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FUNCTIONAL ANALYSIS

On Some Classes of Operators on $C(K,X)$

by

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Summary. Suppose X and Y are Banach spaces, K is a compact Hausdorff space, Σ is the σ-algebra of Borel subsets of K, $C(K,X)$ is the Banach space of all continuous X-valued functions (with the supremum norm), and $T : C(K,X) \to Y$ is a strongly bounded operator with representing measure $m : \Sigma \to L(X,Y)$.

We show that if T is a strongly bounded operator and $\hat{T} : B(K,X) \to Y$ is its extension, then T is limited if and only if its extension $\hat{T}$ is limited, and that T^* is completely continuous (resp. unconditionally converging) if and only if $\hat{T}^*$ is completely continuous (resp. unconditionally converging).

We prove that if K is a dispersed compact Hausdorff space and T is a strongly bounded operator, then T is limited (resp. weakly precompact, has a completely continuous adjoint, has an unconditionally converging adjoint) whenever $m(A) : X \to Y$ is limited (resp. weakly precompact, has a completely continuous adjoint, has an unconditionally converging adjoint) for each $A \in \Sigma$.

1. Introduction. Suppose K is a compact Hausdorff space, X and Y are Banach spaces, $C(K,X)$ is the Banach space of all continuous X-valued functions (with the supremum norm), and Σ is the σ-algebra of Borel subsets of K.

Every continuous linear function $T : C(K,X) \to Y$ may be represented by a vector measure $m : \Sigma \to L(X,Y^{**})$ of finite semivariation [11], [13, p. 182] such that

$$T(f) = \int_K f\,dm, \qquad f \in C(K,X), \qquad \|T\| = \tilde{m}(\Omega),$$

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and $T^*(y^*) = m_{y^*}$ for $y^* \in Y^*$, where $\tilde{m}$ denotes the semivariation of m. For each $y^* \in Y^*$, the vector measure $m_{y^*} = y^*m : \Sigma \to X^*$ defined by $\langle m_{y^*}(A), x\rangle = \langle m(A)(x), y^*\rangle$ for $A \in \Sigma$ and $x \in X$ is a regular countably additive measure of bounded variation. We denote this correspondence $m \leftrightarrow T$. If we denote by $|y^*m|$ the variation of the measure y^*m, then for $E \in \Sigma$, the semivariation $\tilde{m}(E)$ is given by

$$\tilde{m}(E) = \sup\{|y^*m|(E) : y^* \in Y^*,\ \|y^*\| \le 1\}.$$

We note that for $f \in C(K, X)$, $\int_K f\,dm \in Y$ even if m is not $L(X, Y)$-valued. A representing measure m is called *strongly bounded* if $\tilde{m}(A_i) \to 0$ for every decreasing sequence $A_i \to \emptyset$ in Σ, and an operator $m \leftrightarrow T : C(K, X) \to Y$ is called strongly bounded if m is strongly bounded [11]. By [11, Theorem 4.4], a strongly bounded representing measure takes its values in $L(X, Y)$. If m is a strongly bounded representing measure, then there is a nonnegative regular Borel measure λ such that $\tilde{m}(A) \to 0$ as $\lambda(A) \to 0$. We call λ the *control measure* for m. If T is unconditionally converging, then m is strongly bounded [15].

Let χ_A denote the characteristic function of a set A, and $B(K, X)$ denote the space of all bounded, Σ-measurable functions on K with separable range in X and the sup norm. Clearly, $C(K, X)$ is contained isometrically in $B(K, X)$. Further, $B(K, X)$ embeds isometrically in $C(K, X)^{**}$ (see e.g. [11]). The reader should note that if $m \leftrightarrow T$, then $m(A)x = T^{**}(\chi_A x)$ for all $A \in \Sigma$ and $x \in X$. If $f \in B(K, X)$, then f is the uniform limit of X-valued simple functions, $\int_K f\,dm$ is well-defined and defines an extension $\hat{T}$ of T (see e.g. [14]). Theorem 2 of [6] shows that $\hat{T}$ maps $B(K, X)$ into Y if and only if the representing measure m of T is $L(X, Y)$-valued. If $T : C(K, X) \to Y$ is strongly bounded, then m is $L(X, Y)$-valued [11], and thus $\hat{T} : B(K, X) \to Y$. Since $\hat{T}$ is the restriction to $B(K, X)$ of the operator T^{**}, it is clear that an operator $T : C(K, X) \to Y$ is compact (resp. weakly compact) if and only if its extension $\hat{T} : B(K, X) \to Y$ is compact (resp. weakly compact).

Several authors have found the study of $\hat{T}$ to be quite helpful. We mention the work of [6], [8], [9], and [18]. In these papers it has been proved that if m is strongly bounded, then $T : C(K, X) \to Y$ is weakly compact, compact, Dunford–Pettis, Dieudonné, unconditionally converging, strictly singular, strictly cosingular, weakly precompact, and has a weakly precompact adjoint if and only if its extension $\hat{T} : B(K, X) \to Y$ has the same property. We show that if $T : C(K, X) \to Y$ is a strongly bounded operator and $\hat{T} : B(K, X) \to Y$ is its extension, then T is limited if and only if $\hat{T}$ is limited, and that T^* is completely continuous (resp. unconditionally converging) if and only if $\hat{T}^*$ is completely continuous (resp. unconditionally converging).

