FUNDAMENTA
MATHEMATICAE
234 (2016)

Amenability, extreme amenability, model-theoretic stability, and dependence property in integral logic

by

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Abstract. This paper has three parts. First, we study and characterize amenable and extremely amenable topological semigroups in terms of invariant measures using integral logic. We prove definability of some properties of a topological semigroup such as amenability and the fixed point on compacta property. Second, we define types and develop local stability in the framework of integral logic. For a stable formula ϕ, we prove definability of all complete ϕ-types over models and deduce from this the fundamental theorem of stability. Third, we study an important property in measure theory, Talagrand's stability. We point out the connection between Talagrand's stability and dependence property (NIP), and prove a measure-theoretic version of definability of types for NIP formulas.

1. Introduction. Probability logics are logics of probabilistic reasoning. A model-theoretic approach aiming to study probability structures by logical tools was started by Hoover and Keisler (see [H, K] for a survey). Among several variants of this logic, they introduced integral logic $L_\int$ as an equivalent 'Daniell integral' presentation for $L_{\omega_1 P}$. Integral logic uses the language of measure theory, i.e., that of measurable functions and integration. The resulting framework is close to the usual language of probability theory and allows the formalization of much of probability. Bagheri and Pourmahdian [BP] developed a finitary version of integration logic, and proved appropriate versions of the compactness theorem and elementary JEP/AP. The intended models are graded probability structures introduced by Hoover [H], and in addition to random variables over probability spaces, they include dynamical systems and other interesting structures from real analysis. In [KB] the authors showed that many interesting notions such as probability independence, martingale property, and some special cases of the notion

2010 *Mathematics Subject Classification*: Primary 03B48, 03C45, 28C10, 43A07.
Key words and phrases: amenability, extreme amenability, integral logic, local stability, dependence property, continuous logic.
Received 10 September 2014; revised 30 December 2015.
Published online 2 May 2016.

DOI: 10.4064/fm208-1-2016

of conditional expectation (as in martingales) are expressible. Also, Kolmogorov's extension theorem was deduced from the compactness property of model theory. In [KA] the authors further used the logical tools to study invariant measures on compact Hausdorff spaces. Consequently, they gave two proofs of the existence of Haar measure on compact groups. One might therefore hope to obtain other applications of the compactness theorem.

Historically one of the great successes of model theory has been Shelah's stability theory. Essentially the success of the program is largely due to the fact that certain (local) combinatorial properties of formulas determine the corresponding global properties. On the other hand, a general trend in model theory is to generalize these model-theoretic notions and tools to frameworks that go beyond that of first order logic and elementary classes.

In the present paper, on the one hand, we study some analytic concepts, namely amenability and extreme amenability, using integral logic. On the other hand, we study types and local stability in this logic. This approach has two advantages. First, we underline the strengths of application of logical methods to other fields of mathematics. Second, the results obtained by these methods provide a new view on the related subjects in analysis and logic, and open some fruitful areas of research on similar questions.

To summarize the results of this paper, in the first part (Section 4), we consider an arbitrary topological semigroup S and any compact Hausdorff space X such that S acts continuously on X from the left. Let $\mathrm{Inv}_X(S)$ be the set of all Radon probability measures on X which are left invariant under elements of S. It is shown that the nonemptiness of $\mathrm{Inv}_X(S)$ is expressible by a theory $T_{S,X}$ in integral logic. We then present a characterization of amenable topological semigroups in terms of invariant measures (Fact 4.5). Using the compactness theorem, we give a proof of the fundamental result that goes back to N. N. Bogolyubov and N. M. Krylov (Theorem 4.11). The interesting fact is that for a topological semigroup S the amenability of S is expressible by a theory T_S in the framework of integral logic. Some other new results and different proofs of some known results are given for extremely amenable topological semigroups (Fact 4.20 and Proposition 4.22).

Although most of the results in the first part of the paper are standard, the study of amenable and extremely amenable semigroups is necessary because it leads us to the "true and correct" notion of type in integral logic. In fact, types are known mathematical objects, Riesz homomorphisms. Thus, for a complete theory T, the space of complete types $S(T)$ can be represented by the *spectrum* of T. Thereby, in the second part of the paper (Section 5), we define types and develop local stability. For a stable formula ϕ, we prove that all complete ϕ-types over models are definable, and we deduce from this the fundamental theorem of stability (Corollary 5.13). We show that a formula ϕ is stable if and only if its Cantor–Bendixson rank is finite.

In the third part of the paper (Section 6), we study a form of the dependence property, which is an important measure-theoretic property, *Talagrand's stability*. Then we prove that for an *almost dependent formula* ϕ, all ϕ-types are *almost* definable (Theorem 6.5). We then study the Cantor–Bendixson rank in almost dependent theories.

It is worth recalling another line of research arisen from ideas of Chang and Keisler [CK], namely continuous logic. The idea was recently refined and developed in [BBHU] and [BU] by Ben Yaacov, Berenstein, Henson, and Usvyatsov for the class of metric structures which include such important classes of structures as Banach spaces and measure algebras. Although some results in the present paper (cf. Section 5) are similar to those in [BU], in some senses they are different: (i) Our approach can be used to generalize the results in [BU] and [Mo] (see Remark 5.16). (ii) In [B3] and [B2], Ben Yaacov proved that the theory ARV and the category of probability algebras are $\aleph_0$-stable. Note that in this paper we do not study probability measure algebras or L^1-spaces, but we study measurable functions. In contrast to [B3] and [B2], the theory of a probability structure is not necessarily stable. This leads us to the dichotomy between stable probability structures and unstable probability structures. (iii) Some analytic properties such as probability independence, amenability, extreme amenability and the existence of invariant measures on compact spaces are expressible in the framework of integral logic.

After the submission of the present paper we came to know that, independently, similar ideas were used by P. Simon [S] in classical logic. We note that the argument for almost definability in the case of a dependent formula is truly measure-theoretic and can be used to prove some new results in classical logic. We will study this in a future work.

The organization of the paper is as follows. In the next section we review some basic notions from measure theory. In Section 3 a summary of results on integral logic from [BP] is given. In Section 4, we study amenable and extremely amenable topological semigroups, and give a characterization of [extreme] amenability in terms of [multiplicative] invariant measures. A proof of the Bogolyubov–Krylov theorem is given in Section 4. It is shown that the [extreme] amenability of a topological semigroup S is expressible by a theory T_S [$\mathtt{T}_S$] within integral logic. In Section 5, we conclude with the development of local stability, and we prove the fundamental theory of stability. In Section 6, we study NIP theories and give some results.

2. Preliminaries from topological measure theory. In this section we review some basic notions from measure theory. Further details can be found in [F, Fr1, Fr3]. Let X be a compact Hausdorff space. The space $C(X,\mathbb{R})$ of continuous real-valued functions on X is denoted by $C(X)$. Since

X is a compact space, every $f \in C(X)$ is bounded and $C(X)$ is a normed vector space with the uniform norm.

The class of *Baire sets* is defined to be the smallest σ-algebra $\mathcal{B}$ of subsets of X such that each function in $C(X)$ is measurable with respect to $\mathcal{B}$. The smallest σ-algebra containing the open sets is called the class of *Borel sets*. Clearly, every Baire set is a Borel set, but there are compact spaces where the class of Borel sets is larger than the class of Baire sets. By a *Baire* [*Borel*] *measure* on X we mean a finite measure defined for all Baire [Borel] sets. A *Radon measure* on X is a Borel measure which is regular. It is known that every Baire measure on a compact space is regular and has a unique extension to a Radon measure.

A *topological semigroup* is a semigroup S endowed with a Hausdorff topology such that the operation $(x,y) \mapsto xy$ is continuous from $S \times S$ to S. By a *topological group* we mean a group G endowed with a Hausdorff topology such that the group operations $(x,y) \mapsto xy$ and $x \mapsto x^{-1}$ are continuous from $G \times G$ and G to G. A topological group whose topology is [locally] compact and Hausdorff is called a [*locally*] *compact group*.

A topological semigroup S is said to *act on a topological space X from the left* if there is a map $S \times X \to X$ (denoted by $(s,x) \mapsto s \cdot x$ for each $(s,x) \in S \times X$) such that (a) the map $x \mapsto s \cdot x$ is continuous for each $s \in S$, (b) for $s, s' \in S$, $(ss') \cdot x = s \cdot (s' \cdot x)$ for each $x \in X$, and (c) if S has the identity e, then $e \cdot x = x$ for each $x \in X$. In addition, the left action is said to be *continuous* if $(s,x) \mapsto s \cdot x$ is a continuous map from $S \times X$ to X. Similarly one can define a right (continuous) action. If S acts on a topological space X from the left [right] and $E \subseteq X$ and $s \in S$, we define

$$s \cdot E = \{s \cdot x : x \in E\} \quad [E \cdot s = \{x \cdot s : x \in E\}].$$

If f is a continuous real-valued function on a topological space X and $s \in S$, we define the *left* [*right*] *translate of f by s* as follows:

$$(f \cdot s)(x) = f(s \cdot x) \quad [(s \cdot f)(x) = f(x \cdot s)].$$

The point of the above definition is to make $f \cdot (ss') = (f \cdot s) \cdot s'$ $[(ss') \cdot f = s \cdot (s' \cdot f)]$.

If a topological semigroup S acts on a space X from the left [right], then a measure μ on X is *left* [*right*] *S-invariant* if $\mu(s \cdot E)$ $(\mu(E \cdot s))$ is defined and equal to $\mu(E)$ whenever $s \in S$ and E is μ-measurable. If X is a compact Hausdorff space, then a linear functional I on $C(X)$ is called *left* [*right*] *S-invariant* if $I(f \cdot s) = I(f)$ $[I(s \cdot f) = I(f)]$ for all s in S and f in $C(X)$.

A *left* [*right*] *Haar measure* on a compact group G is a nonzero left (right) G-invariant Radon measure μ on G.

PROPOSITION 2.1 ([Fr3, Proposition 441L]). *Let X be a Hausdorff compact space and S a topological semigroup which acts on X. A nonzero Radon* *measure μ on X is a left [right] S-invariant measure iff* $\int f\,d\mu = \int (f\cdot s)\,d\mu$ [$\int f\,d\mu = \int (s\cdot f)\,d\mu$] *for all* $f\in C(X)$ *and* $s\in S$.

If G is a compact group, then a left Haar measure on G is also a right Haar measure. Moreover, the Haar measure is unique up to a positive scalar multiple, i.e. if μ and ν are Haar measures on a compact group G, there exists $c>0$ such that $\mu = c\nu$.

THE RIESZ REPRESENTATION THEOREM. *Let X be a locally compact Hausdorff space and $C_c(X)$ the space of continuous real-valued functions on X with compact support.*

(a) ([F, p. 212]) *If I is a positive linear functional on $C_c(X)$, there is a unique Radon measure μ on X such that* $I(f) = \int f\,d\mu$ *for all* $f\in C_c(X)$.
(b) ([R, p. 358]) *If X is compact, then the dual of $C(X)$ is* (*isometrically isomorphic to*) *the space of all finite signed Baire measures on X with norm defined by* $\|\mu\| = |\mu|(X)$.

THE HAHN–BANACH THEOREM ([F, p. 159]). *Let $\mathcal{N}$ be a normed vector space. If $\mathcal{M}$ is a closed subspace of $\mathcal{N}$ and $x\in\mathcal{N}\setminus\mathcal{M}$, then there exists a bounded linear functional I on $\mathcal{N}$ such that* $I|_{\mathcal{M}}=0$, $\|I\|=1$ *and* $I(x)=\inf_{y\in\mathcal{M}}\|x-y\|$.

Let $(M,\mathcal{B},\mu)$ be a measure space and μ^* its associated outer measure defined by

$$\mu^*(X) = \inf\{\mu(A) : X\subseteq A\in\mathcal{B}\}.$$

If $N\subseteq M$, then $\mathcal{B}_N = \{A\cap N : A\in\mathcal{B}\}$ is a σ-algebra and $\mu_N = \mu^*\restriction\mathcal{B}_N$ is a measure on N. We call μ_N the *subspace measure* on N. Furthermore, a *measurable envelope* for N is a measurable set $E\in\mathcal{B}$ such that $N\subseteq E$ and $\mu(E\cap A) = \mu^*(N\cap A)$ for any $A\in\mathcal{B}$. Every $N\subseteq M$ of finite outer measure has an envelope (e.g. take $E\in\mathcal{B}$ containing N with $\mu(E)=\mu^*(N)$). If $f: M\to\mathbb{R}$ is measurable, $\int_N f$ abbreviates $\int_N (f\restriction N)\,d\mu_N$.

PROPOSITION 2.2 ([Fr1, p. 38]). *Let $(M,\mathcal{B},\mu)$ be a measure space, $N\subseteq M$, and f be an integrable function defined on M.*

(a) *If f is nonnegative then $f\restriction N$ is μ_N-integrable and* $\int_N f\le\int f$.
(b) *If either N is of full outer measure in M or f is zero almost everywhere on $M-N$, then* $\int_N f = \int_M f$.

3. Integral logic. In this section we give a brief review of integral logic from [BP, KA]. Results from those papers are stated without proof. All languages are assumed to contain a unary relation and constant symbols. Let L be a language. To each relation symbol $R\in L$ we assign a nonnegative real number $\flat_R \geq 0$ called the *universal bound* of R. The terms are just the constant symbols and the variables.

Definition 3.1. The family of *L-formulas* and their *universal bounds* is defined as follows:

(1) If R is a relation symbol and t is a term, then $R(t)$ is an atomic formula with bound $\flat_R$.
(2) If ϕ and ψ are formulas and $r, s \in \mathbb{R}$, then $r\phi + s\psi$ and $\phi \times \psi$ are formulas with bounds $|r|\flat_\phi + |s|\flat_\psi$ and $\flat_\phi \flat_\psi$, respectively.
(3) If ϕ is a formula, then $|\phi|$ is a formula with bound $\flat_\phi$.
(4) If ϕ is a formula and x is a variable, then $\int \phi\, dx$ is a formula with bound $\flat_\phi$.

Note that $\phi^+ = \frac{1}{2}(\phi + |\phi|)$ and $\max(\phi, \psi) = (\phi - \psi)^+ + \psi$, and similarly ϕ^- and $\min(\phi, \psi)$, are formulas.

