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Virtual Legendrian isotopy

by

Vladimir Chernov (Hanover, NH) and
Rustam Sadykov (Manhattan, KS)

Abstract. An *elementary stabilization* of a Legendrian knot L in the spherical cotangent bundle ST^*M of a surface M is a surgery that results in attaching a handle to M along two discs away from the image in M of the projection of the knot L. A virtual Legendrian isotopy is a composition of stabilizations, destabilizations and Legendrian isotopies. A class of virtual Legendrian isotopy is called a virtual Legendrian knot.

In contrast to Legendrian knots, virtual Legendrian knots enjoy the property that there is a bijective correspondence between the virtual Legendrian knots and the equivalence classes of Gauss diagrams.

We study virtual Legendrian knots and show that every such class contains a unique irreducible representative. In particular we get a solution to the following conjecture of Cahn, Levi and the first author: two Legendrian knots in ST^*S^2 that are isotopic as virtual Legendrian knots must be Legendrian isotopic in ST^*S^2.

1. Introduction. Let M be a closed oriented surface, possibly non-connected, and L a Legendrian link in the total space of the spherical cotangent bundle $\pi : ST^*M \to M$ of M. An *elementary stabilization* of L is a surgery that results in cutting out from M two discs away from the image πL of the projection of L to M, and attaching a handle to M along the created boundary components. The converse operation is called an *elementary destabilization*. More precisely, let A be a simple connected closed curve in M in the complement to πL. The *elementary destabilization* of L along A consists of cutting M open along A and then capping the resulting boundary circles.

An elementary destabilization of a link is *trivial* if it chops off a sphere containing no components of L. We say that a Legendrian link is *irreducible* if it does not allow non-trivial destabilizations.

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A *virtual Legendrian isotopy* [1] is a composition of elementary stabilizations, destabilizations, and Legendrian isotopies. A virtual Legendrian isotopy class of a Legendrian link (respectively Legendrian knot) is called a *virtual Legendrian link* (respectively *virtual Legendrian knot*). In contrast to Legendrian knots, virtual Legendrian knots enjoy the property [1] that there is a bijective correspondence between the virtual Legendrian knots and equivalence classes of Gauss diagrams ([1]).

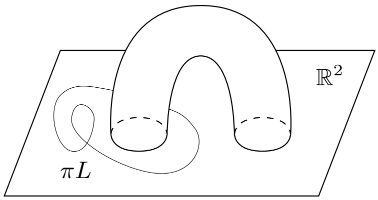

Fig. 1. An elementary stabilization of a Legendrian curve in the spherical cotangent bundle of $\mathbb{R}^2$

Our main result is Theorem 1.1, which should be compared to the Kuperberg Theorem on virtual links [3, Theorem 1]. Note that the proof of Kuperberg's results does not seem to generalize to the category of virtual Legendrian knots.

THEOREM 1.1. *Every virtual isotopy class of Legendrian links contains a unique irreducible representative. The irreducible representative can be obtained from any representative of the virtual Legendrian isotopy class by a composition of destabilizations and isotopies.*

The second assertion of Theorem 1.1 is immediate from the first one. Indeed, each destabilization increases the Euler characteristic of the surface by two. On the other hand, if we disregard surface components with no components of the link projection, then the number of components of the surface is bounded by the number of link components. And therefore, the Euler characteristic of the surface is bounded by twice the number of link components. Thus, for any Legendrian link in the given virtual Legendrian

([1]) Similar to Kauffman's [2] theory of ordinary virtual knots, the theory of virtual Legendrian knots can be reformulated in three equivalent ways:

(1) As the theory of Legendrian knots in ST^*M modulo stabilization, destabilization and Legendrian isotopy.
(2) As the theory of virtual front diagrams on $\mathbb{R}^2$ modulo the standard front moves and the virtual front moves (see [1, Sections 2 and 7]).
(3) As the theory of front Gauss diagrams modulo the modifications of Gauss diagrams (see [1, Sections 4 and 7]). Note that not every Gauss diagram corresponds to an ordinary Legendrian knot.

isotopy class, only finitely many consecutive non-trivial destabilizations are possible. Thus, after finitely many destabilizations we obtain an irreducible representative. It is the unique irreducible representative claimed in the first assertion of Theorem 1.1.

