so (b) holds. To prove (c), we let

$$(x_1,\ldots,x_N)^T = [A-I]U_\sigma^T, \qquad (y_1,\ldots,y_n)^T = \bar{A}^*V_\sigma^T$$

for simplicity. Then, since

$$x_i = \sum_{j=1}^{N} a_{ij}u_{\sigma(j)} - u_{\sigma(i)}$$

for each i, we have

$$\sum_{i=1}^{N} H_{x_i} T_{v_{\sigma(i)}} = \sum_{i=1}^{N} \Big[\sum_{j=1}^{N} a_{ij} H_{u_{\sigma(j)}} - H_{u_{\sigma(i)}} \Big] T_{v_{\sigma(i)}} \tag{9}$$
$$= \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} H_{u_{\sigma(j)}} T_{v_{\sigma(i)}} - \sum_{i=1}^{N} H_{u_{\sigma(i)}} T_{v_{\sigma(i)}}$$
$$= \sum_{j=1}^{N} H_{u_{\sigma(j)}} T_{\sum_{i=1}^{N} a_{ij} v_{\sigma(i)}} - \sum_{i=1}^{N} H_{u_i} T_{v_i}$$
$$= \sum_{j=1}^{N} H_{u_{\sigma(j)}} T_{y_j} - \sum_{i=1}^{N} H_{u_i} T_{v_i}.$$

Since $x_j, y_j \in H^\infty$ by (b), we have $H_{x_j} = 0$ by (3) and $H_{u_{\sigma(j)}} T_{y_j} = H_{u_{\sigma(j)} y_j}$ by Proposition 2.2(c) for each j. Also, since

$$\sum_{j=1}^{N} H_{u_j} T_{v_j} = 0$$

by assumption, it follows from (9) that

$$0 = \sum_{j=1}^{N} H_{u_{\sigma(j)}} T_{y_j} = \sum_{j=1}^{N} H_{u_{\sigma(j)} y_j} = H_{\sum_{j=1}^{N} u_{\sigma(j)} y_j},$$

so that $\sum_{j=1}^{N} u_{\sigma(j)} y_j \in H^\infty$ by (3). On the other hand, since $y_i = \sum_{j=1}^{N} a_{ji} v_{\sigma(j)}$ for each i, we have

$$\sum_{i=1}^{N} u_{\sigma(i)} y_i = \sum_{i=1}^{N} \sum_{j=1}^{N} u_{\sigma(i)} a_{ji} v_{\sigma(j)} = \sum_{j=1}^{N} v_{\sigma(j)} \sum_{i=1}^{N} a_{ji} u_{\sigma(i)} = V_\sigma A U_\sigma^T \tag{10}$$

and thus (c) follows.

Now suppose (a)–(c) hold. Let

$$(x_1, \ldots, x_N)^T = [A - I] U_\sigma^T, \qquad (y_1, \ldots, y_n)^T = \bar{A}^* V_\sigma^T,$$

as before. Note that $x_j, y_j \in H^\infty$ for each j by (a) and (b). Hence $H_{x_j} = 0$ and $H_{u_{\sigma(j)}} T_{y_j} = H_{u_{\sigma(j)} y_j}$ for each j as before. It follows from (9) and (10) that

$$\sum_{i=1}^{N} H_{u_i} T_{v_i} = \sum_{j=1}^{N} H_{u_{\sigma(j)} y_j} = H_{\sum_{j=1}^{N} u_{\sigma(j)} y_j} = H_{V_\sigma A U_\sigma^T} = 0$$

by condition (c) together with (3). Thus (6) holds, as desired. ∎

As the special case when $N = 2$ in Theorem 3.1, we obtain a more concrete characterization.

COROLLARY 3.2. *Let $u, v, \varphi, \psi \in L^\infty(\mathbb{T})$. Then $H_u T_v = H_\varphi T_\psi$ on H^2 if and only if one of the following statements holds:*

(a) $u, \varphi \in H^\infty$.
(b) $v, \psi, uv - \varphi\psi \in H^\infty$.
(c) $v, \varphi, uv \in H^\infty$.
(d) $u, \varphi\psi, \psi \in H^\infty$.
(e) $u + \beta\varphi, \varphi(\psi + \beta v), \psi + \beta v \in H^\infty$ *for some nonzero constant* β.