Definition 3.2. An *L-structure* is a probability measure space $\mathsf{M} = (M, \mathcal{B}, \mu)$ equipped with:

- for each constant symbol $c \in L$, an element $c^{\mathsf{M}} \in M$;
- for each relation symbol $R \in L$, a measurable map $R^{\mathsf{M}} : M \to [-\flat_R, \flat_R]$.

L-structures are denoted by M, N etc. The notion of free variable is defined as usual, and one writes $\phi(\bar{x})$ (or $\phi(x_1, \ldots, x_n)$) to display them. If M is an L-structure, for each formula $\phi(x_1, \ldots, x_n)$ and $\bar{a} \in M^n$, $\phi^{\mathsf{M}}(\bar{a})$ is defined inductively starting from atomic formulas. In particular,

$$\Big(\int \phi(\bar{x}, y)\, dy\Big)^{\mathsf{M}}(\bar{a}) = \int \phi^{\mathsf{M}}(\bar{a}, y)\, dy.$$

An easy induction shows that every $\phi^{\mathsf{M}}(\bar{x})$ is a well-defined measurable function from M^n to $[-\flat_\phi, \flat_\phi]$. Indeed, for every $\phi(\bar{x}, \bar{y})$ and $\bar{a}$, $\phi^{\mathsf{M}}(\bar{a}, \bar{y})$ is measurable. Moreover, we have $\int\int \phi\, dx\, dy = \int\int \phi\, dy\, dx$.

A formula is *closed* if no free variable occurs in it. A *statement* is an expression of the form $\phi(\bar{x}) \geq r$ or $\phi(\bar{x}) = r$. Closed statements are defined similarly. Any set of closed statements is called a *theory*. The theory of a structure M is the collection of closed statements satisfied in it. Such theories are called *complete*. We call M, N *elementarily equivalent* (written $\mathsf{M} \equiv \mathsf{N}$) if they have the same theory. The notion $\mathsf{M} \vDash \Gamma$ is defined in the obvious way. If T is an L-theory, two formulas $\phi(\bar{x}), \psi(\bar{x})$ are said to be *T-equivalent* if the statement $\phi = \psi$ a.e. is satisfied in every model of T. We say T has *quantifier elimination* if every formula is T-equivalent to a quantifier-free formula (i.e. without $\int$).

The ultaproduct of a family M_i, $i \in I$, of structures over an ultrafilter $\mathcal{D}$ is an L-structure and is denoted by $\mathsf{M} = \prod_{\mathcal{D}} \mathsf{M}_i$ (cf. [BP, KA]).

THEOREM 3.3 (Fundamental theorem). *For each $\phi(\bar{x})$ and $[a_i^1],\dots,[a_i^n]$ in M we have*

$$\phi^{\mathsf{M}}([a_i^1],\dots,[a_i^n]) = \lim_{\mathcal{D}} \phi^{\mathsf{M}_i}(a_i^1,\dots,a_i^n).$$

An immediate consequence of the fundamental theorem is the following; the proof is just a modification of its analog in the usual first order logic.

THEOREM 3.4 (Compactness theorem). *Any finitely satisfiable set of closed statements is satisfiable.*

DEFINITION 3.5. (i) If $M \subseteq N$, then M is a *substructure* of N, written $\mathsf{M} \subseteq \mathsf{N}$, if M has the subspace measure and for each $R \in L$ and $\bar{a} \in M$, $R^{\mathsf{M}}(\bar{a}) = R^{\mathsf{N}}(a)$. If these equalities hold for almost all $\bar{a}$, then M is called an *almost substructure* of N, written $\mathsf{M} \subseteq_a \mathsf{N}$.

(ii) An injection $f : \mathsf{M} \to \mathsf{N}$ is called an *elementary embedding* if for each ϕ and $\bar{a} \in M$, $\phi^{\mathsf{M}}(\bar{a}) = \phi^{\mathsf{N}}(f(\bar{a}))$. It is an *almost elementary embedding* if for each ϕ this holds almost surely for $\bar{a} \in M$. If f is the inclusion, these are respectively denoted by $\mathsf{M} \preceq \mathsf{N}$ and $\mathsf{M} \preceq_a \mathsf{N}$. We say f is *almost surjective* if its range has full measure. One also defines *isomorphism* (resp. *almost isomorphism*) as a surjective (resp. almost surjective) elementary (resp. almost elementary) embedding.

The fact that $\preceq$ (resp. $\preceq_a$) is stronger than $\subseteq$ (resp. $\subseteq_a$) is a consequence of the Tarski–Vaught test (see below). Of the two notions of isomorphism, almost isomorphism is more useful for us, although exact isomorphism appears naturally in some cases. In ergodic theory, a map which is an (exact) isomorphism after removing some negligible sets from its domain and codomain is called an isomorphism. This notion is equivalent to our notion of almost isomorphism.

A structure is called *minimal* if it has no redundant measurable sets, i.e., for any substructure $\mathsf{M}' = (M, \mathcal{A}, \mu{\restriction}\mathcal{A})$ where $\mathcal{A} \subseteq \mathcal{B}$, one has $\mathcal{A} = \mathcal{B}$. In fact, every structure is isomorphic to a minimal structure, which can be explicitly described.

PROPOSITION 3.6. *Let $\mathsf{M} = (M, \mathcal{B}, \mu)$ be an L-structure, and $\mathcal{A}$ be the σ-algebra generated by the sets of the form $\{x \in M : \phi^{\mathsf{M}}(x) > 0\}$ where ϕ is any formula with parameters in M. Then $\mathsf{M}' = (M, \mathcal{A}, \mu{\restriction}\mathcal{A})$ is a minimal measure L-structure isomorphic to M.*

PROPOSITION 3.7 (Tarski–Vaught test for $\preceq$). *Let M, N be minimal. If $M \subseteq N$ then $\mathsf{M} \preceq \mathsf{N}$ if and only if for each $\phi(\bar{a}, x)$, where $\bar{a} \in M$, the intersection of the set $\{x \in N : \phi^{\mathsf{N}}(\bar{a}, x) > 0\}$ with M is μ_{M}-measurable and has the same measure. A similar statement holds for $\mathsf{M} \preceq_a \mathsf{N}$ with "for almost all $\bar{a}$" in place of "for each $\bar{a}$". In both cases, $\mu_{\mathsf{M}} = \mu_{\mathsf{N}}{\restriction}M$.*

Next we are going to prove a key result, which plays an important role in the rest of this paper. Assume that X is a compact Hausdorff space. Let L_X be the language consisting of a unary relation symbol R_f for each $f \in C(X)$ and a constant symbol c_a for each $a \in X$. Let M be an L_X-structure with the following properties:

- $X \subseteq M$;
- the restriction of R_f^{M} to X is f, in particular $R_1^{\mathsf{M}} = \mathbf{1}$;
- $R_{f+g}^{\mathsf{M}} = R_f^{\mathsf{M}} + R_g^{\mathsf{M}}$ and $R_{r\times f}^{\mathsf{M}} = r \cdot R_f^{\mathsf{M}}$ for all $f, g \in C(X)$ and real numbers r;
- $R_{f\times g}^{\mathsf{M}} = R_f^{\mathsf{M}} \times R_g^{\mathsf{M}}$ for all $f, g \in C(X)$;
- $R_{\max(f,g)}^{\mathsf{M}} = \max(R_f^{\mathsf{M}}, R_g^{\mathsf{M}})$ for all $f, g \in C(X)$.

The next proposition shows that the subspace measure μ_X on X behaves like the measure μ on M. In fact, $(X, \mathcal{B}_X, \mu_X)$ with the natural interpretation of relation and constant symbols is an elementary substructure of M.

Proposition 3.8. *Assume that X and M are as above.*

(a) *The subspace measure μ_X on X is a regular Baire measure such that $\int f\, d\mu_X = \int R_f^{\mathsf{M}}\, d\mu$ for each $f \in C(X)$.*
(b) *There exists a Radon measure $\bar{\mu}_X$ on X such that $\int f\, d\bar{\mu}_X = \int R_f^{\mathsf{M}}\, d\mu$ for each $f \in C(X)$.*

Proof. (a) By Proposition 2.2, it suffices to show that X is of full outer measure in M. We assume that M is minimal. By Proposition 3.6,

$$\mu_X(X) = \inf\Big\{\sum_{k=1}^{\infty} \mu(A_k) : X \subseteq \bigcup_{k=1}^{\infty} A_k\Big\}$$

where $A_k = (R_{f_k}^{\mathsf{M}})^{-1}(0,\infty)$ for an $f_k \in C(X)$ because every formula ϕ is equal to a relation symbol R_f. We show that $\mu(\bigcup_k A_k) = 1$ for every sequence $\langle A_k\rangle_{k\in\mathbb{N}}$ such that $X \subseteq \bigcup_{k=1}^{\infty} A_k$. If $X \subseteq \bigcup_k f_k^{-1}(0,\infty)$, then there exist $f_1, \dots, f_n$ such that $X = \bigcup_{k=1}^{n} f_k^{-1}(0,\infty)$ because X is compact. If $f = \max(f_1, \dots, f_n)$, then $X = f^{-1}(0,\infty)$. Thus, $X \subseteq (R_f^{\mathsf{M}})^{-1}(0,\infty)$ because $R_f^{\mathsf{M}} = \max(R_{f_1}^{\mathsf{M}}, \dots, R_{f_n}^{\mathsf{M}})$. Since X is compact and f is continuous, there exist real numbers $s \geq r > 0$ such that $X = f^{-1}[r,s]$. Also, we can easily check that $M = (R_f^{\mathsf{M}})^{-1}(0,\infty)$ since $R_f^{\mathsf{M}} \geq r$. Thus, $\mu(\bigcup_k A_k) \geq \mu((R_f^{\mathsf{M}})^{-1}(0,\infty)) = 1$, i.e. $\mu_X(X) = 1$. We may assume that μ_X is a Baire measure. Also, we know that every Baire measure on a compact space is regular.

(b) It is known that every Baire regular measure on a compact space has a unique extension to a Radon measure (cf. [R, p. 341]). Let $\bar{\mu}_X$ be the unique extension of μ_X to a Radon measure on X. Since only the values of $\bar{\mu}_X$ on Baire sets matter for $\int f\,d\bar{\mu}_X$, we have $\int f\,d\bar{\mu}_X = \int f\,d\mu_X$ for each $f \in C(X)$. ∎

4. Amenability and extreme amenability. In this section we study and characterize amenable and extremely amenable topological semigroups in terms of invariant measures using integral logic. First, we give two conditions equivalent to the existence of measures on a compact Hausdorff space X invariant under a semigroup S which acts on X from the left. We then characterize [extremely] amenable topological semigroups in terms of [multiplicative] invariant measures. It is shown that all compact groups, all abelian topological semigroups, and all locally finite topological groups are amenable. An interesting fact is that for a topological semigroup S the [extreme] amenability of S is expressible by a theory T_S [$\mathtt{T}_S$] in the framework of integral logic. Therefore, it is shown that a locally compact group G has no Borel paradoxical decomposition iff the theory T_G is satisfiable.

Let X be a compact Hausdorff space and S be a semigroup which acts on X from the left. Let L_X be the language consisting of a unary relation symbol R_f for each $f \in C(X)$ and a constant symbol c_a for each $a \in X$, and let $T_{S,X}$ be the theory with the following axioms:

(1) $R_1 = \mathbf{1}$,
(2) $\int R_1\,dx = 1$,
(3) $R_f(c_a) = f(a)$ for all $R_f, c_a \in L_X$,
(4) $R_{f+g} = R_f + R_g$ for all $R_f, R_g \in L_X$,
(5) $R_{r\times f} = r \times R_f$ for all $R_f \in L_X$ and $r \in \mathbb{R}$,
(6) $R_{f\times g} = R_f \times R_g$ for all $R_f, R_g \in L_X$,
(7) $R_{\max(f,g)} = \max(R_f, R_g)$ for all $R_f, R_g \in L_X$,
(8) $\int R_f(x)\,dx = \int R_{(f\cdot s)}(x)\,dx$ for all $R_f \in L_X$ and $s \in S$, where $(f\cdot s)(x) = f(s\cdot x)$.

Note that (1) says that the interpretation of R_1 is the constant function $\mathbf{1}$, (2) means that we have a probability measure, (3) says that f is a subset of the interpretation of R_f, (4)–(7) that the family of the interpretations of relation symbols is a vector lattice, and (8) means that the measure is left S-invariant. We call $T_{S,X}$ the theory of *left S-invariant measures on X*.

As a consequence of the compactness theorem we give conditions equivalent to the existence of a left S-invariant Radon measure on X. Later, we give results based on these conditions. Recall $\mathrm{Inv}_X(S)$ is the set of all regular Borel probability measures on X which are left S-invariant.

PROPOSITION 4.1. *Assume that S, X and $T_{S,X}$ are as above. Then the following are equivalent:*

(i) $\mathrm{Inv}_X(S) \neq \emptyset$.
(ii) $T_{S,X}$ *is satisfiable.*

Proof. (i)⇒(ii) is obvious. For the converse, let M be a model of $T_{S,X}$. By Urysohn's lemma, one can easily verify that $X \subseteq M$. By Proposition 3.8(b), there exists a Radon measure $\bar{\mu}_X$ on X such that $\int f\,d\bar{\mu}_X = \int R_f^{\mathsf{M}}\,d\mu$ for each $f \in C(X)$. Therefore, $\bar{\mu}_X$ is a nonzero regular Borel left S-invariant measure on X. ∎

The following classical result gives a condition equivalent to the existence of a left S-invariant Radon measure on X (see [HR, Theorem 17.15]).

Fact 4.2. *Let S be a semigroup with identity. If S acts from the left on a compact Hausdorff space X, then the following are equivalent:*

(i) $\mathrm{Inv}_X(S) \neq \emptyset$.
(ii) *For all $s_1, \dots, s_n \in S$ and $f_1, \dots, f_n \in C(X)$ we have*

$$\Big\|\mathbf{1} - \sum_{i=1}^{n}(f_i \cdot s_i - f_i)\Big\| \geq 1.$$

Proof. (i)⇒(ii). Let $h = \sum_{i=1}^{n}(f_i \cdot s_i - f_i)$. If $\sup_{x\in X}|\mathbf{1} - h(x)| = 1 - \epsilon$ where ϵ is a positive real number, then $\epsilon < h(x) < 2$ for all $x \in X$, so $\int h\,d\mu > \epsilon$ for every probability measure μ on X, i.e., $\mathrm{Inv}_X(S) = \emptyset$.