The main consequence of Theorem 1.1 is Corollary 1.2.

Corollary 1.2. *Virtual Legendrian isotopy classes of irreducible Legendrian links in ST^*M of a surface M are in bijective correspondence with isotopy classes of irreducible Legendrian links in ST^*M.*

Note that, in general, virtual Legendrian isotopy classes of (reducible) links are not in bijective correspondence with Legendrian isotopy classes of links.

In view of Corollary 1.2, we get the solutions to the following two Conjectures 1.3 and 1.4 formulated by P. Cahn, A. Levi and the first author [1, Conjectures 1.5 and 1.4].

Conjecture 1.3. Let K_1 and K_2 be two Legendrian knots in ST^*M that are isotopic as virtual Legendrian knots, and suppose that M is the surface of smallest genus realizing knots in the virtual Legendrian isotopy class of K_1 and K_2. Then, possibly after a contactomorphism of ST^*M, K_1 and K_2 are Legendrian isotopic in ST^*M.

Conjecture 1.4. Two Legendrian knots in ST^*S^2 that are isotopic as virtual Legendrian knots must be Legendrian isotopic in ST^*S^2.

In [1, Conjecture 1.4 and p. 25] a similar fact is also conjectured for virtual Legendrian knots in ST^*S^n, $n \geq 3$, and $ST^*\mathbb{R}^n$, $n \geq 2$. These conjectures are still open.

In view of [1], another immediate corollary of Theorem 1.1 is Corollary 1.5.

Corollary 1.5. *Every Gauss diagram can be represented by a unique irreducible Legendrian knot in ST^*M for some surface M.*

The proof of Theorem 1.1 consists of several steps, and besides the general case there are two exceptional ones that do not fit the general setting. In Section 3 we deal with the exceptional cases. In Section 4 we list the steps; these are Lemmas 4.1–4.3. Lemma 4.3 is proved in Section 4, while Lemmas 4.1 and 4.2 are postponed until Section 5 after we present a necessary auxiliary construction.

2. A reformulation of Theorem 1.1.

We say that two links L_1 in ST^*M_1 and L_2 in ST^*M_2 are *descent-equivalent* if after a composition of destabilizations and isotopies of L_1 and L_2 they become the same.

Suppose that, contrary to the statement of Theorem 1.1, there is a Legendrian link L in ST^*M that has two different irreducible descendants. Among all such pairs (L, M) we may choose a pair such that

- the surface M has no naked sphere components, and
- the genus of M is minimal.

In particular, every link obtained from L by an elementary non-trivial destabilization (or a composition of elementary non-trivial destabilizations) has a unique descendant.

Then there are two Legendrian links L_1 and L_2 in ST^*M, both isotopic to L, and two simple closed connected curves A_1 and A_2 in M such that

- each A_i is disjoint from πL_i, and
- the elementary destabilizations of L_1 along A_1 and of L_2 along A_2 are not descent-equivalent.

Note that the second condition implies that both destabilizations are non-trivial.

Theorem 1.1 is equivalent to the statement that a tuple (M, L_1, L_2, A_1, A_2) as above does not exist. In the following sections we will assume that such an *exceptional* tuple exists and arrive at a contradiction.

3. Exceptional cases. We will often require that the manifold M is distinct from a sphere, and that neither A_1 nor A_2 bounds a disc; our general argument does not work in these exceptional cases, see Remark 5.6 below.

In this section we show that the assumptions that $M \neq S^2$ and that A_1 and A_2 are non-contractible are not restrictive (Lemmas 3.1 and 3.2).

