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Almost maximal topologies on groups

by

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Abstract. Let G be a countably infinite group. We show that for every finite absolute coretract S, there is a regular left invariant topology on G whose ultrafilter semigroup is isomorphic to S. As consequences we prove that (1) there is a right maximal idempotent in $\beta G\setminus G$ which is not strongly right maximal, and (2) for each combination of the properties of being extremally disconnected, irresolvable, and nodec, except for the combination $(-,-,+)$, there is a corresponding regular almost maximal left invariant topology on G.

1. Introduction. A topological space is called *maximal* if its topology is maximal among all dense in itself topologies. A dense in itself Hausdorff space X is maximal if and only if for every $x \in X$ there is only one nonprincipal ultrafilter on X converging to x. We say that a space X is *almost maximal* if it is dense in itself and for every $x \in X$ there are only finitely many ultrafilters on X converging to x. In [8], assuming Martin's Axiom (MA), an exhaustive construction of countable almost maximal topological groups and countable regular almost maximal left topological groups was given. Recall that a group endowed with a topology is called *left topological* and the topology itself *left invariant* if left translations are continuous. All topologies in the present paper are assumed to satisfy the T_1 separation axiom. The existence of a countable almost maximal topological group cannot be established in ZFC, the system of usual axioms of set theory [6]. In this paper we give an exhaustive construction in ZFC of countable regular almost maximal left topological groups.

Throughout the paper, G will be an arbitrary countably infinite discrete group.

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The operation of G extends to the Stone–Čech compactification βG of G so that, for each $a \in G$, the left translation $\beta G \ni x \mapsto ax \in \beta G$ is continuous, and for each $q \in \beta G$, the right translation $\beta G \ni x \mapsto xq \in \beta G$ is continuous.

We take the points of βG to be the ultrafilters on G, the principal ultrafilters being identified with the points of G, and $G^* = \beta G \setminus G$. The topology of βG is generated by taking as a base the subsets $\overline{A} = \{p \in \beta G : A \in p\}$, where $A \subseteq G$. For $p, q \in \beta G$, the ultrafilter pq has a base consisting of subsets $\bigcup_{x \in A} xB_x$, where $A \in p$ and $B_x \in q$. See [1] for more information about βG.

For every left invariant topology $\mathcal{T}$ on G,

$$\mathrm{Ult}(\mathcal{T}) = \{p \in G^* : p \text{ converges to } 1 \text{ in } \mathcal{T}\}$$

is a closed subsemigroup of G^* called the *ultrafilter semigroup* of $\mathcal{T}$ [2, 3]. Not each closed subsemigroup of G^* is the ultrafilter semigroup of a left invariant topology. However, every finite subsemigroup is [8, Proposition 2.4]. Notice that a left invariant topology is *maximal* [*almost maximal*] if and only if its ultrafilter semigroup is a singleton [finite].

Of special interest are regular almost maximal left invariant topologies. If T is a finite subsemigroup of G^* and $\mathcal{T}$ is the left invariant topology on G with $\mathrm{Ult}(\mathcal{T}) = T$, then $\mathcal{T}$ is regular if and only if

(i) for every $p \in G^* \setminus T$, $(pT) \cap T = \emptyset$, and
(ii) for every $a \in G \setminus \{1\}$, $(aT) \cap T = \emptyset$ ($= \mathcal{T}$ is Hausdorff)

[8, Proposition 2.12]. A subsemigroup T of G^* satisfying conditions (i) and (ii) is called *left saturated*. Notice that (ii) is always satisfied if T is a singleton [1, Theorem 3.34] or T is a finite *band* (= semigroup of idempotents) and G can be embedded algebraically in a compact group [9, Lemma 7.10]. Recall that an element p of a semigroup is an *idempotent* if $pp = p$.

The simplest examples of bands are left zero semigroups ($xy = x$), right zero semigroups ($xy = y$), chains of idempotents ($x \leq y$ if and only if $xy = yx = x$), and rectangular bands (= direct products of a left zero semigroup and a right zero semigroup). Each band is a disjoint union of its maximal rectangular subsemigroups and these are partially ordered by $X \leq Y$ if and only if $XY \subseteq X$, equivalently $YX \subseteq X$.

