BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 63, No. 2, 2015

The Hyers–Ulam–Aoki Type Stability of Some Functional Equations on Banach Lattices

by

Nutefe Kwami AGBEKO

Presented by Stanisław KWAPIEŃ

Summary. In Agbeko (2012) the Hyers–Ulam–Aoki stability problem was posed in Banach lattice environments with the addition in the Cauchy functional equation replaced by supremum. In the present note we restate the problem so that it relates not only to supremum but also to infimum and their various combinations. We then propose some sufficient conditions which guarantee its solution.

1. Introduction. In what follows, $(\mathcal{X}, \wedge_{\mathcal{X}}, \vee_{\mathcal{X}})$ will denote a vector lattice and $(\mathcal{Y}, \wedge_{\mathcal{Y}}, \vee_{\mathcal{Y}})$ a Banach lattice with $\mathcal{X}^+$ and $\mathcal{Y}^+$ their respective positive cones.

In the style of the Cauchy functional equation, consider the following operator equation:

$$T(|x| \vartriangle_{\mathcal{X}}^{*} |y|) \vartriangle_{\mathcal{Y}}^{*} T(|x| \vartriangle_{\mathcal{X}}^{**} |y|) = T(|x|) \vartriangle_{\mathcal{Y}}^{**} T(|y|) \tag{1.1}$$

for all $x, y \in \mathcal{X}$, where $\vartriangle_{\mathcal{X}}^{*}, \vartriangle_{\mathcal{X}}^{**} \in \{\wedge_{\mathcal{X}}, \vee_{\mathcal{X}}\}$ and $\vartriangle_{\mathcal{Y}}^{*}, \vartriangle_{\mathcal{Y}}^{**} \in \{\wedge_{\mathcal{Y}}, \vee_{\mathcal{Y}}\}$ are fixed lattice operations and $T : \mathcal{X} \to \mathcal{Y}$ is a map.

Note that if the above four lattice operations are all the supremum (join) or the infimum (meet), then the functional equation (1.1) is just the definition of a join-homomorphism or a meet-homomorphism. Moreover, if $\vartriangle_{\mathcal{X}}^{*}$ and $\vartriangle_{\mathcal{X}}^{**}$ are the same, then the left hand side of (1.1) is the map of the meets or the joins.

Problem. Given lattice operations $\vartriangle_{\mathcal{X}}^{*}, \vartriangle_{\mathcal{X}}^{**} \in \{\wedge_{\mathcal{X}}, \vee_{\mathcal{X}}\}$ and $\vartriangle_{\mathcal{Y}}^{*}, \vartriangle_{\mathcal{Y}}^{**} \in \{\wedge_{\mathcal{Y}}, \vee_{\mathcal{Y}}\}$, a vector lattice G_1, a vector lattice G_2 endowed with a metric $d(\cdot,\cdot)$ and a positive number ε, does there exist some $\delta > 0$ such that, if a

2010 *Mathematics Subject Classification*: Primary 39B82; Secondary 46B30, 46B42.
Key words and phrases: Banach lattices, Hyers–Ulam–Aoki stability.

DOI: 10.4064/ba63-2-6

mapping $F : G_1 \to G_2$ satisfies

$$d(F(|x| \vartriangle^*_{\mathcal{X}} |y|) \vartriangle^*_{\mathcal{Y}} F(|x| \vartriangle^{**}_{\mathcal{X}} |y|), F(|x|) \vartriangle^{**}_{\mathcal{Y}} F(|y|)) \leq \delta$$

for all $x, y \in G_1$, then there exists an operation-preserving functional $T : G_1 \to G_2$ with

$$d(T(x), F(x)) \leq \varepsilon$$

for all $x \in G_1$?

This problem can be viewed as a lattice version of Ulam's stability problem formulated in [12].

Our aim is to provide some conditions that ensure that there exists a unique solution to the above problem, but from the Hyers–Ulam–Aoki stability point of view (cf. [5, 6, 8, 10, 11]).

The motivation of dealing with functional equations and inequalities in lattice environments lies in the fact that many addition-related results or theorems can be extended and proved mutatis mutandis. For more references about the earliest extensions of this kind we refer the reader to [1, 2, 4]. Let us recall the following definition.

