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Complete noncompact submanifolds with flat normal bundle

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Abstract. Let M^n ($n \geq 3$) be an n-dimensional complete super stable minimal submanifold in $\mathbb{R}^{n+p}$ with flat normal bundle. We prove that if the second fundamental form A of M satisfies $\int_M |A|^\alpha < \infty$, where $\alpha \in [2(1-\sqrt{2/n}), 2(1+\sqrt{2/n})]$, then M is an affine n-dimensional plane. In particular, if $n \leq 8$ and $\int_M |A|^d < \infty$, $d = 1, 3$, then M is an affine n-dimensional plane. Moreover, complete strongly stable hypersurfaces with constant mean curvature and finite L^α-norm curvature in $\mathbb{R}^7$ are considered.

1. Introduction. Let N^{n+1} be an oriented $(n+1)$-dimensional Riemannian manifold and $i\colon M^n \to N^{n+1}$ be an isometric immersion of a connected orientable n-dimensional manifold M with constant mean curvature H. Denote by H and A the mean curvature and the second fundamental form of M, respectively. It is convenient to introduce the trace-free second fundamental form of M, i.e., $\phi := A - HI$, where I denotes the identity. Thus $|A|^2 = |\phi|^2 + nH^2$. When N^{n+1} is the simply connected space form $\mathbb{Q}^{n+1}(c)$ with constant curvature $c \in \{-1, 0, 1\}$, i.e., the hyperbolic space $\mathbb{H}^{n+1}$, Euclidean space $\mathbb{R}^{n+1}$ or the standard sphere $\mathbb{S}^{n+1}$, Cheung and Zhou [4] obtained the following Simon inequality:

$$|\phi|\Delta|\phi| \geq \frac{2}{n}\big|\nabla|\phi|\big|^2 - |\phi|^4 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\phi|^3 + n(H^2+c)|\phi|^2. \tag{1.1}$$

Definition 1.1. The immersion i is called *weakly stable* if

$$\int_M [|\nabla f|^2 - (\mathrm{Ric}(\nu,\nu) + |A|^2)f^2] \geq 0 \tag{1.2}$$

for any $f \in C_0^\infty(M)$ satisfying $\int_M f = 0$, where ∇f is the gradient of f in the induced metric of M, while i is called *strongly stable* if (1.2) holds for any $f \in C_0^\infty(M)$. The immersion i is simply called *stable* if $H \neq 0$ and i is weakly stable, or if $H = 0$ (i.e., M is minimal) and i is strongly stable.

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It is known that a complete stable minimal surface in $\mathbb{R}^3$ must be a plane, which was proved by Do Carmo and Peng [6], and independently by Fischer-Colbrie and Schoen [10]. Do Carmo and Peng [7] showed that if M is a stable complete minimal hypersurface in $\mathbb{R}^{n+1}$ and

$$\lim_{R\to\infty} \frac{1}{R^{2+2q}} \int_{B(2R)\setminus B(R)} |A|^2 = 0 \quad \text{for some } q < \sqrt{2/n},$$

then M is a hyperplane. Shen and Zhu [14] showed that if M is a complete stable minimal hypersurface in $\mathbb{R}^{n+1}$ with finite total curvature, that is,

$$\int_M |A|^n < \infty,$$

then M is a hyperplane.

Let M^n be an n-minimal submanifold in $\mathbb{R}^{n+p}$. Spruck [18] proved that for a variation vector field $E = f\upsilon$, the second variation of $\mathrm{Vol}(M_t)$ satisfies

$$\frac{d^2\,\mathrm{Vol}(M_t)}{dt^2} \ge \int_M (|\nabla f|^2 - |A|^2 f^2),$$

where υ is a unit normal vector field and $f \in C_0^\infty(M)$. Motivated by this, Wang [20] introduced the concept of super stability for minimal submanifolds. M is said to be *super stable* if

$$0 \le \int_M (|\nabla f|^2 - |A|^2 f^2), \quad \forall f \in C_0^\infty(M). \tag{1.3}$$

When $p = 1$, the definition of super stability is exactly the same as that of stability, and the normal bundle is trivially flat. Wang [20] proved that a complete super stable minimal submanifold in $\mathbb{R}^{n+p}$ with finite total curvature is an affine plane.