An object P in some category is a *projective* if for every morphism $f : P \to Q$ and for every surjective morphism $g : R \to Q$, there exists a morphism $h : P \to R$ such that $g \circ h = f$. We say that an object P is an *absolute coretract* if for every surjective morphism $g : R \to P$ there exists a morphism $h : P \to R$ such that $g \circ h = \mathrm{id}_P$. Obviously, each projective is an absolute coretract. In many categories these notions coincide but not in all. Let $\mathfrak{F}$ and $\mathfrak{C}$ denote the categories of finite semigroups and compact Hausdorff right topological semigroups, respectively. Then the finite absolute coretracts and the finite projectives in $\mathfrak{C}$ and in $\mathfrak{F}$ are the same objects, and these are certain chains of rectangular bands; in particular, the finite left (right) zero semigroups and chains of idempotents are such [7].

For every regular almost maximal left invariant topology $\mathcal{T}$ on G, $T = \mathrm{Ult}(\mathcal{T})$ is a projective in $\mathfrak{F}$ [8, Theorem 4.1]. Assuming MA, for every finite absolute coretract S in $\mathfrak{C}$, there is a regular left invariant topology $\mathcal{T}$ on G with $\mathrm{Ult}(\mathcal{T})$ isomorphic to S, and in the case $G = \bigoplus_\omega \mathbb{Z}_2$, $\mathcal{T}$ can be chosen to be a group topology [8, Theorem 5.2 and Lemma 6.10]. Every countable almost maximal topological group contains an open Boolean subgroup, and its existence cannot be established in ZFC [6] (see also [9, Theorem 10.15 and Corollary 10.17]). However, there is in ZFC a regular maximal left invariant topology on G [4]. More generally, for every $n \in \mathbb{N}$, there is in ZFC a regular left invariant topology $\mathcal{T}$ on G with $\mathrm{Ult}(\mathcal{T})$ being a chain of n idempotents [8, Theorem 6.1].

In this paper we prove (in ZFC) the following result.

Theorem 1.1. *For every finite absolute coretract S in $\mathfrak{C}$, there is a regular left invariant topology $\mathcal{T}$ on G with* $\mathrm{Ult}(\mathcal{T})$ *isomorphic to S.*

Theorem 1.1 can be rephrased as follows:

For every finite absolute coretract S in $\mathfrak{C}$, there is a left saturated subsemigroup T of G^* isomorphic to S.

Theorem 1.1 is the complete solution to [8, Question 6] (see also [9, Problem 17]). In fact, this question goes back to the late 1990's, when most of the relevant results had already been proved [5, 6, 4].

From Theorem 1.1 two corollaries follow. To state these, we present some terminology. An idempotent $p \in G^*$ is called

- *right maximal* if for every idempotent $q \in G^*$, $qp = p$ implies $pq = q$,
- *strongly right maximal* if the equation $xp = p$ has the unique solution $x = p$ in G^*.

Taking the 2-element right zero semigroup as S, from Theorem 1.1 we deduce

Corollary 1.2. *There is a right maximal idempotent in G^* which is not strongly right maximal.*

Corollary 1.2 is the answer to a question in [1, p. 192].

A space is called

- *extremally disconnected* if the closure of an open set is open,
- *irresolvable* if it cannot be partitioned into two disjoint dense subsets,
- *nodec* if every nowhere dense subset is closed.

An almost maximal left invariant topology $\mathcal{T}$ on G is

- extremally disconnected if and only if $T = \mathrm{Ult}(\mathcal{T})$ has only one minimal right ideal,
- irresolvable if and only if the smallest ideal $K(T)$ of T is a left zero semigroup,
- nodec if and only if $K(T) = T$

(see [9, Proposition 7.7]).

Corollary 1.3. *For each combination of the properties of being extremally disconnected, irresolvable, and nodec, except for the combination* $(-,-,+)$*, there is a corresponding regular almost maximal left invariant topology on* G*. There is no countable regular almost maximal left topological group corresponding to the combination* $(-,-,+)$.

Corollary 1.3 is a ZFC version of [9, Corollary 10.39]. The proof is the same. In particular, for the combination $(-,+,+)$, apply Theorem 1.1 to the 2-element left zero semigroup.