Proof. First suppose $H_u T_v = H_\varphi T_\psi$. By Theorem 3.1 (with σ being the identity permutation without loss of generality), we have

$$
\begin{aligned}
(a-1)u - b\varphi &\in H^\infty,\\
cu - (d-1)\varphi &\in H^\infty,\\
c\psi + av &\in H^\infty,\\
d\psi + bv &\in H^\infty
\end{aligned}
\tag{11}
$$

for some constants a, b, c and d. If $u \in H^\infty$ and $b \neq 0$, then the first line above shows $\varphi \in H^\infty$ and (a) holds. If $u \in H^\infty$, $b = 0$ and $d \neq 0$, then the fourth line above shows $\psi \in H^\infty$. By (3) and Proposition 2.2(c), $H_{\varphi\psi} = 0$ and so $\varphi\psi \in H^\infty$. Thus (d) holds. If $u \in H^\infty$ and $b = d = 0$, then the second line above shows $\varphi \in H^\infty$, so (a) holds. Therefore, if $u \in H^\infty$, then (a) or (d) holds. Similarly, if $v \in H^\infty$, then (b) or (c) holds. Also, if $\varphi \in H^\infty$, then (a) or (c) holds. Finally, if $\psi \in H^\infty$, then (b) or (d) holds.

Now, assume $u, \varphi \notin H^\infty$ and $v, \psi \notin H^\infty$. If $a-1 = b = c = d-1 = 0$, then the third and fourth conditions in (11) show $v, \psi \in H^\infty$, which is impossible. Thus one of $a-1, b, c, d-1$ is nonzero. On the other hand, using the first two conditions in (11), we see that $a \neq 1$ if and only if $b \neq 0$, and $c \neq 0$ if and only if $d \neq 1$. Thus we have $u + \epsilon\varphi \in H^\infty$ where $\epsilon = -b/(a-1)$ or $\epsilon = -(d-1)/c$. Also, if $a = b = c = d = 0$, then the first two conditions in (11) show $u, \varphi \in H^\infty$, which is impossible as well. So, one of a, b, c, d is nonzero. By the same argument as above we see that $\psi + \delta v \in H^\infty$, where $\delta = a/c$ or $\delta = b/d$. By (11), we have $(a-1)(d-1) = bc = ad$ and hence $a + d = 1$. Using this fact, we see that $\epsilon = \delta$ for any $\epsilon \in \{-b/(a-1), -(d-1)/c\}$ and $\delta \in \{a/c, b/d\}$. Since $u + \epsilon\varphi \in H^\infty$ and $\psi + \delta v \in H^\infty$, we have $H_{u+\epsilon\varphi} = 0$ and $H_\varphi T_{\psi+\delta v} = H_{\varphi(\psi+\delta v)}$ by (3) and Proposition 2.2(c). It follows that

$$
\begin{aligned}
H_u T_v &= (H_{u+\epsilon\varphi} - \epsilon H_\varphi) T_v = -\epsilon H_\varphi T_v,\\
H_\varphi T_\psi &= H_\varphi (T_{\psi+\delta v} - \delta T_v) = H_{\varphi(\psi+\delta v)} - \delta H_\varphi T_v.
\end{aligned}
\tag{12}
$$

As $H_u T_v = H_\varphi T_\psi$ and $\epsilon = \delta$, we have $H_{\varphi(\psi+\delta v)} = 0$ and so $\varphi(\psi + \delta v) \in H^\infty$ by (3). So (e) follows with $\beta = \epsilon = \delta$.