LEMMA 3.1. *Suppose that A_1 bounds a disc. Then the destabilization of L_1 along A_1 is descent-equivalent to the destabilization of L_2 along A_2.*

Proof. We will show that we can assume that the intersection of A_1 and A_2 is empty and hence the destabilizations along A_1 and A_2 are descent-equivalent. In more detail, we assume that πL_1 is located close to the center of the disk D bounded by A_1; the case where πL_1 is located outside D is entirely similar. If A_1 and A_2 intersect, take a pair of curves with the minimal number of intersection points among those pairs of curves such that the destabilizations along them are not descent-equivalent.

We show that the number of intersection points may be further decreased, yielding a contradiction. The curve A_2 subdivides the disk D into regions; by induction at least two of these regions are bigons and one of them does not contain the center of D (with πL_1 in its small neighborhood). The latter bigon is bounded by an arc α_2 of A_2 and α_1 of A_1 (see Figure 2). Since this bigon does not contain any components of πL_1, we can compress the arc α_2 along this bigon in such a way that during the compression it does

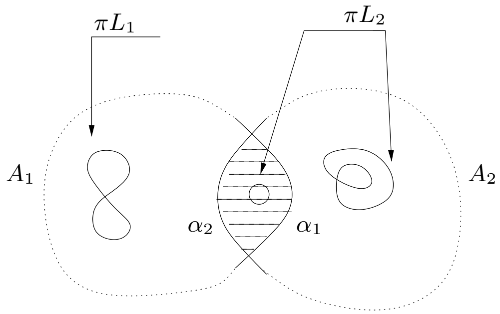

Fig. 2. The bigon bounded by the arcs α_1 and α_2

not intersect πL_1. If this bigon contains curves of πL_2 they will be pushed out through α_1 during the compression. ∎

Lemma 3.2 below immediately follows from Lemma 3.1 since every connected simple closed curve on a sphere bounds a disc.

LEMMA 3.2. *There exists no exceptional tuple with $M = S^2$, i.e., the statement of Theorem* 1.1 *is true for links in the spherical cotangent bundle of a sphere.*

4. Proof of Theorem 1.1. In view of Lemmas 3.2 and 3.1, we may (and will) assume that A_1 and A_2 are not contractible, and that the surface M is not homeomorphic to a sphere.

Proof of Theorem 1.1. Recall that each πL_i is disjoint from A_i. The following lemma essentially asserts that we may also assume that πL_1 is disjoint from both A_1 and A_2. The proof is postponed till Section 5.

LEMMA 4.1. *Suppose that A_1, A_2 are not null-homotopic and that the surface M is distinct from a sphere. Then there is an isotopy of L_1 whose projection does not intersect A_1 and that takes L_1 to a curve whose projection is disjoint from A_2.*

Next, we show that not only can we assume that L_1 is disjoint from A_1 and A_2, but that, in fact, $L_1 = L_2$. The proof will also be given in Section 5.

LEMMA 4.2. *Let (M, L_1, L_2, A_1, A_2) be an exceptional tuple, and suppose that $M \neq S^2$ and A_1, A_2 are not null-homotopic. If πL_1 does not intersect A_2, then (M, L_1, L_1, A_1, A_2) is also an exceptional tuple.*

The next lemma completes the proof of Theorem 1.1 since its conclusion contradicts the minimality of the genus of M.

LEMMA 4.3. *If $L_1 = L_2 = L$, then the genus of M is not minimal.*

Proof. The argument is similar to that by Greg Kuperberg. Namely, assume, contrary to the statement, that $L_1 = L_2$ and the genus of M is minimal.

It follows that the intersection $A_1 \cap A_2$ is non-empty; otherwise the destabilizations of L along A_1 and A_2 are descent-equivalent. Without loss of generality we may assume that A_1 and A_2 intersect in the minimal number of points among pairs of simple connected closed curves such that the destabilizations along them are not descent-equivalent.