Because the normal bundle becomes complicated in higher codimension, we consider the simplest case when the normal bundle is flat. In 2006, Smoczyk, Wang and Xin [16] proved the Bernstein type theorem for minimal submanifolds in $\mathbb{R}^{n+p}$ with flat normal bundle under a certain growth condition. In 2008, Seo [13] showed that if M is a complete super stable minimal submanifold in $\mathbb{R}^{n+p}$ with flat normal bundle and $\int_M |A|^2 < \infty$, then M is an affine plane. Recently, the present author [11], [12] proved that a complete super stable n-minimal submanifold in $\mathbb{R}^{n+p}$ ($n \le 7$) with flat normal bundle which satisfies $\int_M |A|^3 < \infty$ is an affine plane, and a complete super stable n-minimal submanifold in $\mathbb{R}^{n+p}$ ($n \le 5$) with $\int_M |A| < \infty$ is an affine plane.

In this paper we study super stable minimal submanifolds in $\mathbb{R}^{n+p}$ with flat normal bundle, and prove

THEOREM 1.2. *Let M^n $(n \geq 3)$ be a super stable complete minimal submanifold in $\mathbb{R}^{n+p}$ with flat normal bundle. If*

$$\lim_{R\to\infty} \frac{1}{R^2} \int_{B(2R)\setminus B(R)} |A|^{2\alpha} = 0 \quad \text{for some } \alpha \in [1-\sqrt{2/n}, 1+\sqrt{2/n}],$$

then M is an affine n-dimensional plane.

COROLLARY 1.3. *Let M^n $(3 \leq n \leq 8)$ be a super stable complete immersed minimal submanifold in $\mathbb{R}^{n+p}$ with flat normal bundle. If*

$$\int_M |A|^d < \infty, \qquad d = 1, 3,$$

then M is an affine n-dimensional plane.

REMARK 1.4. Theorem 1.2 and Corollary 1.3 can be regarded as generalizations of the results due to Do Carmo and Peng [7], Seo [13] and Fu [11].

Shen and Zhu [15] proved that any complete noncompact strongly stable hypersurface with constant mean curvature and finite total curvature in $\mathbb{R}^{n+1}$ must be a hyperplane. Alencar and Do Carmo [1] showed that any complete noncompact strongly stable hypersurface with constant mean curvature and finite L^2 norm of traceless second fundamental form in $\mathbb{R}^{n+1}$ $(n \leq 5)$ is a hyperplane. The author [12] proved that any complete noncompact stable hypersurface with constant mean curvature and finite L^d $(d = 1, 3)$ norm of the traceless second fundamental form in $\mathbb{R}^{n+1}$ $(n \leq 5)$ is a hyperplane.

Here we prove the following

THEOREM 1.5. *Let M be a strongly stable complete noncompact hypersurface in $\mathbb{R}^7$ with constant mean curvature. If*

$$\int_M |\phi|^{2\alpha} < \infty \quad \text{for some } \alpha \in \left(1 - \sqrt{\frac{1}{3}}, \frac{\sqrt{5}+\sqrt{5-2\sqrt{5}}}{3}\right],$$

then M is a hyperplane.

REMARK 1.6. Theorem 1.5 extends Theorems 4.1 of [15]. In [8], it is claimed that the dimension condition of the result due to Alencar and Do Carmo [1] is improved from $n \leq 5$ to $n \leq 6$. Unfortunately, there is an error in the proof of [8, Theorem 3] (see [15, Remark 4.2]). For $n = 3, 4$, Cheng [2] proved that any complete noncompact stable hypersurface in $\mathbb{R}^{n+1}$ with constant mean curvature is minimal. For $n = 5$, the present author proved a similar result [12].

2. Complete minimal submanifolds with flat normal bundle. We follow the notation of Chern–Do Carmo–Kobayashi [3].

Let M^n be an n-minimal submanifold in $\mathbb{R}^{n+p}$. We choose an orthonormal frame $e_1, \dots, e_{n+p}$ in $\mathbb{R}^{n+p}$ such that, restricted to M, the vectors $e_1, \dots, e_n$ are tangent to M. We shall denote the second fundamental form by h_{ij}^α. Then $|A|^2 = \sum (h_{ij}^\alpha)^2$ and

$$2|A|\Delta|A| + 2\big|\nabla|A|\big|^2 = \Delta|A|^2 = 2\sum (h_{ijk}^\alpha)^2 + 2\sum (h_{ij}^\alpha)\Delta h_{ij}^\alpha. \tag{2.1}$$