Conversely, suppose one of the conditions (a)–(e) holds. If one of (a)–(d) holds, we have $H_u T_v = H_\varphi T_\psi$ by Proposition 2.2(c) and (3). If (e) holds, then (12) with $\beta = \epsilon = \delta$ shows that $H_u T_v = H_\varphi T_\psi$. ∎

Note T_1 is the identity operator. Taking $\psi = 1$ in Corollary 3.2, we characterize when the product of a Hankel operator and a Toeplitz operator is another Hankel operator. Also, if we take $\varphi = 0$, we characterize the zero product of a Hankel operator and a Toeplitz operator. The following recovers Theorem 3.2 of [2].

Corollary 3.3. *Let $u, v, \varphi \in L^\infty(\mathbb{T})$. Then:*

(a) *$H_u T_v = H_\varphi$ on H^2 if and only if either $u, \varphi \in H^\infty$ or $v, uv-\varphi \in H^\infty$.*
(b) *$H_u T_v = 0$ on H^2 if and only if either $u \in H^\infty$ or $v, uv \in H^\infty$.*

Proof. If $H_u T_v = H_\varphi$, then one of the conditions (a)–(e) in Corollary 3.2 (with $\psi = 1$) holds. If one of (a)–(d) holds, we can easily see that either $u, \varphi \in H^\infty$ or $v, uv - \varphi \in H^\infty$. If (e) holds, there exists a nonzero constant β such that $u + \beta\varphi, \varphi(1 + \beta v), 1 + \beta v \in H^\infty$. Hence $v \in H^\infty$ and then $uv + \beta v\varphi \in H^\infty$. Since $\varphi(1 + \beta v) \in H^\infty$, we have $v, uv - \varphi \in H^\infty$.

The converse follows from (3) and Proposition 2.2(c). ∎

We now consider operators which are sums of operators of the form $T_u H_v$. In the following, $\overline{H^\infty}$ denotes the set of all functions $\bar{f}$ for $f \in H^\infty$. Note that $\hat{f} \in H^\infty$ if and only if $f \in \overline{H^\infty}$.

Theorem 3.4. *Let $u_j, v_j \in L^\infty(\mathbb{T})$ for $j = 1, \ldots, N$. Then*

$$\sum_{j=1}^{N} T_{u_j} H_{v_j} = 0 \tag{13}$$

on H^2 if and only if there exist $A \in \mathbb{M}_N$ and $\sigma \in \mathbb{S}_N$ such that:

(a) *$[A - I]U_\sigma^T \in (\overline{H^\infty})^N$.*
(b) *$\bar{A}^* V_\sigma^T \in (H^\infty)^N$.*
(c) *$V_\sigma [A - I]\hat{U}_\sigma^T \in H^\infty$.*

Here $U_\sigma = (u_{\sigma(1)}, \ldots, u_{\sigma(N)})$, $V_\sigma = (v_{\sigma(1)}, \ldots, v_{\sigma(N)})$ and $\hat{U}_\sigma = (\hat{u}_{\sigma(1)}, \ldots, \hat{u}_{\sigma(N)})$.

Proof. First assume (13) holds. By Proposition 2.3(b), we have

$$T_{u_j} H_{v_j} T_z = T_{\bar{z}} T_{u_j} H_{v_j} - (H_{\hat{u}_j} 1) \otimes (H_{v_j}^* 1)$$

for each j and so by (13),

$$\sum_{j=1}^{N} H_{\hat{u}_j} 1 \otimes H_{v_j}^* 1 = 0.$$

Thus, by Lemma 2.1, there exist $A = [a_{ij}] \in \mathbb{M}_N$ and $\sigma \in \mathbb{S}_N$ such that

$$[A - I](H_{\hat{u}_{\sigma(1)}} 1, \ldots, H_{\hat{u}_{\sigma(N)}} 1)^T = 0, \tag{14}$$

$$A^*(H_{v_{\sigma(1)}}^* 1, \ldots, H_{v_{\sigma(N)}}^* 1)^T = 0. \tag{15}$$

Using (14), we see

$$H_{\sum_{j=1}^N a_{ij}\hat{u}_{\sigma(j)}}1 = \sum_{j=1}^N a_{ij} H_{\hat{u}_{\sigma(j)}}1 = H_{\hat{u}_{\sigma(i)}}1,$$