A topological space S is called *dispersed* (or *scattered*) if every nonempty closed subset of S has an isolated point [24]. A compact Hausdorff space K is dispersed if and only if $\ell_1 \not\hookrightarrow C(K)$ [20].

Bombal and Cembranos [8] showed that if K is a dispersed compact Hausdorff space and $m \leftrightarrow T : C(K, X) \to Y$ is an operator, then T is unconditionally converging (resp. completely continuous, Dieudonné, weakly compact) if and only if m is strongly bounded and $m(A) : X \to Y$ is unconditionally converging (resp. completely continuous, Dieudonné, weakly compact) for every $A \in \Sigma$. We prove that if K is a dispersed compact Hausdorff space and $m \leftrightarrow T : C(K, X) \to Y$ is a strongly bounded operator, then T is limited (resp. weakly precompact, compact, has a completely continuous adjoint, has an unconditionally converging adjoint) if and only if for every $A \in \Sigma$, $m(A) : X \to Y$ is limited (resp. weakly precompact, compact, has a completely continuous adjoint, has an unconditionally converging adjoint).

An operator $T : X \to Y$ is *completely continuous* (or *Dunford–Pettis*) if it maps weakly convergent sequences to convergent sequences.

A subset S of X is said to be *weakly precompact* provided that every bounded sequence from S has a weakly Cauchy subsequence [5]. An operator $T : X \to Y$ is *weakly precompact* (or *almost weakly compact*) if $T(B_X)$ is weakly precompact.

A bounded subset A of a Banach space X is called a *limited* (resp. *Dunford–Pettis* (DP)) subset of X if every w^*-null (resp. weakly null) sequence (x_n^*) in X^* tends to 0 uniformly on A, i.e.,

$$\lim_n (\sup\{|x_n^*(x)| : x \in A\}) = 0.$$

Every limited subset of X is weakly precompact [10]. Every DP subset of X is weakly precompact (see e.g. [2] and [21, p. 377]). An operator $T : X \to Y$ is called *limited* if $T(B_X)$ is limited. We note that T is limited if and only if T^* is w^*-norm sequentially continuous.

A series $\sum x_n$ of elements of X is *weakly unconditionally convergent* (wuc) if $\sum |x^*(x_n)| < \infty$ for each $x^* \in X^*$. An operator $T : X \to Y$ is *unconditionally converging* if it maps weakly unconditionally convergent series to convergent ones.

A bounded subset A of X (resp. A of X^*) is called a *V^*-subset of X* (resp. a *V-subset of X^**) provided that

$$\lim_n (\sup\{|x_n^*(x)| : x \in A\}) = 0$$
$$(\text{resp. } \lim_n (\sup\{|x^*(x_n)| : x^* \in A\}) = 0)$$

for each wuc series $\sum x_n^*$ in X^* (resp. wuc series $\sum x_n$ in X).

A bounded subset A of X^* is called an *L-subset of X^** if each weakly null sequence (x_n) in X tends to 0 uniformly on A, i.e.,

$$\lim_n(\sup\{|x^*(x_n)| : x^* \in A\}) = 0.$$

A Banach space X has *property weak* (V) (wV) if any V-subset of X^* is weakly precompact [22].

2. Main results. We begin with the following lemma. If $T : X \to Y^*$ is an operator, then $T^*|_Y$ denotes the restriction of T^* to Y.

LEMMA 1.

(i) *If $T : X \to Y$ is an operator, then $T(B_X)$ is a DP subset of Y if and only if $T^* : Y^* \to X^*$ is completely continuous.*
(ii) *If $T : X \to Y$ is an operator, then $T(B_X)$ is a V^*-subset of Y if and only if $T^* : Y^* \to X^*$ is unconditionally converging.*
(iii) *If $T : X \to Y^*$ is an operator, then $T(B_X)$ is a V-subset of Y^* if and only if $T^*|_Y : Y \to X^*$ is unconditionally converging.*
(iv) *If $T : X \to Y^*$ is an operator, then $T(B_X)$ is an L-subset of Y^* if and only if $T^*|_Y : Y \to X^*$ is completely continuous.*

Proof. (i) Suppose $T(B_X)$ is a DP subset of Y and $T^* : Y^* \to X^*$ is not completely continuous. Let (y_n^*) be weakly null in Y^* such that $\|T^*(y_n^*)\| \nrightarrow 0$. Choose a sequence (x_n) in B_X and $\epsilon > 0$ such that $\langle T^*(y_n^*), x_n\rangle > \epsilon$ for all n. Then $\langle y_n^*, T(x_n)\rangle = \langle T^*(y_n^*), x_n\rangle > \epsilon$ for all n, contrary to $T(B_X)$ being a DP set.