(ii)⇒(i). Let L_X and $T_{S,X}$ be as above. By Proposition 4.1, it suffices to show that the theory $T_{S,X}$ is finitely satisfiable. Assume that Γ is a finite subset of $T_{S,X}$ such that for each $i \leq n$ and $j \leq m$ the statement $\int R_{f_i}\,dx = \int R_{f_i\cdot s_j}\,dx$ is in Γ. Thus, $f_1, \dots, f_n$ are in $C(X)$ and $s_1, \dots, s_m$ are in S. Let M be the closure of the subspace generated by $f_i - f_i \cdot s_j$ for each $i \leq n$ and $j \leq m$. Since S has an identity, clearly $\inf_{h\in M}\|\mathbf{1} - h\| = 1$. Let K be a subspace of $C(X)$ such that $M + K = C(X)$ and $M \cap K = 0$. By the Hahn–Banach theorem, define I to be 0 on M and a nonzero bounded linear functional on K such that $I(\mathbf{1}) = \|I\| = 1$. By the Riesz representation theorem, there exists a signed Baire measure μ on X such that $\int (f_i - f_i \cdot s_j)\,d\mu = 0$ for each $i \leq n$ and $j \leq m$. Also, μ is a nonzero positive measure because $\mu(X) = \int \mathbf{1}\,d\mu = I(\mathbf{1}) = \|I\| = |\mu|(X)$. Hence (X, μ) with the natural interpretation of relation and constant symbols is a model of Γ. ∎

4.1. Amenability. In this subsection we define amenable topological semigroups and characterize them in terms of invariant measures. Also, we show that all compact groups and locally finite topological groups are amenable. Let S be a topological semigroup, and $C_b(S)$ the Banach space of all bounded real-valued continuous functions on S with the usual supremum norm. For $s \in S$ and $f \in C_b(S)$, let $f \cdot s$ and $s \cdot f$ be the elements in $C_b(S)$

defined by

$$(f \cdot s)(t) = f(st) \quad \text{and} \quad (s \cdot f)(t) = f(ts), \qquad t \in S,$$

respectively. A subspace E of $C_b(S)$ is *left* [*right*] *invariant* if $f \cdot s \in E$ [$s \cdot f \in E$] for all $s \in S$ and $f \in E$. If E is both left and right invariant, then E is called *invariant.*

Let E be a left invariant closed subspace of $C_b(S)$ that contains $\mathbf{1}$, the constant 1 function on S. A *mean* on E is a linear functional I on E such that

(1) $I(\mathbf{1}) = 1$,
(2) $I(f) \geq 0$ if $f \geq 0$.

A mean I on a left [right] invariant closed subspace E of $C_b(S)$ that contains $\mathbf{1}$ is said to be *left* [*right*] *invariant* if $I(f \cdot s) = I(f)$ [$I(s \cdot f) = I(f)$] for all $f \in E$ and $s \in S$.

We define the subspace $\mathrm{LUC}(S)$ of all *left uniformly continuous* functions in $C_b(S)$, which plays an important role in the rest of this paper. For a topological semigroup S set

$$\mathrm{LUC}(S) = \{f \in C_b(S) : \text{the map } s \mapsto f \cdot s \text{ is (norm) continuous from } S \text{ to } C_b(S)\}.$$

Similarly one can define the subspace $\mathrm{RUC}(S)$ of all *right uniformly continuous* functions in $C_b(S)$. It is known that $\mathrm{LUC}(S)$ and $\mathrm{RUC}(S)$ are closed and invariant subalgebras of $C_b(S)$. They are also closed under the lattice operations (cf. [N, Lemmas 1.1 and 1.2]). Therefore, $\mathrm{LUC}(S)$ and $\mathrm{RUC}(S)$ are M-spaces with the unit $\mathbf{1}$.

Definition 4.3. A topological semigroup S is said to be *left* [*right*] *amenable* if $\mathrm{LUC}(S)$ [$\mathrm{RUC}(S)$] admits a left [right] invariant mean. A topological semigroup S is called *amenable* if it is both left and right amenable.

We now characterize amenable topological semigroups in terms of invariant measures, for which we need the following lemma.

Lemma 4.4. *Let S be a topological semigroup.*

(i) *If X is a closed and invariant subset of $\{I \in \mathrm{LUC}(S)^* : \|I\| = 1\}$, then the natural action of S on X is continuous.*
(ii) *If X is a compact Hausdorff space and $\cdot$ is a continuous action of S on X (from the left), then for each $f \in C(X)$ the map $s \mapsto f \cdot s$ from S to $C(X)$ is (norm) continuous.*

Proof. (i) Assume that $s, s' \in S$ and $I, I' \in X$. Then for each f in $\mathrm{LUC}(S)$ we have

$$\begin{aligned}|(s'\cdot I')(f)-(s\cdot I)(f)| &= |I'(f\cdot s')-I(f\cdot s)|\\ &\le |I'(f\cdot s')-I'(f\cdot s)|+|I'(f\cdot s)-I(f\cdot s)|\\ &= |I'(f\cdot s'-f\cdot s)|+|I'(f\cdot s)-I(f\cdot s)|\\ &\le \|I'\|\times\|f\cdot s'-f\cdot s\|+|I'(s\cdot f)-I(s\cdot f)|\\ &= \|f\cdot s'-f\cdot s\|+|I'(f\cdot s)-I(f\cdot s)|.\end{aligned}$$

Therefore the continuity of $(s, I)\mapsto I\cdot s$ follows from that of $s\mapsto f\cdot s$.

(ii) Let $f\in C(X)$, $s_0\in S$ and $\epsilon>0$, and let U be the subset of $S\times X$ given by $U=\{(s,x):|f(s_0\cdot x)-f(s\cdot x)|<\epsilon\}$. Then U is open and $\{s_0\}\times X\subseteq U$. Hence there is a neighborhood V of s_0 such that $V\times X\subseteq U$, and it follows that $\|f\cdot s_0-f\cdot s\|<\epsilon$ whenever $s\in V$. ∎

We now give a classical result.

FACT 4.5. *Let S be a topological semigroup with identity. Then the following are equivalent:*

(i) *S is left amenable.*
(ii) *Whenever X is a nonempty compact Hausdorff space and $\cdot$ is a continuous action of S on X (from the left), then $\mathrm{Inv}_X(S)\neq\emptyset$.*

Proof. (i)⇒(ii). By Fact 4.2 it suffices to show that $\sup_{x\in X}|\mathbf{1}-h(x)|\ge 1$ for h of the form $\sum_{i=1}^n(f_i\cdot s_i-f_i)$ where $s_1,\dots,s_n\in S$ and $f_1,\dots,f_n\in C(X)$. If not, then $\sup_{x\in X}h(x)<0$. Let I be a left invariant mean on $\mathrm{LUC}(S)$. Fix a positive linear functional Λ on $C(X)$. Define $\widetilde{f}:S\to\mathbb{R}$ by $\widetilde{f}(s)=\Lambda(f\cdot s)$ for each $f\in C(X)$. We claim that $\widetilde{f}\in\mathrm{LUC}(S)$. By Lemma 4.4(ii), the map $s\mapsto f\cdot s$ is norm continuous from S to $C(X)$. It is easy to verify that the continuity of $s\mapsto\widetilde{f}\cdot s$ follows from that of $s\mapsto f\cdot s$. Define $J:C(X)\to\mathbb{R}$ by $J(f)=I(\widetilde{f})$. Obviously J is a left invariant positive functional on $C(X)$. Therefore, $J(h)=0$ since J is invariant. But $J(h)<0$ since J is positive and $h<0$.

(ii)⇒(i). It is easy to check that the set $M_U(S)$ of all means on $\mathrm{LUC}(S)$ is a weak* compact subset of $\mathrm{LUC}(S)^*$. Note that by Lemma 4.4(i), the natural action of S from the left on $M_U(S)$ is continuous. Let μ be a left S-invariant Radon probability measure on $M_U(S)$. Define $I_\mu:\mathrm{LUC}(S)\to\mathbb{R}$ by $I_\mu(g)=\int\widehat{g}\,d\mu$, where $\widehat{g}:M_U(S)\to\mathbb{R}$ is given by $\widehat{g}(J)=J(g)$. Clearly, I_μ is a left invariant mean on $\mathrm{LUC}(S)$. ∎

REMARK 4.6. If $S=G$ is a locally compact group, then an invariant mean on $\mathrm{LUC}(G)$ extends to an invariant mean on the space $C_b(G)$ of all bounded real-valued continuous functions on G (cf. [Ru, Theorem 1.1.9, p. 21]).

A topological semigroup can be left, but not right, amenable (e.g., consider the semigroup $S=\{a,b\}$ with the following operation: $a\cdot a=b\cdot a=a$,

$a \cdot b = b \cdot b = b$). Of course, if S is a topological group, then S is amenable if and only if it is left (or right) amenable. Basically this depends on the fact that the operation $g \mapsto g^{-1}$ transposes the order of products, and therefore interchanges left and right. Also, we will show that any abelian topological semigroup is (both left and right) amenable (Corollary 4.12).

Thanks to compactness of integral logic we have the following fact.

Proposition 4.7. *Let S be a topological semigroup with identity. Suppose that there is a family $\{S_\alpha\}_{\alpha\in I}$ of subsemigroups of S such that*

(i) *$\bigcup_{\alpha\in I} S_\alpha$ is dense in S;*
(ii) *S_α is an amenable subsemigroup with identity for all $\alpha \in I$;*
(iii) *for any $\alpha_1, \alpha_2 \in I$, there exists $\alpha_3 \in I$ such that $S_{\alpha_1} \cup S_{\alpha_2} \subseteq S_{\alpha_3}$.*

Then S is also amenable.

Proof. Let $S' = \bigcup_{\alpha\in I} S_\alpha$ and X be a compact Hausdorff space and $\cdot$ a left continuous action of S' on X. By assumption, the theory $T_{S',X}$ of left S'-invariant measures on X is finitely satisfiable. By Proposition 4.1, as X and $\cdot$ are arbitrary, S' is amenable. Assume that I is an S'-invariant mean on $\mathrm{LUC}(S')$. Define $J : \mathrm{LUC}(S) \to \mathbb{R}$ by $J(f) = I(f{\restriction}S')$ for each $f \in \mathrm{LUC}(S)$. We can easily check that J is a left invariant mean on $\mathrm{LUC}(S)$ because S' is dense. Similarly, one can show that S is right amenable. ∎

Corollary 4.8. *If every finitely generated subsemigroup* (*with identity*) *of a topological semigroup S is amenable, then S is also amenable.*

Note that the converse may fail. As an example let S' be any finitely generated nonamenable semigroup (e.g., the free group on two generators), and let S be a semigroup containing S' and one new element s_0 such that $s_0 s = s s_0 = s_0 s_0 = s_0$ for all $s \in S'$. Then S has an invariant mean $I(f) = f(s_0)$, but the subsemigroup S' does not.

It is known that every locally compact group has a Haar measure (cf. [F]), but not every locally compact group is amenable. The free group on two generators, with the discrete topology, is a nonamenable locally compact group (cf. [Fr3, Example 449G, p. 399]). Of course, every compact group is amenable. Indeed, assume that G acts continuously from the left on a compact Hausdorff space X. Fix $x_0 \in X$ and set $\phi(a) = a \cdot x_0$ for $a \in G$; then ϕ is continuous. Let μ be the Haar probability measure on G, and ν the Radon probability measure $\mu\phi^{-1}$ on X. Clearly ν is G-invariant. As X and $\cdot$ are arbitrary, we have the following fact.

Fact 4.9. *Every compact group is amenable.*

A group G is called *locally finite* if every finite subset of G generates a finite subgroup of G. An immediate consequence of the above results is the following.

Corollary 4.10. *Let G be a topological group such that the union of the finite subsets of G that generate a compact subgroup is dense. Then G is amenable. In particular, every locally finite topological group is amenable.*

4.2. Commutativity. The usual proof of the Bogolyubov–Krylov theorem uses the Markov–Kakutani fixed point theorem. Now, we give a proof of this theorem by using the compactness theorem and induction.

Theorem 4.11 (Bogolyubov–Krylov). *Assume that S is an abelian semigroup which acts from the left on a compact Hausdorff space X. Then $\mathrm{Inv}_X(S) \neq \emptyset$.*

Proof. By Proposition 4.1, it suffices to consider the case where S is finite. We prove the theorem by induction on the number of elements of S. Let $\mathcal{D}$ be a nonprincipal ultrafilter on $\mathbb{N}$ and x_0 any point of X. If $S = \{s\}$, then define μ_1 by

$$\int f \, d\mu_1 = \lim_{n \to \mathcal{D}} \frac{1}{n+1} \sum_{k=0}^{n} (f \cdot s^k)(x_0) \quad \text{for } f \in C(X).$$

It is easy to check that μ_1 is invariant with respect to s. By induction hypothesis, there exists a measure ν on X which is invariant with respect to $s_1, \ldots, s_{n-1}$. By the Riesz representation theorem, define the measure μ by

$$\int f \, d\mu = \lim_{n \to \mathcal{D}} \frac{1}{n+1} \sum_{k=0}^{n} \int (f \cdot s_n^k) \, d\nu \quad \text{for } f \in C(X).$$

We can easily check that μ is invariant with respect to $s_1, \ldots, s_n$. Indeed, it is easy to verify that μ is s_n-invariant. Also, for each $i \leq n-1$, we have

$$\begin{aligned}
\int (f \cdot s_i) \, d\mu &= \lim_{n \to \mathcal{D}} \frac{1}{n+1} \sum_{k=0}^{n} \int (f \cdot s_i) \cdot s_n^k \, d\nu \\
&= \lim_{n \to \mathcal{D}} \frac{1}{n+1} \sum_{k=0}^{n} \int (f \cdot s_n^k) \cdot s_i \, d\nu \quad \text{(commutativity)} \\
&= \lim_{n \to \mathcal{D}} \frac{1}{n+1} \sum_{k=0}^{n} \int (f \cdot s_n^k) \, d\nu \quad (\nu \text{ is } s_i\text{-invariant}) \\
&= \int f \, d\mu.
\end{aligned}$$

Therefore, μ is the desired measure, so the theorem follows. ∎

An immediate consequence of the Bogolyubov–Krylov theorem is the following.