In fact, we prove a theorem which is a little bit stronger than Theorem 1.1.

Theorem 1.4. *Let* S *be a finite absolute coretract in* $\mathfrak{C}$ *and let* X *be a* G_δ *subset of* G^* *containing an idempotent. Then there is a regular left invariant topology* $\mathcal{T}$ *on* G *such that* $T = \mathrm{Ult}(\mathcal{T})$ *is isomorphic to* S *and* $T \subseteq X$.

The proof of Theorem 1.4 is based on a special construction of regular left invariant topologies and on deep subsets of ω^*.

For every closed subset $Y \subseteq \omega^*$, the *character* of Y in ω^*, denoted $\chi(Y)$, is the minimum cardinality of a family $\mathcal{F}$ of subsets of ω such that $\bigcap_{A\in\mathcal{F}} \overline{A} = Y$. A nonempty closed subset $Z \subseteq \omega^*$ is *deep* if for every closed subset $Y \subseteq \omega^*$ with $\chi(Y) < \mathfrak{c}$, $Y \cap Z$ is either empty or infinite.

Theorem 1.5 ([11, Theorem 3.1]). *There is a deep subset* $Z \subseteq \omega^*$.

As in [11], we use Theorem 1.5 as a replacement of MA.

In Section 2 we discuss first countable regular left invariant topologies. In Section 3 we give that special construction; and in Section 4 we prove Theorem 1.4 itself.

2. First countable regular left invariant topologies

Lemma 2.1. *Let* $\mathcal{T}_0$ *be a Hausdorff* [*regular*] *left invariant topology on* G *and let* $(U_n)_{n<\omega}$ *be any sequence of neighborhoods of* 1 *in* $\mathcal{T}$*. Then* $\mathcal{T}_0$ *can be weakened to a first countable Hausdorff* [*regular*] *left invariant topology* $\mathcal{T}$ *on* G *in which each* U_n *remains a neighborhood of* 1.

Proof. We consider the Hausdorff case; the regular one is [9, Lemma 9.28].

Without loss of generality one may suppose that $U_0 = G$. Enumerate $G \setminus \{1\}$ as $\{x_n : 1 \le n < \omega\}$. Construct inductively a sequence $(V_n)_{n<\omega}$ of open neighborhoods of 1 in $\mathcal{T}_0$ with $V_0 = G$ such that for every $n \ge 1$:

(i) $V_n \subseteq V_{n-1}$,
(ii) $x_n V_n \subseteq V_k$, where $k = \max\{i \le n-1 : x_n \in V_i\}$,
(iii) $(x_n V_n) \cap V_n = \emptyset$, and
(iv) $V_n \subseteq U_n$.

It then follows from (i)–(iii) that there is a Hausdorff left invariant topology $\mathcal{T}$ on G in which $\{V_n : n < \omega\}$ is a neighborhood base at 1 (see [9, Corollary 4.4]), and by (iv), each U_n remains a neighborhood of 1 in $\mathcal{T}$. ∎

For every filter $\mathcal{F}$ on G with $\bigcap \mathcal{F} = \emptyset$, there is a largest left invariant topology $\mathcal{T}[\mathcal{F}]$ on G in which $\mathcal{F}$ converges to 1. The topology $\mathcal{T}[\mathcal{F}]$ has a neighborhood base at 1 consisting of subsets

$$[M] = \{x_0 x_1 \cdots x_n : n < \omega,\ x_0 = 1 \text{ and } x_{i+1} \in M(x_0 \cdots x_i) \text{ for each } i < n\},$$

where $M : G \to \mathcal{F}$ [9, Theorem 4.8].

A filter $\mathcal{F}$ on G is *strongly discrete* if $\bigcap \mathcal{F} = \emptyset$ and there is $M : G \to \mathcal{F}$ such that the subsets $xM(x) \subseteq G$, $x \in G$, are pairwise disjoint.

Theorem 2.2 ([9, Theorem 4.18]). *For every strongly discrete filter $\mathcal{F}$ on G, the topology $\mathcal{T}[\mathcal{F}]$ is regular.*

Lemma 2.3. *Let X be a G_δ subset of G^* containing an idempotent. Then there is a nondiscrete first countable regular left invariant topology $\mathcal{T}$ on G with* $\mathrm{Ult}(\mathcal{T}) \subseteq X$.