Conversely, suppose $T^* : Y^* \to X^*$ is completely continuous. Let (x_n) be a sequence in B_X and (y_n^*) be weakly null in Y^*. Then

$$\langle y_n^*, T(x_n)\rangle = \langle T^*(y_n^*), x_n\rangle \le \|T^*(y_n^*)\| \to 0,$$

and $T(B_X)$ is a DP subset of Y.

(ii) The proof is similar to that of (i).

(iii) Suppose $T(B_X)$ is a V-subset of Y^*. We show that $T^*|_Y : Y \to X^*$ is unconditionally converging. Suppose $\sum y_n$ is wuc in Y. It suffices to show that $\|T^*(y_n)\| \to 0$. Suppose $\|T^*(y_n)\| \nrightarrow 0$. Choose a sequence (x_n) in B_X and $\epsilon > 0$ such that $\langle T^*(y_n), x_n\rangle > \epsilon$ for all n. Then $\langle y_n, T(x_n)\rangle > \epsilon$ for all n, which contradicts $T(B_X)$ being a V-set.

Conversely, suppose $T^*|_Y : Y \to X^*$ is unconditionally converging. Let (x_n) be a sequence in B_X and $\sum y_n$ be wuc in Y. Since $T^*|_Y$ is unconditionally converging,

$$\langle y_n, T(x_n)\rangle = \langle T^*(y_n), x_n\rangle \le \|T^*(y_n)\| \to 0,$$

and $T(B_X)$ is a V-subset of Y^*.

(iv) The proof is similar to that of (iii). ∎

Suppose that $T : C(K,X) \to Y$ is an operator and $\hat{T} : B(K,X) \to Y^{**}$ is its extension to $B(K,X)$. As noted in the Introduction, if $m \leftrightarrow T : C(K,X) \to Y$ is strongly bounded, then m is $L(X,Y)$-valued and $\hat{T}$ maps $B(K,X)$ into Y. Let B_0 denote the unit ball of $C(K,X)$, and B denote the unit ball of $B(K,X)$.

THEOREM 2. *Suppose that $T : C(K,X) \to Y$ is a strongly bounded operator and $\hat{T} : B(K,X) \to Y$ is its extension. Then:*

(i) *T is limited if and only if $\hat{T}$ is limited.*
(ii) *T^* is completely continuous* (*resp. unconditionally converging*) *if and only if $\hat{T}^*$ is completely continuous* (*resp. unconditionally converging*).

Proof. (i) Suppose that $T : C(K,X) \to Y$ is limited and $\hat{T}$ is not. Let (y_n^*) be w^*-null in Y^* and (f_n) be a sequence in the unit ball of $B(K,X)$ such that $\langle y_n^*, \hat{T}(f_n)\rangle = 1$ for all n. Without loss of generality assume $\|y_n^*\| \le 1$ for all n.

Using the existence of a control measure for m and Lusin's theorem, one can find a compact subset K_0 of K such that $\tilde{m}(K \setminus K_0) < 1/4$ and $g_n = f_n|_{K_0}$ is continuous for each $n \in \mathbb{N}$. Let $H = [g_n]$ be the closed linear subspace spanned by (g_n) in $C(K_0,X)$, and $S : H \to C(K,X)$ be the isometric extension operator given by [8, Theorem 1]. If $h_n = S(g_n)$ for each $n \in \mathbb{N}$, then (h_n) is in the unit ball of $C(K,X)$, and

$$\begin{aligned}
|\langle y_n^*, T(h_n)\rangle| &\ge \Big|\Big\langle y_n^*, \int_{K_0} h_n\,dm\Big\rangle\Big| - \Big|\Big\langle y_n^*, \int_{K\setminus K_0} h_n\,dm\Big\rangle\Big| \\
&\ge \Big|\Big\langle y_n^*, \int_{K_0} f_n\,dm\Big\rangle\Big| - 1/4 \\
&\ge \Big|\Big\langle y_n^*, \int_{K} f_n\,dm\Big\rangle\Big| - \Big|\Big\langle y_n^*, \int_{K\setminus K_0} f_n\,dm\Big\rangle\Big| - 1/4 \\
&\ge |\langle y_n^*, \hat{T}(f_n)\rangle| - 1/4 - 1/4 = 1/2.
\end{aligned}$$

This is a contradiction, since $T(B_0)$ is limited.

(ii) By Lemma 1, it is enough to show that $T(B_0)$ is a DP set (resp. a V^*-set) if and only if $\hat{T}(B)$ is a DP set (resp. a V^*-set). Suppose that $T(B_0)$ is a DP set (resp. a V^*-set) and $\hat{T}(B)$ is not a DP set (resp. a V^*-set). Suppose (y_n^*) is weakly null (resp. $\sum y_n^*$ is wuc) in Y^* and (f_n) is a sequence in the unit ball of $B(K,X)$ such that $\langle y_n^*, \hat{T}(f_n)\rangle = 1$ for each n. Continuing as above we find a sequence (h_n) in the unit ball of $C(K,X)$ such that $|\langle y_n^*, T(h_n)\rangle| \ge 1/2$. This is a contradiction, since $T(B_0)$ is a DP set (resp. a V^*-set). ■

Corollary 3. *Suppose that* $m \leftrightarrow T : C(K,X) \to Y$ *is a strongly bounded operator.*

(i) *If* T *is limited, then* $m(A) : X \to Y$ *is limited for each* $A \in \Sigma$.
(ii) *If* T^* *is completely continuous* (*resp. unconditionally converging*), *then for each* $A \in \Sigma$, $m(A)^* : Y^* \to X^*$ *is completely continuous* (*resp. unconditionally converging*).