Corollary 4.12. *Any abelian topological semigroup is amenable.*

Theorem 4.11 gives another proof of the existence of Haar measure on abelian compact groups. By the same method one can also give a functional-analytic proof of the existence of Haar measures on abelian *locally* compact groups. We will present a proof of this theorem using the same method elsewhere.

COROLLARY 4.13 (Mazur–Orlicz). *Let $\mathcal{F}$ be a family of commuting mappings of a set X onto itself. Then there exists a mean on $B(X)$, the space of all bounded real-valued functions on X, which is $\mathcal{F}$-invariant. In particular, every closed linear subspace E of $B(X)$ such that $f \circ h \in E$ whenever $f \in E$ and $h \in \mathcal{F}$ has an $\mathcal{F}$-invariant mean.*

Proof. Use Theorem 4.11. ∎

4.3. Paradoxical decompositions. The problematics of amenability has grown out of the famous Banach–Tarski paradox (which essentially amounts to the nonamenability of the free group on two generators). We continue this paper by looking at the connection between satisfiability and paradoxical decompositions. Let G be a discrete group acting on a nonempty set X. Then $E \subseteq X$ is called *G-paradoxical* if there are pairwise disjoint subsets $A_1, \dots, A_m, B_1, \dots, B_n$ of E along with $g_1, \dots, g_m, h_1, \dots, h_n \in G$ such that $E = \bigcup_{i=1}^m g_i \cdot A_i = \bigcup_{i=1}^n h_i \cdot B_i$. We say X is *G-paradoxical* if it has a G-paradoxical subset. A group G is called *paradoxical* if it is G-paradoxical. Clearly an amenable group is nonparadoxical. A remarkable fact is that the converse is also true, which follows from the following result of Tarski.

THEOREM 4.14 ([Ru, p. 7]). *Assume that G and X are as above. Then there exists a finitely additive, G-invariant measure on X defined for all subsets of X if and only if X is not G-paradoxical.*

A locally compact group G admits a *Borel paradoxical decomposition* if it has a paradoxical decomposition such that the sets $A_1, \dots, A_m, B_1, \dots, B_n$ in the above definition are Borel sets. Paterson [P] proved that a locally compact group G is not amenable if and only if G admits a Borel paradoxical decomposition. The question of whether the nonexistence of such suitable paradoxical decompositions characterizes the amenable topological groups seems to be open (cf. [W]).

Now, we show that the amenability of a topological semigroup is expressible by a theory in integral logic. Note that for a semigroup S the dual of the space $B(S)$ of all bounded real-valued functions on S is the space of all signed charges on all subsets of S (cf. [AB, p. 496]). Therefore, a mean I on $B(S)$ is represented by a (positive) charge ν_I. If ν_I is a charge which is not countably additive, then (S, ν_I) is not a structure in integral logic. Nevertheless, thanks to the representation theorem for M-spaces, the amenability of a topological semigroup is expressible. Indeed, consider a topological semigroup S, and let $\sigma(S)$ $(= \sigma(\mathrm{LUC}(S)))$ be the set of Riesz homomorphisms $h : \mathrm{LUC}(S) \to \mathbb{R}$ such that $h(\mathbf{1}) = 1$ (cf. [Fr2, p. 222]). The set $\sigma(S)$ is sometimes called the *spectrum* of $\mathrm{LUC}(S)$. We will see that $\sigma(S)$ is the space of complete types of a theory (see Proposition 5.6 below). Note that, by [Fr2, Proposition 353P(d), p. 243], $\sigma(S)$ is the set $\mathtt{M}_U(S)$ of all multiplicative means on $\mathrm{LUC}(S)$. First, we remark that $\sigma(S)$ is a weak* compact subset of $\mathrm{LUC}(S)^*$ and $\|h\| = 1$ for every $h \in \sigma(S)$, and hence by Lemma 4.4(i), the natural action of S on $\sigma(S)$ is continuous. The space $\mathrm{LUC}(S)$ can be identified, as a normed Riesz space, with $C(\sigma(S))$, because $\mathrm{LUC}(S)$ is an M-space with standard order unit $\mathbf{1}$ and $\sigma(S)$ is a compact Hausdorff space (cf. [Fr2, Corollary 354L]). The identification is the map $f \mapsto \widehat{f}$ where $\widehat{f}(h) = h(f)$ for $f \in \mathrm{LUC}(S)$ and $h \in \sigma(S)$.

By the Riesz representation theorem, the identification of $\mathrm{LUC}(S)$ with $C(\sigma(S))$ means that we have a one-to-one correspondence $\mu \leftrightarrow I_\mu$ between Radon probability measures μ on $\sigma(S)$ and positive linear functionals I_μ on $\mathrm{LUC}(S)$ such that $I_\mu(\mathbf{1}) = 1$, given by the formula $I_\mu(f) = \int \widehat{f}\, d\mu$ for $f \in \mathrm{LUC}(S)$. Now

$$\begin{aligned}
I_\mu \text{ is invariant } &\Leftrightarrow\ I_\mu(f \cdot s) = I_\mu(f) \text{ for all } f \in \mathrm{LUC}(S) \text{ and } s \in S\\
&\Leftrightarrow\ \int \widehat{f \cdot s}\, d\mu = \int \widehat{f}\, d\mu \text{ for all } f \in \mathrm{LUC}(S) \text{ and } s \in S\\
&\Leftrightarrow\ \int (\widehat{f} \cdot s)\, d\mu = \int \widehat{f}\, d\mu \text{ for all } f \in \mathrm{LUC}(S) \text{ and } s \in S\\
&\Leftrightarrow\ \mu \text{ is invariant.}
\end{aligned}$$

So there is a one-to-one correspondence between Radon probability left S-invariant measures on $\sigma(S)$ and left S-invariant means on $\mathrm{LUC}(S)$. Let $T_S = T_{S,\sigma(S)}$ be the theory of left S-invariant measures on $\sigma(S)$. Summarizing, we have the following.

Proposition 4.15. *Assume that S and T_S are as above. Then the following are equivalent:*

(i) *S is amenable.*
(ii) *T_S is satisfiable.*

If S is a locally compact group, then (i) *and* (ii) *are equivalent to*

(iii) *S is not Borel paradoxical.*

In fact we can say more: if S and T_S are as above, then the cardinality of the set of all left S-invariant means on $\mathrm{LUC}(S)$ is equal to the number of models of T_S up to almost isomorphism. Indeed, if $\mu \neq \nu$ are (left) S-invariant measures on $\sigma(S)$, then $(\sigma(S), \mathcal{B}, \mu)$ and $(\sigma(S), \mathcal{B}, \nu)$ with the natural interpretation of relation and constant symbols are different models of T_S. Conversely, assume that $\mathsf{M} = (M, \mathcal{B}, \mu_{\mathsf{M}})$ is a model of T_S. By Proposition 3.8, the substructure $\mathsf{M}' = (\sigma(S), \mathcal{B}_{\sigma(S)}, \mu_{\mathsf{M}}{\restriction}\sigma(S))$ is also a model of T_S, and the inclusion map $\sigma(S) \to M$ covers a full measure subset of M. Therefore, $\mathsf{M}' \simeq_a \mathsf{M}$. Clearly, the unique extension of $\mu_{\mathsf{M}} \restriction \sigma(S)$ to a Radon measure on $\sigma(S)$ is left S-invariant. To summarize:

PROPOSITION 4.16. *Assume that S and T_S are as above. Then there is a bijection between the set of all models of T_S and the set of all left S-invariant means on* LUC(S).

4.4. Extreme amenability. In this subsection we present some other results for extremely amenable topological semigroups. Most of the proofs are straightforward and we omit some unnecessary details. First, we characterize extremely amenable topological semigroups in terms of multiplicative invariant measures (Fact 4.20). Secondly, we prove that the extreme amenability of a topological semigroup is expressible by a theory in integral logic (Proposition 4.22).

A Radon probability measure μ on a compact Hausdorff space X is *multiplicative* if $\int f \, d\mu \times \int g \, d\mu = \int (f \times g) \, d\mu$ (the pointwise product) for all $f, g \in C(X)$.

Let S be a topological semigroup which acts on a compact hausdorff space X from the left. Let $\mathtt{T}_{S,X}$ be the theory of left S-invariant measures on X with the additional axiom schema

(9) $\int R_{f\times g}(x)\, dx = \int R_f(x)\, dx \times \int R_g(x)\, dx$ for all $R_f, R_g, R_{f\times g} \in L_X$, where $(f \times g)(x) = f(x) \times g(x)$.

Note that (9) says that the measure is multiplicative. We call $\mathtt{T}_{S,X}$ the theory of *multiplicative left S-invariant measures on X*.

Let $\mathrm{MInv}_X(S)$ be the set of all multiplicative, Radon probability measures on X which are left S-invariant. A consequence of the compactness theorem is the following.

PROPOSITION 4.17. *Assume that S, X and $\mathtt{T}_{S,X}$ are as above. Then the following are equivalent:*

(i) $\mathrm{MInv}_X(S) \neq \emptyset$.
(ii) $\mathtt{T}_{S,X}$ *is satisfiable.*

Let S be a topological semigroup. A mean I on LUC(S) is *multiplicative* if $I(f) \times I(g) = I(f \times g)$ (the pointwise product) for all $f, g \in$ LUC(S). We remark that LUC(S) is a closed and invariant subalgebra of $C_b(S)$ (cf. [N, Lemmas 1.1 and 1.2]).

DEFINITION 4.18. A topological semigroup S is said to be *extremely left* (*right*) *amenable* if LUC(S) (RUC(S)) admits a multiplicative left (right) invariant mean. A topological semigroup S is called *extremely amenable* if it is both left and right amenable.

REMARK 4.19. A topological semigroup S has the *left [right] fixed point on compacta property* if every continuous action of S on a compact Hausdorff space from the left [right] has a fixed point. In [M], Mitchell showed that a topological semigroup S has a multiplicative left invariant mean on $\mathrm{LUC}(S)$ iff S has the left fixed point on compacta property. Also, he asked: Is there a nontrivial extremely amenable group at all? Historically the first example of an extremely amenable group was found in [HC]. Many further examples are found in [Pe1, Pe2, Fr3].

The following fact presents a proof of Mitchell's theorem [M, Theorem 1], and it also characterizes extremely amenable topological semigroups in terms of multiplicative invariant measures.

FACT 4.20. *Let S be a topological semigroup with identity. Then the following are equivalent:*

(i) *S is extremely left amenable.*
(ii) *S has the left fixed point on compacta property.*
(iii) *Whenever X is a nonempty compact Hausdorff space and $\cdot$ is a continuous action of S on X from the left, then $\mathrm{MInv}_X(S) \neq \emptyset$.*

Proof. (i)⇔(iii). The set $\mathtt{M}_U(S)$ $(= \sigma(S))$ of all multiplicative means on $\mathrm{LUC}(S)$ is a weak* compact subset of $\mathrm{LUC}(S)^*$. By Lemma 4.4(i), the natural action of S on $\mathtt{M}_U(S)$ (from the left) is continuous. Also, it is easy to verify that $\mathrm{MInv}_X(S) \neq \emptyset$ iff for any $s_1, \ldots, s_n \in S$ and $f_1, \ldots, f_n, g_1, \ldots, g_n \in C(X)$ we have $\|\mathbf{1} - \sum_{i=1}^n g_i \times (f_i \cdot s_i - f_i)\| \geq 1$. (Compare Fact 4.2.) Now, the proof is a simple adaptation of the proof of Fact 4.5.

(ii)⇒(iii). Assume that $x_0 \in X$ is a fixed point, i.e., $s \cdot x_0 = x_0$ for every $s \in S$. Define the measure μ by $\int f \, d\mu = f(x_0)$ for every $f \in C(X)$. Clearly, μ is a multiplicative Radon left S-invariant measure on X.

(iii)⇒(ii). Assume that X is a nonempty compact Hausdorff space and $\cdot$ is a continuous action of S on X from the left. Let μ be a multiplicative left S-invariant Radon probability measure on X. Then the linear functional I defined by $I(f) = \int f \, d\mu$ is multiplicative and invariant. Therefore, by [DS, Lemma 25, p. 278], there is a point x_0 in X such that $I(f) = f(x_0)$ for every $f \in C(X)$. Since $C(X)$ separates points and I is invariant, x_0 is the desired fixed point. ∎

Using the compactness theorem of integral logic, one can prove the following fact.