Proof. Let $e \in X$ be an idempotent. There is a left invariant topology $\mathcal{T}_0$ on G with $\mathrm{Ult}(\mathcal{T}_0) = \{e\}$. By Lemma 2.1, $\mathcal{T}_0$ can be weakened to a first countable Hausdorff left invariant topology $\mathcal{T}_1$ on G with $\mathrm{Ult}(\mathcal{T}_1) \subseteq X$. Let $\{U_n : n < \omega\}$ be a decreasing neighborhood base at 1 in $\mathcal{T}_1$ and enumerate G without repetitions as $\{x_n : n < \omega\}$. Construct inductively a sequence $(a_n)_{n<\omega}$ in G such that

(i) $a_n \in U_n \setminus (\{a_j : j < n\} \cup \{1\})$, and
(ii) the subsets $x_i\{a_j : i \le j \le n\}$, $i \le n$, are pairwise disjoint.

Then $(a_n)_{n<\omega}$ is a one-to-one sequence in $G \setminus \{1\}$ converging to 1 in $\mathcal{T}_1$ and the subsets $x_n A_n$, $n < \omega$, are pairwise disjoint, where $A_n = \{a_j : n \le j < \omega\}$. Consequently, the filter $\mathcal{F}$ on G with a base of subsets A_n, $n < \omega$, is strongly discrete and converges to 1 in $\mathcal{T}_1$. Let $\mathcal{T}_2 = \mathcal{T}[\mathcal{F}]$. By Theorem 2.2, $\mathcal{T}_2$ is regular, and by Lemma 2.1, $\mathcal{T}_2$ can be weakened to a first countable regular left invariant topology $\mathcal{T}$ on G finer than $\mathcal{T}_1$. ∎

Given a left topological group L and a semigroup S, a mapping $h : L \to S$ is a *local homomorphism* if for every $x \in L$, there is a neighborhood U of 1 such that $h(xy) = h(x)h(y)$ for all $y \in U \setminus \{1\}$. If $h : L \to S$ is a local homomorphism, S is finite, and $\overline{h} : \beta L_d \to S$ is the continuous extension of h, then $\overline{h}|_{\mathrm{Ult}(L)} : \mathrm{Ult}(L) \to S$ is a homomorphism [9, Lemma 8.6]. Given left topological groups L and H, a mapping $h : L \to H$ is a *local isomorphism* if h is a homeomorphism with $h(1) = 1$ and a local homomorphism. If $h : L \to H$ is a local isomorphism and $\overline{h} : \beta L_d \to \beta H_d$ is the continuous extension of h, then $\overline{h}|_{\mathrm{Ult}(L)} : \mathrm{Ult}(L) \to \mathrm{Ult}(H)$ is an isomorphism [9, Lemma 8.4]. Homomorphisms and isomorphisms of ultrafilter semigroups induced by local homomorphisms and local isomorphisms are called *proper*. Endow the countably infinite Boolean group $\bigoplus_\omega \mathbb{Z}_2$ with the topology induced by the product topology on $\prod_\omega \mathbb{Z}_2$ and let $\mathbb{H}$ denote its ultrafilter semigroup. For every countable nondiscrete regular left topological group L, there is a local isomorphism of L onto $\bigoplus_\omega \mathbb{Z}_2$, and consequently there is a proper isomorphism of $\mathrm{Ult}(L)$ onto $\mathbb{H}$ [9, Corollary 8.11].

LEMMA 2.4. *Let $\mathcal{T}$ be a nondiscrete first countable regular left invariant topology on G and let $T = \mathrm{Ult}(\mathcal{T})$. Then T admits a proper homomorphism onto any finite semigroup.*

Proof. Let S be a finite semigroup. Pick a local isomorphism $h : (G, \mathcal{T}) \to \bigoplus_\omega \mathbb{Z}_2$. It is easy to construct a local homomorphism $g : \bigoplus_\omega \mathbb{Z}_2 \to S$ such that for every neighborhood U of 0, $g(U \setminus \{0\}) = S$ (see the proof of [9, Theorem 7.24]). Then $g \circ h : (G, \mathcal{T}) \to S$ is a local homomorphism with the same property, and so $\overline{g \circ h}|_T$ is a proper homomorphism of T onto S. ∎

REMARK 2.5. Lemma 2.4 remains true with "any finite semigroup" replaced by "any compact Hausdorff right topological semigroup R whose topological center contains a countable dense subset of R" (see the proof of [9, Theorem 7.24]).