Proof. We will only consider the case of limited operators. The proof of (ii) is similar. If $A \in \Sigma$, $A \neq \emptyset$, define $\theta_A : X \to B(K,X)$ by $\theta_A(x) = \chi_A x$. Then θ_A is an isomorphic isometric embedding of X into $B(K,X)$ and $\hat{T}\theta_A = m(A)$. By Theorem 2, $\hat{T}$ is limited, and thus $m(A)$ is. ∎

The proofs of the following results are similar to those of Theorem 2 and Corollary 3 and will be omitted.

Theorem 4. *Suppose that* $T : C(K,X) \to Y^*$ *is a strongly bounded operator and* $\hat{T} : B(K,X) \to Y^*$ *is its extension. Then* $T^*|_Y$ *is completely continuous* (*resp. unconditionally converging*) *if and only if* $\hat{T}^*|_Y$ *is completely continuous* (*resp. unconditionally converging*).

Corollary 5. *Suppose that* $m \leftrightarrow T : C(K,X) \to Y^*$ *is a strongly bounded operator. If* $T^*|_Y$ *is completely continuous* (*resp. unconditionally converging*), *then for each* $A \in \Sigma$, $m(A)^*|_Y$ *is completely continuous* (*resp. unconditionally converging*).

Next we study the properties of the compact space K for which an operator $T : C(K,X) \to Y$ with representing measure m is limited (resp. weakly precompact, compact, has a completely continuous adjoint, has an unconditionally converging adjoint) whenever m is strongly bounded and $m(A) : X \to Y$ is limited (resp. weakly precompact, compact, has a completely continuous adjoint, has an unconditionally converging adjoint) for each $A \in \Sigma$.

If $T : C(K,X) \to Y$ is an operator, $\bar{K}$ is a metrizable compact space, and $\pi : K \to \bar{K}$ a continuous map which is onto, we will call $\bar{K}$ a *quotient* of K. The map $\bar{\pi} : C(\bar{K}) \to C(K)$ given by $\bar{\pi}\bar{f} = \bar{f}\pi$ defines an isometric embedding of $C(\bar{K})$ into $C(K)$. Let $\bar{T} : C(\bar{K},X) \to Y$ be the operator defined by $\bar{T}(\bar{f}) = T(\bar{f}\pi)$, where $\bar{f} \in C(\bar{K},X)$ and $\pi : K \to \bar{K}$ is the canonical mapping.

The following results will be useful in our study.

Lemma 6.

(i) *An operator* $T : C(K,X) \to Y$ *is limited* (*resp. weakly precompact, compact*) *if and only if, for each metrizable quotient* $\bar{K}$ *of* K, *the operator* $\bar{T} : C(\bar{K},X) \to Y$ *defined as above is limited* (*resp. weakly precompact, compact*).

(ii) *If $T : C(K,X) \to Y$ is an operator, then T^* is completely continuous* (*resp. unconditionally converging*) *if and only if, for each metrizable quotient $\bar{K}$ of K, $\bar{T}^*$ is completely continuous* (*resp. unconditionally converging*), *where $\bar{T} : C(\bar{K},X) \to Y$ is defined as above.*

Proof. We will only consider the case of limited operators. The proof for the other operators is similar. Suppose that $T : C(K,X) \to Y$ is limited and $\bar{K}$ is a metrizable quotient of K. Then $\bar{T}$ is limited.

Conversely, let $T : C(K,X) \to Y$ be an operator and let (f_n) be a sequence in the unit ball of $C(K,X)$. It is known (see [6]) that there exists a metrizable quotient $\bar{K}$ of K and a sequence $(\bar{f}_n)$ in $C(\bar{K},X)$ defined by $\bar{f}_n(\pi(t)) = f_n(t)$ for all $t \in K$ and $n \in \mathbb{N}$. Define $\bar{T} : C(\bar{K},X) \to Y$ by $\bar{T}(\bar{f}) = T(\bar{f}\pi)$, where $\pi : K \to \bar{K}$ is the canonical mapping. By assumption, $\bar{T}$ is limited. Then $(\bar{T}(\bar{f}_n)) = (T(f_n))$ is limited. ∎

Similarly, we obtain the following result.