PROPOSITION 4.21. *If S is a topological semigroup with a dense subset $\bigcup_{\alpha \in I} S_\alpha$ where S_α are extremely amenable semigroups and for any $\alpha_1, \alpha_2 \in I$, $S_{\alpha_1} \cup S_{\alpha_2} \subseteq S_{\alpha_3}$ for some $\alpha_3 \in I$, then S is extremely amenable.*

To end this section, we show that the extreme amenability of a topological semigroup is expressible by a theory in integral logic. Let S be a topological semigroup and $\mathtt{T}_S = \mathtt{T}_{S,\sigma(S)}$ be the theory of multiplicative left S-invariant measures on $\sigma(S)$. In fact, we show that the cardinal of $\mathrm{MInv}_X(S)$ is equal to the number of models of $\mathtt{T}_S$. By Propositions 4.15 and 4.16, it suffices to show that there is a one-to-one correspondence between multiplicative Radon probability measures on $\sigma(S)$ and multiplicative means on $\mathrm{LUC}(S)$. Note that the identification of $\mathrm{LUC}(S)$ and $C(\sigma(S))$ is algebraic, i.e., $\widehat{f\times g} = \widehat{f}\times\widehat{g}$ for all $f,g\in\mathrm{LUC}(S)$ (cf. [Fr2, Prop. 353P(d), p. 243]). Now

$$\begin{aligned}
&I_\mu \text{ is multiplicative}\\
&\qquad\Leftrightarrow\ I_\mu(f\times g) = I_\mu(f)\times I_\mu(g) \text{ for all } f,g\in\mathrm{LUC}(S)\\
&\qquad\Leftrightarrow\ \int\widehat{f\times g}\,d\mu = \int\widehat{f}\,d\mu\times\int\widehat{g}\,d\mu \text{ for all } f,g\in\mathrm{LUC}(S)\\
&\qquad\Leftrightarrow\ \int(\widehat{f}\times\widehat{g})\,d\mu = \int\widehat{f}\,d\mu\times\int\widehat{g}\,d\mu \text{ for all } f,g\in\mathrm{LUC}(S)\\
&\qquad\Leftrightarrow\ \mu \text{ is multiplicative.}
\end{aligned}$$

To summarize:

Proposition 4.22. *Assume that S and $\mathtt{T}_S$ are as above. Then there is a bijection between the set of all models of $\mathtt{T}_S$ and the set of all multiplicative left S-invariant means on* $\mathrm{LUC}(S)$. *In particular, S is extremely left amenable iff $\mathtt{T}_S$ is satisfiable.*

5. Types and stability. In classical model theory, a complete type determines a finitely additive 0-1-valued measure on the formulas. Actually, one can say more, i.e., a complete type is a 0-1-valued Riesz homomorphism on the formulas. Indeed, let L be a first order language, M an L-structure, a an element of M, and $\mathrm{tp}^{\mathsf{M}}(a)$ the complete type of a in M. For each L-formula $\phi(x)$, define $f_\phi : M\to\{0,1\}$ by $f_\phi(b) = 1$ if $\mathsf{M}\vDash\phi(b)$, and $f_\phi(b) = 0$ otherwise. Let $V = \{f_\phi : \phi\in L\}$. One can easily check that V is an (Archimedean) Riesz space (see Definitions 5.1 and 5.2 below). For this we define $f_\phi + f_\psi := f_{\phi\vee\psi}$, $-f_\phi := f_{\neg\phi}$, and for each $r\in\mathbb{R}$, $r\cdot f_\phi := f_\phi$ if $r>0$, $r\cdot f_\phi := f_{\neg\phi}$ if $r<0$, and $r\cdot f_\phi := \mathbf{0}$ if $r=0$. Also, $f_\phi\le f_\psi$ if $f_\phi(b)\le f_\psi(b)$ for each $b\in M$. Clearly, V with this structure is a Riesz space, i.e., a partially ordered linear space which is a lattice. Now, for $a\in M$, define the Riesz homomorphism $I_a : V\to\{0,1\}$ by $I_a(f_\phi) = 1$ if $f_\phi(a) = 1$, and $I_a(f_\phi) = 0$ otherwise, i.e., $I_a(f_\phi) = 1$ iff $\phi\in\mathrm{tp}^{\mathsf{M}}(a)$. In other words, I_a can be interpreted as playing the role of $\mathrm{tp}^{\mathsf{M}}(a)$.

More generally, we consider real-valued Riesz homomorphisms. Indeed, consider an arbitrary partially ordered set $\mathcal{L} = \{f_\phi : M\to\mathbb{R}\mid\phi\in L\}$ such that

$$\forall b \in M : f_\phi(b) \leq f_\psi(b) \;\Leftrightarrow\; \models \phi(b) \to \psi(b),$$
$$\forall b \in M : f_\phi(b) < f_\psi(b) \;\Leftrightarrow\; \models \neg\phi(b) \wedge \psi(b).$$

Let V be the linear space generated by $\mathcal{L}$. Again, V is an Archimedean Riesz space. Define the Riesz homomorphism $I_a : V \to \mathbb{R}$ by $I_a(f) = f(a)$. It is easy to verify that $\phi \in \mathrm{tp}^{\mathsf{M}}(a)$ iff $I_a(f_{\phi\vee\neg\phi}) \leq I_a(f_\phi)$. Therefore it is natural to conjecture that real-valued Riesz homomorphisms on measurable functions should play the role of complete types in the framework of integral logic. Our next goal is to convince the reader that this is indeed the case.

5.1. Types. Let us now return to integral logic. Suppose that L is an arbitrary language, possibly with n-ary relation symbols and n-ary function symbols. Let M be a *graded* L-structure as discussed in [BP], $A \subseteq M$ and $T_A = \mathrm{Th}(\mathsf{M}, a)_{a\in A}$. Let $p(x)$ be a set of $L(A)$-statements in a free variable x. We shall say that $p(x)$ is a *type over* A if $p(x) \cup T_A$ is satisfiable. A *complete type over* A is a maximal type over A. We let $S^{\mathsf{M}}(A)$ be the set of all complete types over A. The *type of* a *in* M *over* A, denoted by $\mathrm{tp}^{\mathsf{M}}(a/A)$, is the set of all $L(A)$-statements satisfied in M by a. For $\phi(x)$ an $L(A)$-formula, we let

$$[\phi > 0] = \{p \in S^{\mathsf{M}}(A) : \text{for some } \epsilon > 0 \text{ the statement } (\phi \geq \epsilon) \text{ is in } p\}.$$

The *logic topology* (or the *Stone topology*) on $S^{\mathsf{M}}(A)$ is the topology generated by taking the sets $[\phi > 0]$ as basic open sets. We will give a characterization of the complete types. First, we need some notions from functional analysis.

Definition 5.1 (Riesz space). A *Riesz space* or *vector lattice* is a partially ordered linear space which is a lattice. A Riesz space $\mathcal{L}$ is called *Archimedean* if $\inf_{\delta>0} \delta f = \mathbf{0}$ for each $f \geq \mathbf{0}$ in $\mathcal{L}$. An element $\mathbf{1} \geq \mathbf{0}$ of $\mathcal{L}$ is an *order unit* in $\mathcal{L}$ if for every $f \in \mathcal{L}$ there is an $n \in \mathbb{N}$ such that $|f| \leq n\mathbf{1}$.

The following notion will play a fundamental role in what follows.

Definition 5.2 (Riesz homomorphism). Let $\mathcal{L}, \mathcal{L}'$ be partially ordered linear spaces. A *Riesz homomorphism* from $\mathcal{L}$ to $\mathcal{L}'$ is a linear operator $T : \mathcal{L} \to \mathcal{L}'$ such that whenever $A \subset \mathcal{L}$ is a finite nonempty set and $\inf A = \mathbf{0}$ in $\mathcal{L}$, then $\inf T[A] = \mathbf{0}$ in $\mathcal{L}'$.

Any Riesz homomorphism is a *positive* linear operator, i.e. $T(f) \geq \mathbf{0}$ for all $f \geq \mathbf{0}$ (see [Fr2, 351H(b)]).

Fact 5.3 ([Fr2, 354K]). *Let $\mathcal{L}$ be an Archimedean Riesz space with order unit* $\mathbf{1}$. *Then it can be embedded as an order-dense and norm-dense Riesz subspace of $C(X)$, where X is a compact Hausdorff space, in such a way that* $\mathbf{1}$ *corresponds to χ_X; moreover, this embedding is essentially unique.*

The compact space X in Fact 5.3 is the set of Riesz homomorphisms I from $\mathcal{L}$ to $\mathbb{R}$ such that $I(\mathbf{1}) = 1$, and the embedding is the map $T : \mathcal{L} \to \mathbb{R}^X$ defined by setting $(Tf)(I) = I(f)$ for any $I \in X$ and $f \in \mathcal{L}$ (see the proof of [Fr2, Theorem 353M]).

Let M be an L-structure and A a subset of M. We define $\mathcal{L}_A$ to be the family of all measurable functions ϕ^{M} where ϕ is an $L(A)$-formula with a free variable x (see the paragraph after Definition 3.2). Then $\mathcal{L}_A$ has a natural Riesz space structure given by $(\phi^{\mathsf{M}} + \psi^{\mathsf{M}})(a) = \phi^{\mathsf{M}}(a) + \psi^{\mathsf{M}}(a)$, $(r\phi^{\mathsf{M}})(a) = r\phi^{\mathsf{M}}(a)$ for all $a \in M$, and $\phi^{\mathsf{M}} \geq \psi^{\mathsf{M}}$ iff $\phi^{\mathsf{M}}(a) \geq \psi^{\mathsf{M}}(a)$ for all $a \in M$. Also, $|\phi^{\mathsf{M}}|(a) = |\phi^{\mathsf{M}}(a)|$ for all $a \in M$, $\min(\phi^{\mathsf{M}}, \psi^{\mathsf{M}})$, $\max(\phi^{\mathsf{M}}, \psi^{\mathsf{M}})$ are in $\mathcal{L}_A$, and $\|\phi^{\mathsf{M}}\| = \sup_{a\in M} |\phi^{\mathsf{M}}(a)|$. Clearly, $\mathcal{L}_A$ is Archimedean. The constant function $\mathbf{1}$ is an order unit, and the uniform norm is its order-unit norm (see [Fr2, 354G(a)]).

Let $\sigma_A(\mathsf{M})$ be the set of Riesz homomorphisms $I : \mathcal{L}_A \to \mathbb{R}$ such that $I(\mathbf{1}) = 1$. This set is called the *spectrum* of T_A. Since $\mathcal{L}_A$ is a normed linear space (with the uniform norm), the unit ball $B^* = \{I \in \mathcal{L}_A^* : \|I\| \leq 1\}$ in $\mathcal{L}_A^*$ is compact in the weak* topology by Alaoglu's theorem. Also, we know that $\sigma_A(\mathsf{M})$ is the set of positive *extreme points* of the unit ball B^*, i.e. $\sigma_A(\mathsf{M}) = \{I \in B^* : \|I\| = 1 \text{ and } I \text{ is positive}\}$ (see [Fr2, 354Y(j)]). Since $\sigma_A(\mathsf{M}) \subseteq B^*$ is weak* closed, it is weak* compact. (We remark that the weak* topology on $\sigma_A(\mathsf{M})$ is simply the topology of pointwise convergence: $I_\alpha \to I$ in the weak* topology iff $I_\alpha(\phi^{\mathsf{M}}) \to I(\phi^{\mathsf{M}})$ for all $\phi^{\mathsf{M}} \in \mathcal{L}_A$; see [F, p. 169] for details.)

The next propositions show that a complete type can be coded by a Riesz homomorphism and give a characterization of complete types. The key idea behind these propositions is a construction which allows us to consider M as an elementary submodel of the type space $S^{\mathsf{M}}(M)$ with the appropriate structure.

Definition 5.4 ($\sigma_M(\mathsf{M})$ as an elementary extension). Assume that M is an L-structure and μ is the measure on M. By Fact 5.3, the space $\mathcal{L}_M$ can be embedded as an order-dense and norm-dense Riesz subspace of $C(\sigma_M(\mathsf{M}))$. The embedding is the map $T : \mathcal{L}_M \to \mathbb{R}^{\sigma_M(\mathsf{M})}$ defined by setting $(T\phi^{\mathsf{M}})(I) = I(\phi^{\mathsf{M}})$ for all $I \in \sigma_M(\mathsf{M})$ and $\phi^{\mathsf{M}} \in \mathcal{L}_M$. We define the elementary extension $\mathsf{N} = (\sigma_M(\mathsf{M}), \nu, T\phi^{\mathsf{M}})_{\phi^{\mathsf{M}}\in\mathcal{L}_M}$ of M with the natural interpretations of symbols and measure as follows:

First, we can easily see that $M \subseteq \sigma_M(\mathsf{M})$. Indeed, for each $a \in M$, define $I_a : \mathcal{L}_M \to \mathbb{R}$ by $I_a(\phi^{\mathsf{M}}) = \phi^{\mathsf{M}}(a)$ for $\phi^{\mathsf{M}} \in \mathcal{L}_M$. Now, one can assume that the language has a 2-ary relation symbol $\mathbf{e}$ with the interpretation $\mathbf{e}(a, b) = 1$ if $a = b$, and $\mathbf{e}(a, b) = 0$ otherwise (cf. [BP, p. 469]). Therefore, $I_a \neq I_b$ if $a \neq b \in M$. More generally, if $\mathcal{L}_M$ separates M, i.e. for all $a \neq b \in M$ there is $\phi^{\mathsf{M}} \in \mathcal{L}_M$ such that $\phi^{\mathsf{M}}(a) \neq \phi^{\mathsf{M}}(b)$, then $I_a \neq I_b$. To summarize, the map

$M \hookrightarrow \sigma_M(\mathsf{M})$ defined by $a \mapsto I_a$ is injective, and so we can assume that $a = I_a$ and $M \subseteq \sigma_M(\mathsf{M})$.

Second, define $\nu\{T\phi^{\mathsf{M}} > 0\} := \mu\{\phi^{\mathsf{M}} > 0\}$ for all $\phi^{\mathsf{M}} \in \mathcal{L}_M$. (Recall that $\{T\phi^{\mathsf{M}} > 0\}$ is the set $\{I \in \sigma_M(\mathsf{M}) : T\phi^{\mathsf{M}}(I) > 0\}$.) Then ν is a premeasure on the algebra $\mathcal{A} = \{\{T\phi^{\mathsf{M}} > 0\} : \phi^{\mathsf{M}} \in \mathcal{L}_M\}$. By Carathéodory's theorem, ν has a unique extension to a measure on the σ-algebra generated by $\mathcal{A}$, still denoted by ν. Also, we can assume that M is ν-measurable and $\nu(N \setminus M) = 0$, i.e. M has full measure.

Third, for each formula $\phi(x, y_1, \ldots, y_n)$ and any $a_1, \ldots, a_n \in M$, define $\phi^{\mathsf{N}}(x, I_{a_1}, \ldots, I_{a_n}) : \sigma_M(\mathsf{M}) \to \mathbb{R}$ by $\phi^{\mathsf{N}}(x, I_{a_1}, \ldots, I_{a_n}) = T\phi^{\mathsf{M}}(x, a_1, \ldots, a_n)$. Then for each $b \in M$ we have

$$\begin{aligned}\phi^{\mathsf{N}}(I_b, I_{a_1}, \ldots, I_{a_n}) &= T\phi^{\mathsf{M}}(x, a_1, \ldots, a_n)(I_b) = I_b(\phi^{\mathsf{M}}(x, a_1, \ldots, a_n)) \\ &= \phi^{\mathsf{M}}(b, a_1, \ldots, a_n).\end{aligned}$$

Also, for a formula $\phi(x_1, x_2)$, define $\phi^{\mathsf{N}}(x_1, x_2) : (\sigma_M(\mathsf{M}))^2 \to \mathbb{R}$ by $\phi^{\mathsf{N}}(I_a, I) = T\phi^{\mathsf{M}}(a, y)(I)$ and $\phi^{\mathsf{N}}(I, I_b) = T\phi^{\mathsf{M}}(x, b)(I)$, where $a, b \in M$ and $I \in N$, and $\phi^{\mathsf{N}}(I, I') = 0$ if $I, I' \in N \setminus M$. Similarly we can define $\phi^{\mathsf{N}}(x_1, \ldots, x_n)$. For a 2-ary function symbol f, define $f^{\mathsf{N}}(I_a, I_b) := f^{\mathsf{M}}(a, b)$ for all $a, b \in M$, and for some $I'' \in N \setminus M$, $f^{\mathsf{N}}(I, I') := I''$ if at least one of I, I' belongs to $N \setminus M$. Similarly we can define $f^{\mathsf{N}}(x_1, \ldots, x_n)$. Moreover, we can assume that the n-ary relations and functions on N are ν_n-measurable. In fact, our definitions are not important on the set $N^n \setminus M^n$, because $\nu_n(N^n \setminus M^n) = 0$ and we can take an appropriate σ-algebra on N^n.