If the two curves A_1 and A_2 intersect at only one point, then take the boundary A_3 of a neighborhood of $A_1 \cup A_2$. Note that the destabilization along A_3 is not trivial: it chops off a naked torus. The destabilizations along A_1 and A_3 are descent-equivalent since the curves are disjoint. Similarly for A_2 and A_3. Therefore the destabilizations along A_1 and A_2 are descent-equivalent, contrary to the assumption.

Finally, suppose that A_1 and A_2 have at least two common points. Let D_1 be an interval in A_1 bounded by two intersection points and containing no other points of A_2. Compress A_2 along D_1, i.e., remove small arcs of A_2 intersecting A_1, and then join the two pairs of boundary points of A_2 by two new arcs parallel to D_1. Then A_2 turns into two new connected curves A_2' and A_2'' in M (see Figure 3). A destabilization along at least one of these components, say A_2', is non-trivial. Observe that the destabilizations of L_2 along A_2 and A_2' are equivalent since both are disjoint from πL_2 and have no common points (after a small displacement of one of them along a vector field orthogonal to the curve). On the other hand, the curve A_2' has fewer intersection points with A_1. Therefore the destabilizations along A_1, A_2' and A_2 are descent-equivalent. ∎

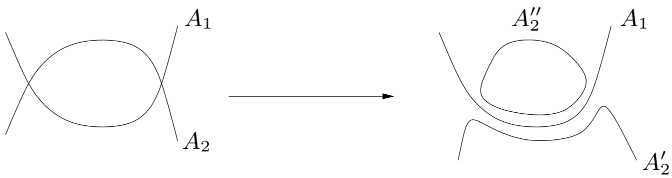

Fig. 3. Compression of A_2 along an arc

This completes the proof of Theorem 1.1 assuming Lemmas 4.1 and 4.2. ∎

5. Proof of Lemmas 4.1 and 4.2. The main tool here is Theorem 5.2. To motivate its proof let us prove Lemma 5.1. We will not use the lemma in what follows. However, this short Lemma 5.1 explains well the counter-intuitive phenomenon that stable Legendrian isotopy in certain cases reduces to Legendrian isotopy.

Lemma 5.1. *Let M be a hyperbolic surface. Let L_1 and L_2 be two Legendrian links in ST^*M whose projections belong to an open disc $D \subset M$. Then L_1 and L_2 are isotopic in ST^*M if and only if they are isotopic in ST^*D.*

Proof. Clearly, if L_1 and L_2 are isotopic in ST^*D, then they are isotopic in ST^*M. Let us prove the converse implication.

Let $p : \mathbb{R}^2 \to M$ denote the universal covering of M. There exist lifts L_1' and L_2' of L_1 and L_2 respectively such that the isotopy of L_1 to L_2 lifts to an isotopy of L_1' to L_2' in $ST^*\mathbb{R}^2$. Choose an arbitrary diffeomorphism $\varphi : \mathbb{R}^2 \to D^2$. It lifts to a contactomorphism $\tilde{\varphi}$ of spherical cotangent bundles. Thus, we obtain a Legendrian isotopy of $\tilde{\varphi}(L_1')$ to $\tilde{\varphi}(L_2')$. It remains to show that L_1 admits a Legendrian isotopy to $\tilde{\varphi}(L_1')$, and L_2 admits a Legendrian isotopy to $\tilde{\varphi}(L_2')$.

We may assume that both L_1 and L_2 are links whose images with respect to π are located in a small neighborhood U of a point in D. Furthermore, we may choose φ so that the composition of a lift of D and φ is the identity map on U so that $\tilde{\varphi}(L_i) = L_i$, $i = 1, 2$. Then it remains to show that, for any link L in ST^*D and any lifts L' and L'' in $ST^*\mathbb{R}^2$, the link $\tilde{\varphi}(L')$ admits a Legendrian isotopy to $\tilde{\varphi}(L'')$. Choose a Legendrian isotopy γ from L' to L'' in $ST^*\mathbb{R}^2$. The desired Legendrian isotopy is $\tilde{\varphi}(\gamma)$. ∎