REMARK 2.6. The existence of a nondiscrete first countable regular left invariant topology $\mathcal{T}$ on G such that $\mathrm{Ult}(\mathcal{T}) \subseteq X$ and $(G, \mathcal{T})$ is locally isomorphic to $\bigoplus_\omega \mathbb{Z}_2$ can be established directly (similarly to the proof of [9, Theorem 7.26]), not involving strongly discrete filters and the local isomorphism theorem, but this direct proof is a little bit longer.

3. Strongly discrete filters. By [10, Lemma 6], there is a surjective finite-to-one function $f : G \to \omega$ such that

(1) $f(1) = 0$,
(2) for every $x \in G$, $f(x) = f(x^{-1})$, and
(3) for all $x, y \in G$, $f(xy) \le \max\{f(x), f(y)\}+1$, and if $|f(x)-f(y)| \ge 2$, then $f(xy) \ge \max\{f(x), f(y)\} - 1$.

The function $f : G \to \omega$ extends continuously to $\beta G \to \beta\omega$. We use the same letter f to denote this extension. Notice that for any $p \in \beta G$ and $q \in G^*$, $f(pq) = f(q) + i$ for some $i \in \{-1, 0, 1\}$.

THEOREM 3.1. *Let $\mathcal{T}$ be a Hausdorff left invariant topology on G and let $(\mathcal{F}_n)_{n<\omega}$ be a sequence of filters on G converging to 1 in $\mathcal{T}$. Suppose that*

(i) *there is a neighborhood U of 1 in $\mathcal{T}$ such that the subsets $f(U\setminus\{1\})+i \subseteq \omega$, $i \in \{-1, 0, 1\}$, are pairwise disjoint,*
(ii) *for every $n < \omega$, there is $A_n \in \mathcal{F}_n$ such that the subsets $f(A_n) \subseteq \omega$, $n < \omega$, are pairwise disjoint.*

Let $\mathcal{F}$ be the filter on G with a base of subsets $\bigcup_{n\le i<\omega} B_i$, where $n < \omega$ and $B_i \in \mathcal{F}_i$. Then $\mathcal{F}$ is strongly discrete.

Proof. For every $n < \omega$, choose a neighborhood U_n of 1 in $\mathcal{T}$ such that

(a) the subsets xU_n, where $x \in G$ with $f(x) \le n$, are pairwise disjoint,

and choose $C_n \in \mathcal{F}_n$ such that

(b) $C_n \subseteq U_n$,
(c) for every $x \in C_n$, $f(x) \ge n + 2$, and
(d) $C_n \subseteq U \cap A_n$.

We claim that the subsets

$$x \bigcup_{n\ge f(x)} C_n,$$

where $x \in G$, are pairwise disjoint.

Let $x, y \in G$, $x \ne y$. Since

$$x \bigcup_{n\ge f(x)} C_n = \bigcup_{n\ge f(x)} xC_n, \qquad y \bigcup_{m\ge f(y)} C_m = \bigcup_{m\ge f(y)} yC_m,$$

it suffices to check that the subsets xC_n and yC_m are disjoint for any $n \ge f(x)$, $m \ge f(y)$. If $n = m$, they are disjoint by (a) and (b). Now let $n \ne m$. Then by (c),

$$f(xC_n) \subseteq \bigcup_{i=-1}^{1} (f(C_n) + i), \qquad f(yC_m) \subseteq \bigcup_{j=-1}^{1} (f(C_m) + j),$$

so by (d),

$$f(xC_n) \subseteq \bigcup_{i=-1}^{1} (f(U \cap A_n) + i), \qquad f(yC_m) \subseteq \bigcup_{j=-1}^{1} (f(U \cap A_m) + j).$$