Proposition 5.5. *Assume that* M *and* N *are as above. Then* $\mathsf{M} \preceq \mathsf{N}$.

Proof. Since $M \subseteq \sigma_M(\mathsf{M})$ and $\phi^{\mathsf{N}}(\bar{b}) = \phi^{\mathsf{M}}(\bar{b})$ for all $\bar{b} \in M$ and formulas $\phi(\bar{x})$, we see that M is a substructure of N. Now by the Tarski–Vaught test (Proposition 3.7 above), N is an elementary extension of M. Indeed, we note that $\nu\{\phi^{\mathsf{N}} > 0\} = \mu\{\phi^{\mathsf{M}} > 0\}$ for all $\phi^{\mathsf{M}} \in \mathcal{L}_M$. (See also [BP, Proposition 5.10].) ∎

We will also see that N realizes every type in $S^{\mathsf{M}}(M)$; in fact $S^{\mathsf{M}}(M) = \sigma_M(\mathsf{M})$.

Proposition 5.6. *Assume that* M *is an* L*-structure and* $A \subseteq M$.

(i) *There is a bijection from* $S^{\mathsf{M}}(M)$ *onto* $\sigma_M(\mathsf{M})$.
(ii) $q \in S^{\mathsf{M}}(A)$ *if and only if there is an elementary extension* N *of* M *and* $x_0 \in N$ *such that* $q = \mathrm{tp}^{\mathsf{N}}(x_0/A)$.

Proof. (i) Assume that $p(x)$ is a complete type over M. Define $I_p : \mathcal{L}_M \to \mathbb{R}$ by $I_p(\phi^{\mathsf{M}}) = r$ if the statement $\phi(x) = r$ is in $p(x)$. Clearly, I_p is a Riesz homomorphism on $\mathcal{L}_M$ and $I_p(\mathbf{1}) = 1$. The map $p \mapsto I_p$ is injective, and we may reasonably assume that $p = I_p \in \sigma_M(\mathsf{M})$. In particular, for any $a \in M$, $\mathrm{tp}^{\mathsf{M}}(a/M) = \{\phi(x) = \phi^{\mathsf{M}}(a) : \phi \in \mathcal{L}_M\}$ and $I_{\mathrm{tp}^{\mathsf{M}}(a/M)}(\phi^{\mathsf{M}}) = \phi^{\mathsf{M}}(a)$.

(Earlier we showed that the map $M \hookrightarrow \sigma_M(\mathsf{M})$ defined by $a \mapsto I_{\mathrm{tp}^{\mathsf{M}}(a/M)}$ is injective.)

Now, we prove that the map $p \mapsto I_p$ is surjective. Assume $I \in \sigma_M(\mathsf{M})$. Let $\mathsf{N} = (\sigma_M(\mathsf{M}), \nu, T\phi^{\mathsf{M}})_{\phi^{\mathsf{M}} \in \mathcal{L}_M}$ be the elementary extension of M constructed in Definition 5.4 and $p = \mathrm{tp}^{\mathsf{N}}(I/M)$. Then it is easy to check that $I_p = I$. (Indeed, recall that $\phi^{\mathsf{N}}(I) = T\phi^{\mathsf{M}}(I) = I(\phi^{\mathsf{M}})$ for all $\phi^{\mathsf{M}} \in \mathcal{L}_M$.) Therefore, the map $p \mapsto I_p$ is also surjective.

(ii) Let $q \in S^{\mathsf{M}}(A)$ and N be the elementary extension of M constructed in Definition 5.4. Assume that $p \in S^{\mathsf{N}}(M) = S^{\mathsf{M}}(M)$ is an extension of q. Then there is a point $x_0 \in N$ such that $p = \mathrm{tp}^{\mathsf{N}}(x_0/M)$ (see (i) above). Clearly, $q = \mathrm{tp}^{\mathsf{N}}(x_0/A)$. ∎

Recall that $\sigma_M(\mathsf{M})$ is weak* compact. Since $S^{\mathsf{M}}(M) = \sigma_M(\mathsf{M})$, we can also equip $S^{\mathsf{M}}(M)$ with the weak* topology. It is easy to check that the weak* topology and the logic topology on $S^{\mathsf{M}}(M)$ are the same. Indeed, for each $\phi^{\mathsf{M}} \in \mathcal{L}_M$, define $\phi : S^{\mathsf{M}}(M) \to [-\flat_\phi, \flat_\phi]$ by $p \mapsto I_p(\phi^{\mathsf{M}})$. Then obviously the logic topology on $S^{\mathsf{M}}(M)$ is the weakest topology in which all the functions $p \mapsto \phi(p)$ are continuous. Therefore, for all $p_\alpha, p \in S^{\mathsf{M}}(M)$ we have

$$\begin{aligned} I_{p_\alpha} \to I_p \text{ in the weak* topology } &\Leftrightarrow I_{p_\alpha}(\phi^{\mathsf{M}}) \to I_p(\phi^{\mathsf{M}}) \text{ for all } \phi^{\mathsf{M}} \in \mathcal{L}_M \\ &\Leftrightarrow \phi(p_\alpha) \to \phi(p) \text{ for all } \phi^{\mathsf{M}} \in \mathcal{L}_M \\ &\Leftrightarrow \phi \text{ is continuous for all } \phi^{\mathsf{M}} \in \mathcal{L}_M \\ &\Leftrightarrow p_\alpha \to p \text{ in the logic topology.} \end{aligned}$$

REMARK 5.7. By Proposition 5.6, the elementary extension $\mathsf{N} = (\sigma_M(\mathsf{M}), \nu, \phi^{\mathsf{N}})$, as constructed in Definition 5.4, realizes every type over M. Also, it is easy to verify that M is a dense subset of $N = \sigma_M(\mathsf{M})$. Indeed, if M is not dense in N, there is a nonzero $h \in C(N)$ such that $h(I_a) = 0$ for every $a \in M$; but as the uniform completion $\overline{\mathcal{L}}_M$ of $\mathcal{L}_M$ is identified with $C(N)$ (because $\mathcal{L}_M$ is dense in $C(N)$), there is an $f \in \overline{\mathcal{L}}_M$ such that $I(f) = h(I)$ for every $I \in N$. Assume that $f_n \to f$ uniformly, where $f_n \in \mathcal{L}_M$. Therefore, there are $h_n \in C(N)$ such that $I(f_n) = h_n(I)$ for every $I \in N$. Clearly, $h_n \to h$ uniformly. In this case, f cannot be the zero function, but $f(a) = \lim_n f_n(a) = \lim_n I_a(f_n) = \lim_n h_n(I_a) = h(I_a) = 0$ for every $a \in M$. Thus the image of M is dense, as claimed.

On the other hand, since ϕ^{N}'s are continuous, the natural measure ν on N is Baire and it has a unique extension to a Radon measure, which we again denote by μ. From now on we assume that $\mathsf{N} = (S(\mathsf{M}), \mu)$ with the appropriate structure, where μ is this Radon measure.

COROLLARY 5.8. *Let G be an amenable topological group and T_G the theory of left G-invariant measures on $\sigma(G)$. Then G is extremely amenable* *iff there is a complete type $p \in S(\sigma(G))$ such that $g \cdot p = p$ for each $g \in G$.*

5.2. Definable relations

Definition 5.9. A relation $\xi : M \to [-\flat, \flat]$ is $\emptyset$-*definable* if there is a sequence $\phi_k(\bar{x})$ of formulas such that $\flat_{\phi_k} \leq \flat$ and $\phi_k \to \xi$ pointwise. A subset is *definable* if its characteristic function is definable.

This may also be defined on the basis of other notions of convergence such as almost uniform convergence, convergence in measure, convergence in the mean etc. However, the corresponding definitions are equivalent. For example if ϕ_k converges in measure to ξ, then it has a subsequence which converges to f almost everywhere. So, if R is definable using the first notion of convergence, it is also definable using the second one. In particular, since the measure is finite and $|\phi_k| \leq \flat$, $\phi_k \to \xi$ in measure iff $\phi_k \to \xi$ in the mean iff $\phi_k \to \xi$ pointwise (see [F]). On the other hand, if $\mathsf{M} \preceq \mathsf{N}$ and ξ is definable in M, then there is a corresponding definable relation ξ' in N, and it is not hard to see that $\mathsf{M} \preceq_a \mathsf{N}$. The set of definable relations is a Banach algebra with the norm defined by $\|\phi\| = \sup_x |\phi(x)|$, and this algebra depends only on T. It can be described as the completion of the algebra of formulas with the uniform norm. We denote this completion by $L(T)$. A relation is M-*definable* if it is definable in $\mathrm{Th}(\mathsf{M}, a)_{a \in M}$. So, $L(\mathsf{M})$ is defined in the natural way.

5.3. Local stability. Here and in Section 6 we give two different notions of "stability" of a formula inside a model: a measure-theoretic one and a model-theoretic one. In fact, the measure-theoretic notion (Definition 6.1) is a suitable form of the dependence property in classical model theory.

Let M be a structure and $\phi(x, y)$ a formula. Assume that $\mathsf{N} \succeq \mathsf{M}$ and $a \in N$. Let $p = \mathrm{tp}_\phi^{\mathsf{M}}(a/M)$ be the *complete ϕ-type of a over M*, i.e., a function which associates to each instance $\phi(x, b)$, $b \in M$, the value $\phi(a, b)$, which will then be denoted by $\phi(p, b)$. Note that the complete ϕ-type p uniquely determines a Riesz homomorphism $I_p : \mathcal{L}_\phi \to \mathbb{R}$ where $\mathcal{L}_\phi$ is the Riesz space generated by $\{\phi(x, b) : b \in M\}$, and $I_p(\phi(x, b)) = \phi(p, b)$ for each $b \in M$. We equip $S_\phi(M)$ with the weakest topology in which all functions $p \mapsto \phi(p, b)$, $b \in M$, are continuous. Equivalently, if $\sigma_\phi(M)$ is the spectrum of $T_\phi = \{\phi \geq r : \phi \geq r \text{ is in } T(\mathsf{M}, a)_{a \in M}\}$ (i.e., the set of Riesz homomorphisms $I : \mathcal{L}_\phi \to \mathbb{R}$ such that $I(\mathbf{1}) = 1$), then $S_\phi(M) = \sigma_\phi(M)$ is equipped with the topology induced by the weak* topology on $\mathcal{L}_\phi^*$. Clearly, $S_\phi(M)$ is compact Hausdorff. If ψ is a continuous function on $S_\phi(M)$ such that ψ can be expressed as the pointwise limit of a sequence of algebraic combinations of functions of the form $p \mapsto \phi(p, b)$, $b \in M$, then ψ is called a *ϕ-definable* *relation over* M. A definable relation $\psi(y)$ over M defines $p \in S_\phi(M)$ if $\phi(p, b) = \psi(b)$ for all $b \in M$.

The next notion is more natural and less technically involved than the measure-theoretic one in Definition 6.1 below (see [B1, Definition 7.1]).

DEFINITION 5.10. A formula $\phi(x, y)$ is called *stable in a structure* M if there are no $r > s$ and infinite sequences $a_n, b_n \in M$ such that, for all $i > j$, $\phi(a_i, b_j) \geq r$ and $\phi(a_j, b_i) \leq s$. A formula ϕ is *stable in a theory* T if it is stable in every model of T.

It is easy to verify that $\phi(x, y)$ is stable in M if whenever $a_n, b_n \in M$ form two sequences, then

$$\lim_n \lim_m \phi(a_n, b_m) = \lim_m \lim_n \phi(a_n, b_m),$$

provided both limits exist.

FACT 5.11 (Grothendieck's criterion, [G]). *Let* X *be an arbitrary topological space, and* $X_0 \subseteq X$ *a dense subset. Then the following are equivalent for a subset* $A \subseteq C_b(X)$*:*

(i) *The set* A *is relatively weakly compact in* $C_b(X)$*.*
(ii) *The set* A *is bounded, and if* $f_n \in A$ *and* $x_n \in X_0$ *form two sequences then*

$$\lim_n \lim_m f_n(x_m) = \lim_m \lim_n f_n(x_m),$$

whenever both limits exist.

5.4. Fundamental theorem of stability. In [BU], Ben Yaacov and Usvyatsov proved a continuous version of the definability of types in a stable theory, which is a generalization of the classical one. Roughly speaking, in continuous logic, for a stable formula ϕ, the number of ϕ-types is controlled by the number of continuous functions on the space of ϕ-types. A similar result holds for a stable formula in integral logic. Another result shows that for an almost dependent formula ϕ (see Definition 6.1 below), the number of ϕ-types (up to an equivalence relation) is controlled by the number of measurable functions on the space of ϕ-types.

On the other hand, in [B3] and [B2], Ben Yaacov studied probability algebras and L^1-random variables in the frameworks of compact abstract theories (cats) and of continuous logic. Note that in this paper we shall *not* identify measurable functions with their classes in L^1. Thus, in contrast to [B3] and [B2], the theory of a probability structure is not necessarily stable.

Now, we come quickly to the following theorem. The proof is essentially similar to that in [B1], but it works for measure structures.