LEMMA 7. *If $T : C(K,X) \to Y^*$ is an operator, then $T^*|_Y$ is completely continuous* (*resp. unconditionally converging*) *if and only if, for each metrizable quotient $\bar{K}$ of K, $\bar{T}^*|_Y$ is completely continuous* (*resp. unconditionally converging*), *where $\bar{T} : C(\bar{K},X) \to Y^*$ is defined as above.*

LEMMA 8 ([8, Lemma 5]). *Let K and K_0 be two compact Hausdorff spaces, Σ and Σ_0 the Borel σ-algebras of K and K_0 respectively, and $\alpha : K \to K_0$ a continuous map. If m is the representing measure of an operator $T : C(K,X) \to Y$ and m_0 is the representing measure of the operator $T_0 : C(K_0,X) \to Y$ defined by $T_0(f) = T(f\alpha)$, then $m_0(A) = m(\alpha^{-1}(A))$ for all $A \in \Sigma_0$. Consequently, $\tilde{m}_0(A) \le \tilde{m}(\alpha^{-1}(A))$ for all $A \in \Sigma_0$.*

LEMMA 9 ([23], [17], [12], [7]). *Let H be a bounded subset of X. If for each $\epsilon > 0$ there is a limited* (*resp. weakly precompact, relatively compact, DP, V^**) *subset H_ϵ of X such that $H \subseteq H_\epsilon + \epsilon B_X$, then H is limited* (*resp. weakly precompact, relatively compact, DP, V^**).

LEMMA 10 ([16], [3]). *Let H be a bounded subset of X^*. If for each $\epsilon > 0$ there is an L-subset* (*resp. a V-subset*) *H_ϵ of X^* such that $H \subseteq H_\epsilon + \epsilon B_{X^*}$, then H is an L-set* (*resp. a V-set*).

Abbott [1] gave an example of a pair $m \leftrightarrow T$ such that T is weakly precompact and m is not strongly bounded.

THEOREM 11. *Suppose that K is a dispersed compact Hausdorff space and $m \leftrightarrow T : C(K,X) \to Y$ is a strongly bounded operator. Then:*

(1) *T is weakly precompact* (*resp. limited*) *if and only if $m(A) : X \to Y$ is weakly precompact* (*resp. limited*) *for each $A \in \Sigma$.*

(2) $T^* : Y^* \to C(K,X)^*$ *is completely continuous* (*resp. unconditionally converging*) *if and only if* $m(A)^* : Y^* \to X^*$ *is completely continuous* (*resp. unconditionally converging*) *for each* $A \in \Sigma$.

Proof. Suppose $m \leftrightarrow T : C(K,X) \to Y$ is strongly bounded.

(1) If T is weakly precompact (resp. limited), then for each $A \in \Sigma$, $m(A) : X \to Y$ is weakly precompact (resp. limited) by [18, Corollary 17] (resp. Corollary 3).

Conversely, suppose that $m \leftrightarrow T : C(K,X) \to Y$ is a strongly bounded operator and $m(A) : X \to Y$ is weakly precompact (resp. limited) for each $A \in \Sigma$. From Lemmas 6 and 8 and the fact that a quotient space of a dispersed space is dispersed [24, 8.5.3], we can suppose without loss of generality that K is metrizable. Since K is dispersed and metrizable, it is countable [24, 8.5.5]. Suppose that $K = \{t_i : i \in \mathbb{N}\}$. Let (f_n) be a sequence in the unit ball of $C(K,X)$. For each $i \in \mathbb{N}$, the set $\{f_n(t_i) : n \in \mathbb{N}\}$ is bounded in X. Then the set

$$H_i = \{m(\{t_i\})(f_n(t_i)) : n \in \mathbb{N}\}$$

is weakly precompact (resp. limited) for each $i \in \mathbb{N}$. Let $A_i = \{t_j : j > i\}$ for $i \in \mathbb{N}$. Then (A_i) is a decreasing sequence of sets. Let $\epsilon > 0$. Since m is strongly bounded, there is a $k \in \mathbb{N}$ such that $\tilde{m}(A_k) < \epsilon$. For each $n \in \mathbb{N}$,

$$T(f_n) = \int_K f_n \, dm = \sum_{i=1}^{k} m(\{t_i\})(f_n(t_i)) + \int_{A_k} f_n \, dm.$$

Further, $\|\int_{A_k} f_n \, dm\| \le \tilde{m}(A_k) < \epsilon$. Therefore

$$T(f_n) \in H_1 + \cdots + H_k + \epsilon B_Y.$$

Since $H_1 + \cdots + H_k$ is weakly precompact (resp. limited), by Lemma 9 the set $\{T(f_n) : n \in \mathbb{N}\}$ is weakly precompact (resp. limited). Thus T is weakly precompact (resp. limited).

(2) If $T^* : Y^* \to C(K,X)^*$ is completely continuous (resp. unconditionally converging), then for each $A \in \Sigma$, $m(A)^* : Y^* \to X^*$ is completely continuous (resp. unconditionally converging) by Corollary 3.