But by (i) and (ii),

$$\bigcup_{i=-1}^{1} (f(U \cap A_n) + i) \quad \text{and} \quad \bigcup_{j=-1}^{1} (f(U \cap A_m) + j)$$

are disjoint. Consequently, $f(xC_n)$ and $f(yC_m)$ are disjoint, and so are xC_n and yC_m. ■

4. Proof of Theorem 1.4. Let $e \in X$ be an idempotent. Pick $A \in e$ such that the subsets $f(A)+i \subseteq \omega$, $i \in \{-1,0,1\}$, are pairwise disjoint. By Lemma 2.3, there is a nondiscrete first countable regular left invariant topology $\mathcal{T}_0$ on G such that

$$T_0 = \mathrm{Ult}(\mathcal{T}_0) \subseteq X \cap \overline{A}.$$

Since $T_0 \subseteq \overline{A}$, we see that for any $p, q \in T_0$, $f(pq) = f(q)$. By Lemma 2.4, there is a surjective proper homomorphism $\pi : T_0 \to S$. For each $s \in S$, let $X_s = \pi^{-1}(s)$. Notice that X_s is a G_δ subset of G^*. Pick an infinite $D_s \subseteq \omega$ with $D_s^* \subseteq f(X_s)$. By Theorem 1.5, there is a deep subset $Z_s \subseteq D_s^*$. Let

$$J = f^{-1}\Big(\bigcup_{s\in S} Z_s\Big) \cap T_0.$$

Then

(i) J is a closed left ideal of T_0,
(ii) for each $s \in S$, $J \cap X_s \neq \emptyset$,
(iii) $f(J) \subseteq \omega^*$ is deep, and
(iv) $J = f^{-1}(f(J)) \cap T_0$.

Next, enumerate the subsets of G as $\{C_\alpha : \alpha < \mathfrak{c}\}$ with $C_0 = G$, and inductively, for every $\alpha > 0$, construct a first countable regular left invariant topology $\mathcal{T}_\alpha$ on G such that

(1) for each $s \in S$, either $T_\alpha \cap X_s \subseteq \overline{C_\alpha}$ or $T_\alpha \cap X_s \subseteq \overline{G \setminus C_\alpha}$, where $T_\alpha = \mathrm{Ult}(\mathcal{T}_\alpha)$, and
(2) for each $s \in S$, $\bigcap_{\gamma\le\alpha} T_\gamma \cap X_s \cap J \neq \emptyset$.

Fix $\alpha > 0$ and suppose that we have already constructed $\mathcal{T}_\gamma$ for all $\gamma < \alpha$ as required. Let

$$P_\alpha = \bigcap_{\gamma<\alpha} T_\gamma \cap J.$$

By (i), P_α is a closed subsemigroup of T_0, and by (ii) and (2), $\pi(P_\alpha) = S$. Since S is an absolute coretract, there is a homomorphism $\varepsilon_\alpha : S \to P_\alpha$ such that $\pi \circ \varepsilon_\alpha = \mathrm{id}_S$. Let $\mathcal{T}'_\alpha$ be the left invariant topology on G with $\mathrm{Ult}(\mathcal{T}'_\alpha) = \varepsilon_\alpha(S)$. For each $s \in S$, pick $D_{\alpha,s} \in \varepsilon_\alpha(s)$ such that either $D_{\alpha,s} \subseteq C_\alpha$ or $D_{\alpha,s} \subseteq G \setminus C_\alpha$, and let $D_\alpha = \bigcup_{s\in S} D_{\alpha,s}$. By Lemma 2.1, $\mathcal{T}'_\alpha$ can be weakened to a first countable Hausdorff left invariant topology $\mathcal{T}''_\alpha$ such that $T''_\alpha = \mathrm{Ult}(\mathcal{T}''_\alpha) \subseteq \overline{D_\alpha}$. Let

$$Q_\alpha = \bigcap_{\gamma<\alpha} T_\gamma \cap T''_\alpha.$$

For each $s \in S$, $\varepsilon_\alpha(s) \in Q_\alpha \cap X_s \cap J$ and $\chi(Q_\alpha \cap X_s) \le |\alpha| + \omega < \mathfrak{c}$, so by (iii), $f(Q_\alpha \cap X_s) \cap f(J)$ is infinite. For every $n < \omega$ and $s \in S$, choose

$$u^n_{\alpha,s} \in f(Q_\alpha \cap X_s) \cap f(J)$$

and $E^n_{\alpha,s} \in u^n_{\alpha,s}$ such that the subsets $E^n_{\alpha,s} \subseteq \omega$, $n < \omega$ and $s \in S$, are pairwise disjoint.