and hence

$$H_{[\sum_{j=1}^N a_{ij}\widehat{u_{\sigma(j)} - u_{\sigma(i)}}]}1 = 0$$

for each $i = 1, \dots, N$. Thus, by (3), we have

$$\sum_{j=1}^N a_{ij}u_{\sigma(j)} - u_{\sigma(i)} \in \overline{H^\infty}$$

for each i, and (a) follows. Recall $H^*_{\alpha g} = \bar{\alpha} H^*_g$ for all $g \in L^\infty(\mathbb{T})$ and $\alpha \in \mathbb{C}$. Letting $V_\sigma = (v_{\sigma(1)}, \dots, v_{\sigma(N)})$ in (15) gives

$$H^*_{\sum_{i=1}^N a_{ij}v_{\sigma(i)}}1 = \sum_{i=1}^N \overline{a_{ij}} H^*_{v_{\sigma(i)}}1 = 0$$

for every j, which implies $\bar{A}^* V_\sigma^T \in (H^\infty)^N$ by (3) again, thus (b) holds. To show (c), let $(x_1, \dots, x_N)^T = [A - I]U_\sigma^T$ and $(y_1, \dots, y_n)^T = \bar{A}^* V_\sigma^T$ as before. Then we obtain

$$\begin{aligned} \sum_{i=1}^N T_{x_i} H_{v_{\sigma(i)}} &= \sum_{i=1}^N \Big[\sum_{j=1}^N a_{ij} T_{u_{\sigma(j)}} - T_{u_{\sigma(i)}} \Big] H_{v_{\sigma(i)}} \\ &= \sum_{i=1}^N \sum_{j=1}^N a_{ij} T_{u_{\sigma(j)}} H_{v_{\sigma(i)}} - \sum_{i=1}^N T_{u_{\sigma(i)}} H_{v_{\sigma(i)}} \\ &= \sum_{j=1}^N T_{u_{\sigma(j)}} H_{\sum_{i=1}^N a_{ij} v_{\sigma(i)}} - \sum_{i=1}^N T_{u_i} H_{v_i} \\ &= \sum_{j=1}^N T_{u_{\sigma(j)}} H_{y_j} - \sum_{i=1}^N T_{u_i} H_{v_i}. \end{aligned} \tag{16}$$

Since $x_j \in \overline{H^\infty}$ by (a) and $y_j \in H^\infty$ by (b), we have $H_{y_j} = 0$ and $T_{x_i} H_{v_{\sigma(i)}} = H_{\hat{x}_i v_{\sigma(i)}}$ for each j by (3) and Proposition 2.2(c), respectively. It follows from (13) and (16) that

$$0 = \sum_{i=1}^N T_{x_i} H_{v_{\sigma(i)}} = \sum_{i=1}^N H_{\hat{x}_i v_{\sigma(i)}} = H_{\sum_{i=1}^N \hat{x}_i v_{\sigma(i)}}.$$

Thus $\sum_{i=1}^N \hat{x}_i v_{\sigma(i)} \in H^\infty$ by (3). On the other hand, since

$$\hat{x}_i = \sum_{j=1}^N a_{ij} \hat{u}_{\sigma(j)} - \hat{u}_{\sigma(i)}$$

for each i, we have

$$\sum_{i=1}^{N} \hat{x}_i v_{\sigma(i)} = V_\sigma [A - I] \hat{U}_\sigma^T \tag{17}$$

and thus (c) holds.

To prove the converse, suppose that (a)–(c) hold. Let $(x_1, \dots, x_N)^T = [A - I]U_\sigma^T$ and $(y_1, \dots, y_n)^T = \bar{A}^* V_\sigma^T$ as before. Notice that $x_j \in \overline{H^\infty}$ and $y_j \in H^\infty$ for each j by (a) and (b). Hence $H_{y_j} = 0$ and $T_{x_i} H_{v_{\sigma(i)}} = H_{\hat{x}_i v_{\sigma(i)}}$ for each j as before. It follows from (16) and (17) that

$$\sum_{i=1}^{N} T_{u_i} H_{v_i} = -\sum_{i=1}^{N} H_{\hat{x}_i v_{\sigma(i)}} = -H_{\sum_{i=1}^{N} \hat{x}_i v_{\sigma(i)}} = -H_{V_\sigma [A-I] \hat{U}_\sigma^T} = 0$$

by (c) together with (3). Thus (13) holds and the proof is complete. ■

As before, in the special case $N = 2$ in Theorem 3.4, we have the following characterization.