THEOREM 5.12 (Definability of types). *Let* $\phi(x, y)$ *be a formula stable in a structure* M*. Then every* $p \in S_\phi(M)$ *is definable by a unique* $\tilde{\phi}$*-definable relation* $\psi(y)$ *over* M*, where* $\tilde{\phi}(y, x) = \phi(x, y)$*.*

Proof. Let $X = S_\phi(M)$, and let $X_0 \subseteq X$ be the collection of those types realized in M, which is dense in X. Since X is compact, the weak topology on $C(X)$ coincides with the topology of pointwise convergence. Since every formula is bounded, the set $A = \{\phi^a : p \mapsto \phi(a,p) \mid a \in M\} \subseteq C(X)$ is bounded. By Fact 5.11, since ϕ is stable in M, we see that A is relatively poinwise compact in $C(X)$. Let $p(x) \in S_\phi(M)$, and let $a_i \in M$ be any net such that $\lim_i \mathrm{tp}_\phi(a_i/M) = p$. Since A is relatively pointwise compact, there is a $\psi \in C(X)$ such that $\lim_i \phi^{a_i}(y) = \psi(y)$. By [KN, Theorem 8.20], ψ is the closure point of a sequence $\phi^{a_n}(y)$ of the family $\{\phi^{a_i}(y)\}_i$, and there is a subsequence $\phi^{a_{n_k}}(y)$ such that $\lim_k \phi^{a_{n_k}}(y) = \psi(y)$. Clearly, $\psi(y)$ is a $\tilde{\phi}$-definable relation over M, and for $b \in M$ we have $\phi(p,b) = \lim_k \phi(a_{n_k}, b) = \psi(b)$. Therefore, p is definable by a $\tilde{\phi}$-definable relation ψ over M. If p is definable by ψ_1, ψ_2, then $\psi_1(b) = \psi_2(b)$ for all $b \in M$. Since $X_0 \subseteq X$ is dense, $\psi_1 = \psi_2$. ∎

We are now ready to prove the main theorem of this section.

Corollary 5.13 (Fundamental theorem of stability). *Let $\phi(x,y)$ be a formula and T a theory. Then the following are equivalent:*

(i) *The formula ϕ is stable in T.*
(ii) *For every model $\mathsf{M} \vDash T$, every ϕ-type over M is definable by a $\tilde{\phi}$-predicate over M.*
(iii) *For each cardinal $\lambda = \kappa^{\aleph_0} \geq |T|$, and each model $\mathsf{M} \vDash T$ with $|M| \leq \lambda$, we have $|S_\phi(M)| \leq \lambda$.*
(iv) *There exists a cardinal $\lambda = \kappa^{\aleph_0} \geq |T|$ such that for every model $\mathsf{M} \vDash T$, if $|M| \leq \lambda$ then $|S_\phi(M)| \leq \lambda$.*

Proof. We proved (i)⇒(ii) in Theorem 5.12. The implications (ii)⇒(iii) ⇒(iv) are clear. For (iv)⇒(i), use a many-type argument and the downward Löwenheim–Skolem theorem [BP, Proposition 5.13]. ∎

5.5. Cantor–Bendixson rank. Let M be a structure. By Remark 5.7, $\mathsf{N} = (S(\mathsf{M}), \mu)$ is an elementary extension of M, and a very unlikely one from the point of view of classical model theory. Moreover, N is a topological measure space, N is compact, and μ is a Radon measure. Similarly, for a formula $\phi(x,y)$, the structure $\mathsf{N}_\phi = (S_\phi(\mathsf{M}), \mu_\phi)$ also has these properties. In fact, N_ϕ has further structures:

Definition 5.14 ([BU]). A (compact) *topometric* space is a triplet (X, τ, d), where τ is a (compact) Hausdorff topology and d a metric on X, satisfying:

(i) the metric topology refines the topology;
(ii) for every closed $F \subseteq X$ and $\epsilon > 0$, the closed ϵ-neighbourhood of F is closed in X as well.

FACT 5.15. N_ϕ *is a compact topometric space.*

Proof. For $p, q \in S_\phi(M)$, define $d(p,q) = \sup\{|\phi(p,a) - \phi(q,a)| : a \in M\}$. Clearly, d is a metric on $S_\phi(M)$; the topology generated by d is sometimes called the *uniform topology.* On the other hand, we know that $p_\alpha \to p$ in the logic topology τ iff $\phi^{p_\alpha} \to \phi^p$ in the topology of pointwise convergence, or equivalently, $\phi^{p_\alpha} \to \phi^p$ in the weak topology. Now, it is easy to verify that $(S_\phi(M), \tau, d)$ is a compact topometric space. ∎

REMARK 5.16. Let U be an Archimedean Riesz space with order unit e. Then it can be embedded as an order-dense and norm-dense Riesz subspace of $C(X)$, where X is a compact Hausdorff space (see Fact 5.3). For $a, b \in X$, define $d(a,b) = \sup\{|f(a) - f(b)| : f \in C(X)\}$. Clearly, (X, d) is a compact topometric space. Therefore, all results in this paper can be extended to Archimedean Riesz spaces with order unit, and our approach is appropriate for continuous logics as well as operator logics (cf. [Mo]).

We have the following continuous version of the Cantor–Bendixson rank.

DEFINITION 5.17 ([BU]). Let X be a compact topometric space. For a fixed $\epsilon > 0$, we define a decreasing sequence of closed subsets $X_{\epsilon,\alpha}$ by induction:

$$\begin{aligned}
X_{\epsilon,0} &= X,\\
X_{\epsilon,\alpha} &= \bigcap_{\beta<\alpha} X_{\epsilon,\beta} \quad \text{for } \alpha \text{ a limit ordinal},\\
X_{\epsilon,\alpha+1} &= \bigcap\{F \subseteq X_{\epsilon,\alpha} : F \text{ is closed and } \operatorname{diam}(X_{\epsilon,\alpha} \setminus F) \le \epsilon\},\\
X_{\epsilon,\infty} &= \bigcap_{\alpha} X_{\epsilon,\alpha},
\end{aligned}$$

where the *diameter* of a subset $U \subseteq X$ is defined by

$$\operatorname{diam}(U) = \sup\{d(x,y) : x, y \in U\}.$$

For any nonempty subset $U \subseteq X$ we define its *ϵ-Cantor–Bendixson rank in X* as

$$\mathrm{CB}_{X,\epsilon}(U) = \sup\{\alpha : U \cap X_{\epsilon,\alpha} \neq \emptyset\} \subseteq \mathrm{Ord} \cup \{\infty\}.$$

The next result characterizes stability in terms of CB-ranks. We remark that a structure M is *ω-saturated* if every 1-type over a finite tuple in M is realized in M.

PROPOSITION 5.18 (cf. [BU]). *ϕ is stable iff for any ω-saturated model $\mathsf{M} \models T$ where $|M| = (|T| + \kappa)^{\aleph_0}$ we have*

$$\mathrm{CB}_{S_\phi(M),\epsilon}(S_\phi(M)) < \infty \quad \textit{for all } \epsilon.$$

Proof. Let $\kappa > |T|$ be any cardinal such that $\kappa = \kappa^{\aleph_0}$. Let λ be the least cardinal such that $2^\lambda > \kappa$. Assume that $Y = S_\phi(M)_{\epsilon,\infty}$ is nonempty.

Then Y is compact, and if $U \subseteq Y$ is relatively open and nonempty then $\operatorname{diam}(U) > \epsilon$. We can therefore find nonempty open sets U_0, U_1 such that $\bar{U}_0, \bar{U}_1 \subseteq U$ and $d(U_0, U_1) > \epsilon$. Now, if $p \in U_0$, $q \in U_1$ then $d(p,q) > \epsilon$. Proceed by induction. If M is $2^{<\lambda}$-saturated and $(2^{<\lambda})^{\aleph_0} = 2^{<\lambda}$ then we can find a model $\mathsf{M}_0 \preceq \mathsf{M}$ of cardinality $2^{<\lambda}$ and types $\{p_\alpha\}_{\alpha<2^\lambda} \subseteq S_\phi(M_0)$ such that $d(p_\alpha, p_{\alpha'}) > \epsilon$ for all $\alpha \neq \alpha'$. Therefore, $\|S_\phi(M_0)\| > |M_0|$, i.e., the density character of $S_\phi(M_0)$ is greater than the cardinality of M_0.

The converse is standard. ∎

5.6. Stability and amenability. Now we return to analytic concepts. A topological group is called *precompact* if it is isomorphic to a subgroup of a compact group. Assume that G acts on a set X. Then a bounded function f on X is called *weakly almost periodic* if the G-orbit of f is weakly relatively compact in the Banach space $l^\infty(X)$ of all bounded real-valued functions on X equipped with the supremum norm. For a topological group G, denote by $\mathrm{WAP}(G)$ the space of all continuous weakly almost periodic functions on G.

FACT 5.19. *Assume that G is a topological group, and its theory, T_G, is satisfiable. Then the following are equivalent:*

(i) *T_G is stable.*
(ii) *G is precompact.*

Proof. We know that T_G is stable (i.e., $\mathrm{LUC}(G)$ is weakly compact) if and only if $\mathrm{LUC}(G) = \mathrm{WAP}(G)$. By [MPU, Theorem 4.5], $\mathrm{LUC}(G) = \mathrm{WAP}(G)$ if and only if G is precompact. ∎

COROLLARY 5.20. *Assume that G and T_G are as above. If T_G is stable, then G is uniquely amenable.*

Proof. It is known that for every precompact group G, the algebras $\mathrm{LUC}(G)$ and $\mathrm{LUC}(\widehat{G})$ are canonically isomorphic, where $\widehat{G}$ denotes the compact completion of G. Also, every compact group provides an obvious example of a uniquely amenable group for which the unique invariant mean comes from the Haar measure. So G is uniquely amenable since $\widehat{G}$ is. ∎

6. NIP. Talagrand [T] gave the first explicit definition of a stable set of functions. In fact, the notion of a stable set of functions [Fr3, 465B] is a measure-theoretic version of a well-known model-theoretic property, the dependence property. The definition is not obvious, but the basic properties of stable sets listed in [Fr3, 465C] are natural and easy to check, and we come quickly to the fact that (for complete locally determined spaces) pointwise bounded stable sets are relatively pointwise compact sets of measurable functions (Fact 6.3). We are now ready for the main definition, which is an adapted version of [Fr3, Definition 465B].

6.1. Almost dependence property

DEFINITION 6.1. A formula $\phi(x, y)$ has the *almost dependence property*, or is *almost dependent*, in a structure M if the set $A = \{\phi(x, b), \phi(a, y) : a, b \in M\}$ is a stable set of functions in the sense of [Fr3, Definition 465B], that is, whenever $E \subseteq M$ is measurable, $\mu(E) > 0$ and $s < r$ in $\mathbb{R}$, then there is some $k \geq 1$ such that $(\mu^{2k})^* D_k(A, E, s, r) < (\mu E)^{2k}$ where

$$D_k(A, E, s, r) = \bigcup_{f \in A} \{w \in E^{2k} : f(w_{2i}) \leq s,\ f(w_{2i+1}) \geq r \text{ for } i < k\}.$$

A formula ϕ has the *almost dependence property* in a theory T if it has the almost dependence property in every model of T.

NOTE 6.2. Assume that for each $s < r$ and $k \in \mathbb{N}$ the set $D_k(A, E, s, r)$ is measurable in M. Then it is easy to verify that $\phi(x, y)$ fails to be almost dependent in M if and only if there exist $E \subseteq M$ with $\mu(E) > 0$ and $s < r$ in $\mathbb{R}$ such that for each $k \geq 1$, almost each $w \in E^k$, and each $I \subseteq \{1, \ldots, k\}$, there is $f \in A$ with $f(w_i) \leq s$ for $i \in I$ and $f(w_i) \geq r$ for $i \notin I$ (see [T, Proposition 4]).

In the above definition, if $\mu(E) \geq \epsilon > 0$ then we say that ϕ fails to be almost ϵ-dependent, or that it has the *ϵ-FD property*. It is an easy exercise to show that the ϵ-FD property is a *first order* property (in integral logic), or equivalently, it is expressible. Clearly, ϕ has the almost dependence property if it fails to have the ϵ-FD property for all $\epsilon > 0$.

Note that the sets $A_1 = \{\phi(a, y) : a \in M\}$ and $A_2 = \{\phi(x, b) : b \in M\}$ are dependent if and only if $A = A_1 \cup A_2$ is dependent (cf. [Fr3, Proposition 465C(a), (d)]). On the other hand, one can easily define the (exact) *dependence property*. For this, we say ϕ fails to be *dependent*, or is *independent*, in M if there exist $s < r$ in $\mathbb{R}$ such that for each $k \geq 1$ there are $w_1, \ldots, w_k \in M$ such that for each $I \subseteq \{1, \ldots, k\}$ there is $f \in A$ with $f(w_i) \leq s$ for $i \in I$ and $f(w_i) \geq r$ for $i \notin I$. Clearly, a dependent formula (or theory) is necessarily almost dependent.

We immediately arrive at the following fact, which is an adapted version of [Fr3, Proposition 465D].

FACT 6.3. *Let $\mathsf{M} = (M, \Sigma, \mu)$ be a structure such that μ is a complete locally determined measure space, and $\phi(x, y)$ an almost dependent formula. Since every formula is bounded, so is ϕ. Therefore, $A = \{\phi(x, b), \phi(a, y) : a, b \in M\}$ is relatively compact in the space of measurable functions for the topology of pointwise convergence.*

We compare our notions:

PROPOSITION 6.4. *Let $\phi(x, y)$ be a stable formula in a theory T. Then ϕ is almost dependent in T.*

Proof. Assume that ϕ fails to be almost dependent. Therefore, there is a model $\mathsf{M} \vDash T$, $E \subseteq M$, with $\mu(E) > 0$, and $r > s$ in $\mathbb{R}$ such that $(\mu^{2k})^* D_k(A, E, s, r) = (\mu E)^{2k}$ for each k. Then it is easy to verify that for each k there are finite sequences $a_n, b_n \in E$, $n \leq k$, such that for all $j < i \leq k$ we have $\phi(a_i, b_j) \geq r$ and $\phi(a_j, b_i) \leq s$. Now, by the compactness theorem of model theory, there is an elementary extension $\mathsf{N} \succ \mathsf{M}$ such that ϕ is not stable in N. Thus, ϕ is not stable in T. ∎

To summarize:

$$\phi \text{ is stable} \ \Rightarrow\ \phi \text{ is dependent} \ \Rightarrow\ \phi \text{ is almost dependent.}$$

By a result of Bourgain, Fremlin and Talagrand [BFT, Theorem 2F], one can easily check that a formula ϕ is (exactly) dependent if and only if it is almost dependent for each Radon measure. We will study this connection in a future work.