By [3, (2.23)], we have

$$\sum (h_{ij}^\alpha)\Delta h_{ij}^\alpha = -\sum (h_{ik}^\alpha h_{jk}^\beta - h_{jk}^\alpha h_{ik}^\beta)(h_{il}^\alpha h_{jl}^\beta - h_{jl}^\alpha h_{il}^\beta) - \sum h_{ij}^\alpha h_{kl}^\alpha h_{ij}^\beta h_{kl}^\beta.$$

Since M has flat normal bundle, we have $h_{ik}^\alpha h_{jk}^\beta - h_{jk}^\alpha h_{ik}^\beta = 0$. Therefore,

$$\sum (h_{ij}^\alpha)\Delta h_{ij}^\alpha = -\sum h_{ij}^\alpha h_{kl}^\alpha h_{ij}^\beta h_{kl}^\beta.$$

For each α, let H_α denote the symmetric matrix (h_{ij}^α), and set $S_{\alpha\beta} = \sum h_{ij}^\alpha h_{ij}^\beta$. Then the $p \times p$ matrix $(S_{\alpha\beta})$ is symmetric and can be assumed to be diagonal for a suitable choice of $e_{n+1}, \dots, e_{n+p}$. Thus

$$\sum (h_{ij}^\alpha)\Delta h_{ij}^\alpha = -\sum S_{\alpha\alpha}^2 = -\sum_\alpha \Big(\sum_{i,j} (h_{ij}^\alpha)^2\Big)^2. \tag{2.2}$$

Moreover,

$$|A|^4 = (|A|^2)^2 = \Big(\sum_\alpha \sum_{i,j} (h_{ij}^\alpha)^2\Big)^2 \ge \sum_\alpha \Big(\sum_{i,j} (h_{ij}^\alpha)^2\Big)^2. \tag{2.3}$$

Hence from (2.1)–(2.3) we have

$$2|A|\Delta|A| + 2\big|\nabla|A|\big|^2 \ge 2\sum (h_{ijk}^\alpha)^2 - 2|A|^4.$$

Since $\sum (h_{ijk}^\alpha)^2 = |\nabla A|^2$, we get

$$|A|\Delta|A| + \big|\nabla|A|\big|^2 \ge |\nabla A|^2 - |A|^4. \tag{2.4}$$

From (2.4) and the curvature estimate by Y. Xin [21, Lemma 3.1], we obtain

$$|A|\Delta|A| + |A|^4 \ge \frac{2}{n}\big|\nabla|A|\big|^2. \tag{2.5}$$

Proof of Theorem 1.2. By (2.5), we compute, for any positive constant α,

$$\begin{aligned} |A|^\alpha\Delta|A|^\alpha &= |A|^\alpha\big(\alpha(\alpha-1)|A|^{\alpha-2}\big|\nabla|A|\big|^2 + \alpha|A|^{\alpha-1}\Delta|A|\big) \\ &= \frac{\alpha-1}{\alpha}\big|\nabla|A|^\alpha\big|^2 + \alpha|A|^{2\alpha-2}|A|\Delta|A| \\ &\ge \frac{\alpha-1}{\alpha}\big|\nabla|A|^\alpha\big|^2 + \frac{2\alpha}{n}|A|^{2\alpha-2}|\nabla|A||^2 - \alpha|A|^{2\alpha+2} \\ &= \frac{\alpha-1}{\alpha}\big|\nabla|A|^\alpha\big|^2 + \frac{2}{n\alpha}|\nabla|A|^\alpha|^2 - \alpha|A|^{2\alpha+2} \\ &= \Big(1 - \frac{n-2}{n\alpha}\Big)\big|\nabla|A|^\alpha\big|^2 - \alpha|A|^{2\alpha+2}. \end{aligned} \tag{2.6}$$