Corollary 3.5. *Let $u, v, \varphi, \psi \in L^\infty(\mathbb{T})$. Then $T_u H_v = T_\varphi H_\psi$ if and only if one of the following statements holds:*

(a) $u, \varphi \in \overline{H^\infty}$ *and* $\hat{u}v - \hat{\varphi}\psi \in H^\infty$.
(b) $u \in \overline{H^\infty}$ *and* $\psi, \hat{u}v \in H^\infty$.
(c) $v, \psi \in H^\infty$.
(d) $v, \hat{\varphi}\psi \in H^\infty$ *and* $\varphi \in \overline{H^\infty}$.
(e) $u + \epsilon\varphi \in \overline{H^\infty}$ *and* $\psi + \epsilon v, (\widehat{u + \epsilon\varphi})v \in H^\infty$ *for a nonzero constant* ϵ.

Proof. Suppose first $T_u H_v = T_\varphi H_\psi$. By Theorem 3.4 (with σ being the identity permutation without loss of generality), we have

$$\begin{aligned} (a-1)u - b\varphi &\in \overline{H^\infty}, \\ cu - (d-1)\varphi &\in \overline{H^\infty}, \\ c\psi + av &\in H^\infty, \\ d\psi + bv &\in H^\infty \end{aligned} \tag{18}$$

for some constants a, b, c and d. If $u \in \overline{H^\infty}$ and $b \neq 0$, then the first line above shows $\varphi \in \overline{H^\infty}$ and so (a) holds by (3) and Proposition 2.2. If $u \in \overline{H^\infty}$, $b = 0$, and $d \neq 0$, then the last line above shows $\psi \in H^\infty$. So, by (3) and Proposition 2.2, $H_{\hat{u}v} = 0$ and so $\hat{u}v \in H^\infty$. Thus (b) holds. If $u \in \overline{H^\infty}$ and $b = d = 0$, then the second line above shows $\varphi \in \overline{H^\infty}$ and so (a) holds by (3) and Proposition 2.2 again. Therefore, if $u \in \overline{H^\infty}$, then (a) or (b) holds. Similarly, if $v \in H^\infty$, then (c) or (d) holds; if $\varphi \in \overline{H^\infty}$, then (a) or (d) holds; and if $\psi \in H^\infty$, then (b) or (c) holds.

Now, assume $u, \varphi \notin \overline{H^\infty}$ and $v, \psi \notin H^\infty$. By the same argument used in Corollary 3.2, we see that $u + \epsilon\varphi \in \overline{H^\infty}$ and $\psi + \epsilon v \in H^\infty$ for some constant

$\epsilon \neq 0$. It follows from Proposition 2.2(c) that

$$\begin{aligned} T_u H_v &= (T_{u+\epsilon\varphi} - \epsilon T_\varphi) H_v = H_{\widehat{(u+\epsilon\varphi)}v} - \epsilon T_\varphi H_v, \\ T_\varphi H_\psi &= T_\varphi (H_{\psi+\epsilon v} - \epsilon H_v) = -\epsilon T_\varphi H_v. \end{aligned} \tag{19}$$

Since $H_u T_v = H_\varphi T_\psi$, we have $H_{\widehat{(u+\epsilon\varphi)}v} = 0$ and so $(\widehat{u+\epsilon\varphi})v \in H^\infty$ by (3). Thus (e) follows.

Conversely, if one of (a)–(d) holds, we have $T_u H_v = T_\varphi H_\psi$ by Proposition 2.2 and (3). If (e) holds, (19) shows $T_u H_v = T_\varphi H_\psi$. ∎

Taking $\varphi = 1$ in Corollary 3.5, we have the following companion result of Corollary 3.3 with a similar proof.