Conversely, suppose $m(A)^* : Y^* \to X^*$ is completely continuous (resp. unconditionally converging) for each $A \in \Sigma$. By Lemma 1, $m(A)(B_X)$ is a DP set (resp. a V^*-set) for each $A \in \Sigma$. Let (f_n) be a sequence in the unit ball of $C(K,X)$. Using an argument similar to the one above, we can show that $\{T(f_n) : n \in \mathbb{N}\}$ is a DP set (resp. a V^*-set). By Lemma 1, $T^* : Y^* \to C(K,X)^*$ is completely continuous (resp. unconditionally converging). ∎

Remark 1. It is known that if $m \leftrightarrow T : C(K,X) \to Y$ is a compact operator, then m is strongly bounded and for each $A \in \Sigma$, $m(A) : X \to Y$ is compact [11]. The proof of Theorem 11 shows that the following result holds: Suppose that K is a dispersed compact Hausdorff space and $m \leftrightarrow T : C(K,X) \to Y$ is a strongly bounded operator. If (f_n) is a bounded sequence in $C(K,X)$ and for all $A \in \Sigma$ and $t \in K$, $m(A)(\{f_n(t) : n \in \mathbb{N}\})$ is relatively compact, then $\{T(f_n) : n \in \mathbb{N}\}$ is relatively compact. It follows that if K is dispersed and $m \leftrightarrow T : C(K,X) \to Y$ is an operator, then T is compact if and only if m is strongly bounded and $m(A) : X \to Y$ is compact for each $A \in \Sigma$.

THEOREM 12. *Suppose that K is a dispersed compact Hausdorff space and $m \leftrightarrow T : C(K,X) \to Y^*$ is a strongly bounded operator. Then $T^*|_Y : Y \to C(K,X)^*$ is completely continuous* (*resp. unconditionally converging*) *if and only if for each $A \in \Sigma$, $m(A)^*|_Y : Y \to X^*$ is completely continuous* (*resp. unconditionally converging*).

Proof. The proof is similar to the proof of Theorem 11 and uses Lemmas 1, 7, 8, and 10. ∎

COROLLARY 13. *Suppose that K is a dispersed compact Hausdorff space.*

(i) *If every unconditionally converging* (*resp. completely continuous*) *operator $S : X \to Y$ is weakly precompact, then every unconditionally converging* (*resp. completely continuous*) *operator $T : C(K,X) \to Y$ is weakly precompact.*
(ii) *If X has property (wV), then every unconditionally converging operator $T : C(K,X) \to Y$ is weakly precompact.*

Proof. (i) If $m \leftrightarrow T : C(K,X) \to Y$ is an unconditionally converging operator, then m is strongly bounded and $m(A) : X \to Y$ is unconditionally converging for each $A \in \Sigma$ [15]. If $m \leftrightarrow T : C(K,X) \to Y$ is completely continuous, then m is strongly bounded and $m(A) : X \to Y$ is completely continuous for each $A \in \Sigma$ (this can be shown as in [15]). Hence m is strongly bounded and $m(A) : X \to Y$ is weakly precompact for each $A \in \Sigma$. Then T is weakly precompact by Theorem 11.

(ii) Suppose X has property (wV). Then every unconditionally operator on X has a weakly precompact adjoint [22, p. 529], and thus is weakly precompact, by [4, Corollary 2]. Apply (i). ∎

COROLLARY 14. *Suppose that K is a dispersed compact Hausdorff space. Suppose $m \leftrightarrow T : C(K,X) \to Y$ is an operator such that m is strongly bounded and $m(A)^* : Y^* \to X^*$ is weakly precompact for each $A \in \Sigma$. Then T is unconditionally converging and weakly precompact. In addition, if X^* is weakly sequentially complete, then T is weakly compact.*

Proof. For each $A \in \Sigma$, $m(A) : X \to Y$ is unconditionally converging and weakly precompact, by [4, Corollary 2]. Then T is unconditionally converging and weakly precompact by [8, Theorem 9] and Theorem 11.

Moreover, if X^* is weakly sequentially complete, then $m(A)^* : Y^* \to X^*$ is weakly compact for each $A \in \Sigma$. Hence $m(A) : X \to Y$ is weakly compact for each $A \in \Sigma$. By [8, Theorem 7], T is weakly compact. ■

The following theorem gives a characterization of Dunford–Pettis sets.

Theorem 15. *Supppose A is a bounded subset of a Banach space X. Then the following assertions are equivalent:*

(i) *A is a DP set.*
(ii) *If $T : X \to Y$ is an operator with weakly precompact adjoint, then $T(A)$ is relatively compact.*
(iii) *If $T : X \to c_0$ is an operator with weakly precompact adjoint, then $T(A)$ is relatively compact.*
(iv) *If $T : X \to c_0$ is a weakly compact operator, then $T(A)$ is relatively compact.*
(v) *If (x_n^*) is a weakly null sequence in X^* and (x_n) is a sequence in A, then $\lim x_n^*(x_n) = 0$.*

Proof. (i)⇒(ii). Suppose that A is a DP set and let $T : X \to Y$ be an operator such that T^* is weakly precompact. Let (x_n) be a sequence in A. Without loss of generality we may assume that (x_n) is weakly Cauchy [21], [2].