We first consider the case of $\alpha \in (1-\sqrt{2/n}, 1+\sqrt{2/n})$. Let $\varphi \in C_0^\infty(M)$. Multiplying (2.6) by φ^2 and integrating on M, we obtain

$$\begin{aligned}\left(1-\frac{n-2}{n\alpha}\right)\int_M \big|\nabla|A|^\alpha\big|^2\varphi^2 &\le \int_M |A|^\alpha \Delta |A|^\alpha \varphi^2 + \alpha \int_M |A|^2|A|^{2\alpha}\varphi^2 \\ &= \alpha \int_M |A|^2|A|^{2\alpha}\varphi^2 - \int_M \big|\nabla|A|^\alpha\big|^2\varphi^2 - 2\int_M |A|^\alpha \varphi\langle\nabla\varphi, \nabla|A|^\alpha\rangle,\end{aligned}$$

which gives

$$\begin{aligned}\left(2-\frac{n-2}{n\alpha}\right)\int_M |\nabla|A|^\alpha|^2\varphi^2 \le \alpha \int_M |A|^2|A|^{2\alpha}\varphi^2 \\ - 2\int_M |A|^\alpha\varphi\langle\nabla\varphi, \nabla|A|^\alpha\rangle.\end{aligned} \tag{2.7}$$

Using the Cauchy–Schwarz inequality, we can rewrite (2.7) as

$$\begin{aligned}\left(2-\frac{n-2}{n\alpha}-\epsilon\right)\int_M \varphi^2\big|\nabla|A|^\alpha\big|^2 \le \frac{1}{\epsilon}\int_M |A|^{2\alpha}|\nabla\varphi|^2 \\ + \alpha\int_M |A|^2|A|^{2\alpha}\varphi^2\end{aligned} \tag{2.8}$$

for any $\epsilon > 0$.

On the other hand, replacing f by $|A|^\alpha\varphi$ in (1.3), we get

$$\begin{aligned}\int_M |A|^2|A|^{2\alpha}\varphi^2 \le \int_M \big|\nabla|A|^\alpha\big|^2\varphi^2 + \int_M |A|^{2\alpha}|\nabla\varphi|^2 \\ +2\int_M |A|^\alpha\varphi\langle\nabla\varphi, \nabla|A|^\alpha\rangle,\end{aligned} \tag{2.9}$$

which gives

$$\begin{aligned}\int_M |A|^2|A|^{2\alpha}\varphi^2 \le (1+\epsilon)\int_M \big|\nabla|A|^\alpha\big|^2\varphi^2 \\ +\left(1+\frac{1}{\epsilon}\right)\int_M |A|^{2\alpha}|\nabla\varphi|^2.\end{aligned} \tag{2.10}$$

If $2-\frac{n-2}{n\alpha}-\epsilon > 0$, then inserting (2.10) to (2.8), we obtain

$$B\int_M |\nabla|A|^\alpha|^2\varphi^2 \le D\int_M |A|^{2\alpha}|\nabla\varphi|^2, \tag{2.11}$$

where

$$B = \left(2-\frac{n-2}{n\alpha}-\epsilon\right) - (1+\epsilon)\alpha, \qquad D = \frac{1}{\epsilon} + \frac{\alpha(1+\epsilon)}{\epsilon}.$$

For $\alpha \in (1-\sqrt{2/n}, 1+\sqrt{2/n})$, it is easy to see that $2-\frac{n-2}{n\alpha} > 0$ and $2-\frac{n-1}{n\alpha}-\alpha > 0$. Then we can choose $\epsilon > 0$ sufficiently small so that

$2-\frac{n-2}{n\alpha}-\epsilon>0$ and $B>0$. It follows from (2.11) that

$$\int_M \big|\nabla|A|^\alpha\big|^2\varphi^2 \le C\int_M |A|^{2\alpha}|\nabla\varphi|^2, \tag{2.12}$$

where C is a constant that depends on α and ϵ. Let φ be a smooth function on $[0,\infty)$ such that $\varphi\ge 0$, $\varphi=1$ on $[0,R]$ and $\varphi=0$ in $[2R,\infty)$ with $|\varphi'|\le 1/R$. Then considering $\varphi\circ r$, where r is the function in the definition of $B(R)$, from (2.12) we have

$$\int_M \big|\nabla|A|^\alpha\big|^2\varphi^2 \le \frac{C}{R^2}\int_{B(2R)\backslash B(R)} |A|^{2\alpha}. \tag{2.13}$$

By the assumption that $\lim_{R\to\infty}\frac{1}{R^2}\int_{B(2R)\backslash B(R)}|A|^{2\alpha}=0$, letting $R\to\infty$ in (2.13) we conclude that $\nabla|A|^\alpha=0$, and $|A|$ is constant. Thus it follows by substituting the above $|A|$ into (2.9) that

$$\int_{B(R)} |A|^2|A|^{2\alpha} \le \int_M |A|^2|A|^{2\alpha}\varphi^2 \le \frac{1}{R^2}\int_{B(2R)}|A|^{2\alpha}.$$

So letting $R\to\infty$, we get $|A|\equiv 0$. Hence M is totally geodesic, i.e., M is an affine n-dimensional plane.