Theorem 5.2. *Let L_1 and L_2 be isotopic Legendrian links in the spherical cotangent bundle ST^*M of a connected closed surface $M \neq S^2$, and let A_1 and $A_2 = A$ be simple, connected, not null-homotopic, closed curves in M such that (M, L_1, L_2, A_1, A_2) is an exceptional tuple. Suppose that A is disjoint from πL_1 and πL_2. If A breaks M into two surfaces, suppose that πL_1 and πL_2 belong to the same path component of $M \setminus A$. Then the tuple (M, L_1, L_1, A_1, A_2) is also exceptional.*

Before proving Theorem 5.2, let us construct an (in general, non-regular) covering of M by a surface $\tilde{M}$ homeomorphic to the connected component of $M \setminus A$ which contains $\pi(L_1)$ and $\pi(L_2)$. In fact we will give three equivalent definitions; each has its advantage.

Definition 5.3 (First definition). Choose a base point in M in the path component of $M \setminus A$ that contains πL_1 and πL_2. We say that an element in the fundamental group $\pi_1 M$ *avoids* A if it admits a representing curve that does not intersect A. The subset of elements in $\pi_1 M$ avoiding A forms a subgroup. Let $\tilde{M} \to M$ be the covering corresponding to the subgroup of $\pi_1 M$ of elements avoiding A.

Definition 5.4 (Second definition). Since M is distinct from a sphere, it admits a universal covering $u : \mathbb{R}^2 \to M$. We choose a base point in $\mathbb{R}^2$ that projects to the base point in M. Then every point x in the universal covering space can be identified with the pair of a point $y = u(x)$ and the homotopy class of the projection in M of the curve in $\mathbb{R}^2$ from the base point to x. The manifold $\tilde{M}$ is the quotient of $\mathbb{R}^2$ by the relation that identifies (y, γ_1) with (y, γ_2) whenever $\gamma_1 \gamma_2^{-1}$ contains a loop that does not intersect A.

Definition 5.5 (Third definition). Suppose that A does not separate M. Since M is either a torus or hyperbolic, there is an infinite covering $\mathbb{H} \to M$ (or $\mathbb{R}^2 \to M$), and we may assume that a lift $\tilde{A}$ of A is a geodesic (every simple non-contractible curve on M is isotopic to a unique simple geodesic). There is a monodromy action $\mathbb{Z}$ on $\mathbb{H}$ (or on $\mathbb{R}^2$) corresponding to the loop A; namely, we know that M is the quotient of $\mathbb{H}$ (or of $\mathbb{R}^2$) by the action of $\pi_1 M$, and the above-mentioned monodromy action is the action by the subgroup generated by A. It acts on the geodesic $\tilde{A}$ by translations. Attach $(\mathbb{H} \setminus \tilde{A})/\mathbb{Z}$ (or $(\mathbb{R}^2 \setminus \tilde{A})/\mathbb{Z}$) to $M \setminus A$ so that the projections of the two cylinders $(\mathbb{H} \setminus \tilde{A})/\mathbb{Z}$ (or of $(\mathbb{R}^2 \setminus \tilde{A})/\mathbb{Z}$) and of the manifold $M \setminus A$ to M form an infinite covering $\tilde{M} \to M$; this is the desired covering.

Suppose now that A separates M into two components M_1 and M_2, where M_1 is the component containing the images of the projections of L_1 and L_2 to M. Again, take a covering $\mathbb{H} \to M$ (respectively $\mathbb{R}^2 \to M$) and cut $\mathbb{H}$ (respectively $\mathbb{R}^2$) along a lift $\tilde{A}$ of A. Attach one component of $(\mathbb{H} \setminus \tilde{A})/\mathbb{Z}$ (respectively of $(\mathbb{R}^2 \setminus \tilde{A})/\mathbb{Z}$) to M_1 so that their projections to M form the desired covering $\tilde{M} \to M$.

Remark 5.6. If A bounds a disc, which is the case that we exclude from consideration, then the first and the second definitions result in the one-sheet covering, while the third definition makes no sense since a lift of a contractible curve A is not a geodesic.