Define $S : \ell_1 \to X$ by $S(b) = \sum b_n x_n$ for $b = (b_n) \in \ell_1$. Since the closed absolutely convex hull of (x_i) is a DP subset of X, $S(B_{\ell_1})$ is a DP set. By Lemma 1, S^* is completely continuous. Since T^* is weakly precompact, S^*T^*, and thus TS, is compact. Then $(T(x_n)) = (TS(e_n^*))$ is relatively compact, and $T(A)$ is relatively compact.

(ii)⇒(iii) and (iii)⇒(iv) are clear.

(iv)⇒(v) and (v)⇒(i) by [2, Theorem 1]. ■

Corollary 16. *Suppose that K is a dispersed compact Hausdorff space and (f_n) is a bounded sequence in $C(K,X)$.*

(i) *If for each $t \in K$, $(f_n(t))$ is a DP set, then (f_n) is a DP set.*
(ii) *If for each $t \in K$, $(f_n(t))$ is a V^*-set, then (f_n) is a V^*-set.*

Proof. (i) Suppose that for each $t \in K$, $(f_n(t))$ is a DP set. Let $m \leftrightarrow T : C(K,X) \to Y$ be an operator such that T^* is weakly precompact. Then T is unconditionally converging by [4, Corollary 2], thus strongly bounded [15]. For each $A \in \Sigma$, $m(A)^* : Y^* \to X^*$ is weakly precompact, by [18, Corollary 20]. Let $A \in \Sigma$ and $t \in K$. By Theorem 15, $m(A)(\{f_n(t) : n \in \mathbb{N}\})$ is relatively compact. By Remark 1, $\{T(f_n) : n \in \mathbb{N}\}$ is relatively compact. Then (f_n) is a DP set, by Theorem 15.

(ii) Suppose that for each $t \in K$, $(f_n(t))$ is a V^*-set. Let $m \leftrightarrow T : C(K,X) \to \ell_1$ be an operator. Then T is unconditionally converging, thus strongly bounded [15]. Let $A \in \Sigma$ and $t \in K$. By [7, Proposition 1.1], $m(A)(\{f_n(t) : n \in \mathbb{N}\})$ is relatively compact. By Remark 1, $\{T(f_n) : n \in \mathbb{N}\}$ is relatively compact. Then (f_n) is a V^*-set, by [7, Proposition 1.1]. ∎

Next we produce operators $m \leftrightarrow T : C(K,X) \to Y$ such that m is strongly bounded, $m(A)$ is compact for each $A \in \Sigma$, yet T fails to be compact. In the following two results the unit vector basis of c_0 is denoted by (e_n) and the unit vector basis of ℓ_1 is denoted by (e_n^*).

Let Δ be the Cantor set $\{-1,1\}^{\mathbb{N}}$, and let λ be the Haar measure on Δ. Let C_{ni}, $1 \le i \le 2^n$, denote the dyadic partition at the nth stage, so that for example $C_{11} = \{(t_n) : t_1 = 1\}$ and $C_{12} = \{(t_n) : t_1 = -1\}$. Let (r_n) in $C(\Delta)$ be the sequence of Rademacher functions on Δ, i.e., $r_n(t) = t_n$, for $t \in \Delta$.

THEOREM 17. *Suppose X is an infinite-dimensional Banach space. Then there is a nonlimited and noncompact operator $m \leftrightarrow T : C(\Delta,X) \to c_0$ such that m is strongly bounded and $m(A) : X \to c_0$ is compact for every $A \in \Sigma$.*

Proof. Use the Josefson–Nissenzweig theorem to choose a w^*-null sequence (x_n^*) in X^* with $\|x_n^*\| = 1$ for all n. For each n, choose x_n in B_X such that $\langle x_n^*, x_n\rangle > 1/2$. Define $T : C(\Delta,X) \to c_0$ by

$$T(f) = \Big(\int_{\Delta} \langle x_n^*, f(t)\rangle r_n(t)\, d\lambda\Big)_n, \qquad f \in C(\Delta,X).$$

If $f \in C(\Delta)$ and $x \in X$, let $f \otimes x$ be the element of $C(\Delta,X)$ defined by $(f \otimes x)(t) = f(t)\,x$. Then

$$T(f \otimes x) = \Big(\int_{\Delta} \langle x_n^*, x\rangle f(t) r_n(t)\, d\lambda\Big)_n.$$

Since $\|x_n^*\| = 1$ and $(\int_\Delta f(t) r_n(t)\, d\lambda) \to 0$, we have $T(f\otimes x) \in c_0$ for all $f \in C(\Delta)$ and $x \in X$. Therefore $T(f) \in c_0$ for every $f \in C(\Delta,X)$. The representing measure m of T is given by

$$m(A)(x) = \Big(\langle x_n^*, x\rangle \int_A r_n(t)\, d\lambda\Big)_n$$

for $A \in \Sigma$ and $x \in X$. Since $\int_A r_n(t)\, d\lambda \to 0$ for all $A \in \Sigma$ and $\{\langle x_n^*, x\rangle : n \in \mathbb{N}, x \in B_X\}$ is bounded, it follows that $m(A)(x) \in c_0$. Further, $\|m(A)\| \le \lambda(A)$, m is a dominated representing measure [14], [11, p. 148], and thus strongly bounded. If $(b_n) \in \ell_1$, then

$$\langle m(A)^*, (b_n)\rangle = \sum b_n \Big(\int_A r_n(t)\, d\lambda\Big) x_n^*.$$

Note that $m(A)^*$ maps the unit ball of ℓ_1 into the absolute closed convex hull of $\{(\int_A r_n(t)\, d\lambda) x_n^* : n \in \mathbb{N}\}$, which is a compact set (since $\|(\int_A r_n(t)\, d\lambda) x_n^*\| \le |\int_A r_n(t)\, d\lambda| \to 0$).