We say that a map $T : \mathcal{X} \to \mathcal{Y}$ is *cone-related* if

$$T(\mathcal{X}^+) = \{T(|x|) : x \in \mathcal{X}\} \subset \mathcal{Y}^+$$

(for more about this notion see [3, 4]).

Our theorems will be deduced from the following result of Forti [7].

Theorem 1.1 (Forti). *Let (X, d) be a complete metric space and S a set. Assume that a function $f : S \to X$ satisfies*

$$d(H(f(G(x))), f(x)) \leq \delta(x) \tag{1.2}$$

for all $x \in S$, where $G : X \to X$ and $\delta : S \to [0, \infty)$ are some maps and $H : X \to X$ is a continuous map satisfying

$$d(H(u), H(v)) \leq \varphi(d(u, v)), \quad u, v \in X, \tag{1.3}$$

for a certain non-decreasing subadditive function $\varphi : [0, \infty) \to [0, \infty)$ such that the series

$$\sum_{j=0}^{\infty} \varphi^j(\delta(G^j(x))) \tag{1.4}$$

is convergent for every $x \in S$. Then there exists a unique solution $F : S \to X$ of the functional equation

$$H(F(G(x))) = F(x), \quad x \in S, \tag{1.5}$$

satisfying

$$d(F(x), f(x)) \leq \sum_{j=0}^{\infty} \varphi^j(\delta(G^j(x))). \tag{1.6}$$

The map F is given by

$$F(x) = \lim_{n\to\infty} H^n(f(G^n(x))). \tag{1.7}$$

2. The main results

THEOREM 2.1. *Consider a cone-related mapping $F : \mathcal{X} \to \mathcal{Y}$ for which there are $\vartheta > 0$ and $\alpha \in (-\infty, 1)$ such that*

$$\left\| \frac{F(|x| \bigtriangleup_{\mathcal{X}}^{*} |y|) \bigtriangleup_{\mathcal{Y}}^{*} F(|x| \bigtriangleup_{\mathcal{X}}^{**} |y|)}{\tau} - F\left(\frac{|x|}{\tau}\right) \bigtriangleup_{\mathcal{Y}}^{**} F\left(\frac{|y|}{\tau}\right) \right\| \le \frac{\vartheta}{4}(\|x\|^{\alpha} + \|y\|^{\alpha}) \tag{2.1}$$

for all $x, y \in \mathcal{X}$ and $\tau \in (0, \infty)$, where $\bigtriangleup_{\mathcal{X}}^{*}, \bigtriangleup_{\mathcal{X}}^{**} \in \{\wedge_{\mathcal{X}}, \vee_{\mathcal{X}}\}$ and $\bigtriangleup_{\mathcal{Y}}^{*}, \bigtriangleup_{\mathcal{Y}}^{**} \in \{\wedge_{\mathcal{Y}}, \vee_{\mathcal{Y}}\}$ are fixed lattice operations. Then $(2^{-n}F(2^n|x|))_n$ is a Cauchy sequence for every $x \in \mathcal{X}$. Moreover, define $T : \mathcal{X} \to \mathcal{Y}$ by

$$T(|x|) = \lim_{n\to\infty} 2^{-n}F(2^n|x|). \tag{2.2}$$

Then

(a) *T is semi-homogeneous, i.e. $T(\gamma|x|) = \gamma T(|x|)$ for all $x \in \mathcal{X}$ and all $\gamma \in [0, \infty)$;*

(b) *T is the unique cone-related map satisfying both (1.1) and*

$$\|T(|x|) - F(|x|)\| \le \frac{2^{\alpha}\vartheta}{2 - 2^{\alpha}}\|x\|^{\alpha} \tag{2.3}$$

for every $x \in \mathcal{X}$.