COROLLARY 3.6. *Let $u, v, \psi \in L^\infty(\mathbb{T})$. Then:*

(a) $T_u H_v = H_\psi$ *if and only if either* $v, \psi \in H^\infty$ *or* $u \in \overline{H^\infty}$, $\hat{u}v - \psi \in H^\infty$.
(b) $T_u H_v = 0$ *if and only if either* $v \in H^\infty$ *or* $u \in \overline{H^\infty}$, $\hat{u}v \in H^\infty$.

In connection with Corollaries 3.2 and 3.5, we next consider operators of the form $H_u T_v + T_\varphi H_\psi$ and characterize when such an operator is zero.

THEOREM 3.7. *Let $u, v, \varphi, \psi \in L^\infty(\mathbb{T})$. Then $H_u T_v + T_\varphi H_\psi = 0$ if and only if one of the following statements holds:*

(a) $u, \hat{\varphi}\psi \in H^\infty$ *and* $\varphi \in \overline{H^\infty}$.
(b) $u, \psi \in H^\infty$.
(c) $v, uv + \hat{\varphi}\psi \in H^\infty, \varphi \in \overline{H^\infty}$.
(d) $v, \psi, uv \in H^\infty$.
(e) $v\hat{\varphi}, u - \alpha\hat{\varphi}, \psi - \alpha v \in H^\infty$ *for some nonzero constant* α.

Proof. First assume $H_u T_v + T_\varphi H_\psi = 0$. By Proposition 2.3, we have

$$H_u 1 \otimes H_v^* 1 = H_{\hat{\varphi}} 1 \otimes H_\psi^* 1. \tag{20}$$

If $H_u 1 = 0$, then $H_{\hat{\varphi}} 1 = 0$ or $H_\psi^* 1 = 0$. By (3), we have either $u \in H^\infty, \varphi \in \overline{H^\infty}$ or $u, \psi \in H^\infty$. If $u \in H^\infty$ and $\varphi \in \overline{H^\infty}$, then $0 = T_\varphi H_\psi = H_{\hat{\varphi}\psi}$ by (3) and Proposition 2.2. Hence $u, \hat{\varphi}\psi \in H^\infty$ and $\varphi \in \overline{H^\infty}$ by (3) again, so (a) or (b) holds. By similar arguments, $H_v^* 1 = 0$ implies (c) or (d); $H_{\hat{\varphi}} 1 = 0$ implies (a) or (c); and $H_\psi^* 1 = 0$ implies (b) or (d).

If none of $H_u 1, H_v^* 1, H_{\hat{\varphi}} 1, H_\psi^* 1$ is zero, (20) implies that there exists a nonzero constant α for which $H_u 1 = \alpha H_{\hat{\varphi}} 1$ and $H_\psi^* 1 = \bar{\alpha} H_v^* 1$. Since $\bar{\alpha} H_v^* 1 = H_{\alpha v}^* 1$, we have $u - \alpha\hat{\varphi}, \psi - \alpha v \in H^\infty$ by (3). It follows that

$$\begin{aligned} H_u T_v &= [H_{u-\alpha\hat{\varphi}} + \alpha H_{\hat{\varphi}}] T_v = \alpha H_{\hat{\varphi}} T_v, \\ T_\varphi H_\psi &= T_\varphi [H_{\psi-\alpha v} + \alpha H_v] = \alpha T_\varphi H_v. \end{aligned} \tag{21}$$

Since $\alpha \neq 0$ and $H_u T_v + T_\varphi H_\psi = 0$ by assumption, by Proposition 2.2(b) we have

$$0 = H_{\hat{\varphi}} T_v + T_\varphi H_v = H_{\hat{\varphi} v}.$$

Thus $v\hat{\varphi} \in H^\infty$ and (e) follows.