Let M, A be as in Theorem 5.2 and $\tilde{M} \to M$ be the covering from Definitions 5.3–5.5. If A does not separate M, let M_1 denote $M \setminus A$. If A does separate M, let M_1 denote the connected component of $M \setminus A$ that contains the projections of L_1, L_2.

Lemma 5.7. *The surface $\tilde{M}$ is homeomorphic to M_1.*

Proof. This immediately follows from Definition 5.5. Indeed, the manifold $\tilde{M}$ is obtained from M_1 by attaching one or two cylinders depending on whether M_1 has one or two ends. ∎

To summarize, we have constructed a covering $\tilde{M} \to M$ by a surface homeomorphic to M_1.

Proof of Theorem 5.2. Since πL_1 is disjoint from A, it lifts to a simple (not connected if L_1 is a link) closed Legendrian curve L_1' in $ST^*M_1 \subset ST^*\tilde{M}$. Furthermore, the Legendrian isotopy of L_1 to L_2 lifts to a Legendrian isotopy of L_1' to L_2' in $ST^*\tilde{M}$.

Let U be a thin neighborhood of ∂M_1 in M disjoint from πL_1 and πL_2. Suppose that L'_2 belongs to the leaf $ST^*M_1 \subset ST^*\tilde{M}$. Then there is an isotopy of the identity map of $ST^*\tilde{M}$ relative to $ST^*(M_1 \setminus U)$ to a map with image in ST^*M_1 and that brings the isotopy of L'_1 to L'_2 into ST^*M_1. This isotopy of $\mathrm{id}_{ST^*\tilde{M}}$ comes from a deformation retraction $\tilde{M} \to M_1$ fixing points in $M_1 \setminus U$.

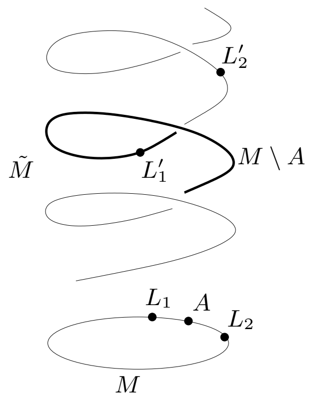

Fig. 4. Lifts of L_1 and L_2 to the covering

Suppose now that L'_2 belongs to a leaf of the covering $ST^*\tilde{M} \to ST^*M$ distinct from the leaf ST^*M_1. In this case, a deformation retraction of $\tilde{M}$ to M_1 moves L'_2, and therefore the above argument does not work. During the isotopy of L'_1 to L'_2 we find that at a certain moment L'_1 leaves ST^*M_1 and, in view of the deformation retraction, in this case we may assume that the projection of L_1 to M belongs to the interior of $U \cap M_1$. (Here U is a thin neighborhood of ∂M_1.) Similarly, by exchanging the roles of L_1 and L_2, we may assume that the projection of L_2 to M belongs to the interior of $U \cap M_1$. Furthermore, we may suppose that the projection of the isotopy from L_1 to L_2 is in U.

Let L''_2 be a link obtained from L_2 by a translation such that $\pi L''_2$ belongs to the same path component of $U \setminus A$ that contains πL_1. Then the destabilization of L''_2 along A is descent-equivalent to the destabilization of L_2 along A. Indeed, after the destabilization along A_2 both πL_2 and $\pi L''_2$ are curves in a neighborhood of a point, and hence both links are descent-equivalent to the same link in ST^*S^2. Therefore we may replace L_2 with L''_2. On the other hand, L''_2 is isotopic to L_1 in $\pi^{-1}(M_1 \cap U)$. ∎

Proof of Lemma 4.2. To simplify notation let us assume that L is a Legendrian knot; the case where L is a link is similar. If L_1 and L_2 belong to the same component of $M \setminus A_2$, then the required Legendrian isotopy exists by Theorem 5.2.