This can be done by induction on n as follows. For each $s \in S$, pick $u^n_{\alpha,s} \in (f(Q_\alpha \cap X_s) \cap f(J)) \setminus \overline{F^{n-1}_\alpha}$ and $E^n_{\alpha,s} \in u^n_{\alpha,s}$, where $F^{n-1}_\alpha = \bigcup_{j \le n-1,\, s \in S} E^j_{\alpha,s}$, such that (a) the subsets $E^n_{\alpha,s}$, $s \in S$, are pairwise disjoint and disjoint from F^{n-1}_α, and (b) $(f(Q_\alpha \cap X_s) \cap f(J)) \setminus \overline{F^n_\alpha} \neq \emptyset$ for each $s \in S$.

For every $n < \omega$ and $s \in S$, pick $q^n_{\alpha,s} \in Q_\alpha \cap X_s$ such that $f(q^n_{\alpha,s}) = u^n_{\alpha,s}$. By (iv), $q^n_{\alpha,s} \in J$, so

$$q^n_{\alpha,s} \in Q_\alpha \cap X_s \cap J.$$

For every $n < \omega$, let $\mathcal{F}^n_\alpha = \bigcap_{s \in S} q^n_{\alpha,s}$ and $A^n_\alpha = \bigcup_{s \in S} f^{-1}(E^n_{\alpha,s})$. Then $A^n_\alpha \in \mathcal{F}^n_\alpha$ and the subsets $f(A^n_\alpha) \subseteq \omega$, $n < \omega$, are pairwise disjoint. Let $\mathcal{F}_\alpha$ be the filter on G with a base consisting of subsets $\bigcup_{n \le i < \omega} B^i_\alpha$, where $n < \omega$ and $B^i_\alpha \in \mathcal{F}^i_\alpha$, and let $\mathcal{T}'''_\alpha = \mathcal{T}[\mathcal{F}_\alpha]$. By Theorem 3.1, $\mathcal{F}_\alpha$ is strongly discrete, so $\mathcal{T}'''_\alpha$ is regular. By Lemma 2.1, $\mathcal{T}'''_\alpha$ can be weakened to a first countable regular left invariant topology $\mathcal{T}_\alpha$ finer than $\mathcal{T}''_\alpha$. Clearly, condition (1) is satisfied. To see (2), let q be any limit point of $\{q^n_{\alpha,s} : n < \omega\}$. Then $\mathcal{F}_\alpha \subseteq q$ and $q \in \bigcap_{\gamma < \alpha} T_\gamma \cap X_s \cap J$, so $q \in \bigcap_{\gamma \le \alpha} T_\gamma \cap X_s \cap J$.

Finally, let $\mathcal{T}$ be the least upper bound of topologies $\mathcal{T}_\alpha$, $\alpha < \mathfrak{c}$. That is, $\mathcal{T}$ is the left invariant topology on G with a neighborhood base at 1 consisting of subsets $\bigcap_{i \le n} U_{\alpha_i}$, where $n < \omega$, $\alpha_0 < \cdots < \alpha_n < \mathfrak{c}$, and U_{α_i} is a neighborhood of 1 in $\mathcal{T}_{\alpha_i}$ for each $i \le n$. Then $T = \mathrm{Ult}(\mathcal{T}) = \bigcap_{\alpha < \mathfrak{c}} T_\alpha$. If each U_{α_i} is closed in $\mathcal{T}_{\alpha_i}$, then $\bigcap_{i \le n} U_{\alpha_i}$ is closed in $\mathcal{T}$. Consequently, $\mathcal{T}$ is regular. Since $T_0 \subseteq X$, one has $T \subseteq X$. By (1) and (2), $T \cap X_s$ is a singleton for each $s \in S$. Hence, $T \ni p \mapsto \pi(p) \in S$ is an isomorphism.

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