THEOREM 2.2. *Consider a cone-related map $F : \mathcal{X} \to \mathcal{Y}$ for which there are $\vartheta > 0$ and $p \in (1, \infty)$ such that*

$$\|\tau(F(|x| \bigtriangleup_{\mathcal{X}}^{*} |y|) \bigtriangleup_{\mathcal{Y}}^{*} F(|x| \bigtriangleup_{\mathcal{X}}^{**} |y|)) - F(\tau|x|) \bigtriangleup_{\mathcal{Y}}^{**} F(\tau|y|)\| \le \vartheta(\|x\|^{p} + \|y\|^{p}) \tag{2.4}$$

for all $x, y \in \mathcal{X}$ and $\tau \in (0, \infty)$, where $\bigtriangleup_{\mathcal{X}}^{*}, \bigtriangleup_{\mathcal{X}}^{**} \in \{\wedge_{\mathcal{X}}, \vee_{\mathcal{X}}\}$ and $\bigtriangleup_{\mathcal{Y}}^{*}, \bigtriangleup_{\mathcal{Y}}^{**} \in \{\wedge_{\mathcal{Y}}, \vee_{\mathcal{Y}}\}$ are fixed lattice operations. Then $(2^{n}F(2^{-n}|x|))_n$ is a Cauchy sequence for every $x \in \mathcal{X}$. Moreover, define $T : \mathcal{X} \to \mathcal{Y}$ by

$$T(|x|) = \lim_{n\to\infty} 2^{n}F(2^{-n}|x|). \tag{2.5}$$

Then

(a) *T is semi-homogeneous, i.e. $T(\gamma|x|) = \gamma T(|x|)$ for all $x \in \mathcal{X}$ and all $\gamma \in [0, \infty)$;*

(b) *T is the unique cone-related map satisfying both (1.1) and*

$$\|T(|x|) - F(|x|)\| \le \frac{2^{p}\vartheta}{2^{p} - 2}\|x\|^{p} \tag{2.6}$$

for every $x \in \mathcal{X}$.

The following obvious remarks will be used repeatedly.

Remark 2.1. If the conditions of Theorem 2.1 or Theorem 2.2 hold true, then $F(0) = 0$.

Remark 2.2. Let Z be a set closed under scalar multiplication, i.e. $bz \in Z$ whenever $b \in \mathbb{R}$ and $z \in Z$. Given $c \in \mathbb{R}$ define $\gamma : Z \to Z$ by $\gamma(z) = cz$. Then the jth iteration of γ, $\gamma^j : Z \to Z$, is given by $\gamma^j(z) = c^j z$ for every $j \geq 2$.

3. The proof of the main theorems. *Proof of Theorem 2.1.* First, if we choose $\tau = 2$, $y = x$ and replace x by $2x$ in (2.1) then obviously

$$\left\| \frac{F(2|x|)}{2} - F(|x|) \right\| \leq \vartheta 2^{\alpha-1} \|x\|^\alpha. \tag{3.1}$$

Next, let us define the following maps:

$$\begin{aligned}
&G : \mathcal{X} \to \mathcal{X}, && G(|x|) = 2|x|,\\
&\delta : \mathcal{X} \to [0,\infty), && \delta(|x|) = \vartheta 2^{\alpha-1}\|x\|^\alpha,\\
&\varphi : [0,\infty) \to [0,\infty), && \varphi(t) = 2^{-1}t,\\
&H : \mathcal{Y} \to \mathcal{Y}, && H(|y|) = 2^{-1}|y|,\\
&d(\cdot,\cdot) : \mathcal{Y}\times\mathcal{Y} \to [0,\infty), && d(y_1,y_2) = \|y_1 - y_2\|.
\end{aligned}$$

We shall verify that all conditions of Forti's theorem are satisfied.

(I) From (3.1) we obviously have

$$d(H(F(G(|x|))), F(|x|)) = \left\| \frac{F(2|x|)}{2} - F(|x|) \right\| \leq \vartheta 2^{\alpha-1}\|x\|^\alpha = \delta(|x|).$$

(II) $d(H(|y_1|), H(|y_2|)) = 2^{-1}\|y_1 - y_2\| = \varphi(d(y_1,y_2))$ for all $y_1, y_2 \in \mathcal{Y}$.