Suppose now that A_2 separates the surface into two components, and that L_1 and L_2 belong to different path components of $M \setminus A_2$. In this case the argument in the proof of Theorem 5.2 shows that we may assume that πL_2 belongs to a neighborhood of A_2.

Let L_2' be the link obtained from L_2 by a translation such that $\pi L_2'$ belongs to the same path component of $M \setminus A_2$ that contains L_1. Then the destabilization of L_2' along A_2 is descent-equivalent to the destabilization of L_2 along A_2.

Thus, we may assume that L_1 and L_2 belong to the same path component of $M \setminus A_2$; this case has been considered above. ∎

Proof of Lemma 4.1. Recall that we assumed that M is not a sphere and there is a Legendrian link L represented by links L_1 and L_2, and there are two simple closed connected curves A_1 and A_2 that are not null-homotopic such that the destabilization of L_1 along A_1 is not descent-equivalent to the destabilization of L_2 along A_2. Furthermore, we may assume that A_1 and A_2 are geodesics. Indeed, there exists an ambient isotopy φ_t, with $t \in [0, 1]$, of the surface M that takes A_1 into a geodesic. The ambient isotopy of the surface lifts to an isotopy $\tilde{\varphi}_t$ of the spherical cotangent bundle of M. Clearly, the destabilization of the Legendrian link $\tilde{\varphi}_1 L_1$ along $\varphi_1 A_1$ is descent-equivalent to the destabilization of L_1 along A_1. Thus we may assume that A_1 is a geodesic. Similarly, we may find an isotopy ψ_t and its lift $\tilde{\psi}_t$ such that $\psi_1 A_2$ is a geodesic, and the destabilization of L_2 along A_2 is descent-equivalent to the destabilization of $\tilde{\psi}_1 A_2$ along $\psi_1 A_2$. If we now replace the original pairs (L_1, A_1) and (L_2, A_2) with the new pairs $(\tilde{\varphi}_1 L_1, \varphi A_1)$ and $(\tilde{\psi}_1 L_2, \psi_1 A_2)$, then we obtain an example as the original one but with the additional property that the destabilizations are performed along geodesics.

As in the proof of Theorem 5.2, consider a covering $\tilde{M} \to M$ by a surface $\tilde{M}$ homeomorphic to M_1. Take the lift of an isotopy from L_1 to L_2 to a covering isotopy from L_1' to L_2' in $ST^*\tilde{M}$. A crucial observation is that the inverse image A_2' of A_2 in $\tilde{M}$ consists of disjoint geodesics. The parts of these geodesics over cylinders attached to M_1 are easy to visualize. There is an isotopy of $\tilde{M}$ to M_1 that at each time takes the geodesics of A_2' to themselves. This isotopy takes L_1 to a curve disjoint both from A_1 and from A_2. ∎

6. Final remarks. It would be interesting to know the relation between the Legendrian isotopy and the virtual Legendrian isotopy in higher dimensions.

QUESTION 6.1. What is the relation between the Legendrian isotopy and the virtual Legendrian isotopy in higher dimensions?

Two Legendrian knots in respectively ST^*M_1 and ST^*M_2 are virtually Legendrian isotopic if one can be obtained from the other by a sequence of

Legendrian isotopies and modifications of contact manifolds ST^*M corresponding to surgeries of the manifold M away from the projection of the knot. The following conjecture formulated by Cahn and Levi [1] is still open.

Conjecture 6.2. Two Legendrian knots in $ST^*\mathbb{R}^m$ or ST^*S^m are virtual Legendrian isotopic if and only if they are Legendrian isotopic.

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Vladimir Chernov
Department of Mathematics
Dartmouth College
6188 Kemeny Hall
Hanover, NH 03755, U.S.A.
E-mail: Vladimir.Chernov@dartmouth.edu

Rustam Sadykov
Department of Mathematics
Kansas State University
138 Cardwell Hall
Manhattan, KS 66506, U.S.A.
E-mail: rstsdk@gmail.com