In the case of $\alpha=1-\sqrt{2/n}$ or $1+\sqrt{2/n}$, from (2.6) we get

$$\begin{aligned}\int_M \varphi^2|A|^\alpha\Delta|A|^\alpha \ge{}& \left(1-\frac{n-2}{n\alpha}\right)\int_M \varphi^2\big|\nabla|A|^\alpha\big|^2\\ &-\alpha\int_M |A|^2\varphi^2|A|^{2\alpha}.\end{aligned} \tag{2.14}$$

We compute

$$\begin{aligned}\int_M \varphi^2|A|^\alpha\Delta|A|^\alpha &= -\int_M \langle\nabla(\varphi^2|A|^\alpha),\nabla|A|^\alpha\rangle\\ &= \int_M \langle|A|^\alpha\nabla\varphi+\nabla(\varphi|A|^\alpha),|A|^\alpha\nabla\varphi-\nabla(\varphi|A|^\alpha)\rangle\\ &= \int_M |A|^{2\alpha}|\nabla\varphi|^2-\int_M|\nabla(\varphi|A|^\alpha)|^2.\end{aligned} \tag{2.15}$$

Combining (2.15) with (2.14), we obtain

$$\begin{aligned}\left(1-\frac{n-2}{n\alpha}\right)\int_M \varphi^2\big|\nabla|A|^\alpha\big|^2+\int_M\big|\nabla(\varphi|A|^\alpha)\big|^2\\ \le \int_M |A|^{2\alpha}|\nabla\varphi|^2+\alpha\int_M |A|^2\varphi^2|A|^{2\alpha}.\end{aligned} \tag{2.16}$$

By the Schwarz inequality,

$$\begin{aligned}(2.17)\qquad \int_M \varphi^2 \big|\nabla|A|^\alpha\big|^2 &= \int_M \langle \nabla(\varphi|A|^\alpha) - |A|^\alpha\nabla\varphi, \nabla(\varphi|A|^\alpha) - |A|^\alpha\nabla\varphi\rangle \\ &= -2\int_M \langle \nabla(\varphi|A|^\alpha), |A|^\alpha\nabla\varphi\rangle + \int_M \big(\big|\nabla(\varphi|A|^\alpha)\big|^2 + |A|^{2\alpha}|\nabla\varphi|^2\big) \\ &\geq \left(1-\frac{1}{\epsilon}\right)\int_M |A|^{2\alpha}|\nabla\varphi|^2 + (1-\epsilon)\int_M |\nabla(\varphi|A|^\alpha)|^2.\end{aligned}$$

Combining (2.17) with (2.16), we get

$$\begin{aligned}(2.18)\qquad \left[1+(1-\epsilon)\left(1-\frac{n-2}{n\alpha}\right)\right]\int_M |\nabla(\varphi|A|^\alpha)|^2 &\leq \alpha\int_M |A|^2\varphi^2|A|^{2\alpha} \\ &\quad + \left[1+\left(\frac{1}{\epsilon}-1\right)\left(1-\frac{n-2}{n\alpha}\right)\right]\int_M |A|^{2\alpha}|\nabla\varphi|^2.\end{aligned}$$

Choosing φ as above, from (2.18) we obtain

$$\begin{aligned}(2.19)\qquad &\left[1+(1-\epsilon)\left(1-\frac{n-2}{n\alpha}\right)\right]\int_{B(R)} \big|\nabla|A|^\alpha\big|^2 \\ &\leq \alpha\int_M |A|^2|A|^{2\alpha} + \left[1+\left(\frac{1}{\epsilon}-1\right)\left(1-\frac{n-2}{n\alpha}\right)\right]\frac{1}{R^2}\int_{B(2R)\setminus B(R)} |A|^{2\alpha}.\end{aligned}$$