(III) Clearly, φ is a non-decreasing subadditive function on $[0,\infty)$, and by applying Remark 2.2 to both the iterations G^j and φ^j of G and φ respectively, one can observe that

$$\sum_{j=0}^{\infty} \varphi^j(\delta(G^j(|x|))) = \vartheta 2^{\alpha-1}\|x\|^\alpha \sum_{j=0}^{\infty} 2^{(\alpha-1)j} = \vartheta\|x\|^\alpha \frac{2^\alpha}{2-2^\alpha} < \infty.$$

Then in view of Forti's theorem, $(H^n(F(G^n|x|)))_n$ is a Cauchy sequence for every $x \in \mathcal{X}$ and thus so is $(2^{-n}F(2^n|x|))_n$; furthermore, (2.2) is the unique mapping which satisfies (2.3).

Next, we prove (1.1). In fact, in (2.1) replace x with $2^n x$ and y with $2^n y$, and also let $\tau = 1$. Then

$$\|F(2^n(|x| \bigtriangleup^*_{\mathcal{X}} |y|)) \bigtriangleup^{**}_{\mathcal{Y}} F(2^n(|x| \bigtriangleup^{**}_{\mathcal{X}} |y|)) - F(2^n|x|) \bigtriangleup^{**}_{\mathcal{Y}} F(2^n|y|)\| \\ \leq \frac{\vartheta}{4} 2^{n\alpha}(\|x\|^\alpha + \|y\|^\alpha).$$

Dividing both sides by 2^n yields

$$\left\| \frac{F(2^n(|x| \triangle_{\mathcal{X}}^{*} |y|)) \triangle_{\mathcal{Y}}^{**} F(2^n(|x| \triangle_{\mathcal{X}}^{**} |y|))}{2^n} - \frac{F(2^n|x|) \triangle_{\mathcal{Y}}^{**} F(2^n|y|)}{2^n} \right\| \leq \frac{\vartheta}{4}(\|x\|^\alpha + \|y\|^\alpha)2^{(\alpha-1)n}. \tag{3.2}$$

Taking the limit in (3.2) we see via (2.2) that

$$\|T(|x| \triangle_{\mathcal{X}}^{*} |y|) \triangle_{\mathcal{Y}}^{*} T(|x| \triangle_{\mathcal{X}}^{**} |y|) - T(|x|) \triangle_{\mathcal{Y}}^{**} T(|y|)\| = 0,$$

which is equivalent to

$$T(|x| \triangle_{\mathcal{X}}^{*} |y|) \triangle_{\mathcal{Y}}^{*} T(|x| \triangle_{\mathcal{X}}^{**} |y|) = T(|x|) \triangle_{\mathcal{Y}}^{**} T(|y|).$$

By Remark 2.1 the identity $\gamma F(|x|) = F(\gamma|x|)$ is trivial for $\gamma = 0$ and all $x \in \mathcal{X}$, and for $x = 0$ and all $\gamma \in [0, \infty)$. Now fix $\gamma \neq 0$ and $x \in \mathcal{X} \setminus \{0\}$. In (2.1) choose $y = x$, $\tau = \gamma^{-1}$ and change x to $2^n x$. Then

$$\|\gamma F(2^n|x|) - F(\gamma 2^n|x|)\| \leq \frac{\vartheta}{2}\|x\|^\alpha 2^{n\alpha}.$$

Divide both sides by 2^n to get

$$\|\gamma 2^{-n} F(2^n|x|) - 2^{-n} F(\gamma 2^n|x|)\| \leq \frac{\vartheta}{2}\|x\|^\alpha 2^{(\alpha-1)n}. \tag{3.3}$$

By taking the limit in (3.3) we find via (2.2) that

$$\|\gamma T(|x|) - T(\gamma|x|)\| = 0,$$

or equivalently

$$T(\gamma|x|) = \gamma T(|x|)$$

for all $x \in \mathcal{X}$. We have thus shown the semi-homogeneity of T. This completes the argument. ∎

Proof of Theorem 2.2. First, if we choose $\tau = 2$, $y = x$ and replace x by $2^{-1}x$ in (2.4) then we obviously have

$$\|2F(2^{-1}|x|) - F(|x|)\| \leq \vartheta 2^{1-p}\|x\|^p. \tag{3.4}$$

Next, let us define the following functions:

$$\begin{aligned}
&G : \mathcal{X} \to \mathcal{X}, && G(|x|) = 2^{-1}|x|,\\
&\delta : \mathcal{X} \to [0, \infty), && \delta(|x|) = \vartheta 2^{1-p}\|x\|^p,\\
&\varphi : [0, \infty) \to [0, \infty), && \varphi(t) = 2t,\\
&H : \mathcal{Y} \to \mathcal{Y}, && H(|y|) = 2|y|,\\
&d(\cdot, \cdot) : \mathcal{Y} \times \mathcal{Y} \to [0, \infty), && d(y_1, y_2) = \|y_1 - y_2\|.
\end{aligned}$$

We now verify that the conditons of Forti's theorem are satisfied.