Conversely, if one of (a)–(d) holds, we deduce from (3) and Proposition 2.2 that $H_u T_v + T_\varphi H_\psi = 0$. Finally, assume (e). By (21), Proposition 2.2(b) and (3) again, we have

$$H_u T_v + T_\varphi H_\psi = \alpha[H_{\hat{\varphi}} T_v + T_\varphi H_v] = \alpha H_{v\hat{\varphi}} = 0$$

and hence $H_u T_v + T_\varphi H_\psi = 0$, as desired. ■

Finally, if we take $\varphi = -v$ and $\psi = u$ in Theorem 3.7, we characterize when a Toeplitz operator and a Hankel operator commute in the following corollary where we recover the result of Martínez-Avendaño [6]. For $f \in H^2$, we note that $\hat{f} = f$ if and only if $\hat{f} \in H^2$ if and only if f is constant.

Corollary 3.8. *Let $u, v \in L^\infty(\mathbb{T})$. Then $H_u T_v = T_v H_u$ if and only if one of the following statements holds:*

(a) *$u \in H^\infty$.*
(b) *$v\hat{v}, v + \hat{v}$ are constants and $v + \alpha u \in H^\infty$ for some constant α.*

Proof. If H_u and T_v commute, then one of the conditions (a)–(e) in Theorem 3.7 holds (with $\varphi = -v, \psi = u$). If one of (a), (b) and (d) holds, we have $u \in H^\infty$. Also, if (c) holds, then v is constant and (b) in Corollary 3.8 holds with $\alpha = 0$. Also, if (e) holds, there exists a constant $\alpha \neq 0$ such that $v\hat{v}, u + \alpha\hat{v}, u - \alpha v \in H^\infty$. Hence $v - \alpha^{-1}u \in H^\infty$ and $\alpha(\hat{v} + v) \in H^\infty$. Since $\alpha \neq 0$, we have $\hat{v} + v \in H^\infty$ and so $\hat{v} + v$ is constant because $\hat{v} + v = \widehat{\hat{v} + v}$. Now, it remains to show $v\hat{v}$ is also constant. To do so, we use an idea in [6]. Let $\hat{v} + v = \delta$ for some constant δ. Then, since $v = \delta - \hat{v}$ and $H_u T_v = T_v H_u$, we have $H_u T_{\hat{v}} = T_{\hat{v}} H_u$ and so $H_u T_{\hat{v}} T_v = T_{\hat{v}} T_v H_u$. Also, $H_u = \alpha H_v$ because $u - \alpha v \in H^\infty$ and $T_{\hat{v}v} = H_v H_v + T_{\hat{v}} T_v$ by Proposition 2.2. It follows that

$$\begin{aligned} H_u T_{\hat{v}v} &= H_u H_v H_v + H_u T_{\hat{v}} T_v = \alpha H_v H_v H_v + T_{\hat{v}} T_v H_u \\ &= [H_v H_v + T_{\hat{v}} T_v] H_u = T_{\hat{v}v} H_u \end{aligned}$$

and hence H_u commutes with $T_{\hat{v}v}$. By the proof just before, either $u \in H^\infty$ or $\hat{v}v = \frac{1}{2}[\hat{v}v + \widehat{\hat{v}v}]$ is constant. Thus (a) or (b) holds.

Conversely, if (a) holds, then $H_u = 0$ by (3) and $H_u T_v = T_v H_u$. Assume (b) holds. If $\alpha = 0$, then $v, \hat{v} \in H^\infty$ and hence v is constant. Then clearly (a) holds. Now assume $\alpha \neq 0$ and let $v + \hat{v} = \epsilon$ for some constant ϵ. Then $0 = H_{v+\alpha u} = H_v + \alpha H_u$ and hence $H_v = -\alpha H_u$. Since $v\hat{v} \in H^\infty$, by (3) and

Proposition 2.2(b), we have

$$\begin{aligned} 0 = H_{v\hat{v}} &= T_v H_v + H_{\hat{v}} T_v = T_v H_v + H_{\epsilon - v} T_v \\ &= T_v H_v - H_v T_v = -\alpha [T_v H_u - H_u T_v] \end{aligned}$$

and thus $H_u T_v = T_v H_u$ because $\alpha \neq 0$. ∎

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