For each n, let $f_n = r_n x_n$ in $C(\Delta, X)$; note that $\|f_n\| \le 1$ and

$$T(f_n) = \Big(\int_\Delta \langle x_i^*, f_n(t)\rangle r_i(t)\, d\lambda\Big)_i$$
$$= \Big(\int_\Delta \langle x_i^*, x_n\rangle r_n(t) r_i(t)\, d\lambda\Big)_i = \langle x_n^*, x_n\rangle e_n.$$

Since $\langle T(f_n), e_n^*\rangle = \langle x_n^*, x_n\rangle > 1/2$, T is nonlimited and noncompact. ∎

Theorem 18. *The following statements are equivalent:*

(i) *K is dispersed.*
(ii) *For any pair of Banach spaces X and Y, a strongly bounded operator $m \leftrightarrow T : C(K, X) \to Y$ is limited if and only if $m(A) : X \to Y$ is limited for every $A \in \Sigma$.*
(iii) *There is a Banach space X such that a strongly bounded operator $m \leftrightarrow T : C(K, X) \to c_0$ is limited if and only if $m(A) : X \to c_0$ is limited for every $A \in \Sigma$.*

Proof. (i)⇒(ii) by Theorem 11. (ii)⇒(iii) is clear.

(iii)⇒(i). Suppose that (iii) holds and K is not dispersed. Then there is a purely nonatomic regular probability Borel measure λ on K [19, Theorem 2.8.10]. Now we can construct a Haar system $\{A_i^n : 1 \le i \le 2^n, n \ge 0\}$ in Σ (that is, $A_1^0 = K$, for each $n \in \mathbb{N}$, $\{A_i^n : 1 \le i \le 2^n\}$ is a partition of K, and $A_i^n = A_{2i-1}^{n+1} \cup A_{2i}^{n+1}$, $1 \le i \le 2^n$, $n \ge 0$) such that $\lambda(A_i^n) = 2^{-n}$ for $1 \le i \le 2^n$ and $n \ge 0$. Let (x_n) be a sequence in X with $\|x_n\| = 1$ for $n \ge 0$. For each $n \ge 0$, choose $x_n^* \in X^*$ such that $\langle x_n^*, x_n\rangle = 1 = \|x_n^*\|$, and let $r_n = \sum_{i=1}^{2^n} (-1)^i \chi_{A_i^n}$. Then (r_n) is orthonormal in $L^2(\lambda)$, and thus weakly null in $L^1(\lambda)$. Define $T : C(K, X) \to c_0$ by

$$T(f) = \Big(\int_K \langle x_n^*, f(t)\rangle r_n(t)\, d\lambda\Big)_{n\ge 0}, \qquad f \in C(K, X).$$

We note that $T(f) \in c_0$ for each $f \in C(K, X)$, and that the representing measure m of T is given by

$$m(A)(x) = \Big(\langle x_n^*, x\rangle \int_A r_n(t)\, d\lambda\Big)_{n\ge 0}$$

for $A \in \Sigma$ and $x \in X$. As in the proof of the previous theorem, we have $\|m(A)\| \le \lambda(A)$, $m(A)(x) \in c_0$ for all $A \in \Sigma$, $x \in X$, and m is strongly bounded. Further, $m(A) : X \to c_0$ is compact, thus limited, for every $A \in \Sigma$. By assumption, T is limited.

Let $\hat{T}$ be the extension of T to $B(K, X)$. For each n, let $f_n = r_n x_n$. Note that $\|f_n\| \le 1$ and $\hat{T}(f_n) = e_n$. Since (e_n) is not limited in c_0, $\hat{T}$ is not limited. By Theorem 2, T is not limited. This contradiction concludes the proof. ∎

We recall that an operator $T : C(K,X) \to Y$ is compact if and only if its extension $\hat{T} : B(K,X) \to Y$ is compact (as noted in the Introduction). Since compact operators are in particular limited, the above argument and Remark 1 also prove the following result.

THEOREM 19. *The following statements are equivalent:*

(i) *K is dispersed.*

(ii) *For any pair of Banach spaces X and Y, an operator $m \leftrightarrow T : C(K,X) \to Y$ is compact if and only if m is strongly bounded and $m(A) : X \to Y$ is compact for every $A \in \Sigma$.*

(iii) *There is a Banach space X such that an operator $m \leftrightarrow T : C(K,X) \to c_0$ is compact if and only if m is strongly bounded and $m(A) : X \to c_0$ is compact for every $A \in \Sigma$.*

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