(I) From (3.4) we obviously have

$$d(H(F(G(|x|))), F(|x|)) = \|2F(2^{-1}|x|) - F(|x|)\| \leq \vartheta 2^{1-p}\|x\|^p = \delta(|x|).$$

(II) $d(H(|y_1|), H(|y_2|)) = 2\|y_1 - y_2\| = \phi(d(y_1, y_2))$ for all $y_1, y_2 \in \mathcal{Y}$.

(III) Clearly, φ is a non-decreasing subadditive function on $[0, \infty)$, and by applying Remark 2.2 to both the iterations G^j and φ^j, one can observe that

$$\sum_{j=0}^{\infty} \varphi^j(\delta(G^j(|x|))) = \vartheta 2^{1-p}\|x\|^p \sum_{j=0}^{\infty} 2^{(1-p)j} = \vartheta\|x\|^p \frac{2^p}{2^p - 2} < \infty.$$

Then by Forti's theorem, $(H^n(F(G^n|x|)))_n$ is a Cauchy sequence for every $x \in \mathcal{X}$, and thus so is $(2^n F(2^{-n}|x|))_n$; furthermore, (2.5) is the unique mapping which satisfies (2.6).

Next, we prove (1.1). In fact, in (2.4) replace x with $2^{-n}x$ and y with $2^{-n}y$, and also let $\tau = 1$. Then

$$\|F(2^{-n}(|x| \vartriangle^*_{\mathcal{X}} |y|)) \vartriangle^{**}_{\mathcal{Y}} F(2^{-n}(|x| \vartriangle^{**}_{\mathcal{X}} |y|)) - F(2^{-n}|x|) \vartriangle^{**}_{\mathcal{Y}} F(2^{-n}|y|)\| \\ \leq \vartheta 2^{-np}(\|x\|^p + \|y\|^p).$$

Multiplying both sides by 2^n yields

$$\begin{aligned} \|2^n(F(2^{-n}(|x| \vartriangle^*_{\mathcal{X}} |y|)) \vartriangle^{**}_{\mathcal{Y}} F(2^{-n}(|x| \vartriangle^{**}_{\mathcal{X}} |y|))) \\ - 2^n(F(2^{-n}|x|) \vartriangle^{**}_{\mathcal{Y}} F(2^{-n}|y|))\| \leq \vartheta(\|x\|^p + \|y\|^p)2^{(1-p)n}. \end{aligned} \tag{3.5}$$

Taking the limit in (3.5) we deduce via (2.5) that

$$\|T(|x| \vartriangle^*_{\mathcal{X}} |y|) \vartriangle^*_{\mathcal{Y}} T(|x| \vartriangle^{**}_{\mathcal{X}} |y|) - T(|x|) \vartriangle^{**}_{\mathcal{Y}} T(|y|)\| = 0,$$

which is equivalent to

$$T(|x| \vartriangle^*_{\mathcal{X}} |y|) \vartriangle^*_{\mathcal{Y}} T(|x| \vartriangle^{**}_{\mathcal{X}} |y|) = T(|x|) \vartriangle^{**}_{\mathcal{Y}} T(|y|).$$

By Remark 2.1, the identity $\gamma F(|x|) = F(\gamma|x|)$ is trivial for $\gamma = 0$ and all $x \in \mathcal{X}$, and for $x = 0$ and all $\gamma \in [0, \infty)$. Fix $\gamma \neq 0$ and $x \in \mathcal{X} \setminus \{0\}$. In (2.4) choose $y = x$, $\tau = \gamma$ and change x to $2^{-n}x$. Then

$$\|\gamma F(2^{-n}|x|) - F(\gamma 2^{-n}|x|)\| \leq \vartheta\|x\|^p 2^{-np}.$$

Multiply both sides by 2^n to get

$$\|\gamma 2^n F(2^{-n}|x|) - 2^n F(\gamma 2^{-n}|x|)\| \leq \vartheta\|x\|^p 2^{(1-p)n}. \tag{3.6}$$