Using the assumption, and letting $R\to\infty$ and $\epsilon\to 0$, from (2.19) we get

$$(2.20)\qquad \int_M \big|\nabla|A|^\alpha\big|^2 \leq \int_M |A|^2|A|^{2\alpha}.$$

Similar to the proof of (2.20), from (2.10) one concludes that

$$\int_M \big|\nabla|A|^\alpha\big|^2 = \int_M |A|^2|A|^{2\alpha}.$$

Hence, either $|A| = 0$, i.e., M is totally geodesic, or equality holds in (2.5). Furthermore, all inequalities leading to (2.5) become equalities. Thus for $\alpha = n+1,\dots,n+p$, $\sum_{i,j}(h_{ij}^\alpha)^2$ has at least $p-1$ zeros. So it is easy to see from a theorem of [9] that M lies in a totally geodesic $\mathbb{R}^{n+1}\hookrightarrow\mathbb{R}^{n+p}$. Hence the eigenvalues of A are λ with multiplicity $n-1$ and $-(n-1)\lambda$ with $\lambda\neq 0$ because $|A|>0$. By a result of Do Carmo and Dajczer [5, Cor. 4.4], this neighborhood is part of a catenoid. Hence $i(M)$ is contained in a catenoid $\mathcal{C}$ by minimality of the immersion. Since M is complete and i is a local isometry into the catenoid $\mathcal{C}$ which is simply connected for $n\geq 3$, $i(M)$ must be an embedding [17, p. 330]. Hence $i(M)$ is a catenoid. Since Tam and Zhou [19] proved that the catenoid is $\frac{n-2}{n}$-stable, this is a contradiction. Hence M is totally geodesic, i.e., M is an affine n-dimensional plane. ∎

3. Complete noncompact strongly stable hypersurfaces in $\mathbb{R}^7$

THEOREM 3.1. *Let M be a strongly stable complete noncompact hypersurface in $\mathbb{R}^7$ with constant mean curvature. If*

$$\lim_{R\to\infty}\frac{1}{R^2}\int_{B(2R)\setminus B(R)}|\phi|^{2\alpha}=0$$

$$\textit{for some } \alpha\in\left[\frac{\sqrt{5}-\sqrt{5-2\sqrt{5}}}{3},\frac{\sqrt{5}+\sqrt{5-2\sqrt{5}}}{3}\right],$$

then M is a hyperplane.

Proof. By (1.1), as in (2.6), we obtain, for any positive constant α,

$$|\phi|^{\alpha}\Delta|\phi|^{\alpha}\ge\left(1-\frac{n-2}{n\alpha}\right)\big|\nabla|\phi|^{\alpha}\big|^2-\alpha|\phi|^{2\alpha+2}$$
$$-\alpha\frac{n(n-2)}{\sqrt{n(n-1)}}H|\phi|^{2\alpha+1}+\alpha nH^2|\phi|^{2\alpha}.$$

We first consider the case of

$$\alpha\in\left(\frac{\sqrt{5}-\sqrt{5-2\sqrt{5}}}{3},\frac{\sqrt{5}+\sqrt{5-2\sqrt{5}}}{3}\right).$$

Using the same argument as in the proof of Theorem 1.2, we obtain $\phi=0$, i.e., M is totally umbilical. Hence M is a hyperplane.

In the case of

$$\alpha=\frac{\sqrt{5}-\sqrt{5-2\sqrt{5}}}{3}\quad\text{or}\quad\frac{\sqrt{5}+\sqrt{5-2\sqrt{5}}}{3},$$

it follows by the same method as employed in Theorem 1.2 that either $\phi=0$, i.e., M is totally umbilical, or the equality (1.1) holds. Furthermore, all inequalities leading to (1.1) become equalities. From the proof of [4, (1.1)], we deduce that $\phi=0$, i.e., M is totally umbilical. Hence M is a hyperplane. ■

PROPOSITION 3.2. *Let M^n $(n\ge 3)$ be a complete stable hypersurface in $\mathbb{Q}^{n+1}(c)$ with constant mean curvature. For $2(1-\sqrt{2/n})<d<2(1+\sqrt{2/n})$,*

$$\textit{if}\quad\int_M|\phi|^d<\infty,\quad\textit{then}\quad\int_M|\phi|^{d+2}<\infty.$$

REMARK 3.3. Proposition 3.2 is proved in [12].

By Theorem 3.1 and Proposition 3.2, we obtain Theorem 1.5.

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