6.2. Almost definability of types. Here, a result similar to the stable case can be proved for the almost dependence property. For this, we need some definitions. Let ψ be a measurable function on $(S_\phi(M), \mu_\phi)$ where μ_ϕ is the unique Radon measure induced by $\phi^{\mathsf{M}}(x, b)$ for all $b \in M$. Then ψ is called an *almost ϕ-definable relation over M* if there is a sequence $g_n : S_\phi(M) \to \mathbb{R}$, $|g_n| \leq |\phi|$, of continuous functions such that $\lim_n g_n(b) = \psi(b)$ for almost all $b \in S_\phi(M)$. (We note that by the Stone–Weierstrass theorem every continuous function $g_n : S_\phi(M) \to \mathbb{R}$ can be expressed as a uniform limit of a sequence of algebraic combinations of functions of the form $p \mapsto \phi(p, b)$, $b \in M$.) An almost definable relation $\psi(y)$ over M defines $p \in S_\phi(M)$ if $\phi(p, b) = \psi(b)$ for almost all $b \in M$, and in this case we say that p is *almost definable.* Assume that every type p in $S_\phi(M)$ is almost definable by a measurable function ψ^p. We say that p is *almost equal to q*, written $p \equiv q$, if $\psi^p(b) = \psi^q(b)$ for almost all $b \in M$. Define $[p] = \{q \in S_\phi(M) : p \equiv q\}$ and $[S_\phi](M) = \{[p] : p \in S_\phi(M)\}$.

THEOREM 6.5 (Almost definability of types). *Let $\phi(x, y)$ be a formula almost dependent in a structure M. Then every $p \in S_\phi(M)$ is almost definable by a (unique up to measure) almost $\tilde{\phi}$-definable relation $\psi(y)$ over M, where $\tilde{\phi}(y, x) = \phi(x, y)$.*

Proof. We know that $(\mathsf{M}, \mu_\phi^{\mathsf{M}}) \preceq (S_\phi(M), \mu_\phi)$. First, assume $(S_\phi(M), \mu_\phi)$ is minimal, i.e., μ_ϕ is Baire and it is not nesessarily Radon. (One can easily verify that the subspace measure $\mu_\phi{\restriction}_M$ is μ_ϕ^{M}. Therefore, by [Fr3, Proposition 465C(n)], since the set $\{\phi(a, y){\restriction}_M : a \in M\} \subseteq \mathbb{R}^M$ is also almost dependent with respect to $\mu_\phi{\restriction}_M$, the set $A = \{\phi(a, y) : a \in M\} \subseteq \mathbb{R}^{S_\phi(M)}$ is almost dependent with respect to μ_ϕ.) By [Fr3, Proposition 465C(i)], the set A is moreover almost dependent with respect to the completion $\hat{\mu}_\phi$ of μ_ϕ.

Now, let $p(x) \in S_\phi(M)$, and let $a_i \in M$ be any net such that $\lim_i \mathrm{tp}_\phi(a_i/M) = p$. Since $\hat{\mu}_\phi$ is complete, by Fact 6.3, there is a $\hat{\mu}_\phi$-measurable function ψ such that $\lim_i \phi^{a_i}(y) = \psi(y)$. Let $\bar{\mu}_\phi$ be the unique extension of μ_ϕ to a Radon measure. Thereby it is also an extension of $\hat{\mu}_\phi$. Since $\bar{\mu}_\phi$ is Radon, by [F, Proposition 7.9] there is a sequence g_n of continuous functions on $S_\phi(M)$ such that $g_n \to \psi$ in $L^1(\bar{\mu}_\phi)$, and hence by [F, Corollary 2.32] a subsequence (still denoted by g_n) that converges $\bar{\mu}_\phi$-a.e. to ψ. Clearly, ψ is unique up to the measure $\bar{\mu}_\phi$. ∎

Corollary 6.6. *Let $\phi(x,y)$ be a formula and T a theory. Then* (i)$\Rightarrow$(ii) $\Rightarrow$(iii)*, where:*

(i) *The formula ϕ is almost dependent in T.*
(ii) *For every model $\mathsf{M} \models T$, every ϕ-type over M is almost definable by a $\tilde{\phi}$-predicate over M.*
(iii) *For each cardinal $\lambda = \kappa^{\aleph_0} \geq |T|$, and each model $\mathsf{M} \models T$ with $|M| \leq \lambda$, we have $|[S_\phi](M)| \leq \lambda$.*

Proof. Clear. ∎

6.3. Almost Cantor–Bendixson rank. A result similar to the Cantor–Bendixson rank for stable formulas holds for the almost dependence property. For this we need some definitions. For a μ_ϕ-measurable function $\xi : S_\phi(M) \to [-\flat_\phi, \flat_\phi]$ where $\flat_\phi$ is the universal bound of ϕ, let

$$[\xi] = \{\chi : S_\phi(M) \to [-\flat_\phi, \flat_\phi] \mid \chi \text{ is } \mu_\phi\text{-measurable and } \chi = \xi \text{ a.e.}\}.$$

Let $L^1_\phi = \{[\xi] \mid \xi : S_\phi(M) \to [-\flat_\phi, \flat_\phi] \text{ is } \mu_\phi\text{-measurable}\}$. We show that L^1_ϕ has a natural compact topometric structure. Indeed, let $\mathfrak{d}([\xi],[\xi']) = \int |\xi - \xi'|\, d\mu_\phi$, and $[\xi_\alpha] \xrightarrow{\mathfrak{T}} [\xi]$ iff $I([\xi_\alpha]) \to I([\xi])$ for all $I \in (L^1)^*$. In fact, the topology generated by the metric $\mathfrak{d}$ is the norm topology on L^1, and $\mathfrak{T}$ is the weak topology generated by $(L^1)^*$. Now, it is easy to verify that $(L^1_\phi, \mathfrak{d}, \mathfrak{T})$ is a compact topometric space. Indeed, since L^1_ϕ is uniformly integrable, by [Fr1, Theorem 247C], L^1_ϕ is relatively weakly compact. Also, L^1_ϕ is closed in the norm topology. It is well-known that for a convex subset of a locally convex space, the weak closure is equal to the norm closure. Therefore, L^1_ϕ is weakly closed, and hence it is weakly compact. On the other hand, it is well-known that the norm L^1 is weakly lower semicontinuous (cf. [AB, Lemma 6.22]). So L^1_ϕ is a compact topometric space, as claimed.

We remark that if the types p, q are definable by measurable functions ψ^p, ψ^q, then $p \equiv q$ iff $\psi^p(b) = \psi^q(b)$ for almost all $b \in M$, or equivalently, $[\psi^p] = [\psi^q]$. (Note that since $\mathsf{M} \preceq (S_\phi(M), \mu_\phi)$, we have $\psi^p(b) = \psi^q(b)$ for almost all $b \in M$ iff $\psi^p = \psi^q$ μ_ϕ-almost everywhere.) Therefore, if $|M| = \kappa^{\aleph_0}$ and ϕ is a formula almost dependent in the structure M, then

$|L^1_\phi| = |[S_\phi](M)| = \kappa^{\aleph_0}$. (See Theorem 6.5 and the definitions before it.) This yields

Proposition 6.7. *If ϕ is almost dependent, then for any ω-saturated model $\mathsf{M} \models T$ where $|M| = (|T| + \kappa)^{\aleph_0}$ we have $\mathrm{CB}_{L^1_\phi,\epsilon}(L^1_\phi) < \infty$ for all ϵ.*

The almost dependence property is linked with a notion from another area. Historically, this property arose naturally when Talagrand and Fremlin were studying pointwise compact sets of measurable functions; they found that in many cases a set of functions was relatively pointwise compact because it was almost dependent. Later it appeared that the concept was connected with *Glivenko–Cantelli classes* in the theory of empirical measures, as explained in [T]. Also, a version of *Vapnik–Chervonenkis dimension* which is suitable for measure structures can be defined, and will be studied in a future work.

7. Conclusion. In the first part of this paper we studied some concrete analytic structures. This study led us to the natural and correct notion of types. The perspective on types in this paper can be used in other logics. For example, this approach seems to be appropriate for continuous logic [BU] as well as operator logics [Mo]. Note that by Remark 5.16, every Archimedean Riesz space with order unit admits a natural compact topometric structure. Therefore, most of the results in this paper can be extended to Archimedean Riesz spaces. Also, the notion of forking and independence, and their connections with measure theory can be studied. On the other hand, one can do much more classifications, e.g. the strict order property and other properties. We will study them elsewhere. Finally, all these results suggest that many interesting analytic concepts may be studied by model-theoretic methods. Moreover, these methods provide a new view on the related subjects in analysis, and open some fruitful areas of research on similar questions.

Acknowledgements. I want to thank John T. Baldwin and Massoud Pourmahdian for their interest in this article and for their comments and corrections. I thank the anonymous referee for his/her detailed suggestions and corrections; they helped to improve the exposition of this paper significantly. I would like to thank the Institute for Basic Sciences (IPM), Tehran, Iran. This research was partly supported by IPM (grant nos. 91030032 and 92030032).

References

[AB] C. D. Aliprantis and K. C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, 3rd ed., Springer, Berlin, 2006.

[BP] S. M. Bagheri and M. Pourmahdian, *The logic of integration*, Arch. Math. Logic 48 (2009), 465–492.

[B1] I. Ben Yaacov, *Model theoretic stability and definability of types, after A. Grothendieck*, Bull. Symbolic Logic 20 (2014), 491–496.

[B2] I. Ben Yaacov, *On theories of random variables*, Israel J. Math. 194 (2013), 957–1012.

[B3] I. Ben Yaacov, *Schrödinger's cat*, Israel J. Math. 153 (2006), 157–191.

[BBHU] I. Ben Yaacov, A. Berenstein, C. W. Henson and A. Usvyatsov, *Model theory for metric structures*, in: Model Theory with Applications to Algebra and Analysis, Vol. 2, Z. Chatzidakis et al. (eds.), London Math. Soc. Lecture Note Ser. 350, Cambridge Univ. Press, Cambridge, 2008, 315–427.

[BU] I. Ben Yaacov and A. Usvyatsov, *Continuous first order logic and local stability*, Trans. Amer. Math. Soc. 362 (2010), 5213–5259.

[BFT] J. Bourgain, D. H. Fremlin and M. Talagrand, *Pointwise compact sets of Baire-measurable functions*, Amer. J. Math. 100 (1978), 845–886.

[CK] C. C. Chang and H. J. Keisler, *Continuous Model Theory*, Princeton Univ. Press, Princeton, NJ, 1966.

[DS] N. Dunford and J. T. Schwartz, *Linear Operators*, Part I, Pure Appl. Math. 7, Interscience, New York, 1958.

[F] G. B. Folland, *Real Analysis. Modern Techniquess and Their Applications*, 2nd ed., Wiley, New York, 1999.

[Fr1] D. H. Fremlin, *Measure Theory, Vol. 2, Broad Foundations*, Torres Fremlin, Colchester, 2003.

[Fr2] D. H. Fremlin, *Measure Theory, Vol. 3, Measure Algebras*, Torres Fremlin, Colchester, 2004.

[Fr3] D. H. Fremlin, *Measure Theory, Vol. 4, Topological Measure Spaces*, Torres Fremlin, Colchester, 2006.

[G] A. Grothendieck, *Critères de compacité dans les espaces fonctionnels généraux*, Amer. J. Math. 74 (1952), 168–186.

[HC] W. Herer and J. P. R. Christensen, *On the existence of pathological submeasures and the construction of exotic topological groups*, Math. Ann. 213 (1975), 203–210.

[HR] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis*, Vol. I, Springer, Berlin, 1963.

[H] D. N. Hoover, *Probability logic*, Ann. Math. Logic 14 (1978), 287–313.

[K] H. J. Keisler, *Probability quantifiers*, in: *Model-Theoretic Logics*, J. Barwise and S. Feferman (eds.), Springer, New York, 1985, 509–556.

[KN] J. L. Kelley and I. Namioka, *Linear Topological Spaces*, Van Nostrand, Princeton, NJ, 1963.

[KA] K. Khanaki and M. Amini, *Haar measure and integral logic*, Math. Logic Quart. 58 (2012), 294–302.

[KB] K. Khanaki and S. M. Bagheri, *Random variables and integral logic*, Math. Logic Quart. 57 (2011), 494–503.

[MPU] M. Megrelishvili, V. G. Pestov and V. V. Uspenskij, *A note on the precompactness of weakly almost periodic groups*, in: Nuclear Groups and Lie Groups, E. Martín Peinador and J. Núñez García (eds.), Res. Exp. Math. 24, Heldermann, Lemgo, 2001, 209–216.

[M] T. Mitchell, *Topological semigroups and fixed points*, Illinois J. Math. 14 (1970), 630–641.

[Mo] A. Mofidi, *Quantified universes and ultraproducts*, Math. Logic Quart. 58 (2012), 63–74.

[N] I. Namioka, *On certain actions of semi-groups on L-spaces*, Studia Math. 29 (1967), 63–77.

[P] A. L. T. Paterson, *Nonamenability and Borel paradoxical decompositions for locally compact groups*, Proc. Amer. Math. Soc. 96 (1986), 89–90.

[Pe1] V. Pestov, *Topological groups: where to from here?*, Topology Proc. 24 (1999), 421–502.

[Pe2] V. Pestov, *Ramsey–Milman phenomenon, Urysohn metric spaces, and extremely amenable groups*, Israel J. Math. 127 (2002), 317–357.

[R] H. L. Royden, *Real Analysis*, 3rd ed., Macmillan, New York, 1988.

[Ru] V. Runde, *Lectures on Amenability*, Lecture Notes in Math. 1774, Springer, Berlin, 2002.

[S] P. Simon, *Rosenthal compacta and NIP formulas*, Fund. Math. 231 (2015), 81–92.

[T] M. Talagrand, *The Glivenko–Cantelli problem*, Ann. Probab. 15 (1987), 837–870.

[W] S. Wagon, *The Banach–Tarski Paradox*, Cambridge Univ. Press, Cambridge, 1985.

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