By taking the limit in (3.6) we conclude via (2.5) that

$$\|\gamma T(|x|) - T(\gamma|x|)\| = 0,$$

or equivalently

$$T(\gamma|x|) = \gamma T(|x|)$$

for all $x \in \mathcal{X}$. We have thus shown the semi-homogeneity of T, finishing the proof. ∎

To end our note, we provide an example showing that if in (2.4) the parameter τ is omitted and $p = 1$, then stability cannot always be guaranteed. We recall that in the addition environments Gajda [8] and Găvruţa [9] gave some interesting examples to show how stability fails when the power of the norms is equal to 1.

Example 1. Consider the Lipschitz-continuous function

$$F : [0,\infty) \to [0,\infty), \qquad F(x) = \sqrt{x^2+1}.$$

Fix $x, y \in [0,\infty)$. Since F is increasing, the first equality in the chain below is valid, implying the subsequent relations:

$$\begin{aligned}
|F(x\vee y) - (F(x)\wedge F(y))| &= |F(x\vee y) - F(x\wedge y)| \\
&= |\sqrt{(x\vee y)^2+1} - \sqrt{(x\wedge y)^2+1}| \\
&= \frac{(x\vee y)^2 - (x\wedge y)^2}{\sqrt{(x\vee y)^2+1} + \sqrt{(x\wedge y)^2+1}} \\
&= |x-y| \cdot \frac{(x\vee y) + (x\wedge y)}{\sqrt{(x\vee y)^2+1} + \sqrt{(x\wedge y)^2+1}} \\
&\le |x-y| \le x+y
\end{aligned}$$

for all $x, y \in [0,\infty)$. Now, let $T : [0,\infty) \to [0,\infty)$ be a function such that $T(x) = xT(1)$ for all $x \in [0,\infty)$. Then a simple argument shows

$$\sup_{x\in(0,\infty)} \frac{|F(x) - T(x)|}{x} = \sup_{x\in(0,\infty)} |\sqrt{1+x^{-2}} - T(1)| = \infty.$$

Acknowledgements. I would like to express my hearfelt thanks and appreciation to the referee for his/her suggestions which led to the improvement of the paper.

References

[1] N. K. Agbeko, *On optimal averages*, Acta Math. Hungar. 63 (1994), 133–147.

[2] N. K. Agbeko, *On the structure of optimal measures and some of its applications*, Publ. Math. Debrecen 46 (1995), 79–87.

[3] N. K. Agbeko, *Stability of maximum preserving functional equations on Banach lattices*, Miskolc Math. Notes 13 (2012), 187–196.

[4] N. K. Agbeko and S. S. Dragomir, *The extension of some Orlicz space results to the theory of optimal measure*, Math. Nachr. 286 (2013), 760–771.

[5] T. Aoki, *Stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan 2 (1950), 64–66.

[6] W. Fechner, *On the Hyers–Ulam stability of functional equations connected with additive and quadratic mappings*, J. Math. Anal. Appl. 322 (2006), 774–786.

[7] G.-L. Forti, *Comments on the core of the direct method for proving Hyers–Ulam stability of functional equations*, J. Math. Anal. Appl. 295 (2004), 127–133.

[8] Z. Gajda, *On stability of additive mappings*, Int. J. Math. Sci. 14 (1991), 431–434.

[9] P. Găvruţa, *On a problem of G. Isac and Th. M. Rassias concerning the stability of mappings*, J. Math. Anal. Appl. 261 (2001), 543–553.

[10] R. Ger and P. Šemrl, *The stability of the exponential equation*, Proc. Amer. Math. Soc. 124 (1996), 779–787.

[11] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. USA 27 (1941), 222–224.
[12] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience, New York, 1960.

Nutefe Kwami Agbeko
Institute of Mathematics
University of Miskolc
3515 Miskolc-Egyetemváros, Hungary
E-mail: matagbek@uni-miskolc.hu

Received July 8, 2015;
received in final form October 13, 2015 (8028)